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Support properties for integral operators in hyperfunctions

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Abstract

The main result of this paper is that the integral operators between spaces of compactly supported hyperfunctions must have properly supported kernels. We also discuss the uniqueness and the regularity of the integral operators in hyperfunction theory.

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1. Statement of the main result

Consider open sets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$, and a hyperfunction $\mathcal{K}$ defined on $V \times U$ satisfying the condition

$$\{(x, y, 0, \eta) \in V \times U \times \mathbb{R}^n \times \mathbb{R}^m; \eta \neq 0\} \cap \mathrm{WF}_A \mathcal{K} = \emptyset, \tag{1.1}$$

where $\mathrm{WF}_A \mathcal{K}$ denotes the analytic wave front set of $\mathcal{K}$. We can then associate with $\mathcal{K}$ a linear operator $T : \mathcal{A}'(U) \to \mathcal{B}(V)$, by

$$(Tu)(x) = \int_U \mathcal{K}(x, y)u(y)\, dy, \quad \text{for } u \in \mathcal{A}'(U). \tag{1.2}$$

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Here $\mathcal{A}'(U)$ denotes the space of real-analytic functionals on U, $\mathcal{B}(V)$ the space of hyperfunctions on V and the meaning of the integral in (1.2) is the one given by microlocal analysis to such expressions. (Cf. [18,9]. We shall have to come back to this in Section 4.) Note that we shall identify $\mathcal{A}'(U)$ with the space $\mathcal{B}_c(U)$ of hyperfunctions on U with compact support. In this setting, T is said to be the integral operator associated with $\mathcal{K}$, and $\mathcal{K}$ is said to be the kernel of T.

Our main result is the following:

Theorem 1.1. *Consider $\mathcal{K} \in \mathcal{B}(V \times U)$ satisfying (1.1) and let $T : \mathcal{A}'(U) \to \mathcal{B}(V)$ be the associated integral operator. The following conditions are equivalent:*

(i) $T(\mathcal{A}'(U)) \subset \mathcal{A}'(V)$.
(ii) *For every compact set $K \subset U$ there is a compact set $L \subset V$ such that* $\operatorname{supp} u \subset K$ *implies* $\operatorname{supp} Tu \subset L$.
(iii) *The map $p_2|_{\operatorname{supp}\mathcal{K}} : \operatorname{supp}\mathcal{K} \to U$ is proper, where p_2 denotes the second projection $V \times U \to U$.*
(iv) *T is a composition of a continuous linear map $\mathcal{A}'(U) \to \mathcal{A}'(V)$ and the inclusion map $\mathcal{A}'(V) \to \mathcal{B}(V)$.*

A kernel $\mathcal{K}$ satisfying (iii) in the theorem above is called a properly supported kernel in this paper.

Theorem 1.1, or more precisely speaking, the implications (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) have been announced in [13] (see Theorem 3.3(1)).

Note that the assumption in (i) is that for every fixed $u \in \mathcal{A}'(U)$ there is a compact set $L \subset V$ with $\operatorname{supp} Tu \subset L$. The new information in (ii) is then just that this compact set L essentially only depends on the support of u and not on u itself. It is nevertheless the implication (i) $\Rightarrow$ (ii) which seems most interesting to us. In fact the main technical difficulty in the proof of this implication is that it is not immediate how to use in a quantitative way (when u is varying) the information that Tu has compact support. At the origin of this is (by the very definition of hyperfunctions) the fact that Tu vanishes in a neighborhood of some point x^0 gives only a cohomological information about the holomorphic representation functions of Tu near x^0.

By contrast, the implications (ii) $\Rightarrow$ (iii), respectively (iii) $\Rightarrow$ (iv), are relatively easy consequences of known results and the fact that (iv) implies (i) is of course trivial. Note also that the implication (iii) $\Rightarrow$ (ii) is a direct corollary of the definition of integration along fibers for hyperfunctions. Moreover, we should mention that using functional analysis it is quite easy to prove directly that (iv) $\Rightarrow$ (ii). (See Proposition 3.6.)

We also mention the following results concerning the uniqueness and the regularity of the kernels.

Theorem 1.2. *Let $\mathcal{K} \in \mathcal{B}(V \times U)$ be a hyperfunction satisfying (1.1) and denote by T the associated operator defined in (1.2). If $Tu = 0$ for every $u \in \mathcal{A}'(U)$, then $\mathcal{K}$ must vanish on $V \times U$.*

Theorem 1.3. *Let $\mathcal{K} \in \mathcal{B}(V \times U)$ be a hyperfunction satisfying (1.1). Assume that the operator $T : \mathcal{A}'(U) \to \mathcal{B}(V)$ defined in (1.2) actually maps $\mathcal{A}'(U)$ into $\mathcal{A}(V)$. Then $\mathcal{K}$ is real-analytic on $V \times U$.*

These results in turn are based on the following three theorems:

Theorem 1.4 (*Kaneko*). *Let p_1 denote the first projection $V \times U \to V$, $\mathcal{K} \in \mathcal{B}(V \times U)$ be a hyperfunction for which the map $p_1|_{\operatorname{supp}\mathcal{K}} : \operatorname{supp}\mathcal{K} \to V$ is proper, and consider the operator $T : \mathcal{A}(U) \to \mathcal{B}(V)$ given by $(Tu)(x) = \int_U \mathcal{K}(x, y)u(y)dy$. Assume that Tu is real-analytic on V for any $u \in \mathcal{A}(U)$. Then $\mathcal{K}$ satisfies*

$$\{(x, y; \xi, 0) \in V \times U \times \mathbb{R}^n \times \mathbb{R}^m; \xi \neq 0\} \cap \mathrm{WF}_A\mathcal{K} = \emptyset. \tag{1.3}$$

Theorem 1.5 (*Bastin–Laubin*). *Let $\mathcal{K} \in \mathcal{A}'(\mathbb{R}^n \times \mathbb{R}^m)$, $y^0 \in \mathbb{R}^m$, $\eta^0 \in \mathbb{R}^m$. In the case $\eta^0 \neq 0$, also assume that*

$$(x, y^0, 0, -\eta^0) \notin \mathrm{WF}_A\mathcal{K} \tag{1.4}$$

for every $x \in \mathbb{R}^n$. If $x^0 \in \mathbb{R}^n, \xi^0 \in \mathbb{R}^n$ and $(x^0, \xi^0) \notin \mathrm{WF}_A(\int \mathcal{K}(x, y)g(y)dy)$ for every $g \in \mathcal{B}(\mathbb{R}^m)$ which satisfies

$$\mathrm{WF}_A g \subset \{(y^0, t\eta^0); t > 0\} \tag{1.5}$$

then $(x^0, y^0, \xi^0, -\eta^0) \notin \mathrm{WF}_A\mathcal{K}$.

Theorem 1.6 (*Oshima–Kataoka*). *Let $\mathcal{K}(x, y) \in \mathcal{B}(V \times U)$ be a hyperfunction with real-analytic parameter y satisfying*

$$\mathcal{K}(x, y)|_{y=y^0} = 0 \quad \text{for any } y^0 \in U.$$

Then $\mathcal{K} = 0$ on $V \times U$.

Recall that by definition $\mathcal{K}(x, y)$ is said to be a “hyperfunction with real-analytic parameters y” precisely when the condition (1.1) holds, and also that in this case the restrictions $\mathcal{K}(x, y)|_{y=y^0}$ are well-defined in microlocal analysis. Cf. [9]. Hyperfunctions with real-analytic parameters y form a subsheaf of $\mathcal{B}$ on $\mathbb{R}^n \times \mathbb{R}^m$.

Remark 1.7. (a) Theorem 1.4 is proved by Kaneko in [8], and also by Bastin and Laubin in [2]. Kaneko used the twisted Radon transform while Bastin–Laubin used the FBI transform. We observe, on the other hand, that the converse is an easy consequence of the definitions. Our next remark is that condition (1.3) is symmetric (when replacing ξ with η) to (1.1). This corresponds to the fact that for continuous operators $T : \mathcal{A}'(U) \to \mathcal{A}'(Q)$ with Q some compact analytic manifold, we can calculate the adjoint operator $T^* : \mathcal{A}(Q) \to \mathcal{A}(U)$ and that then the kernel $\mathcal{K}^*$ of T^* is related to the kernel $\mathcal{K}$ of T by $\mathcal{K}^*(y, x) = \mathcal{K}(x, y)$.

(b) For Theorem 1.5 see [3]. In the case $\eta^0 = 0$, the condition (1.5) reads that the test function g runs through the space $\mathcal{A}(\mathbb{R}^m)$. This special case is the main result of [2], which is a refinement of Theorem 1.4.

(c) Theorem 1.6 is a variant for hyperfunctions with real-analytic parameters of a theorem of Malgrange–Zerner. It was established independently by Kataoka and Oshima. (The only proof in print of this theorem seems to be the one in Kaneko's book [9, Theorem 4.4.7']. For a related result see [14].)

The plan of the paper is the following. In Section 2, we shall prove Theorems 1.2 and 1.3 using Theorems 1.6 and 1.5 respectively. It is clear that once Theorem 1.2 is established, we also have proved the implication (ii) ⇒ (iii) in Theorem 1.1. On the other hand Theorem 1.2 is also an immediate consequence of Theorem 1.3. Also this fact shall be explained in Section 2.

We shall prove (iv) $\Rightarrow$ (ii) in Section 3 and (iii) $\Rightarrow$ (iv) in Section 4, after having previously recalled some elementary facts concerning topologies in spaces of real-analytic functions and functionals in Section 3.

At that moment we will know then already that the conditions (ii)–(iv) are equivalent and that (i) follows from any one of them. To complete the proof of Theorem 1.1, we shall then show that (i) $\Rightarrow$ (iii). The necessary preparations for this will be given in Section 5 and the argument is concluded in Section 6.

2. Uniqueness theorems

Proof of Theorem 1.2. Theorem 1.2 directly follows from Theorem 1.6. In fact, we have $\mathcal{K}(x,y)|_{y=y^0} = (T\delta_{y^0})(x)$, where $\delta_{y^0}(y) := \delta(y-y^0)$ is a translation of the Dirac δ-distribution $\delta(y)$ on $\mathbb{R}^m$. $\square$

As mentioned above, we can also prove Theorem 1.2 starting from Theorem 1.3. This is seen as follows: using Theorem 1.3 it follows that a kernel $\mathcal{K}(x,y)$ as in Theorem 1.2 must be real-analytic. The action of T on δ_{y^0} for any $y^0 \in U$ is then the analytic function $\mathcal{K}(x,y^0)$ and we know that this function vanishes as a hyperfunction. Therefore $\mathcal{K}$ vanishes.

Proof of Theorem 1.3. Let $\mathcal{K} \in \mathcal{B}(V \times U)$ be a hyperfunction satisfying the assumption of Theorem 1.3. We fix an arbitrary point $(x^0, y^0; \xi^0, -\eta^0) \in V \times U \times \dot{\mathbb{R}}^{n+m}$ and will show that $(x^0, y^0; \xi^0, -\eta^0) \notin \mathrm{WF}_A\mathcal{K}$, using Theorem 1.5. Note that $\dot{\mathbb{R}}^p$ denotes $\mathbb{R}^p \setminus \{0\}$ in this paper, and that here we used it in case $p = n+m$.

We may assume from the beginning that $\xi^0 \neq 0$, since $(x^0, y^0; 0, -\eta^0) \notin \mathrm{WF}_A\mathcal{K}$ is a direct consequence of the condition (1.1).

