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ADVANCES IN Mathematics

Advances in Mathematics 231 (2012) 1462–1486

www.elsevier.com/locate/aim

Quasi-selective ultrafilters and asymptotic numerosities

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Received 21 June 2011; accepted 28 June 2012
Available online 10 August 2012

Communicated by H. Jerome Keisler

Abstract

We isolate a new class of ultrafilters on $\mathbb{N}$, called "quasi-selective" because they are intermediate between selective ultrafilters and P-points. (Under the Continuum Hypothesis these three classes are distinct.) The existence of quasi-selective ultrafilters is equivalent to the existence of "asymptotic numerosities" for all sets of tuples $A \subseteq \mathbb{N}^k$. Such numerosities are hypernatural numbers that generalize finite cardinalities to countable point sets. Most notably, they maintain the structure of ordered semiring, and, in a precise sense, they allow for a natural extension of asymptotic density to all sets of tuples of natural numbers.

MSC: 03E65; 03E05; 03C20

Keywords: Ultrafilter; Non-standard model; Numerosity; Semiring

0. Introduction

Special classes of ultrafilters over $\mathbb{N}$ have been introduced and variously applied in the literature, starting from the pioneering work by Choquet [8,9] in the sixties.

In this paper we introduce a new class of ultrafilters, namely the *quasi-selective ultrafilters*, as a tool to generate a good notion of "equinumerosity" on the sets of tuples of natural numbers

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0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.06.021

(or, more generally, on all point sets $A \subseteq \mathcal{L}^k$ over a countable line $\mathcal{L}$). By equinumerosity we mean an equivalence relation that preserves the basic properties of equipotency for finite sets, including the Euclidean principle that "the whole is greater than the part". More precisely, we require that – similarly as finite numbers – the corresponding numerosities be the non-negative part of a discretely ordered ring, where 0 is the numerosity of the empty set, 1 is the numerosity of every singleton, and sums and products of numerosities correspond to disjoint unions and Cartesian products, respectively.

This idea of numerosities that generalize finite cardinalities has been recently investigated by Benci, Di Nasso, and Forti in a series of papers, starting from [1], where a numerosity is assigned to each pair $\langle A, \ell_A \rangle$, depending on the (finite-to-one) "labeling function" $\ell_A : A \to \mathbb{N}$. The existence of a numerosity function for labeled sets turns out to be equivalent to the existence of a selective ultrafilter. That research was then continued by investigating a similar notion of numerosity for sets of arbitrary cardinality, namely: sets of ordinals in [2], subsets of a superstructure in [3], and point sets over the real line in [11]. A related notion of "fine density" for sets of natural numbers is introduced and investigated in [10]. In each of these contexts special classes of ultrafilters over large sets naturally arise.

Here we focus on subsets $A \subseteq \mathbb{N}^k$ of tuples of natural numbers, and we show that the existence of particularly well-behaved equinumerosity relations (which we call "asymptotic") for such sets is equivalent to the existence of another special kind of ultrafilters, named quasi-selective ultrafilters. Such ultrafilters may be of independent interest, because they are closely related (but not equivalent) to other well-known classes of ultrafilters that have been extensively considered in the literature. In fact, on the one hand, all selective ultrafilters are quasi-selective and all quasi-selective ultrafilters are P-points. On the other hand, it is consistent that these three classes of ultrafilters are distinct.

The paper is organized as follows. In Section 1 we introduce the class of quasi-selective ultrafilters on $\mathbb{N}$ and we study their properties, in particular their relationships with P-points and selective ultrafilters. In Section 2, assuming the Continuum Hypothesis, we present a general construction of quasi-selective non-selective ultrafilters, that are also weakly Ramsey in the sense of [4,12]. In Section 3 we introduce axiomatically a general notion of "equinumerosity" for sets of tuples of natural numbers. In Section 4 we show that the resulting numerosities, where sum, product and ordering are defined in the standard Cantorian way, are the non-negative part of an ordered ring. In Section 5 we introduce the special notion of "asymptotic" equinumerosity, which generalizes the fine density of [10]. We show that there is a one-to-one correspondence between asymptotic equinumerosities and quasi-selective ultrafilters, where equinumerosity is witnessed by a special class of bijections depending on the ultrafilter ("$\mathcal{U}$-congruences"). The corresponding semiring of numerosities is isomorphic to an initial cut of the ultrapower $\mathbb{N}^{\mathbb{N}}_{\mathcal{U}}$. In particular, asymptotic numerosities exist if and only if there exist quasi-selective ultrafilters. Final remarks and open questions are contained in the concluding Section 6.

In general, we refer the readers to [7] for definitions and basic facts concerning ultrafilters, ultrapowers, and non-standard models, and to [6] for special ultrafilters over $\mathbb{N}$.

1. Quasi-selective ultrafilters

We denote by $\mathbb{N}$ the set of all *non-negative integers*, and by $\mathbb{N}^+$ the subset of all *positive* integers.

Recall that, if $\mathcal{F}$ is a filter on X, then two functions $f, g : X \to Y$ are called $\mathcal{F}$-*equivalent* if $\{x \in X \mid f(x) = g(x)\} \in \mathcal{F}$. In this case we write $f \equiv_{\mathcal{F}} g$.

Definition 1.1. A non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is called *quasi-selective* if every function f such that $f(n) \leq n$ for all $n \in \mathbb{N}$ is $\mathcal{U}$-equivalent to a non-decreasing one.

The name 'quasi-selective' recalls one of the characterizations of *selective* (or *Ramsey*) ultrafilters (see *e.g.* [1, Proposition 4.1]), namely

- *the ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is selective if and only if every $f : \mathbb{N} \to \mathbb{N}$ is $\mathcal{U}$-equivalent to a non-decreasing function.*

In particular all selective ultrafilters are quasi-selective.

Let us call "interval-to-one" a function $g : \mathbb{N} \to \mathbb{N}$ such that, for all n, $g^{-1}(n)$ is a (possibly infinite) interval of $\mathbb{N}$.

Proposition 1.2. *Let $\mathcal{U}$ be a quasi-selective ultrafilter. Then any partition of $\mathbb{N}$ is an* interval partition *when restricted to a suitable set in $\mathcal{U}$. Hence every $f : \mathbb{N} \to \mathbb{N}$ is $\mathcal{U}$-equivalent to an "interval-to-one" function. In particular all quasi-selective ultrafilters are P-points.*

Proof. Given a partition of $\mathbb{N}$, consider the function f mapping each number to the least element of its class. Then $f(n) \leq n$ for all $n \in \mathbb{N}$, so, by quasi-selectivity, there exists a set $U \in \mathcal{U}$ such that the restriction $f_{\upharpoonright U}$ is non-decreasing. Then the given partition is an interval partition when restricted to U. $\square$

We say that a function $f : \mathbb{N} \to \mathbb{N}$ has *polynomial growth* if it is eventually dominated by some polynomial, *i.e.* if there exist k, m such that for all $n > m$, $f(n) \leq n^k$.

We say that a function $f : \mathbb{N} \to \mathbb{N}$ has *minimal steps* if $|f(n+1) - f(n)| \leq 1$ for all $n \in \mathbb{N}$.

Proposition 1.3. *The following properties are equivalent for a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$:*

(1) *$\mathcal{U}$ is quasi-selective;*
(2) *every function of polynomial growth is $\mathcal{U}$-equivalent to a non-decreasing one;*
(3) *every function with minimal steps is $\mathcal{U}$-equivalent to a non-decreasing one.*[1]

Proof. (1) $\Longrightarrow$ (2). We prove that if every function $f < n^k$ is $\mathcal{U}$-equivalent to a non-decreasing one, then the same property holds for every function $g < n^{2k}$. The thesis then follows by induction on k. Given g, let f be the integral part of the square root of g. So $g < f^2 + 2f + 1$, and hence $g = f^2 + f_1 + f_2$ for suitable functions $f_1, f_2 \leq f < n^k$. By hypothesis we can pick non-decreasing functions f', f_1', f_2' that are $\mathcal{U}$-equivalent to f, f_1, f_2, respectively. Then clearly g is $\mathcal{U}$-equivalent to the non-decreasing function $f'^2 + f_1' + f_2'$.

(2) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (1). We begin by showing that (3) implies the following

Claim. *There exists $U = \{u_1, \ldots, u_n \ldots\} \in \mathcal{U}$ such that $u_{n+1} > 2u_n$.*

Define three minimal step functions f_0, f_1, f_2 as follows, for $k \in \mathbb{N}^+$:

$$\begin{aligned}
f_0(m) &= 2^{k-1} - |3 \cdot 2^{k-1} - m| \quad \text{for } 2^k \leq m \leq 2^{k+1}\\
f_1(m) &= 9 \cdot 2^{2k-1} - |15 \cdot 2^{2k-1} - m| \quad \text{for } 3 \cdot 2^{2k} \leq m \leq 3 \cdot 2^{2k+2}\\
f_2(m) &= 9 \cdot 2^{2k-2} - |15 \cdot 2^{2k-2} - m| \quad \text{for } 3 \cdot 2^{2k-1} \leq m \leq 3 \cdot 2^{2k+1}.
\end{aligned}$$

[1] Ultrafilters satisfying this property are called smooth in [10].

The graphs of these functions are made up of the catheti of isosceles right triangles whose hypotenuses are placed on the horizontal axis. The function f_0 is decreasing in the intervals $[3 \cdot 2^{k-1}, 2^{k+1}]$, whereas in the intervals $[2^k, 3 \cdot 2^{k-1}]$ the function f_1 is decreasing for odd k, and f_2 is decreasing for even k. The ultrafilter $\mathcal{U}$ contains a set V on which all three of these functions are non-decreasing. Such a set V has at most one point in each interval. Starting from each one of the first four points of V, partition V into four parts by taking every fourth point, so as to obtain four sets satisfying the condition of the claim. Then exactly one of the resulting sets belongs to $\mathcal{U}$, and the claim follows.

Now remark that every function $f \leq n$ can be written as a sum $f_1 + f_2$, where $f_1, f_2 \leq \lceil \frac{n}{2} \rceil$. Pick a set $U \in \mathcal{U}$ as given by the claim. Then both functions f_1 and f_2 agree on U with suitable minimal step functions, because $u_{n+1} - u_n > \frac{u_{n+1}}{2}$, whereas $g \leq \lceil \frac{n}{2} \rceil$ implies $|g(u_{n+1}) - g(u_n)| \leq \lceil \frac{u_{n+1}}{2} \rceil$. So f_1, f_2 are equivalent modulo $\mathcal{U}$ to two non-decreasing functions f_1', f_2' respectively, and f is equivalent modulo $\mathcal{U}$ to their sum $f_1' + f_2'$, which is non-decreasing as the sum of non-decreasing functions. $\square$

Theorem 1.4. *Let $\mathcal{U}$ be a quasi-selective ultrafilter, and let $f : \mathbb{N} \to \mathbb{N}$ be non-decreasing. Then the following properties are equivalent:*

(i) *for every function $g \leq f$ there exists a non-decreasing function $h \equiv_{\mathcal{U}} g$;*
(ii) *there exists $U = \{u_n \mid n \in \mathbb{N}\} \in \mathcal{U}$ such that $f(u_n) < u_{n+1} - u_n$.*

Proof. (i) $\Longrightarrow$ (ii). Define inductively the sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ by putting $x_0 = 1$ and $x_{n+1} = f(x_n) + x_n$. Define $g : \mathbb{N} \to \mathbb{N}$ by $g(x_n + h) = f(x_n) - h$ for $0 \leq h < f(x_n)$. Assuming (i), there is a set in $\mathcal{U}$ which meets each interval $[x_n, x_{n+1})$ in one point a_n. So by putting either $u_n = a_{2n}$ or $u_n = a_{2n+1}$ we obtain a set U satisfying the condition (ii). Namely, in the even case we have

$$u_{n+1} - u_n > x_{2n+2} - x_{2n+1} = f(x_{2n+1}) \geq f(u_n),$$

and similarly in the odd case.

