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The Q-curvature on a 4-dimensional Riemannian manifold (M, g) with $\int_M Q dV_g = 8\pi^2$

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Abstract

We deal with the Q-curvature problem on a 4-dimensional compact Riemannian manifold (M, g) with $\int_M Q_g dV_g = 8\pi^2$ and positive Paneitz operator P_g. Let $\tilde{Q}$ be a positive smooth function. The question we consider is, when can we find a metric $\tilde{g}$ which is conformal to g, such that $\tilde{Q}$ is just the Q-curvature of $\tilde{g}$. A sufficient condition to this question is given in this paper.

Keywords: Q-curvature 4-dimensional Riemannian manifold

1. Introduction

One of the most important problems in conformal geometry is the construction of conformal metrics for which a certain curvature quantity equals a prescribed function, e.g. a constant. In two dimensions, the problem of prescribed Gaussian curvature asks the following: given a smooth function K on (M, g_0), can we find a metric g conformal to g_0 such that K is the Gaussian

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curvature of the new metric g? If we let $g = e^{2u} g_0$ for some $u \in C^\infty(M)$, then the problem is equivalent to solving the nonlinear elliptic equation:

$$\Delta u + K e^{2u} - K_0 = 0, \tag{1.1}$$

where Δ denotes the Beltrami–Laplacian of (M, g_0) and K_0 is the Gaussian curvature of g_0.

In dimension four, there is an analogous formulation of Eq. (1.1). Let (M, g) be a compact Riemannian four manifold, and let Ric and R denote respectively the Ricci tensor and the scalar curvature of g. A natural conformal invariant in dimension four is

$$Q = Q_g = -\frac{1}{12}(\Delta R - R^2 + 3|\mathrm{Ric}|^2).$$

Note that, under a conformal change of the metric

$$\tilde{g} = e^{2u} g,$$

the quantity Q transforms according to

$$2Q_{\tilde{g}} = e^{-4u}(Pu + 2Q_g), \tag{1.2}$$

where $P = P_g$ denotes the Paneitz operator with respect to g, introduced in [18]. The operator P_g acts on a smooth function u on M via

$$P_g(u) = \Delta_g^2 u + \mathrm{div}\left(\frac{2}{3} R_g - 2\mathrm{Ric}_g\right) du,$$

which plays a similar role as the Laplace operator in dimension two. Note that the Paneitz operator is conformally invariant in the sense that

$$P_{\tilde{g}} = e^{-4u} P_g$$

for any conformal metric $\tilde{g} = e^{2u} g$.

It follows from (1.2) that the expression $k = k_g := \int_M Q dV_g$ is conformally invariant. A natural problem to propose is to prescribe the Q-curvature: that is, to ask whether on a given four-manifold (M, g) there exists a conformal metric $\tilde{g} := e^{2u} g$ for which the Q-curvature of $\tilde{g}$ equals the prescribed function $\tilde{Q}$. This is related to solving the following equation

$$P_g u + 2Q_g = 2\tilde{Q} e^{4u}. \tag{1.3}$$

This equation is the Euler–Langrange equation of the functional

$$II_g(u) = \int_M u P_g u dV_g + 4\int_M Q_g u dV_g - \left(\int_M Q_g dV_g\right) \log \int_M \tilde{Q} e^{4u} dV_g. \tag{1.4}$$

A partial affirmative answer to the problem (1.3) in the case where $\tilde{Q}$ equals some constant is given by Chang–Yang [3] provided the Paneitz operator is weakly positive and the integral k is less than $8\pi^2$. In view of the result of Gursky [9] the former hypothesis is satisfied whenever $k > 0$ and provided (M, g) is of positive Yamabe type. The result of Chang–Yang has been extended recently by Djadli–Malchiodi [7] to the case in which P_g has no kernel and k is not a positive integer multiple of $8\pi^2$.

In the critical case, when $k = 8\pi^2$, the study of Eq. (1.3) becomes rather delicate. In this case, the functional II_g fails to satisfy standard compactness conditions like the Palais–Smale condition, and generally blow-up may occur. Note that when $(M, g) = (S^4, g_c)$, Eq. (1.3) is reduced to the following one

$$P_g u + 6 = 2\tilde{Q}e^{4u}. \tag{1.5}$$

This is the analogue of the well-known Nirenberg's problem. We should mention that, the blow-up phenomena for the Paneitz operator and other 4-th order elliptic equations have been deeply studied by Druert–Robert [8] and Weinstein–Zhang [21]. For other recent results, one can refer to [1,2,5,4,15,19,20,16]. We remark that, similar to Nirenberg's problem, there are some obstructions for the existence of the solution to Eq. (1.5) in the standard four-sphere case. The Gauss–Bonnet–Chern formula implies that there could not be a solution if $\tilde{Q} \le 0$. On the other hand, one has the identities of Kazdan–Warner type to this equation.

The main goal of this paper is to study Eq. (1.3) with critical value $k = 8\pi^2$ and positive $\tilde{Q}$. We shall pursue a variational approach which was used in [6]. Let (M, g) be any closed four dimensional Riemannian manifold with positive P_g, i.e., $\int_M uP_g u dV_g \ge 0$ and $\ker P_g = \{\text{constants}\}$. Then we have

$$\int_M uP_g u dV_g \ge \lambda \int_M |\nabla_g u|^2 dV_g, \quad \text{when } \int_M u dV_g = 0 \tag{1.6}$$

for some positive λ and the following improved Adams–Fontana inequality [3]:

$$\log \int_M e^{4u} dV_g \le \frac{1}{8\pi^2}\int_M uP_g u dV_g + \frac{1}{2\pi^2}\int_M u dV_g + C, \quad \forall u \in W^{2,2}(M). \tag{1.7}$$

We consider (for any small $\epsilon > 0$)

$$II_\epsilon(u) = \int_M \langle u, u\rangle dV_g + 4\left(1 - \frac{\epsilon}{8\pi^2}\right)\int_M Q_g u dV_g - (8\pi^2 - \epsilon)\log \int_M \tilde{Q}e^{4u} dV_g,$$

where we denote

$$\langle u, v\rangle = \Delta_g u \Delta_g v + \left(\frac{2}{3}R_g(\nabla u, \nabla v) - 2\mathrm{Ric}_g(\nabla u, \nabla v)\right).$$

By using the inequality (1.7), it is not so difficult to prove that

$$\inf II_\epsilon(u) > -\infty, \quad \forall \epsilon > 0, \text{ and moreover, } II_\epsilon \text{ has a minimum point } u_\epsilon.$$

For this minimizing sequence u_ϵ, two possibilities may occur: let $m_\epsilon = u_\epsilon(x_\epsilon) = \max_{x\in M} u_\epsilon(x)$,

(1) $\sup_\epsilon m_\epsilon < +\infty$, then, by passing to a subsequence, $\{u_\epsilon\}$ converges to some u_0 as $\epsilon \to 0$, and u_0 minimizes II;
(2) $m_\epsilon \to +\infty$, as $\epsilon \to 0$; We say, in this case, the u_ϵ blows up.

One of the main concern is to prove that, if the second case happens, then we find an explicit bound for the II_ϵ. More precisely, we have

$$\inf_{u\in W^{2,2}(M)} II(u) \ge \Lambda_g(\tilde{Q}, p), \tag{1.8}$$

where

$$\Lambda_g(\tilde{Q}, p) = -16\pi^2 \log \frac{\sqrt{3\tilde{Q}(p)}}{12} - 8\pi^2 \log 8\pi^2 - 16\pi^2 S_0(p) + 2\int_M Q G_p dV_g + (8/3 - 16)\pi^2,$$

p is the bubble point, and $S_0(p)$ is the constant term of the Green function at point p (see Appendix).

On the other hand, if we can construct some test function sequence ϕ_ϵ, s.t.

$$II(\phi_\epsilon) < \Lambda_g(\tilde{Q}, p),$$

we see that the blow-up does not happen. Therefore, we can get some sufficient condition under which (1.3) has a solution.

One of our main theorems in this paper is as follows.

Theorem 1.1. *Let (M, g) be a closed Riemannian manifold of dimension four, with $k = 8\pi^2$. Suppose P_g is positive and $\tilde{Q} > 0$. If $\inf_{u \in W^{2,2}(M)} II(u)$ is not attained, i.e. Eq. (1.3) has no minimal solution, then*

$$\inf_{u \in W^{2,2}(M)} II(u) = \inf_{p \in M} \Lambda_g(\tilde{Q}, p). \tag{1.9}$$

Now let p' be a point s.t.

$$\Lambda_g(\tilde{Q}, p') = \inf_{x \in M} \Lambda_g(\tilde{Q}, x),$$

we will prove that p' is in fact determined by the conformal class $[g]$ of (M, g).

Another main result in this paper is the existence theorem of Eq. (1.3).

Theorem 1.2. *Let (M, g) be a closed Riemannian manifold of dimension four, with $k = 8\pi^2$. Suppose P_g is positive. Let $\tilde{Q}$ be a positive smooth function on M. Assume that $\Lambda_g(\tilde{Q}, x)$ achieves its minimum at the point p'. If*

$$\tilde{Q}(p')\left(\Delta_g S(p') + 4|\nabla_g S(p')|^2 - \frac{R(p')}{18}\right) + \left[(2\nabla_g S \nabla_g \tilde{Q})(p') + \frac{1}{4}\Delta_g \tilde{Q}(p')\right] > 0,$$

then Eq. (1.3) has a minimal solution.

Corollary 1.3. *Under the assumption as in Theorem 1.2, if*

$$\Delta_g S(p') + 4|\nabla_g S(p')|^2 - \frac{R(p')}{18} > 0,$$

then M has a constant Q-curvature up to conformal transformations.

It is interesting to note that, in the four-dimensional case, the method in [6] cannot be directly used. Since Eq. (1.3) does not satisfy the Maximum Principle, the method used in [6] does not work here to calculate

$$\int_{B_\delta \setminus B_{Lr_\epsilon}(x_\epsilon)} |\Delta_g u_\epsilon|^2 dV_g. \tag{1.10}$$

We will apply the capacity to get the lower bound of (1.10). The usefulness of capacity in similar problems was first discovered by the second author, and has been used in [11,12].

2. Preliminary estimate

In this section we collect some useful preliminary facts and then derive some estimates for the solutions. We start with the following lemma.

Lemma 2.1. *For any $\epsilon > 0$, II_ϵ has a minimum point.*

Proof. By using the inequality (1.7), it is easy to see that, when $\int_M u dV_g = 0$, we have

$$\begin{aligned} II_\epsilon(u) &= \int_M uP_g u dV_g + 4\left(1 - \frac{\epsilon}{8\pi^2}\right)\int_M QudV_g - (8\pi^2 - \epsilon)\log\int_M \tilde{Q}e^{4u}dV_g \\ &\geq C + \frac{\epsilon}{8\pi^2}\int_M uP_g u dV_g + 4\left(1 - \frac{\epsilon}{8\pi^2}\right)\int_M QudV_g \\ &\geq C + \lambda\frac{\epsilon}{8\pi^2}\int_M |\nabla_g u|^2 dV_g + 4\left(1 - \frac{\epsilon}{8\pi^2}\right)\int_M QudV_g. \end{aligned}$$

For any $\epsilon_1 > 0$, we have

$$\int_M QudV_g \leq \epsilon_1\int_M |u|^2 + C_\epsilon \leq \lambda_0\epsilon_1\int_M |\nabla u|^2 dV_g + C_\epsilon,$$

where λ_0 is the first eigenvalue of Δ. Then,

$$\int_M |\nabla_g u|^2 dV_g \leq C(\epsilon)II_\epsilon(u) + C \tag{2.1}$$

and then

$$\int_M |\Delta_g u|^2 dV_g \leq \frac{8\pi}{\epsilon}II_\epsilon(u) + C. \tag{2.2}$$

Let $u_k = u_{\epsilon,k}$ be a minimizing sequence of II_ϵ, i.e.

$$II_\epsilon(u_k) \to \inf II_\epsilon(u) = A,$$

which, together with the above inequality, implies that

$$\int_M |\Delta_g u_k|^2 dV_g \leq C,$$

for some constant C which may depend on ϵ. Therefore, by passing to a subsequence, we have $u_k \rightharpoonup u_\epsilon$ and

$$\int_M |\Delta_g u_k|^2 dV_g \to B.$$

Since the functional II_ϵ is invariant under a translation by a constant, we may assume that $\int_M u_k dV_g = 0$, then by (1.7), we can see that $e^{4u_k} \in L^p$ for any $p > 0$.

