



L^∞ cohomology is intersection cohomology[☆]

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Abstract

Let X be a subanalytic compact pseudomanifold. We show a de Rham theorem for L^∞ forms on the nonsingular part of X . We prove that their cohomology is isomorphic to the intersection cohomology of X in the maximal perversity.

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0. Introduction

During the three last decades, many authors studied L^p differential forms on singular varieties. The history started with Cheeger who computed the cohomology of L^2 forms on pseudomanifolds with metrically conical singularities [4]. He proved in [5] that the L^2 cohomology is actually isomorphic (for pseudomanifolds with metrically conical singularities) to intersection homology in the middle perversity (see also [8]).

Intersection homology was introduced independently by Goresky and MacPherson in [10] in order to study the topology of singular sets. Its main feature is to satisfy Poincaré duality for a large class of singularities, sufficiently general to enclose all the complex projective analytic varieties (see [10,11]).

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Cheeger's de Rham theorem thus provided a means to investigate the topology of singular sets via differential geometry. It also enabled to carry out a Hodge theory on pseudomanifolds with metrically conical singularities, which was developed by Cheeger himself in a series of works [4–7].

L^p cohomology, $p \neq 2$, turned out to be related to intersection homology as well. Let us mention some of the many related works which then appeared. In [28], Youssin computes the L^p cohomology groups of spaces with conical horns. He shows that the L^p cohomology groups are isomorphic to intersection cohomology groups in the so-called L^p perversity $1 < p < \infty$. He also describes quite explicitly the case of f -horns. The so-called f -horns are cones endowed with a metric decreasing at a rate proportional to a function f of the distance to the origin. Saper studies in [19] the L^2 cohomology for sets with isolated singularities with a distinguished Kähler metric.

In [12], the authors focus on normal algebraic complex surfaces (not necessarily metrically conical). They also show that the L^2 cohomology is dual to intersection homology (see also [20]).

In [3], the authors show, on a simplicial complex, an explicit isomorphism between the L^p shadow forms and intersection homology. The shadow forms are smooth forms constructed by the authors in a combinatorial way, like Whitney forms [27].

It is striking that, all the above mentioned de Rham theorems include an assumption on the metric type of the singularities or are devoted to low dimensional singular sets whose metric type is easier to handle. In this paper, we focus on L^∞ forms, i.e., forms having a bounded size. We prove a de Rham theorem for any compact subanalytic pseudomanifold, establishing an isomorphism between L^∞ cohomology and intersection homology in the maximal perversity (Theorem 1.2.2). We also prove that the isomorphism is provided by integration on subanalytic singular chains. The class of subanalytic pseudomanifolds covers a large class of subsets such as all the complex analytic projective varieties. Furthermore, the theory presented in this paper could go over singular subanalytic subsets which are not pseudomanifolds and we could adapt the statement to arbitrary subanalytic subsets.

This theorem, which applies to any compact subanalytic pseudomanifold, is proved by looking in details at the metric structure of subanalytic sets (see Section 2). The sharp description of the metric type of singularities obtained [22,23] will make it possible to work without any extra assumption on the metric type of the singularities.

As a consequence, we immediately see the L^∞ groups are finitely generated. The purpose is also, as in the case of L^2 cohomology, to find a category of forms for which we can carry out a Hodge theory for any compact subanalytic singular variety. Performing analysis or differential geometry on singular spaces is much more challenging than on smooth manifolds because the metric geometry of singular sets is much harder to handle.

1. Definitions and the main result

This paper deals with subanalytic sets. We recall their definition and outline their basic properties in [Appendices A](#) and [B](#) at the end of the paper.

1.1. L^∞ -cohomology groups

Before stating the main result, we need to define the L^∞ cohomology groups.

Definition 1.1.1. Let Y be a C^∞ submanifold of \mathbb{R}^n . As Y is embedded in \mathbb{R}^n , it inherits a natural structure of Riemannian manifold. We say that a j -differential form ω on Y is L^∞ if there exists

a constant C such that for any $x \in Y$:

$$|\omega(x)| \leq C,$$

where $|\omega(x)|$ denotes the norm of $\omega(x)$ (as a linear mapping). We will write d for the exterior differential operator.

We denote by $\Omega_\infty^j(Y)$ the real vector space constituted by all the differential C^∞ j -forms ω such that ω and $d\omega$ are both L^∞ .

Given $\omega \in \Omega_\infty^j(Y)$, we set $|\omega|_\infty := \sup_{x \in Y} |\omega(x)|$.

The cohomology groups of this cochain complex are called the L^∞ **cohomology groups of Y** and will be denoted by $H_\infty^\bullet(Y)$.

1.2. *Intersection homology in the maximal perversity and the main theorem*

We recall below the definition of intersection homology (see [10]). Intersection homology as defined in the latter article depends on a “perversity”. The definition below corresponds to the case of the maximal perversity $t = (0, 1, \dots, l - 2)$ (the letter t stands for “top” perversity).

As we will be interested in the only case of the maximal perversity, we specify this particular case in the definition and shall not introduce the technical notion of perversity. But this accounts for the notation $I^t C_j(X)$ (which is the notation of [10]) used below.

Given a subanalytic set X , we denote by X_{reg} the set of points of X at which X is locally a C^∞ manifold (without boundary, of any dimension) and we will write X_{sing} for the complement of X_{reg} in X .

Subanalytic singular simplices are subanalytic continuous maps $c : T_j \rightarrow X$, T_j being the oriented j -simplex spanned by $0, e_1, \dots, e_j$ where e_1, \dots, e_j is the canonical basis of \mathbb{R}^j . Given a globally subanalytic set $X \subset \mathbb{R}^n$ we denote by $C_\bullet(X)$ the resulting chain complex (with coefficients in \mathbb{R}). We will write $|c|$ for the support of a chain c and by ∂c the boundary of c .

Definition 1.2.1. An l -dimensional pseudomanifold is a globally subanalytic locally closed set $X \subset \mathbb{R}^n$ such that X_{reg} is a manifold of dimension l and $\dim X_{sing} \leq l - 2$ (see Appendix A for the definition of the dimension).

A globally subanalytic subset $Y \subset X$ is called $(t; i)$ -allowable if $\dim Y \cap X_{sing} < i - 1$. Define $I^t C_i(X)$ as the subgroup of $C_i(X)$ consisting of those chains ξ such that $|\xi|$ is (t, i) -allowable and $|\partial \xi|$ is $(t, i - 1)$ -allowable.

The j th **intersection cohomology group of maximal perversity**, denoted by $I^t H^j(X)$, is the j th cohomology group of the cochain complex $I^t C^\bullet(X) = Hom(I^t C_\bullet(X); \mathbb{R})$.

In this paper, we prove the following.

Theorem 1.2.2. *Let X be a compact subanalytic pseudomanifold. For any j :*

$$H_\infty^j(X_{reg}) \simeq I^t H^j(X).$$

This theorem is proved in Section 4. This requires to investigate in details the metric type of subanalytic singular sets. This is accomplished in Section 2.

We then briefly recall the notion of normalization of pseudomanifolds in Section 3.

We will also show that the isomorphism is given by integration on simplices (Section 4.3). Simplices are singular and lie in X (whereas forms are only defined on X_{reg}) but integration is well defined and gives rise to a cochain map if the simplices are subanalytic (see Section 4.3 for details).

Notations and conventions. We denote by $B^n(x; \varepsilon)$ the ball of radius ε centered at $x \in \mathbb{R}^n$ while $S^{n-1}(x; \varepsilon)$ will stand for the corresponding sphere. The symbol $|\cdot|$ will denote the Euclidean norm while $d(\cdot, \cdot)$ will stand for the Euclidean distance.

We denote by $k(0_+)$ the field of Puiseux series $\sum_{i \geq m} a_i T^{\frac{i}{p}}$, $p \in \mathbb{N}$, $a_i \in \mathbb{R}$, $i, m \in \mathbb{Z}$, with $\sum_{i \in \mathbb{N}} a_i t^i$ convergent for t in a neighborhood of zero (see [Appendix B](#)). We can order this field by setting $f \leq g$ in $k(0_+)$ if $f(t) \leq g(t)$ for t positive real number in a neighborhood of zero. We write T for the indeterminate. The motivation for considering this field is clarified in [Appendix B](#).

Let R stand for either \mathbb{R} or $k(0_+)$. By **Lipschitz function**, we will mean a function $f : A \rightarrow R$, $A \subset \mathbb{R}^n$, satisfying for some integer N :

$$|f(x) - f(x')| \leq N|x - x'|,$$

for all x and x' in A . It is important to notice that we require the constant to be an integer for $k(0_+)$ is not Archimedean (see again [Appendix B](#)). A map $h : A \rightarrow \mathbb{R}^m$ is Lipschitz if so are all its components; a homeomorphism h is **bi-Lipschitz** if h and h^{-1} are Lipschitz.

Given two functions $f, g : A \rightarrow R$, we write $f \sim g$ (and say that f is **equivalent** to g) if there exists a positive integer C such that $\frac{f}{C} \leq g \leq Cf$.

Given a function $\xi : A \rightarrow \mathbb{R}$, we denote by T_ξ its graph and by $\xi|_B$ its restriction to a subset B of A . Given two functions ζ and ξ on a set $A \subset \mathbb{R}^n$ with $\xi \leq \zeta$, we define the **closed interval** as the set:

$$[\xi; \zeta] := \{(x; y) \in A \times \mathbb{R} : \xi(x) \leq y \leq \zeta(x)\}.$$

The open and semi-open intervals are then defined analogously.

Given $A \subset \mathbb{R}^n$, we respectively write $cl(A)$ and $Int(A)$ for the closure and the interior of A (with respect to the Euclidean topology). We also define the (topological) **boundary** of A by $\delta A := cl(A) \setminus Int(A)$.

Convention. All the sets and mappings considered in this paper will be assumed to be globally subanalytic (if not otherwise specified), except the differential forms.

For the convenience of the reader, all the necessary definitions and basic facts of subanalytic geometry may be found in two Appendices at the end of the paper, where references of proofs are also provided.

2. Lipschitz retractions

This section provides some results about the metric geometry of globally subanalytic sets. These results will be very important to compute the L^∞ cohomology groups later on. Given a germ of subanalytic set X at x_0 , we shall construct a Lipschitz strong deformation retraction r_t , $t \in [0, 1]$, $r_0 \equiv x_0$, $r_1 = Id$, of this germ onto x_0 ([Theorem 2.3.1](#)). It is the main result of this section.

By way of motivation for all the results of this section, let us briefly outline the strategy of the proof of [Theorem 2.3.1](#). Let X be the germ of a singular (subanalytic) set. Replacing X by \hat{X} (see (2.7) for \hat{X}), we may estimate the distance to the origin by the first coordinate. We proceed by induction on n , if $X \subset \mathbb{R}^n$. The result is therefore true for $\pi_n(X)$ if $\pi_n : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ is the canonical projection. By [Corollary 2.1.4](#) and [Lemma 2.1.5](#), up to a bi-Lipschitz homeomorphism preserving the first coordinate, we know that \hat{X} may be included in the graphs of finitely many Lipschitz functions. We thus can lift the retraction obtained by induction, making use of the estimates of [Lemma 2.2.3](#) so as to establish its Lipschitz character.