We take open sets V_0, V_1, U_1 with $x^0 \in V_0 \Subset V_1 \Subset V$ and $y^0 \in U_1 \Subset U$. Then from the flabbiness of the sheaf of microfunctions and the existence of a global hyperfunction representative for any microfunction, there exists a hyperfunction $\mathcal{K}_1 \in \mathcal{B}(V \times U)$ such that

$$(\mathcal{K}_1 - \mathcal{K})|_{V_0 \times U} \in \mathcal{A}(V_0 \times U), \tag{2.1}$$

and that

$$\mathrm{WF}_A\mathcal{K}_1 = \overline{\mathrm{WF}_A\mathcal{K} \cap (V_0 \times U) \times \dot{\mathbb{R}}^{n+m}}, \tag{2.2}$$

where the closure in the right hand side is taken in $(V \times U) \times \dot{\mathbb{R}}^{n+m}$. From (2.2), we have an estimate $\mathrm{WF}_A\mathcal{K}_1 \subset \mathrm{WF}_A\mathcal{K} \cap (\overline{V}_0 \times U) \times \dot{\mathbb{R}}^{n+m}$, which in particular implies that $\mathcal{K}_1$ also satisfies the wave front set estimate (1.1). Moreover $\mathcal{K}_1$ is real-analytic on $(V \setminus \overline{V}_0) \times U$.

Therefore we can define a hyperfunction $\tilde{\mathcal{K}} \in \mathcal{B}(V \times U)$ with compact support by

$$\tilde{\mathcal{K}}(x,y) := \mathcal{K}_1(x,y)\chi_{V_1}(x)\chi_{U_1}(y),$$

where χ_{V_1} and χ_{U_1} denote the characteristic functions of V_1 and U_1 respectively. Note that the product in the right hand side is well-defined. Moreover in the case of $\eta^0 \neq 0$, we can easily see that $\tilde{\mathcal{K}}$ satisfies (1.4). Now we shall show that $(x^0; \xi^0) \notin \mathrm{WF}_A(\int \tilde{\mathcal{K}}(x,y)g(y)dy)$ for $g \in \mathcal{B}(\mathbb{R}^m)$ with (1.5). In the formula

$$\begin{aligned}\int \tilde{\mathcal{K}}(x,y)g(y)dy &= \int \mathcal{K}_1(x,y)\chi_{V_1}(x)\chi_{U_1}(y)g(y)dy \\ &= \chi_{V_1}(x)(T(\chi_{U_1}g))(x) + \chi_{V_1}(x)\int \big(\mathcal{K}_1(x,y) - \mathcal{K}(x,y)\big)(\chi_{U_1}(y)g(y))dy,\end{aligned}$$

we have that $(T(\chi_{U_1}g))(x) \in \mathcal{A}(V)$ from the assumption of Theorem 1.3, and that the second term is real-analytic on V_0 from (2.1). Applying Theorem 1.5, we have $(x^0, y^0; \xi^0, -\eta^0) \notin \mathrm{WF}_A\tilde{\mathcal{K}}$. Again from (2.1) and the definition of $\tilde{\mathcal{K}}$, it follows that $(x^0, y^0; \xi^0, -\eta^0) \notin \mathrm{WF}_A\mathcal{K}$. □

3. Topologies

The purpose of this section is to review some results on the topology of the space $\mathcal{A}(V)$ of real-analytic functions on an open set $V \subset \mathbb{R}^n$ and on the topology of its strong dual space $\mathcal{A}'(V)$. Most of the results which we need are known, but since there are scattered through the literature, we thought it useful to collect them in a single section.

As a preliminary remark we consider an open set $W \subset \mathbb{C}^n$ and the space $\mathcal{O}(W)$ of holomorphic functions on W, endowed with the topology of uniform convergence on compact sets in W. Then $\mathcal{O}(W)$ is clearly a Fréchet space. A set $\mathcal{M} \subset \mathcal{O}(W)$ is bounded in $\mathcal{O}(W)$ if for every increasing sequence of compact sets $K_j \Subset W$ which exhausts W (in the sense that $\bigcup_j K_j = W$) there exist constants c_j such that $|f(z)| \le c_j$, for all $z \in K_j$ and all $f \in \mathcal{M}$.

Let us turn back to $\mathcal{A}(V)$. There are three natural topologies on $\mathcal{A}(V)$, and we shall briefly recall two of them. (As for the "third" natural topology on $\mathcal{A}(V)$ given by explicit semi-norms, we refer to [12] and references therein.) The equivalence of these two topologies is shown in [16]. The first is when we write $\mathcal{A}(V) = \varinjlim_{\Omega} \mathcal{O}(\Omega)$, where Ω runs through open sets in $\mathbb{C}^n$ satisfying $\Omega \cap \mathbb{R}^n = V$. We call such Ω a complex neighborhood of V. It is then natural to endow $\mathcal{A}(V)$ with the inductive limit topology of the spaces $\mathcal{O}(\Omega)$. The second topology is a little bit more involved, but is better adapted to the topological study of the spaces $\mathcal{A}(V)$.

We start by considering the space $\mathcal{A}(K)$ ($\simeq \mathcal{O}(K)$) for a compact set K in $\mathbb{R}^n$, which we regard as the (non-strict) inductive limit of the spaces $\mathcal{O}(K_d)$ where the K_d are defined by

$$K_d = \{z \in \mathbb{C}^n; \operatorname{dist}(z, K) < d\}. \tag{3.1}$$

While it is only a non-strict limit of Fréchet spaces, it is known that it is also a DFS space. (DFS = dual of Fréchet–Schwartz.)

We now want to recall the "second" topology on $\mathcal{A}(V)$. Since $\mathcal{A}(V)$ can be identified in a natural way with $\mathcal{A}(V) = \varprojlim_{K \Subset V} \mathcal{A}(K)$, it is indeed also natural to consider $\mathcal{A}(V)$ endowed with the projective limit topology of the spaces $\mathcal{A}(K)$. The limit is here of course essentially for countably many K. As mentioned above the two topologies coincide.

We also want to see how bounded sets in $\mathcal{A}(K)$ and $\mathcal{A}(V)$ can be characterized. We start with the following result of Martineau:

Theorem 3.1 (*[15, Lemma A in p. 14]*)**.** *Let $\mathcal{M}$ be bounded in $\mathcal{A}(K)$. Then there is $d > 0$ so that $\mathcal{M} \subset \mathcal{O}(K_d)$ and so that $\mathcal{M}$ is bounded in $\mathcal{O}(K_d)$.*

Since we already know how to characterize bounded sets in $\mathcal{O}(K_d)$ we can conclude the following:

Remark 3.2. A set $\mathcal{N} \subset \mathcal{A}(K)$ is bounded precisely if there is $d > 0$ so that all functions in $\mathcal{N}$ have analytic extensions to $\mathcal{O}(K_d)$ and such that for every $d' < d$ there is a constant $c_{d'} > 0$ for which

$$|f(z)| \le c_{d'}, \quad \forall z \in K_{d'}, \forall f \in \mathcal{N}. \tag{3.2}$$

Thus, it is also equivalent to the condition that there is $d > 0$ and $c > 0$ so that $\mathcal{N} \subset \mathcal{O}(K_d)$ and that $\sup_{z \in K_d} |f(z)| \le c$ for any $f \in \mathcal{N}$.

We can now study bounded sets in $\mathcal{A}(V)$ for $V \subset \mathbb{R}^n$. Since $\mathcal{A}(V)$ is the projective limit of the $\mathcal{A}(K)$, a set $\mathcal{M} \subset \mathcal{A}(V)$ will be bounded if the image of $\mathcal{M}$ under the restriction map $\mathcal{A}(V) \to \mathcal{A}(K)$ is bounded in $\mathcal{A}(K)$ for every compact K in V. If we now take into account also the characterization of bounded sets in $\mathcal{A}(K)$ mentioned above, we can conclude that a set $\mathcal{M}$ is bounded precisely when there is an open neighborhood $\Omega \subset \mathbb{C}^n$ of V such that all functions f in $\mathcal{M}$ admit analytic extensions to functions in $\mathcal{O}(\Omega)$ and that $\mathcal{M}$ considered as a subset in $\mathcal{O}(\Omega)$ is bounded there. In fact, if we fix an increasing sequence of compact sets $K_j \subset V$ which exhausts V, then there are positive constants c_j and d_j such that any $f \in \mathcal{M}$ admits an analytic extension to the complex d_j-neighborhood Ω_j of K_j and satisfies $\sup_{z\in\Omega_j} |f(z)| \le c_j$. By shrinking d_j, we may assume that $\Omega_j \cap \mathbb{R}^n \subset V$. Therefore, $\Omega := \bigcup_j \Omega_j$ becomes a complex neighborhood of V and we have that

$$\mathcal{M} \subset \{f \in \mathcal{O}(\Omega);\ \sup_{z\in\Omega_j} |f(z)| \le c_j,\ \text{for any } j\}.$$

It is in particular clear from this that the domains of analyticity of the functions in $\mathcal{M}$ all contain some neighborhood of V in $\mathbb{C}^n$.

Theorem 3.3 (*Martineau [16, Proposition 1.7]*)**.** *The space of real-analytic functions on V is a complete Schwartz space (for the definition of Schwartz spaces cf.* [19, *p.* 112]*) and ultrabornological.*

We now consider some statements on the topology of $\mathcal{A}'(K)$ and of $\mathcal{A}'(V)$, the strong dual spaces of $\mathcal{A}(K)$, respectively $\mathcal{A}(V)$. At first we study semi-norms in $\mathcal{A}'(K)$. In view of the characterization of bounded sets in $\mathcal{A}(K)$ a fundamental system of semi-norms in $\mathcal{A}'(K)$ is given as $\{\|\cdot\|_{-d,K}\}_{d>0}$ by the expressions

$$\|u\|_{-d,K} = \sup_{g\in\mathcal{N}} |u(g)|, \tag{3.3}$$

where $\mathcal{N} = \{g \in \mathcal{O}(K_d), |g(z)| \le 1, \forall z \in K_d\}$ and K_d is given by (3.1). We explicitly mention that $\|u\|_{-d,K}$ also defines by the same expression a continuous semi-norm on $\mathcal{O}'(K_d)$.

We also recall the following well-known fact. (See [16, Proposition 1.4 and Théorème 1.2(a)].)

Proposition 3.4. *$\mathcal{A}'(V)$ is the inductive limit of the $\mathcal{A}'(K)$.*

We conclude the section with some statements on continuous linear maps between spaces of analytic functionals. We recall at first the following result in abstract functional analysis:

Theorem 3.5 (*[6, p. 198, Chapter 4, Part 1, Section 5, Theorem 1]*)**.** *Let $\cdots \to X_i \to X_{i+1} \to \cdots$ be a sequence of Fréchet spaces and continuous maps. Denote by X the inductive limit of the X_i, by $f_i : X_i \to X$ the natural maps and consider a continuous linear map $T : F \to X$ where F is a Fréchet space. Assume that X is Hausdorff. Then there is an index i^0 such that $T(F) \subset f_{i^0}(X_{i^0})$. Moreover if f_{i^0} is injective, then there is a continuous map $T^0 : F \to X_{i^0}$ such that T is factorized into $F \xrightarrow{T^0} X_{i^0} \xrightarrow{f_{i^0}} X$.*

As a consequence of Theorem 3.5 we can give a direct proof of the implication (iv) $\Rightarrow$ (ii) in Theorem 1.1.

Proposition 3.6. *Let $T : \mathcal{A}'(U) \to \mathcal{A}'(V)$ be a continuous linear map. Then for every $K \Subset U$ there is $L \Subset V$ such that*

$$T(\mathcal{A}'(K)) \subset \mathcal{A}'(L).$$

4. Defining functions of kernels

In this section, we give the proof of the implication (iii) $\Rightarrow$ (iv) in Theorem 1.1. In the argument we must understand the structure of a (properly supported) kernel which satisfies the condition (1.1) in terms of defining functions. The first result in this direction is Lemma 4.2, which we shall use again in Section 6. Note that the last statement regarding C^∞ kernels is used only there. See Remark 6.2.