(ii) $\Longrightarrow$ (i). By (ii) we may pick $U \in \mathcal{U}$ such that $x < y$ both in U implies $y > f(x)$. Given $g \leq f$, partition U as follows

$$U_1 = \{u \in U \mid \forall x \in U \ (x < u \Longrightarrow g(x) \leq g(u))\}$$
$$U_2 = \{u \in U \mid \exists x \in U \ (x < u \text{ and } g(x) > g(u))\}.$$

Then g is non-decreasing on U_1, so we are done when U_1 belongs to $\mathcal{U}$. Otherwise $U_2 \in \mathcal{U}$ and we have $g(u) < u$ for all $u \in U_2$. In fact, given $u \in U_2$, pick $x \in U$ such that $x < u$ and $g(x) > g(u)$: then

$$g(u) < g(x) \leq f(x) < u.$$

Then (i) follows by quasi-selectivity of $\mathcal{U}$. $\square$

Let us denote by $F_{\mathcal{U}}$ the class of all functions f satisfying the equivalent conditions of the above theorem

$$F_{\mathcal{U}} = \{f : \mathbb{N} \to \mathbb{N} \text{ non-decreasing } \mid g \leq f \Longrightarrow \exists h \text{ non-decreasing s.t. } h \equiv_{\mathcal{U}} g\}.$$

Recall that, if $\mathcal{U}$ is quasi-selective, then every function is $\mathcal{U}$-equivalent to an "interval-to-one" function. As a consequence, the class $F_{\mathcal{U}}$ "measures the selectivity" of quasi-selective ultrafilters, according to the following proposition.

Proposition 1.5. *Let $\mathcal{U}$ be a quasi-selective ultrafilter and let g be an unbounded interval-to-one function. Define the function g^+ by*

$$g^+(n) = \max\{x \mid g(x) = g(n)\},$$

and let e_g be the function enumerating the range of g^+. Then the following are equivalent:

(1) *g is $\mathcal{U}$-equivalent to a one-to-one function;*
(2) *g^+ belongs to $F_\mathcal{U}$;*
(3) *there exists $U = \{u_0 < u_1 < \cdots < u_n < \cdots\} \in \mathcal{U}$ such that $u_n > e_g(n)$.*

Proof.

(1) $\Longrightarrow$ (2). Let g be one-to-one on $U \in \mathcal{U}$. Then each interval where g is constant contains at most one point of U. Hence g^+ is increasing on U. Let $h \leq g^+$ be given, and put

$$U_1 = \{u \in U \mid h(u) \leq u\}, \qquad U_2 = \{u \in U \mid h(u) > u\}.$$

Then $u < h(u) \leq g^+(u)$ for $u \in U_2$, and hence h is increasing when restricted to U_2. So we are done when $U_2 \in \mathcal{U}$. On the other hand, when $U_1 \in \mathcal{U}$, the function h is $\mathcal{U}$-equivalent to a non-decreasing one, by quasi-selectivity.

(2) $\Longrightarrow$ (3). By Theorem 1.4, there exists a set $U = \{u_n \mid n \in \mathbb{N}\} \in \mathcal{U}$ such that $g^+(u_n) < u_{n+1} - u_n$. Suppose that $g(u_n) = g(u_{n+1})$ for some n; then $g^+(u_n) \geq u_{n+1} \geq u_{n+1} - u_n > g^+(u_n)$, a contradiction. Hence g is one-to-one on U, and so before u_{n+1} there are at least n intervals of the form $g^{-1}(k)$. Therefore $u_{n+1} > e_g(n)$, and $U \setminus \{u_0\}$ satisfies condition (3).

(3) $\Longrightarrow$ (1). For each $n \in \mathbb{N}$ let k be the unique number such that u_n lies in the interval $[e_g(n+k-1), e_g(n+k))$. Then let h be the unique number such that $e_g(n+k)$ lies in the interval $(u_{n+h-1}, u_{n+h}]$. Thus we have

$$e_g(n+k-1) \leq u_n \leq u_{n+h-1} < e_g(n+k) \leq u_{n+h}.$$

Then $k \geq h$ and $u_n \geq k$, and we can define the function f on U by $f(u_{n+i}) = k - i$, for $0 \leq i < h$. Since $f(u) \leq u$ for all $u \in U$ there exists a set $V \in \mathcal{U}$ on which f is non-decreasing. Then g is one-to-one on $V \cap U$. $\square$

Recall that the ultrafilter $\mathcal{U}$ is *rapid* if for every increasing function f there exists $U = \{u_0 < u_1 < \cdots < u_n < \cdots\} \in \mathcal{U}$ such that $u_n > f(n)$. The equivalence of the conditions in the above proposition yields the following corollary.

Corollary 1.6. *A quasi-selective ultrafilter is selective if and only if it is rapid.* $\square$

The class of functions $F_\mathcal{U}$ has the following closure properties.

Proposition 1.7. *Let $\mathcal{U}$ be a quasi-selective ultrafilter, and let*

$$F_\mathcal{U} = \{f : \mathbb{N} \to \mathbb{N} \text{ non-decreasing } \mid g \leq f \Longrightarrow \exists h \text{ non-decreasing s.t. } h \equiv_\mathcal{U} g\}.$$

Then

(1) *for all $f \in F_\mathcal{U}$ also $\widetilde{f} \in F_\mathcal{U}$, where*

$$\widetilde{f}(n) = f^{\circ f(n)}(n) = \underbrace{(f \circ f \circ \cdots \circ f)}_{f(n) \text{ times}}(n)^{2};$$

2 Here we agree that $f^{\circ 0}(n) = n$.

(2) *every sequence* $\langle f_n \mid n \in \mathbb{N}\rangle$ *in* $F_{\mathcal{U}}$ *is dominated by a function* $f_\omega \in F_{\mathcal{U}}$, *i.e. for all n there exists* k_n *such that* $f_\omega(m) > f_n(m)$ *for all* $m > k_n$.

In particular the left cofinality of the gap determined by $F_{\mathcal{U}}$ *in the ultrapower* $\mathbb{N}^{\mathbb{N}}_{\mathcal{U}}$ *is uncountable.*

Proof. We prove first that $\widetilde{f}$ fulfills property (ii) of Theorem 1.4, provided f fulfills both properties (i) and (ii) of the same theorem. By possibly replacing f by $\max\{f, \mathrm{id}\}$, we may assume without loss of generality that $f(n) \geq n$ for all $n \in \mathbb{N}$.

Let $U = \{u_0 < u_1 < \cdots < u_n \cdots\} \in \mathcal{U}$ be given by property (ii) for f. Define inductively the sequence $\sigma : \mathbb{N} \to \mathbb{N}$ by

$$\sigma(0) = 1 \quad \text{and} \quad \sigma(n+1) = \sigma(n) + f(u_{\sigma(n)}).$$

Define the function g on U by

$$g(u_{\sigma(n)+j}) = f(u_{\sigma(n)}) - j \quad \text{for } 0 \leq j < f(u_{\sigma(n)}).$$

Since $g \leq f$ on U, there exists a subset $V \in \mathcal{U}$ on which g is non-decreasing. For each n, such a V contains at most one point $v_n = u_{\tau(n)}$ with $\sigma(n) \leq \tau(n) < \sigma(n+1)$. Assume without loss of generality that the set $\{v_{2n} \mid n \in \mathbb{N}\} \in \mathcal{U}$. We shall complete the proof by showing that $v_{2n+2} - v_{2n} > \widetilde{f}(v_{2n})$. Put $k = \tau(2n+2) - \tau(2n)$; then

$$\begin{aligned} v_{2n+2} - v_{2n} &= \sum_{i=0}^{k-1}(u_{\tau(2n)+i+1} - u_{\tau(2n)+i}) > \sum_{i=0}^{k-1} f(u_{\tau(2n)+i}) \\ &\geq f(u_{\tau(2n)+k-1}) \geq f(f(u_{\tau(2n)+k-2})) \geq \cdots \geq f^{\circ k}(u_{\tau(2n)}). \end{aligned}$$

Now

$$k = \tau(2n+2) - \tau(2n) > \sigma(2n+2) - \sigma(2n+1) = f(u_{\sigma(2n+1)}) \geq f(v_{2n}).$$

Hence

$$v_{2n+2} - v_{2n} > f^{\circ k}(u_{\tau(2n)}) \geq f^{\circ f(v_{2n})}(v_{2n}) = \widetilde{f}(v_{2n}).$$

This completes the proof of (1).

In order to prove point (2), let sets $U_n \in \mathcal{U}$ be chosen so as to satisfy the property (ii) with respect to the function f_n. As $\mathcal{U}$ is a P-point, we can take $V \in \mathcal{U}$ almost included in every U_n, *i.e.* $V \setminus U_n$ finite for all $n \in \mathbb{N}$. Define the function f_ω by

$$f_\omega(m) = \min\{v' - v \mid v', v \in V,\ v' > v \geq m\}.$$

Let k_n be such that, for all $v \in V$, $v \geq k_n$ implies $v \in U_n$. Given $m > k_n$ let $f_\omega(m) = v' - v$, with $m < v < v'$ as required by the definition of f_ω, and let u be the successor of v in U_n. Then

$$f_n(m) \leq f_n(v) < u - v \leq v' - v = f_\omega(m),$$

and so f_ω dominates every f_n. Now if $V = \{v_0 < v_1 < \cdots\}$, then $f_\omega(v_k) < v_{k+2} - v_k$. So either $U = \{v_{2n} \mid n \in \mathbb{N}\}$ or $U' = \{v_{2n+1} \mid n \in \mathbb{N}\}$ witnesses the property (ii) of Theorem 1.4 for f_ω, and (2) follows. $\square$

Remark that all *Ackermann functions*[3] $f_m(n) = A(m, n)$ belong to $F_{\mathcal{U}}$, because $f_{m+1} \le \widetilde{f_m}$. Since every primitive recursive function is eventually dominated by some f_m, we obtain the following property of "primitive recursive rapidity".

Corollary 1.8. *Let $\mathcal{U}$ be a quasi-selective ultrafilter, and let $f : \mathbb{N} \to \mathbb{N}$ be primitive recursive. Then f is non-decreasing modulo $\mathcal{U}$, and there exists a set $U = \{u_0 < u_1 < \cdots < u_n \cdots\} \in \mathcal{U}$ with $u_{n+1} > f(u_n)$.* □

As proved at the beginning of this section, one has the following implications

selective $\Longrightarrow$ *quasi-selective* $\Longrightarrow$ *P-point.*

Not even the existence of P-points can be proven in ZFC (see *e.g.* [17,16]), so the question as to whether the above three classes of ultrafilters are distinct only makes sense under additional hypotheses.

However the following proposition[4] holds in ZFC.

Proposition 1.9. *Assume that the ultrafilter $\mathcal{U}$ is not a Q-point. Then there exists an ultrafilter $\mathcal{U}' \cong \mathcal{U}$ that is not quasi-selective.*

Proof. Let $\mathcal{U}$ be an ultrafilter that is not a Q-point, so there is a partition of $\mathbb{N}$ into finite sets F_n $(n \in \omega)$ such that every set in $\mathcal{U}$ meets some F_n in more than one point. Inductively choose pairwise disjoint sets $G_n \subseteq \mathbb{N}$ such that, for each n,

$$\min(G_n) > |G_n| = |F_n|.$$

Let $f : \mathbb{N} \to \bigcup_n G_n$ be such that the restrictions $f \restriction F_n$ are bijections $f \restriction F_n : F_n \to G_n$ for all n. Since the G_n are pairwise disjoint, f is one-to-one, and therefore the ultrafilter $\mathcal{V} = f(\mathcal{U})$ is isomorphic to $\mathcal{U}$. We shall complete the proof by showing that $\mathcal{V}$ is not quasi-selective.