The techniques of this section, especially [Theorem 2.3.1](#), can have other applications (see [\[21\]](#)). We start by recalling some results of [\[22,23\]](#).

Given $n > 1$ and a positive constant M we set:

$$\mathcal{C}_n(M) := \{(x_1; x') \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 \leq |x'| \leq Mx_1\}.$$

For $n = 1$, we just set $\mathcal{C}_1(M) = \mathbb{R}$.

2.1. Regular vectors

In the definition below, R stands for either $k(0_+)$ or \mathbb{R} .

Definition 2.1.1. Let X be a subset of \mathbb{R}^n . An element λ of S^{n-1} is said to be **regular for X** if there is a positive real number α such that:

$$d(\lambda; T_x X_{reg}) \geq \alpha,$$

for any x in X_{reg} .

Recall that the order relation in $k(0_+)$ was defined by comparing the series on a right-hand-side neighborhood of zero. Therefore, in the above definition, the inequality means in the case $R = k(0_+)$ that for $x \in X_{reg}$ the limit at zero of the Puiseux series $d(\lambda; T_x X_{reg})$ cannot be smaller than $\alpha > 0$. It is important to notice that α is required to be a positive *real number* and not a Puiseux series: it implies that the Puiseux series $d(\lambda; T_x X_{reg})$ may not tend to zero at zero.

Regular vectors do not always exist, as it is shown by the simple example of a circle. Nevertheless, we can get a regular vector without affecting the metric type of a subanalytic set.

Theorem 2.1.2 ([\[22\]](#)). *Let X be a subset of $k(0_+)^n$ of empty interior. Then there exists a bi-Lipschitz homeomorphism $h : k(0_+)^n \rightarrow k(0_+)^n$ such that e_n is regular for $h(X)$.*

For instance, if X is the circle (in $k(0_+)^2$) defined by $x^2 + y^2 = 1$ then the provided bi-Lipschitz homeomorphism may send X onto a triangle (in $k(0_+)^2$). We see (intuitively at least) that it is not possible to require $h(X)$ to be a smooth manifold even if so is X . Such a mapping h is the generic fiber (see [Appendix B](#)) of a family of homeomorphisms sending the cylinder $(0, \varepsilon) \times C$, where ε is a positive real number and C denotes the unit circle in \mathbb{R}^2 , onto the product of a triangle with the interval $(0, \varepsilon)$. The situation gets more difficult when X is singular since it may have many different limits of tangent spaces at a singular point.

Definition 2.1.3. A map $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is x_1 -**preserving** if it preserves the first coordinate in the canonical basis of \mathbb{R}^n .

It is shown in [\[23\]](#) that, if the considered subset lies in $\mathcal{C}_n(M)$, then the homeomorphism of [Theorem 2.1.2](#) may be chosen x_1 -preserving. In [\[23\]](#), the result was for semialgebraic sets. Below, we prove it in the subanalytic framework.

In the proof below, we consider subsets of \mathbb{R}^n as families of subsets of \mathbb{R}^{n-1} parameterized by the first coordinate. Given $t \in \mathbb{R}$, we write X_t for the set of points of X having their first coordinate equal to t .

Corollary 2.1.4. *Let X be the germ at 0 of a subset of $\mathcal{C}_n(M)$ of empty interior, $M > 0$. There exists a germ of x_1 -preserving bi-Lipschitz homeomorphism (onto its image) $h : \mathcal{C}_n(M) \rightarrow \mathcal{C}_n(M)$ such that e_n is regular for $h(X)$.*

Proof. Apply [Theorem 2.1.2](#) to the generic fiber:

$$X_{0_+} := \{x : (T; x) \in X_{k(0_+)}\},$$

where $X_{k(0_+)}$ denotes the extension of the set X to $k(0_+)$ (see [Appendix B](#)). This provides a bi-Lipschitz homeomorphism $H : k(0_+)^{n-1} \rightarrow k(0_+)^{n-1}$ which immediately gives rise (via the so-called transfer principle, see again [Appendix B](#)) to a x_1 -preserving bi-Lipschitz homeomorphism $h : (0; \varepsilon) \times \mathbb{R}^{n-1} \rightarrow (0; \varepsilon) \times \mathbb{R}^{n-1}$, $(t, x) \mapsto (t, h_t(x))$, with h_t bi-Lipschitz (with the same constant as H) for every $t < \varepsilon$ and such that there is a real number $\alpha > 0$ such that

$$d(e_n, T_x h(X_t)) \geq \alpha, \tag{2.1}$$

for any $x \in h(X_t)_{reg}$ and t positive small enough. Up to a translation, we may assume that $h_t(0) \equiv 0$ so that h maps $\mathcal{C}_n(M)$ into $\mathcal{C}_n(M')$, for some M' . Up to a x_1 -preserving linear mapping, we may assume $M = M'$.

We now check that e_n is regular for the germ of $Y := h(X)$. Suppose not. It means that the element $(0, e_n)$ belongs to the closure of the set:

$$\{(x, u) : x \in Y_{reg} \text{ and } u \in T_x Y_{reg}\}.$$

As a matter of fact, by curve selection lemma (see [Appendix A](#)), there exists an analytic arc $\gamma : [0; \varepsilon] \rightarrow Y_{reg}$ with $\gamma(0) = 0$ and $e_n \in \tau := \lim_{t \rightarrow 0} T_{\gamma(t)} Y_{reg}$. On the other hand, by (2.1), we have $e_n \notin \lim_{t \rightarrow 0} T_{\gamma(t)} Y_{\gamma_1(t)}$. This implies that

$$\tau \cap \langle e_1 \rangle^\perp \neq \lim_{t \rightarrow 0} (T_{\gamma(t)} Y_{reg} \cap \langle e_1 \rangle^\perp),$$

and consequently τ may not be transverse to $\langle e_1 \rangle^\perp$ (since otherwise the intersection with the limit would be the limit of the intersection), which means that $\tau \subseteq \langle e_1 \rangle^\perp$. This implies that the limit vector $\lim_{t \rightarrow 0} \frac{\gamma(t)}{|\gamma(t)|} = \lim_{t \rightarrow 0} \frac{\gamma'(t)}{|\gamma'(t)|} \in \tau$ is orthogonal to e_1 . Therefore,

$$\lim_{t \rightarrow 0} \frac{\gamma_1(t)}{|\gamma(t)|} = 0,$$

in contradiction with $\gamma(t) \in \mathcal{C}_n(M)$.

Let us now show that h is also Lipschitz with respect to the parameter x_1 . Suppose that the germ of h fails to be Lipschitz. In this case, the element $(0, 0, 0)$ belongs to the closure of the set germ:

$$\left\{ (p, q, z) : p \in \mathcal{C}_n(M), q \in \mathcal{C}_n(M), p \neq q, z = \frac{|p - q|}{|h(p) - h(q)|} \right\}.$$

Then, by curve selection lemma (see [Appendix A](#)), we can find two analytic arcs in $\mathcal{C}_n(M)$, say $p(t)$ and $q(t)$, tending to zero and along which:

$$|p(t) - q(t)| \ll |h(p(t)) - h(q(t))|. \tag{2.2}$$

Recall that h preserves the fibers of π_1 , the projection onto the first coordinate. We may assume that $p(t)$ (and thus $h(p(t))$ too) is parameterized by its x_1 -coordinate, i.e., we may assume $\pi_1(p(t)) = t$, $t > 0$ small ($f(t) := \pi_1(p(t))$ being a real analytic function, it induces a homeomorphism in a right-hand-side neighborhood of the origin whose inverse f^{-1} is a Puiseux series). As $p(t)$ and $h(p(t))$ are Puiseux arcs in $\mathcal{C}_n(M)$ we have:

$$|h(p(t)) - h(p(t'))| \sim |t - t'| \tag{2.3}$$

and

$$|p(t) - p(t')| \sim |t - t'| \leq |p(t) - q(t)|, \tag{2.4}$$

where t' denotes the first coordinate of $q(t)$.

Therefore, by (2.2)–(2.4) we have for some constant $C \in \mathbb{R}$:

$$|h(p(t)) - h(q(t))| \sim |h(p(t')) - h(q(t))| \sim |p(t') - q(t)| \leq C|p(t) - q(t)|,$$

a contradiction. Arguing in the same way on h^{-1} , we could show that h is bi-Lipschitz. \square

There is a close interplay between Lipschitz functions and regular vectors.

Lemma 2.1.5. *Assume that e_n is regular for a set $X \subset \mathbb{R}^n$. Then X is contained in the union of the respective graphs of some Lipschitz functions $\xi_i : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, $i = 1, \dots, k$.*

Proof. Take a cell decomposition compatible with X . Since e_n is regular for X , the set X is the union of some cells which are graphs (not bands, see Definition A.2.1) of some analytic functions $\eta_i : D_i \rightarrow \mathbb{R}$, $i = 1, \dots, k$, where $D_i \subset \mathbb{R}^{n-1}$. Observe that, because e_n is regular for their graph, the η_i 's have bounded derivatives.

By Theorem 1.2 of [13], there is a finite partition of every D_i into analytic manifolds, say $D_{i,1}, \dots, D_{i,m_i}$, and a constant M such that any given two points x and y in the same $D_{i,j}$ may be joint by an arc whose length does not exceed $M|x - y|$. This implies that any given smooth function $f : D_{i,j} \rightarrow \mathbb{R}$, $j \leq m_i$, which has bounded derivatives is Lipschitz. In particular, η_i induces a Lipschitz function on every $D_{i,j}$, say $\eta_{i,j}$.

Now, the lemma follows from the fact that we can extend each $\eta_{i,j} : D_{i,j} \rightarrow \mathbb{R}$ to a Lipschitz function $\xi_{i,j} : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ by setting:

$$\xi_{i,j}(x) := \inf\{\eta_{i,j}(y) + L_{i,j}|x - y| : y \in D_{i,j}\},$$

where $L_{i,j}$ denotes the Lipschitz constant of $\eta_{i,j}$. \square

2.2. Some preliminaries

Before constructing the desired retraction, we need to put the set in a nice position. For this purpose, we will need yet another result whose proof may be found in [22] as well (Proposition 2.2.1 below). It is a consequence of the preparation theorem [16,14,25].

Basically, this proposition says that distance functions (i.e. functions of type $x \mapsto d(x, W)$, $W \subset \mathbb{R}^n$) may be used as a “basis of valuations”, in the sense that every (globally subanalytic) nonnegative function may be compared (up to constants) to a product of powers of distance functions (after a partition).

We recall that, except the differential forms, all the sets and functions of this paper are assumed to be globally subanalytic.

Proposition 2.2.1. *Let $X \subset \mathbb{R}^n$ and let $\xi : X \rightarrow \mathbb{R}$ be a nonnegative function. There exists a finite partition of X such that over each element of this partition the function ξ is \sim to a product of powers of distances to subsets of X .*

The powers involved in the above proposition are always rational numbers.

Remark 2.2.2. We now would like to formulate two observations that will be useful in the proof of the next lemma.