Recall that the dual cone $G^\perp \subset \mathbb{R}^p$ of a cone $G \subset \mathbb{R}^p$ is defined by $G^\perp := \{\xi \in \mathbb{R}^p; \langle s, \xi\rangle \geq 0,\ \forall s \in G\}$. Also recall the twisted Radon kernel $W(z, \Delta)$ of Bony-type on $\mathbb{R}^p$ with respect to a proper cone $\Delta \subset \mathbb{R}^p$. It is defined by

$$W(z, \Delta) := \int_{S^{p-1}\cap\Delta} W(z, \xi)d\omega(\xi), \tag{4.1}$$

where

$$W(z, \xi) := \frac{(p-1)!}{(-2\pi i)^p} \cdot \frac{1 + i\langle z, \xi\rangle}{(\langle z, \xi\rangle + i\langle z, z\rangle)^p},$$

for $(z, \xi) \in \mathbb{C}^p \times S^{p-1}$ with $|\operatorname{Im} z - \xi/2|^2 < 1/4 + |\operatorname{Re} z|^2$, and where $\omega(\xi)$ denotes the standard surface element of S^{p-1}. (See (A.8.1) in [4].) We will use W in the proof of Lemma 4.2, in case of $p = n + m$ with z replaced by (z, w). For the theory of the twisted Radon transform, we refer to [10,9,1]. Here we only add the following preparation, which is an analogue for Bony-type kernels of Lemma 2.3.5 in [9].

Lemma 4.1. *For $f(x) \in C_0^\infty(\mathbb{R}^p)$, the function*

$$\tilde{f}(z, \xi) := \int_{\mathbb{R}^p} W(z - x, \xi) f(x)dx$$

defined on $\{(z, \xi) \in \mathbb{C}^p \times S^{p-1}; |\operatorname{Im} z - \xi/2| < 1/2\}$ can be extended continuously to

$$\{(z, \xi) \in \mathbb{C}^p \times S^{p-1}; |\operatorname{Im} z - \xi/2| \leq 1/2, |\operatorname{Im} z| < 1/2\}.$$

Moreover the derivatives $\partial_z^\alpha \tilde{f}(z, \xi)$ in z admits the same domain of continuity.

Proof. We define $\Phi(z, \xi) := \langle z, \xi\rangle + i\langle z, z\rangle$, $\Psi(z, \xi) := \Phi(z, \xi)\big(\log \Phi(z, \xi) - 1\big)$, and

$$P(z, \xi, \partial_z) := \frac{(-1)^{p-1}(1 + i\langle z, \xi\rangle)}{(-2\pi i)^p} \left(\frac{1}{1 + 2i\langle z, \xi\rangle}\langle \xi, \partial_z\rangle\right)^{p+1}.$$

Here we regard P as a differential operator of order $p + 1$ with holomorphic coefficients defined on $\{|\operatorname{Im} z| < 1/2\}$. Note that Ψ is continuous on $\{|\operatorname{Im} z - \xi/2| \leq 1/2\}$ since $\operatorname{Im} \Phi \geq 0$ holds there. With these notations, we have $W(z, \xi) = P(z, \xi, \partial_z)\Psi(z, \xi)$, and therefore $\tilde{f}$ can be written as

$$\tilde{f}(z, \xi) = \int_{\mathbb{R}^p} \Psi(z - x, \xi)Q(x, z, \xi, \partial_x) f(x)dx,$$

where $Q(x, z, \xi, \partial_x)$ is the adjoint operator of $P(z - x, \xi, -\partial_x)$. The conclusion for $\tilde{f}$ follows from the hypothesis $f \in C_0^\infty(\mathbb{R}^p)$ and the continuity of Ψ. As for its k-th order derivatives, we repeat the same argument, with Ψ replaced by a C^k-function $\Phi^{k+1}(\log \Phi - \sum_{j=1}^{k+1} 1/j)/(k+1)!$ on $\operatorname{Im} \Phi \geq 0$, and with the power $p + 1$ by $p + 1 + k$ in the definition of P. $\square$

Lemma 4.2. *Let $\mathcal{K} \in \mathcal{B}(V \times U)$ be a kernel satisfying (1.1), and consider proper open convex cones $G_j \subset \mathbb{R}^n$ $(j = 1, \dots, J)$ such that the interiors of their dual cones form a covering of $\dot{\mathbb{R}}^n$. Then there exist open sets $\Omega_j \subset \mathbb{C}^{n+m}_{z,w}$ and holomorphic functions $F_j \in \mathcal{O}(\Omega_j)$, $j = 1, 2, \dots, J$ satisfying the following properties:*

(1) *For any relatively compact open set $M \Subset V \times U$, we can find open convex cones $\Lambda_j \subset \mathbb{R}^{n+m}$ $(j = 1, \dots, J)$ with $\Lambda_j \supset (\overline{G}_j \setminus \{0\}) \times \{0\}$ and a positive constant $\varepsilon > 0$ for which*

$$\{(z, w) \in M + i\Lambda_j;\ |\mathrm{Im}\, z| < \varepsilon, |\mathrm{Im}\, w| < \varepsilon\} \subset \Omega_j, \quad \forall j.$$

(2) $(V \times U) \setminus \mathrm{singsupp}_A \mathcal{K} \subset \Omega_j$, $\forall j$, *where* $\mathrm{singsupp}_A \mathcal{K}$ *denotes the analytic singular support of $\mathcal{K}$. In other words, when $M \Subset (V \times U) \setminus \mathrm{singsupp}_A \mathcal{K}$ in the situation* (1)*, then we can take $\Lambda_j = \mathbb{R}^{n+m}$ for any j. In particular, every F_j extends holomorphically to a complex neighborhood of* $(V \times U) \setminus \mathrm{singsupp}_A \mathcal{K}$.

(3) *$\mathcal{K}$ is the sum of the boundary values of the F_j's, i.e.,*

$$\mathcal{K} = \sum_j b(F_j) \quad \text{on } V \times U.$$

Here the $b(F_j)$ are defined, locally in the variables $(x, y) = \mathrm{Re}(z, w) \in M$, as the hyperfunctional boundary values of the F_j from the region $\mathrm{Im}(z, w) \in \Lambda_j$. *($M$ and Λ_j are as in* (1)*).*

Moreover, if $\mathcal{K} \in C^\infty(V \times U)$, then we can take $F_j \in \mathcal{O}(\Omega_j)$ with the additional property

(4) *each $F_j(z, w)$ can be continued to a continuous function on $\Omega_j \cup V \times U$, and the continuations satisfy $\sum_{j=1}^{J} F_j(x, y) = \mathcal{K}(x, y)$ on $V \times U$.*

Remark 4.3. We want to make a comment on the geometry of the domains appearing in the lemma: for any $M \Subset V \times U$, the functions F_j are holomorphic in a set of form

$$\{(z, w) \in M + i(G_j \times \mathbb{R}^m);\ |\mathrm{Im}\, w| < \delta|\mathrm{Im}\, z|,\ |\mathrm{Im}\, z| < \varepsilon,\ |\mathrm{Im}\, w| < \varepsilon\}$$

with some positive δ and ε which may depend on the choice of M.

Proof of Lemma 4.2. We take larger proper open convex cones $G'_j \supset \overline{G_j} \setminus \{0\}$ such that the interiors of their duals form a covering of $\dot{\mathbb{R}}^n$, and also take a covering $\bigcup_{j=1}^{J} \Gamma_j = \dot{\mathbb{R}}^n$ consisting of proper closed convex cones Γ_j in $\dot{\mathbb{R}}^n$, such that their interiors are mutually disjoint and that $\Gamma_j \subset \mathrm{Int}\, G'^{\perp}_j$. For each $j = 1, 2, \dots, J$, we fix some vector $s^j \in G'_j$ and define open convex cones $\Lambda_{j,k}$ in $\mathbb{R}^{n+m}$ by

$$\Lambda_{j,k} := \{(s, 0) + r(s^j, t) \in \mathbb{R}^n \times \mathbb{R}^m;\ s \in G'_j, r > 0, |t| < 1/k\},$$

for $k \in \mathbb{N}$. By this definition, we can see that

$$\mathrm{Int}\, \Lambda^{\perp}_{j,k} = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m;\ \xi \in \mathrm{Int}\, G'^{\perp}_j, |\eta| < k\langle s^j, \xi\rangle\},$$

and that $\mathrm{Int}\, \Lambda^{\perp}_{j,k}$ for $k = 1, 2, \dots$ form an absorbing family of increasing open conic subsets of $\mathrm{Int}\, G'^{\perp}_j \times \mathbb{R}^m$.

Now for an arbitrarily fixed relatively compact open set $M \Subset V \times U$, we associate with M holomorphic functions $F_{j,M}$ as follows. Since $\mathcal{K}$ satisfies (1.1), $\mathrm{WF}_A \mathcal{K}$ is included in $V \times U \times \dot{\mathbb{R}}^n \times \mathbb{R}^m = \bigcup_{j=1}^{J} V \times U \times \mathrm{Int}\, G'^{\perp}_j \times \mathbb{R}^m = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{J} V \times U \times \mathrm{Int}\, \Lambda^{\perp}_{j,k}$. Moreover since $\overline{M}$ is compact and since $\mathrm{WF}_A\mathcal{K}$ is conic, there exists a constant k_M such that

$$(\overline{M} \times \dot{\mathbb{R}}^{n+m}) \cap \mathrm{WF}_A\mathcal{K} \subset \overline{M} \times \bigcup_{j=1}^{J} \mathrm{Int}\, \Lambda_{j,k_M}^{\perp}.$$

Then we define a closed convex cone $\Delta_{j,M} \subset \dot{\mathbb{R}}^{n+m}$ by

$$\Delta_{j,M} := (\Gamma_j \times \mathbb{R}^m) \cap \Lambda_{j,k_M}^{\perp},$$

and it trivially follows that

$$(\overline{M} \times \Gamma_j \times \mathbb{R}^m) \cap \mathrm{WF}_A\mathcal{K} \subset \overline{M} \times \Delta_{j,M}.$$

Note that $\Delta_{j,M}$ is included in $\mathrm{Int}\, \Lambda_{j,k_M+1}^{\perp}$, which can be seen by a straightforward argument. Next we take a hyperfunction $\mathcal{L}_M \in \mathcal{B}(\mathbb{R}^{n+m})$ with compact support, which coincides with $\mathcal{K}$ in a neighborhood of $\overline{M}$. This can be in general done using the flabbiness of $\mathcal{B}$, but when $\mathcal{K} \in C^\infty(V \times U)$, we can take $\mathcal{L}_M \in C_0^\infty(\mathbb{R}^{n+m})$. Finally we define $F_{j,M}$ by

$$F_{j,M}(z, w) := \int_{\mathbb{R}^{n+m}} W(z - x, w - y, \Delta_{j,M}) \mathcal{L}_M(x, y)\,dx\,dy,$$

where $W(z, w, \Delta_{j,M})$ is given by (4.1) with z replaced by (z, w). Then, we have the following:

- Each $F_{j,M}$ is holomorphic on
$$\tilde{\Omega}_{j,M} := \{(z, w) \in \mathbb{R}^{n+m} + i\Lambda_{j,k_M+1};\, |\mathrm{Im}\, z| < \varepsilon_M,\, |\mathrm{Im}\, w| < \varepsilon_M\}$$
with some $\varepsilon_M > 0$, since $\Delta_{j,M} \subset \mathrm{Int}\, \Lambda_{j,k_M+1}^{\perp}$.
- Each $F_{j,M}$ can be continued holomorphically to a complex neighborhood of $\mathbb{R}^{n+m} \setminus \mathrm{singsupp}_A \mathcal{L}_M$, in particular, to a complex neighborhood of $M \setminus \mathrm{singsupp}_A \mathcal{K}$.
- $\mathcal{L}_M - \sum_{j=1}^{J} b(F_{j,M}) \in \mathcal{A}(\mathbb{R}^{n+m})$. Therefore $\mathcal{K} - \sum_{j=1}^{J} b(F_{j,M}) \in \mathcal{A}(M)$.
- If $\mathcal{L}_M \in C_0^\infty(\mathbb{R}^{n+m})$, then $F_{j,M}$ can be continued to a continuous function on $\tilde{\Omega}_{j,M} \cup \mathbb{R}^{n+m}$.