Set

$$II_\epsilon(u_k) := \int_M |\Delta_g u_k|^2 dV_g + \int_M F(u_k)dV_g,$$

then we have,

$$\lim_{k\to+\infty}\int_M F(u_k)dV_g = A - B, \quad \text{and}$$

$$\lim_{k\to+\infty,m\to+\infty}\int_M F\left(\frac{u_k+u_m}{2}\right)dV_g = A - B.$$

Since $II_\epsilon(\frac{u_k+u_m}{2}) \geq A$, we have

$$\frac{1}{4}\int_M (|\Delta_g u_k|^2 + |\Delta_g u_m|^2)dV_g + \frac{1}{2}\int_M \Delta_g u_k \Delta_g u_m dV_g \geq B.$$

Hence

$$\lim_{k\to+\infty,m\to+\infty}\int_M \Delta_g u_k \Delta_g u_m dV_g \geq B.$$

Then

$$\begin{aligned}&\lim_{k\to+\infty,m\to+\infty}\int_M |\Delta_g(u_k-u_m)|^2 dV_g\\ &= \lim_{k\to+\infty,m\to+\infty}\left(\int_M |\Delta_g u_k|^2 dV_g + \int_M |\Delta_g u_m|^2 dV_g - 2\int_M \Delta_g u_k \Delta_g u_m dV_g\right)\\ &\leq 0.\end{aligned}$$

Therefore, $\{u_k\}$ is a Cauchy sequence in $W^{2,2}(M)$. □

Lemma 2.2. *We have*

$$\lim_{\epsilon\to 0}\inf II_\epsilon = \inf II.$$

Proof. Obviously,

$$\begin{aligned}II_\epsilon(u) &= \int_M uP_g u dV_g + 4\left(1-\frac{\epsilon}{8\pi^2}\right)\int_M Qu dV_g - (8\pi^2-\epsilon)\log\int_M \tilde{Q}e^{4u}dV_g\\ &= \int_M uP_g u dV_g + 4\int_M Qu dV_g - 8\pi^2\log\int_M \tilde{Q}e^{4u}dV_g\\ &\quad - \frac{4\epsilon}{8\pi^2}\int_M Qu dV_g + \epsilon\log\int_M \tilde{Q}e^{4u}dV_g\\ &= II(u) - \frac{4\epsilon}{8\pi^2}\int_M Qu dV_g + \epsilon\log\int_M \tilde{Q}e^{4u}dV_g.\end{aligned}$$

Let u_k satisfy

$$\lim_{k\to+\infty} II(u_k) = \inf II.$$

Then for any $\epsilon > 0$ and fixed u_k, we have

$$\inf II_\epsilon \leq II_\epsilon(u_k) = II(u_k) - \frac{4\epsilon}{8\pi^2}\int_M Q_g u_k dV_g + \epsilon\log\int_M \tilde{Q}e^{4u_k}.$$

Letting $\epsilon\to 0$, we get

$$\overline{\lim_{\epsilon\to 0}}(\inf II_\epsilon) \leq II(u_k).$$

Then letting $k \to +\infty$, we get

$$\overline{\lim_{\epsilon\to 0}}(\inf II_\epsilon) \le \inf II.$$

Next, we prove

$$\underline{\lim}_{\epsilon\to 0}(\inf II_\epsilon) \ge \inf II. \tag{2.3}$$

Let u_ϵ attain $\inf II_\epsilon$. Since $II_\epsilon(u+c) = II_\epsilon(u)$, we may assume $\int_M u_\epsilon dV_g = 0$. Obviously,

$$II_\epsilon(u_\epsilon) = \left(1 - \frac{\epsilon}{8\pi^2}\right) II(u_\epsilon) + \frac{\epsilon}{8\pi^2}\int_M u_\epsilon P_g u_\epsilon.$$

By (1.6), we have

$$\inf II_\epsilon = II_\epsilon(u_\epsilon) \ge \left(1 - \frac{\epsilon}{8\pi^2}\right) II(u_\epsilon) \ge \left(1 - \frac{\epsilon}{8\pi^2}\right)\inf II.$$

Letting $\epsilon \to 0$, we get (2.3). □

Now let u_ϵ be the minimum point of II_ϵ. It is clear that u_ϵ satisfies the following equation:

$$\begin{cases} P_g u_\epsilon + 2\left(1 - \frac{\epsilon}{8\pi^2}\right) Q_g = 2\left(1 - \frac{\epsilon}{8\pi^2}\right)\tilde{Q} e^{4u_\epsilon} \\ \int_M \tilde{Q} e^{4u_\epsilon} dV_g = 8\pi^2. \end{cases}$$

The same proof of Lemma 2.3 in [14] yields the following.

Lemma 2.3. *There are constants $C_1(q), C_2(q), C_3(q)$ depending only on p and M such that, for r sufficiently small and for any $x \in M$ there holds*

$$\int_{B_r(x)} |\nabla^3 u_\epsilon|^q dV_g \le C_1(q) r^{4-3q}, \qquad \int_{B_r(x)} |\nabla^2 u_\epsilon|^q dV_g \le C_2(q) r^{4-2q},$$

and

$$\int_{B_r(x)} |\nabla u_\epsilon|^q dV_g \le C_3(q) r^{4-q}$$

where, respectively, $q < \frac{4}{3}$, $q < 2$, and $q < 4$.

3. The proof of Theorem 1.1

Let x_ϵ be the maximum point of u_ϵ. Assume $m_\epsilon = u_\epsilon(x_\epsilon)$, $r_\epsilon = e^{-m_\epsilon}$, and $x_\epsilon \to p$. Let $\{e_i(x)\}$ be an orthonormal basis of TM near p and $\exp_x : T_x M \to M$ be the exponential mapping. The smooth mapping $E : B_\delta(p) \times B_r \to M$ is defined as follows,

$$E(x, y) = \exp_x(y^i e_i(x)),$$

where B_r is a small ball in $\mathbb{R}^n$. Note that $E(x, \cdot) : T_x M \to M$ are all differential homeomorphism if r is sufficiently small.

We set

$$g_{ij}(x, y) = \left\langle (\exp_x)_* \frac{\partial}{\partial y^i}, (\exp_x)_* \frac{\partial}{\partial y^j} \right\rangle_{E(x,y)}.$$

It is well-known that $g = (g_{ij})$ is smooth, and $g(x, y) = I + O(|y|^2)$ for any fixed x. That is, we are able to find a constant K, s.t.

$$\|g(x, y) - I\|_{C^0(B_\delta(p)\times B_r)} \le K|y|^2$$

when δ and r are sufficiently small. Moreover, for any $\varphi \in C^\infty(B_\rho(x_k))$ we have

$$\Delta_g u_\epsilon = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^k}\left(\sqrt{|g|}g^{km}\frac{\partial u_\epsilon(E(x_\epsilon, x))}{\partial x^m}\right),$$
$$|\nabla u_\epsilon|^2 = g^{pq}\frac{\partial u_\epsilon(E(x_\epsilon, x))}{\partial x^p}\frac{\partial u_\epsilon(E(x_\epsilon, x))}{\partial x^q},$$

and

$$\int_{B_\delta(x_k)} \varphi dV_g = \int_{E^{-1}(x_k,y)B_{\delta(x_k)}} \varphi(E^{-1}(x_k, y))\sqrt{|g|}dy.$$

We define

$$\tilde{u}_\epsilon(x) = u_\epsilon(E(x_\epsilon, x)),$$

and

$$v_\epsilon(x) = \tilde{u}_\epsilon(r_\epsilon x), \qquad v'_\epsilon = v_\epsilon - m_\epsilon. \tag{3.1}$$

Now v_ϵ, v'_ϵ are functions defined on $B_{\frac{r}{2r_\epsilon}} \subset \mathbb{R}^n$.

We have

$$\Delta^2_{g_\epsilon} v'_\epsilon = r_\epsilon^2 O(|\nabla^2 v'_\epsilon|) + r_\epsilon^3 O(\nabla v'_\epsilon) + \tilde{Q}_g(E(x_\epsilon, r_\epsilon x))e^{4v'_\epsilon}. \tag{3.2}$$

It follows from Lemma 2.3 that,

$$\|\nabla^2 v'_\epsilon\|_{L^q(B_L)} \le C(L, q) \quad \text{and} \quad \|\nabla v'_\epsilon\|_{L^q(B_L)} \le C'(L, q) \quad \text{for any } q \in (1, 2).$$

Then (3.2) implies that

$$\|\Delta_{g_\epsilon}(\Delta_{g_\epsilon} v'_\epsilon)\|_{L^q(B_L)} \le C'(L).$$

Using the standard elliptic estimates, we get

$$\|\Delta_{g_k} v'_\epsilon\|_{W^{2,q}(B_L)} \le C_2(L).$$

The Sobolev inequality then yields,

$$\|\Delta_{g_\epsilon} v'_\epsilon\|_{L^q(B_L)} \le C_3(q, L) \quad \text{for any } q \in (0, 4).$$

We therefore have

$$\|v'_\epsilon\|_{W^{2,q}(B_L)} \le C_4(L).$$

Hence, by using the standard elliptic estimates, we see that v'_ϵ converge smoothly to w, which satisfies

$$\Delta_0^2 w = 2\tilde{Q}(p)e^{4w},$$

where Δ_0 is the Laplace operator in $\mathbb{R}^4$. Moreover, it is easy to check that

$$\int_{B_L} \tilde{Q}(p)e^{4w}dx \le 8\pi^2$$

for any $L > 0$. By the result of [13], we have

(a) $w = -\log(1 + \frac{\sqrt{3\tilde{Q}(p)}}{12}|x|^2)$, with

$$\tilde{Q}(p)\int_{\mathbb{R}^4} e^{4w} dV_g = 8\pi^2,$$

or

(b) w has the following asymptotic behavior:

$$-\Delta w \to a > 0 \quad \text{as } |x| \to +\infty.$$

We claim that (b) does not happen. If it does, then we have

$$\lim_{\epsilon\to+0}\int_{B_R} -\Delta_g v_\epsilon \sim \frac{\omega_3}{4} a R^4.$$

However, it follows from Lemma 2.3 that

$$\int_{B_R} |\Delta_{g_\epsilon} v'_\epsilon| dV_g \le CR^2.$$

This shows that the case (b) does not happen.

For simplicity, let $\lambda = \frac{\sqrt{3Q(p)}}{12}$, so that we have

$$w = -\log(1 + \lambda|x|^2).$$

Now, we consider the convergence of u_ϵ outside the bubble. By Lemma 2.3, u_ϵ is bounded in $W^{3,q}$ for any $q < \frac{4}{3}$. Then, it is easy to check that $u_\epsilon - \overline{u}_\epsilon \rightharpoonup G_p$, where $\overline{u}_\epsilon = \frac{1}{|M|}\int_M u_\epsilon dV_g$ and

$$P_g G_p + 2Q_g = 16\pi^2 \delta_p, \qquad \int_M G_p dV_g = 0.$$

To prove the strong convergence of $u_\epsilon - \overline{u}_\epsilon$, we first show the following lemma.

Lemma 3.1. *Given $\Omega \subset\subset M \setminus \{p\}$, there holds*

$$\int_\Omega e^{q(u_\epsilon - \overline{u}_\epsilon)} dV_g < C(\Omega, q)$$

for any $q > 0$.