- (1) If X is the union of the graphs of finitely many Lipschitz functions ξ_1, \dots, ξ_k over \mathbb{R}^n then, using the operators \min and \max , we may find an ordered family of Lipschitz functions $\theta_1 \leq \dots \leq \theta_k$ such that X is the union of the graphs of these functions.
- (2) Given a family of Lipschitz functions f_1, \dots, f_k defined over \mathbb{R}^{n-1} , we can find some Lipschitz functions $\xi_1 \leq \dots \leq \xi_m$ on \mathbb{R}^{n-1} and a cell decomposition \mathcal{D} of \mathbb{R}^{n-1} such that over each $[\xi_i|_D; \xi_{i+1}|_D]$, where $D \in \mathcal{D}$, the family of functions

$$|y - f_1(x)|, \dots, |y - f_k(x)|, f_1(x), \dots, f_k(x),$$

(for $(x; y) \in [\xi_i|_D; \xi_{i+1}|_D]$) is totally ordered (for relation \leq). Indeed, it suffices to choose a cell decomposition \mathcal{D} of \mathbb{R}^{n-1} compatible with the sets $f_i = f_j$ and to apply (1) to the functions $f_i, (f_i - f_j), (f_i + f_j)$, and $\frac{f_i + f_j}{2}, i \leq k, j \leq k$.

The lemma below somehow combines Corollary 2.1.4 and Proposition 2.2.1 in a single statement. We denote by $\pi_n : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ the orthogonal projection onto \mathbb{R}^{n-1} .

Lemma 2.2.3. *Given some germs $X_1, \dots, X_s \subset C_n(M)$ at 0, there exist a germ of x_1 -preserving bi-Lipschitz homeomorphism (onto its image) $h : C_n(M) \rightarrow C_n(M)$ and a cell decomposition \mathcal{E} of \mathbb{R}^n such that for some representatives of the germs:*

- (1) \mathcal{E} is compatible with $h(X_1), \dots, h(X_s)$ and $h(C_n(M))$.
- (2) e_n is regular for any cell of \mathcal{E} which is a graph (not a band, see Definition A.2.1).
- (3) Given finitely many nonnegative functions ξ_1, \dots, ξ_m on $C_n(M)$, we may assume that on each cell $E \subset h(C_n(M))$ of \mathcal{E} , each function $\xi_i \circ h^{-1}$ is \sim to a function of the form:

$$|y - \theta(x)|^r a(x) \tag{2.5}$$

(for $(x; y) \in \mathbb{R}^{n-1} \times \mathbb{R}$) where $a, \theta : \pi_n(E) \rightarrow \mathbb{R}$ are functions with θ Lipschitz, $r \in \mathbb{Q}$.

Proof. It will be convenient to complete the family X_1, \dots, X_s by setting $X_{s+1} := C_n(M)$. Apply Proposition 2.2.1 to the functions $\xi_j, j = 1, \dots, m$. This provides a partition E_1, \dots, E_b of $C_n(M)$ together with some subsets of $C_n(M)$, say W_1, \dots, W_c , such that on each $E_i, i \leq b$, each function $\xi_j, j \leq m$, is equivalent to a product of powers of functions of type $q \mapsto d(q; W_k), k \leq c$.

Possibly refining the partition E_i , we may assume that the W_k 's are unions of some elements of this partition (thanks to existence of cell decompositions, see Appendix A). Hence, on every E_i , if $d(x, W_k)$ is not identically zero, then it is nowhere zero and $d(x, W_k)$ is equivalent to $d(x, \delta W_k)$. Therefore, we may assume that the W_k 's have empty interior, possibly replacing them with their boundaries (if a function ξ_j is identically zero on E_i then (3) is trivial on E_i).

Apply now Corollary 2.1.4 to the union of the δX_i 's, the δE_i 's, and the W_k 's. This provides a germ of x_1 -preserving bi-Lipschitz homeomorphism $h : C_n(M) \rightarrow C_n(M)$ which maps the latter subsets into the union of the graphs of some Lipschitz functions $\theta_1, \dots, \theta_d$.

By Remark 2.2.2(2) applied to the family of functions constituted by the θ_i 's together with all the $(n - 1)$ -variable functions $x \mapsto d(x; \pi_n(W_k \cap \Gamma_{\theta_v})), v \leq d, k \leq c$, we know that there exist a finite number of functions $\eta_1 \leq \dots \leq \eta_p$ and a cell decomposition \mathcal{D} of \mathbb{R}^{n-1} such that for every $D \in \mathcal{D}$, over each $[\eta_i|_D; \eta_{i+1}|_D], i < p$, the family constituted by all the n -variable functions $|y - \theta_v(x)|, v \leq d$, together with the functions

$$x \mapsto d(x; \pi_n(W_k \cap \Gamma_{\theta_v})), \quad v \leq d, k \leq c$$

is totally ordered (for order relation \leq , considering the latter functions as n -variable functions). By (1) of Remark 2.2.2, we can find a totally ordered finite family $\sigma_1 \leq \dots \leq \sigma_\mu$ such that $\cup_{i=1}^\mu \Gamma_{\sigma_i}$ contains both the graphs of the θ_i 's and the graphs of the η_i 's.

Consider a cell decomposition \mathcal{D}' of \mathbb{R}^n compatible with the cells of \mathcal{D} , the sets defined by all the equations $\sigma_j = \sigma_i, i \leq \mu, j \leq \mu$, as well as all the sets $h(X_j) \cap \Gamma_{\sigma_i}, j \leq s + 1, i \leq \mu$. The graphs of the respective restrictions of the functions $\sigma_1, \dots, \sigma_\mu$, to the sets $\pi_n(E), E \in \mathcal{D}'$, define a cell decomposition \mathcal{E} of \mathbb{R}^n .

For a proof of (1), take a cell $E \in \mathcal{E}, E \subset C_n(M)$. If E is a graph (not a band) then (1) for E follows from the fact that \mathcal{D}' is compatible with the $h(X_j) \cap \Gamma_{\sigma_i}$'s. Assume thus that E is a band, say $(\sigma_{i|D}, \sigma_{i+1|D})$ where $i < \mu, D \subset \mathbb{R}^{n-1}$. As $\delta h(X_j) \subset \cup_{k=1}^\mu \Gamma_{\sigma_k}$, for all j , the set $E \cap h(X_j)$ is open and closed in E . Hence, if $E \cap h(X_j)$ is nonempty it is equal to E (E is connected). This yields (1).

Observe that e_n is regular for any cell of \mathcal{E} which is a graph, since the σ_i 's are Lipschitz functions. This already proves that (2) holds.

To prove (3), fix a cell $E \subset h(C_n(M))$ of \mathcal{E} which is a band, say $(\sigma_{k|D}, \sigma_{k+1|D})$ where $k < \mu, D \subset \mathbb{R}^{n-1}$ ((3) is trivial if E is a graph). We first check that E is included in $h(E_i)$, for some i . As $\delta h(E_i) \subset \cup_{k=1}^\mu \Gamma_{\sigma_k}$, for each i , the set $E \cap h(E_i)$ is open and closed in E . Hence, if $E \cap h(E_i)$ is nonempty it is equal to E . As $h(E_1), \dots, h(E_b)$ constitute a partition of $h(C_n(M))$, this shows that $E \subset h(E_i)$, for some i .

Consequently, as h is bi-Lipschitz, each $\xi_j \circ h^{-1}$ is equivalent to a product of powers of functions of type $q \mapsto d(q; h(W_i)), i \leq c$. It is thus enough to show (2.5) for these latter functions.

As the θ_v 's are Lipschitz functions, we have for any $v \in \{1, \dots, d\}$:

$$d(q; h(W_i) \cap \Gamma_{\theta_v}) \sim |y - \theta_v(x)| + d(x; \pi_n(h(W_i) \cap \Gamma_{\theta_v})) \tag{2.6}$$

where $q = (x; y)$ in $\mathbb{R}^{n-1} \times \mathbb{R}$.

By construction $E \subset [\eta_{k,|A}, \eta_{k+1,|A}]$, for some $k < p$ and $A \subset \mathbb{R}^{n-1}$. As a matter of fact, for every i , the terms of the right-hand-side are comparable with each other (for partial order relation \leq) over the cell E . Therefore, the left-hand-side is \sim to one of them on E .

Note that, as each $h(W_i)$ is included in the union of the graphs of the θ_v 's, we have:

$$d(q; h(W_i)) = \min_{1 \leq v \leq d} d(q; h(W_i) \cap \Gamma_{\theta_v}).$$

The latter family of functions is totally ordered over E . Hence, by (2.6), each function $d(q; h(W_i))$ is equivalent over E either to one of the functions $x \mapsto d(x; \pi_n(h(W_i) \cap \Gamma_{\theta_v}))$, or to some function $(x; y) \mapsto |y - \theta_v(x)|, v \in \{1, \dots, d\}$. Thus, (3) holds. \square

2.3. Lipschitz retractions of subanalytic germs

We are now ready to construct the desired strong deformation retraction. Given $X \subset \mathbb{R}^n$ we define:

$$\hat{X} := \{(y; x) \in \mathbb{R} \times X : |x| = y\}. \tag{2.7}$$

Observe that \hat{X} is a subset of $C_{n+1}(1)$.

In the theorem below we write $d_x r_t$ for the derivative of r_t which exists for x generic although r_t is not smooth since, like all the mappings in this paper, r is implicitly assumed to be subanalytic and thus smooth on a (subanalytic) dense subset.

By **Lipschitz deformation retraction onto** x_0 , we mean a Lipschitz family of maps r_t with $r_0(x) \equiv x_0$ and $r_1(x) \equiv x$.

Theorem 2.3.1. *Let $X \subset \mathbb{R}^n$ be locally closed and let $x_0 \in X$. Then, for any $\varepsilon > 0$ small enough there exists a Lipschitz deformation retraction*

$$r : X \cap B^n(x_0; \varepsilon) \times [0; 1] \rightarrow X \cap B^n(x_0; \varepsilon), \quad (x, t) \mapsto r_t(x),$$

onto x_0 , preserving X_{reg} for $t > 0$.

Furthermore, the derivative $d_x r_t$ tends to 0 as $t \rightarrow 0$ for any x generic in X_{reg} .

Proof. We will assume for simplicity that $x_0 = 0$. We will actually prove by induction on n the following statements.

(A_n) Let X_1, \dots, X_s be finitely many subsets of $\mathcal{C}_n(M)$ and let ξ_1, \dots, ξ_m be bounded functions on $\mathcal{C}_n(M)$, with $M > 0$. There exists $\varepsilon > 0$ such that if we set $U_\varepsilon := \{x \in \mathcal{C}_n(M) : 0 \leq x_1 < \varepsilon\}$, there is a Lipschitz strong deformation retraction of U_ε

$$r : U_\varepsilon \times [0; 1] \rightarrow U_\varepsilon, \quad (x, t) \mapsto r_t(x),$$

onto 0 such that for any $j \leq s$:

- (1) r_t preserves $X_j \cap U_\varepsilon$ for $t \in (0; 1]$.
- (2) $d_x r_t$ goes to zero as t tends to 0 for any x generic in $X_j \cap U_\varepsilon$.
- (3) There is a constant C such that for any i and any $0 < t \leq 1$ we have for all $x \in U_\varepsilon$:

$$\xi_i(r_t(x)) \leq C \xi_i(x). \tag{2.8}$$

Before proving these statements, let us make it clear that this implies the desired result. If $X \subset \mathbb{R}^n$ then \hat{X} (see (2.7) for \hat{X}) is a subset of $\mathcal{C}_{n+1}(1)$ to which we can apply (A_{n+1}). Then, as \hat{X} is bi-Lipschitz equivalent to X , the result immediately ensues. Thanks to (1), we may assume that the retraction preserves X_{reg} .