Here we used Lemma 4.1 for the last property. We shrink $\tilde{\Omega}_{j,M}$ to

$$\Omega_{j,M} := \{(z, w) \in M + i\Lambda_{j,k_M+1};\, |\mathrm{Im}\, z| < \varepsilon_M,\, |\mathrm{Im}\, w| < \varepsilon_M\} \tag{4.2}$$

and regard $F_{j,M}$ and $b(F_{j,M})$ as holomorphic functions on $\Omega_{j,M}$ and hyperfunctions on M respectively.

Now we fix j and take two relatively compact open subsets M and N in $V \times U$, and compare $F_{j,M}$ and $F_{j,N}$. We may assume without loss of generality that $k_M \le k_N$, then we have $\Delta_{j,M} \subset \Delta_{j,N}$ and

$$\begin{aligned}(F_{j,N} - F_{j,M})(z, w) = &\int_{\mathbb{R}^{n+m}} W(z - x, w - y, \Delta_{j,N} \setminus \Delta_{j,M}) \mathcal{L}_N(x, y)\,dx\,dy \\ &+ \int_{\mathbb{R}^{n+m}} W(z - x, w - y, \Delta_{j,M}) (\mathcal{L}_N - \mathcal{L}_M)(x, y)\,dx\,dy.\end{aligned}$$

The first term in the right hand side extends holomorphically to a complex neighborhood of $M \cap N$, since $(M \cap N) \times \overline{(\Delta_{j,N} \setminus \Delta_{j,M})} \cap \mathrm{WF}_A\mathcal{L}_N = (M \cap N) \times \overline{(\Delta_{j,N} \setminus \Delta_{j,M})} \cap \mathrm{WF}_A\mathcal{K} = \emptyset$. The second term also extends holomorphically to a complex neighborhood of $M \cap N$, since

$\mathcal{L}_N - \mathcal{L}_M = 0$ on $M \cap N$. Therefore, $F_{j,N} - F_{j,M}$ also extends to a complex neighborhood of $M \cap N$, which implies that $b(F_{j,N}) - b(F_{j,M})$ is real-analytic on $M \cap N$. In view of this, the family $b(F_{j,N}) - b(F_{j,M})$ on $M \cap N$ forms for $M, N \Subset V \times U$ a cocycle with real-analytic coefficients associated with the open covering $\{M\}_{M \Subset V\times U}$ of $V \times U$, and we can find, (using the cohomological triviality of $\mathcal{A}$), $H_{j,M} \in \mathcal{A}(M)$ for $M \Subset V \times U$ such that $b(F_{j,N}) - b(F_{j,M}) = H_{j,N} - H_{j,M}$ on $M \cap N$. We denote the holomorphic extension of $H_{j,M}$ by the same symbol $H_{j,M}$, and then the last equality shows that $b(F_{j,N} - H_{j,N}) = b(F_{j,M} - H_{j,M})$ on $M \cap N$. Since an equality of two single boundary value expressions implies the equality of their defining functions, $\{F_{j,M} - H_{j,M}\}_{M \Subset V\times U}$ can be glued together to a holomorphic function F_j.

Let us next show that we can find Ω_j as in the relations (1)–(3) in the lemma. For any $M \Subset V \times U$, we can take $N \Subset V \times U$ satisfying $M \Subset N$. Then we have the equality $F_j = F_{j,N} - H_{j,N}$ on the intersection of $\Omega_{j,N}$ and a complex neighborhood of N. Since a complex neighborhood of N must include a set of form $\{(z, w) \in M + i\mathbb{R}^{n+m}; |\mathrm{Im}\, z| < \varepsilon', |\mathrm{Im}\, w| < \varepsilon'\}$ with some $\varepsilon' > 0$, F_j is holomorphic on

$$\{(z, w) \in M + i\Lambda_{j,k_N+1}; |\mathrm{Im}\, z| < \varepsilon, |\mathrm{Im}\, w| < \varepsilon\},$$

where $\varepsilon := \min\{\varepsilon_N, \varepsilon'\}$, and by taking the union of such open sets for $M \Subset V \times U$, we get Ω_j satisfying the property (1) and $F_j \in \mathcal{O}(\Omega_j)$.

We can add a complex neighborhood of $V \times U \setminus \mathrm{singsupp}_A \mathcal{K}$ to Ω_j since $F_{j,M}$ extends to a neighborhood of $M \setminus \mathrm{singsupp}_A \mathcal{K}$.

At this moment, the property (3) does not yet hold. However, since $\mathcal{K} - \sum_{j=1}^{J} b(F_{j,M}) \in \mathcal{A}(M)$ and $b(F_{j,M}) - b(F_j) \in \mathcal{A}(M)$ hold for any M, we have $H := \mathcal{K} - \sum_{j=1}^{J} b(F_j) \in \mathcal{A}(V \times U)$. By adding H to F_1, then (3) will now hold. Note that by this change it might be necessary to shrink Ω_1, but both (1) and (2) will still remain valid.

Finally we consider the case $\mathcal{K} \in C^\infty(V \times U)$. Then $F_{j,M}$ extends to a continuous function up to $\mathbb{R}^{n+m}$, which implies also that $F_{j,M} - H_{j,M}$ does so up to M. Therefore we can define the extension of F_j to $\Omega_j \cup V \times U$, and the continuity holds since continuity is a local property. This proves the first part of (4), and the second part follows from the first part and (3). $\square$

The decomposition in the lemma gives us the possibility to write an integral operator as in (1.2) in an explicit way. We shall start by decomposing $\mathcal{K}$ into the sum $\sum_{j=1}^{J} \mathcal{K}_j$, where $\mathcal{K}_j = b(F_j)$ with F_j as in Lemma 4.2. This leads us to a decomposition of the operator $T = \sum_{j=1}^{J} T_j$ with $T_j : \mathcal{A}'(U) \to \mathcal{B}(V)$ given by

$$(T_j u)(x) := \int \mathcal{K}_j(x, y)u(y)dy.$$

For any given $u \in \mathcal{A}'(U)$, we take $M := V_0 \times U_0 \Subset V \times U$ with $\mathrm{supp}\, u \subset U_0$. Then, F_j is holomorphic in a set of form (4.2). (The notations are as in the proof of Lemma 4.2.) Thus the defining function of $T_j u$ is given by

$$\int F_j(z, y)u(y)dy,$$

which is holomorphic on $\{z \in V_0 + iG_j; |\mathrm{Im}\, z| < \varepsilon\}$. We note here that the constant $\varepsilon > 0$ may depend on V_0 (and on $\mathrm{supp}\, u$). But, since the choice of $V_0 \Subset V$ was arbitrary, the boundary value

$b_{G_j}\big(\int F_j(z,y)u(y)dy\big)$ is an element of $\mathcal{B}(V)$. Therefore we have the formula

$$(Tu)(x) = \int \mathcal{K}(x,y)u(y)dy = \sum_{j=1}^{J} b_{G_j}\bigg(\int F_j(z,y)u(y)dy\bigg), \tag{4.3}$$

for a kernel $\mathcal{K} \in \mathcal{B}(V \times U)$ satisfying (1.1) and $u \in \mathcal{A}'(U)$.

Consider next the case when $\mathcal{K} \in \mathcal{B}(V \times U)$ is a properly supported kernel, i.e., assume that the projection $p_2|_{\operatorname{supp}\mathcal{K}} : \operatorname{supp}\mathcal{K} \to U$ is proper. We fix an open set $U_0 \Subset U$. Then there exists a compact set $L \subset V$ such that

$$(V \times \overline{U_0}) \cap \operatorname{supp}\mathcal{K} \subset L \times \overline{U_0}. \tag{4.4}$$

For these U_0 and L, we take a compact set L_0 and an open set V_0 in V such that $L \Subset L_0 \Subset V_0 \Subset V$. In this situation, the property (2) of Lemma 4.2 gives also the following estimate of the domains of holomorphy of the F_j's: there exist positive constants ε and δ such that each F_j is holomorphic in the union

$$\begin{aligned}&\{(z,w) \in (V_0 \times U_0) + i(G_j \times \mathbb{R}^m);\ |\operatorname{Im} w| < \delta|\operatorname{Im} z|,\ |\operatorname{Im} z| < \varepsilon,\ |\operatorname{Im} w| < \varepsilon\}\\ &\quad\cup \{(z,w) \in ((V_0 \setminus L_0) \times U_0) + i\mathbb{R}^{n+m};\ |\operatorname{Im} z| < \varepsilon,\ |\operatorname{Im} w| < \varepsilon\},\end{aligned} \tag{4.5}$$

and the sum $\sum_{j=1}^{J} F_j(z,w)$ vanishes on the set $\{(z,w) \in ((V_0 \setminus L_0) \times U_0) + i\mathbb{R}^{n+m};\ |\operatorname{Im} z| < \varepsilon,\ |\operatorname{Im} w| < \varepsilon\}$.

We now apply this for the operator in (1.2). For a hyperfunction u with $\operatorname{supp} u \subset U_0$, the functions

$$H_j(z) := \int F_j(z,y)u(y)dy,$$

are holomorphic in

$$\{z \in V_0 + iG_j;\ |\operatorname{Im} z| < \varepsilon\} \cup \{z \in (V_0 \setminus L_0) + i\mathbb{R}^n;\ |\operatorname{Im} z| < \varepsilon\}, \tag{4.6}$$

and the sum $\sum_j H_j$ vanishes on $\{z \in (V_0 \setminus L_0) + i\mathbb{R}^n;\ |\operatorname{Im} z| < \varepsilon\}$. The duality pairing of $Tu = \sum_j T_j u$ with a real-analytic function $\varphi(x)$ defined in a neighborhood of L can be calculated as follows: take the V_0 and L_0 as above such that V_0 is included in the domain of definition of φ, and also take a compact set Z with piecewise smooth boundary satisfying $L_0 \Subset Z \Subset V_0$. We choose vectors $s^j \in G_j$ and define contours γ_j's by

$$\gamma_j := \{x + i\psi(x)s^j;\ x \in Z\}.$$

Here ψ is a non negative function satisfying $\psi(x) = 0$ on ∂Z, chosen such that γ_j is included in the set (4.6) and in the domain of analyticity of φ. Then $\int (Tu)(x)\varphi(x)dx$ is given by

$$\sum_{j=1}^{J} \int_{\gamma_j} H_j(z)\varphi(z)dz.$$

Note that the single integrals depend on the choice of Z but the sum does not.

In this way, we have an explicit integration formula

$$\langle Tu, \varphi\rangle = \sum_{j=1}^{J} \int_{\gamma_j} \int \varphi(z)F_j(z,y)u(y)dydz, \quad \forall u \in \mathcal{A}'(U_0), \forall \varphi \in \mathcal{A}(V_0), \tag{4.7}$$

for fixed U_0 and V_0. This formula now shows that the operator T is the formal adjoint of the operator $S : \mathcal{A}(V) \to \mathcal{A}(U)$ defined in the following way: if $\varphi \in \mathcal{A}(V)$ is given and if we fix some $U_0 \Subset U$, then the value of $S\varphi$ for $y \in U_0$ is given by

$$(S\varphi)(y) = \sum_{j=1}^{J} \int_{\gamma_j} \varphi(z) F_j(z, y) dz, \tag{4.8}$$

where the γ_j are associated with φ as above.

