Available online at www.sciencedirect.com
SciVerse ScienceDirect
Advances in Mathematics 231 (2012) 1598–1618
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

The Alexandroff one-point compactification as a prototype for extensions

Gerald Beer[a,*], Maria Cristina Vipera[b]

[a] *Department of Mathematics, California State University Los Angeles, 5151 State University Drive, Los Angeles, CA 90032, USA*
[b] *Dipartimento di Matematica, Università di Perugia, Via Vanvitelli 1, 06123 Perugia, Italy*

Received 3 November 2011; accepted 12 July 2012
Available online 10 August 2012

Communicated by the Managing Editors of AIM

Abstract

Using the Alexandroff one-point compactification as a point of departure, we study a general procedure for building an extension $\langle X \cup I, \tau_0 \rangle$ of a topological space $\langle X, \tau \rangle$, given a family $\{\mathscr{B}_i : i \in I\}$ of nontrivial closed ideals on X, indexed by the intended remainder I.

MSC: primary 54D35; 46A17

Keywords: Bornological extension; Strict extension; Ideal; Bornology; One-point extension; Alexandroff compactification; Wallman extension; Stone–Čech compactification; Metric completion; Semiregular space

1. Introduction

Casual students of point-set topology may not retain much from an introduction to the subject, but among those constructions that surely have staying power is the Alexandroff one-point compactification of a locally compact noncompact Hausdorff space $\langle X, \tau \rangle$. One adjoins an ideal point i to X and takes as open sets all elements of τ plus all sets of the form $\{X \setminus E \cup \{i\} : E \text{ is compact}\}$ [23, p. 136]. In contrast, the Stone–Čech compactification of

* Corresponding author.
E-mail addresses: gbeer@cslanet.calstatela.edu (G. Beer), vipera@dmi.unipg.it (M.C. Vipera).

0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.07.008

a noncompact Tychonoff space, whether it is approached through the Tychonoff embedding theorem or ultrafilters of zero sets, seems not only more complicated but far removed from the one-point compactification. While the topology of the completion of a noncomplete metric space is much more accessible, this too seems to be fundamentally unrelated to the one-point compactification construction.

The aim of this paper is to provide a framework to understand the most important extension topologies — including those just mentioned — as refinements of the one-point compactification construction. Throughout, we reassert the naive approach to extensions, that of adjoining points to an initial space, as an alternative to the modern approach of embedding an initial space into a larger structure, e.g., a filter system (see, e.g., [19, Chapter 17] or [17, Section 7.1]).

The point of departure to this understanding is to index a base for the one-point compactification by the τ-closed subsets of X. Given $E \subseteq X$ with $X \setminus E \in \tau$, define $W_E \subseteq X \cup \{i\}$ by

$$W_E = \begin{cases} X \setminus E \cup \{i\} & \text{if } E \text{ is compact} \\ X \setminus E & \text{otherwise.} \end{cases}$$

Clearly $\{W_E : E\ \tau\text{-closed}\}$ contains all open neighborhoods of the ideal point plus some members of τ. To show that it forms a base for the extension topology, we need only show that it contains this base for τ: the subfamily of open sets with compact closure. But if $V \in \tau$ has compact closure, then since X is assumed noncompact, $X \setminus V$ cannot be compact, whence $V = W_{X\setminus V}$.

Now in the present context, the family $\mathscr{A}$ of relatively compact subsets (i) forms an hereditary family closed under finite unions, (ii) $A \in \mathscr{A} \Rightarrow \mathrm{cl}(A) \in \mathscr{A}$, and (iii) $X \notin \mathscr{A}$. With respect to conventional usage, this means that $\mathscr{A}$ is a nontrivial ideal of subsets of X with a closed base. For our basic open sets for the extension topology, we adjoin i to the complement of a τ-closed set E if and only if E belongs to the ideal. How should we organize things more generally if we have many ideal points, each corresponding to such an ideal?

Let $\langle X, \tau\rangle$ be a topological space and let I be a nonempty set disjoint from X. Suppose $\{\mathscr{A}_i : i \in I\}$ is a family of nontrivial ideals on X such that for each $i \in I$ and for each $E \in \mathscr{A}_i$, $\mathrm{cl}_\tau(E) \in \mathscr{A}_i$. Then all sets of the form

$$(X \setminus E) \cup \{i \in I : E \in \mathscr{A}_i\} \quad (E\,\tau\text{-closed})$$

form a base for an extension topology τ_0 on $X \cup I$. It turns out that different families of ideals determine distinct extension topologies. In fact, we can recover the original family of ideals from the extension topology that it determines, in a way that we now describe.

Each extension $\langle X \cup I, \sigma\rangle$ of $\langle X, \tau\rangle$ naturally induces such a family of ideals. For each $i \in I$, let $\mathscr{W}_i$ be the neighborhood system at i as determined by σ. Upon taking relative complements of the members of each $\mathscr{W}_i$ in X, we obtain for each $i \in I$ an ideal $\mathscr{B}_i(\sigma)$ of proper subsets of X which is stable under taking closures of its elements. When the extension is T_1, each ideal is a bornology on X, that is, an ideal that is also a cover of X.

We can define a second extension topology $\tau_0(\sigma)$ by performing the procedure outlined in the first paragraph on the family of ideals $\{\mathscr{B}_i(\sigma) : i \in I\}$. Indeed, this is a well-known procedure perhaps first explicitly described in dual form by Banaschewski using filters (see, e.g., [3,17,19,20]). It can be shown that the basic open set determined by a τ-closed set E is the largest σ-open set whose intersection with X yields $X \setminus E$ and, as a result, the topology $\tau_0(\sigma)$ is weaker than σ. The key fact here is this: if σ happens to arise from $\{\mathscr{A}_i : i \in I\}$ through our procedure, then for each $i \in I$, we must have $\mathscr{B}_i(\sigma) = \mathscr{A}_i$.

We call an extension $\langle X \cup I, \sigma\rangle$ that can be obtained from a family of ideals on X via our construction a *bornological extension*. By the concluding remark of the last paragraph, this is equivalent to the property $\sigma = \tau_0(\sigma)$; in Banaschewski's terminology, our bornological extensions must be strict extensions. We choose the descriptor "bornological" because the most important extensions are T_1.

We take care to explain how standard approaches to various classical extensions fall within our program when seen in the proper light: the Wallman extension of a T_1 space, the Stone–Čech compactification of a Tychonoff space, the completion of a metric space, and the Dedekind–MacNeille compactification of the order topology. Particular attention is paid to one-point (not necessarily compact) extensions.

We give several characterizations of bornological extensions. Most notable is the following one stated in terms of an approximation property for closed sets: an extension $\langle X \cup I, \sigma\rangle$ is bornological if and only if each σ-closed subset C can be expressed as $\cap\{\mathrm{cl}_\sigma(A) : A \subseteq X \text{ and } C \subseteq \mathrm{cl}_\sigma(A)\}$. From a slight variant of this, we obtain an attractive proof of the following old result of Stone [3,17,18]: all semiregular extensions $\langle X \cup I, \sigma\rangle$ are strict extensions. From another characterization, we easily describe compactness of the extension.

It can happen that a topological space is a bornological extension of some of its dense subspaces but not of the others. In fact, the binary relation of one space being a bornological extension of a second space fails to be transitive! We characterize those spaces that are bornological extensions of each of its dense subspaces.

While it can happen that our construction produces an extension topology τ_0 that contains our initial topology τ — as is the case with the Alexandroff one-point compactification — this is not to be expected. An extension of $\langle X, \tau\rangle$ is called a *strong bornological extension* if its topology σ is generated by $\tau \cup \tau_0(\sigma)$. We study these as well.

At the end of the paper, we consider the lower separation axioms for bornological extensions up through regularity.

2. Preliminaries

Let $\langle X, \tau\rangle$ be a topological space. We write $\mathscr{P}(X)$ for the power set of X and $\mathscr{C}(X)$ for the family of closed subsets of X. As frequently more than one topology will be under consideration at a given moment, if $E \subseteq X$, we denote its closure and interior with respect to τ by $\mathrm{cl}_\tau(E)$ and $\mathrm{int}_\tau(E)$, respectively.

A nonempty hereditary family of subsets of X closed under finite unions is called an *ideal* or a *boundedness* [8,13] Evidently, each ideal must contain the empty set. Notice that if an ideal contains X, then it must coincide with $\mathscr{P}(X)$. We call $\mathscr{P}(X)$ the *trivial ideal*.

Given a nonempty family of subsets $\mathscr{A}$ of X, there is a smallest ideal containing $\mathscr{A}$. Writing $\downarrow \mathscr{A}$ for $\{B \in \mathscr{P}(X) : \exists A \in \mathscr{A} \text{ with } B \subseteq A\}$ and $\Sigma(\mathscr{A})$ for the family of all finite unions of members of $\mathscr{A}$, this smallest ideal can be expressed as either $\downarrow \Sigma(\mathscr{A})$ or $\Sigma(\downarrow \mathscr{A})$. As expected, this is called the *ideal generated by* $\mathscr{A}$.

We call a subfamily $\mathscr{B}_0$ of an ideal $\mathscr{B}$ a *base* for $\mathscr{B}$ if $\mathscr{B} =\downarrow \mathscr{B}_0$. We say that an ideal is *closed* (resp. *open*) if it has a closed (resp. open) base. Denoting the closed members of $\mathscr{B}$ by $\widehat{\mathscr{B}}$, to say that $\mathscr{B}$ is closed means that $\mathscr{B} =\downarrow \widehat{\mathscr{B}}$.

By a *bornology* $\mathscr{B}$ on a set X, we mean an ideal of subsets that also forms a cover of X [4,6,12,14,21]. Clearly a bornology must contain all singleton subsets as well as the empty set. The finite subsets of X form the smallest bornology on X while the power set is the largest bornology. We call a bornology $\mathscr{B}$ *local* if it contains a neighborhood of each point of X.

Some notable bornologies are: (1) the bornology of relatively compact subsets of a topological space; (2) the bornology of metrically bounded subsets of a metric space (this bornology is both closed and open); (3) the bornology of totally bounded subsets of a uniform space (this bornology is closed but may not be open); (4) the bornology of Bourbaki bounded subsets of a uniform space [7,21]; (5) the bornology of all subsets of a set X on which $f : X \to \mathbb{R}$ is bounded. The nowhere dense subsets of a topological space X form an ideal that is a bornology if and only if each point of X is a limit point of X.

3. The bornological extension topologies

Suppose $\langle X, \tau\rangle$ is a topological space and $\langle X \cup I, \sigma\rangle$ is an extension of $\langle X, \tau\rangle$. For complete clarity, this means that I is a nonempty set disjoint from X, each nonempty element of σ hits X, and the relative topology that X inherits from $\langle X \cup I, \sigma\rangle$ coincides with τ. For each $i \in I$,

$$\{X \setminus W : W \in \sigma, i \in W\}$$

is a closed base for an ideal $\mathscr{B}_i(\sigma)$ on X, and $\mathscr{B}_i(\sigma) = \{E \subseteq X : i \notin \mathrm{cl}_\sigma(E)\} = \{E \subseteq X : \exists W \in \sigma \text{ such that } i \in W,\ W \cap E = \emptyset\}$. By the denseness of X, the ideal is nontrivial.

If the extension is T_1, the ideal becomes a bornology. More generally this is true, if given any point x of X, there is a neighborhood of i that is disjoint from $\{x\}$.