Define $g : \bigcup_n G_n \to \mathbb{N}$ by requiring that, for each n, the restriction $g \restriction G_n$ is the unique strictly decreasing bijection from G_n to the initial segment $[0, |G_n|)$ of $\mathbb{N}$. Notice that the values g takes on G_n are all $< \min(G_n)$; thus $g(x) < x$ for all $x \in G_n$. Extend g to all of $\mathbb{N}$ by setting $g(x) = 0$ for $x \notin \bigcup_n G_n$. Now $g : \mathbb{N} \to \mathbb{N}$ and $g(x) \le x$ for all x. If $\mathcal{V}$ were quasi-selective, there would be a set $A \in \mathcal{V}$ on which g is non-decreasing. Since g is strictly decreasing on each G_n, each intersection $A \cap G_n$ would contain at most one point. Therefore each of the pre-images $f^{-1}(A) \cap F_n$ would contain at most one point. But from $A \in \mathcal{V}$, we infer that $f^{-1}(A)$ is in $\mathcal{U}$ and therefore meets some F_n in at least two points. This contradiction shows that $\mathcal{V}$ is not quasi-selective. □

Hence the mere existence of a non-selective P-point yields also the existence of non-quasi-selective P-points. So the second implication can be reversed only if the three classes

[3] Recall that $f_m(n) = A(m, n)$ can be inductively defined by

$$f_0(n) = A(0, n) = n + 1, \qquad f_{m+1}(n) = A(m + 1, n) = \underbrace{(f_m \circ f_m \circ \cdots \circ f_m)}_{n+1 \text{ times}}(1).$$

[4] Recall that a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is called a *Q-point* if every finite-to-one function on $\mathbb{N}$ is one-to-one on some set in $\mathcal{U}$. Recall also that two ultrafilters, $\mathcal{U}$ and $\mathcal{U}'$, are *isomorphic*, written $\mathcal{U} \cong \mathcal{U}'$, if they are related by a bijection from $\mathbb{N}$ to itself. It is easy to check that the properties "selective", "P-point", and "Q-point" are all invariant under isomorphism. Finally, recall that if f is one-to-one on a set in $\mathcal{U}$, then there is a bijection $\mathbb{N} \to \mathbb{N}$ that agrees with f on a set in $\mathcal{U}$, and therefore $\mathcal{U} \cong f(\mathcal{U})$.

are the same. Recall that this possibility has been shown consistent by Shelah (see [16, Section XVIII.4]).

We conclude this section by stating a theorem that settles the question under the Continuum Hypothesis CH.

Theorem 1.10. *Assume* CH. *Then there exist* $2^{\mathfrak{c}}$ *pairwise non-isomorphic* P*-points that are not quasi-selective, and* $2^{\mathfrak{c}}$ *pairwise non-isomorphic quasi-selective ultrafilters that are not selective.*

The first assertion of this theorem follows by combining Proposition 1.9 with the known fact that CH implies the existence of $2^{\mathfrak{c}}$ non-isomorphic non-selective P-points. The rather technical proof of the second assertion of the theorem is contained in the next section. A few open questions involving quasi-selective ultrafilters are to be found in Section 6.

2. A construction of quasi-selective ultrafilters

This section is entirely devoted to the proof of the following theorem, which in turn will yield the second assertion of Theorem 1.10.

Theorem 2.1. *Assume* CH. *For every selective ultrafilter* $\mathcal{U}$*, there is a non-selective but quasi-selective ultrafilter* $\mathcal{V}$ *above* $\mathcal{U}$ *in the Rudin–Keisler ordering.*[5] *Furthermore,* $\mathcal{V}$ *can be chosen to satisfy the partition relation* $\mathbb{N} \to [\mathcal{V}]^2_3$.

The square-bracket partition relation in the theorem means that, if $[\mathbb{N}]^2$ is partitioned into 3 pieces, then there is a set $H \in \mathcal{V}$ such that $[H]^2$ meets at most 2 of the pieces. This easily implies by induction that, if $[\mathbb{N}]^2$ is partitioned into any finite number of pieces, then there is a set $H \in \mathcal{V}$ such that $[H]^2$ meets at most 2 of the pieces. It is also known [4] to imply that $\mathcal{V}$ is a P-point and that $\mathcal{U}$ is, up to isomorphism, the only non-principal ultrafilter strictly below $\mathcal{V}$ in the Rudin–Keisler ordering.

It will be convenient to record some preliminary information before starting the proof of the theorem. Suppose $\mathcal{X}$ is an upward-closed (with respect to $\subseteq$) family of finite subsets of $\mathbb{N}$. Call $\mathcal{X}$ *rich* if every infinite subset of $\mathbb{N}$ has an initial segment in $\mathcal{X}$. (This notion resembles Nash-Williams notion of a *barrier*, but it is not the same.) Define $\rho\mathcal{X}$ to be the family of those finite $A \subseteq \mathbb{N}$ such that $A \to (\mathcal{X})^2_2$, i.e., every partition of $[A]^2$ into two parts has a homogeneous set in $\mathcal{X}$. (The notation ρ stands for "Ramsey".)

Lemma 2.2. *If* $\mathcal{X}$ *is rich, then so is* $\rho\mathcal{X}$.

Proof. This is a standard compactness argument, but we present it for the sake of completeness. Let S be any infinite subset of $\mathbb{N}$, and, for each $n \in \mathbb{N}$, let S_n be the set of the first n elements of S. We must show that $S_n \in \rho\mathcal{X}$ for some n. Suppose not. Then, for each n, there are counterexamples, i.e., partitions $F : [S_n]^2 \to 2$ with no homogeneous set in $\mathcal{X}$. These counterexamples form a tree, in which the predecessors of any F are its restrictions to $[S_m]^2$ for smaller m. This tree is infinite but finitely branching, so König's infinity lemma gives us a path through it. The union of all the partitions along this path is a partition $G : [S]^2 \to 2$, and by Ramsey's theorem it has an infinite homogeneous set $H \subseteq S$. Since $\mathcal{X}$ is rich, it contains $H \cap S_n$ for some n. But then one of our counterexamples, namely $G \restriction [S_n]^2$, has a homogeneous set in $\mathcal{X}$, so it is not really a counterexample. This contradiction completes the proof of the lemma. □

5 Recall that $\mathcal{V}$ is above $\mathcal{U}$ in the Rudin–Keisler (pre)ordering if there exists a function f such that $\mathcal{U} = f(\mathcal{V})$.

Of course, we can iterate the operation ρ. The lemma implies that, if $\mathcal{X}$ is rich, then so is $\rho^n \mathcal{X}$ for any finite n.

Notice that, for any upward-closed $\mathcal{X}$, we have $\rho\mathcal{X} \subseteq \mathcal{X}$. Indeed, if $A \in \rho\mathcal{X}$, then applying the definition of ρ with an arbitrary partition, we obtain a subset of A that is in $\mathcal{X}$; then by upward-closure, A itself is in $\mathcal{X}$. Notice also that $\rho\mathcal{X}$ is always upward closed. So we can iterate the preceding observation to obtain $\mathcal{X} \supseteq \rho\mathcal{X} \supseteq \rho^2\mathcal{X} \supseteq \cdots$.

We shall apply all this information to a particular $\mathcal{X}$, namely

$$\mathcal{L} = \{A \text{ finite, non-empty } \subseteq \mathbb{N} \mid \min(A) + 2 < |A|\},$$

which is obviously rich. Observe that any $A \in \mathcal{L}$ has $|A| \geq 3$. It easily follows that any $A \in \rho^n\mathcal{L}$ has $|A| \geq 3 + n$. (In fact the sizes of sets in $\rho^n\mathcal{L}$ grow very rapidly, but we do not need this fact here.) In particular, no finite set can belong to $\rho^n\mathcal{L}$ for arbitrarily large n, and so we can define a norm for finite sets by

$$\nu : [\mathbb{N}]^{<\omega} \to \mathbb{N} : A \mapsto \text{ least } n \text{ such that } A \notin \rho^n\mathcal{L}.$$

Because each $\rho^n\mathcal{L}$ is rich, we can partition $\mathbb{N}$ into consecutive finite intervals I_n such that $I_n \in \rho^n\mathcal{L}$ for each n. Define $p : \mathbb{N} \to \mathbb{N}$ to be the function sending all elements of any I_n to n; so $p^{-1}[B] = \bigcup_{n\in B} I_n$ for all $B \subseteq \mathbb{N}$.

For any $X \subseteq \mathbb{N}$, define its growth $\gamma(X) : \mathbb{N} \to \mathbb{N}$ to be the sequence of norms of its intersections with the I_n's:

$$\gamma(X)(n) = \nu(X \cap I_n).$$

Notice that, by our choice of the I_n, $\gamma(\mathbb{N})(n) > n$ for all n.

With these preliminaries, we are ready to return to ultrafilters and prove the theorem. The proof uses ideas from [4,15], but some modifications are needed, and so we present the proof in detail.

Proof of Theorem 2.1. Assume that CH holds, and let $\mathcal{U}$ be an arbitrary selective ultrafilter on $\mathbb{N}$. We adopt the quantifier notation for ultrafilters:

$$(\mathcal{U}n)\, \varphi(n) \text{ means "for } \mathcal{U}\text{-almost all } n, \varphi(n) \text{ holds", i.e., } \{n \mid \varphi(n)\} \in \mathcal{U}.$$

Call a subset X of $\mathbb{N}$ *large* if

$$(\forall k \in \mathbb{N}) \quad (\mathcal{U}n)\, \gamma(X)(n) > \sqrt{n} + k;$$

equivalently, in the ultrapower of $\mathbb{N}$ by $\mathcal{U}$, $[\gamma(X)]$ is infinitely larger than $[\lceil\sqrt{n}\,\rceil]$. Since n is asymptotically much larger than $\sqrt{n}$, we have that $\mathbb{N}$ is large.

Using CH, list all partitions $F : [\mathbb{N}]^2 \to \{0, 1\}$ in a sequence $\langle F_\alpha \mid \alpha < \aleph_1\rangle$ of length $\aleph_1$. We intend to build a sequence $\langle A_\alpha \mid \alpha < \aleph_1\rangle$ of subsets of $\mathbb{N}$ with the following properties.

(1) Each A_α is a large subset of $\mathbb{N}$.
(2) If $\alpha < \beta$, then $A_\beta \subseteq A_\alpha$ modulo $\mathcal{U}$, *i.e.*, $p[A_\beta - A_\alpha] \notin \mathcal{U}$, *i.e.*, $(\mathcal{U}n)\, A_\beta \cap I_n \subseteq A_\alpha$.
(3) For each n, F_α is constant on $[A_{\alpha+1} \cap I_n]^2$.

After constructing this sequence, we shall show how it yields the desired ultrafilter $\mathcal{V}$.

We construct A_α by induction on α, starting with $A_0 = \mathbb{N}$. We have already observed that requirement (1) is satisfied by $\mathbb{N}$; the other two requirements are vacuous at this stage.

Before continuing the construction, notice that the relation of inclusion modulo $\mathcal{U}$ is transitive.

At a successor step, we are given the large set A_α and we must find a large $A_{\alpha+1} \subseteq A_\alpha$ modulo $\mathcal{U}$ such that F_α is constant on each $[A_{\alpha+1} \cap I_n]^2$. Transitivity and the induction hypothesis then ensure that $A_{\alpha+1}$ is included in each earlier A_ξ modulo $\mathcal{U}$. (We shall actually get $A_{\alpha+1} \subseteq A_\alpha$, not just modulo $\mathcal{U}$, but the inclusions in the earlier A_ξ's will generally be only modulo $\mathcal{U}$.) We define $A_{\alpha+1}$ by defining its intersection with each I_n; then of course $A_{\alpha+1}$ will be the union of all these intersections.

If $\gamma(A_\alpha)(n) \leq 1$, then set $A_{\alpha+1} \cap I_n = \varnothing$. Note that the set of all such n's is not in $\mathcal{U}$, because A_α is large. If $\gamma(A_\alpha)(n) > 1$, then $A_\alpha \cap I_n$ is in $\rho^{\gamma(A_\alpha)(n)-1}\mathcal{L}$, so it has a subset that is homogeneous for $F_\alpha \restriction [A_\alpha \cap I_n]^2$ and is in $\rho^{\gamma(A_\alpha)(n)-2}\mathcal{L}$; let $A_{\alpha+1} \cap I_n$ be such a subset.

This choice of $A_{\alpha+1} \cap I_n$ (for each n) clearly ensures that requirements (2) and (3) are preserved. For requirement (1), simply observe that $(\mathcal{U}n)\, \gamma(A_{\alpha+1})(n) \geq \gamma(A_\alpha)(n) - 1$.