As the theorem obviously holds in the case where $n = 1$ (with $r_t(x) = tx$), we fix some $n > 1$. We also fix some subsets X_1, \dots, X_s of $\mathcal{C}_n(M)$, for $M > 0$, and some bounded functions $\xi_1, \dots, \xi_m : \mathcal{C}_n(M) \rightarrow \mathbb{R}$.

Before defining the desired map, we need some preliminaries: we first construct a family of bounded $(n - 1)$ -variable functions $\sigma_1, \dots, \sigma_p$ to which we will apply (3) of (A_{n-1}).

Apply Lemma 2.2.3 to the family constituted by the germs of the X_i 's and the zero loci of the ξ_i 's. We get a x_1 -preserving bi-Lipschitz map $h : \mathcal{C}_n(M) \rightarrow \mathcal{C}_n(M)$ and a cell decomposition \mathcal{E} such that (1) and (2) of the latter lemma hold. Moreover, thanks to (3) of the latter lemma, we may also assume that the ξ_i 's are like in (2.5) on every cell. As we may work up to a x_1 -preserving bi-Lipschitz map we will identify h with the identity map.

By Lemma 2.1.5, the union of the cells of \mathcal{E} for which e_n is regular may be included in the union of the graphs of finitely many Lipschitz functions η_1, \dots, η_v . Moreover, by Remark 2.2.2(1), we can assume $\eta_1 \leq \dots \leq \eta_v$.

In order to define the desired functions $\sigma_1, \dots, \sigma_p$, let us fix a cell A of \mathcal{E} , and set $A' := \pi_n(A)$, $\pi_n : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ denoting the projection onto the $(n - 1)$ first coordinates. Choose then $j < v$ and set $D := (\eta_j|_{A'}; \eta_{j+1}|_{A'})$. By construction, D is included in a cell of \mathcal{E} .

Since the ξ_k 's are like in (2.5) on D , for every $k = 1, \dots, m$, there exist some $(n - 1)$ -variable functions on A' , say θ_k and a_k , such that for $(x; y) \in D \subset \mathbb{R}^{n-1} \times \mathbb{R}$:

$$\xi_k(x; y) \sim |y - \theta_k(x)|^{\alpha_k} a_k(x), \tag{2.9}$$

where α_k is a rational number (possibly negative).

As \mathcal{E} is compatible with the zero loci of the ξ_k 's, we have on A' : if ξ_k is not identically zero on D then either $\theta_k \leq \eta_j$ or $\theta_k \geq \eta_{j+1}$. Fix k with $\xi_k \neq 0$ on D . We will assume for simplicity that $\theta_k \leq \eta_j$.

It means that on D :

$$\xi_k(x; y) \sim \min((y - \eta_j(x))^{\alpha_k} a_k(x); (\eta_j(x) - \theta_k(x))^{\alpha_k} a_k(x)), \tag{2.10}$$

if α_k is negative, and

$$\xi_k(x; y) \sim \max((y - \eta_j(x))^{\alpha_k} a_k(x); (\eta_j(x) - \theta_k(x))^{\alpha_k} a_k(x)), \tag{2.11}$$

in the case where α_k is nonnegative.

We are now ready to define the desired family $\sigma_1, \dots, \sigma_p$ of $(n - 1)$ -variable functions. We first set for $\xi_k \neq 0$ on D :

$$\kappa_k(x) := |\eta_j(x) - \theta_k(x)|^{\alpha_k} a_k(x). \tag{2.12}$$

Since ξ_k is bounded, by (2.5), this defines a bounded function. Complete the family κ by adding the functions $\min(f; 1)$ where f describes all the $(\eta_{j+1} - \eta_j)a_k$'s.

Doing this for all the cells $A \in \mathcal{E}$ and integers $j < v$, and collecting all the respective families κ obtained in this way, we eventually get a family of bounded functions $\sigma_1, \dots, \sigma_p$.

We now turn to the construction of the desired retraction. Consider a cylindrical cell decomposition \mathcal{D} compatible with the cells of \mathcal{E} and the graphs of the η_j 's. Apply the induction hypothesis to the family of sets $\pi_n(D) \cap C_{n-1}(M)$, $D \in \mathcal{D}$. This provides a deformation retraction $r : V_\varepsilon \times [0; 1] \rightarrow V_\varepsilon$, $\varepsilon > 0$, where $V_\varepsilon := \{x \in C_{n-1}(M) : 0 \leq x_1 < \varepsilon\}$.

We are going to lift r to a retraction of $[\eta_1|_{V_\varepsilon}, \eta_v|_{V_\varepsilon}]$. Thanks to the induction hypothesis, we may assume that the functions $\sigma_1, \dots, \sigma_p$, as well as the $(\eta_{j+1} - \eta_j)$'s and the functions $x \mapsto \xi_i(x; \eta_j(x))$ satisfy (2.8).

Now, we may lift r as follows. On $(\eta_j; \eta_{j+1})$, $j = 1, \dots, v - 1$, we set

$$v(q) := \frac{y - \eta_j(x)}{\eta_{j+1}(x) - \eta_j(x)},$$

if $q = (x; y) \in (\eta_j; \eta_{j+1}) \subset \mathbb{R}^{n-1} \times \mathbb{R}$, and then

$$\tilde{r}_t(q) := (r_t(x); v(q)(\eta_{j+1}(r_t(x)) - \eta_j(r_t(x))) + \eta_j(r_t(x))).$$

This mapping is then easily extended continuously on each Γ_{η_j} by setting if $q = (x; \eta_j(x))$:

$$\tilde{r}_t(q) := (r_t(x); \eta_j(r_t(x))).$$

For any j , the mapping \tilde{r} maps linearly the segment $[\eta_j(x); \eta_{j+1}(x)]$ onto the segment $[\eta_j(r_t(x)); \eta_{j+1}(r_t(x))]$. Thanks to the induction hypothesis, the inequality (2.8) is fulfilled by the function $(\eta_{j+1} - \eta_j)$. Therefore, as r is Lipschitz, we see that \tilde{r} is Lipschitz as well. As \tilde{r} preserves the cells of \mathcal{E} , it preserves the X_j 's, the zero loci of the ξ_k 's, and U_ε .

We have to check that the ξ_k 's fulfill (2.8) along the trajectories of \tilde{r} . We check it on a given cell E of \mathcal{D} . If $E \subset \Gamma_{\eta_j}$ for some j , this follows from the induction hypothesis since we have assumed that the functions $x \mapsto \xi_k(x; \eta_j(x))$ satisfy (2.8) on E .

Otherwise, there exists j such that E sits in $(\eta_j; \eta_{j+1})$. Fix an integer $1 \leq k \leq m$. On the cell E , the function ξ_k may be estimated as in (2.9). By the induction hypothesis we know that κ_k (see (2.12), if $\xi_k = 0$ on E then (2.8) is trivial for ξ_k) satisfies (2.8).

If a function ξ is bounded and if $\min(\xi; 1)$ satisfies (2.8) then ξ satisfies this inequality as well. We will therefore check (2.8) for $\min(\xi_k; 1)$.

Observe also that if two given functions ξ and ζ both satisfy (2.8) then $\min(\xi; \zeta)$ and $\max(\xi; \zeta)$ both satisfy this inequality as well. Hence, by (2.10) and (2.11), it is enough to show that the functions $\min((y - \eta_j(x))^{\alpha_k} a_k(x); 1)$ and the functions $|\theta_k - \eta_j|^{\alpha_k} a_k$ satisfy (2.8). The latter functions are nothing but the κ_k 's for which we already have seen that this inequality is true. Let us focus on the former functions.

For simplicity we set

$$F(x; y) := (y - \eta_j(x))^{\alpha_k} a_k(x)$$

and

$$G(x) := (\eta_{j+1} - \eta_j)(x)^{\alpha_k} a_k(x).$$

We have to show the desired inequality for $\min(F; 1)$. We have:

$$F(x; y) = v(q)^{\alpha_k} \cdot G(x). \tag{2.13}$$

Remark that the function $v(\tilde{r}_t(q))$ is constant with respect to t . This implies that:

$$F(\tilde{r}_t(q)) = v(q)^{\alpha_k} \cdot G(r_t(x)). \tag{2.14}$$

We assume first that α_k is negative. Thanks to the induction hypothesis ($\min(G; 1)$ is one of the σ_i 's) we know that for some constant C we have for all x in $\pi_n(E)$:

$$\min(G(r_t(x)); 1) \leq C \min(G(x); 1).$$

This implies (multiplying by v^{α_k} and applying (2.13) and (2.14)) that for $q \in E$:

$$\min(F(\tilde{r}_t(q)); v^{\alpha_k}(q); 1) \leq C \min(F(q); v^{\alpha_k}(q); 1).$$

But, as α_k is negative, $\min(F; v^{\alpha_k}; 1) = \min(F; 1)$, which yields the desired inequality for $\min(F; 1)$, as required.

We now assume that α_k is nonnegative. Thanks to (2.13) and (2.14), it actually suffices to show the desired inequality for G . But, as ξ_k is bounded, by (2.11) so is G , and the result follows from the induction hypothesis since $\min(G; 1)$ is one of the σ_i 's (as G is bounded and $\min(G, 1)$ satisfies (2.8) then G satisfies this inequality as well). This yields (2.8) along the trajectories of \tilde{r} .

We now check that $d_q \tilde{r}_t$ tends to zero when t goes to zero. It follows from the induction hypothesis that $d_x r_t$ goes to zero as t goes to zero, for any x generic. As the η_i 's have bounded derivatives, this already proves for almost every x :

$$\lim_{t \rightarrow 0} d_x [(\eta_i - \eta_{i+1}) \circ r_t] = 0. \tag{2.15}$$

On the other hand, a straightforward computation shows that for $q = (x; y)$:

$$|d_{\tilde{r}_t(q)} v| \leq \frac{C}{|\eta_i(x) - \eta_{i+1}(x)|},$$

which, together with (2.15) and (2.8) for $(\eta_{i+1} - \eta_i)$, implies that $d_q \tilde{r}_t$ tends to zero as t goes to zero. \square

Remark 2.3.2. The mapping r_t could be proved to be bi-Lipschitz for every $t > 0$. The Lipschitz constant of r_t^{-1} may of course tend to infinity as t goes to zero. We nevertheless could have a control on the way distances are contracted by r_t , similarly as in [22,23]. We could show that for

a suitable basis of unit 1-forms $\theta_1, \dots, \theta_n$ and some functions $\varphi_1, \dots, \varphi_n$ on \mathbb{R}^n such that almost everywhere on $X_{reg} \cap B^n(x_0; \varepsilon) \times [0, 1]$:

$$r_t^* \rho(x) \approx \sum_{i=1}^n \varphi_i(x; t)^2 \theta_i^2(x),$$

where ρ is the metric of \mathbb{R}^n and $r_t^* \rho$ its pull-back. Similarly as in [22], the functions φ_i , which are the contractions of the metric that r operates, could be expressed as powers, products, and sums of distance functions in X (i.e. $x \mapsto d(x; W)$ with $W \subset X \cap B^n(0; \varepsilon)$) and the function $(x; t) \mapsto t$. These powers may be negative which makes it difficult to get decreasing functions and accounts for the difficulty we have in the proof of the above theorem.