Thus, each extension $\langle X \cup I, \sigma\rangle$ of $\langle X, \tau\rangle$ gives rise to a family of nontrivial closed ideals $\{\mathscr{B}_i(\sigma) : i \in I\}$ on X. A basic question to be addressed is the following: when does this family of ideals determine σ in a natural way?

We will now describe a natural way to associate an extension of X to every family of nontrivial closed ideals on X. Let $\langle X, \tau\rangle$ be a topological space and let I be a nonempty set disjoint from X. Suppose $\{\mathscr{B}_i : i \in I\}$ is a family of nontrivial closed ideals on X. For each $E \in \mathscr{C}(X)$, put

$$U_E := \{i \in I : E \in \mathscr{B}_i\}, \qquad V_E = X \setminus E.$$

Lemma 3.1. *Let $\{\mathscr{B}_i : i \in I\}$ be a family of nontrivial closed ideals on $\langle X, \tau\rangle$. Then the family $\{U_E \cup V_E : E \in \mathscr{C}(X)\}$ is closed under finite intersections.*

Proof. For E, F closed in X, one has

$$U_E \cap U_F = \{i \in I : E \in \mathscr{B}_i \text{ and } F \in \mathscr{B}_i\} = \{i \in I : E \cup F \in \mathscr{B}_i\} = U_{E\cup F};$$

hence,

$$(U_E \cup V_E) \cap (U_F \cup V_F) = (U_E \cap U_F) \cup (V_E \cap V_F) = U_{E\cup F} \cup V_{E\cup F}. \quad \square$$

Since each ideal contains the empty set, it is clear that $U_\emptyset \cup V_\emptyset = X \cup I$. Since $\{U_E \cup V_E : E \in \mathscr{C}(X)\}$ is closed under finite intersections, this family forms a base for a topology $\tau_0(\{\mathscr{B}_i : i \in I\})$ on $X \cup I$ [23, Theorem 5.3]. When no ambiguity can result, we just write τ_0 for this topology.

Proposition 3.2. *Let $\langle X, \tau\rangle$ be a topological space, and let $\{\mathscr{B}_i : i \in I\}$ be a family of nontrivial closed ideals on X. Then $\langle X \cup I, \tau_0(\{\mathscr{B}_i : i \in I\})\rangle$ is an extension of X.*

Proof. The relative topology on X determined by τ_0 is contained in τ as the intersection of each basic open set with X evidently lies in τ. To show the reverse inclusion, let $W \in \tau$ be a nonempty proper subset of X. Evidently,

$$W = (U_{X\setminus W} \cup V_{X\setminus W}) \cap X$$

which shows that W is relatively open.

To see that X is dense, suppose $i \in I$ belongs to some basic open set $U_E \cup V_E$. Then $i \in U_E$ so that $E \in \mathscr{B}_i$ which means $E \neq X$ and thus $(U_E \cup V_E) \cap X = V_E \cap X \neq \emptyset$. $\square$

We will call $\langle X \cup I, \tau_0(\{\mathscr{B}_i : i \in I\})\rangle$ the *bornological extension* of X determined by $\{\mathscr{B}_i : i \in I\}$ and τ_0 the associated *bornological extension topology*.

The family $\{U_E \cup V_E : E \in \mathscr{C}(X)\} \cup \tau$ is also closed under finite intersections and thus forms a base for a formally stronger topology τ_0^s on $X \cup I$. We call $\langle X \cup I, \tau_0^s\rangle$ the *strong bornological extension* of X with respect to $\{\mathscr{B}_i : i \in I\}$. In this extension in which X is also dense, the remainder is closed. Notice that for each $i \in I$, $\{W \in \tau_0 : i \in W\}$ is a local base for τ_0^s at i, and in particular, the two topologies restricted to the remainder agree. Presently, we will see when the two extensions coincide.

Definition 3.3. We will say that the extension $\langle X \cup I, \sigma\rangle$ of X is a *bornological extension* (resp. *strong bornological extension*) if there exists a family $\{\mathscr{B}_i : i \in I\}$ of nontrivial closed ideals on X such that $\sigma = \tau_0(\{\mathscr{B}_i : i \in I\})$ (resp. $\sigma = \tau_0^s(\{\mathscr{B}_i : i \in I\})$).

For an extension $\langle X \cup I, \sigma\rangle$, recall that we denoted by $\{\mathscr{B}_i(\sigma) : i \in I\}$ the family of closed ideals naturally determined by σ. We will denote by $\tau_0(\sigma)$ the bornological extension topology $\tau_0(\{\mathscr{B}_i(\sigma) : i \in I\})$ and by $\tau_0^s(\sigma)$ the corresponding strong bornological extension topology.

Given a bornological extension $\langle X \cup I, \tau_0(\{\mathscr{B}_i : i \in I\})\rangle$, it would seem possible that there could be a different family of closed nontrivial ideals $\{\mathscr{A}_i : i \in I\}$ on X such that $\tau_0(\{\mathscr{B}_i : i \in I\}) = \tau_0(\{\mathscr{A}_i : i \in I\})$. The next result rules this out.

Proposition 3.4. *Let $\{\mathscr{A}_i : i \in I\}$ be a family of nontrivial closed ideals on $\langle X, \tau\rangle$ and let τ_0 be the topology on $X \cup I$ that it induces. Then $\forall i \in I$, we have $\mathscr{A}_i = \mathscr{B}_i(\tau_0)$.*

Proof. Since both $\mathscr{A}_i$ and $\mathscr{B}_i(\tau_0)$ have closed bases, we need only show that for each index i, we have $\widehat{\mathscr{A}_i} = \widehat{\mathscr{B}_i}(\tau_0)$.

Suppose first that $E \in \widehat{\mathscr{B}_i}(\tau_0)$; by definition, there exists $W \in \tau_0$ with $i \in W$ and $W \cap E = \emptyset$. In view of the standard base for τ_0, $\exists F \in \mathscr{C}(X)$ such that $i \in U_F \cup V_F$ and $(U_F \cup V_F) \cap E = \emptyset$. It follows that $E \subseteq F$, and thus, $E \in \mathscr{A}_i$ because $F \in \mathscr{A}_i$ and $\mathscr{A}_i$ is hereditary. We conclude that $E \in \widehat{\mathscr{A}_i}$ because E is τ-closed.

For the reverse inclusion, suppose $E \in \widehat{\mathscr{A}_i}$ so that $i \in U_E$. Then $U_E \cup V_E$ is a τ_0-neighborhood of i disjoint from E, and so $E \in \widehat{\mathscr{B}_i}(\tau_0)$. $\square$

Corollary 3.5. *Suppose $\{\mathscr{B}_i : i \in I\}$ is a family of nontrivial closed ideals on $\langle X, \tau\rangle$. Then $\tau_0(\tau_0(\{\mathscr{B}_i : i \in I\})) = \tau_0(\{\mathscr{B}_i : i \in I\})$.*

Banaschewski [3] called an extension topology *strict* if it satisfies condition (2) of the next summary result, which says that all extensions that arise from our construction are strict extensions in his sense and conversely. It is anticipated by a parallel understanding in the context of filter systems (see, e.g., [3, pp. 4–8] or [19, pp. 133-137]).

Theorem 3.6. *Let $\langle X \cup I, \sigma\rangle$ be an extension of $\langle X, \tau\rangle$. The following conditions are equivalent:*

(1) *$\langle X \cup I, \sigma\rangle$ is a bornological extension;*
(2) *$\sigma = \tau_0(\sigma)$;*
(3) *there exists an extension $\langle X \cup I, \mu\rangle$ such that $\sigma = \tau_0(\mu)$.*

Proof. (1) $\Rightarrow$ (2) follows immediately from Corollary 3.5 and (2) $\Rightarrow$ (3) is trivial. For (3) $\Rightarrow$ (1), we simply recall that $\tau_0(\mu) = \tau_0(\{\mathscr{B}_i(\mu) : i \in I\})$. $\square$

Of course $\langle X \cup I, \sigma\rangle$ will be a strong bornological extension of $\langle X, \tau\rangle$ if and only if $\sigma = \tau_0^s(\sigma)$, or, equivalently, if σ is generated by $\tau_0(\sigma) \cup \tau$.

Let $\mathfrak{E}$ denote the family of all topologies σ on $X \cup I$ that determine extensions of $\langle X, \tau\rangle$. Define an operator $\Phi : \mathfrak{E} \to \mathfrak{E}$ by $\Phi(\sigma) = \tau_0(\sigma)$. From Corollary 3.5 we obtain, in particular, $\tau_0(\tau_0(\sigma)) = \tau_0(\sigma)$; that is, Φ is an idempotent operator whose range is the family of bornological extension topologies for $\langle X, \tau\rangle$.

In general, $\tau_0(\sigma)$ will be weaker than σ [3, p. 5]. This is an immediate consequence of the following key lemma, the second part of which describes in a tangible way how the standard basic open sets for $\tau_0(\sigma)$ arise.

Lemma 3.7. *Let $\langle X \cup I, \sigma\rangle$ be an extension of $\langle X, \tau\rangle$, and $\forall i \in I$, let $\mathscr{B}_i(\sigma)$ be the ideal generated by $\{X \setminus W : W \in \sigma, i \in W\}$. Then for each $E \in \mathscr{C}(X)$, one has*

$$(X \cup I) \setminus cl_\sigma(E) = U_E \cup V_E.$$

Thus, $U_E \cup V_E$ is the largest element of σ whose intersection with X is $X \setminus E$.

Proof. Clearly, the points in X that are in both sets are the same and comprise $X \setminus E$. Put $W = (X \cup I) \setminus \mathrm{cl}_\sigma(E)$, and suppose $i \in I \cap W$. Then $E = X \setminus W \in \widehat{\mathscr{B}_i}(\sigma)$; hence $i \in U_E$. Conversely, let $i \in U_E$. Then $E \in \mathscr{B}_i(\sigma)$ and so there exists $W \in \sigma$ with $E \cap W = \emptyset$ and $i \in W$. Since $i \in W$, one has $i \in (X \cup I) \setminus \mathrm{cl}_\sigma(E)$.

From the equality we have proved, it follows that $U_E \cup V_E$ is an element of σ and clearly its intersection with X is $X \setminus E = V_E$. If $W \in \sigma$ and $W \cap X = X \setminus E$, then $W \cap E = \emptyset$; hence $W \subseteq (X \cup I) \setminus \mathrm{cl}_\sigma(E) = U_E \cup V_E$. □

Corollary 3.8. *Let $\langle X \cup I, \sigma\rangle$ be an extension of $\langle X, \tau\rangle$. Then $\tau_0(\sigma) \subseteq \sigma$.*

We now give some easy examples that consider whether or not a particular extension is a bornological extension.

Example 3.9. We give here an example of a one-point extension that is a bornological extension. Let $X = \{x_1, x_2\}$ and let $I = \{i\}$. Our extension topology is given by

$$\sigma = \{\emptyset, X \cup I, \{x_2, i\}\}.$$

This induces the relative topology $\tau = \{\emptyset, X, \{x_2\}\}$ on X, and we also have $\mathscr{B}_i(\sigma) = \{\emptyset, \{x_1\}\}$. As $\mathscr{C}(X) = \{\emptyset, X, \{x_1\}\}$, and $U_\emptyset \cup V_\emptyset = X \cup I$, $U_X \cup V_X = \emptyset$, and $U_{\{x_1\}} \cup V_{\{x_1\}} = \{x_2, i\}$, the extension topology σ agrees with $\tau_0(\sigma)$.

Note that $\langle X \cup I, \sigma\rangle$ fails to be a strong bornological extension (cf. Proposition 5.14 below), and for future reference, that σ is not regular as $\{x_2, i\}$ contains no closed neighborhood of x_2. In fact σ fails to have a local base at x_2 consisting of regular open sets.