For the limit step of the induction, suppose β is a countable limit ordinal and we already have A_α for all $\alpha < \beta$. Choose an increasing $\mathbb{N}$-sequence of ordinals $\langle \alpha_i : i \in \mathbb{N} \rangle$ with limit β. Let $A'_n = \bigcap_{i \leq n} A_{\alpha_i}$. So the sets A'_n form a decreasing sequence. Because of induction hypothesis (2), each A'_n is equal modulo $\mathcal{U}$ to A_{α_n} (i.e., each includes the other modulo $\mathcal{U}$) and is therefore large. We shall find a large set A_β that is included modulo $\mathcal{U}$ in all of the A'_n, hence in all the A_{α_n}, and hence in all the A_α for $\alpha < \beta$. Thus, we shall preserve induction hypotheses (1) and (2); requirement (3) is vacuous at limit stages.

We shall obtain the desired A_β by defining its intersection with every I_n.

Because the ultrapower of $\mathbb{N}$ by $\mathcal{U}$ is countably saturated, we can fix a function $g : \mathbb{N} \to \mathbb{N}$ such that its equivalence class $[g]$ in the ultrapower is below each $[\gamma(A'_n)]$ but above $[x \mapsto \lceil \sqrt{x} \rceil + k]$ for every $k \in \mathbb{N}$. For each $x \in \mathbb{N}$, define $h(x)$ to be the largest number $q \leq x$ such that $g(x) \leq \gamma(A'_q)(x)$, or 0 if there is no such q. Finally, set $A_\beta \cap I_x = A'_{h(x)} \cap I_x$. We verify that this choice of A_β does what we wanted.

For any fixed n, $\mathcal{U}$-almost all x satisfy $g(x) \leq \gamma(A'_n)(x)$, and $x \geq n$, and therefore $h(x) \geq n$, and therefore $A'_{h(x)} \cap I_x \subseteq A'_n \cap I_x$. Thus, $A_\beta \subseteq A'_n$ modulo $\mathcal{U}$ for each n. As we saw earlier, this implies requirement (2).

Furthermore, for $\mathcal{U}$-almost all x,

$$\gamma(A_\beta)(x) = \gamma(A'_{h(x)})(x) \geq g(x),$$

and so our choice of g ensures that A_β is large.

This completes the construction of the sequence $\langle A_\alpha : \alpha < \aleph_1 \rangle$ and the verification of properties (1), (2), and (3). We shall now use this sequence to construct the desired ultrafilter.

We claim first that the sets $A_\alpha \cap p^{-1}[B]$, where α ranges over $\aleph_1$ and B ranges over $\mathcal{U}$, constitute a filter base. Indeed, the intersection of any two of them, say

$$A_\alpha \cap p^{-1}[B] \cap A_{\alpha'} \cap p^{-1}[B']$$

with, say, $\alpha \leq \alpha'$, includes $A_{\alpha'} \cap p^{-1}[B \cap B' \cap C]$, where, thanks to requirement (2) above, C is a set in $\mathcal{U}$ such that $A_{\alpha'} \cap I_n \subseteq A_\alpha$ for all $n \in C$.

Let $\mathcal{V}$ be the filter generated by this filterbase. Because each A_α is large, because $[\gamma(A_\alpha \cap p^{-1}[B])] = [\gamma(A_\alpha)]$ in the $\mathcal{U}$-ultrapower for any $B \in \mathcal{U}$, and because largeness is obviously preserved by supersets, we know that every set in $\mathcal{V}$ is large.

We claim next that $\mathcal{V}$ is an ultrafilter. To see this, let $X \subseteq \mathbb{N}$ be arbitrary, and consider the following partition $F : [\mathbb{N}]^2 \to \{0, 1\}$. If X contains both or neither of x and y, then $F(\{x, y\}) = 0$; otherwise $F(\{x, y\}) = 1$. Requirement (3) of our construction provides an $A_\alpha \in \mathcal{V}$ such that F is constant on each $[A_\alpha \cap I_n]^2$, say with value $f(n)$. A set on which F is constant with value 1 obviously contains at most two points, one in X and one outside X. As

A_α is large, we infer that $(\mathcal{U}n)\, f(n) = 0$. That is, for $\mathcal{U}$-almost all n, $A_\alpha \cap I_n$ is included in either X or $\mathbb{N} - X$. As $\mathcal{U}$ is an ultrafilter, it contains a set B such that either $A_\alpha \cap I_n \subseteq X$ for all $n \in B$ or $A_\alpha \cap I_n \subseteq \mathbb{N} - X$ for all $n \in B$. Then $A_\alpha \cap p^{-1}[B] \in \mathcal{V}$ is either included in or disjoint from X. Since X was arbitrary, this completes the proof that $\mathcal{V}$ is an ultrafilter.

Since all sets in $\mathcal{V}$ are large, the finite-to-one function p is not one-to-one on any set in $\mathcal{V}$. Thus $\mathcal{V}$ is not selective, in fact not even a Q-point.

Our next goal is to prove that $\mathbb{N} \to [\mathcal{V}]^2_3$. Let an arbitrary $F : [\mathbb{N}]^2 \to \{0, 1, 2\}$ be given. We follow the custom of calling the values of F colors, and we use the notation $\{a < b\}$ to mean the set $\{a, b\}$ and to indicate the notational convention that $a < b$. We shall find two sets $X, Y \in \mathcal{V}$ such that all pairs $\{a < b\} \in [X]^2$ with $p(a) = p(b)$ have a single color and all pairs $\{a < b\} \in [Y]^2$ with $p(a) \neq p(b)$ have a single color (possibly different from the previous color). Then $X \cap Y \in \mathcal{V}$ has the weak homogeneity property required by $\mathbb{N} \to [\mathcal{V}]^2_3$.

To construct X, begin by considering the partition $G : [\mathbb{N}]^2 \to \{0, 1\}$ obtained from F by identifying the color 2 with 1. By our construction of $\mathcal{V}$, it contains a set A_α such that, for each n, all pairs in $[A_\alpha \cap I_n]^2$ are sent to the same color $g(n)$ by G. $\mathcal{U}$, being an ultrafilter, contains a set B on which g is constant. Then $A_\alpha \cap p^{-1}(B)$ is a set in $\mathcal{V}$ such that all pairs $\{a < b\}$ in $A_\alpha \cap p^{-1}(B)$ with $p(a) = p(b)$ have the same G-color. So the F-colors of these pairs are either all 0 or all in $\{1, 2\}$. If they are all 0, then $A_\alpha \cap p^{-1}(B)$ serves as the desired X. If they are all in $\{1, 2\}$, then we repeat the argument using G', obtained from F by identifying 2 with 0. We obtain a set $A_{\alpha'} \cap p^{-1}(B') \in \mathcal{V}$ such that the F-colors of its pairs with $p(a) = p(b)$ are either all 1 or all in $\{0, 2\}$. Then $A_\alpha \cap p^{-1}(B) \cap A_{\alpha'} \cap p^{-1}(B') \in \mathcal{V}$ serves as the desired X.

It remains to construct Y. It is well known that selective ultrafilters have the Ramsey property. So we can find a set $B \in \mathcal{U}$ such that all or none of the pairs $\{x < y\} \in [B]^2$ satisfy the inequality

$$9^{|\bigcup_{z \le x} I_z|} < y. \tag{†}$$

If we had the "none" alternative here, then all elements $y \in B$ would be bounded by $9^{|\bigcup_{z \le x} I_z|}$ where x is the smallest element of B. That is absurd, as B is infinite, so we must have the "all" alternative.

For each $b \in p^{-1}[B]$, let f_b be the function telling how b is related by F to elements in earlier fibers over B. That is, if $p(b) = n \in B$, let the domain of f_b be $\{a \in \mathbb{N} : p(a) < n$ and $p(a) \in B\}$, and define f_b on this domain by $f_b(a) = F(\{a, b\})$. Notice that the domain of f_b has cardinality at most $|\bigcup_{z \le m} I_z|$ where m is the last element of B before n (or 0 if n is the first element of B). Since f_b takes values in 3, the number of possible f_b's, for $p(b) = n \in B$, is at most

$$3^{|\bigcup_{z \le m} I_z|} < \sqrt{n},$$

in view of the inequality (†) satisfied by all pairs from B.

Define a new partition, $H : [\mathbb{N}]^2 \to \{0, 1\}$ by setting $H(\{b < c\}) = 0$ if f_b and f_c are defined and equal, and $H(\{b < c\}) = 1$ otherwise. Proceeding as in the first part of the construction of X above, we obtain a set $Z \in \mathcal{V}$ such that all pairs $\{b < c\} \in [Z]^2$ with $p(b) = p(c)$ have the same H-color. That is, in each of the sets $Z \cap I_n$ $(n \in B)$, either all the points b have the same f_b or they all have different f_b's. But Z is large so the number of such points is, for $\mathcal{U}$-almost all n, larger than $\sqrt{n}$. So, by the estimate above of the number of f_b's, there are not enough of these functions for every b to have a different f_b. Thus, for $\mathcal{U}$-almost all $n \in B$, all $b \in Z \cap I_n$ have the same f_b. Shrinking B to a smaller set in $\mathcal{U}$ (which we still call B to avoid extra notation), we can assume that, for all $n \in B$, f_b depends only on $p(b)$ as long as $b \in Z$. Let us also shrink Z to $Z \cap p^{-1}[B]$, which is of course still in $\mathcal{V}$.

Going back to the original partition F, we have that the color of a pair $\{a < b\} \in [Z]^2$ with $p(a) < p(b)$ depends only on a and $p(b)$, because this color is $F(\{a < b\}) = f_b(a)$ and f_b depends only on $p(b)$.

For each $a \in Z$, let g_a be the function, with domain equal to the part of B after $p(a)$, such that $g_a(n)$ is the common value of $F(\{a < b\})$ for all $b \in Z \cap I_n$. This g_a maps a set in $\mathcal{U}$ (namely a final segment of B) into 3, so it is constant, say with value $j(a)$, on some set $C_a \in \mathcal{U}$.

Using again the Ramsey property of $\mathcal{U}$, we obtain a set $D \in \mathcal{U}$ such that $D \subseteq B$ and all or none of the pairs $\{x < y\} \in [D]^2$ satisfy

$$(\forall a \in Z \cap I_x) \quad y \in C_a.$$

If we had the "none" alternative, then, letting x be the first element of D, we would have that

$$D \cap \bigcap_{a \in Z \cap I_x} C_a = \emptyset.$$

But this is the intersection of finitely many sets from $\mathcal{U}$, so it cannot be empty. This contradiction shows that we must have the "all" alternative. In view of the definition of C_a, this means that, when $a < b$ are in $Z \cap p^{-1}[D]$ and $p(a) < p(b)$, the F-color of $\{a < b\}$ depends only on a, not on b. Since there are only finitely many possible colors and since $\mathcal{V}$ is an ultrafilter, we can shrink $Z \cap p^{-1}[D]$ to a set $Y \in \mathcal{V}$ on which the F-color of all such pairs is the same.

This completes the construction of the desired Y and thus the proof that $\mathbb{N} \to [\mathcal{V}]^2_3$.

An easy induction argument shows that the following result, which looks stronger at first sight, is in fact a consequence of the partition relation $\mathbb{N} \to [\mathcal{V}]^2_3$. For any partition of $[\mathbb{N}]^2$ into any finite number of pieces, there is a set $H \in \mathcal{V}$ such that $[H]^2$ is included in the union of two of the pieces. We shall need this for a partition into 6 pieces.