After these preparations, we can now prove the implication (iii) $\Rightarrow$ (iv). To do so, we recall that a fundamental system of semi-norms on $\mathcal{A}'(V)$ is given by the expressions $\|v\|_{\mathcal{M}} = \sup_{f \in \mathcal{M}} |v(f)|$ where $\mathcal{M}$ runs through the family of bounded sets in $\mathcal{A}(V)$. We fix such a bounded set $\mathcal{M}$ and want to show that there is c and a bounded set $\mathcal{N} \subset \mathcal{A}(U)$ such that $\|Tu\|_{\mathcal{M}} \le c\|u\|_{\mathcal{N}}$ for every $u \in \mathcal{A}'(U)$. In view of the duality (4.7), it shall suffice to show then that the operator S defined in (4.8) maps bounded sets in $\mathcal{A}(V)$ to bounded sets in $\mathcal{A}(U)$.

Lemma 4.4. *Let $\mathcal{M}$ be a bounded set in $\mathcal{A}(V)$. Define $\mathcal{N} \subset \mathcal{A}(U)$ by $\mathcal{N} = \{\int_V \mathcal{K}(x, y)\varphi(x)dx; \varphi \in \mathcal{M}\}$. Then $\mathcal{N}$ is bounded in $\mathcal{A}(U)$.*

The proof of Lemma 4.4 is by direct inspection of (4.8), taking also into account that the functions in $\mathcal{M}$ admit a common domain of definition. Indeed, it follows for example from (4.8) that if the functions $\varphi \in \mathcal{A}(V)$ are analytic near the set $\{z \in \mathbb{C}^n; \operatorname{Re} z \in V_0, |\operatorname{Im} z| < \varepsilon\}$, then $S\varphi$ is analytic on $\{w \in \mathbb{C}^m; \operatorname{Re} w \in U_0, |\operatorname{Im} w| < \min\{\delta\varepsilon, \varepsilon\}\}$. (See (4.5).) It then just remains to estimate the functions $S\varphi$. We omit further details.

5. Preparations for the proof of (i) $\Rightarrow$ (iii)

In the remaining part of the paper we shall essentially deal with the implication (i) $\Rightarrow$ (iii) in Theorem 1.1.

We start the argument with a review of terminology and a preliminary result. In describing terminology, we shall work in n variables, but similar notations shall be used also in related situations. A function $\ell : \mathbb{R}^n \to \mathbb{R}_+$ is said to be "sublinear" if for every $\varepsilon > 0$ there is c such that $\ell(\xi) \le c + \varepsilon|\xi|$, $\forall \xi \in \mathbb{R}^n$. A measurable function $f : \mathbb{R}^n \to \mathbb{C}$ shall be called of "infra-exponential type" if there is a sublinear function $\ell(\xi)$ such that $|f(\xi)| \le \exp[\ell(\xi)]$, $\forall \xi \in \mathbb{R}^n$. Similarly, a measurable function $f : \mathbb{C}^n \to \mathbb{C}$ shall be called of infra-exponential type if $\log(1 + |f(\zeta)|)$ is sublinear on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. We also mention the following simple lemma on sublinear functions.

Lemma 5.1. *For every sequence of positive sublinear functions ℓ_j, $j \in \mathbb{N}$, there is a sublinear function ℓ and constants c'_j such that $\ell_j(\xi) \le \ell(\xi) + c'_j$ holds for any $j \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$.*

Consider next an entire function $P(\zeta)$ on $\mathbb{C}^n$ of infra-exponential type. In particular, we can expand P into a power series $P(\zeta) = \sum_{|\alpha|<\infty} a_\alpha \zeta^\alpha$, $\zeta \in \mathbb{C}^n$. We can associate with it an "infinite order partial differential operator of infra-exponential type", $P(D)$, first on holomorphic functions $F(z)$ defined on open sets $\Omega \subset \mathbb{C}^n$, and then on hyperfunctions $u(x)$ on open sets $V \subset \mathbb{R}^n$ in the following way: $P(D_z)F(z) = \sum_{|\alpha|<\infty} a_\alpha (-i\partial/\partial z)^\alpha F(z)$, and to define $P(D_x)u$, we write u locally near some point $x^0 \in V$ as the sum of hyperfunctional boundary values of some holomorphic functions F_j, defined on wedges which live near x^0, in notations, $u = \sum_{j=1}^{J} b(F_j)$, and then we set $P(D_x)u = \sum_{j=1}^{J} b(P(D_z)F_j)$. (We have written $P(D_z)$

when $P(D)$ acts on holomorphic functions, since it then corresponds to $\sum_{|\alpha|<\infty} a_\alpha(-i\partial/\partial z)^\alpha$, and we have written $P(D_x)$ when it acts on hyperfunctions, since it then corresponds formally to $\sum_{|\alpha|<\infty} a_\alpha(-i\partial/\partial x)^\alpha$.)

We next recall how one can define the Fourier inverse transform of a measurable infra-exponential function $f(\xi)$ on $\mathbb{R}^n$ by what we call "Carleman regularization". (cf. [9].) Our aim is to give a meaning in hyperfunctions to the integral $\int_{\mathbb{R}^n} \exp[i\langle x,\xi\rangle] f(\xi)d\xi$. For this purpose we fix open convex cones $\Gamma_j \subset \dot{\mathbb{R}}^n$, $j = 1,\dots,J$, such that $\dot{\mathbb{R}}^n = \bigcup_{j=1}^J \Gamma_j$. Also consider a splitting of f into a sum of measurable infra-exponential functions $\sum_{j=1}^J f_j$ such that $\operatorname{supp} f_j \subset \Gamma_j$. The functions $F_j(z) = \int_{\mathbb{R}^n} \exp[i\langle z,\xi\rangle] f_j(\xi)d\xi$ are then holomorphic on the set $\mathbb{R}^n + i\operatorname{Int}\Gamma_j^\perp$ and therefore admit hyperfunctional boundary values to $\mathbb{R}^n$. We denote these hyperfunctional boundary values by $b(F_j)$. The Fourier inverse transform of f is then by definition

$$\mathcal{F}^{-1}(f) = (2\pi)^{-n}\sum_{j=1}^J b(F_j). \tag{5.1}$$

It is easy to see that this definition does not depend on the way we have split f into the sum $f = \sum_{j=1}^J f_j$. Moreover, Proposition 5.2 is well-known and follows from the definition. Note that the Fourier transform $\mathcal{F}u$ of a hyperfunction u with compact support is defined by

$$(\mathcal{F}u)(\xi) := \int u(x)\exp[-i\langle x,\xi\rangle]dx,$$

and is an infra-exponential function on $\mathbb{R}^n$.

Proposition 5.2. *Let f be a measurable infra-exponential function on $\mathbb{R}^n$.*

(i) *If f is equal to the Fourier transform $\mathcal{F}u$ of a hyperfunction u with compact support, then its Fourier inverse transform $\mathcal{F}^{-1}f = \mathcal{F}^{-1}\mathcal{F}u$ is equal to u.*

(ii) *Let $\Gamma \subset \mathbb{R}^n$ be a closed cone. If f satisfies*

$$|f(\xi)| \le Ce^{-\varepsilon|\xi|} \quad \textit{for } \xi \notin \Gamma,$$

with some positive constants C and ε, then

$$\mathrm{WF}_A\mathcal{F}^{-1}(f) \subset \mathbb{R}^n \times \Gamma. \tag{5.2}$$

In particular, (5.2) holds when $\operatorname{supp} f \subset \Gamma$.

(iii) *If $|\xi|^k f(\xi)$ is bounded for any k, then $\mathcal{F}^{-1}(f) \in C^\infty(\mathbb{R}^n)$.*

(iv) *When $P(D)$ is an infra-exponential differential operator constructed as above, then*

$$P(D)\mathcal{F}^{-1}(f) = \mathcal{F}^{-1}(P(\xi)f(\xi)). \tag{5.3}$$

After these remarks concerning terminology, we now turn to the preparations part. We need in fact to show that hyperfunctions can be represented modulo real-analytic functions as images of (say) C^∞ functions by a partial differential operator of infinite order. For an early instance of such a representation, we refer to [7]. (Infinite order infra-exponential operators as used in the present argument were used at about the same time by Kawai [11] and Boutet de Monvel [5]. Since in [7] no real-analytic parameters y were used, there was no need there for real-analytic "remainder terms".)

As a consequence, we shall have to deal with the analytic singular supports of (the hyperfunctions in) the ranges of integral operators with differentiable kernels, rather than with the supports of the ranges of operators with hyperfunction kernels.

Theorem 5.3. *Let $\mathcal{K} \in \mathcal{B}(V \times U)$ be a kernel satisfying the wave front set estimate (1.1). Then there are an elliptic (infinite order) partial differential operator $P(D_x)$ with constant coefficients in the variables x, a C^∞ function $\mathcal{K}'$ on $V \times U$, which also satisfies the wave front set estimate*

$$\{(x, y; 0, \eta) \in V \times U \times \mathbb{R}^n \times \mathbb{R}^m; \eta \neq 0\} \cap \mathrm{WF}_A \mathcal{K}' = \emptyset, \tag{5.4}$$

and a real-analytic function $\mathcal{K}''$ on $V \times U$, such that $\mathcal{K} = P(D_x)\mathcal{K}' + \mathcal{K}''$.

Proof. We start by taking an increasing sequence $K_1 \subset K_2 \subset \cdots$ of compact subsets in $V \times U$ satisfying $\bigcup_j \mathrm{Int}\, K_j = V \times U$, and consider for each K_j a hyperfunction $\mathcal{K}_j$ with compact support which coincides with $\mathcal{K}$ in a neighborhood of K_j.

We shall prove Theorem 5.3 in three steps. In the first, our main goal is to find an elliptic operator $P(D_x)$, whose symbol increases sufficiently rapidly on $\mathbb{R}^n$, depending on the growth orders of the Fourier transforms of the $\mathcal{K}_j$ and the wave front set estimates of the $\mathcal{K}_j$'s over the sets K_j's. In the second step, we shall construct functions $\mathcal{K}'_j \in C^\infty(\mathrm{Int}\, K_j)$ and $\mathcal{K}''_j \in \mathcal{A}(\mathrm{Int}\, K_j)$ for $j = 1, 2, \ldots$ satisfying the desired properties for $\mathcal{K}_j$ on $\mathrm{Int}\, K_j$. The final step is then to glue the various functions and hyperfunctions together.

Step 1. Since $\mathcal{K}$ satisfies (1.1), there exist positive constants c_j, $j = 1, 2, \ldots$, such that

$$\mathrm{WF}_A \mathcal{K}_j \cap \{(x, y; \xi, \eta); (x, y) \in K_j\} \subset \{(x, y; \xi, \eta); |\eta| \leq c_j |\xi|\}. \tag{5.5}$$

Moreover, the $\mathcal{K}_j$ being compactly supported hyperfunctions we can find sublinear functions $\ell_j : \mathbb{R}^{n+m} \to \mathbb{R}_+$ such that their Fourier transforms $\mathcal{F}\mathcal{K}_j$ can be estimated in the form

$$|\mathcal{F}\mathcal{K}_j(\xi, \eta)| \leq \exp \ell_j(\xi, \eta), \quad \forall (\xi, \eta) \in \mathbb{R}^{n+m}. \tag{5.6}$$

Actually, we shall not need this estimate in the regions (in the phase space) where there is no wave front set (over K_j: see (5.5)), and in the regions where we want to use (5.6) we want to replace the sequence of functions ℓ_j by a single sublinear function which depends only on ξ. To achieve the latter goal, we consider at first the sublinear functions

$$\tilde{\ell}_j(\xi) = \sup_{|\eta| \leq 2c_j |\xi|} \left(\ell_j(\xi, \eta) + |(\xi, \eta)|^{1/2}\right) \tag{5.7}$$

and then we take a sublinear function $\ell : \mathbb{R}^n \to \mathbb{R}_+$ and constants c'_j such that $\tilde{\ell}_j(\xi) \leq c'_j + \ell(\xi)$, $\forall \xi \in \mathbb{R}^n$. (See Lemma 5.1.) It follows that we have

$$|\mathcal{F}\mathcal{K}_j(\xi, \eta)| \leq A_j \exp\left(\ell(\xi) - |(\xi, \eta)|^{1/2}\right) \quad \text{if } |\eta| \leq 2c_j |\xi|,$$

with some positive constants A_j. Finally, to conclude Step 1, we take an infra-exponential elliptic symbol $P(\zeta)$ ($\zeta \in \mathbb{C}^n$) and a positive constant c satisfying

$$|P(\zeta)| \geq e^{\ell(\mathrm{Re}\, \zeta)} \quad \text{if } |\mathrm{Im}\, \zeta| < c|\mathrm{Re}\, \zeta|. \tag{5.8}$$

(See [11,7].)