Example 3.10. Next is an example of an extension with a two-point closed remainder that is not a bornological extension. Let $X = \{x\}$, let $I = \{i_1, i_2\}$ and let

$$\sigma = \{\emptyset, X \cup I, \{x, i_1\}, \{x, i_2\}, \{x\}\}.$$

Here, $\mathscr{B}_{i_1}(\sigma) = \mathscr{B}_{i_2}(\sigma) = \{\emptyset\}$, and the relative topology is $\{\emptyset, X\}$. For $\tau_0(\sigma)$, we only get the indiscrete topology $\{\emptyset, X \cup I\}$. Note that $\tau_0^s(\sigma) = \{\emptyset, X, X \cup I\}$ is properly weaker than σ as well.

In the last two examples, the induced family of ideals $\{\mathscr{B}_i(\sigma) : i \in I\}$ failed to be a family of bornologies. In the next two examples, we do obtain bornologies, as both the extensions satisfy the T_1 separation axiom.

Example 3.11. Let $\langle X \cup I, \sigma\rangle$ be a pseudo-metrizable extension of $\langle X, \tau\rangle$. We will show that $\langle X \cup I, \sigma\rangle$ is a bornological extension of X. As announced in the Introduction, we will strengthen this result in the sequel, but using very different machinery.

Let d be a compatible pseudo-metric for σ. For each $i \in I$, it is clear that $\mathscr{B}_i(\sigma) = \{E \in \mathscr{P}(X) : d(i, E) > 0\}$, where the distance from any point to the empty set is understood to be ∞. To show $\sigma \subseteq \tau_0(\sigma)$, let $w \in W \in \sigma$, and choose $n \in \mathbb{N}$ with

$$\left\{u \in X \cup I : d(u, w) < \frac{1}{n}\right\} \subseteq W.$$

Put $E = \{x \in X : d(x, w) \geq \frac{1}{2n}\}$. We claim that $w \in U_E \cup V_E \subseteq W$. If $w \in X$, since $V_E = X \setminus E$ and $d(w, w) = 0 < \frac{1}{2n}$, we get $w \in V_E$. On the other hand, if $w \in I$, then $d(w, E) > 0$ gives $E \in \mathscr{B}_w(\sigma)$ which means $w \in U_E$.

For the inclusion, it is obvious that $V_E \subseteq W$: if $x \in V_E = X \setminus E$, we have $d(x, w) < \frac{1}{2n} < \frac{1}{n}$. To show $U_E \subseteq W$, we show that if $i \in I \setminus W$, then $i \notin U_E$. Since $d(i, w) \geq \frac{1}{n}$ and X is dense in the extension, we can find a sequence $\langle x_k\rangle$ in X that is convergent to i such that for all k, $d(x_k, w) > \frac{1}{2n}$. By definition, each x_k lies in E, and so $d(i, E) = 0$ which means that $i \notin U_E$, as required.

Example 3.12. Let X be an infinite set and let I be nonempty. Equipping $X \cup I$ with the cofinite topology σ [23, p. 26] produces an extension of X, the relative topology τ being the cofinite topology on X. Note that for each $i \in I$, we have

$$\mathscr{B}_i(\sigma) = \{E \subseteq X : E \text{ is finite}\},$$

and

$$\tau_0(\sigma) = \{\emptyset, X \cup I\} \cup \{I \cup (X \setminus E) : E \subseteq X \text{ and } E \text{ is finite}\}.$$

If I is a singleton, then $X \in \sigma \setminus \tau_0(\sigma)$. On the other hand, if I contains at least two distinct members, let I_1 be a nonempty proper subset of I that is cofinite in I. Then $X \cup I_1 \in \sigma \setminus \tau_0(\sigma)$. Thus the cofinite topology on $X \cup I$ fails to be a bornological extension.

The following result allows us to easily prove by an example that there are Hausdorff extensions which are not bornological, while, as we will prove in Section 5, all regular extensions are bornological extensions.

Proposition 3.13. *Let $\langle X \cup I, \sigma\rangle$ and $\langle X \cup I, \mu\rangle$ be extensions of $\langle X, \tau\rangle$. Suppose for each $i \in I$, the relative topology that $X \cup \{i\}$ inherits from σ coincides with the one inherited from μ. Then $\tau_0(\sigma) = \tau_0(\mu)$. Moreover, if $\langle X \cup I, \sigma\rangle$ is a bornological extension, then $\sigma \subseteq \mu$.*

Proof. By Lemma 3.7, a closed subset E of X is in $\widehat{\mathscr{B}_i}(\sigma)$ if and only if $i \notin \mathrm{cl}_\sigma(E)$ that is, if and only if E is closed in $X \cup \{i\}$. Thus if σ and μ induce the same topology on $X \cup \{i\}$, then $\widehat{\mathscr{B}_i}(\sigma) = \widehat{\mathscr{B}_i}(\mu)$. Therefore, under our assumption, $\tau_0(\sigma) = \tau_0(\mu)$.

If σ is a bornological extension topology, using Theorem 3.6 and Corollary 3.8, we obtain

$$\sigma = \tau_0(\sigma) = \tau_0(\mu) \subseteq \mu. \quad \square$$

Example 3.14. Let X be the open upper half plane in $\mathbb{R} \times \mathbb{R}$, and let I be the x-axis. Let σ be the Euclidean topology on $X \cup I$ and let τ be the relative topology on X. For every $i \in I$, and $r \in (0, \infty)$, put

$$U_{i,r} = \{p \in X : d(p, i) < r\},$$

where d is the Euclidean metric. Then the family of subsets of $X \cup I$

$$\tau \cup \{\{i\} \cup U_{i,r} : i \in I,\ r > 0\}$$

is clearly a base for a finer Hausdorff topology μ on $X \cup I$ which induces the discrete topology on I and which is also an extension topology for $\langle X, \tau\rangle$. As $\{\{i\} \cup U_{i,r} : i \in I,\ r > 0\}$ forms a local base at i for both relative topologies on $X \cup \{i\}$, the one-point extensions agree. By metrizability of σ, we have $\tau_0(\sigma) = \sigma$. Since $\tau \subseteq \sigma$, we also have $\sigma = \tau_0^s(\sigma)$. From the previous proposition we obtain $\tau_0(\mu) = \tau_0(\sigma) = \sigma$. Moreover, $\tau_0^s(\mu)$, being generated by $\tau \cup \tau_0(\sigma) = \sigma$ is in fact, equal to σ; hence, $\langle X \cup I, \mu\rangle$ is neither a bornological nor a strong bornological extension.

The remainder of this section contains assorted other results regarding bornological extensions. The next result will be applied in Section 5.

Proposition 3.15. *Let $\langle X, \tau\rangle$ be a topological space, and let $\langle X \cup I, \tau_0\rangle$ be determined by a family $\{\mathscr{B}_i : i \in I\}$ of nontrivial closed ideals on X. Then a subset W of X belongs to τ_0 if and only $\forall x \in W,\ \exists E_x \in \mathscr{C}(X)$ such that $E_x \notin \cup_{i \in I} \mathscr{B}_i$ and $x \in X \setminus E_x \subseteq W$.*

Proof. We begin with necessity. If $W = \emptyset$, the condition is satisfied vacuously. Otherwise, let $x \in W$ be arbitrary. In consideration of the standard base for the extension, there exists $E_x \in \mathscr{C}(X)$ with

$$x \in U_{E_x} \cup V_{E_x} \subseteq W.$$

Since $U_{E_x} \cap X = \emptyset$, it follows that $\forall i \in I,\ E_x \notin \mathscr{B}_i$ and $x \in X \setminus E_x \subseteq W$.

For sufficiency, simply observe that for all $x \in W,\ U_{E_x} = \emptyset$, and we get

$$W = \bigcup_{x \in W} X \setminus E_x = \bigcup_{x \in W} V_{E_x} = \bigcup_{x \in W} (U_{E_x} \cup V_{E_x}) \in \tau_0. \quad \square$$

Using this proposition we can immediately describe when the remainder is closed, i.e., when the bornological extension agrees with the strong bornological extension.

Corollary 3.16. *Let $\langle X, \tau\rangle$ be a topological space, and let $\langle X \cup I, \tau_0\rangle$ be determined by a family $\{\mathscr{B}_i : i \in I\}$ of nontrivial closed ideals on X. The following conditions are equivalent:*

(i) $\tau_0 = \tau_0^s$;
(ii) $X \in \tau_0$;
(iii) *there exists $\Delta \subseteq \mathscr{C}(X)$ with $\cap\Delta = \emptyset$ and $\forall E \in \Delta,\ \forall i \in I,\ E \notin \mathscr{B}_i$.*

Proof. Conditions (i) and (ii) are equivalent because τ is the relative topology on X and so $X \in \tau_0$ is equivalent to $\tau \subseteq \tau_0$. In view of Proposition 3.15, conditions (ii) and (iii) are equivalent because $\cap\Delta = \emptyset$ means $\cup_{E \in \Delta} X \setminus E = X$. $\square$

In a bornological extension of $\langle X, \tau\rangle$, it is always the case that X must be dense. We now see when the remainder is dense (note that the remainder cannot be dense in a strong bornological extension).

Proposition 3.17. *Let $\langle X, \tau\rangle$ be a topological space, and let $\langle X \cup I, \tau_0\rangle$ be determined by a family $\{\mathscr{B}_i : i \in I\}$ of nontrivial closed ideals on X. Then I is τ_0-dense if and only if whenever $E \in \mathscr{C}(X)$ and $E \neq X$, then $\exists i \in I$ with $E \in \mathscr{B}_i$.*

Proof. For sufficiency, suppose $E \in \mathscr{C}(X)$ and $U_E \cup V_E$ is nonempty. This means that E is a proper subset of X, and by choosing i with $E \in \mathscr{B}_i$, we have $i \in U_E \subseteq U_E \cup V_E$. Thus, I is dense. For necessity, suppose I is dense and E is a closed proper subset of X. Then $U_E \cup V_E$ is nonempty as $X \setminus E$ is nonempty, so for some $i \in I$, we have $i \in U_E$, i.e., $E \in \mathscr{B}_i$. □

Let $\{\mathscr{B}_i : i \in I\}$ be a family of nontrivial closed ideals in $\langle X, \tau\rangle$. If we replace our index set $\mathscr{C}(X)$ for our construction by a closed base $\mathscr{C}$ that is stable under finite unions and contains $\emptyset$, then the proofs of Lemma 3.1 and Proposition 3.2 go through with almost no modification to show that $\{U_E \cup V_E : E \in \mathscr{C}\}$ is a base for an extension topology τ_1 on $X \cup I$. In the next proposition, which we will presently apply to construct the Stone–Čech compactification within our framework, we give sufficient conditions on the family of ideals so that $\{U_E \cup V_E : E \in \mathscr{C}(X)\}$ generates no finer topology on $X \cup I$.