Proof. Let $f_\epsilon = \tilde{Q}_g e^{4u_\epsilon}$. For any $x \in \Omega$, we have the following representation formula,

$$u_\epsilon(x) - \overline{u}_\epsilon = -\int_M G(x,y) Q_g dV_{g,y} + \int_M G(x,y) f_\epsilon.$$

Hence, if we let $\Omega_\epsilon = M \setminus B_{Lr_\epsilon}(x_\epsilon)$, and $\mu_\epsilon = 1/\int_{\Omega_\epsilon} |f_\epsilon| dV_g$, we have, for any $q' > 0$,

$$e^{q'\mu_\epsilon(u_\epsilon - \overline{u}_\epsilon + \int_M G(x,y)Q_g dV_g)} = e^{\int_{\Omega_\epsilon} q' G(x,y)\mu_\epsilon f_\epsilon(y) dV_{g,y} + \int_{B_{Lr_\epsilon}} q' G(x,y)\mu_\epsilon f_\epsilon(y) dV_{g,y}}.$$

Notice that for any $x \in \Omega$ and $y \in B_{Lr_\epsilon}(x_\epsilon)$, $|G(x,y)| < C(\Omega, L)$. We have

$$\int_{B_{Lr_\epsilon}(x_\epsilon)} q'|G(x,y)|\mu_\epsilon f_\epsilon(y) dV_{g,y} \le C_1(L)\int_{B_{Lr_\epsilon}(x_\epsilon)} f_\epsilon(y) dV_g \le C_2(L),$$

and

$$e^{\int_{\Omega_\epsilon} q'G(x,y)\mu_\epsilon f_\epsilon(y)dV_{g,y}} \le \int_{\Omega_\epsilon} \frac{f_\epsilon(y)}{\|f_\epsilon\|_{L^1(\Omega_\epsilon)}} e^{q'G(x,y)} dV_{g,y}.$$

Therefore, by using Jensen's inequality and Fubini's theorem, we obtain

$$\begin{aligned}\int_\Omega e^{\int_{\Omega_\epsilon} q'G(x,y)\mu_\epsilon f_\epsilon(y)dV_{g,y}} dV_g &\le \int_\Omega \frac{f_\epsilon(y)}{\|f_\epsilon\|_{L^1(\Omega_\epsilon)}} \left(\int_{\Omega_\epsilon} e^{q'G(x,y)} dV_{g,x}\right) dV_{g,y} \\ &\le C\int_\Omega \frac{f_\epsilon(y)}{\|f_\epsilon\|_{L^1(\Omega_\epsilon)}} \left(\int_{\Omega_\epsilon} \frac{1}{|x-y|^{\frac{q'}{8\pi^2}}} dV_{g,x}\right) dV_{g,y}.\end{aligned}$$

The last integral is finite provided $q' < 32\pi^2$. Hence, for any $q > 0$, if ϵ is sufficiently small so that $q \le q'\mu_\epsilon$ we have

$$\begin{aligned}\int_\Omega e^{q(u_\epsilon(x)-\overline{u}_\epsilon)} dx &\le \int_\Omega e^{q'\mu_\epsilon(u_\epsilon(x)-\overline{u}_\epsilon)} dx \\ &\le C\int_\Omega e^{\int_{\Omega_\epsilon} q'G(x,y)\mu_\epsilon f_\epsilon(y)dV_{g,y}} dV_g \le C. \quad \square\end{aligned}$$

As a consequence of the above lemma, we have the following lemma.

Lemma 3.2. *Let $\Omega \subset\subset M \setminus \{x_0\}$. Then $u_\epsilon - \overline{u}_\epsilon$ converges to G_{x_0} in $C^k(\Omega)$ as $\epsilon \to 0$.*

Proof. It is easy to see that $\overline{u}_\epsilon < C$. Then the lemma follows. $\square$

Remark 3.3. In B_{δ_0}, using the above coordinates, we set $p = y_\epsilon$ for any ϵ. Clearly, $y_\epsilon \to 0$. Then we also have $u_\epsilon(E(p,x)) - \overline{u}_\epsilon \to G_p(E(p,x))$. Moreover, we may write

$$G(E(p,x)) = -2\log|x| + S_0(p) + S_1(x),$$

where $S_0(p)$ is a constant and $S_1 = O(r)$. It is easy to check $\tilde{u}_\epsilon - \overline{u}_\epsilon \to G(E(p,x))$ smoothly in $B_{\delta_0} \setminus B_\delta$ for any fixed δ.

Now, we give a lower bound of $\lim_{\epsilon\to 0}\int_M \langle u_\epsilon, u_\epsilon\rangle dV_g$. We write

$$\int_M \langle u_\epsilon, u_\epsilon\rangle dV_g = I_1 + I_2 + I_3,$$

where I_1, I_2, I_3 denote the integrals on $M \setminus B_\delta(x_\epsilon)$, $B_{Lr_\epsilon}(x_\epsilon)$ and $B_\delta \setminus B_{Lr_\epsilon}(x_\epsilon)$ (any fixed L and δ) respectively. We remark that the integral I_1, I_2 can be easily treated due to the above lemmas. On the other hand, by Lemma 2.3, we have

$$\int_{B_\delta\setminus B_{Lr_\epsilon}(x_\epsilon)} |\nabla_g u_\epsilon|^2 dV_g \to \int_{B_\delta(p)} |\nabla_g G|^2 = O(\delta^2).$$

So, the key point is to calculate

$$\int_{B_\delta(x_\epsilon)\setminus B_{Lr_\epsilon}(x_\epsilon)} |\Delta_g u_\epsilon|^2 dV_g.$$

We are going to prove the following lemma.

Lemma 3.4. *We have*

$$\int_{B_\delta(x_\epsilon)\setminus B_{Lr_\epsilon}(x_\epsilon)} |\Delta_g u_\epsilon|^2 dV_g \geq \int_{B_\delta\setminus B_{Lr_\epsilon}} |(1-b|x|^2)\Delta_0\tilde{u}_\epsilon|^2 dx + J(L,\epsilon,\delta),$$

for some $b>0$, where

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} J(L,\epsilon,\delta)=0.$$

Proof. Since we have

$$\begin{aligned}|\Delta_g u_\epsilon|^2 &= \left|g^{km}\frac{\partial^2\tilde{u}_\epsilon}{\partial x^k\partial x^m} + O(|\nabla\tilde{u}_\epsilon|)\right|^2\\ &= \left|g^{km}\frac{\partial^2\tilde{u}_\epsilon}{\partial x^k\partial x^m}\right|^2 + O(|\nabla^2\tilde{u}_\epsilon|(|\nabla\tilde{u}_\epsilon|)) + O((|\nabla\tilde{u}_\epsilon|^2)),\end{aligned}$$

and since $\tilde{u}_\epsilon - \overline{u}_\epsilon$ converges to $G_p(E(p,x))$ in $W^{3,q}$ for any $q<\frac{4}{3}$, we get

$$\begin{aligned}&\int_{B_\delta\setminus B_{Lr_\epsilon}} O|\nabla^2\tilde{u}_\epsilon|(|\nabla\tilde{u}_\epsilon|) + O(|\nabla\tilde{u}_\epsilon|^2)\\ &\quad\leq C(\|\nabla^2 G_p\|_{L^q(B_\delta\setminus B_{Lr_\epsilon})}\|\nabla_g G_p\|_{L^{q'}}(B_\delta\setminus B_{Lr_\epsilon}) + \|G_p\|_{W^{1,2}(B_\delta\setminus B_{Lr_\epsilon})})\\ &\quad= J(L,\epsilon,\delta),\end{aligned}$$

where $\frac{3}{2}<q<2$, and $\frac{1}{q'}+\frac{1}{q}=1$.

Let $g^{km}=\delta^{km}+A^{km}$, with $|A^{km}|\leq K|x|^2$ for any ϵ,k,m. Consequently, we have

$$\left|g^{km}\frac{\partial^2\tilde{u}_\epsilon}{\partial x^k\partial x^m}\right|^2 = |\Delta_0\tilde{u}_\epsilon|^2 + 2\sum_{s,t}A^{st}\Delta_0\tilde{u}_\epsilon\frac{\partial^2\tilde{u}_\epsilon}{\partial x^s\partial x^t} + \sum_{k,m,s,t}A^{km}A^{st}\frac{\partial^2\tilde{u}_\epsilon}{\partial x^k\partial x^m}\frac{\partial^2\tilde{u}_\epsilon}{\partial x^s\partial x^t}.$$

It is clear that

$$2\int_{B_\delta\setminus B_{Lr_\epsilon}}\left|A^{st}\Delta_0\tilde{u}_\epsilon\frac{\partial^2\tilde{u}_\epsilon}{\partial x^s\partial x^t}\right| \leq K\int_{B_\delta\setminus B_{Lr_\epsilon}}\left(|x|^2|\Delta_0\tilde{u}_\epsilon|^2 + |x|^2\left|\frac{\partial^2\tilde{u}_\epsilon}{\partial x^s\partial x^t}\right|^2\right)dx,$$

and

$$\begin{aligned}\int_{B_\delta\setminus B_{Lr_\epsilon}}|x|^2\left|\frac{\partial^2\tilde{u}_\epsilon}{\partial x^s\partial x^t}\right|^2 dx &= \int_{B_\delta\setminus B_{Lr_\epsilon}}|x|^2\frac{\partial^2\tilde{u}_\epsilon}{\partial x^t\partial x^t}\frac{\partial^2\tilde{u}_\epsilon}{\partial x^s\partial x^s}dx\\ &\quad+\int_{B_\delta\setminus B_{Lr_\epsilon}} O(|x|\,|\nabla\tilde{u}_\epsilon|\,|\nabla^2\tilde{u}_\epsilon|)dx\\ &\quad+\int_{\partial(B_\delta\setminus B_{Lr_\epsilon})}|x|^2\frac{\partial\tilde{u}_\epsilon}{\partial x^t}\frac{\partial^2\tilde{u}_\epsilon}{\partial x^s\partial x^t}\left\langle\frac{\partial}{\partial x^t},\frac{\partial}{\partial r}\right\rangle ds\\ &\quad+\int_{\partial(B_\delta\setminus B_{Lr_\epsilon})}|x|^2\left(\frac{\partial\tilde{u}_\epsilon}{\partial x^t}\frac{\partial^2\tilde{u}_\epsilon}{\partial x^s\partial x^s}\left\langle\frac{\partial}{\partial x^s},\frac{\partial}{\partial r}\right\rangle\right)ds\\ &= \int_{B_\delta\setminus B_{Lr_\epsilon}}|x|^2\frac{\partial^2\tilde{u}_\epsilon}{\partial x^t\partial x^t}\frac{\partial^2\tilde{u}_\epsilon}{\partial x^s\partial x^s}dx + J(L,\epsilon,\delta).\end{aligned}$$

On $\partial B_\delta(x_\epsilon)$, since $\tilde{u}_\epsilon - \overline{u}_\epsilon \to G_p(E(p,x))$, as $\epsilon \to 0$, we have

$$\begin{aligned}
&\int_{\partial B_\delta} |x|^2 \frac{\partial \tilde{u}_\epsilon}{\partial x^i} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^j \partial x^k} \left\langle \frac{\partial}{\partial x^s}, \frac{\partial}{\partial r} \right\rangle ds \\
&\quad \to \int_{\partial B_\delta} |x|^2 \left(\frac{\partial G_p(E(p,x))}{\partial x^i} \frac{\partial^2 G_p(E(p,x))}{\partial x^j \partial x^k} \left\langle \frac{\partial}{\partial x^s}, \frac{\partial}{\partial r} \right\rangle \right) ds \\
&\quad = \int_{\partial B_\delta} O\left(\frac{1}{\delta}\right) ds \\
&\quad = O(\delta^2).
\end{aligned}$$

On ∂B_{Lr_ϵ}, since $\tilde{u}_k(r_\epsilon x) - m_\epsilon \to \omega$ as $\epsilon \to 0$, we have

$$\frac{1}{r_\epsilon^2} \int_{\partial B_{Lr_\epsilon}} |x|^2 \frac{\partial \tilde{u}_\epsilon}{\partial x^i} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^j \partial x^k} \left\langle \frac{\partial}{\partial x^s}, \frac{\partial}{\partial r} \right\rangle ds \to \int_{\partial B_L} |x|^2 \frac{\partial \omega}{\partial x^i} \frac{\partial^2 \omega}{\partial x^j \partial x^k} \left\langle \frac{\partial}{\partial x^s}, \frac{\partial}{\partial r} \right\rangle ds.$$