Step 2. We define measurable functions $Q_j(\xi, \eta)$ and $R_j(\xi, \eta)$ by

$$Q_j(\xi, \eta) = \begin{cases} \mathcal{F}\mathcal{K}_j(\xi, \eta)/P(\xi) & \text{if } |\eta| \leq 2c_j |\xi|, \\ 0 & \text{if } |\eta| > 2c_j |\xi|, \end{cases}$$

and

$$R_j(\xi,\eta) = \mathcal{F}\mathcal{K}_j(\xi,\eta) - P(\xi)Q_j(\xi,\eta).$$

In view of the definitions, we also have

$$\operatorname{supp} Q_j \subset \{(\xi,\eta);\ |\eta| \le 2c_j|\xi|\}, \qquad \operatorname{supp} R_j \subset \{(\xi,\eta);\ |\eta| \ge 2c_j|\xi|\},$$
$$|Q_j(\xi,\eta)| \le A_j \exp\bigl(-|(\xi,\eta)|^{1/2}\bigr) \quad \text{for } (\xi,\eta) \in \mathbb{R}^{n+m}. \tag{5.9}$$

Both Q_j and R_j are infra-exponential functions, so their Fourier inverse transforms are well-defined. It follows from (i) and (iv) of Proposition 5.2 that

$$\mathcal{K}_j(x,y) = P(D_x)\mathcal{F}^{-1}Q_j(x,y) + \mathcal{F}^{-1}R_j(x,y).$$

It is also immediate from (ii) and (iii) of Proposition 5.2 that

$$\mathrm{WF}_A\mathcal{F}^{-1}Q_j \subset \{(x,y;\xi,\eta);\ |\eta| \le 2c_j|\xi|\},$$
$$\mathrm{WF}_A\mathcal{F}^{-1}R_j \subset \{(x,y;\xi,\eta);\ |\eta| \ge 2c_j|\xi|\},$$
$$\mathcal{F}^{-1}Q_j \in C^\infty(\mathbb{R}^{n+m}).$$

(The last statement follows from (5.9), which actually shows that $\mathcal{F}^{-1}Q_j$ is a Gevrey-2 function.)

Moreover if we take into account (5.5) and the microlocal ellipticity of P away from $\{(\xi,\eta) \in \mathbb{R}^{n+m};\ \xi = 0\}$, we have

$$\mathrm{WF}_A\mathcal{F}^{-1}Q_j \cap \{(x,y;\xi,\eta);\ (x,y) \in K_j\} \subset \{|\eta| \le c_j|\xi|\} \cup \{|\eta| = 2c_j|\xi|\},$$
$$\mathrm{WF}_A\mathcal{F}^{-1}R_j \cap \{(x,y;\xi,\eta);\ (x,y) \in K_j\} \subset \{|\eta| = 2c_j|\xi|\}.$$

Now we claim that for each j, there exists a C^∞ function $\mathcal{K}'_j$ on $\operatorname{Int} K_j$ satisfying

$$\mathrm{WF}_A\mathcal{K}'_j \subset \{(x,y;\xi,\eta);\ |\eta| \le c_j|\xi|\}, \tag{5.10}$$

$$\mathrm{WF}_A(\mathcal{F}^{-1}Q_j - \mathcal{K}'_j) \subset \{(x,y;\xi,\eta);\ |\eta| = 2c_j|\xi|\}. \tag{5.11}$$

In fact, we define $\mathcal{K}'_j$ as the restriction to $\operatorname{Int} K_j$ of the iterated integral

$$\int_{|\eta| \le \frac{3}{2}c_j|\xi|} d\omega(\xi,\eta) \int_{\mathbb{R}^{n+m}} W(x-\tilde{x}, y-\tilde{y}, \xi, \eta)(\chi \cdot \mathcal{F}^{-1}Q_j)(\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y},$$

with some $\chi \in C_0^\infty(\mathbb{R}^{n+m})$, $\chi \equiv 1$ on K_j. Precisely speaking, we first integrate for complex (x,y), and then take a continuous extension to $(x,y) \in \mathbb{R}^{n+m}$. The existence of such an extension and its differentiability in (x,y) follow from Lemma 4.1. On the other hand, by a standard argument concerning the twisted Radon kernel, (5.10) and (5.11) follow from the estimates of $\mathrm{WF}_A(\mathcal{F}^{-1}Q_j)$.

The estimate (5.10) directly gives the wave front set estimate (5.4) for $\mathcal{K}'_j$. Moreover if we look at the estimates of the wave front sets on both sides of the equality

$$\mathcal{K}_j - P(D_x)\mathcal{K}'_j = P(D_x)(\mathcal{F}^{-1}Q_j - \mathcal{K}'_j) + \mathcal{F}^{-1}R_j,$$

we can see easily that $\mathcal{K}''_j := \mathcal{K}_j - P(D_x)\mathcal{K}'_j$ is real-analytic on $\operatorname{Int} K_j$.

Step 3. For any $j < k$, $\mathcal{K}'_j - \mathcal{K}'_k$ is real-analytic on the common part $\operatorname{Int} K_j \cap \operatorname{Int} K_k = \operatorname{Int} K_j$ of the domains of definition. In fact, $P(D_x)(\mathcal{K}'_j - \mathcal{K}'_k) = -\mathcal{K}''_j + \mathcal{K}''_k \in \mathcal{A}(\operatorname{Int} K_j)$ and the microlocal ellipticity of $P(D_x)$ outside $\{(x,y;\xi,\eta);\ \xi = 0\}$ implies $\mathrm{WF}_A(\mathcal{K}'_j - \mathcal{K}'_k) \subset \{\xi = 0\}$, while (5.10) implies $\mathrm{WF}_A(\mathcal{K}'_j - \mathcal{K}'_k) \subset \{\xi \neq 0\}$. Thus the family $(\mathcal{K}'_j - \mathcal{K}'_k)_{j,k}$ forms a cocycle with real-analytic coefficients associated with the open covering $\{\mathrm{Int}\, K_j\}_j$ of $V \times U$, and we can take $\mathcal{L}_j \in \mathcal{A}(\mathrm{Int}\, K_j)$ $(j \in \mathbb{N})$ such that $\mathcal{K}'_j - \mathcal{K}'_k = \mathcal{L}_j - \mathcal{L}_k$. Then the family $(\mathcal{K}'_j - \mathcal{L}_j)_j$ defines a global section $\mathcal{K}' \in C^\infty(V \times U)$. It is easily seen that $\mathcal{K}'$ and $\mathcal{K}'' := \mathcal{K} - P(D_x)\mathcal{K}'$ satisfy all the desired properties. □

Remark 5.4. Under the hypothesis of (i) of Theorem 1.1, we can apply Theorem 5.3 to the kernel of T and get $\mathcal{K}'$, $\mathcal{K}''$ and $P(D_x)$ as in that theorem. Denote by T' the operator

$$(T'u)(x) = \int_U \mathcal{K}'(x, y)u(y)\, dy.$$

If $u \in \mathcal{A}'(U)$, then $T'u$ is real-analytic outside some compact set (which depends on u). In fact, in view of $Tu = P(D_x)(T'u) + \int \mathcal{K}''(x, y)u(y)dy$, we have $\mathrm{singsupp}_A Tu = \mathrm{singsupp}_A P(D_x)(T'u) = \mathrm{singsupp}_A T'u$, since $\int \mathcal{K}''(x, y)u(y)dy$ is real-analytic and since $P(D_x)$ is elliptic.

6. End of the proof of Theorem 1.1

In view of the discussions in the preceding section, we can easily prove that (i) ⇒ (iii) will follow if we can prove the following:

Theorem 6.1. *Let $\mathcal{K} \in C^\infty(V \times U)$ be such that the analytic singular support of $(Tu)(x) = \int_U \mathcal{K}(x, y)u(y)\, dy$ is compact for every compactly supported Radon measure u. Then for every $K \Subset U$ there is $L \Subset V$ such that Tu is analytic outside L if u is a Radon measure with* $\mathrm{supp}\, u \subset K$.

Remark 6.2. (i) Radon measures on U are elements in the dual of $C(U)$, where the latter is endowed with the topology of uniform convergence on compact sets in U. When we speak about Radon measures on a compact set K, we mean elements in the dual of $C(K)$, where $C(K)$ is the space of continuous functions on K with the topology of uniform convergence on K. The dual of $C(K)$ shall be denoted by $C'(K)$.

(ii) Consider the explicit formula (4.3) for the situation of Theorem 6.1, that is, $\mathcal{K} \in C^\infty(V \times U)$ and $u \in C'(K)$ with $K \Subset U$. Now we use the notations G_j, Ω_j, F_j, and so on, as in Lemma 4.2. As is seen from Lemma 4.2, defining functions $F_j \in \mathcal{O}(\Omega_j)$ of $\mathcal{K}$ can be chosen so as to be continuous up to $V \times U$. In this case, the integral $H_j(z) := \int F_j(z, y)u(y)dy$, which is a priori holomorphic on $\{z \in V_0 + iG_j;\, |\mathrm{Im}\, z| < \varepsilon\}$ for any $V_0 \Subset V$, extends continuously to $\{z \in V_0 + iG_j;\, |\mathrm{Im}\, z| < \varepsilon\} \cup V_0$, and also $H_j(x) = \int F_j(x, y)u(y)dy$ holds. Therefore the hyperfunction $b(H_j)$ is equal to a continuous function $H_j(x)$, and by summing up, we have that the hyperfunction Tu is equal to a continuous function $x \mapsto \int \mathcal{K}(x, y)u(y)dy$, where the right hand side is calculated, for any fixed x, as the Radon measure u evaluated at the continuous function $y \mapsto \mathcal{K}(x, y)$.

Proof of Theorem 1.1. (i) ⇒ (iii) using Theorem 6.1. Under the hypothesis of (i) of Theorem 1.1, we can apply Remark 5.4 to $\mathcal{K}$ and get a decomposition $\mathcal{K} = P(D_x)\mathcal{K}' + \mathcal{K}''$ where $\mathcal{K}'$ satisfies the hypothesis of Theorem 6.1. Then, Theorem 6.1 guarantees that for every $K \Subset U$, there exists $L \Subset V$ such that $u \in C'(K)$ implies $\mathrm{singsupp}_A \int \mathcal{K}'(x, y)u(y)dy \subset L$, which also implies $\mathrm{singsupp}_A Tu \subset L$.

Now we denote by $\hat{L}$ the union of L and the connected components of $V \setminus L$ which are relatively compact in V. Then $\hat{L}$ is also compact and no component of $V \setminus \hat{L}$ is relatively compact in V.

We claim here that $\operatorname{supp} Tu \subset \hat{L}$ for any $u \in C'(K)$. In fact, assume by contradiction that $\operatorname{supp} Tu$ contains a point x^0 in $V \setminus \hat{L}$. Since Tu is real-analytic outside $\hat{L}$, the component V_0 of $V \setminus \hat{L}$ containing x^0 is included in $\operatorname{supp} Tu$. On the other hand, again from the hypothesis of (i) of Theorem 1.1, the support of Tu is compact, which contradicts the facts that $\operatorname{supp} Tu \supset V_0$ and that V_0 is not relatively compact in V.