Proposition 3.18. *Let $\mathscr{C}$ be a closed base for $\langle X, \tau\rangle$ that is stable under finite unions and contains the empty set. Suppose $\{\mathscr{B}_i : i \in I\}$ is a family of nontrivial ideals each having a base consisting of elements of $\mathscr{C}$. Let $\tau_1(\{\mathscr{B}_i : i \in I\})$ be the topology on $X \cup I$ generated by $\{U_E \cup V_E : E \in \mathscr{C}\}$. Then $\tau_1(\{\mathscr{B}_i : i \in I\}) = \tau_0(\{\mathscr{B}_i : i \in I\})$.*

Proof. We need only show that $\forall E \in \mathscr{C}(X)$, we have $U_E \cup V_E \in \tau_1$. By assumption, this is true if $E = \emptyset$, so, suppose E is nonempty. First, suppose $x \in V_E$. By the definition of a closed base, $\exists F \in \mathscr{C}$ with $x \in X \setminus F \subseteq V_E$. Then $E \subseteq F$, and we have $x \in U_F \cup V_F \subseteq U_E \cup V_E$. On the other hand, if $i \in U_E$, then $E \in \mathscr{B}_i$. By assumption, we can find $H \in \mathscr{C} \cap \mathscr{B}_i$ with $E \subseteq H$. Clearly, $i \in U_H \cup V_H \subseteq U_E \cup V_E$. We have shown that $U_E \cup V_E$ contains a τ_1-neighborhood of each of its points, as required. □

4. Classical extensions viewed as bornological extensions

The aim of this section is to show how our construction can unify some different methods for building extensions of a space. We will explicitly describe how several important extensions can be obtained as bornological extensions, that is, as extensions arising from a family of nontrivial closed ideals. Our treatment is intended to be representative rather than exhaustive.

The first example presents a construction of historical significance in a new light: the ultrafilter construction of the Wallman extension [9,22,23].

Example 4.1. The classical construction of the Wallman extension of a T_1 space $\langle X, \tau\rangle$ regards the remainder I as the set of all free order ultrafilters on $\mathscr{C}(X)$, that is, exclusive of those of the form $\{E \in \mathscr{C}(X) : p \in E\}$ where p runs over X, as these correspond to points of X. Let us denote by $\{\mathscr{F}_i : i \in I\}$ the family of free order ultrafilters on $\mathscr{C}(X)$. We will explain here how this classical construction can be understood as an application of our construction.

A subfamily $\mathscr{F}$ of $\mathscr{C}(X)$ is an order ultrafilter as the term is usually understood [23, p. 83] if and only if $\mathscr{C}(X) \setminus \mathscr{F}$ is a minimal prime order ideal (cf. [1, p. 154]). This means that $\mathscr{C}(X) \setminus \mathscr{F}$ is minimal among the subfamilies $\mathscr{I}$ of $\mathscr{C}(X)$ satisfying the four properties below:

(1) $X \notin \mathscr{I}$;
(2) whenever $E_1, E_2 \in \mathscr{I}$, then $E_1 \cup E_2 \in \mathscr{I}$;

(3) whenever $E_1 \in \mathscr{I}$ and $E_2 \in \mathscr{C}(X)$ with $E_2 \subseteq E_1$, then $E_2 \in \mathscr{I}$;
(4) whenever $E_1, E_2 \in \mathscr{C}(X)$ and $E_1 \cap E_2 \in \mathscr{I}$, then either $E_1 \in \mathscr{I}$ or $E_2 \in \mathscr{I}$.

Since an ultrafilter on $\mathscr{C}(X)$ is free if and only if it does not contain singletons, the complement of a free ultrafilter $\mathscr{F}$ is a prime ideal which contains all singletons. Then $\downarrow (\mathscr{C}(X) \setminus \mathscr{F})$, taken within $\mathscr{P}(X)$, is a nontrivial bornology on X with closed base. We are in a position to apply our construction to the induced family of nontrivial bornologies $\{\mathscr{B}_i : i \in I\}$, where for each $i \in I$, $\mathscr{B}_i = \downarrow (\mathscr{C}(X) \setminus \mathscr{F}_i)$.

We can consider on $X \cup I$ the bornological extension topology τ_0 and the Wallman topology τ_w, identifying each $\mathscr{F}_i$ with its index i. It is easy to see that $\tau_0 = \tau_w$. As is well-known [23, p. 142], a base for the closed subsets of τ_w consists of all subsets of $X \cup I$ of the form

$$E \cup \{i \in I : E \in \mathscr{F}_i\}, \quad E \in \mathscr{C}(X).$$

As a result, a base for the open subsets of τ_w consists of all sets of the form

$$(X \setminus E) \cup \{i \in I : E \notin \mathscr{F}_i\} = (X \setminus E) \cup \{i \in I : E \in \mathscr{B}_i\} = U_E \cup V_E,$$

with $E \in \mathscr{C}(X)$.

To recapitulate: to obtain the Wallman extension, the family of nontrivial ideals $\{\mathscr{B}_i : i \in I\}$ corresponding to the remainder are the bornologies having a closed base forming a minimal prime order ideal in $\mathscr{C}(X)$, other than those of form

$$\downarrow \{E \in \mathscr{C}(X) : x \notin E\} \quad (x \in X),$$

which correspond to points of X.

Example 4.2. The classical ultrafilter construction of the Stone–Čech compactification of a Tychonoff space [11,23] exactly parallels the construction of the Wallman extension, where $\mathscr{C}(X)$ is replaced by the family of zero-sets which is at once a closed base and a lattice (see, e.g., [9, pp. 64–65]). Denoting the zero-sets by $\mathscr{Z}(X)$, the arguments in the last example can be easily adapted to show that a base for the compactification topology is $\{U_E \cup V_E : E \in \mathscr{Z}(X)\}$ with respect to the family of all bornologies $\{\mathscr{B}_i : i \in I\}$ on X having as a base a minimal prime order ideal of zero-sets, other than those of the form $\downarrow \{E \in \mathscr{Z}(X) : x \notin E\}$. But by Proposition 3.18, the Stone–Čech compactification topology is equally well generated by the larger collection

$$\{U_E \cup V_E : E \in \mathscr{C}(X)\}.$$

More generally, this program can be pursued for extensions of the Wallman type (see, e.g., [2,10,16,23]).

Example 4.3. A standard way to present the completion of a noncomplete metric space $\langle X, d\rangle$ is to equip equivalence classes of Cauchy sequences in X with a natural metric [23, p. 176]. Two Cauchy sequences g and h are declared equivalent if

$$\lim_{n\to\infty} d(h(n), g(n)) = 0,$$

in which case we write $g \sim h$. If g is convergent, then $g \sim h_a$ for some $a \in X$ where $\forall n \in \mathbb{N},\ h_a(n) = a$. Equivalence classes of nonconvergent Cauchy sequences correspond to points of the remainder I. The (well-defined) complete metric on pairs of equivalence classes is defined by $\overline{d}(g, h) := \lim_{n\to\infty} d(h(n), g(n))$ where g and h are representatives of each equivalence class.

We now explain how to externally construct the topology σ of the completion on $X \cup I$ within our framework as $\tau_0(\{\mathscr{B}_i : i \in I\})$. We cheat a little, in that the family of ideals — actually bornologies — comes from an understanding of $\{\mathscr{B}_i(\sigma) : i \in I\}$ more refined than the more general understanding gained from Example 3.11. Now if $E \subseteq X$ and $i \in I$, we have

$$E \notin \mathscr{B}_i(\sigma) \Leftrightarrow i \in \mathrm{cl}_\sigma(E) \Leftrightarrow \forall n \in \mathbb{N}, \quad \exists a_n \in E \text{ with } \overline{d}(h_{a_n}, i) < \frac{1}{n}$$
$$\Leftrightarrow \exists \text{ a Cauchy sequence } h \text{ in } E \text{ with } h \sim i.$$

In view of Proposition 3.4, the family $\{\mathscr{B}_i : i \in I\}$ must be described by the condition: $E \in \mathscr{B}_i$ if and only if no Cauchy sequence in E is equivalent to i. Further, for each $E \in \mathscr{C}(X)$, the basic open set in the completion topology that E determines adjoins to $X \setminus E$ all points in I that fail to be equivalent to any Cauchy sequence in E.

Example 4.4. There is more than one way to order embed a lattice $\langle X, \preceq\rangle$ into a second lattice in which suprema and infima of arbitrary subsets exist, i.e., into a complete lattice. It is well-known that a linearly ordered set is a complete lattice if and only if its order topology is compact (see, e.g. [23, pp. 124–125]). In this example, we explain how we obtain a base for the topology for the most important of these extensions of a linearly ordered set that is not already a complete lattice.

The smallest complete order extension of a noncomplete lattice is the Dedekind–MacNeille completion, characterized by the property that each element of the completion can be realized as the supremum of a subset of X and the infimum of a subset of X as well (cf. [1, p. 237]). Given a subset A of X, let A^u be its set of upper bounds and let A^l be its set of lower bounds. The points Δ of the completion consist of those subsets D for which $(D^u)^l = D$. The family Δ is closed under arbitrary intersections and so forms a complete lattice with respect to inclusion (see [1, p. 46]). The join operation is more complicated [15]: $\bigvee_{i\in I} D_i = \left(\bigcap_{i\in I} D_i^u\right)^l$.

Let us now focus our attention on the special case of a linearly ordered set $\langle X, \preceq\rangle$ which we assume not to be a complete lattice. In view of the characteristic property of the Dedekind–MacNeille completion mentioned above, the assignment $a \mapsto D_a := \{x \in X : x \preceq a\}$ densely embeds $\langle X, \preceq\rangle$ into $\langle \Delta, \subseteq\rangle$ where both linearly ordered sets are equipped with the order topology.

Denote the compact Hausdorff order topology on Δ by σ, and let I denote the remainder. Notice that $\emptyset \in I$ if and only if X has no smallest element, while $X \in I$ if and only if X has no largest element. There are there possibilities for the bornology $\mathscr{B}_i(\sigma)$ for $i \in I$. If $i = \emptyset$, then $E \in \mathscr{B}_i(\sigma)$ if and only if $\{D_e : e \in E\}$ is disjoint from some order interval of the form $[\emptyset, D_a)$. If $i = X$, then $E \in \mathscr{B}_i(\sigma)$ if and only if $\{D_e : e \in E\}$ is disjoint from some order interval of the form $(D_b, X]$. If $i \in I$ is some proper nonempty subset of X, then $E \in \mathscr{B}_i(\sigma)$ if and only if $\{D_e : e \in E\}$ is disjoint from some order interval of the form (D_a, D_b) containing i.

For each closed subset E of X, our construction produces the basic open set that adjoins to $X \setminus E$ (i) the empty set provided X has no smallest element and E is lower bounded in X, (ii) X provided X has no largest element and E is upper bounded in X and (iii) all points of the remainder located in some order interval (D_a, D_b) where $E \cap \{x \in X : a \prec x \prec b\} = \emptyset$.

5. When is an extension a bornological extension?

As usual, let $\langle X \cup I, \sigma\rangle$ be an extension of $\langle X, \tau\rangle$, and for each $i \in I$, let $\mathscr{B}_i(\sigma)$ be the ideal generated by $\{X \setminus W : W \in \sigma, i \in W\}$. Our goal in this section is to produce sufficient (that may also be necessary) conditions for the extension to be either a bornological extension or strong bornological extension of $\langle X, \tau\rangle$. By Theorem 3.6, this amounts to showing that σ agrees with $\tau_0(\sigma)$ or $\tau_0^s(\sigma)$ as determined by the family of ideals $\{\mathscr{B}_i(\sigma) : i \in I\}$.