3. Normal pseudomanifolds

In this section, we shall also deal with *topological* pseudomanifolds. Given $X \subset \mathbb{R}^n$, denote by X_{reg}^0 the set of points of X near which X is a C^0 -manifold (of any dimension). We say that a locally closed set X is an l -dimensional **topological pseudomanifold** if $\dim X \setminus X_{reg}^0 < l - 1$ and if X_{reg}^0 is an l -dimensional manifold.

Definition 3.0.3. An l -dimensional topological pseudomanifold X is called **normal** if for any x in X , $\dim H_l(X; X \setminus \{x\}) = 1$.

We shall recall some basic facts about normal pseudomanifolds. These may be found in [10] (Section 4) and make normalizations very useful to investigate intersection homology in the maximal perversity. Observe that if $X \subset \mathbb{R}^n$ is a normal topological pseudomanifold which is connected then $H_l(X) = \mathbb{R}$, since if there were two generators, say σ and τ , $\dim |\sigma| \cap |\tau| < l$, we would have $H_l(X; X \setminus x) \neq \mathbb{R}$ at any point of the intersection of the supports.

The main interest of normal spaces lies in the following lemma. Denote by $L(x; X_{reg})$ the set $S^{n-1}(x; \varepsilon) \cap X_{reg}$. It is well known that the topology of $L(x; X_{reg})$ is independent of $\varepsilon > 0$ small enough.

Lemma 3.0.4 ([10]). *A topological pseudomanifold $X \subset \mathbb{R}^n$ is normal if and only if $L(x; X_{reg}^0)$ is connected at any point of $X \setminus X_{reg}^0$.*

See for instance [10] Section 4 for a proof. The very significant advantage of normal pseudomanifolds lies in the following proposition.

Proposition 3.0.5 ([10]). *Let X be a normal topological pseudomanifold. The mapping $\alpha : I^l H_j(X) \rightarrow H_j(X)$, induced by the inclusion between the chain complexes, is an isomorphism for all j .*

3.1. Normalizations of pseudomanifolds

We shall need some basic facts about normalizations.

Definition 3.1.1. A **normalization** of the topological pseudomanifold X is a normal topological pseudomanifold \tilde{X} together with a finite-to-one continuous mapping $\pi : \tilde{X} \rightarrow X$ such that, for any p in X ,

$$\pi_* : \bigoplus_{q \in \pi^{-1}(p)} H_l(\tilde{X}; \tilde{X} \setminus q) \rightarrow H_l(X; X \setminus p)$$

is an isomorphism.

We are going to see that normalizations are useful to compute intersection homology in the maximal perversity.

Proposition 3.1.2. *Every pseudomanifold X admits a normalization $\pi : \tilde{X} \rightarrow X$. The mapping π then induces a homeomorphism above the regular locus of X .*

Proof. We follow the construction of [10]. Consider a triangulation of X (since X is globally subanalytic, it admits a C^0 triangulation, see Appendix A), i.e., a homeomorphism $T : K \rightarrow X$, with K finite union of open simplices.

Let L be the disjoint union of all the closures in K of the l -dimensional open simplices of K (where l is the dimension of X). Identify the closure in K of two $(l - 1)$ open faces of two elements of L if these two faces coincide in K . This provides a simplicial complex \tilde{X} . Denote then by $\pi : \tilde{X} \rightarrow X$ the map induced by T .

Observe that by construction the mapping π is a homeomorphism on the complement in X of the $(l - 2)$ -skeleton. It is thus easily checked from the definition that the mapping π induces a homeomorphism above X_{reg} and that $L(x, \tilde{X}_{reg}^0)$ is connected at singular points. \square

Remark 3.1.3. It is possible to see that the normalization of a pseudomanifold is unique, up to a homeomorphism.

It is not difficult to see from their construction that normalizations must identify $(t; i)$ -allowable chains of \tilde{X} with $(t; i)$ -allowable chains of X , which implies that they yield an isomorphism between the intersection homology groups (see [10]).

Proposition 3.1.4 ([10]). *Let $\pi : \tilde{X} \rightarrow X$ be a normalization of X . Then, for any j the induced map $\pi_* : I^l H_j(\tilde{X}) \rightarrow I^l H_j(X)$ is an isomorphism.*

4. Computation of the L^∞ cohomology groups

This section proves the main result of this paper, Theorem 1.2.2.

4.1. Weakly differentiable forms

For technical reasons, we will need to work with non smooth forms, which are weakly differentiable, i.e., differentiable as distributions. Therefore, the first step is to prove that the bounded weakly differentiable forms give rise to the same cohomology theory. We will follow an argument similar to the one used by Youssin in [28].

Let M be a smooth manifold. We denote by $\Lambda_0^j(M)$ the set of C^2 j -forms on M with compact support.

Definition 4.1.1. Let U be an open subset of \mathbb{R}^n . A continuous differential j -form α on U is called **weakly differentiable** if there exists a continuous $(j + 1)$ -form ω such that for any form $\varphi \in \Lambda_0^{l-j-1}(U)$

$$\int_U \alpha \wedge d\varphi = (-1)^{j+1} \int_U \omega \wedge \varphi.$$

The form ω is then called **the weak exterior differential of α** and we write $\omega = \bar{d}\alpha$. A continuous differential j -form α on M is called **weakly differentiable** if it gives rise to weakly differentiable forms via the coordinate systems of M .

We denote by $\overline{\Omega}_\infty^j(M)$ the set of weakly differentiable forms which are bounded and which have a bounded weak exterior differential. They constitute a cochain complex whose coboundary operator is \bar{d} . We denote by $\overline{H}_\infty^\bullet(M)$ the resulting cohomology groups.

It is well known that if ω is smooth then it is weakly differentiable and $d\omega = \bar{d}\omega$. Therefore $\Omega_\infty^j(M) \subset \overline{\Omega}_\infty^j(M)$. Moreover, every L^∞ weakly differentiable form may be approximated (for the L^∞ norm) by smooth bounded forms (with approximation of the differential if it is L^∞). Consequently, any weakly differentiable 0-form ω satisfying $\bar{d}\omega = 0$ is constant.

We shall see that smooth and weakly differentiable forms give rise to isomorphic cohomology theories. The lemma below addresses the case of compact manifolds with boundary.

Given a smooth manifold with boundary K , we write $H_{dR}^j(K)$ for the de Rham cohomology of K , i.e., the cohomology of the C^∞ differential forms on K .

Lemma 4.1.2. *Let K be a compact manifold with boundary. The mapping $H_{dR}^j(K) \rightarrow \overline{H}_\infty^j(K \setminus \partial K)$ induced by the inclusion between the respective cochain complexes is an isomorphism.*

Proof. As the smooth forms on K satisfy Poincaré Lemma (see for instance [2]), they give rise to a fine torsionless resolution of the constant sheaf. By the uniqueness of the map between sheaf cohomology theories with coefficient in sheaves of \mathbb{R} -modules, it is enough to show Poincaré Lemma for weakly differentiable forms, i.e., it is enough to show that every point of K has a contractible neighborhood U in K such that for any $\omega \in \overline{\Omega}_\infty^j(U \setminus \partial K)$, $j > 0$, there is $\alpha \in \overline{\Omega}_\infty^{j-1}(U \setminus \partial K)$, such that $\bar{d}\alpha = \omega$.

Poincaré Lemma for $\overline{\Omega}_\infty^j(K \setminus \partial K)$ may be either derived by following the same argument as for the smooth forms on compact manifolds with boundary or directly deduced from the proof of Theorem 4.2.1 which actually applies to any weakly differentiable bounded j -form ω (this theorem indeed states a more difficult result since it deals with every subanalytic set, possibly singular). \square

For noncompact manifolds, we can now prove the following.

Proposition 4.1.3. *For any C^∞ manifold M (without boundary), the inclusion $\Omega_\infty^\bullet(M) \hookrightarrow \overline{\Omega}_\infty^\bullet(M)$ induces isomorphisms on the cohomology groups.*

Proof. It is enough to show that, for any form $\alpha \in \overline{\Omega}_\infty^j(M)$ with $\bar{d}\alpha \in \Omega_\infty^{j+1}(M)$ (i.e. α is weakly differentiable and $\bar{d}\alpha$ is smooth), there exists $\theta \in \overline{\Omega}_\infty^{j-1}(M)$ such that $(\alpha + \bar{d}\theta)$ is C^∞ (if $j = 0$ then $\theta \equiv 0$).

Choose a sequence of compact smooth manifolds with boundary $K_i \subset M$, $i \in \mathbb{N}$, such that for each $i \geq 0$, K_i is included in the interior of K_{i+1} and $\cup K_i = M$.

Fix a form $\alpha \in \overline{\Omega}_\infty^j(M)$ with $\bar{d}\alpha \in \Omega_\infty^{j+1}(M)$. We are going to construct a sequence $(\theta_i)_{i \in \mathbb{N}}$ in $\overline{\Omega}_\infty^{j-1}(M)$ such that for every $i \in \mathbb{N}$, we have $\text{supp } \theta_i \subset \text{Int}(K_i) \setminus K_{i-2}$ as well as $|\theta_i|_\infty + |\bar{d}\theta_i|_\infty \leq 1$ and such that the form $\alpha_i := \alpha + \sum_{k=0}^i \bar{d}\theta_k$ is smooth in a neighborhood of K_{i-1} .

Before defining inductively the θ_i 's, observe that $\theta := \sum_{i=0}^\infty \theta_i$ is the desired form (this sum is locally finite).

We now define the θ_i 's by induction on i . Let us assume that $\theta_0, \dots, \theta_{i-1}$ have been constructed, $i \geq 1$ (we may set $K_{-1} := K_{-2} := \emptyset$). We will also argue by induction on j . For $j = -1$, both cochain complexes vanish and the result is clear.

Observe that by Lemma 4.1.2, there exists a smooth j -form β on K_i such that $d\beta = \bar{d}\alpha_{i-1}$. It means that $(\alpha_{i-1} - \beta)$ is \bar{d} -closed, and thus again by Lemma 4.1.2 there is a smooth j -form β' on K_i such that

$$\alpha_{i-1} - \beta = \beta' + \bar{d}\gamma, \tag{4.16}$$

with $\gamma \in \bar{\Omega}_\infty^{(j-1)}(K_i)$.