We can use Theorem 1.6 for $\mathcal{K}(x, y)$ restricted to $(V \setminus \hat{L}) \times \operatorname{Int} K$. In fact, for any $y^0 \in \operatorname{Int} K$, δ_{y^0} belongs to $C'(K)$ and we have

$$\mathcal{K}(x, y^0) = T\delta_{y^0} = 0 \quad \text{on } V \setminus \hat{L}.$$

Therefore $\mathcal{K}$ vanishes on $(V \setminus \hat{L}) \times \operatorname{Int} K$. Since the choice of $K \Subset U$ was arbitrary, this implies (iii). $\square$

It is clear from Remark 6.2 that, for any fixed compact $K \subset U$, the linear map T restricted on $C'(K)$ is a continuous map from the Banach space $C'(K)$ to $C^\infty(V)$. Therefore, Theorem 6.1 follows directly from the following.

Theorem 6.3. *Let $T : X \to C^\infty(V)$ be a continuous linear map from some Fréchet space X to $C^\infty(V)$. Assume that for any $u \in X$, the analytic singular support* $\operatorname{singsupp}_A Tu$ *of Tu is compact. Then there exists a compact set $L \subset V$ such that* $\operatorname{singsupp}_A Tu \subset L$ *holds for any $u \in X$.*

In the present paper, there is no need to work for an abstract Fréchet space in this Theorem 6.3, but this formulation costs no additional effort and will be useful in a forthcoming paper.

In the proof of Theorem 6.3, we shall use the following remark about "non-analyticity".

Lemma 6.4. *Assume that $f \in C^\infty(V)$ is not real-analytic at x^0. Then there is a sequence of points x^i in V converging to x^0 and a sequence of multiindices α^i such that*

$$|D^{\alpha^i} f(x^i)/\alpha^i!|^{1/|\alpha^i|} \geq i^2. \tag{6.1}$$

(The "square" in i^2 in (6.1) is for later convenience.)

Proof. The real-analyticity of f at x^0 can be characterized by the existence of constants $\varepsilon > 0$ and c such that

$$\sup_{|x-x^0|<\varepsilon} \sup_\alpha |D^\alpha f(x)/\alpha!|^{1/|\alpha|} \leq c.$$

The opposite is at first that for every $\ell \in \mathbb{N}$ we have

$$\sup_{|x-x^0|<1/\ell} \sup_\alpha |D^\alpha f(x)/\alpha!|^{1/|\alpha|} = \infty.$$

We can then, in a second step, find a sequence of points x^i and of multiindices α^i such that (for example)

$$|x^i - x^0| \leq 1/i, \qquad |D^{\alpha^i} f(x^i)/\alpha^i!|^{1/|\alpha^i|} \geq i^2. \quad \square$$

Our next lemma is as follows.

Lemma 6.5. *Let T be as in Theorem 6.3. Assume that for every compact set $L \subset V$ we can find $u \in X$ satisfying* $\operatorname{singsupp}_A Tu \not\subset L$*. Then we can find a sequence of compact sets $L_k \subset V$, of points $x^k \in V$ and of elements $u_k \in X$, $k = 1, 2, \ldots$, with the following properties:*

- $L_k \subset L_{k+1}$, $\bigcup_{k=1}^{\infty} \operatorname{Int} L_k = V$,
- $x^k \in V \setminus L_k$,
- Tu_i, $i = 1, \ldots, k-1$, *are real-analytic on $V \setminus L_k$, whereas Tu_k is not real-analytic at x_k.*
- $\sum_{k=1}^{\infty} c_k u_k$ *converges in X for any bounded sequence $\{c_k\}_k$ in $\mathbb{C}$.*

Proof. At first we consider a sequence of compact sets $\{L'_k\}_k$ in V satisfying the condition $\bigcup_{k=1}^{\infty} \operatorname{Int} L'_k = V$, then we define L_k with $L_k \supset L'_k$ and u_k recursively for $k = 1, 2, \ldots$ as follows. Putting $L_1 = L'_1$, we can find $u_1 \in X$ satisfying $\operatorname{singsupp}_A Tu_1 \not\subset L_1$. After defining L_j and u_j for $j = 1, 2, \ldots, k-1$, we define

$$L_k = L_{k-1} \cup L'_k \cup \bigcup_{j=1}^{k-1} \operatorname{singsupp}_A Tu_j,$$

which is compact since so is every $\operatorname{singsupp}_A Tu_j$, and pick $u_k \in X$ satisfying $\operatorname{singsupp}_A Tu_k \not\subset L_k$. Finally we consider a sequence of points $x^k \in \operatorname{singsupp}_A Tu_k \setminus L_k$. Now all the requirements except the last one are fulfilled.

Let $\{\| \cdot \|_j\}_{j \in \mathbb{N}}$ be a countable system of semi-norms on X which defines the Fréchet topology of X. Then we take positive constants λ_k such that $\lambda_k > 2^k \max_{j=1,\ldots,k} \|u_k\|_j$. The remaining thing to do is to replace u_k by u_k/λ_k, so that

$$\sum_k \|u_k\|_j \le \sum_{k<j} \|u_k\|_j + \sum_{k \ge j} 2^{-k} < +\infty \quad \text{for any } j \in \mathbb{N},$$

which implies the last property in Lemma 6.5. □

Proof of Theorem 6.3. We argue by contradiction and can therefore find L_k, u_k, x^k, $k = 1, 2, \ldots$ as in the preceding Lemma 6.5.

We shall achieve a contradiction by choosing constants c_p such that $u = \sum_{p=1}^{\infty} c_p u_p$ converges in X, but such that Tu is not real-analytic at any of the points x^k. For the first condition, we will choose c_p such that $|c_p| \le 1$.

Since the functions Tu_p, $p \le k-1$, are real-analytic in a neighborhood of x^k, we can find constants $\mu_k \ge 0$ and $\nu_k > 0$ such that

$$|D^\alpha Tu_p(x)| \le \mu_k^{|\alpha|+1} \alpha! \quad \text{for } |x - x^k| \le \nu_k,\ p \le k-1,\ \alpha \in \mathbb{N}_0^n. \tag{6.2}$$

According to Lemma 6.4 we can also find sequences $i \mapsto x^{k,i}$ and $i \mapsto \alpha^{k,i}$ such that $\lim_{i \to \infty} x^{k,i} = x^k$ and that

$$|D^{\alpha^{k,i}} Tu_k(x^{k,i})| \ge i^{2|\alpha^{k,i}|} \alpha^{k,i}! \ge i^{1+|\alpha^{k,i}|} \alpha^{k,i}!, \quad \forall i. \tag{6.3}$$

The main task in the argument is to find the correct constants c_p. They shall be found iteratively. The index in the iteration shall be denoted by "j". The choice of $c_1, \ldots, c_{j-1}$ will be done in such a way that most of the work related to the points $x^1, \ldots, x^{j-1}$ is already done. When we look for a new c_j we shall have to take care not to ruin what has been achieved for $x^1, \ldots, x^{j-1}$, but also to take care of what is required for x^j.

The c_j shall be found iteratively together with some subsequences $i \mapsto z^{j,k,i}$, $i \mapsto \beta^{j,k,i}$, $k \le j$, of the initial sequences $i \mapsto x^{k,i}$, $i \mapsto \alpha^{k,i}$, such that the following inductive statement holds:

Lemma 6.6. *For every $j \in \mathbb{N}$ and every $k \le j$ we can find a constant c_j (which we shall always choose with the property $|c_j| \le 1$) and sequences $i \mapsto z^{j,k,i}$, $i \mapsto \beta^{j,k,i}$, $k \le j$, satisfying the following properties:*

(a) *For every j and every $k \le j$, the sequence $i \mapsto (z^{j,k,i}, \beta^{j,k,i})$ is a subsequence of $i \mapsto (x^{k,i}, \alpha^{k,i})$. By this we mean that for every j and every k there is a strictly increasing sequence of natural numbers $i \mapsto q_i$ such that $(z^{j,k,i}, \beta^{j,k,i}) = (x^{k,q_i}, \alpha^{k,q_i})$. In particular, $\lim_{i\to\infty} z^{j,k,i} = x^k$, $\forall k$, $\forall j$. Also note that by writing $i \mapsto (z^{j,k,i}, \beta^{j,k,i})$ and similar expressions, we want to make it notationally clear that choices for subsequences are made simultaneously for the "z" and for the "β".*

(b) *For $j \ge 2$, $i \le j-1$, $k \le j-1$, we have*

$$(z^{j,k,i}, \beta^{j,k,i}) = (z^{j-1,k,i}, \beta^{j-1,k,i}). \tag{6.4}$$

(c) *For every $j \ge 2$ and every $k \le j-1$, the sequence $i \mapsto (z^{j,k,i}, \beta^{j,k,i})$ is a subsequence of the sequence $i \mapsto (z^{j-1,k,i}, \beta^{j-1,k,i})$. This means in analogy with the above that for every j and every k there is a strictly increasing sequence of natural numbers q_i such that $(z^{j,k,i}, \beta^{j,k,i}) = (z^{j-1,k,q_i}, \beta^{j-1,k,q_i})$, $\forall i$, $\forall k$. Also note that the index j is associated with the iteration step and the index "k" relates our choices to the point x^k. In particular, sequences with different k are not related to each other.*

(d) *For every j, every $k \le j$ and every i,*

$$\left| D^{\beta^{j,k,i}} T\left(\sum_{p=1}^{j} c_p u_p\right)(z^{j,k,i}) \right| \ge (1 + 2^{-j}) i^{|\beta^{j,k,i}|} \beta^{j,k,i}!. \tag{6.5}$$

(Observe that we do not consider sequences $i \mapsto (z^{j,k,i}, \beta^{j,k,i})$ for $k > j$.)

Proof of Lemma 6.6 (*Beginning*)**.** We denote the conditions (a)–(d) with arbitrarily fixed j by (a_j), (b_j), (c_j), and (d_j) respectively. For $j = 1$ we choose $c_1 = 1$, $z^{1,1,i} = x^{1,i}$, $\beta^{1,1,i} = \alpha^{1,i}$. (a_1) is then trivial, (b_1) and (c_1) are void and (d_1) follows from (6.3).

Assume now by induction that for some fixed j we have already found c_p, $p \le j-1$, and sequences $i \mapsto (z^{p,k,i}, \beta^{p,k,i})$, $k \le p \le j-1$, which satisfy (a_{j-1}), (b_{j-1}), (c_{j-1}) and (d_{j-1}). Our aim is to find a constant c_j with $0 \ne |c_j| \le 1$, and sequences $i \mapsto (z^{j,k,i}, \beta^{j,k,i})$, $k \le j$, which have the properties (a_j), (b_j), (c_j) and (d_j).

The first remark is, if we want to have (b_j), that we must set

$$(z^{j,k,i}, \beta^{j,k,i}) := (z^{j-1,k,i}, \beta^{j-1,k,i}) \quad \text{for } i \le j-1, k \le j-1. \tag{6.6}$$

We next fix a constant $d > 0$ for which the conclusion in the following remark holds.

Remark 6.7. Assume that

$$\left| D^{\beta^{j-1,k,i}} T\left(\sum_{p=1}^{j-1} c_p u_p\right)(z^{j-1,k,i}) \right| \ge (1 + 2^{-j+1}) i^{|\beta^{j-1,k,i}|} \beta^{j-1,k,i}!,$$
$$\forall i \le j-1,\ \forall k \le j-1, \tag{6.7}$$

and (6.6). Then there exists a constant $d > 0$ such that the inequality (6.5) will hold for $i \leq j-1, k \leq j-1$, and for any c_j satisfying $|c_j| \leq d$.