Theorem 5.1. *Let $\langle X \cup I, \sigma\rangle$ be an extension of $\langle X, \tau\rangle$. The following conditions are equivalent.*

(1) $\sigma = \tau_0(\sigma)$;
(2) $\{(X \cup I) \setminus cl_\sigma(E) : E \in \mathscr{C}(X)\}$ *is a base for* σ;
(3) *whenever C is σ-closed,* $\exists\{E_\lambda : \lambda \in \Lambda\} \subseteq \mathscr{C}(X)$ *such that* $C = \bigcap_{\lambda\in\Lambda} cl_\sigma(E_\lambda)$;
(4) *whenever C is σ-closed,* $C = \bigcap\{cl_\sigma(A) : A \subseteq X \text{ and } C \subseteq cl_\sigma(A)\}$;
(5) *whenever $A \subseteq I$ and $p \in (X \cup I) \setminus cl_\sigma(A)$, there exists $B \subseteq X$ with $p \notin cl_\sigma(B)$ and $cl_\sigma(A) \subseteq cl_\sigma(B)$.*

Proof. By Lemma 3.7, condition (2) says that the standard base for $\tau_0(\sigma)$ is a base for σ as well. Thus, conditions (1) and (2) are equivalent. Condition (3) says that $\{\mathrm{cl}_\sigma(E) : E \in \mathscr{C}(X)\}$ is a closed base associated with σ, and so conditions (2) and (3) are equivalent. Clearly, condition (3) implies (4), and condition (4) implies condition (5). It remains to prove if condition (5) holds, then (3) holds.

To this end, let $C \subseteq X \cup I$ be closed in the extension. We have the decomposition $C = \mathrm{cl}_\sigma(C \cap X) \cup \mathrm{cl}_\sigma(C \cap I)$. Suppose $p \notin C$; as $p \notin \mathrm{cl}_\sigma(C \cap I)$, by condition (5), $\exists B_p \subseteq X$ with

$$\mathrm{cl}_\sigma(C \cap I) \subseteq \mathrm{cl}_\sigma(B_p) \quad \text{and} \quad p \notin \mathrm{cl}_\sigma(B_p).$$

For each $p \in (X \cup I) \setminus C$, put $E_p := \mathrm{cl}_\tau((C \cap X) \cup B_p) \in \mathscr{C}(X)$. We compute

$$\mathrm{cl}_\sigma(E_p) = \mathrm{cl}_\sigma(C \cap X) \cup \mathrm{cl}_\sigma(B_p) \supseteq C,$$

and so by construction, $C = \bigcap_{p\notin C} \mathrm{cl}_\sigma(E_p)$ as required. □

Note that Stone [18, p. 120] originally defined strictness of an extension by an opaque condition on the structure of its open subsets that can be viewed as a translation of our transparent condition (4). We next use condition (2) to characterize compactness of a bornological extension.

Theorem 5.2. *Let $\langle X, \tau\rangle$ be a topological space and let $\{\mathscr{B}_i : i \in I\}$ be a family of closed nontrivial ideals on X. Then the bornological extension $\langle X \cup I, \tau_0\rangle$ determined by the family of ideals is compact if and only if whenever $\{F_\lambda : \lambda \in \Lambda\}$ is a family of closed subsets of X such that for any finite subfamily, either their intersection is nonempty or there exists an element i of the remainder such that each member of the finite subfamily fails to lie in $\mathscr{B}_i$, then the same is true for the entire family.*

Proof. The condition asserted to be equivalent to compactness says that whenever $\{F_\lambda : \lambda \in \Lambda\}$ is a family of closed subsets of X such that $\{\mathrm{cl}_{\tau_0}(F_\lambda) : \lambda \in \Lambda\}$ has the finite intersection property, then $\{\mathrm{cl}_{\tau_0}(F_\lambda) : \lambda \in \Lambda\}$ has nonempty intersection. But by condition (2), $\{\mathrm{cl}_{\tau_0}(F) : F \in \mathscr{C}(X)\}$ forms a closed base for the extension topology, so the condition is equivalent to compactness. □

While condition (3) is not so transparent, we intend now to apply it to obtain an elegant proof of a classical result of Stone [18, Theorem 65] asserting that any extension having a base of regular open sets is strict in his sense. A space with this property is called *semiregular* [23]. Obviously, each regular space is semiregular, and so all Hausdorff compactifications, all real compactifications, and all pseudo-metrizable extensions are bornological extensions.

Theorem 5.3. *Let $\langle X \cup I, \sigma\rangle$ be a semiregular extension of $\langle X, \tau\rangle$. Then $\sigma = \tau_0(\sigma)$.*

Proof. Let C be σ-closed. For each $p \in (X \cup I) \setminus C$ choose by semiregularity $W_p \in \sigma$ such that $W \cap C = \emptyset$ and $W_p = \mathrm{int}_\sigma \mathrm{cl}_\sigma(W_p)$. Using the fact that X is dense in each open subset of the extension, we compute

$$\begin{aligned}(X \cup I) \setminus W_p &= \mathrm{cl}_\sigma((X \cup I) \setminus \mathrm{cl}_\sigma(W_p)) = \mathrm{cl}_\sigma(\mathrm{int}_\sigma((X \cup I) \setminus W_p)) \\ &= \mathrm{cl}_\sigma(X \cap \mathrm{int}_\sigma((X \cup I) \setminus W_p)) \\ &= \mathrm{cl}_\sigma(\mathrm{cl}_\tau(X \cap \mathrm{int}_\sigma((X \cup I) \setminus W_p)))\end{aligned}$$

and the desired subfamily of $\mathscr{C}(X)$ with respect to condition (3) of the last theorem is $\{\mathrm{cl}_\tau(X \cap \mathrm{int}_\sigma((X \cup I) \setminus W_p)) : p \notin C\}$. $\square$

Example 3.9 shows that semiregularity of an extension is not necessary for it to be a bornological extension.

We now turn to strong bornological extensions. Theorem 5.1 immediately yields the following.

Theorem 5.4. *Let $\langle X \cup I, \sigma\rangle$ be an extension of $\langle X, \tau\rangle$. Then $\sigma = \tau_0^s(\sigma)$ if and only if*

$$\tau \cup \{(X \cup I) \setminus cl_\sigma(E) : E \in \mathscr{C}(X)\}$$

is a base for σ.

Corollary 5.5. *Let $\langle X \cup I, \sigma\rangle$ be an extension of $\langle X, \tau\rangle$. Then $\sigma = \tau_0^s(\sigma)$ if and only if $X \in \sigma$ and $\forall i \in I$, $\{(X \cup I) \setminus cl_\sigma(E) : E \in \widehat{\mathscr{B}_i}(\sigma)\}$ is a local base for σ at i.*

Proof. Suppose that $\sigma = \tau_0^s(\sigma)$. Since $X \in \tau_0^s(\sigma)$, we get $X \in \sigma$. Now if $i \in I$, since i belongs to no member of τ, a local base for σ at i is

$$\{(X \cup I) \setminus \mathrm{cl}_\sigma(E) : i \in (X \cup I) \setminus \mathrm{cl}_\sigma(E),\ E \in \mathscr{C}(X)\}.$$

But if $i \in (X \cup I) \setminus \mathrm{cl}_\sigma(E)$ where $E \in \mathscr{C}(X)$, then i has a σ-neighborhood disjoint from E, which means $E \in \mathscr{B}_i(\sigma)$, and so $E \in \widehat{\mathscr{B}_i}(\sigma)$. This proves necessity of the conditions.

For sufficiency, first note that $X \in \sigma$ and the fact that σ is an extension of τ guarantees $\tau \subseteq \sigma$. As always we have $\tau_0(\sigma) \subseteq \sigma$, we conclude that $\tau \cup \tau_0(\sigma) \subseteq \sigma$, and so $\tau_0^s(\sigma) \subseteq \sigma$.

For the reverse inclusion, we must show that each $W \in \sigma$ contains a $\tau_0^s(\sigma)$-neighborhood of each of its points. Let $w \in W$ be arbitrary. If $w \in X$, then $W \cap X$ lies in $\tau \subseteq \tau_0^s(\sigma)$ and clearly $w \in W \cap X \subseteq W$. On the other hand, if $w \in I$, we can find $E \in \widehat{\mathscr{B}_w}(\sigma)$ such that $w \in (X \cup I) \setminus \mathrm{cl}_\sigma(E) \subseteq W$. Then $E \in \mathscr{C}(X)$, and by Lemma 3.7, $w \in U_E \cup V_E \subseteq W$. $\square$

Obviously, if σ contains X and σ is regular, then $\sigma = \tau_0(\sigma) = \tau_0^s(\sigma)$. But just as in the previous result, a weaker statement guarantees coincidence. We leave the proof of the following result to the reader.

Proposition 5.6. *Let $\langle X \cup I, \sigma\rangle$ be an extension of $\langle X, \tau\rangle$ such that $X \in \sigma$. Then $\sigma = \tau_0(\sigma) = \tau_0^s(\sigma)$ provided σ has a closed local base at each point of the remainder I.*

Our next goal is to produce conditions of a different flavor that ensure that an extension topology σ reduces to $\tau_0(\sigma)$. Necessity of these conditions will be established by the next proposition along with Proposition 3.15.

Proposition 5.7. *Let $\langle X, \tau\rangle$ be a topological space, and let $\langle X \cup I, \tau_0\rangle$ be determined by a family $\{\mathscr{B}_i : i \in I\}$ of nontrivial closed ideals on X. Then there is a base $\mathscr{W}$ for τ_0 such that whenever $i \in I$ and $x \in X$ and $\{i, x\} \subseteq W \in \mathscr{W}$, there exists $W_1 \in \tau_0$ such that $\{i, x\} \subseteq W_1 \subseteq W$ and $U_{X\setminus W_1} \subseteq W$.*

Proof. The standard base for τ_0 satisfies the conditions, for if $W = U_E \cup V_E$ where U_E is nonempty, then we can take $W_1 = W$ whatever x and i may be in W. $\square$

Theorem 5.8. *Let $\langle X \cup I, \sigma\rangle$ be an extension of $\langle X, \tau\rangle$, and $\forall i \in I$, let $\mathscr{B}_i(\sigma)$ be the ideal generated by $\{X \setminus W : W \in \sigma, i \in W\}$. Then $\sigma = \tau_0(\sigma)$ if and only if σ has a base $\mathscr{W}$ with the following properties:*

(P1) *whenever $W \in \mathscr{W}$ and $W \subseteq X$, $\forall x \in W$, $\exists E_x \in \mathscr{C}(X)$ such that $E_x \notin \cup_{i\in I} \mathscr{B}_i(\sigma)$ and $x \in X \setminus E_x \subseteq W$;*

(P2) *whenever $W \in \mathscr{W}$ and $W \cap I \neq \emptyset$, then for each $i \in I$ and $x \in X$ with $\{i, x\} \subseteq W$, there exists $W_1 \in \sigma$ such that $\{i, x\} \subseteq W_1 \subseteq W$ and $U_{X\setminus W_1} \subseteq W$.*

Proof. Necessity of the conditions follows from Propositions 3.15 and 5.7 above. For sufficiency, we need only show that $\mathscr{W} \subseteq \tau_0(\sigma)$ provided (P1) and (P2) hold because always $\tau_0(\sigma) \subseteq \sigma$.

Suppose $W \in \mathscr{W}$ and $W \subseteq X$. By (P1) for each $x \in W$, we have $U_{E_x} = \emptyset$, and so the condition $\forall x \in W,\ x \in X \setminus E_x \subseteq W$ can be rewritten as

$$\forall x \in W, \quad x \in U_{E_x} \cup V_{E_x} \subseteq W,$$

and we see that W contains a $\tau_0(\sigma)$-neighborhood of each of its points.