Then we get

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} \int_{\partial(B_\delta \setminus B_{Lr_\epsilon})} |x|^2 \frac{\partial \tilde{u}_\epsilon}{\partial x^i} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^j \partial x^k} \left\langle \frac{\partial}{\partial x^s}, \frac{\partial}{\partial r} \right\rangle ds = 0.$$

Moreover,

$$2 \sum_{k,s,t} \int_{B_\delta \setminus B_{Lr_\epsilon}} \left| A^{st} \Delta_0 \tilde{u}_\epsilon \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t} \right| \le 4K \int_{B_\delta \setminus B_{Lr_\epsilon}} |x|^2 |\Delta_0 \tilde{u}_\epsilon|^2 dx + J(L,\epsilon,\delta).$$

A similar argument as above then gives

$$\int_{B_\delta \setminus B_{Lr_\epsilon}} \sum_{k,m,s,t} A^{km} A^{st} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^k \partial x^m} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t} \le K^2 \int_{B_\delta \setminus B_{Lr_\epsilon}} |x|^4 |\Delta_0 \tilde{u}_\epsilon|^2 dx + J(L,\epsilon,\delta).$$

This proves the lemma. □

Lemma 3.5. *There is a function sequence $U_\epsilon \in W^{2,2}(B_\delta \setminus B_{Lr_\epsilon})$ s.t.*

$$\begin{aligned}
&U_\epsilon|_{\partial B_\delta} = -2\log\delta + S_0(p) + \overline{u}_\epsilon, \qquad U_\epsilon|_{\partial B_{Lr_\epsilon}} = w(L) + m_\epsilon \\
&\left.\frac{\partial U_\epsilon}{\partial r}\right|_{\partial B_\delta} = -\frac{2}{\delta}, \qquad \left.\frac{\partial U_\epsilon}{\partial r}\right|_{\partial B_{Lr_\epsilon}} = w'(L)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_\delta \setminus B_{Lr_\epsilon}} \left| \Delta_0 \left((1 - b|x|^2)(U_\epsilon - \overline{u}_\epsilon) \right) \right|^2 dx \\
&\quad = \int_{B_\delta \setminus B_{Lr_\epsilon}} |(1 - b|x|^2) \Delta_0 \tilde{u}_\epsilon|^2 dx + J(L,\epsilon,\delta),
\end{aligned}$$

where

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} J(L,\epsilon,\delta) = 0.$$

Proof. Let u'_ϵ be the solution of

$$\begin{cases} \Delta_0^2 u'_\epsilon = \Delta_0^2 v_\epsilon \\ \left.\dfrac{\partial u'_\epsilon}{\partial n}\right|_{\partial B_{2L}} = \left.\dfrac{\partial v_\epsilon}{\partial n}\right|_{\partial B_{2L}}, \qquad u'_\epsilon|_{\partial B_{2L}} = v_\epsilon|_{\partial B_{2L}} \\ \left.\dfrac{\partial u'_\epsilon}{\partial n}\right|_{\partial B_L} = \left.\dfrac{\partial w}{\partial n}\right|_{\partial B_L}, \qquad u'_\epsilon|_{\partial B_L} = m_\epsilon + w|_{\partial B_L}, \end{cases}$$

where v_ϵ is defined by (3.1). We set

$$U'_\epsilon = \begin{cases} u'_\epsilon\left(\dfrac{x}{r_\epsilon}\right) & Lr_\epsilon \le |x| \le 2Lr_\epsilon \\ \tilde{u}_\epsilon(x) & 2Lr_\epsilon \le |x|. \end{cases}$$

It is easy to see that $u'_\epsilon - m_\epsilon$ converges to w smoothly on $B_{2L} \setminus B_L$; then we have

$$\lim_{\epsilon\to 0}\int_{B_{2Lr_\epsilon}\setminus B_{Lr_\epsilon}} (1-b|x|^2)^2(|\Delta_0 U'_\epsilon|^2 - |\Delta_0\tilde{u}_\epsilon|^2)dx = 0.$$

Let η be a smooth function which satisfies:

$$\eta(t) = \begin{cases} 1 & t \le 1/2 \\ 0 & t > 2/3. \end{cases}$$

Set $G_\epsilon = \eta(\frac{|x|}{\delta})(\tilde{u}_\epsilon - S_0(p) + 2\log|x|^2 - \overline{u}_\epsilon) - 2\log|x|^2 + S_0(p)$. Recall that $u_\epsilon - \overline{u}_\epsilon$ converges to G_p smoothly on $M \setminus B_{\frac{\delta}{2}}(p)$; then we have

$$G_\epsilon \to -2\log|x|^2 + S_0(p) + \eta\left(\frac{|x|}{\delta}\right)S_1(x),$$
$$\tilde{u}_\epsilon - G_\epsilon - \overline{u}_\epsilon \to \left(\eta\left(\frac{|x|}{\delta}\right) - 1\right)S_1(x).$$

Therefore

$$\begin{aligned} &\lim_{\epsilon\to 0}\left|\int_{B_\delta\setminus B_{\delta/2}} |\Delta_0\tilde{u}_\epsilon|^2 dx - \int_{B_\delta\setminus B_{\delta/2}} |\Delta_0 G_\epsilon|^2 dx\right| \\ &= \left|\int_{B_\delta\setminus B_{\delta/2}} |\Delta_0 G_p(E(p,x))|^2 dx - \int_{B_\delta\setminus B_{\delta/2}} |\Delta_0 G_\epsilon|^2 dx\right| \\ &= \left|\int_{B_\delta\setminus B_{\delta/2}} \Delta_0 G_p(E(p,x) + G_\epsilon)dx \int_{B_\delta\setminus B_{\delta/2}} \Delta_0(G_0(E(p,x)) - G_\epsilon)dx\right| \\ &\le \sqrt{\int_{B_\delta\setminus B_{\delta/2}} \left|\Delta_0\left(\eta\left(\frac{|x|}{\delta}\right)-1\right)S_1(x)\right|^2 dx \int_{B_\delta\setminus B_{\delta/2}} \left|\Delta_0\left(G_p - 2\log|x|^2 + \eta\left(\frac{|x|}{\delta}\right)S_1(x)\right)\right|^2 dx} \\ &\le C\sqrt{\delta|\log\delta|}. \end{aligned}$$

Now set

$$U_\epsilon = \begin{cases} U'_\epsilon(x) & |x| \le \dfrac{\delta}{2} \\ G_\epsilon(x) + \overline{u}_\epsilon & \delta/2 \le |x| \le \delta. \end{cases}$$

We then have,

$$\int_{B_\delta\setminus B_{L\epsilon}}|(1-B|x|^2)\Delta_0(U_\epsilon-\overline{u}_\epsilon)|^2dx=\int_{B_\delta\setminus B_{Lr_\epsilon}}|\Delta_0(1-B|x|^2)(U_\epsilon-\overline{u}_\epsilon)|^2dx$$
$$+\int_{B_\delta\setminus B_{Lr_\epsilon}}O(|\nabla U_\epsilon|^2+|U_\epsilon-\overline{u}_\epsilon|^2)dV_g.$$

To complete the proof, we only need to prove

$$\lim_{L\to+\infty}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\|U_\epsilon-\overline{u}_\epsilon\|_{W^{1,2}(B_\delta\setminus B_{Lr_\epsilon})}=0. \tag{3.3}$$

We have

$$\begin{aligned}\|U_\epsilon-\overline{u}_\epsilon\|^2_{W^{1,2}(B_\delta\setminus B_{Lr_\epsilon})}&=\|U_\epsilon-\overline{u}_\epsilon\|^2_{W^{1,2}(B_\delta\setminus B_{\delta/2})}+\|U_\epsilon-\overline{u}_\epsilon\|^2_{W^{1,2}(B_{\delta/2}\setminus B_{2Lr_\epsilon})}\\&\quad+\|U_\epsilon-\overline{u}_\epsilon\|^2_{W^{1,2}(B_{\delta/2}\setminus B_{2Lr_\epsilon})}+\|U_\epsilon-\overline{u}_\epsilon\|^2_{W^{1,2}(B_{2Lr_\epsilon}\setminus B_{Lr_\epsilon})}\\&=\|G_\epsilon\|^2_{W^{1,2}(B_\delta\setminus B_{\delta/2})}+\|\tilde{u}_\epsilon-\overline{u}_\epsilon\|^2_{W^{1,2}(B_{\delta/2}\setminus B_{2Lr_\epsilon})}\\&\quad+\|\tilde{u}_\epsilon-\overline{u}_\epsilon\|^2_{W^{1,2}(B_{2Lr_\epsilon}\setminus B_{Lr_\epsilon})}+\|U'_\epsilon-\tilde{u}_\epsilon\|^2_{W^{1,2}(B_{2Lr_\epsilon}\setminus B_{Lr_\epsilon})}\\&\le\|G_\epsilon\|^2_{W^{1,2}(B_\delta\setminus B_{\delta/2})}+\|\tilde{u}_\epsilon-\overline{u}_\epsilon\|^2_{W^{1,2}(B_{\delta/2})}\\&\quad+\|U'_\epsilon-\tilde{u}_\epsilon\|^2_{W^{1,2}(B_{2Lr_\epsilon}\setminus B_{Lr_\epsilon})}.\end{aligned}$$

It is easy to check that

$$\lim_{\epsilon\to 0}\|U'_\epsilon-\tilde{u}_\epsilon\|^2_{W^{1,2}(B_{2Lr_\epsilon}\setminus B_{Lr_\epsilon}(x_\epsilon))}=0.$$

Recall $\tilde{u}_\epsilon-\overline{u}_\epsilon\to G_p(E(p,x))$. We get (3.3). □

Now, we are going to apply capacity estimate to derive the lower bound for

$$\int_{B_\delta\setminus B_{Lr_\epsilon}}\left|\Delta_0\left((1-b|x|^2)(U_\epsilon-\overline{u}_\epsilon)\right)\right|^2dx.$$

First we need to calculate

$$\inf_{\Phi|_{\partial B_r}=P_1,\,\Phi|_{\partial B_R}=P_2,\,\frac{\partial\Phi}{\partial r}\big|_{\partial B_r}=Q_1,\,\frac{\partial\Phi}{\partial r}\big|_{\partial B_R}=Q_2}\int_{B_R\setminus B_r}|\Delta_0\Phi|^2dx,$$

where P_1, P_2, Q_1, Q_2 are constants. Obviously, the minimum can be attained by the function Φ which satisfies

$$\begin{cases}\Delta_0^2\Phi=0\\ \Phi|_{\partial B_r}=P_1,\qquad \Phi|_{\partial B_R}=P_2,\qquad \left.\dfrac{\partial\Phi}{\partial r}\right|_{\partial B_r}=Q_1,\qquad \left.\dfrac{\partial\Phi}{\partial r}\right|_{\partial B_R}=Q_2.\end{cases}$$

Clearly, we can set

$$\Phi=A\log r+Br^2+\frac{C}{r^2}+D,$$

where A, B, C, D are all constants. Then we have

$$\begin{cases} A\log r + Br^2 + \dfrac{C}{r^2} + D = P_1 \\ A\log R + BR^2 + \dfrac{C}{R^2} + D = P_2 \\ \dfrac{A}{r} + 2Br - 2\dfrac{C}{r^3} = Q_1 \\ \dfrac{A}{R} + 2BR - 2\dfrac{C}{R^3} = Q_2. \end{cases}$$

We have

$$\begin{cases} A = \dfrac{P_1 - P_2 + \frac{\varrho}{2} r Q_1 + \frac{\varrho}{2} R Q_2}{\log r/R + \varrho} \\ B = \dfrac{-2P_1 + 2P_2 - rQ_1\left(1 + \frac{2r^2}{R^2-r^2}\log r/R\right) + RQ_2\left(1 + \frac{2R^2}{R^2-r^2}\log r/R\right)}{4(R^2+r^2)(\log r/R + \varrho)}, \end{cases}$$

where $\varrho = \frac{R^2-r^2}{R^2+r^2}$. Furthermore,

$$\int_{B_R\setminus B_r} |\Delta_0 \Phi|^2 dx = -8\pi^2 A^2 \log r/R + 32\pi^2 AB(R^2 - r^2) + 32\pi^2 B^2(R^4 - r^4).$$