To see this, we use the trivial inequalities

$$(1+2^{-j+1}) i^{|\beta^{j,k,i}|} \beta^{j,k,i}! > (1+2^{-j}) i^{|\beta^{j,k,i}|} \beta^{j,k,i}!, \quad \text{for } i \leq j-1, \tag{6.8}$$

which give

$$\left| D^{\beta^{j,k,i}} T \left(\sum_{p=1}^{j-1} c_p u_p \right) (z^{j,k,i}) \right| > (1+2^{-j}) i^{|\beta^{j,k,i}|} \beta^{j,k,i}!, \quad \text{for } i \leq j-1,\ k \leq j-1,$$

since $(z^{j,k,i}, \beta^{j,k,i}) := (z^{j-1,k,i}, \beta^{j-1,k,i})$ (in view of (6.6)) for the relevant indices. By adding the term $c_j u_j$ to $\sum_{p=1}^{j-1} c_p u_p$ with some sufficiently small c_j, we can then obtain (6.5) for $i \leq j-1,\ k \leq j-1$, since this involves only finitely many inequalities.

In order to define c_j, and in order to define $(z^{j,k,i}, \beta^{j,k,i})$ for $i \geq j, k \leq j-1$, and for $i \geq 1,\ k = j$, we prepare the following:

Lemma 6.8. *We take the index set $\mathbb{N}_{\geq j} := \{j, j+1, \ldots\}$ and fix a sequence $\hat{\beta} := \{\beta^i\}_{i \geq j}$ of multiindices, where the index i runs through $\mathbb{N}_{\geq j}$. For a sequence $\{f^i\}_{i \geq j}$ of complex numbers, we define*

$$\rho_{\hat{\beta}}(\{f^i\}_i) := \sup_i (|f^i| / \beta^i!)^{1/(1+|\beta^i|)}.$$

Then,

(i) *The condition $\rho_{\hat{\beta}}(\{f^i\}_i) = \infty$ is equivalent to the existence of a strictly increasing sequence $i \mapsto q_i \in \mathbb{N}_{\geq j}$ such that*

$$|f^{q_i}| \geq 2 i^{1+|\beta^{q_i}|} \beta^{q_i}!, \quad i = 1, 2, \ldots.$$

(ii) *Let $\{f^i\}_i$ and $\{g^i\}_i$ be two sequences and assume that at least one of the two quantities $\rho_{\hat{\beta}}(\{f^i\}_i)$ and $\rho_{\hat{\beta}}(\{g^i\}_i)$ is not finite. Then $\rho_{\hat{\beta}}(\{f^i + cg^i\}_i) = \infty$ except for at most one complex value $c \in \mathbb{C}$. (That is, the set $\{c \in \mathbb{C};\ \rho_{\hat{\beta}}(\{f^i + cg^i\}_i) < +\infty\}$ is either empty, or is a set with precisely one element.)*

Proof. (i) Can be seen by a straightforward argument. To check (ii), it suffices to pay attention to the fact that the set $Y := \{\{f^i\}_{i \geq j};\ \rho_{\hat{\beta}}(\{f^i\}_i) < +\infty\}$ becomes a subspace of the vector space $\mathbb{C}^{\mathbb{N}_{\geq j}}$ of the sequences of complex numbers indexed by $\mathbb{N}_{\geq j}$. In fact, assume by contradiction that there were two distinct numbers $c_1 \neq c_2$ for which $\rho_{\hat{\beta}}(\{f^i + c_p g^i\}_i) < +\infty$ $(p = 1, 2)$, i.e., with $\{f^i + c_p g^i\}_i \in Y$. Then it would follow that we had $\{g^i\}_i = \frac{\{f^i + c_1 g^i\}_i - \{f^i + c_2 g^i\}_i}{c_1 - c_2} \in Y$ and also that $\{f^i\}_i = \{f^i + c_1 g^i\}_i - c_1 \{g^i\}_i \in Y$. This would violate the assumption that $\rho_{\hat{\beta}}(\{f^i\}_i) + \rho_{\hat{\beta}}(\{g^i\}_i) = \infty$. □

We now return to the proof of Lemma 6.6. Recall that the index j is fixed, and that the constants c_p $(p \leq j-1)$ and the sequences $\{(z^{p,k,i}, \beta^{p,k,i})\}_i$ $(k \leq p \leq j-1)$ have been found.

For each fixed $k = 1, \ldots, j-1$, we can apply (ii) of Lemma 6.8 for the case that the index set is $\mathbb{N}_{\geq j}$, the sequence of multiindices is $\hat{\beta}^{j-1,k} := \{\beta^{j-1,k,i}\}_{i \geq j}$, and two sequences $\{f^i\}_{i \geq j}, \{g^i\}_{i \geq j}$ are given by

$$f^i := D^{\beta^{j-1,k,i}} T \left(\sum_{p=1}^{j-1} c_p u_p \right) (z^{j-1,k,i}), \qquad g^i := D^{\beta^{j-1,k,i}} T(u_j)(z^{j-1,k,i}).$$

Note that in this case $\rho_{\hat\beta^{j-1,k}}(\{f^i\}_i)=\infty$. Then we get at most one exceptional value for c_j, for which $\rho_{\hat\beta^{j-1,k}}(\{f^i+c_jg^i\}_i)=\infty$ fails. Since k can have only the values $1,\dots,j-1$, we get a set E' of at most $j-1$ exceptional values for c_j, such that if $c_j\notin E'$, then

$$\rho_{\hat\beta^{j-1,k}}\left(\left\{D^{\beta^{j-1,k,i}}T\left(\sum_{p=1}^{j-1}c_pu_p+c_ju_j\right)(z^{j-1,k,i})\right\}_i\right)=\infty,$$
$$\text{if } k=1,\dots,j-1. \tag{6.9}$$

For $k=j$, we again apply (ii) of Lemma 6.8, now for the case that the index set is $\mathbb{N}_{\geq 1}$, the sequence of multiindices is $\hat\alpha^j:=\{\alpha^{j,i}\}_i$, and two sequences $\{f^i\}_{i\geq1}$, $\{g^i\}_{i\geq1}$ are given by

$$f^i:=D^{\alpha^{j,i}}T\left(\sum_{p=1}^{j-1}c_pu_p\right)(x^{j,i}),\qquad g^i:=D^{\alpha^{j,i}}T(u_j)(x^{j,i}).$$

Note that in this case $\rho_{\hat\alpha^j}(\{g^i\}_i)=\infty$ by (6.3). Then we get at most one exceptional value for c_j. Explicitly, we obtain that if c_j does not take this exceptional value, then

$$\rho_{\hat\alpha^j}\left(\left\{D^{\alpha^{j,i}}T\left(\sum_{p=1}^{j-1}c_pu_p+c_ju_j\right)(x^{j,i})\right\}_i\right)=\infty. \tag{6.10}$$

Since $T(\sum_{p=1}^{j-1}c_pu_p)$ is real-analytic at x^j, we know that $\rho_{\hat\alpha^j}(\{f^i\}_i)<\infty$ and therefore that the exceptional value for (6.10) is 0. But we do not need this explicit information in this context.

We have thus proved the following intermediate statement:

> There is an exceptional set $E\subset\mathbb{C}$ of at most j elements such that (6.9) and (6.10) hold if $c_j\notin E$.

We are now ready to specify the choice of c_j: we ask for the condition $|c_j|\leq 1$ to guarantee the convergence of $\sum_{p=1}^\infty c_pu_p$, for the condition $|c_j|<d$ with d given by Remark 6.7, and for the condition $c_j\notin E$ with E given as above. Since E is a finite set, we can take c_j satisfying these conditions. Note that (6.5) holds for $k=1,\dots,j-1$ and $i\leq j-1$ in view of Remark 6.7.

Once we have chosen c_j, we define $(z^{j,k,i},\beta^{j,k,i})$ for $i\geq j$, $k\leq j-1$ and for $i\geq1$, $k=j$, as follows.

For each fixed k with $k\leq j-1$, we now use (6.9) and the part (i) of Lemma 6.8. We can then take a (simultaneous) subsequence $\{(z^{j,k,i},\beta^{j,k,i})\}_{i\geq j}$ of $\{(z^{j-1,k,i},\beta^{j-1,k,i})\}_{i\geq j}$, such that (6.5) holds also for $i\geq j$. This shows that (c_j) and (d_j) hold for k with $k=1,\dots,j-1$. Note that (c_j) and (a_{j-1}) imply (a_j).

As for the case $k=j$, we use (6.10), in combination again with (i) of Lemma 6.8. This shows that if $\{(z^{j,j,i},\beta^{j,j,i})\}_{i\geq1}$ is a suitable subsequence of $\{(x^{j,i},\alpha^{j,i})\}_{i\geq1}$ then (6.5) with $k=j$ will hold. Therefore (a_j) and (d_j) hold also in the case $k=j$. Note that (c_j) with $k=j$ is void.

As mentioned in the beginning of the proof, we can now argue by induction in j, and complete the proof of Lemma 6.6. □

Corollary 6.9. *For j,k,i satisfying $k\leq i\leq j$, we have*

$$(z^{j,k,i},\beta^{j,k,i})=(z^{i,k,i},\beta^{i,k,i}), \tag{6.11}$$

$$\left|D^{\beta^{i,k,i}}T\left(\sum_{p=1}^{j}c_pu_p\right)(z^{i,k,i})\right|\geq(1+2^{-j})i^{|\beta^{i,k,i}|}\beta^{i,k,i}!. \tag{6.12}$$

In fact, (6.11) is a consequence of repeated use of Lemma 6.6(b), and (6.12) follows from (6.11) and (6.5), as

$$\left| D^{\beta^{i,k,i}} T\left(\sum_{p=1}^{j} c_p u_p\right)(z^{i,k,i})\right| = \left| D^{\beta^{j,k,i}} T\left(\sum_{p=1}^{j} c_p u_p\right)(z^{j,k,i})\right|$$
$$\geq (1+2^{-j}) i^{|\beta^{j,k,i}|} \beta^{j,k,i}! = (1+2^{-j}) i^{|\beta^{i,k,i}|} \beta^{i,k,i}!.$$

We now return to the proof of Theorem 6.3. Let c_j for $j \in \mathbb{N}$ and $i \mapsto (z^{j,k,i}, \beta^{j,k,i})$ for $k \leq j$ be constants and sequences given by Lemma 6.6. Now we consider for every k the "diagonal" sequence $i \mapsto (z^{i,k,i}, \beta^{i,k,i})$ and $u = \sum_{p=1}^{\infty} c_p u_p$. We can see from Lemma 6.6(a) and (c), that the diagonal sequence $i \mapsto (z^{i,k,i}, \beta^{i,k,i})$ form a subsequence of $i \mapsto (x^{k,i}, \alpha^{k,i})$ for any k, which in particular shows that $\lim_{i\to\infty} z^{i,k,i} = x^k$.

By passing to the limit $j \to \infty$ in (6.12) in Corollary 6.9, we now have that

$$|D^{\beta^{i,k,i}} Tu(z^{i,k,i})| \geq i^{|\beta^{i,k,i}|} \beta^{i,k,i}!, \quad \text{if } i \geq k. \tag{6.13}$$

Here we used the convergence of $\sum_{p=1}^{\infty} c_p u_p$ to u in the Fréchet space X and the continuity of the map $T : X \to C^\infty(V)$. Since $\lim_{i\to\infty} z^{i,k,i} = x^k$, the estimate (6.13) shows that Tu is not real-analytic at every x^k, and that $\operatorname{singsupp}_A Tu \not\subset L_k$ for any k. Since $\bigcup_k \operatorname{Int} L_k = V$, this leads to a contradiction in that $\operatorname{singsupp}_A Tu$ is not compact, concluding the proof of Theorem 6.3. □

Remark 6.10. We mention that there is some analogy to the proof of a theorem of J. Peetre in distribution theory which states that a linear operator from $\mathcal{D}(U)$ to $\mathcal{D}'(U)$ which shrinks supports must be continuous outside a locally finite set of "points of discontinuity", and is therefore (by a theorem of Schwartz) a linear partial differential operator with $\mathcal{D}'(U)$-coefficients outside that same set. (This is a vague statement, cf. [17] for details. Also recall that using this theorem, Peetre proved that a linear operator $\mathcal{D}(U) \to \mathcal{D}(U)$ which shrinks supports is a linear partial differential operator with C^∞-coefficients.)

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