Now suppose $W \in \mathscr{W}$ and $W \cap I \neq \emptyset$, so that by density of X, we have $W \cap X \neq \emptyset$ as well. Let $i \in I$ and $x \in X$ be arbitrary with $\{i, x\} \subseteq W$ and let $W_1 \in \sigma$ be as guaranteed by (P2). Since $V_{X\setminus W_1} = W_1 \cap X$, we obtain

$$\{i, x\} \subseteq U_{X\setminus W_1} \cup V_{X\setminus W_1} \subseteq W,$$

and this shows W contains a $\tau_0(\sigma)$-neighborhood of each of its points in the second case. $\square$

The second condition (P2) in the Theorem 5.8 is superfluous in the case of one-point extensions.

Proposition 5.9. *Let $\langle X \cup \{i\}, \sigma\rangle$ be a one-point extension of $\langle X, \tau\rangle$. Then $\sigma = \tau_0(\sigma)$ if and only if whenever $w \in W \in \sigma$ and $W \subseteq X$, there exists $E \in \mathscr{C}(X)$ such that $w \in X \setminus E \subseteq W$ and $i \in cl_\sigma(E)$.*

Proof. Condition (P1) holds for a particular base $\mathscr{W}$ if and only if it holds for σ itself. The stated condition is just a reformulation of condition (P1) in the context of one-point extensions when $\mathscr{W} = \sigma$ and so it is necessary. To show it is sufficient, we need only observe that condition (P2) is automatically satisfied in a one-point extension when $\mathscr{W} = \sigma$ as then (P1) and (P2) hold with respect to this largest possible base: if $\{i, x\} \subseteq W \in \mathscr{W}$, then with $W_1 = W$, we have $\{i, x\} \subseteq W_1 \subseteq W$ and $\{i\} = U_{X\setminus W_1} \subseteq W$. $\square$

Example 5.10. Let A and B be nonempty disjoint sets and let $X = A \cup B$ and $I = \{i\}$. For our extension topology σ, take $\{\emptyset, X \cup \{i\}, B, B \cup \{i\}\}$. Now the only bases for σ are σ and $\{X \cup \{i\}, B, B \cup \{i\}\}$, and the condition of Proposition 5.9 fails for $W = B$; so such an extension cannot be a bornological extension.

Corollary 5.11. *Let $\langle X \cup \{i\}, \sigma\rangle$ be a one-point extension of $\langle X, \tau\rangle$ such that $\forall x \in X$, x and i have disjoint neighborhoods. Then the extension is bornological. In particular, this is true if σ is Hausdorff.*

Proof. Let $\mathscr{B} = \{E \subseteq X : i \notin \mathrm{cl}_\sigma(E)\}$ so that $\tau_0(\sigma) = \tau_0(\mathscr{B})$. Suppose $w \in W \in \sigma$ where $W \subseteq X$. Choose $G_w \in \sigma$ and $G_i \in \sigma$ with $w \in G_w, i \in G_i$ and $G_w \cap G_i = \emptyset$. Then $G_w \cap W \in \mathscr{B}$ so that $E := X \setminus (G_w \cap W) \notin \mathscr{B}$. Clearly, $w \in X \setminus E \subseteq W$ and $i \in \mathrm{cl}_\sigma(E)$. Apply Proposition 5.9. □

It is natural to ask whether a bornological extension of a bornological extension need be a bornological extension of the initial space. We are now in a position to answer this question in the negative.

Example 5.12. We produce spaces $\langle X, \tau\rangle$, $\langle Y, \sigma\rangle$, and $\langle W, \mu\rangle$ such that the second is a bornological extension of the first, the third is a bornological extension of the second, but $\langle W, \mu\rangle$ fails to be a bornological extension of $\langle X, \tau\rangle$. We start with the largest space: let $W = \mathbb{R}$ be equipped with the Hausdorff topology μ generated by $\{(a, b) : a \in \mathbb{R}, b \in \mathbb{R}$ and $a < b\} \cup \left\{\mathbb{R} \setminus \left\{\frac{1}{n} : n \in \mathbb{N}\right\}\right\}$ (see, e.g., [23, Example 14.2]). Let $Y = W \setminus \{0\}$ be equipped with the relative topology it inherits from $\langle W, \mu\rangle$ which is nothing but the relative usual topology of the line. Finally, let $X = Y \setminus \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ be equipped with the relative topology τ. Clearly, X is dense in Y and Y is dense in W. Further, by regularity of $\langle Y, \sigma\rangle$, the second space is a bornological extension of the first, and by Corollary 5.11, the third is a bornological extension of the second. But viewing $\langle W, \mu\rangle$ as an extension of $\langle X, \tau\rangle$, it is clear that condition (4) of Theorem 5.1 fails with respect to $C = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, for if the μ-closure of a subset of X were to contain C, it would also contain 0.

The last example shows in particular that a topological space can be a bornological extension of some dense subspaces but not of others. Criterion (5) of Theorem 5.1 was included precisely because it leads to a simple answer that we now give to the following natural question: when is a topological space a bornological extension of each of its dense subspaces?

Theorem 5.13. *Let $\langle Y, \sigma\rangle$ be a topological space. Then $\langle Y, \sigma\rangle$ is a bornological extension of each of its dense subspaces if and only if whenever I is a nonempty subset of Y with empty interior, $A \subseteq I$ and $p \in Y \setminus cl_\sigma(A)$, there exists $B \subseteq Y \setminus I$ with $p \notin cl_\sigma(B)$ and $cl_\sigma(A) \subseteq cl_\sigma(B)$.*

Proof. Evidently, $Y \setminus I$ is dense in Y if and only if $\mathrm{int}_\sigma(I) = \emptyset$. The result immediately follows from criterion (5) of Theorem 5.1 for a space to be a bornological extension of a dense subspace. □

For completeness, we include one further result on one-point extensions.

Proposition 5.14. *Let $\langle X \cup \{i\}, \sigma\rangle$ be a one-point extension of $\langle X, \tau\rangle$. Then*

(1) *$\tau_0^s(\sigma) = \sigma$ if and only if $X \in \sigma$;*
(2) *$\sigma = \tau_0(\sigma) = \tau_0^s(\sigma)$ if and only if τ contains a subfamily $\mathscr{A}$ such that $\forall A \in \mathscr{A}$, $i \in cl_\sigma(X \setminus A)$ and $\cup\mathscr{A} = X$.*

Proof. Necessity in (1) is obvious. For sufficiency, we have $\tau \cup \tau_0(\sigma) \subseteq \sigma$, and so $\tau_0^s(\sigma) \subseteq \sigma$. For the reverse inclusion, let $W \in \sigma$. If $W \subseteq X$, then $W \in \tau \subseteq \tau_0^s(\sigma)$. Otherwise, $i \in W$, and $W = U_{X\setminus W} \cup V_{X\setminus W} \in \tau_0(\sigma) \subseteq \tau_0^s(\sigma)$.

For statement (2), we note that the condition is a rephrasing of condition (iii) in Corollary 3.16, which guarantees $X \in \tau_0(\sigma)$ and the equality of $\tau_0(\sigma)$ with $\tau_0^s(\sigma)$. But since $X \in \tau_0(\sigma) \subseteq \sigma$ gives $X \in \sigma$, by statement (1), we have $\tau_0^s(\sigma) = \sigma$. In view of Corollary 3.16, the converse holds just knowing $\tau_0(\sigma) = \tau_0^s(\sigma)$. □

Statement (1) in the above proposition implies the known fact that every T_1 one-point extension is a strong bornological extension (cf. [8, Section 2])

An example of a one-point extension which is a strong bornological extension but not a bornological extension is $X \cup \{i\}$, where X is infinite, equipped with the cofinite topology (see Example 3.12).

Theorem 5.15. *Let $\langle X \cup I, \sigma\rangle$ be an extension of $\langle X, \tau\rangle$ such that $\sigma \supseteq \tau$. The following conditions are equivalent:*

(1) *$\langle X \cup I, \sigma\rangle$ is a strong bornological extension;*
(2) *$\langle X \cup I, \sigma\rangle$ is the weakest extension topology μ on $X \cup I$ containing τ such that for each $i \in I$, the relative topology on $X \cup \{i\}$ inherited from σ coincides with the one inherited from μ.*

Proof. If condition (1) holds, Proposition 3.13 guarantees that condition (2) also holds. Suppose that condition (2) holds. We claim that $\tau_0^s(\sigma)$ has the same trace on each one point extension as does σ. For each $i \in I$, let $\langle X \cup \{i\}, \sigma_i\rangle$ be the one point extension induced by $\langle X \cup I, \sigma\rangle$. By Proposition 5.14, we have $\sigma_i = \tau_0^s(\sigma_i)$, and upon reflection, $\tau_0^s(\sigma_i)$ is seen to be the relative topology on $X \cup \{i\}$ induced by $\tau_0^s(\sigma)$. By condition (2), we have $\sigma \subseteq \tau_0^s(\sigma)$. But $\sigma \supseteq \tau$ and Corollary 3.8 force the reverse inclusion. Thus, $\sigma = \tau_0^s(\sigma)$ as required. □

Condition (P2) in Theorem 5.8 was formulated in the symmetric way it was with a particular application in mind. Suppose $\langle X \cup I, \sigma\rangle$ is an extension of $\langle X, \tau\rangle$ in which both X and I are dense. Then we can ask this question: when is $\langle X \cup I, \sigma\rangle$ a bornological extension of both X and I separately, equipped with their relative topologies? Let us denote the relative topology for I by μ, and let $\mu_0(\sigma)$ be the induced bornological extension on $X \cup I$ where now I is viewed as the primal space and X is viewed as the remainder. Further, for H a μ-closed subset of I, let U'_H denote $\{x \in X : x \notin \mathrm{cl}_\sigma(H)\}$.

Theorem 5.16. *Let $\langle X \cup I, \sigma\rangle$ be a topological space in which both X and I are dense, and let τ (resp. μ) be the relative topology on X (resp. I). Then $\sigma = \tau_0(\sigma) = \mu_0(\sigma)$ provided σ has a base $\mathscr{W}$ such that whenever $W \in \mathscr{W}$ and $\{i, x\} \subseteq W$, there exists $W_1 \in \sigma$ such that $\{i, x\} \subseteq W_1 \subseteq W$ and $U_{X\setminus W_1} \cup U'_{I\setminus W_1} \subseteq W$.*

Proof. This follows from the fact that each member of σ hits both X and I so that condition (P1) in Theorem 5.8 never comes into play. □

6. The separation axioms in bornological extensions

Proposition 6.1. *Let $\langle X, \tau\rangle$ be a T_0-space and let $\{\mathscr{B}_i : i \in I\}$ be a family of nontrivial closed ideals. Then $\langle X \cup I, \tau_0\rangle$ is T_0 if and only if*

(1) *$\mathscr{B}_i \neq \mathscr{B}_j$ for $i \neq j$;*
(2) *for every $x \in X$ and $i \in I$, whenever $\{x\} \notin \mathscr{B}_i$, there exists a closed subset $F = F_{x,i}$ of X such that $F \notin \mathscr{B}_i$ and $x \notin F$.*

Proof. Suppose that $\langle X \cup I, \tau_0 \rangle$ is T_0. Let $i, j \in I$, $i \neq j$. Without loss of generality, we can suppose that there is a basic open set $U_E \cup V_E$ such that $i \in U_E$ and $j \notin U_E$. This implies $E \in \mathscr{B}_i \setminus \mathscr{B}_j$ and so $\mathscr{B}_i \neq \mathscr{B}_j$. To prove (2), suppose $\{x\} \notin \mathscr{B}_i$ and let $i \in U_E \cup V_E$. Then $E \in \mathscr{B}_i$ and, by our assumption, $x \notin E$, so $x \in V_E$. Therefore, every open set containing i also contains x. The T_0 assumption for the extension gives a basic open set $U_F \cup V_F$ such that $x \in U_F \cup V_F$ and $i \notin U_F \cup V_F$. This means $x \notin F$ and $F \notin \mathscr{B}_i$.