In our case, $R = \delta$, $r = Lr_\epsilon$,

$$P_1 = (1 - B|x|^2)U_\epsilon|_{\partial B_{Lr_\epsilon}} = m_\epsilon - \overline{u}_\epsilon + w(L) + O(r_\epsilon \overline{u}_\epsilon),$$
$$P_2 = (1 - B|x|^2)U_\epsilon|_{\partial B_\delta} = -2\log\delta + S_0(p) + O(\delta\log\delta),$$
$$Q_1 = \left.\frac{\partial(1 - B|x|^2)U_\epsilon}{\partial r}\right|_{\partial B_{Lr_\epsilon}} = \frac{2\lambda L}{r_\epsilon(1+\lambda L^2)},$$
$$Q_2 = \left.\frac{\partial(1 - B|x|^2)U_\epsilon}{\partial r}\right|_{\partial B_\delta} = -\frac{2}{\delta} + O(\delta\log\delta).$$

If we define

$$\begin{aligned} N(L,\epsilon,\delta) &= w(L) + 2\log\delta - S_0 - \frac{\varrho}{2}\frac{2\lambda L^2}{1+\lambda L^2} \\ &= w(L) + 2\log\delta - S_0 - 2 + O(\delta\log\delta) + O\left(\frac{1}{L^2}\right) + O(Lr_\epsilon), \end{aligned}$$

and

$$P = \log\delta - \log L,$$

then we have

$$\begin{aligned} A^2(-\log Lr_\epsilon/\delta) &= \left(\frac{m_\epsilon - \overline{u}_\epsilon + N(L,\epsilon,\delta)}{m_\epsilon + P - \varrho}\right)^2 (m_\epsilon + P) \\ &= \left(1 + \frac{P-\varrho}{m_\epsilon}\right)^{-2}\left(1 + \frac{P}{m_\epsilon}\right) m_\epsilon \left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon} + \frac{N(L,\epsilon,\delta)}{m_\epsilon}\right)^2 \end{aligned}$$

$$= \left(1 - 2\frac{P-\varrho}{m_\epsilon} + O\left(\frac{1}{m_\epsilon^2}\right)\right)\left(1 + \frac{P}{m_\epsilon}\right) m_\epsilon$$
$$\times \left[\left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon}\right)^2 + 2\left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon}\right)\frac{N(L,\epsilon,\delta)}{m_\epsilon}\right.$$
$$\left. + O\left(\frac{1}{m_\epsilon^2}\right) + O(e^{-m_\epsilon} m_\epsilon)\frac{\overline{u}_\epsilon}{m_\epsilon}\right]$$
$$= m_\epsilon \left(1 - \frac{\overline{u}_\epsilon}{u_\epsilon}\right)^2 + 2\left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon}\right) N(L,\epsilon,\delta)$$
$$- (P - 2\varrho)\left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon}\right)^2 + O\left(\frac{1}{m_\epsilon}\right)\left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon}\right)^2 + O\left(\frac{1}{m_\epsilon}\right),$$

and

$$A = -\frac{m_\epsilon - \overline{u}_\epsilon + N(L,\epsilon,\delta)}{m_\epsilon - \log L + \log \delta + \varrho} = -\left(1 - O\left(\frac{1}{m_\epsilon}\right)\right)^{-1}\left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon} + O\left(\frac{1}{m_\epsilon}\right)\right)$$
$$= -1 + \frac{\overline{u}_\epsilon}{m_\epsilon} + O\left(\frac{1}{m_\epsilon}\right).$$

Notice that $r_\epsilon m_\epsilon \to 0$ as $\epsilon \to 0$, we have

$$B = \frac{-2m_\epsilon + 2\overline{u}_\epsilon + O(1) + \left(2\frac{2\delta^2}{\delta^2 - (Lr_\epsilon)^2} + O(\delta \log \delta)\right) m_\epsilon}{4(\delta^2 + (Lr_\epsilon)^2)(\log L - m_\epsilon - \log \delta + \varrho)}$$
$$= -\frac{1}{2\delta^2}\left(1 + \frac{\overline{u}_\epsilon}{m_\epsilon} + O\left(\frac{1}{m_\epsilon}\right)\right)\left(1 - O\left(\frac{1}{m_\epsilon}\right)\right)^{-1}$$
$$= -\frac{1}{2\delta^2}\left(1 + \frac{\overline{u}_\epsilon}{m_\epsilon} + O\left(\frac{1}{m_\epsilon}\right)\right).$$

It concludes that

$$\int_{B_\delta \setminus B_{Lr_\epsilon}} |\Delta_0(1 - b|x|^2)(U_\epsilon - \overline{u}_\epsilon)|^2 dx$$
$$\geq 8\pi^2 m_\epsilon \left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon}\right)^2 + 16\pi^2\left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon}\right) N(L,\epsilon,\delta) - 8\pi^2(P - 2\varrho)\left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon}\right)^2$$
$$+ 16\pi^2\left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon}\right)\left(1 + \frac{\overline{u}_\epsilon}{m_\epsilon}\right) + 8\pi^2\left(1 + \frac{\overline{u}_\epsilon}{m_\epsilon}\right)^2$$
$$+ O\left(\frac{1}{m_\epsilon}\right)\left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon}\right)^2 + O\left(\frac{1}{m_\epsilon}\right) + J_6(L,\epsilon,\delta).$$

Using the fact that $\overline{u}_\epsilon \leq C$, we have

$$(8\pi^2 - \epsilon)\overline{u}_\epsilon > 8\pi^2 \overline{u}_\epsilon + \epsilon C.$$

Therefore

$$\begin{aligned} II_\epsilon(u_\epsilon) &\geq \int_{B_{Lr_\epsilon}(x_\epsilon)} |\Delta_g u_\epsilon|^2 dV_g + \int_{B_\delta \setminus B_{Lr_\epsilon}} |\Delta_0(1-|B|^2)(U_\epsilon - \overline{u}_\epsilon)|^2 dx + 8\pi^2 \overline{u}_\epsilon \\ &\quad + \int_{M\setminus B_\delta(x_0)} \langle G_p, G_p\rangle + 4\int_M \tilde{Q} G_p dV_g + J(L,\epsilon,\delta) \\ &\geq 8\pi^2(m_\epsilon + C_1)\left(1 + \frac{\overline{u}_\epsilon}{m_\epsilon}\right)^2 + C_2\left(1 + \frac{\overline{u}_\epsilon}{m_\epsilon}\right) + C_3 \end{aligned}$$

where C_1, C_2, C_3 are some constants. Since $II_\epsilon(u_\epsilon) = \inf II_\epsilon < C' < \infty$, we must have $(1 + \frac{\overline{u}_\epsilon}{m_\epsilon}) \to 0$ as $\epsilon \to 0$, i.e. $\frac{\overline{u}_\epsilon}{m_\epsilon} \to -1$.

Consequently, we have

$$\begin{aligned} &\int_{B_\delta\setminus B_{Lr_\epsilon}} |\Delta_0(1-b|x|^2)(U_\epsilon - \overline{u}_\epsilon)|^2 dx + 8\pi^2 \overline{u}_\epsilon \\ &\quad \geq 8\pi^2 m_\epsilon \left(1 + \frac{\overline{u}_\epsilon}{m_\epsilon}\right)^2 + 16\pi^2 N(L,\epsilon,\delta)\left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon}\right) \\ &\quad\quad -8\pi^2(\log\delta - \log L - 2\varrho)\left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon}\right)^2 + J(L,\epsilon,\delta) \\ &\quad \geq 16\pi^2\left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon}\right) N(L,\epsilon,\delta) - 8\pi^2(\log\delta - \log L - 2\varrho)\left(1 - \frac{\overline{u}_\epsilon}{m_\epsilon}\right)^2 + J(L,\epsilon,\delta). \end{aligned} \tag{3.4}$$

Since we have

$$\Delta_0 w = \frac{4\lambda^2|x|^2}{(1+\lambda|x|^2)^2} - \frac{8\lambda}{1+\lambda|x|^2},$$

a direct calculation yields that

$$\int_{B_L} |\Delta_0 w|^2 dx = 16\pi^2 \log(1+\lambda L^2) + \frac{8\pi^2}{3} + O\left(\frac{\log L}{L^2}\right).$$

On the other hand, it is obvious to see that,

$$\int_{B_\delta(x_\epsilon)} |\nabla u_\epsilon|^2 \to \int_{B_\delta(x_\epsilon)} |\nabla G_p|^2 = O(\delta\log\delta), \tag{3.5}$$

and

$$\begin{aligned} &\int_{M\setminus B_\delta(x_0)} \langle G_p, G_p\rangle dV_g \\ &\quad = \int_{M\setminus B_\delta(x_0)} G_p P_g G_p dV_g - \int_{\partial B_\delta} \frac{\partial G_p}{\partial r}\Delta_g G_p dV_g + \int_{\partial B_\delta} G_p \frac{\partial \Delta G_p}{\partial r} dV_g \\ &\quad\quad + \int_{\partial B_\delta}\left(\frac{2}{3}RG\frac{\partial G}{\partial r} - 2G\mathrm{Ric}(dG, dr)\right) dS_g \\ &\quad = -2\int_M Q_g G_p dV_g - 16\pi^2 + 16\pi^2(-2\log\delta + S_0(p)) + O(\delta\log\delta). \end{aligned} \tag{3.6}$$

Together with Lemmas 3.4 and 3.5, (3.4)–(3.6), we have

$$\begin{aligned}\lim_{\epsilon\to 0} II_\epsilon \geq\ & 32\pi^2 \lim_{\epsilon\to 0} N(L,\epsilon,\delta) - 32\pi^2(\log\delta - \log L - 2) + 16\pi^2\log(1+\lambda L^2)\\ &+ \frac{8\pi^2}{3} + (-2\log\delta + S_0(p))16\pi^2 + 2\int_M Q_g G_p dV_g - 8\pi^2\log 8\pi^2\\ &+ O(\delta\log\delta) + O\left(\frac{\log L}{L^2}\right)\\ =\ & -16\pi^2\log\frac{1+\lambda L^2}{L^2} + \frac{8\pi^2}{3} - 16\pi^2 S_0(p) - 16\pi^2\\ &+ 2\int_M Q_g G_p dV_g - 8\pi^2\log 8\pi^2 + O(\delta\log\delta) + O\left(\frac{\log L}{L^2}\right).\end{aligned}$$

Letting first $\delta\to 0$, then $L\to+\infty$, we get

$$\lim_{\epsilon\to 0} II_\epsilon \geq -16\pi^2\log\lambda - 8\pi^2\log 8\pi^2 - 16\pi^2 S_0 + (8/3-16)\pi^2 + 2\int_M Q_g G_p dV_g.$$

This shows the first part of Theorem 1.1, that is

$$\inf_{u\in W^{2,2}(M)} II(u) \geq \inf_{p\in M} \Lambda_g(\tilde{Q}, p).$$

The second part

$$\inf_{u\in W^{2,2}(M)} II(u) \leq \inf_{p\in M} \Lambda_g(\tilde{Q}, p)$$

follows from the proof of Theorem 1.2 in the next section.

To end this section, we will prove a conformal property of $\Lambda_g(\tilde{Q}, p)$.