Finally, we prove that $\mathcal{V}$ is quasi-selective. Consider an arbitrary $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \le n$ for all n; we seek a set in $\mathcal{V}$ on which f is non-decreasing. Define a partition $F : [\mathbb{N}]^2 \to 6$ by setting

$$F(\{a < b\}) = \begin{cases} 0 & \text{if } p(a) = p(b) \text{ and } f(a) < f(b) \\ 1 & \text{if } p(a) = p(b) \text{ and } f(a) = f(b) \\ 2 & \text{if } p(a) = p(b) \text{ and } f(a) > f(b) \\ 3 & \text{if } p(a) < p(b) \text{ and } f(a) < f(b) \\ 4 & \text{if } p(a) < p(b) \text{ and } f(a) = f(b) \\ 5 & \text{if } p(a) < p(b) \text{ and } f(a) > f(b). \end{cases}$$

Since p is non-decreasing, the six cases cover all the possibilities. Fix a set $H \in \mathcal{V}$ on whose pairs F takes only two values. The first of those two values must be in $\{0, 1, 2\}$ and the second in $\{3, 4, 5\}$ because p is neither one-to-one nor constant on any set in $\mathcal{V}$.

If the second value were 5, then by choosing an infinite sequence $a_0 < a_1 < \cdots$ in H with $p(a_0) < p(a_1) < \cdots$, we would get an infinite decreasing sequence $f(a_0) > f(a_1) > \cdots$ of natural numbers. Since this is absurd, the second value must be 3 or 4.

If the second value is 4, then any two elements of H from different I_n's have the same f value. But then the same is true also for any two elements of H from the same I_n, because we can compare them with a third element of H chosen from a different I_m. So in this case f is constant on H; in particular, it is non-decreasing, as desired.

So from now on, we may assume that the second value is 3; f is increasing on pairs in H from different I_n's.

Thus, if the first value is either 0 or 1, then f is non-decreasing on all of H, as desired. It remains only to handle the case that the first value is 2; we shall show that this case is impossible, thereby completing the proof of the theorem.

Suppose, toward a contradiction, that the first value were 2. This means that, for each n, the restriction of f to $H \cap I_n$ is strictly decreasing. Temporarily fix n, and let b be the smallest element of $H \cap I_n$. Then $f(b)$ is the largest value taken by f on $H \cap I_n$, and there are exactly $|H \cap I_n|$ such values. Therefore, $f(b) \geq |H \cap I_n| - 1$. On the other hand, by the hypothesis on f, we have $f(b) \leq b$, and therefore $b \geq |H \cap I_n| - 1$.

Now un-fix n. We have just shown that

$$\min(H \cap I_n) \geq |H \cap I_n| - 1,$$

and so $H \cap I_n \notin \mathcal{L}$. That is,

$$\gamma(H)(n) = \nu(H \cap I_n) = 0$$

for all n. That contradicts the fact that, like all sets in $\mathcal{V}$, H is large, and the proof of the theorem is complete. $\square$

Now, in order to deduce Theorem 1.10, we have only to recall the well known fact that, under CH, there are $2^{\mathfrak{c}}$ pairwise non-isomorphic selective ultrafilters. For each of them, say $\mathcal{U}$, Theorem 2.1 provides a non-selective quasi-selective $\mathcal{V}$ that is Rudin–Keisler above $\mathcal{U}$. As remarked at the beginning of this section, according to [4], there is a unique isomorphism class of selective ultrafilters Rudin–Keisler below $\mathcal{V}$, because $\mathcal{V}$ was constructed so as to satisfy the partition relation $\mathbb{N} \to [\mathcal{V}]^2_3$. So all these $\mathcal{V}$'s are pairwise non-isomorphic.

Finally, as remarked at the end of Section 1, Proposition 1.9 allows for associating to each $\mathcal{V}$ an isomorphic non-quasi-selective P-point $\mathcal{V}'$.

3. Equinumerosity of point sets

Numerosity is a generalization of finite cardinality that has been recently introduced and investigated by V. Benci, M. Di Nasso and M. Forti in a series of papers, aiming at preserving the ancient principle that *the whole is larger than part*. A short review is in order.

In [1], a numerosity $\mathfrak{n}(A)$ is assigned to each (countable) "labeled set", namely a pair $\langle A, \ell_A \rangle$ where the finite-to-one "labeling function" $\ell_A : A \to \mathbb{N}$ makes it possible to "count" the elements of the set A by means of finite approximations $A_n = \{a \in A \mid \ell_A(a) \leq n\}$. Numerosities are assumed to be linearly ordered, and it is postulated that, whenever $|A_n| \leq |B_n|$ for all n, then $\mathfrak{n}(A) \leq \mathfrak{n}(B)$. Two more axioms are also added so as to incorporate the whole-part property, and to govern suitable notions of sum and product of labeled sets. The existence of a numerosity function $\mathfrak{n}$ for labeled sets turns out to be equivalent to the existence of a *selective* ultrafilter.

That research was then extended by considering sets of arbitrary cardinality. In [2], a notion of numerosity is studied for sets of ordinals, which satisfies some natural "geometric" properties, including invariance under suitable translations. In [3], a numerosity function taking values in an ordered ring is introduced in the general framework of a "mathematical universe", namely a superstructure $V_\omega(X)$ over a given set of atoms X. In the definition, it is postulated that disjoint sums and Cartesian products correspond to sums and products of numerosities, respectively.

The following paper [11] is focused on point sets $A \subseteq \mathbb{R}^k$ over the real line. In addition to the axioms as given in [3], a natural property is also considered to govern differences of numerosities, namely: $\mathfrak{n}(X) < \mathfrak{n}(Y)$ if and only if $\mathfrak{n}(X) = \mathfrak{n}(Z)$ for some proper subset $Z \subset Y$. It turns out that such a property seems difficult to fulfill. For example, it can be realized for countable sets under mild set-theoretic hypotheses such as[6] $\mathbf{cov}(\mathcal{B}) = \mathfrak{c} < \aleph_\omega$; but it is an open problem whether one can consistently assume it in general. In [10], a notion of "fine density" for sets of natural numbers is introduced, which is closely related to numerosities, but where no notion of product is available. Besides the usual properties for disjoint unions and for differences, a natural asymptotic property is also assumed, namely that $\mathfrak{n}(A) \leq \mathfrak{n}(B)$ whenever $|A \cap [1, n]| \leq |B \cap [1, n]|$ for all n. With the addition of a "compatibility" property for pairs of sets, the existence of fine densities is equivalent to the existence of a special kind of P-points named *smooth* ultrafilters, which eventually turned out to coincide with the quasi-selective ultrafilters studied in this paper.

In this section we study a notion of "numerosity" for *point sets of natural numbers*, *i.e.* for subsets of the spaces of k-tuples $\mathbb{N}^k$. Contrarily to the previous approaches mentioned above, here numerosity will be defined by postulating axioms for an equivalence relation of "*equinumerosity*", in a similar way as Cantorian cardinalities are defined by means of the equipotency relation.

For simplicity we follow the usual practice and we identify Cartesian products with the corresponding "concatenations". That is, for every $A \subseteq \mathbb{N}^k$ and for every $B \subseteq \mathbb{N}^h$, we identify $A \times B = \{(\vec{a}, \vec{b}) \mid \vec{a} \in A, \vec{b} \in B\}$ with

$$A \times B = \{(a_1, \dots, a_k, b_1, \dots, b_h) \mid (a_1, \dots, a_k) \in A \text{ and } (b_1, \dots, b_h) \in B\}.$$

Definition 3.1. We call *equinumerosity* an equivalence relation $\approx$ that satisfies the following properties for all point sets A, B of natural numbers.

(E1) $A \approx B$ if and only if $A \setminus B \approx B \setminus A$.
(E2) Exactly one of the following three conditions holds:
 (a) $A \approx B$;
 (b) $A' \approx B$ for some proper subset $A' \subset A$;
 (c) $A \approx B'$ for some proper subset $B' \subset B$.
(E3) $A \times \{P\} \approx \{P\} \times A \approx A$ for every point $P \in \mathbb{N}$.
(E4) $A \approx A'$ & $B \approx B' \;\Rightarrow\; A \times B \approx A' \times B'$.

Some comments are in order. Axiom (E1) is but a compact equivalent reformulation of the second and third common notions of Euclid's Elements (see [13]):

"*If equals be added to equals, the wholes are equal*",

and

"*if equals be subtracted from equals, the remainders are equal*".

(A precise statement of this equivalence is given in Proposition 3.2.) Notice that the first common notion

"*Things which are equal to the same thing are also equal to one another*"

is already secured by the assumption that equinumerosity is an equivalence relation.

The *trichotomy property* of axiom (E2) combines two natural ideas: first that, given two sets, one is equinumerous to some subset of the other, and second that no proper subset is

6 $\mathbf{cov}(\mathcal{B})$ denotes the minimum number of nowhere dense sets needed to cover the real line.

equinumerous to the set itself. So (E2) allows for a natural ordering of sizes that satisfies the implicit assumption of the classical theory that *(homogeneous) magnitudes are always comparable*, as well as the fifth Euclidean common notion

"*The whole is greater than the part*".

We remark that both properties (E1) and (E2) hold for equipotency between finite sets, while both fail badly for equipotency between infinite sets.

The third axiom (E3) can be viewed as an instance of the fourth Euclidean common notion

"*Things applying [exactly] onto one another are equal to one another*".[7]

In particular (E3) incorporates the idea that any set has equinumerous "lifted copies" in any higher dimension (see Proposition 3.4).

Axiom (E4) is postulated so as to allow for the natural definition of *multiplication* of numerosities, which admits the numerosity of every singleton as an identity by axiom (E3). This multiplication, together with the natural addition of numerosities, as given by *disjoint union*, satisfies the properties of *discretely ordered semirings* (see Theorem 4.2).

Remark that we do not postulate here *commutativity* of product. On the one hand, this assumption is unnecessary for the general treatment of numerosities; on the other hand, commutativity follows from the given axioms in the case of *asymptotic* numerosities (see Section 5).

Proposition 3.2. *Axiom* (E1) *is equivalent to the conjunction of the following two principles*[8].

(1) Sum Principle. *Let* A, A', B, B' *be such that* $A \cap B = \emptyset$ *and* $A' \cap B' = \emptyset$.
If $A \approx A'$ *and* $B \approx B'$, *then* $A \cup B \approx A' \cup B'$.
(2) Difference Principle. *Let* A, A', C, C' *be such that* $A \subseteq C$ *and* $A' \subseteq C'$.
If $A \approx A'$ *and* $C \approx C'$, *then* $C \setminus A \approx C' \setminus A'$.

Proof. We begin by proving that (E1) follows from the conjunction of (1) and (2). In fact, if $A \setminus B \approx B \setminus A$, then

$$A = (A \setminus B) \cup (A \cap B) \approx (B \setminus A) \cup (A \cap B) = B,$$

by (1). Conversely, if $A \approx B$, then

$$A \setminus B = A \setminus (A \cap B) \approx B \setminus (A \cap B) = B \setminus A,$$

by (2).

Now remark that, if we put $A \cup B = C$ and $A' \cup B' = C'$ in (1), then both principles are immediate consequences of the following statement.

(E1)* *Let* A, A', C, C' *be such that* $A \subseteq C$ *and* $A' \subseteq C'$. *If* $A \approx A'$, *then*

$$C \setminus A \approx C' \setminus A' \Longleftrightarrow C \approx C'.$$

[7] Equicardinality is characterized by equipotency; similarly one might desire that equinumerosity be characterized by "isometry" witnessed by a suitable class of bijections. This assumption seems *prima facie* very demanding. However, we shall see that it is fulfilled by the *asymptotic equinumerosities* of Section 5.

[8] We implicitly assume that $A, B \subseteq \mathbb{N}^k$ and $A', B' \subseteq \mathbb{N}^{k'}$ are homogeneous pairs (*i.e.* sets of the same dimension); otherwise $A \cup B$, $A' \cup B'$, $A \setminus B$, $A' \setminus B'$ are not point sets.

We are left to show that (E1) implies (E1)*.