Now suppose conditions (1) and (2) hold. For the T_0 property, if both points lie in X, we can use the fact that X is T_0. Next, suppose $x \in X, i \in I$. If $\{x\} \in \mathscr{B}_i$, putting $E = \mathrm{cl}_\tau(\{x\}) \in \mathscr{B}_i$, one has $x \notin U_E \cup V_E$ and $i \in U_E \cup V_E$. If $\{x\} \notin \mathscr{B}_i$, let $F = F_{x,i}$ be as in condition (2). Then $x \in U_F \cup V_F$ and $i \notin U_F \cup V_F$. Finally, if $i, j \in I$ with $i \neq j$, we can suppose $\mathscr{B}_i \not\subseteq \mathscr{B}_j$. If $E \in \mathscr{B}_i \setminus \mathscr{B}_j$, then $i \in U_E \cup V_E$ and $j \notin U_E \cup V_E$. □

Proposition 6.2. *Let $\langle X, \tau \rangle$ be a T_0-space and let $\{\mathscr{B}_i : i \in I\}$ be a family of nontrivial closed ideals. Then $\langle X \cup I, \tau_0^s \rangle$ is T_0 if and only if $\mathscr{B}_i \neq \mathscr{B}_j$ for $i \neq j$.*

Proof. The condition is sufficient. Let w_1, w_2 be distinct points in $X \cup I$. The case when both points are in X is trivial, since τ is assumed to be T_0. When exactly one of the points, say w_1, is in X, then $w_1 \in X \in \tau_0^s$ and $w_2 \notin X$. The case $w_1, w_2 \in I$ can be handled just as in the previous proposition. The proof of necessity is also the same as before. □

Proposition 6.3. *Let $\langle X, \tau \rangle$ be a T_1-space and let $\{\mathscr{B}_i : i \in I\}$ be a family of nontrivial closed ideals. Then $\langle X \cup I, \tau_0 \rangle$ is T_1 if and only if*

(1) *each $\mathscr{B}_i$ is a bornology;*
(2) *$\mathscr{B}_i \not\subseteq \mathscr{B}_j$ for $i \neq j$;*
(3) *for every $x \in X$ and $i \in I$, there exists $F = F_{x,i} \in \mathscr{C}(X)$ such that $F \notin \mathscr{B}_i$ and $x \notin F$.*

Proof. Assume that $\langle X \cup I, \tau_0 \rangle$ is T_1. As mentioned before, each $\mathscr{B}_i$ is a bornology because for each $x \in X$ and $i \in I$, we can find a neighborhood of i that does not contain x. To prove (2) we can use the same argument used to prove (1) in Proposition 6.1. As for (3), if $x \in X$ and $i \in I$, there is a closed subset F of X such that $x \in U_F \cup V_F$, $i \notin U_F \cup V_F$, that is, $F \notin \mathscr{B}_i$ and $x \notin F$.

Conversely, assume that the conditions hold. We consider cases on distinct points of $X \cup I$. We need only look at distinct points where at least one lies in I. If $x \in X$ and $i \in I$, then by condition (1), $i \in U_{\{x\}} \cup V_{\{x\}}$ which does not contain x. On the other hand, taking $F = F_{x,i}$, one has $x \in U_F \cup V_F$ which does not contain i. If $i, j \in I$ with $i \neq j$, we can appropriate the argument in the final lines of the proof of Proposition 6.1. □

Proposition 6.4. *Let $\langle X, \tau \rangle$ be a T_1-space and let $\{\mathscr{B}_i : i \in I\}$ be a family of nontrivial closed ideals. Then $\langle X \cup I, \tau_0^s \rangle$ is T_1 if and only if*

(1) *each $\mathscr{B}_i$ is a bornology;*
(2) *$\mathscr{B}_i \not\subseteq \mathscr{B}_j$ for $i \neq j$;.*

Proof. Necessity is established as in Proposition 6.3. For sufficiency, the only difference comes when $x \in X$ and $i \in I$: $i \in U_{\{x\}} \cup V_{\{x\}}$ which does not contain x, while $x \in X$ which does not contain i. □

Proposition 6.5. *Let $\langle X, \tau \rangle$ be a Hausdorff space and let $\{\mathscr{B}_i : i \in I\}$ be a family of nontrivial closed ideals. The following are equivalent:*

(1) *$\langle X \cup I, \tau_0 \rangle$ is Hausdorff;*

(2) *$\langle X \cup I, \tau_0^s\rangle$ is Hausdorff;*

(3) *each $\mathscr{B}_i$ is a local bornology, and for distinct i, j in I, there exists $E \in \mathscr{B}_i$, $F \in \mathscr{B}_j$ with $E \cup F = X$.*

Proof. (1) $\Rightarrow$ (2). This is obvious because $\tau_0 \subseteq \tau_0^s$.

(2) $\Rightarrow$ (3). By the last proposition, every $\mathscr{B}_i$ is a bornology. We now prove that each $\mathscr{B}_i$ is local. Let $x \in X$. There exist $W \in \tau_0^s$ with $x \in W$ and $U_E \cup V_E$ with $E \in \widehat{\mathscr{B}_i}$, such that $W \cap (U_E \cup V_E) = \emptyset$. This implies $W_1 := W \cap X \subseteq E$, and so $x \in W_1 \in \mathscr{B}_i$. Finally, let $i, j \in I, i \neq j$ and let $U_E \cup V_E$ and $U_F \cup V_F$ be disjoint basic neighborhoods of i and j, respectively. Then $E \in \mathscr{B}_i$, $F \in \mathscr{B}_j$ and $V_E \cap V_F = \emptyset$, that is, $E \cup F = X$.

(3) $\Rightarrow$ (1). Assuming (3), we must separate distinct points of $X \cup I$ by disjoint τ_0-neighborhoods.

First, let $x, y \in X$ with $x \neq y$. Choose A and B to be disjoint open subsets of X containing x and y, respectively. Putting $E = X \setminus A$ and $F = X \setminus B$, one has $E \cup F = X$. We now obtain

$$\begin{aligned}(U_E \cup V_E) \cap (U_F \cup V_F) &= (U_E \cup A) \cap (U_F \cup B)\\ &= (U_E \cap U_F) \cup (A \cap B) = U_{E\cup F} \cup \emptyset\\ &= U_X \cup \emptyset = \emptyset,\end{aligned}$$

and we have produced disjoint τ_0-neighborhoods of x and y.

Next, let $x \in X$ and $i \in I$. Since $\mathscr{B}_i$ is local, we can find $W \in \tau$ with $x \in W \in \mathscr{B}_i$. Put $E = X \setminus W$ and $F = \mathrm{cl}_\tau(W)$. Since the bornology is closed, we have $F \in \mathscr{B}_i$. Since $E \cup F = X$, again one has $U_E \cap U_F = \emptyset$, and $V_E \cap V_F = X \setminus (E \cup F) = \emptyset$. Clearly $x \in U_E \cup V_E$ and $i \in U_F \cup V_F$, and these neighborhoods are disjoint.

Finally, let $i, j \in I$ with $i \neq j$. Since $\mathscr{B}_i$ and $\mathscr{B}_j$ are closed bornologies, we can choose $E \in \widehat{\mathscr{B}_i}$, $F \in \widehat{\mathscr{B}_j}$ such that $E \cup F = X$. Then $U_E \cup V_E$ and $U_F \cup V_F$ are disjoint members of τ_0 containing i and j, respectively. □

Proposition 6.6. *Let $\langle X, \tau\rangle$ be a regular space and let $\{\mathscr{B}_i : i \in I\}$ be a family of closed nontrivial ideals on X. Then the bornological extension $\langle X \cup I, \tau_0\rangle$ is regular if and only if the following three conditions hold:*

(1) *for every $x \in X$ and for every $E \in \mathscr{C}(X)$ such that $x \notin E$, there is a τ-neighborhood $W_{x,E}$ of x such that: $\forall i \in I$, $W_{x,E} \notin \mathscr{B}_i \Rightarrow E \in \mathscr{B}_i$;*

(2) *each ideal $\mathscr{B}_i$ is open;*

(3) *for every $i \in I$ and for every closed $E \in \mathscr{B}_i$, there is a family $\{A_\lambda : \lambda \in \Lambda\}$ of closed subsets of X such that $X \setminus \left[\bigcap_{\lambda\in\Lambda} A_\lambda\right] \in \mathscr{B}_i$, and $\forall j \in I$, $\{A_\lambda : \lambda \in \Lambda\} \cap \mathscr{B}_j = \emptyset \Rightarrow E \in \mathscr{B}_j$.*

Proof. We first prove necessity which is the easier direction. Let $x \in X$ and $E \in \mathscr{C}(X)$ with $x \notin E$. By regularity of the extension, there exists a τ_0-neighborhood W of x such that $\mathrm{cl}_{\tau_0}(W) \cap \mathrm{cl}_{\tau_0}(E) = \emptyset$. Put $W_{x,E} = W \cap X$. If $W_{x,E} \notin \mathscr{B}_i$ then $i \in \mathrm{cl}_{\tau_0}(W_{x,E}) \subseteq \mathrm{cl}_{\tau_0}(W)$. Then $i \notin \mathrm{cl}_{\tau_0}(E)$, that is, $E \in \mathscr{B}_i$. Thus, condition (1) holds.

To prove (2), let $E \in \mathscr{B}_i$. Then there exists an open τ_0-neighborhood W of i disjoint from E. By regularity, there is $W_1 \in \tau_0$ such that $i \in W_1 \subseteq \mathrm{cl}_{\tau_0}(W_1) \subseteq W$. Then one has

$$E \subseteq X \setminus W \subseteq X \setminus \mathrm{cl}_{\tau_0}(W_1) \subseteq X \setminus W_1.$$

Clearly $X \setminus W_1 \in \widehat{\mathscr{B}_i}$; hence $X \setminus \mathrm{cl}_{\tau_0}(W_1)$ is an open member of $\mathscr{B}_i$ containing E.

Now we prove (3). Let $i \in I$ and let $E \in \mathscr{B}_i$. The neighborhood $U_E \cup V_E$ of i must contain a closed neighborhood. So there is a τ_0-closed subset C of $X \cup I$ and a closed member F of $\mathscr{B}_i$ such that

$$i \in U_F \cup V_F \subseteq C \subseteq U_E \cup V_E.$$

By Theorem 5.1, $\{\mathrm{cl}_{\tau_0}(A) : A \in \mathscr{C}(X)\}$ forms a base for the closed subsets of $X \cup I$. Thus, we can find $\{A_\lambda : \lambda \in \Lambda\} \subseteq \mathscr{C}(X)$ with $C = \cap_{\lambda \in \Lambda} \mathrm{cl}_{\tau_0}(A_\lambda)$, and so

$$U_F \cup V_F \subseteq \bigcap_{\lambda \in \Lambda} \mathrm{cl}_{\tau_0}(A_\lambda) \subseteq U_E \cup V_E.$$

Taking the intersections with X, we obtain

$$V_F \subseteq \bigcap_{\lambda \in \Lambda} A_\lambda \subseteq V_E.$$

Taking complements, $X \setminus \left(\bigcap_{\lambda \in \Lambda} A_\lambda\right) \subseteq F$ and this implies $X \setminus \left(\bigcap_{\lambda \in \Lambda} A_\lambda\right) \in \mathscr{B}_i$.