Lemma 3.6. *Letting $\tilde{g}\in[g] : \tilde{g} = e^{2v}g$ for some $v\in C^\infty(M)$, we have*

$$II_{\tilde{g}}(u) = II_g(u+v) - \int_M \langle v, v\rangle dV_g - 4\int_M Q v dV_g.$$

If we set

$$P_{\tilde{g}}\tilde{G}_y + 2Q_{\tilde{g}} = 16\pi^2\delta_y,$$

then

$$\tilde{G}_y = G_y - v, \quad \textit{and} \quad \tilde{S}_0(y) = S_0(y) + v(y).$$

Proof. Since $P_{\tilde{g}} = e^{-4v}P_g$, $2Q_{\tilde{g}} = e^{-4v}(P_g v + 2Q_g)$, we get

$$\begin{aligned}II_{\tilde{g}}(u) &= \int_M \langle u, u\rangle dV_g + 2\int_M (P_g v + 2Q_g)u dV_g - 8\pi^2\log\int_M \tilde{Q}e^{4(u+v)}dV_g\\ &= \int_M \langle u+v, u+v\rangle dV_g + 4\int_M Q_g u dV_g\\ &\quad - 8\pi^2\log\int_M \tilde{Q}e^{4(u+v)}dV_g - \int_M \langle v, v\rangle dV_g\\ &= II_g(u+v) - \int_M \langle v, v\rangle dV_g - 4\int_M Q v dV_g.\end{aligned}$$

On the other hand, we have

$$P_{\tilde{g}}(G-v)+2Q_{\tilde{g}}=e^{-4v}(P_gG+2Q_g)=16\pi^2e^{-4v}\delta_{y,g}=16\pi^2\delta_{y,\tilde{g}}.$$

Since $\mathrm{dist}_{\tilde{g}}(y,x)=e^{v(y)}\mathrm{dist}_g(y,x)+O(\mathrm{dist}_g(y,x))^2$, we have

$$\begin{aligned}\tilde{G}_y &= G_y-v\\ &= -2\log\mathrm{dist}_g(y,x)+S_0(y)-v(y)+O(\mathrm{dist}(y,x))\\ &= -2\log\mathrm{dist}_{\tilde{g}}(y,x)+v(y)+S_0(y)+O(\mathrm{dist}(y,x)).\end{aligned}$$

Thus $\tilde{S}_0(y)=S_0(y)+v(y)$. □

4. Testing function

In this section, we will construct a blow up sequence ϕ_ϵ s.t.

$$II(\phi_\epsilon)<\inf_{x\in M}\Lambda(x).$$

We use standard notation from [10]. In a normal geodesic coordinate system $\{x^i\}$, we denote

$$R_{ijkl}=\langle R(\partial_k,\partial_l)\partial_j,\partial_i\rangle,\qquad R_{ij}=-g^{jk}R_{ijkl},$$

where R is the curvature operator, defined as follows,

$$R(X,Y)=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}.$$

Suppose that p' is a point such that $\Lambda(p')=\inf_{x\in M}\Lambda(x)$.

We know that, locally we have

$$\begin{aligned}g_{pq} &= \delta_{pq}+\frac{1}{3}R_{pijq}(p')x^ix^j+\frac{1}{6}R_{pijq,k}(p')x^ix^jx^k\\ &\quad+\left(\frac{1}{20}R_{pijq,kl}+\frac{2}{45}R_{pijm}(p')R_{qklm}(p')\right)x^ix^jx^kx^l+O(r^5).\\ |g| &= 1-\frac{1}{3}R_{ij}x^{ij}-\frac{1}{6}R_{ij,k}(p')x^{ijk}\\ &\quad-\left(\frac{1}{20}R_{ij,kl}(p')+\frac{1}{90}R_{hijm}(p')R_{hklm}(p')\right)x^ix^jx^kx^m+O(r^5).\end{aligned}$$

In the sequel, let us denote

$$x^{i_1\cdots i_m}_{j_1\cdots j_n}=x^{i_1\cdots i_m j_1\cdots j_n},\quad\text{and}\quad \alpha^{i_1\cdots i_m}_{j_1\cdots j_n}=\frac{1}{2\pi^2}\int_{S^3}x^{i_1\cdots i_m j_1\cdots j_n}ds;$$

then around the point p' we write

$$\begin{aligned}&g^{km}=\delta^{km}+M^{km}=\delta^{km}+M^{ij}_{km}x^{km}+M^{ij}_{kms}x^{kms}+M^{ij}_{kmst}x^{kmst}+O(r^5)\\ &M=M^{ij}\delta_{ij}=M_{km}x^{km}+M_{kms}x^{kms}+M_{kmst}x^{kmst}+O(r^5),\\ &\sqrt{|g|}=1-\frac{1}{6}R_{ij}x^{ij}+K_{ijk}x^{ijk}+K_{ijkm}x^{ijkm}+O(r^5).\\ &N^k=-g^{ij}\Gamma^k_{ij}=N^k_ix^i+N^k_{ij}x^{ij}+N^k_{ijm}x^{ijm}+O(r^5).\end{aligned}$$

It is easy to check that $M^{ij}_{km}=-\frac{1}{3}R_{ikmj}(p')$, $M_{km}=\frac{1}{3}R_{ij}(p')$ and $N^k_i=-\frac{2}{3}R_{ik}(p')$.

We prove the following lemma.

Lemma 4.1. *We have*

$$\frac{1}{18}R_{ij}(p')R_{km}(p')\alpha^{ijkm} + N_{ijk}^{m}\alpha_{m}^{ijk} + M_{ijkm}\alpha^{ijkm} = 4K_{ijkm}\alpha^{ijkm}. \tag{4.1}$$

Proof. We have, for any small $t > 0$,

$$\begin{aligned}
&\int_{B_t} \Delta_g r^2 dV_g \\
&\quad = \int_{B_t} \left(8 - \frac{2}{3}R_{ij}x^{ij} + 2M_{ijk}x^{ijk} + 2M_{ijkm}x^{ijkm} + 2N_{ij}^{k}x_{k}^{ij} + 2N_{ijk}^{p}x_{p}^{ijk}\right) \\
&\qquad \times \left(1 - \frac{1}{6}R_{ij}x^{ij} + K_{ijk}x^{ijk} + K_{ijkm}x^{ijkm}\right) dx + o(t^8) \\
&\quad = 4\pi^2 t^4 - 2R_{ij}\alpha^{ij} \times 2\pi^2 \frac{t^6}{6} \\
&\qquad + \left(\frac{1}{9}R_{ij}R_{km}\alpha^{ijkm} + 2M_{ijkm}\alpha^{ijkm} + 2N_{ijk}^{p}\alpha_{p}^{ijk} + 8K_{ijkm}\alpha^{ijkm}\right) 2\pi^2 \frac{t^8}{8} + o(t^8);
\end{aligned}$$

on the other hand, we have

$$\begin{aligned}
\int_{\partial B_t} 2r\, ds_g &= \int_{\partial B_t} 2r \left(1 - \frac{1}{6}R_{ij}x^{ij} + K_{ijkm}x^{ijkm} + O(r^5)\right) ds_0 \\
&= 4\pi^2 t^4 - 4\pi^2 \frac{R_{ij}}{6}\alpha^{ij} t^6 + 2K_{ijkm}\alpha^{ijkm} 2\pi^2 t^8 + o(t^8).
\end{aligned}$$

Now the conclusion follows from Stokes' theorem. □

Note that locally, we may write (see Lemma A.1 in the Appendix),

$$G_{p'} = -2\log r + S,$$

with

$$S = S_0(p') + a_i x^i + \frac{a_{ij}}{2}x^{ij} + O(r^{2+\alpha}).$$

We define

$$\varphi_\epsilon = -\log\left(1 + \lambda\left|\frac{x}{\epsilon}\right|^2\right) + C_\epsilon + \mu|x|^2, \quad x \in B_{L\epsilon}$$

where

$$\mu = -\frac{1}{L^2\epsilon^2(1+\lambda L^2)}, \quad \lambda = \frac{\sqrt{3\tilde{Q}(p')}}{12}$$

and

$$C_\epsilon = \log(1+\lambda L^2) - 2\log L\epsilon - \mu L^2\epsilon^2.$$

We set

$$\phi_\epsilon = \begin{cases} G + \varphi_\epsilon + 2\log r & x \in B_{L\epsilon} \\ G & x \notin B_{L\epsilon}, \end{cases}$$

then, in $B_{L\epsilon}$, we have

$$\phi_\epsilon = -\log\left(1+\lambda\left|\frac{x}{\epsilon}\right|^2\right) + C_\epsilon + S + \mu|x|^2 = \varphi_\epsilon + S. \tag{4.2}$$

Hence, it is easy to check that $\phi_\epsilon \in W^{2,p}(M)$ for any $p > 0$.

We write

$$\begin{aligned} II(\phi_\epsilon) &:= \int_M \langle \phi_\epsilon, \phi_\epsilon\rangle dV_g + 4\int_M Q_g \phi_\epsilon dV_g - 8\pi^2 \log \int_M \tilde{Q} e^{4\phi_\epsilon} dV_g \\ &= II_1 + II_2 + II_3. \end{aligned}$$

First we will calculate the term II_3. In the small neighborhood around the point p', we set

$$\tilde{Q} = \tilde{Q}(p') + b_i x^i + \frac{b_{ij}}{2} x^{ij} + O(r^3),$$

then we have

$$\begin{aligned} \tilde{Q} e^{4\phi_\epsilon}\sqrt{|g|} &= \frac{e^{4C_\epsilon + 4S_0}}{\epsilon^4\left(1+\lambda\left|\frac{x}{\epsilon}\right|^2\right)^4}\Bigg[(1 + 4a_i x^i + 2a_{ij}x^{ij} + 8a_i a_j x^{ij} + 4\mu r^2)\tilde{Q}(p') \\ &\quad + b_i x^i + \frac{b_{ij}}{2}x^{ij} + 4a_i b_i x^{ij} + O(r^{2+\alpha}) + O\left(\frac{r^2\epsilon^2}{L^8}\right)\Bigg] \\ &\quad \times \left(1 - \frac{R_{ij}x^{ij}}{6} + O(r^3)\right) \\ &= \frac{e^{4C_\epsilon + 4S_0}}{\epsilon^4\left(1+\lambda\left|\frac{x}{\epsilon}\right|^2\right)^4}\Bigg[\left(1 + 4a_i x^i + 2a_{ij}x^{ij} + 8a_i a_j x^{ij} + 4\mu r^2 - \frac{R_{ij}x^{ij}}{6}\right) \\ &\quad \times \tilde{Q}(p') + b_i x^i + \frac{b_{ij}}{2}x^{ij} + 4a_i b_i x^{ij} + O(r^{2+\alpha}) + O\left(\frac{r^2}{L^8}\right)\Bigg]. \end{aligned}$$

Therefore, by using the symmetry of the ball and the fact that $\alpha_{ij} = \frac{1}{4}\delta_{ij}$, we have

$$\begin{aligned} &\int_{B_{L\epsilon}} \tilde{Q} e^{4\phi_\epsilon}\sqrt{|g|} dV_g \\ &= 2\pi^2 e^{4C_\epsilon + 4S_0(p')}\epsilon^4 \int_0^L \frac{1}{(1+\lambda r^2)^4}\Bigg[\tilde{Q}(p')\Bigg(1 + \epsilon^2 r^2\Bigg(\sum_i \left(\frac{a_{ii}}{2} + 2a_i^2\right) \\ &\quad + 4\mu - \frac{R(p')}{24}\Bigg)\Bigg) + \sum_i \left(a_i b_i + \frac{b_{ii}}{8}\right)\epsilon^2 r^2 + O(\epsilon r)^{2+\alpha} + O\left(\frac{r^2}{L^4}\right)\Bigg] r^3 dr. \end{aligned}$$

A direct calculation then yields that

$$\begin{aligned} 2\pi^2 \int_0^L \frac{r^3 dr}{(1+\lambda r^2)^4} &= \frac{\pi^2}{6\lambda^2} + O\left(\frac{1}{L^4}\right), \\ 2\pi^2 \int_0^L \frac{r^5 dr}{(1+\lambda r^2)^4} &= \frac{\pi^2}{3\lambda^3} + O\left(\frac{1}{L^2}\right), \end{aligned}$$

and

$$4\mu\epsilon^2 \times 2\pi^2 \int_0^L \frac{r^5 dr}{(1+\lambda r^2)^4} = O\left(\frac{1}{L^4}\right).$$