Thanks to the induction on i , we know that there exists an open neighborhood V of K_{i-2} on which α_{i-1} is smooth. This implies that $\bar{d}\gamma$ is smooth on V . Therefore, applying the induction hypothesis to γ (which is a $(j - 1)$ -form), we can add a weakly exact form $\bar{d}\sigma$ to γ to get a form smooth on a neighborhood of K_{i-2} . Multiplying σ by a smooth function which has compact support included in V and which is 1 on a neighborhood $W \subset V$ of K_{i-2} , we get a form σ' on M such that $(\bar{d}\sigma' + \gamma)$ is smooth on W . It means that we can assume that γ is smooth on an open neighborhood W of K_{i-2} . We will assume this fact without changing notations.

By means of convolution products with a bump function, for any $\varepsilon > 0$, we may construct a smooth form γ_ε such that $|\gamma_\varepsilon - \gamma|_\infty \leq \varepsilon$ and $|d\gamma_\varepsilon - \bar{d}\gamma|_\infty \leq \varepsilon$ on K_i .

Consider a smooth function ϕ which is 1 on a neighborhood of $(M \setminus W) \cap K_{i-1}$ and with support in $(K_i \setminus \partial K_i) \setminus K_{i-2}$. Then, set:

$$\theta_i(x) := \phi(x)(\gamma_\varepsilon - \gamma)(x). \tag{4.17}$$

If ε is chosen small enough $|\theta_i|_\infty + |\bar{d}\theta_i|_\infty \leq 1$. On a neighborhood of $(M \setminus W) \cap K_{i-1}$, because $\phi \equiv 1$, we have by (4.16) and (4.17): $\alpha_{i-1} + \bar{d}\theta_i = \beta + \beta' + d\gamma_\varepsilon$ which is clearly smooth. The form $(\alpha_{i-1} + \bar{d}\theta_i)$ is smooth on W as well, since α_{i-1} and θ_i are both smooth. \square

4.2. Proof of the de Rham Theorem for L^∞ cohomology

We are now ready to prove Poincaré Lemma for L^∞ cohomology.

Theorem 4.2.1 (Poincaré Lemma for L^∞ Cohomology). *Let $X \subset \mathbb{R}^n$ be locally closed and let $x_0 \in X$. There exists $\varepsilon > 0$ such that for any closed form $\omega \in \Omega_\infty^j(B^n(x_0, \varepsilon) \cap X_{reg})$, $j \geq 1$, we can find $\alpha \in \bar{\Omega}_\infty^{j-1}(B^n(x_0, \varepsilon) \cap X_{reg})$, such that $\omega = d\alpha$.*

Proof. Let $r : X \cap B^n(x_0, \varepsilon) \times [0; 1] \rightarrow X \cap B^n(x_0, \varepsilon)$ be the map obtained by applying Theorem 2.3.1 to X . For simplicity, as our problem is local, we will identify X with the subset $X \cap B^n(x_0, \varepsilon)$. Let $\omega \in \Omega_\infty^j(X_{reg})$, with $j \geq 1$ and $d\omega = 0$. By Proposition 4.1.3, it is enough to find $\alpha \in \bar{\Omega}_\infty^{j-1}(X_{reg})$ satisfying $\bar{d}\alpha = \omega$.

The problem is that r may fail to be weakly smooth. To overcome this difficulty, we shall work with an approximation of r . We need to be particular since we wish to preserve the property that the derivative of r_t goes to zero (pointwise and generically) as t goes to zero.

Consider a sequence of compact subsets, $(K_i)_{i \in \mathbb{N}}$, such that $\cup K_i = X_{reg} \times (0; 1)$ and $K_i \subset \text{Int}(K_{i+1})$, for any i . Let Y be the set of points of $X_{reg} \times (0; 1)$ at which r fails to be smooth. Define then a sequence of compact subsets for $i \geq 1$:

$$L_i := \{q \in K_i : d(q; Y) \geq 1/i\}.$$

Define also

$$X' := \text{cl}(Y) \cap (X_{reg} \times \{0\})$$

and observe that, since Y is of positive codimension in $X_{reg} \times (0, 1)$, X' is of positive codimension in X_{reg} (we will consider it as a subset of X_{reg}).

As r is continuous, we may choose, for a given $\varepsilon_i > 0$, a C^∞ approximation r_i (not necessarily subanalytic) of r on K_i satisfying for any x in this set:

$$|r_i(x) - r_{i,t}(x)| \leq \varepsilon_i.$$

Furthermore, as r is smooth on L_i , we may require that on this set

$$|d_x r_{i,t} - d_x r_i| \leq \varepsilon_i. \tag{4.18}$$

Moreover, as the first derivative of r is bounded, the first derivative of r_i may be assumed to be bounded as well.

Let $(\varphi_i)_{i \in \mathbb{N}}$ be a partition of unity subordinated to the covering $(Int(K_{i+2}) \setminus K_i)_{i \in \mathbb{N}}$ of $X_{reg} \times (0; 1)$. Set $r' := \sum \varphi_i r_i$. If the sequence ε_i is decreasing fast enough, then a straightforward computation shows that the first derivative of r' is bounded above.

Furthermore, given any positive continuous function $\varepsilon : X_{reg} \times (0; 1) \rightarrow \mathbb{R}$, we can have if the sequence ε_i is decreasing fast enough:

$$|r'_i(x) - r_i(x)| \leq \varepsilon(x; t). \tag{4.19}$$

Finally, we shall check that $d_x r'_i$ tends to zero as t goes to zero for $x \notin X'$. Fix x in $X_{reg} \setminus X'$. There exists $a > 0$ such that $\{x\} \times [0; a]$ does not meet Y . It means that for any i large enough

$$K_i \cap (\{x\} \times [0; a]) = L_i \cap (\{x\} \times [0; a]). \tag{4.20}$$

For t small enough, if $\varphi_i(x; t) \neq 0$ then $(x; t) \in K_{i+2}$ and thus by (4.20) belongs to L_{i+2} . By (4.18), this implies that if ε_i tends to zero fast enough, $\lim_{t \rightarrow 0} |d_x r'_i| = 0$ for every $x \in X_{reg} \setminus X'$. We also see for the same reason that $d_x r'_i$ tends to the identity as t goes to 1 (for almost every x).

Let $\pi : W \rightarrow X_{reg}$ be a retraction where W is a tubular neighborhood of X_{reg} . Taking W small enough, we may assume that π has bounded first partial derivatives. By (4.19), $r'_i(x)$ belongs to W if the function ε is decreasing fast enough. Hence, composing r' with π if necessary we may assume that r' preserves X_{reg} . We will assume this without changing notations.

Define two L^∞ forms ω_1 and ω_2 on $X_{reg} \times (0; 1]$ by:

$$r'^* \omega := \omega_1 + dt \wedge \omega_2.$$

Now, we may set:

$$\alpha(x) := \int_0^1 \omega_2(x; t) dt.$$

As ω is L^∞ and r' has bounded first partial derivatives, the form α is clearly bounded. By Lebesgue's dominated convergence theorem, it is continuous. We claim that it is weakly differentiable and that $\vec{d}\alpha = \omega$.

Let us fix a C^2 -form $\varphi \in \Lambda_0^{m-j}(X_{reg})$ with compact support. We have, by definition of α :

$$\int_{X_{reg}} \alpha \wedge d\varphi = \int_{X_{reg}} \int_0^1 \omega_2 \wedge d\varphi = \lim_{t \rightarrow 0} \int_{X_{reg} \times [t; 1]} r'^* \omega \wedge d\varphi. \tag{4.21}$$

As $r'^* \omega$ is closed, by Stokes' formula we have:

$$\int_{X_{reg} \times [t; 1]} r'^* \omega \wedge d\varphi = (-1)^j \int_{x \in X_{reg}} \omega(x) \wedge \varphi(x) - (-1)^j \int_{x \in X_{reg}} \omega_1(x; t) \wedge \varphi(x),$$

since $\lim_{t \rightarrow 1} r^{*k} \omega(x; t) = \omega(x)$ for any $x \in X_{reg}$. Recall that $d_x r_t'$ tends to zero as t goes to 0 for almost every x . This implies that $\omega_1(x; t)$ goes to zero as t goes to 0. Hence, passing to the limit we get:

$$\int_{X_{reg}} \alpha \wedge d\varphi = (-1)^j \int_{X_{reg}} \omega \wedge \varphi,$$

as required. \square

Proof of Theorem 1.2.2. Let $\pi : \tilde{X} \rightarrow X$ be a normalization of X (see Proposition 3.1.2). Let us define a presheaf on \tilde{X} by

$$\tilde{\Omega}_{\infty}^j(U) := \Omega_{\infty}^j(\pi(U) \cap X_{reg}),$$

for every open set U of \tilde{X} (π is a homeomorphism above X_{reg}). For every j , this presheaf immediately gives rise to a sheaf that we will denote by \mathcal{F}_{∞}^j . We will write $\mathcal{F}_{\infty, x_0}^j$ for the stalk of \mathcal{F}_{∞}^j at $x_0 \in \tilde{X}$, i.e., the vector space obtained after identifying two sections which coincide near x_0 . As \tilde{X} is compact, any global section of \mathcal{F}_{∞}^j is bounded, so that, since π induces a homeomorphism above X_{reg} :

$$\mathcal{F}_{\infty}^{\bullet}(\tilde{X}) \simeq \Omega_{\infty}^{\bullet}(X_{reg}), \tag{4.22}$$

as cochain complexes.

We denote by \mathbb{R}_X the constant sheaf on X . Let $x_0 \in \tilde{X}$ and set $U^{\varepsilon} := B^n(x_0, \varepsilon) \cap \tilde{X}$. As π is a normalization, $\pi(U^{\varepsilon}) \cap X_{reg}$ is connected, which means that $H_{\infty}^0(X_{reg} \cap \pi(U^{\varepsilon})) = \mathbb{R}$, for $\varepsilon > 0$ small enough.

Moreover, by Theorem 4.2.1, for $j > 0$, the germ at $\pi(x_0)$ of a smooth bounded closed j -form ω on $\pi(U^{\varepsilon}) \cap X_{reg}$ is the exterior differential of the germ of a form $\alpha \in \mathcal{F}_{\infty, x_0}^{j-1}$. Therefore, the sequence:

$$0 \longrightarrow \mathbb{R}_X \xrightarrow{d} \mathcal{F}_{\infty}^0 \xrightarrow{d} \mathcal{F}_{\infty}^1 \xrightarrow{d} \dots \tag{4.23}$$

is a fine torsionless resolution of the constant sheaf. By classical arguments of sheaf theory (see for instance [26]), the latter exact sequence of sheaves implies via (4.22) that $H_{\infty}^j(X_{reg})$ is isomorphic to the singular cohomology of \tilde{X} . But then, by Propositions 3.0.5 and 3.1.4, we get:

$$H_{\infty}^j(X_{reg}) \simeq H^j(\tilde{X}) \simeq I^t H^j(\tilde{X}) \simeq I^t H^j(X). \quad \square \tag{4.24}$$

4.3. Integration on subanalytic singular simplices

We are going to prove that the isomorphism is provided by integrating the forms on the allowable chains. We first check that integration gives rise to a well defined cochain map. This may be done because we restrict ourselves to the t -allowable *subanalytic* singular cochains. Let X be a compact pseudomanifold.