For the last part of condition (3), suppose $\mathscr{B}_j$ does not contain A_λ for any λ. Then

$$j \in \left(\bigcap_{\lambda \in \Lambda} \mathrm{cl}_{\tau_0}(A_\lambda)\right) \cap I \subseteq U_E,$$

which means $E \in \mathscr{B}_j$ as required. We have proved (3).

Now we turn to sufficiency. First, let $x \in X$ and suppose $U_E \cup V_E$ is a basic open neighborhood of x. By regularity of $\langle X, \tau\rangle$, let W be a τ-open neighborhood of x such that $\mathrm{cl}_\tau(W) \subseteq W_{x,E} \cap V_E$, where $W_{x,E}$ is chosen to satisfy condition (1). We put $T = X \setminus W$ and $F = \mathrm{cl}_\tau(W)$. Clearly $U_T \cup V_T$ is a neighborhood of x. We want to prove that

$$U_T \cup V_T \subseteq \mathrm{cl}_{\tau_0}(F) \subseteq U_E \cup V_E.$$

For the inclusion $U_T \cup V_T \subseteq \mathrm{cl}_{\tau_0}(F)$, clearly, $V_T = W \subseteq F \subseteq \mathrm{cl}_{\tau_0}(F)$. If $i \in U_T$, then $T \in \mathscr{B}_i$; being that $T \cup F = X$, $F \notin \mathscr{B}_i$. This implies $i \in \mathrm{cl}_{\tau_0}(F)$. For the second inclusion, if $x \in \mathrm{cl}_{\tau_0}(F) \cap X$, then $x \in \mathrm{cl}_\tau(W) \subseteq V_E$. If $i \in \mathrm{cl}_{\tau_0}(F) \cap I$, then $F \notin \mathscr{B}_i$. Since $F \subseteq W_{x,E}$, this implies that $W_{x,E} \notin \mathscr{B}_i$ as well. Hence, by condition (1), we have $E \in \mathscr{B}_i$, that is, $i \in U_E$.

We finally consider points of the remainder which proves to be quite delicate. Let $i \in I$ and suppose $U_E \cup V_E$ is a basic open neighborhood of i. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ satisfy condition (3) (with respect to i and E) and put $W = X \setminus \left(\bigcap_{\lambda \in \Lambda} A_\lambda\right)$. By condition (2), let W_1 be an open member of $\mathscr{B}_i$ containing E and put $F = \mathrm{cl}_\tau(W \cup W_1) \in \mathscr{B}_i$. Clearly $U_F \cup V_F$ is a τ_0-neighborhood of i. Put

$$C = \left[\mathrm{cl}_{\tau_0}\left(\bigcap_{\lambda \in \Lambda} A_\lambda\right)\right] \cap \mathrm{cl}_{\tau_0}(X \setminus W_1).$$

Evidently, C is τ_0-closed. We will be done if we can verify that

$$U_F \cup V_F \subseteq C \subseteq U_E \cup V_E.$$

For the first inclusion, let $x \in V_F = X \setminus F$, then clearly $x \notin W \cup W_1$; hence $x \in \left[\bigcap_{\lambda \in \Lambda} A_\lambda\right] \cap (X \setminus W_1) \subseteq C$. If $j \in U_F$, then $F \in \mathscr{B}_j$. Now by the definition of W, for each $\lambda \in \Lambda$, F contains $X \setminus A_\lambda$. As a result, $A_\lambda \cup F = X$ for every λ, and so $A_\lambda \notin \mathscr{B}_j$. This implies $\forall \lambda \in \Lambda$, $j \in \mathrm{cl}_{\tau_0}(A_\lambda)$. Similarly, $X \setminus W_1 \notin \mathscr{B}_j$; hence, $j \in \mathrm{cl}_{\tau_0}(X \setminus W_1)$.

For the inclusion $C \subseteq U_E \cup V_E$, let $x \in C \cap X$. Then $x \notin W_1$ and so $x \in V_E$. If $j \in C \cap I$, then $j \in \bigcap_{\lambda \in \lambda} \mathrm{cl}_{\tau_0}(A_\lambda)$. This implies $\forall \lambda \in \Lambda,\ A_\lambda \notin \mathscr{B}_j$. Applying condition (3), $E \in \mathscr{B}_j$ and this implies $j \in U_E$. □

Our regularity result for strong bornological extensions involves a uniform localness condition which implies that each ideal is a bornology.

Proposition 6.7. *Let $\langle X, \tau\rangle$ be a regular space and let $\{\mathscr{B}_i : i \in I\}$ be a family of closed nontrivial ideals on X. Then the strong bornological extension $\langle X \cup I, \tau_0^s\rangle$ is regular if and only if the following hold:*

(1) *each x in X has a τ-neighborhood W_x that belongs to every $\mathscr{B}_i$;*
(2) *each ideal $\mathscr{B}_i$ is open;*
(3) *for every $i \in I$ and for every closed $E \in \mathscr{B}_i$, there is a family $\{A_\lambda : \lambda \in \Lambda\}$ of closed subsets of X such that $X \setminus [\bigcap_{\lambda\in\Lambda} A_\lambda] \in \mathscr{B}_i$, and $\forall j \in I, \{A_\lambda : \lambda \in \Lambda\} \cap \mathscr{B}_j = \emptyset \Rightarrow E \in \mathscr{B}_j$.*

Proof. Suppose that the three conditions hold. As condition (1) above ensures that condition (1) of Proposition 6.6 holds (take $W_{x,E} = W_x$ whatever E may be), the bornological extension is regular. Let $i \in I$; since $\{U_E \cup V_E : E \in \widehat{\mathscr{B}_i}\}$ remains a local base for τ_0^s at i, and since $\mathrm{cl}_{\tau_0^s}(U_E \cup V_E) \subseteq \mathrm{cl}_{\tau_0}(U_E \cup V_E)$, each τ_0^s-neighborhood of i contains a closed τ_0^s-neighborhood.

Next let $x \in X$ and let W be an open τ_0^s-neighborhood of x. Then $W \cap W_x \in \tau$ and since $W \cap W_x \in \cap_{i\in I} \mathscr{B}_i$, each point of I has a τ_0-neighborhood disjoint from $W \cap W_x$. By regularity of X, $\exists W_1 \in \tau$ with

$$x \in W_1 \subseteq \mathrm{cl}_\tau(W_1) \subseteq W \cap W_x.$$

Clearly, each point of $X \setminus W$ has a τ-neighborhood disjoint from W_1. In summary, each point of $(X \cup I) \setminus W$ has a τ_0^s-neighborhood disjoint from W_1, and so

$$x \in W_1 \subseteq \mathrm{cl}_{\tau_0^s}(W_1) \subseteq W.$$

Conversely, if $\langle X \cup I, \tau_0^s\rangle$ is assumed regular, then by Theorem 5.2, $\langle X \cup I, \tau_0^s\rangle$ is a bornological extension, that is, $\tau_0^s = \tau_0$. By Proposition 6.6, conditions (2) and (3) hold. To prove (1), we observe that, since $X \in \tau_0^s$, every $x \in X$ has a τ-neighborhood W_x such that $\mathrm{cl}_{\tau_0^s}(W_x)$ is disjoint from I. This means that each $i \in I$ has a τ_0-neighborhood disjoint from W_x so that W_x belongs to each $\mathscr{B}_i$. □

Corollary 6.8. *Let $\langle X, \tau\rangle$ be a regular space and let $\{\mathscr{B}_i : i \in I\}$ be a family of closed nontrivial ideals on X. Then the strong bornological extension $\langle X \cup I, \tau_0^s\rangle$ is regular if and only if $\tau_0 = \tau_0^s$ and $\langle X \cup I, \tau_0\rangle$ is regular.*

In all of the above results, the separation property under consideration is assumed for the primal space $\langle X, \tau\rangle$ rather than listed as a condition to be satisfied. This is a matter of taste, given that each property is hereditary.

7. Concluding remarks

While our paper can be viewed as contained within the framework of general topology, it is just as importantly a paper on structures that represent larger phenomena, the carriers of which are ideals and bornologies. General topology, with its focus on nearness, is simply not equipped to explicitly deal with large phenomena, e.g., coercivity of functions as it is understood in optimization theory (see, e.g., [4]). With this deficiency in mind, Hu in his seminal paper [13] conceived of a space as consisting of a set equipped with a topology and an ideal with some interplay between them. He coined the term *universe* to describe such a triple, and as a main result, gave necessary and sufficient conditions on a bornology relative to a pseudo-metrizable topology so that it is the bornology of d-bounded subsets with respect to some compatible pseudo-metric d (a parallel exercise for bornologies of Bourbaki bounded sets in a uniform space was undertaken in [21], whereas bornologies of totally bounded subsets of a metrizable space were recently characterized in [5]).

Our results suggest a broader possible definition for a universe: a set equipped with a topology and a family of ideals.

References

[1] R. Balbes, P. Dwinger, Distributive Lattices, University of Missouri Press, Columbia, MO, 1974.
[2] B. Banaschewski, Normal systems of sets, Math. Nachr. 24 (1962) 53–75.
[3] B. Banaschewski, Extensions of topological spaces, Canad. Math. Bull. 7 (1964) 1–22.
[4] G. Beer, Embeddings of bornological universes, Set-Valued Anal. 16 (2008) 477–488.
[5] G. Beer, C. Costantini, S. Levi, Total boundedness in metrizable spaces, Houston J. Math. 37 (2011) 1347–1362.
[6] G. Beer, S. Levi, Total boundedness and bornologies, Topology Appl. 156 (2009) 1271–1288.
[7] N. Bourbaki, Topologie Générale, Hermann, Paris, 1950.
[8] A. Caterino, T. Panduri, M. Vipera, Boundedness, one-pont extensions, and B-extensions, Math. Slovaca 58 (1) (2008) 101–114.
[9] R. Engelking, General Topology, Polish Scientific Publishers, Warsaw, 1977.
[10] O. Frink, Compactifications and and semi-normal spaces, Amer. J. Math. 86 (1964) 602–607.
[11] L. Gillman, M. Jerison, Rings of Continuous Functions, Van Nostrand, Princeton, NJ, 1960.
[12] H. Hogbe-Nlend, Bornologies and Functional Analysis, North-Holland, Amsterdam, 1977.
[13] S.-T. Hu, Boundedness in a topological space, J. Math. Pures Appl. 228 (1949) 287–320.
[14] A. Lechicki, S. Levi, A. Spakowski, Bornological convergences, J. Math. Anal. Appl. 297 (2004) 751–770.
[15] H. MacNeille, Partially ordered sets, Trans. Amer. Math. Soc. 42 (1937) 416–460.
[16] O. Njastad, On Wallman-type compactifications, Math. Z. 91 (1966) 267–276.
[17] J. Porter, R.G. Woods, Extensions and Absolutes of Hausdorff Spaces, Springer, New York, 1988.
[18] M.H. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc. 41 (1937) 374–481.
[19] W. Thron, Topological Stuctures, Holt, Rinehart and Winston, New York, 1966.
[20] M. Tikoo, The Banaschewski–Fomin–Shanin extension μX, Topology Proc. 10 (1985) 187–206.
[21] T. Vroegrijk, Uniformizable and realcompact bornological universes, Appl. Gen. Topol. 10 (2009) 277–287.
[22] H. Wallman, Lattices and topological spaces, Ann. of Math. 39 (1938) 112–126.
[23] S. Willard, General Topology, Addison-Wesley, Reading, MA, 1970.