Assume that the equivalence relation $\approx$ satisfies (E1), and put

$$A_0 = A \setminus C', \qquad A'_0 = A' \setminus C, \qquad D = A \cap A', \qquad A_1 = A \setminus (A_0 \cup D),$$
$$A'_1 = A' \setminus (A'_0 \cup D),$$
$$C_0 = C \setminus (A \cup C'), \qquad C'_0 = C' \setminus (A' \cup C), \qquad E = (C \cap C') \setminus (A \cup A'),$$

so as to obtain pairwise disjoint sets $A_0, A_1, A'_0, A'_1, D, C_0, C'_0, E$ such that

$$A = A_0 \cup A_1 \cup D, \qquad A' = A'_0 \cup A'_1 \cup D,$$
$$C \setminus C' = A_0 \cup C_0, \qquad C' \setminus C = A'_0 \cup C'_0,$$
$$C \setminus A = C_0 \cup E \cup A'_1, \qquad C' \setminus A' = C'_0 \cup E \cup A_1.$$

Hence

$$C \approx C' \Longleftrightarrow C_0 \cup A_0 \approx C'_0 \cup A'_0$$

and

$$C \setminus A \approx C' \setminus A' \Longleftrightarrow C_0 \cup A'_1 \approx C'_0 \cup A_1.$$

Since $A \approx A'$ we have $A_0 \cup A_1 \approx A'_0 \cup A'_1$, whence

$$C_0 \cup A_0 \cup A_1 \approx C_0 \cup A'_0 \cup A'_1.$$

So $C \setminus A \approx C' \setminus A'$ implies $C_0 \cup A_0 \cup A_1 \approx C'_0 \cup A_1 \cup A'_0$, whence $C \approx C'$. Conversely, $C \approx C'$ implies $C_0 \cup A_0 \cup A_1 \approx C'_0 \cup A'_0 \cup A_1$, whence $C \setminus A \approx C' \setminus A'$. $\square$

Thus equinumerosity behaves coherently with respect to the operations of *disjoint union* and *set difference*. We can now prove that our notion of equinumerosity satisfies the basic requirement that *finite point sets are equinumerous if and only if they have the same "number of elements"*.

Proposition 3.3. *Assume* (E1) *and* (E2)*, and let A, B be finite sets. Then*

$$A \approx B \Longleftrightarrow |A| = |B|.$$

Proof. We begin by proving that (E2) implies that any two singletons are equinumerous. In fact the only proper subset of any singleton is the empty set, and no non-empty set can be equinumerous to $\emptyset$, by (E2).

Given non-empty finite sets A, B, pick $a \in A, b \in B$ and put $A' = A \setminus \{a\}, B' = B \setminus \{b\}$. Then, by (E1)*, $A' \approx B' \Longleftrightarrow A \approx B$, because $\{a\} \approx \{b\}$, and the thesis follows by induction on $n = |A|$. $\square$

By the above proposition, we can identify each natural number $n \in \mathbb{N}$ with the equivalence class of all those point sets that have finite cardinality n.

Remark that trichotomy is not essential in order to obtain Proposition 3.3. In fact, let $\approx$ be non-trivial and satisfy (E1), (E3), and (E4). Then, by (E3), we have

$$\{x\} \approx \{y\} \times \{x\} \approx \{x\} \times \{y\} \approx \{y\}$$

for all $x \in \mathbb{N}^h$ and all $y \in \mathbb{N}^k$, and so all singletons are equinumerous. On the other hand, if any singleton is equinumerous to $\emptyset$, then all sets are, because of (E4).

Clearly (E3) formalizes the natural idea that singletons have "unitary" numerosity. A trivial but important consequence of this axiom, already mentioned above, is the existence of infinitely many pairwise disjoint equinumerous copies of any given point set in every higher dimension. Namely, we have the following proposition.

Proposition 3.4. *Assume* (E3)*, and let* $A \subseteq \mathbb{N}^k$ *be a k-dimensional point set. If* $h > k$ *let* $P \in \mathbb{N}^{h-k}$ *be any* $(h-k)$*-dimensional point. Then* $\{P\} \times A \subseteq \mathbb{N}^h$ *is equinumerous to* A. □

4. The algebra of numerosities

Starting from the equivalence relation of *equipotency*, Cantor introduced the algebra of cardinals by means of disjoint unions and Cartesian products. Our axioms have been chosen so as to allow for the introduction of an "algebra of numerosities" in the same Cantorian manner.

Definition 4.1. Let $\mathbf{F} = \bigcup_{k \in \mathbb{N}^+} \mathcal{P}(\mathbb{N}^k)$ be the family of all point sets over $\mathbb{N}$, and let $\approx$ be an equinumerosity relation on $\mathbf{F}$.

- The *numerosity* of $A \in \mathbf{F}$ (with respect to $\approx$) is the equivalence class of all point sets equinumerous to A

  $$\mathfrak{n}_{\approx}(A) = [A]_{\approx} = \{B \in \mathbf{F} \mid B \approx A\}.$$

- The *set of numerosities* of $\approx$ is the quotient set $\mathfrak{N}_{\approx} = \mathbf{F}/\approx$.
- The *numerosity function* associated to $\approx$ is the canonical map $\mathfrak{n}_{\approx} : \mathbf{F} \to \mathfrak{N}_{\approx}$.

In the sequel we shall drop the subscript $\approx$ whenever the equinumerosity relation is fixed. Numerosities will be usually denoted by *Frakturen* $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$, *etc.*

The given axioms guarantee that numerosities are naturally equipped with a "nice" algebraic structure. (This has to be contrasted with the awkward cardinal algebra, where *e.g.* $\kappa + \mu = \kappa \cdot \mu = \max\{\kappa, \mu\}$ for all infinite κ, μ.)

Theorem 4.2. *Let* $\mathfrak{N}$ *be the set of numerosities of the equinumerosity relation* $\approx$*. Then there exist unique operations* $+$ *and* $\cdot$*, and a unique linear order* $<$ *on* $\mathfrak{N}$*, such that for all point sets* A, B*:*

(1) $\mathfrak{n}(A) + \mathfrak{n}(B) = \mathfrak{n}(A \cup B)$ *whenever* $A \cap B = \emptyset$[9]*;*
(2) $\mathfrak{n}(A) \cdot \mathfrak{n}(B) = \mathfrak{n}(A \times B)$*;*
(3) $\mathfrak{n}(A) < \mathfrak{n}(B)$ *if and only if* $A \approx B'$ *for some proper subset* $B' \subset B$.

The resulting structure on $\mathfrak{N}$ *is the non-negative part of a discretely ordered ring* $(\mathfrak{R}, 0, 1, +, \cdot, <)$*. Moreover, if the fundamental subring of* $\mathfrak{R}$ *is identified with* $\mathbb{Z}$*, then* $\mathfrak{n}(A) = |A|$ *for every finite point set* A.

Proof. We begin with the ordering. Trivially $A \approx A$, and so the trichotomy property (E2) implies the irreflexivity $\mathfrak{n}(A) \not< \mathfrak{n}(A)$. In order to prove transitivity, we show first the following property.

($\star$) If $A \approx B$, then for any $X \subset A$ there exists $Y \subset B$ such that $X \approx Y$.

Since $A \approx B$, the proper subset $X \subset A$ cannot be equinumerous to B. Similarly, B cannot be equinumerous to a proper subset $X' \subset X \subset A$. We conclude that $X \approx Y$ for some $Y \subset B$, and ($\star$) is proved.

Now assume $\mathfrak{n}(A) < \mathfrak{n}(B) < \mathfrak{n}(C)$. Pick proper subsets $A' \subset B$ and $B' \subset C$ such that $A \approx A'$ and $B \approx B'$. By ($\star$), there exists $A'' \subset B'$ such that $A' \approx A''$. So, $\mathfrak{n}(A) < \mathfrak{n}(C)$ holds because

[9] Again, here we implicitly assume that $A, B \subseteq \mathbb{N}^k$ have the same dimension k, as otherwise their union $A \cup B$ would not be a point set.

$A \approx A'' \subset C$, and transitivity follows. Finally, again by trichotomy, we get that the order $<$ is linear.

We now define a sum for numerosities. Given $\mathfrak{x}, \mathfrak{y} \in \mathfrak{N}$, there exist $k > 0$ and disjoint point sets $A, B \subseteq \mathbb{N}^k$ such that $\mathfrak{n}(A) = \mathfrak{x}$ and $\mathfrak{n}(B) = \mathfrak{y}$, by Proposition 3.4. Then put $\mathfrak{x}+\mathfrak{y} = \mathfrak{n}(A \cup B)$. This addition is independent of the choice of A and B, by Proposition 3.2. Commutativity and associativity trivially follow from the corresponding properties of disjoint unions.

Clearly $0 = \mathfrak{n}(\emptyset)$ is the (unique) neutral element. Moreover Proposition 3.2 directly yields the cancellation law

$$\mathfrak{x} + \mathfrak{z} = \mathfrak{x} + \mathfrak{z}' \iff \mathfrak{z} = \mathfrak{z}'.$$

By definition, $\mathfrak{x} \leq \mathfrak{y}$ holds if and only if there exists $\mathfrak{z}$ such that $\mathfrak{y} = \mathfrak{x} + \mathfrak{z}$. Such a $\mathfrak{z}$ is unique by the cancellation law, and so the monotonicity property with respect to addition follows from the equivalences

$$\mathfrak{w} + \mathfrak{x} \leq \mathfrak{w} + \mathfrak{y} \iff \exists \mathfrak{z}\, (\mathfrak{w} + \mathfrak{x} + \mathfrak{z} = \mathfrak{w} + \mathfrak{y}) \iff \exists \mathfrak{z}\, (\mathfrak{x} + \mathfrak{z} = \mathfrak{y}) \iff \mathfrak{x} \leq \mathfrak{y}.$$

The multiplication of numerosities given by condition (2) is well-defined by axiom $(\mathsf{E4})$. Associativity follows from the corresponding property of concatenation (and this is the reason for our convention on Cartesian products). Distributivity is inherited from the corresponding property of Cartesian products with respect to disjoint unions. Moreover $1 = \mathfrak{n}(\{P\})$ is an identity, by $(\mathsf{E3})$. Finally, by definition, $\mathfrak{x} \cdot \mathfrak{y} = 0$ if and only if $\mathfrak{x} = 0$ or $\mathfrak{y} = 0$.

Therefore $\langle \mathfrak{N}, +, \cdot, < \rangle$ is *the non-negative part of a linearly ordered ring* $\mathfrak{R}$, say.[10] By Proposition 3.3, $\mathbb{N}$ can be identified with the set of the numerosities of finite sets; hence it is an initial segment of $\mathfrak{N}$. It follows that the ordering of $\mathfrak{R}$ is discrete. $\square$

Remark that the previous presentations [2,3,11] were different, in that they were grounded on the primitive notion of *numerosity function* in place of the equinumerosity relation. In particular, the ordered ring properties of the set of numerosities were directly included in the definition. Of course, by appropriate choices of the axioms, the two approaches are equivalent.

5. Asymptotic numerosities and quasi-selective ultrafilters

Various measures of size for infinite sets of positive integers $A \subseteq \mathbb{N}^+$, commonly used in number theory, are obtained by considering the sequence of ratios

$$\frac{|\{a \in A \mid a \leq n\}|}{n}.$$

For example, the *(upper, lower) asymptotic density* of A are defined as the limit (superior, inferior) of this sequence. This procedure might be viewed as measuring *the ratio between the numerosities of* A *and* $\mathbb{N}^+$. In this perspective, one should assume that $\mathfrak{n}(A) \leq \mathfrak{n}(B)$ whenever the sequence of ratios for B dominates that for A. So one is led to introduce the following notion of "asymptotic" equinumerosity relation.

Definition 5.1. For $X \subseteq \mathbb{N}^k$ put

$$X_n = \{x \in X \mid x_i \leq n \text{ for } i = 1, \ldots, k\}.$$

[10] $\mathfrak{R}$ can be defined as usual by mimicking the construction of $\mathbb{Z}$ from $\mathbb{N}$. Take the quotient of $\mathfrak{N} \times \mathfrak{N}$ modulo the equivalence $(x, y) \equiv (x', y') \Leftrightarrow x + y' = x' + y$, and define the operations in the obvious way.

The equinumerosity relation $\approx$ is *asymptotic* if:

(E0) if $|X_n| \leq |Y_n|$ for all $n \in \mathbb{N}$, then there exists $Z \subseteq Y$ such that $X \approx Z$.

Remark that sets X and Y are not assumed to be of the *same dimension*. In fact, at the end of this section we shall use the condition (E0) to give a notion of "quasi-numerosity" that is defined on *all sets of tuples* of natural numbers.