Hence we get

$$\begin{aligned}
&\int_{B_{L\epsilon}} \tilde{Q} e^{4\phi_\epsilon} \sqrt{|g|} dx \\
&= e^{4C_\epsilon + 4S_0} \epsilon^4 \left[8\pi^2 - \frac{24\pi^2}{\lambda^2 L^4} + \frac{\pi^2}{3\lambda^3} \epsilon^2 \left(\sum_i \left(\frac{a_{ii}}{2} + 2a_i^2 \right) \tilde{Q}(p') - \frac{R(p')}{24} \tilde{Q}(p') \right.\right. \\
&\quad \left.\left. + \sum_i \left(a_i b_i + \frac{b_{ii}}{8} \right) \right) + O\left(\frac{1}{L^4}\right) + O(\epsilon^{2+\alpha}) + O\left(\frac{\epsilon^2}{L^2}\right) \right].
\end{aligned}$$

On the other hand, it is not difficult to check that

$$\begin{aligned}
\int_{M \setminus B_{L\epsilon}} \tilde{Q} e^{4\phi_\epsilon} \sqrt{|g|} dx &= \int_{L\epsilon}^{\delta} \tilde{Q}(p') \frac{e^{4S_0}}{r^5} 2\pi^2 dr + O\left(\frac{1}{L^2 \epsilon^2}\right) \\
&= e^{4C_\epsilon + 4S_0} \epsilon^4 \left(\frac{24\pi^2}{\lambda^2 L^4} + O\left(\frac{\epsilon^2}{L^2}\right) \right).
\end{aligned}$$

In conclusion, we have

$$\begin{aligned}
&8\pi^2 \log \int_M \tilde{Q} e^{4\phi_\epsilon} \sqrt{|g|} dx \\
&= 8\pi^2 [\log 8\pi^2 + 4(C_\epsilon + \log \epsilon + S_0)] \\
&\quad + \frac{\pi^2}{3\lambda^3} \left[\tilde{Q}(p') \sum_i \left(\frac{a_{ii}}{2} + 2a_i^2 \right) + \sum_i \left(a_i b_i + \frac{b_{ii}}{8} \right) - \frac{R(p')}{24} \tilde{Q}(p') \right] \epsilon^2 \\
&\quad + O(\epsilon^{2+\alpha}) + O\left(\frac{\epsilon^2}{L^2}\right) + O\left(\frac{1}{L^4}\right).
\end{aligned} \tag{4.3}$$

Next, we calculate II_1: first of all, by (4.2) we have

$$\begin{aligned}
\int_M \langle \phi_\epsilon, \phi_\epsilon \rangle dV_g &= \int_{M \setminus B_{L\epsilon}} \langle \phi_\epsilon, \phi_\epsilon \rangle dV_g + \int_{B_{L\epsilon}} \langle \phi_\epsilon, \phi_\epsilon \rangle dV_g \\
&= \int_{M \setminus B_{L\epsilon}} \langle G, \phi_\epsilon \rangle dV_g + \int_{B_{L\epsilon}} \langle G, \phi_\epsilon \rangle dV_g \\
&\quad + \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2\log r, \phi_\epsilon \rangle dV_g \\
&= \int_M \langle G, \phi_\epsilon \rangle dV_g + \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2\log r, \phi_\epsilon \rangle dV_g \\
&= 16\pi^2 (C_\epsilon + S_0(p')) - 2\int_M Q\phi_\epsilon dV_g \\
&\quad + \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2\log r, \varphi_\epsilon + S \rangle dV_g.
\end{aligned} \tag{4.4}$$

We set η to be a cut-off function which is 0 at 1 and 1 in [0, 1/4] with $\eta'(1) = 1$, and

$$h_\tau = \begin{cases} \eta\left(\frac{|x|}{\tau}\right) + \log\tau & |x| \le \tau \\ \log r & |x| \ge \tau. \end{cases}$$

Then for fixed ϵ and L, we have

$$\lim_{\tau\to 0}\int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2h_\tau, \varphi_\epsilon + S\rangle dV_g = \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2\log r, \varphi_\epsilon + S\rangle dV_g.$$

On the other hand, we have

$$\begin{aligned}
&\int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2h_\tau, \varphi_\epsilon + S\rangle dV_g \\
&\quad= \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2h_\tau, G\rangle dV_g + \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2h_\tau, \varphi_\epsilon + 2\log r\rangle dV_g \\
&\quad= 16\pi^2 C_\epsilon + 32\pi^2\eta(0) + 32\pi^2\log\tau - 2\int_{B_{L\epsilon}} Q_g(\varphi_\epsilon + 2h_\tau) \\
&\qquad+ \int_{B_{L\epsilon}} \langle \varphi_\epsilon, \varphi_\epsilon\rangle dV_g + \int_{B_{L\epsilon}} \langle \varphi_\epsilon, 2\log r + 2h_\tau\rangle dV_g + \int_{B_{L\epsilon}} \langle 2\log r, 2h_\tau\rangle dV_g.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2\log r, \varphi_\epsilon + S\rangle dV_g \\
&\quad= 32\pi^2\eta(0) - 2\int_{B_{L\epsilon}} Q_g(\varphi_\epsilon + 2\log r) + \int_{B_{L\epsilon}} \langle \varphi_\epsilon, \varphi_\epsilon\rangle dV_g \\
&\qquad+ \int_{B_{L\epsilon}} \langle \varphi_\epsilon, 4\log r\rangle dV_g + \lim_{\tau\to 0}\left(\int_{B_{L\epsilon}} \langle 2\log r, 2h_\tau\rangle dV_g + 32\pi^2\log\tau\right) \\
&\quad= 32\pi^2\eta(0) - 2\int_{B_{L\epsilon}} Q_g(\varphi_\epsilon + 2\log r) + \int_{B_{L\epsilon}} \Delta_g\varphi_\epsilon\Delta_g\varphi_\epsilon dV_g \\
&\qquad+ 4\int_{B_{L\epsilon}} \Delta_g\varphi_\epsilon\Delta_g\log r dV_g + \lim_{\tau\to 0}\left(\int_{B_{L\epsilon}} \Delta_g 2\log r\Delta_g 2h_\tau dV_g + 32\pi^2\log\delta\right) \\
&\qquad+ \int_{B_{L\epsilon}} \frac{2}{3}R\langle d(\varphi_\epsilon + 2\log r), d(\varphi_\epsilon + 2\log r)\rangle dV_g \\
&\qquad- \int_{B_{L\epsilon}} 2\mathrm{Ric}(d(\varphi_\epsilon + 2\log r), d(\varphi_\epsilon + 2\log r))dV_g.
\end{aligned} \tag{4.5}$$

By a simple calculation, one gets

$$\begin{aligned}
\int_{B_\tau} (\Delta_g 2\log r)\Delta_g(2h_\tau)dV_g &= \int_{B_\tau} \Delta_0(2\log r)\Delta_0\left(2\eta\left(\frac{|x|}{\tau}\right)\right)dx + O(\tau) \\
&= -32\pi^2\eta(0) + 16\pi^2 + O(\tau).
\end{aligned} \tag{4.6}$$

To compute $\int_{B_{L\epsilon}\setminus B_\delta}\Delta_g \log r\,\Delta_g \log r$, we first verify that, for any function f which is smooth on $[t_0, t_1]$, where $t_0 < t_1$, we have

$$\begin{aligned}\Delta_g f(r) &= (\delta_{km} + M_{ij}^{km}x^{ij} + M_{ijs}^{km}x^{ijs} + M_{ijst}^{km}x^{ijst} + O(r^5)) \\ &\quad\times\left(f''\frac{x_{km}}{r^2} + f'\frac{\delta_{km}}{r} - f'\frac{x_{km}}{r^3}\right) + N^k\frac{x_k}{r}f' \\ &= f'' + f'\left(\frac{3}{r} - \frac{R_{ij}x^{ij}}{3r} + \frac{M_{ijk}x^{ijk} + N_{ij}^k x_k^{ij}}{r} + \frac{M_{ijkm}x^{ijkm} + N_{ijk}^m x_m^{ijk}}{r}\right) \\ &\quad + O(r^5|f''|) + O(r^4|f'|).\end{aligned}$$

Here, we use the fact that $M_{ij}^{km}x_{km}^{ij} = M_{ijst}^{km}x_{km}^{ijst} = 0$. Then, applying Lemma 4.1, for any f_1 and f_2 which are smooth in $[t_0, t_1]$, we have

$$\begin{aligned}&\int_{B_{t_1}\setminus B_{t_0}} \Delta_g f_1(|x|)\Delta_g f_2(|x|)dV_g \\ &= \int_{t_0}^{t_1} f_1'' f_2''\left(1 - \frac{R}{24}r^2 + K_{ijkm}\alpha^{ijkm}r^4\right)2\pi^2 r^3 dr \\ &\quad + \int_{t_0}^{t_1}(f_1' f_2'' + f'' f_2')\frac{1}{r}\left(3 - \frac{5R}{24}r^2 + 7K_{ijkm}\alpha^{ijkm}r^4\right)2\pi^2 r^3 dr \\ &\quad + \int_{t_0}^{t_1} f_1' f_2'\frac{1}{r^2}\left(9 + 33K_{ijkm}\alpha^{ijkm}r^4 - \frac{7R}{8}r^2 + \frac{1}{9}R_{ij}R_{km}\alpha^{ijkm}r^2\right)2\pi^2 r^3 dr \\ &\quad + \int_{t_0}^{t_1}\left(O(r^8|f_1'' f_2''|) + O(r^7(|f_1'' f_2'| + |f_1'|\,|f_2''|)) + O(r^6|f_1' f_2'|)\right) \\ &= \int_{t_0}^{t_1}(f_1'' f_2'' + (f_1' f_2'' + f_1'' f_2')\frac{3}{r} + f_1' f_2'\frac{9}{r^2})2\pi^2 r^3 \\ &\quad + R\int_{t_0}^{t_1}\left(-f_1'' f_2''\frac{r^2}{24} - \frac{5r}{24}(f_1' f_2'' + f'' f_2') - \frac{7}{8}f_1' f_2'\right)2\pi^2 r^3 \\ &\quad + K_{ijkm}\alpha^{ijkm}\int_{t_0}^{t_1}(f_1'' f_2'' r^4 + 7(f_1' f_2'' + f_1'' f_2')r^3 + 33 f_1' f_2' r^2)2\pi^2 r^3 dr \\ &\quad + R_{ij}R_{km}\alpha^{ijkm}\int_{t_0}^{t_1}\frac{1}{9}f_1' f_2' r^2 2\pi^2 r^3 dr \\ &\quad + \int_{t_0}^{t_1}\left(O(r^8|f_1'' f_2''|) + O(r^7(|f_1'' f_2'| + |f_1'|\,|f_2''|)) + O(r^6|f_1' f_2'|)\right)dr. \qquad (4.7)\end{aligned}$$

Then, choosing $f_1 = f_2 = 2\log r$, $t_1 = L\epsilon$, $t_0 = \tau$, we get

$$\begin{aligned}\int_{B_{L\epsilon}\setminus B_\tau} \Delta_g(2\log r)\Delta_g(2h_\tau)dV_g &= \int_{B_{L\epsilon}\setminus B_\tau} \Delta_g(2\log r)\Delta_g(2\log r)dV_g \\ &= 40K_{ijkm}\alpha^{ijkm}\pi^2(L\epsilon)^4 + \frac{2\pi^2}{9}R_{ij}R_{km}\alpha^{ijkm}(L\epsilon)^4 \\ &\quad -2R\pi^2(L\epsilon)^2 + 32\pi^2\log L\epsilon - 32\pi^2\log\tau \\ &\quad + O(\tau) + O(L\epsilon)^5. \end{aligned} \tag{4.8}$$

Now we will calculate the term $\int_{B_{L\epsilon}} \Delta_g\varphi_\epsilon\Delta_g(\varphi_\epsilon + 4\log r)dV_g$: in (4.7), we choose $f_1 = \varphi_\epsilon$, $f_2 = \varphi_\epsilon + 4\log r$, $t_0 = 0$, $t_1 = L\epsilon$ then we get

$$\begin{aligned}\int_{B_{L\epsilon}} \Delta_g\varphi_\epsilon\Delta_g(\varphi_\epsilon + 4\log r)dV_g &= -\frac{88}{3}\pi^2 + \frac{16\pi^2}{\lambda L^2} - 16\pi^2\log(1+\lambda L^2) \\ &- R\epsilon^2\frac{8\pi^2}{9\lambda} + 2\pi^2R(L\epsilon)^2 - 40K_{ijkm}\alpha^{ijkm}\pi^2(L\epsilon)^4 \\ &- \frac{2\pi^2}{9}R_{ij}R_{km}\alpha^{ijkm}(L\epsilon)^4 + O(\epsilon^4L^2) + \frac{\epsilon^2}{L^2} + O(L\epsilon)^5. \end{aligned} \tag{4.9}$$