Let $L \subset X_{reg}$ be an oriented manifold of dimension j with $cl(L)$ $(t; j)$ -allowable i.e.:

$$\dim cl(L) \cap X_{sing} \leq (j - 2).$$

Set $\partial L := cl(L) \setminus L$. Then, for any given ω in $\Omega_{\infty}^{j-1}(X_{reg})$, $\int_L d\omega$ and $\int_{(\partial L)_{reg}} \omega$ are well defined since ω is continuous almost everywhere on $(\partial L)_{reg}$ and bounded. We start by recalling a version of Stokes' formula proved by Łojasiewicz in [15] who generalized a formula of Pawłucki [17].

Lemma 4.3.1 ([15]). *Take L and ω as in the above paragraph. Then:*

$$\int_{(\partial L)_{reg}} \omega = \int_L d\omega. \tag{4.25}$$

Łojasiewicz’s formula is actually devoted to bounded subanalytic forms, but the required property is indeed that they are bounded and extend continuously almost everywhere on the closure of the manifold L , which obviously holds true when the form is L^∞ and $cl(L)$ is $(t; j)$ -allowable.

Next we turn to see that integration is well defined for any $(t; j)$ allowable *subanalytic singular simplex*. Let $\sigma : \Delta_j \rightarrow X$ be an oriented $(t; j)$ -allowable (subanalytic) simplex. Denote by σ_{reg} the set of points in $\sigma^{-1}(X_{reg})$ near which σ induces a smooth mapping. Observe that the complement of σ_{reg} in Δ_j has Lebesgue measure zero. Hence, it makes sense to set for $\omega \in \Omega_\infty^j(X_{reg})$:

$$\int_\sigma \omega := \int_{\Delta_j} \sigma^* \omega = \int_{\sigma_{reg}} \sigma^* \omega. \tag{4.26}$$

Stokes’ formula continues to hold for subanalytic singular $(t; j)$ -allowable simplices.

Lemma 4.3.2. *Let $\sigma \in I^1 C_j(X)$ and $\omega \in \Omega_\infty^{j-1}(X_{reg})$. Then the integral defined in (4.26) is finite and:*

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega. \tag{4.27}$$

Proof. Let

$$\Gamma := \{(x; y) \in X \times \Delta_j : x = \sigma(y)\},$$

and consider a cell decomposition of \mathbb{R}^{n+j} compatible with Γ . Refining it, we can assume that the boundary of a cell is a union of cells. For simplicity, we will identify Δ_j with Γ and assume that σ is the canonical projection (restricted to Γ).

Let $C \subset \Gamma$ be a cell of this cell decomposition and let $i := \dim C$. Observe that either $\sigma|_C$ is a diffeomorphism or $\dim \sigma(C) < i$. In the former case, if we endow $\sigma(C)$ with the orientation induced by Δ_j via σ , we have by definition:

$$\int_C \sigma^* \alpha = \int_{\sigma(C)} \alpha \tag{4.28}$$

for any $\alpha \in \Omega_\infty^i(X_{reg})$ (we shall need both the cases $\alpha = \omega$ and $\alpha = d\omega$). If $\dim \sigma(C) < i$, then both vanish and this remains true. Note that, as a cell decomposition is a finite partition, the latter formula already shows that the integral defined in (4.26) is finite.

By Lemma 4.3.1, if C is a cell of dimension j :

$$\int_C \sigma^* d\omega \stackrel{(4.28)}{=} \int_{\sigma(C)} d\omega \stackrel{(4.25)}{=} \int_{\partial(\sigma(C))} \omega = \int_{\sigma(\partial C)} \omega \stackrel{(4.28)}{=} \int_{\partial C} \sigma^* \omega, \tag{4.29}$$

the third equality being true because σ is identified with a linear mapping (a projection) on the cell C . As Δ_j is a union of cells, the latter equality still holds if we replace C with Δ_j . We

conclude that for relevant orientations:

$$\int_{\sigma} d\omega = \int_{\Delta_j} \sigma^* d\omega \stackrel{(4.29)}{=} \int_{\partial\Delta_j} \sigma^* \omega = \int_{\partial\sigma} \omega. \quad \square$$

In conclusion, we get that the isomorphism of [Theorem 1.2.2](#) is given by integrating the differential forms on the simplices.

Theorem 4.3.3. *Let X be a compact pseudomanifold. The cochain maps*

$$\begin{aligned} \psi_X^j : \Omega_{\infty}^j(X_{reg}) &\rightarrow I^j C^j(X), \\ \omega &\mapsto \left[\sigma \mapsto \int_{\sigma} \omega \right], \end{aligned}$$

induce isomorphisms between the cohomology groups.

To prove it, observe that the cochain map ψ_X induces a sheaf homomorphism (recall that the L^{∞} forms give rise to sheaf on the normalization of X , see the proof of [Theorem 1.2.2](#)). By the uniqueness of the map between sheaf cohomology theories with coefficient in sheaves of \mathbb{R} -modules, this map must coincide with the isomorphism [\(4.24\)](#).

Remarks 4.3.4. [Theorem 1.2.2](#) still holds if X is a pseudomanifold with boundary [\[10\]](#) (indeed, our Poincaré Lemma for L^{∞} cohomology does not assume that X is a pseudomanifold). The relative version is then true as well, by the five lemma. Again, the isomorphism is provided by integration of forms on allowable chains.

It is worthy of notice that the arguments of the proof of [Theorem 1.2.2](#) also apply in the noncompact case, establishing an isomorphism between the cohomology of the *locally* bounded forms (locally in X , not in X_{reg}) and intersection homology in the maximal perversity.

The results of this paper remain true if we replace the subanalytic category with a polynomially bounded o -minimal structure [\[24\]](#). We need the structure to be polynomially bounded for we made use of the preparation theorem for proving [Proposition 2.2.1](#). It is unclear (but not impossible) whether the results of this paper, especially [Theorem 2.3.1](#), are valid on a non polynomially bounded o -minimal structure, especially for the log – exp sets, on which a generalized preparation theorem holds [\[14\]](#).

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Appendix A. Globally subanalytic sets

A.1. Basic definitions

Let N be an analytic manifold. Recall that a subset $E \subset N$ is called **semianalytic** if it is locally defined by finitely many real analytic equations and inequalities. More precisely, for each $p \in N$, there is a neighborhood U of p in N , and real analytic functions f_i, g_{ij} on U , where $i = 1, \dots, r, j = 1, \dots, s$, such that

$$E \cap U = \bigcup_{i=1}^r \bigcap_{j=1}^s \{x \in U : g_{ij}(x) > 0 \text{ and } f_i(x) = 0\}.$$

A subset $E \subset N$ is **subanalytic** if it can be locally represented as the projection of a semianalytic set. More precisely, for every $p \in N$, there exist a neighborhood U of p in N , an analytic manifold P , and a relatively compact semianalytic set $Z \subset N \times P$ such that $E \cap U = \pi(Z)$, where $\pi : N \times P \rightarrow N$ is the natural projection. In particular, semianalytic sets are subanalytic.

A subset of \mathbb{R}^n is **globally subanalytic** if it coincides with a subanalytic subset of \mathbb{P}^n after identifying \mathbb{R}^n with an open subset of \mathbb{P}^n via:

$$(y_1, \dots, y_n) \rightarrow (1 : y_1 : \dots : y_n) : \mathbb{R}^n \rightarrow \mathbb{P}^n.$$

We will denote by \mathcal{S}_n the set of globally subanalytic subsets of \mathbb{R}^n .

Clearly, a bounded subset is subanalytic if and only if it is globally subanalytic. We say that a **function is globally subanalytic** if its graph is globally subanalytic.

A.2. Basic properties of globally subanalytic sets

Any real algebraic set is globally subanalytic. Furthermore, globally subanalytic sets have the following very useful properties (see [24]):

- (1) \mathcal{S}_n is stable under unions, intersections and complement.
- (2) If $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$ then $A \times B \in \mathcal{S}_{m+n}$.
- (3) If $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is the projection on the first n coordinates and $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$.
- (4) The elements of \mathcal{S}_1 are precisely the finite unions of points and intervals.

When a family of sets has these properties, we say that it constitutes an *o-minimal structure*. These properties are indeed all the basic properties we need to do most of geometric constructions (such as triangulations, stratifications, retracts, ...). Property (3) is the motivation for introducing subanalytic sets: semianalytic sets are not stable under projection and thus do not fulfill (3).

Property (4) is a finiteness assumption which makes it possible to derive all the finiteness properties of globally subanalytic sets. The first one and the most important is existence of cell decompositions (see definition below). Most of the results we give below are not really proper to globally subanalytic sets and are shared by all the sets definable in an *o-minimal structure*. We will therefore often refer to [9] for proofs.

Definition A.2.1. A **cell decomposition of \mathbb{R}^n** is a finite partition of \mathbb{R}^n into globally subanalytic sets $(C_i)_{i \in I}$, called **cells**, satisfying certain properties explained below.

$n = 1$: A **cell decomposition of \mathbb{R}** is given by a finite subdivision $a_1 < \dots < a_l$ of \mathbb{R} . The cells of \mathbb{R} are the singletons $\{a_i\}$, $0 < i \leq l$, and the intervals (a_i, a_{i+1}) , $0 \leq i < l$, where $a_0 = -\infty$ and $a_{l+1} = +\infty$.

$n > 1$: A **cell decomposition of \mathbb{R}^n** is the data of a cell decomposition of \mathbb{R}^{n-1} and, for each cell D of \mathbb{R}^{n-1} , some globally subanalytic functions analytic on D (which is an analytic manifold):

$$\zeta_{D,1} < \dots < \zeta_{D,l(D)} : D \rightarrow \mathbb{R}.$$

The **cells of \mathbb{R}^n** are the **graphs**

$$\{(x, \zeta_{D,i}(x)) : x \in D\}, \quad 0 < i \leq l(D),$$

and the **bands**

$$(\zeta_{D,i}, \zeta_{D,i+1}) := \{(x, y) : x \in D \text{ and } \zeta_{D,i}(x) < y < \zeta_{D,i+1}(x)\},$$

for $0 \leq i < l(D)$, where $\zeta_{D,0} = -\infty$ and $\zeta_{D,l(D)+1} = +\infty$.

A cell decomposition is said to be **compatible with finitely many sets** A_1, \dots, A_k if the A_i 's are unions of cells.

Given some globally subanalytic sets A_1, \dots, A_k , it is always possible to find a cell decomposition compatible with this family of sets. Detailed proofs may be found in [9].

This already describes very precisely the geometry of globally subanalytic sets. Below, we list some related extra basic properties, useful for us.