According to this definition, if $\mathfrak{n} : \mathbf{F} \to \mathfrak{N}$ is the numerosity function associated to an equinumerosity $\approx$, then $\approx$ is asymptotic if and only if $\mathfrak{n}$ satisfies the following property for all $X, Y \in \mathbf{F}$:

$$|X_n| \leq |Y_n| \quad \text{for all } n \in \mathbb{N} \Longrightarrow \mathfrak{n}(X) \leq \mathfrak{n}(Y).$$

Actually, an apparently stronger property holds for asymptotic numerosities, namely

$$|X_n| \leq |Y_n| \quad \text{for all } n \geq m \Longrightarrow \mathfrak{n}(X) \leq \mathfrak{n}(Y).$$

In fact, let $Z \subseteq Y_m$ be such that $|Z| = |X_m|$, and put $X' = X \setminus X_m$ and $Y' = Y \setminus Z$; then $|X'_n| \leq |Y'_n|$ for all $n \in \mathbb{N}$, and

$$\mathfrak{n}(X) = \mathfrak{n}(X') + |X_m| \leq \mathfrak{n}(Y') + |Z| = \mathfrak{n}(Y).$$

In the following theorem we give a nice algebraic characterization of asymptotic numerosities.

Theorem 5.2. *Define the map $\Phi : \mathbf{F} \to \mathbb{Z}^{\mathbb{N}}$ by $\Phi(X) = \langle\, |X_n|\, \rangle_{n \in \mathbb{N}}$. Then we have the following.*

(i) *The ring additively generated by the range of Φ is the subring $\mathbb{P}$ of $\mathbb{Z}^{\mathbb{N}}$ consisting of all polynomially bounded sequences*

$$\mathbb{P} = \{g : \mathbb{N} \to \mathbb{Z} \mid \exists k, m\ \forall n > m |g(n)| < n^k\} \subseteq \mathbb{Z}^{\mathbb{N}}.$$

(ii) *Let $\mathfrak{n} : \mathbf{F} \to \mathfrak{N}$ be the numerosity function associated to an asymptotic equinumerosity $\approx$. Then there exists a unique homomorphism ψ from the partially ordered ring $\mathbb{P}$ onto the ordered ring $\mathfrak{R}$ generated by $\mathfrak{N}$ such that the following diagram commutes*

$$\begin{tikzcd} \mathbf{F} \arrow[r, "\Phi"] \arrow[d, "\mathfrak{n}"'] & \mathbb{P} \arrow[d, "\psi"] \\ \mathfrak{N} \arrow[r, "j"] & \mathfrak{R} \end{tikzcd}$$

(where j is the natural embedding).

(iii) *There exists a quasi-selective ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that the sequence $g \in \mathbb{P}$ belongs to the kernel of ψ if and only if its zero-set $Z(g) = \{n \mid g(n) = 0\}$ belongs to $\mathcal{U}$. In particular*

$$X \approx Y \Longleftrightarrow \{n \in \mathbb{N} \mid |X_n| = |Y_n|\} \in \mathcal{U}. \qquad (\sharp)$$

Proof.

(i) In order to characterize the range of Φ, recall that there are $(n+1)^k$-many k-tuples of non-negative integers not exceeding n. So $\Phi(X)$ belongs to $\mathbb{P}$ for all $X \in \mathbf{F}$. On the other hand, let $g \in \mathbb{P}$. By suitably increasing k, we may assume that $|g(n) - g(n-1)| < (n+1)^k$ for $n > 0$.

For $n > 0$ let $Y^{(n)}$ be a set of $|g(n) - g(n-1)|$-many k-tuples of non-negative integers not exceeding n, and put

$$Z = \bigcup_{g(n)>g(n-1)} \{n\} \times Y^{(n)} \quad \text{and} \quad W = \bigcup_{g(n)<g(n-1)} \{n\} \times Y^{(n)}.$$

Then

$$g = \Phi(Z) - \Phi(W) + g(0) \cdot \Phi(\{0\}^{k+1})$$

belongs to the ring generated by $\Phi[\mathbf{F}]$, as a linear combination of values of Φ. The set $\mathbb{P}$ being a ring, the proof of (i) is complete.

(ii) Given $g \in \mathbb{P}$, let k, Z, and W be chosen as done sub (i) above, and put

$$\psi(g) = j(\mathfrak{n}(Z)) - j(\mathfrak{n}(W)) + g(0).$$

This definition is well-posed. In fact, for any other choice of k', Z', and W' one has

$$|Z_n| = \sum_{i=1}^{n} \max\{g(i) - g(i-1), 0\} = |Z'_n|,$$

and

$$|W_n| = \sum_{i=1}^{n} \max\{g(i-1) - g(i), 0\} = |W'_n|.$$

Hence $\mathfrak{n}(Z) = \mathfrak{n}(Z')$ and $\mathfrak{n}(W) = \mathfrak{n}(W')$, because $\approx$ is asymptotic. Remark that the above definition is the unique possible one, in order to have a homomorphism satisfying $\psi(\Phi(X)) = j(\mathfrak{n}(X))$ for all $X \in \mathbf{F}$.

The map ψ is a ring homomorphism, as a straightforward consequence of the following trivial identities:

(1) $\Phi(X) + \Phi(Y) = \Phi(X \cup Y) + \Phi(X \cap Y)$ for all $X, Y \in \mathbb{N}^k$, and
(2) $\Phi(X) \cdot \Phi(Y) = \Phi(X \times Y)$ for all $X, Y \in \mathbf{F}$.

It remains to prove that $g \geq 0$ implies $\psi(g) \geq 0$. In fact, $g = \Phi(Z) - \Phi(W) + g(0) \cdot \Phi(\{0\}^{k+1})$ with $Z_n - W_n + g(0) \geq 0$ for all n. Assume without loss of generality that k has been chosen such that $2^k > \max\{g(1), g(0)\}$, so we can pick a set F of $g(0)$-many k-tuples of zeros and ones, such that $(Z \cup W) \cap F = \emptyset$. Put $Z' = Z \cup F$: then $g' = \Phi(Z') - \Phi(W)$ satisfies the conditions $g'(n) = g(n)$ for $n > 0$, and $g'(0) = 0$. Then $|Z'_n| \geq |W_n|$ for all n, and so $\mathfrak{n}(Z') \geq \mathfrak{n}(W)$. Hence

$$\psi(g) = \mathfrak{n}(Z) + g(0) - \mathfrak{n}(W) = \mathfrak{n}(Z') - \mathfrak{n}(W) \geq 0.$$

(iii) The kernel $\ker \psi = \mathbb{I}$ is a prime ideal of $\mathbb{P}$, because $\mathfrak{R}$ is a domain. The idempotents of $\mathbb{P}$ are all and only the characteristic functions χ_A of subsets $A \subseteq \mathbb{N}$ *i.e.* $\chi_A(n) = 1$ if $n \in A$ and $\chi_A(n) = 0$ if $n \notin A$. Hence, for every $A \subseteq \mathbb{N}$, the ideal $\mathbb{I}$ contains exactly one of the two complementary idempotents $\chi_A, 1 - \chi_A = \chi_{\mathbb{N}\setminus A}$. Moreover $1 - \chi_{A\cap B} = 1 - \chi_A\chi_B = 1 - \chi_A + \chi_A(1 - \chi_B)$, and $A \subseteq B$ implies $1 - \chi_B = (1 - \chi_A)(1 - \chi_B)$. Hence the set

$$\mathcal{U} = \{A \subseteq \mathbb{N} \mid 1 - \chi_A \in \mathbb{I}\}$$

is an ultrafilter on $\mathbb{N}$.

Now $g = g\cdot(1-\chi_{Z(g)})$, so g belongs to $\mathbb{I}$ whenever its zero-set $Z(g)$ belongs to U. Conversely, if $Z(g) \notin \mathcal{U}$, then $\psi(\chi_{Z(g)}) = 0$ and $g' = g^2 + \chi_{Z(g)} - 1$ is non-negative, so also $\psi(g')$ is non-negative. It follows that $\psi(g) \neq 0$. In particular

$$X \approx Y \Longleftrightarrow \Phi(X) - \Phi(Y) \in \ker\psi \Longleftrightarrow \{n \in \mathbb{N} \mid |X_n| - |Y_n| = 0\} \in \mathcal{U}.$$

It remains to show that $\mathcal{U}$ is quasi-selective. To this aim, remark that every element of $\mathfrak{N}$ is of the form $\pm\mathfrak{n}(X)$, for some $X \in \mathbf{F}$. So every element of $\mathbb{P}$ is congruent to $\pm\Phi(X)$ modulo $\mathbb{I}$. In particular, all non-negative elements of $\mathbb{P}$ are equivalent to some $\Phi(X)$, which is non-decreasing. So we may conclude that every polynomially bounded non-negative sequence is $\mathcal{U}$-equivalent to a non-decreasing one. Hence $\mathcal{U}$ is quasi-selective. $\square$

This construction allows for classifying all asymptotic equinumerosity relations by means of quasi-selective ultrafilters on $\mathbb{N}$ as follows.

Corollary 5.3. *There exists a biunique correspondence between asymptotic equinumerosity relations on the space of point sets over $\mathbb{N}$ and quasi-selective ultrafilters on $\mathbb{N}$. In this correspondence, if the equinumerosity $\approx$ corresponds to the ultrafilter $\mathcal{U}$, then*

$$X \approx Y \Longleftrightarrow \{n \in \mathbb{N} \mid |X_n| = |Y_n|\} \in \mathcal{U}. \quad (\sharp)$$

More precisely, let $\mathfrak{N}$ be the set of numerosities of $\approx$, and let $\mathfrak{n}$ be the corresponding numerosity function. Then the map

$$\varphi : \mathfrak{n}(X) \mapsto [\langle |X_n| \rangle_{n\in\mathbb{N}}]_{\mathcal{U}}$$

is an isomorphism of ordered semirings between $\mathfrak{N}$ and the initial segment $\mathbb{P}_{\mathcal{U}}$ of the ultrapower $\mathbb{N}^{\mathbb{N}}_{\mathcal{U}}$ that contains the classes of all polynomially bounded sequences. In particular, asymptotic numerosities are, up to isomorphism, non-standard integers.

Proof. Given an asymptotic equinumerosity $\approx$, let Φ, ψ, and the ultrafilter $\mathcal{U} = \mathcal{U}_{\approx}$ be as in Theorem 5.2. Then $\mathcal{U}$ is quasi-selective, and

$$X \approx Y \Longleftrightarrow \Phi(X) - \Phi(Y) \in \ker\psi \Longleftrightarrow \{n \in \mathbb{N} \mid |X_n| - |Y_n| = 0\} \in \mathcal{U},$$

that is ($\sharp$).

Conversely, given a quasi-selective ultrafilter $\mathcal{U}$ on $\mathbb{N}$, define the equivalence $\approx_{\mathcal{U}}$ on $\mathbf{F}$ by ($\sharp$). Then (E1), (E3), and (E4) are immediate. We prove now the following strong form of (E0) that implies also (E2):

$$\{n \in \mathbb{N} \mid |X_n| \leq |Y_n|\} \in \mathcal{U} \Longrightarrow \exists Z \subseteq Y \quad \text{s.t. } Z \approx_{\mathcal{U}} X.$$

Assume that $\{n \in \mathbb{N} \mid |X_n| \leq |Y_n|\} = U \in \mathcal{U}$: by quasi-selectivity there exists $V \in \mathcal{U}$ such that $|X_n|$, $|Y_n|$, and $|Y_n| - |X_n|$ are non-decreasing on V. Then one can isolate from Y a subset $Z \approx X$ in the following way: if m, m' are consecutive elements of $U \cap V$, put in Z exactly $|X_{m'}| - |X_m|$ elements of $Y_{m'} \setminus Y_m$.