By a direct calculation, we have

$$\begin{aligned}&\int_{B_{L\epsilon}} \frac{2}{3}R(\nabla_g(\varphi_\epsilon + 2\log r), \nabla_g(\varphi_\epsilon + 2\log r))dV_g \\ &= \frac{2}{3}\int_0^{L\epsilon} R(p')\left(\frac{2\epsilon^2}{(\epsilon^2+\lambda r^2)r} + 2\mu r\right)^2 2\pi^2r^3 \\ &\quad + \frac{2}{3}\int_{B_{L\epsilon}} (R_{,i}(p')x^i + O(r^2))\left(\frac{2\epsilon^2}{(\epsilon^2+\lambda r^2)r} + 2\mu r\right)^2(1+O(r^3))dx \\ &= \frac{8}{3\lambda}R(p')\pi^2\epsilon^2 + \int_{B_{L\epsilon}}\left(\frac{2\epsilon^2}{(\epsilon^2+\lambda r^2)r} + 2\mu r\right)^2 O(r^2)dx \\ &= \frac{8}{3\lambda}R(p')\pi^2\epsilon^2 + O(\epsilon^4L^2) + O\left(\frac{\epsilon^2}{L^2}\right), \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}&\int_{B_{L\epsilon}} 2\mathrm{Ric}(\nabla_g(\varphi_\epsilon + 2\log r), \nabla_g(\varphi_\epsilon + 2\log r))dV_g \\ &= \frac{1}{2}R(p')\int_0^{L\epsilon}\left(\frac{2\epsilon^2}{(\epsilon^2+\lambda r^2)r} + 2\mu r\right)^2 2\pi^2r^3dr \\ &\quad + 2\int_{B_{L\epsilon}} g^{is}g^{jt}(R_{ij,k}(p')x^k + O(r^2))\left(\frac{2\epsilon^2}{(\epsilon^2+\lambda r^2)r^2} + 2\mu\right)^2 x_{st}(1+O(r^3))dx \\ &= \frac{2}{\lambda}R(p')\pi^2\epsilon^2 + 2\int_{B_{L\epsilon}}(R_{ij,k}(p')x^k + O(r^2)) \\ &\quad \times\left(\frac{2\epsilon^2}{(\epsilon^2+\lambda r^2)r^2} + 2\mu\right)^2 x^{ij}(1+O(r^3))dx \end{aligned}$$

$$= \frac{2}{\lambda} R(p')\pi^2\epsilon^2 + \int_{B_{L\epsilon}} \left(\frac{2\epsilon^2}{(\epsilon^2 + \lambda r^2) r^2} + 2\mu \right)^2 O(r^4) dx$$

$$= \frac{2}{\lambda} R(p')\pi^2\epsilon^2 + O(\epsilon^4 L^2) + O\left(\frac{\epsilon^2}{L^2}\right). \tag{4.11}$$

Together with (4.4)–(4.6) and (4.8)–(4.11), we obtain the following identity

$$\begin{aligned} II_\epsilon(u_\epsilon) &= II_1 + II_2 + II_3 \\ &= -16\pi^2 \log\lambda - 8\pi^2 \log 8\pi^2 + \frac{8\pi^2}{3} - 16\pi^2 + 2\int_M QG - 16\pi^2 S_0 \\ &\quad - \frac{\epsilon^2\pi^2}{3\lambda^3}\left(\tilde{Q}(p') \sum_i \left(\frac{a_{ii}}{2} + 2a_i^2 \right) + \sum_i \left(a_i b_i + \frac{b_{ii}}{8} \right) - \frac{R(p')}{36} \tilde{Q}(p') \right) \\ &\quad + O\left(\frac{\epsilon^2}{L^2}\right) + O(\epsilon^{2+\alpha}) + O\left(\frac{1}{L^4}\right) + O(\epsilon^4 L^2) + O((L\epsilon)^5). \end{aligned} \tag{4.12}$$

Proof of Theorem 1.2. We set $L = \frac{\log \frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$, then

$$\epsilon^2 \gg O\left(\frac{\epsilon^2}{L^2}\right) + O(\epsilon^{2+\alpha}) + O\left(\frac{1}{L^4}\right) + O(\epsilon^4 L^2) + O((L\epsilon)^5)$$

when ϵ is very small. Therefore, we get Theorem 1.2. □

5. The local conformally case

In this section, we will discuss the local conformally flat case of Theorem 1.2.

In this situation, locally we may write

$$g = e^{2f} \sum_i dx^i \otimes dx^i \quad \text{with } f = c_i x^i + \frac{1}{2} c_{ij} x^{ij} + O(r^3),$$

and

$$\tilde{Q} = \tilde{Q}(p') + b_i x^i + \frac{1}{2} b_{ij} x^{ij} + O(r^3).$$

Note that by the conformal property of P_g, the corresponding Green function has the following local expression:

$$G = -2\log|x| + S_0(p') + a_i x^i + \frac{1}{2} a_{ij} x^{ij} + O(r^3).$$

When $f = 0$, we can use Theorem 1.2 to obtain: if

$$\sum_i \left(\frac{a_{ii}}{2} + 2a_i^2 + \frac{1}{\tilde{Q}(p')} \left(a_i b_i + \frac{b_{ii}}{8} \right) \right) > 0,$$

then (1.3) has a solution.

For the general case, we set $g' = e^{-2f} g$, then applying Lemma 3.6, we get $G'_{p'} = G + f$, and then

$$a'_i = a_i + c_i, \quad \text{and} \quad a'_{ii} = a_{ii} + c_{ii}.$$

Thus we have the following results.

Theorem 5.1. *Let (M, g) be a closed 4-dimensional manifold with $k = 8\pi^2$ and let P_g be positive. Suppose further that it is locally conformal flat near p'. If*

$$\sum_i \frac{a_{ii} + c_{ii}}{2} + 2(a_i + c_i)^2 + \frac{1}{\tilde{Q}(p')}\left((a_i + c_i)b_i + \frac{b_{ii}}{8}\right) > 0,$$

then Eq. (1.3) has a minimal solution.

As a corollary, we have the following.

Corollary 5.2. *With the same assumption as in Theorem 5.1. If*

$$\sum_i \frac{a_{ii} + c_{ii}}{2} + 2(a_i + c_i)^2 > 0,$$

then in the conformal class of (M, g) there is a constant Q-curvature.

To end this section, we propose the following conjecture.

Conjecture. *Let (M, g) be a locally conformal flat closed Riemannian manifold of dimension four, with $k = 8\pi^2$ and let P_g be positive. Then we have*

$$\sum_i \left(\frac{a_{ii} + c_{ii}}{2} + 2(a_i + c_i)^2\right) \geq 0, \quad \textit{at the point } p' \text{ where } \Lambda_g(p') = \min_{x \in M} \Lambda_g(8\pi^2, x),$$

and the equality holds if and only if (M, g) is in the conformal class of the standard 4-sphere.

Let $\tilde{g} = e^{2G} g$; then we have

$$Q_{\tilde{g}}(x) = 0$$

for any $x \neq p$. Near p, we can write

$$\tilde{g} = \frac{e^{S_0(p) + (c_i + a_i)x^i + (c_{ij} + a_{ij})x^{ij}}}{r^2} = \frac{e^{S_0(p)}}{r^2}(\theta_i x^i + \theta_{ij} x^{ij} + O(|x|^3)).$$

So the above conjecture is equivalent to

$$\sum_i \theta_{ii} > 0$$

when $M \neq S^4$. So, this problem is very similar to the positive mass problem.

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Appendix

Suppose $\mathrm{Ker} P_g = \{\text{constant}\}$. Let G be the Green function which satisfies

$$P_g G + 2Q_g = 16\pi^2 \delta_p.$$

As a corollary of a result in [17], we have the following.

Lemma A.1. *In a normal coordinate system of p, we have*

$$G = -2\log r + S_0 + a_i x^i + a_{ij} x^{ij} + O(r^{2+\alpha}).$$

However, for the reader's sake, we give a brief proof of this lemma here.

Proof. In a normal coordinate system, we set

$$|g| = 1 - \frac{1}{3} R_{ij} x^{ij} + O(r^3), \quad \text{and} \quad g^{km} = \delta^{km} - \frac{1}{3} R_{kijm} x^{ij} + O(r^3)$$

where φ_{ijk} and θ_{ijk} are smooth.

Given a smooth function F, we have

$$\begin{aligned}
\Delta_g F(|x|) &= \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^k} \left(\sqrt{|g|} g^{km} \frac{\partial}{\partial x^m} F \right) \\
&= \frac{\partial}{\partial x_k} \left(g^{km} F' \frac{x_m}{r} \right) + \frac{1}{2} g^{km} F_m \frac{\partial}{\partial x_k} \log |g| \\
&= \frac{\partial}{\partial x_k} \left(F' \frac{x_k}{r} - \frac{1}{3} R_{kijm} F' \frac{x^{kij}}{r} + F' O(r^3) \right) - \frac{1}{3} R_{ij} F' \frac{x^{ij}}{r} + O(F' r^2) \\
&= \frac{\partial}{\partial x_k} \left(F' \frac{x_k}{r} + F' O(r^3) \right) - \frac{1}{3} R_{ij} F' \frac{x^{ij}}{r} + O(F' r^2) \\
&= \Delta_0 F - \frac{1}{3} R_{ij} F' \frac{x^{ij}}{r} + O(F' r^2) + O(F'' r^3).
\end{aligned}$$

Then

$$\Delta_g(-2\log r) = -\frac{4}{r^2} + \frac{2}{3} R_{ij} \frac{x^{ij}}{r^2} + O(r)$$

and

$$\Delta_g \left(-\frac{4}{r^2} \right) = \Delta_0 \left(-\frac{4}{r^2} \right) - \frac{8 R_{ij} x^{ij}}{3r^4} + O\left(\frac{1}{r} \right) = 16\pi^2 \delta_0 - \frac{8 R_{ij} x^{ij}}{3r^4} + O\left(\frac{1}{r} \right).$$

It is easy to check that

$$\Delta_g \frac{2}{3} R_{ij} \frac{x^{ij}}{r^2} = \Delta_0 \frac{2}{3} R_{ij} \frac{x^{ij}}{r^2} + O\left(\frac{1}{r} \right) = \frac{4R}{3r^2} - \frac{16 R_{ij} x^{ij}}{3r^4}.$$

Hence, we get

$$\Delta_g^2(-2\log r) = 16\pi^2 \delta_p + \frac{4R}{3r^2} - 8 \frac{R_{ij} x^{ij}}{r^4} + O\left(\frac{1}{r} \right).$$

Moreover, we have

$$\begin{aligned}
&\operatorname{div}\left(\frac{2}{3}R_g(-d2\log r) - 2\mathrm{Ric}_g\langle d(-2\log r), \cdot\rangle\right)\\
&\quad = \frac{2}{3}R_p(p')(2\log r)_{kk} - 2R_{km}(p')(2\log r)_{km} + O\left(\frac{1}{r}\right)\\
&\quad = \frac{2}{3}R_g(p')\frac{4}{r^2} - 4R_g(p')\frac{1}{r^2} + 8R_{km}\frac{x^{km}}{r^4} + O\left(\frac{1}{r}\right).
\end{aligned}$$

We therefore have

$$P_g(-2\log r) = 16\pi^2\delta_0 + O\left(\frac{1}{r}\right).$$

We set

$$G = -2\log r + S$$

where $S \in C^{1,\alpha}$. Then, we get

$$\Delta_g^2 S = P_g S + O\left(\frac{1}{r}\right) = P_g G + 2P_g \log r + O\left(\frac{1}{r}\right) = O\left(\frac{1}{r}\right).$$

This proves the lemma. □

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