- ◇ (*curve selection lemma*) Let $A \in \mathcal{S}_n$ and $b \in cl(A)$. There is an analytic map $\gamma : (-1, 1) \rightarrow \mathbb{R}^n$ such that $\gamma(0) = b$ and $\gamma((0, 1)) \subset A$.
- ◇ (*subanalyticity of the connected components*) Subanalytic sets have only finitely many connected components. They are subanalytic.
- ◇ (*uniform bound*) Let $f : A \rightarrow B$ be a globally subanalytic map, with finite fibers. There is $k \in \mathbb{N}$ such that $card f^{-1}(b) \leq k$ for any b .
- ◇ (*subanalytic choice*) Any globally subanalytic map $f : A \rightarrow B$ (not necessarily continuous) admits a globally subanalytic section, i.e., a globally subanalytic map $s : B \rightarrow A$ such that $f(s(b)) = b$.
- ◇ (*subanalyticity of the regular locus*) Let $X \in \mathcal{S}_n$. Then X_{reg} is a finite union of analytic manifolds; it is globally subanalytic and dense in X .

For a proof of curve selection lemma or subanalyticity of the regular locus we refer the reader to [1]. A proof of all the other statements may be found in [9].

The set X_{reg} is the union of finitely many analytic manifolds. The **dimension of X** , denoted by $\dim X$, is the maximal dimension of these manifolds.

Appendix B. Some basic model theoretic principles

B.1. Formulas

We shall need some basic facts of model theory. We first define what we call \mathcal{L} -formulas. Basically, it is a sequence constituted by quantifier and some symbols, like for instance $\forall x, \exists y, x \leq 2yz$. More precisely, \mathcal{L} -**formulas** are defined inductively as follows:

- (1) If f is a globally subanalytic function then $f(x) > 0$ and $f(x) = 0$ are \mathcal{L} -formulas.
- (2) If $\Phi(x_1, \dots, x_n)$ and $\Psi(x_1, \dots, x_n)$ are \mathcal{L} -formulas, then “ Φ and Ψ ”, “ Φ or Ψ ”, and “not Φ ”, are \mathcal{L} -formulas as well.
- (3) If $\Phi(y, x)$ is an \mathcal{L} -formula, then $\exists x, \Phi(y, x)$ and $\forall x, \Phi(y, x)$ are \mathcal{L} -formulas.

The parameters $x = (x_1, \dots, x_n)$ in $\Phi(x)$ denote the free variables (those which are not quantified). The symbol \mathcal{L} stands for language: \mathcal{L} -formulas are sentences “in the language of subanalytic geometry”. Roughly speaking, \mathcal{L} -formulas are all the mathematical sentences that one can write using quantifiers, globally subanalytic functions, equalities and inequalities.

As an example, consider the formula $\Phi(x)$:

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall y, \quad |x - y| \leq \alpha \Rightarrow |f(x) - f(y)| < \varepsilon.$$

However, the formula $\exists n, n \in \mathbb{N}$ and $y = nx$ is *not* an \mathcal{L} -formula (\mathbb{N} may not be described by an \mathcal{L} -formula). Observe that this formula does not define a subanalytic set of \mathbb{R}^2 . Indeed, the following fundamental result relates \mathcal{L} -formulas to subanalytic sets.

Proposition B.1.1. *If $\Phi(x)$ is an \mathcal{L} -formula, then the set $\{x \in \mathbb{R}^n : \Phi(x)\}$ is globally subanalytic.*

To briefly account for this proposition, let us point out that the sentence $\exists y, f(x, y) = 0$ defines the projection of the set defined by $f(x, y) = 0$. Thus, for this sentence the result follows from Property (3) of [Appendix A.2](#). The proposition is then showed by induction on the number of quantifiers (the case of the universal quantifier may be reduced to the existential by considering the negation of the sentence, for more details see [9], Theorem 1.13).

This proposition shows how important it is to work with a category of sets which is stable under projection. As a consequence of this proposition, the interior and the closure of a globally subanalytic set are globally subanalytic.

Another consequence of this proposition is that if the graph of a function is defined by an \mathcal{L} -formula, then this function is globally subanalytic. It enables to establish that a function is subanalytic without much work.

For instance, if $\xi : \mathbb{R}^n \times [0, 1] \rightarrow [0, 1]$ is a globally subanalytic function then the function defined by $\zeta(x) := \inf_{t \in [0, 1]} \xi(x, t)$ is globally subanalytic. It is then easy to check that if A denotes a subanalytic set then the function $x \mapsto d(x, A)$, which assigns to x the Euclidean distance from x to A , is globally subanalytic.

B.2. Field extensions

Consider all the one variable globally subanalytic functions which are defined in a right-hand-side neighborhood of the origin and identify any two of them which coincide in a small right-hand-side neighborhood of the origin. It follows from Puiseux Lemma [18] that this set is indeed the field of real Puiseux series $\sum_{i \geq m} a_i T^{\frac{i}{k}}, m \in \mathbb{Z}, a_i \in \mathbb{R}$, with $\sum_{i \in \mathbb{N}} a_i t^i$ convergent for t in a neighborhood of zero. We shall denote this field $k(0_+)$. We may embed $\mathbb{R} \hookrightarrow k(0_+)$, sending every real number onto the corresponding constant series.

We can order this field by setting $f \leq g$ in $k(0_+)$ if $f(t) \leq g(t)$ for t in a right-hand-side neighborhood of the origin. Observe that the indeterminate T is smaller than any positive real number in $k(0_+)$. Consequently, this field is not Archimedean.

We may also define the **Euclidean norm** on $k(0_+)^n$, by $|x| := \sum_{i=1}^n x_i^2 \in k(0_+)$. This gives rise to a topology. A good reference for all the results of this section is [9].

Extension of sets and functions. As the composite of globally subanalytic mappings is globally subanalytic, any globally subanalytic function $\xi : \mathbb{R}^n \rightarrow \mathbb{R}$ may be extended to a function $\xi_{k(0_+)} : k(0_+)^n \rightarrow k(0_+)$, $\xi_{k(0_+)}(x(T)) := \xi(x(T))$, $x(T) \in k(0_+)$. Any \mathcal{L} -formula may then be also “extended” to $k(0_+)$. For instance the formula $\Phi, \exists x \in \mathbb{R}^n, f(x) \geq 0$, where f is an analytic function, has the following extension $\Phi_{k(0_+)}$ to $k(0_+)$:

$$\exists x \in k(0_+)^n, \quad f_{k(0_+)}(x) \geq 0.$$

In other words, we can extend a formula by extending the functions this formula involves.

It is not very difficult to check that if two formulas define the same set in \mathbb{R}^n then their respective extensions define the same set in $k(0_+)^n$ (see [9]). Hence, we may define the **extension of the set** $A := \{x \in \mathbb{R}^n : \Phi(x)\}$ by setting

$$A_{k(0_+)} := \{x \in k(0_+)^n : \Phi_{k(0_+)}(x)\}.$$

In other words, the extension of a set is obtained by regarding the associated equations and inequalities in the field of real analytic Puiseux series. This set is merely the set of germs of

Puiseux arcs lying on A . For instance, the extension of the sphere S^{n-1} (generally still denoted by S^{n-1}) is the set:

$$\left\{ x \in k(0_+)^n : \sum_{i=1}^n x_i^2 = 1 \right\}.$$

Generic fibers. Let now $A \subset \mathbb{R} \times \mathbb{R}^n$ be a globally subanalytic set. Regarding the first variable as a parameter, we will consider this set as a family. We define the **generic fiber** of this family of sets as (recall that $T \in k(0_+)$ stands for the indeterminate):

$$A_{0_+} := \{x \in k(0_+)^n : (T, x) \in A_{k(0_+)}\}.$$

It is nothing but the set of germs of Puiseux arcs $x(t)$ such that $(t, x(t)) \in A$ for every t positive small enough. Observe that we can also define the **generic fiber of a family of globally subanalytic functions** $f : A \rightarrow \mathbb{R}$, which is the function $f_{0_+} : A_{0_+} \rightarrow k(0_+)$ which assigns to $x \in A_{k(0_+)}$ the value $f_{k(0_+)}(T, x)$.

We then can define the subanalytic sets (resp. mappings) of $k(0_+)^n$ as the collection of all the generic fibers of subanalytic families of sets (resp. mappings). This constitutes a family of Boolean algebras which enjoys the same properties as $(\mathcal{S}_n)_{n \in \mathbb{N}}$. Indeed, as they satisfy (1–4) of [Appendix A.2](#), they then satisfy all the other properties which come down from these properties.

As a matter of fact, [Proposition B.1.1](#) is still true if we replace \mathbb{R} with $k(0_+)$. For example, the function $x \mapsto d(x, A)$ is well defined and globally subanalytic if so is $A \subset k(0_+)^n$.

We can also define the **generic fiber of a formula**. If $\Phi(t, x)$ is a formula, with x and t free variables (t considered as a parameter) we define the generic fiber of $\Phi(t, x)$ as the formula obtained by replacing t with the indeterminate T , i.e. we set $\Phi_{0_+}(x) := \Phi_{k(0_+)}(T, x)$. This of course reduces the number of free variables.

Transfer principle. The study of the generic fiber of a family $A \subset \mathbb{R} \times \mathbb{R}^n$ can provide us information on what happens on the fiber A_t for generic parameter t . More precisely, we have the following very important fact. Let $\Phi(t)$ be an \mathcal{L} -formula. The formula Φ_{0_+} holds true in $k(0_+)$ if and only if $\Phi(t)$ holds for every positive real number t small enough. This is a consequence of a more general theorem sometimes referred as *Łoś's Theorem*.

To make it more concrete, let us illustrate it by some examples. One may easily derive from this fact that if the generic fiber of the family of functions $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded by 1 then there is $\varepsilon > 0$ such that $\sup_{x \in \mathbb{R}^n} f(t, x)$ is not greater than 1 for any $t \in (0, \varepsilon)$. This is due to the fact that $\forall x, f(t, x) \leq 1$ is an \mathcal{L} -formula.

Less easy, but still true, is the fact that if f_{0_+} is continuous then there is $\varepsilon > 0$ such that the restriction $f : (0, \varepsilon) \times \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous (the non obvious part is the continuity with respect to the parameter $t \in (0, \varepsilon)$ which is not guaranteed by the above transfer principle, see [9] Section 5.6 for details). This points out the interplay between the geometry of globally subanalytic sets of $k(0_+)^n$ and the geometry of families of globally subanalytic sets of $\mathbb{R} \times \mathbb{R}^n$, for generic parameters.

To give another illustration, let us focus on the argument of the proof of [Corollary 2.1.4](#). Applying [Theorem 2.1.2](#) provides a bi-Lipschitz globally subanalytic map $H : k(0_+)^{n-1} \rightarrow k(0_+)^{n-1}$. This homeomorphism is the generic fiber of a family of mappings $h : (0, \varepsilon) \times \mathbb{R}^{n-1} \rightarrow (0, \varepsilon) \times \mathbb{R}^{n-1}$. As the desired property (see (2.1)) holds for h_{0_+} , by Łoś's Theorem, it also holds for h_t , for $t > 0$ small enough (since it may be expressed by an \mathcal{L} -formula).

To summarize, we get parameterized versions of theorems just by working with a bigger underlying field. Working with a bigger field often makes no difference as most of the time we simply have to write $k(0_+)$ instead of \mathbb{R} .

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