So $\mathfrak{n}(X) < \mathfrak{n}(Y)$ if and only if $\varphi(\mathfrak{n}(X)) < \varphi(\mathfrak{n}(Y))$ in $\mathbb{P}_{\mathcal{U}}$, and the last assertion of the theorem follows. Finally, remark that every subset $A \subseteq \mathbb{N}$ can be written as $A = \{n \in \mathbb{N} \mid |X_n| = |Y_n|\}$ for suitable point sets X, Y. So the biconditional ($\sharp$) uniquely determines the ultrafilter $\mathcal{U} = \mathcal{U}_{\approx}$, and one has

- $\mathcal{U}_{(\approx_{\mathcal{U}})} = \mathcal{U}$ for all quasi-selective ultrafilters $\mathcal{U}$;
- $\approx_{(\mathcal{U}_{\approx})} = \approx$ for all asymptotic equinumerosity relations $\approx$.

Hence the correspondence between $\approx$ and $\mathcal{U}$ is biunique. $\square$

It is worth remarking that the property (E0) yields at once (E3) and commutativity of product, as well as many other natural instantiations of the fourth Euclidean common notion:

"*Things applying [exactly] to one another are equal to one another*".

More precisely, if the *support* of a tuple is defined by

$$\operatorname{supp}(x_1, \ldots, x_k) = \{x_1, \ldots, x_k\},$$

then all *support preserving bijections* can be taken as "congruences" for asymptotic numerosities, because any such $\sigma : X \to Y$ maps X_n onto Y_n for all $n \in \mathbb{N}$.

Actually, in order to give a "Cantorian" characterization of asymptotic equinumerosities, we isolate a wider class of bijections, namely the following.

Definition 5.4. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. A bijection $\sigma : X \to Y$ is a $\mathcal{U}$-*congruence* if $\{n \in \mathbb{N} \mid \sigma[X_n] = Y_n\} \in \mathcal{U}$.

When the ultrafilter $\mathcal{U}$ is quasi-selective, the $\mathcal{U}$-congruences determine an asymptotic equinumerosity.

Corollary 5.5. *Let $\approx$ be an asymptotic equinumerosity, and let $\mathcal{U}$ be the corresponding quasi-selective ultrafilter. Then $X \approx Y$ if and only if there exists a $\mathcal{U}$-congruence $\sigma : X \to Y$.* $\square$

This point of view allows for an interesting generalization of the notion of asymptotic equinumerosity to *all subsets* of $\mathbb{E} = \bigcup_{k \in \mathbb{N}^+} \mathbb{N}^k$, namely

- put $\mathbb{E}_n = \bigcup_{1 \le k \le n} \{0, \ldots, n\}^k$;
- let $\mathcal{U}$ be a filter on $\mathbb{N}$;
- call a map $\sigma : \mathbb{E} \to \mathbb{E}$ a $\mathcal{U}$-*isometry* if $\{n \mid \sigma[\mathbb{E}_n] = \mathbb{E}_n\} \in \mathcal{U}$;
- let $\mathfrak{S}_{\mathcal{U}}$ be the group of all $\mathcal{U}$-isometries, and for $X, Y \subseteq \mathbb{E}$ put

$$X \approx_{\mathcal{U}} Y \iff \text{there exists } \sigma \in \mathfrak{S}_{\mathcal{U}} \text{ such that } \sigma[X] = Y.$$

If $\mathcal{U}$ is a quasi-selective ultrafilter we can now assign a "quasi-numerosity" to every subset of $\mathbb{E}$, namely its orbit under $\mathfrak{S}_{\mathcal{U}}$:

$$\mathfrak{n}_{\mathcal{U}}(X) = [X]_{\approx_{\mathcal{U}}} = X^{\mathfrak{S}_{\mathcal{U}}} = \{\sigma[X] \mid \sigma \in \mathfrak{S}_{\mathcal{U}}\}.$$

Let $\mathfrak{N}_{\mathcal{U}} = \mathcal{P}(\mathbb{E})/\approx_{\mathcal{U}}$ be the set of quasi-numerosities, and let $\mathbb{S}_{\mathcal{U}}$ be the initial segment of the ultrapower $\mathbb{N}^{\mathbb{N}}_{\mathcal{U}}$ determined by $\mathfrak{n}_{\mathcal{U}}(\mathbb{E}) = [\langle (n+1)\frac{(n+1)^n - 1}{n} \mid n \in \mathbb{N}\rangle]_{\mathcal{U}}$.

Notice that if $X \subseteq \mathbb{N}^k$, then $X_n = X \cap \mathbb{E}_n$ for $n \ge k$. So we have the following theorem.

Theorem 5.6. *The relation $\approx_{\mathcal{U}}$ is an equivalence on $\mathcal{P}(\mathbb{E})$ that satisfies the properties* (E0), (E1), (E2)*, and, when restricted to* **F**, *agrees with the asymptotic equinumerosity corresponding to $\mathcal{U}$. Moreover the map*

$$\varphi_{\mathcal{U}} : \mathfrak{n}_{\mathcal{U}}(X) \mapsto [\langle |X \cap \mathbb{E}_n| \rangle_{n \in \mathbb{N}}]_{\mathcal{U}}$$

preserves sums and is an order isomorphism between $\mathfrak{N}_{\mathcal{U}}$ and $\mathbb{S}_{\mathcal{U}} \subseteq \mathbb{N}^{\mathbb{N}}_{\mathcal{U}}$. $\square$

It is worth mentioning that both the "multiplicative" properties (E3) and (E4) can fail for sets of infinite dimension. *E.g.*

$$\{(0, 1)\} \times \mathbb{E} \subset \{0\} \times \mathbb{E} \subset \mathbb{E}$$

have increasing quasi-numerosities, thus contradicting both (E3) and (E4).

6. Final remarks and open questions

It is interesting to remark that the non-selective quasi-selective ultrafilter $\mathcal{V}$ defined in the proof of Theorem 2.1 satisfies the following "weakly Ramsey" property[11]:

for any finite coloring of $[\mathbb{N}]^2$ *there is* $U \in \mathcal{V}$ *such that* $[U]^2$ *has only two colors.*

It is easily seen that if $\mathcal{V}$ is weakly Ramsey, then every function is either one-to-one or non-decreasing modulo $\mathcal{V}$. So both weakly Ramsey and quasi-selective ultrafilters are P-points. However these two classes are different whenever there exists a non-selective P-point, because the former is closed under isomorphism, whereas the latter is not, by Proposition 1.9.

In ZFC, one can draw the following diagram of implications

$$\begin{array}{ccccc} & & \text{Quasi-selective} & & \\ & \nearrow & & \searrow & \\ \text{Selective} & & & & P\text{-point} \\ & \searrow & & \nearrow & \\ & & \text{Weakly Ramsey} & & \end{array}$$

Recall that, assuming CH, the following facts hold:

- there exist quasi-selective weakly Ramsey ultrafilters that are not selective (Theorem 2.1);
- the class of quasi-selective non-selective ultrafilters is not closed under isomorphisms (Proposition 1.9);
- there are non-weakly-Ramsey P-points (see Theorem 2 of [4]).

It follows that, in the diagram above, no arrow can be reversed nor inserted.

The relationships between quasi-selective and weakly Ramsey ultrafilters are extensively studied in [12]. In particular, it is proved there that both quasi-selective and weakly Ramsey ultrafilters are P-points of a special kind, since they share the property that every function is equivalent to an *interval-to-one* function. So the question naturally arises as to whether this class of "interval P-points" is distinct from either one of the other three classes.

Many weaker conditions than the Continuum Hypothesis have been considered in the literature, in order to get more information about special classes of ultrafilters on $\mathbb{N}$. Of particular interest are (in)equalities among the so called "combinatorial cardinal characteristics of the continuum". (*E.g.* one has that P-points or selective ultrafilters are generic if $\mathfrak{c} = \mathfrak{d}$ or $\mathfrak{c} = \mathbf{cov}(\mathcal{B})$, respectively. Moreover if $\mathbf{cov}(\mathcal{B}) < \mathfrak{d} = \mathfrak{c}$ then there are filters that are included in P-points, but cannot be extended to selective ultrafilters. See the comprehensive survey [5].) We conjecture that similar hypotheses can settle the problems mentioned above.

It is worth mentioning that, given a quasi-selective ultrafilter $\mathcal{U}$, the corresponding asymptotic "quasi-numerosity" $\mathfrak{n}_{\mathcal{U}}$ of Theorem 5.6 can be extended to all subsets of the *algebraic Euclidean space*

$$\mathcal{Q} = \bigcup_{k\in\mathbb{N}^+} \overline{\mathbb{Q}}^k, \quad \text{where } \overline{\mathbb{Q}} \text{ is the field of all algebraic numbers.}$$

To this end, replace the sequence of finite sets $\mathbb{E}_n$ by

$$\mathcal{Q}_n = \left\{ (\alpha_1, \ldots, \alpha_k) \in \mathcal{Q} \;\middle|\; k \leq n, \text{ and } \exists a_{ih} \in \mathbb{Z},\ |a_{ih}| \leq n,\ \sum_{0\leq h\leq n} a_{ih}\alpha_i^h = 0 \right\}.$$

[11] Ultrafilters satisfying this property have been introduced in [4] under the name *weakly Ramsey*, and then generalized to $(n+1)$-Ramsey ultrafilters in [15].

Then extend the definition of *$\mathcal{U}$-isometry* to maps $\sigma : \mathcal{Q} \to \mathcal{Q}$ such that

$$\{n \mid \sigma[\mathcal{Q}_n] = \mathcal{Q}_n\} \in \mathcal{U},$$

and, for $X, Y \subseteq \mathcal{Q}$ put

$$X \approx_{\mathcal{U}} Y \iff \text{there exists a } \mathcal{U}\text{-isometry } \sigma \text{ such that } \sigma[X] = Y.$$

Remark that the sequence $\langle |\mathcal{Q}_n| \rangle_{n\in\mathbb{N}}$ belongs to $F_{\mathcal{U}}$, since it is bounded by $\langle n^{3n^2} \rangle_{n\in\mathbb{N}}$, say. Hence one obtains the following natural extension of Theorem 5.6.

Theorem 6.1. *The relation $\approx_{\mathcal{U}}$ is an equivalence on $\mathcal{P}(\mathcal{Q})$ that satisfies the properties* (E0), (E1), (E2)*, and, when restricted to $\bigcup_{k\in\mathbb{N}} \mathcal{P}(\overline{\mathbb{Q}}^k)$, also* (E3) *and* (E4).

The map

$$\mathfrak{n}_{\mathcal{U}}(X) \mapsto [\langle |X \cap \mathcal{Q}_n| \rangle_{n\in\mathbb{N}}]_{\mathcal{U}}$$

preserves sums and is an order isomorphism between the set of "asymptotic quasi-numerosities" $\mathfrak{N}_{\mathcal{U}} = \mathcal{P}(\mathcal{Q})/\approx_{\mathcal{U}}$ and the initial segment $\mathbb{T}_{\mathcal{U}} \subseteq \mathbb{N}^{\mathbb{N}}_{\mathcal{U}}$ determined by $\mathfrak{n}_{\mathcal{U}}(\mathcal{Q}) = [\langle |\mathcal{Q}_n| \rangle_{n\in\mathbb{N}}]_{\mathcal{U}}$. $\square$

Similar results hold for point sets over any *countable* line $\mathcal{L}$ equipped with a *height function* h, provided that the corresponding function $g : \mathbb{N} \to \mathbb{N}$ defined by $g(n) = |\{x \in \mathcal{L} \mid h(x) \le n\}|$ belongs to the class $F_{\mathcal{U}}$ of Section 1. If this is not the case, one can still maintain a biunique correspondence between asymptotic equinumerosities and ultrafilters, by restricting to ultrafilters $\mathcal{U}$ with the property that every function bounded by g is $\mathcal{U}$-equivalent to a non-decreasing one (*g-quasi-selective* ultrafilters).

The question whether there exist equinumerosities which are not asymptotic with respect to suitable height functions is still open. Of particular interest might be the identification of equinumerosities whose existence is provable in ZFC alone. Actually, the notion of gauge ideal has been introduced in [14] in order to facilitate the investigation of these most general equinumerosity relations.

Acknowledgments

A. Blass has been partially supported by NSF grant DMS-0653696. M. Di Nasso and M. Forti have been supported by MIUR grant PRIN2007.

The authors are grateful to Vieri Benci for many useful discussions and suggestions.

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