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The Kapustin–Li formula revisited

Tobias Dyckerhoff[a,*], Daniel Murfet[b]

[a] Department of Mathematics, Yale University, United States
[b] Department of Mathematics, UCLA, United States

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Abstract

We provide a new perspective on the Kapustin–Li formula for the duality pairing on the morphism complexes in the matrix factorization category of an isolated hypersurface singularity. In our context, the formula arises as an explicit description of a local duality isomorphism, obtained by using the basic perturbation lemma and Grothendieck residues. The non-degeneracy of the pairing becomes apparent in this setting. Further, we show that the pairing lifts to a Calabi–Yau structure on the matrix factorization category. This allows us to define topological quantum field theories with matrix factorizations as boundary conditions.
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Keywords: Dg categories; Matrix factorizations; Topological quantum field theory

Contents

* Corresponding author.
E-mail addresses: tobias.dyckerhoff@yale.edu (T. Dyckerhoff), daniel.murfet@math.ucla.edu (D. Murfet).

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1. Introduction

Let k be a field of characteristic zero and R be a regular local augmented k-algebra with maximal ideal $\mathfrak{m}$. We consider a nonzero element $w \in \mathfrak{m}$ which we assume to have *isolated critical locus*. By this, we mean that there exists a sequence $\{\partial_1, \dots, \partial_n\}$ of k-derivations of R such that $\partial_i w \in \mathfrak{m}$ and the sequence

$$\dim_k R/(\partial_1 w, \partial_2 w, \dots, \partial_n w) < \infty.$$

Note that this condition is equivalent to saying that the sequence $\{\partial_1 w, \dots, \partial_n w\}$ forms a system of parameters for R. The object of our interest is the local germ of the singular hypersurface defined by the equation $w = 0$. More precisely, we study the stable homological algebra of the hypersurface by means of the category of matrix factorizations $\mathrm{MF}(R, w)$. The latter is a differential $\mathbb{Z}/2$-graded category whose objects are given by $\mathbb{Z}/2$-graded finite free R-modules $X = X^0 \oplus X^1$ equipped with an odd endomorphism d satisfying $d^2 = w$. Such an object (X, d) corresponds, after choosing bases for X^0 and X^1, to a pair of square matrices (φ, ψ) satisfying

$$\varphi \circ \psi = \psi \circ \varphi = w \,\mathrm{id},$$

hence the nomenclature. Equivalently, we can combine the matrices into a supermatrix

$$Q = \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix}$$

satisfying $Q^2 = w\,\mathrm{id}$. We refer to d or Q as the twisted differential associated with the matrix factorization X. More details on the category $\mathrm{MF}(R, w)$ as well as an overview of its relevance in terms of homological algebra over R/w can be found in [11,27,10] and the references therein.

Fixing a hypersurface (R, w), we introduce the abbreviation $T = \mathrm{MF}(R, w)$ as well as the symbol $[T]$ for the homotopy category of T. Recall that the category $[T]$ is obtained by applying $\mathrm{H}^0(-)$ to all morphism complexes in T. For matrix factorizations X and Y, the morphism complex in T will be denoted by $T(X, Y)$, the morphisms in the homotopy category by $[T](X, Y)$. As first established by Auslander [1, Proposition 8.8 in Ch. 1 and Proposition 1.3 in Ch. 3], the triangulated category $[T]$ is a Calabi–Yau category, i.e. there exist an integer $n \in \mathbb{Z}$ and a non-degenerate pairing

$$[T](X, Y) \otimes_k [T](Y, X[n]) \to k$$

for every pair of objects X,Y in $[T]$. However, an explicit description of this pairing was not known until this category appeared in the context of topological string theory. Following a proposal by Kontsevich, the physicists Kapustin and Li [15] interpreted the category $[T]$ as the category of boundary conditions in the Landau–Ginzburg B-model corresponding to (R, w). This allowed them to apply path integral methods when $k = \mathbb{C}$ in order to derive a formula for a pairing

$$[T](X, Y) \otimes_k [T](Y, X[n]) \to k, \qquad (F, G) \mapsto \frac{1}{(2\pi i)^n n!} \oint_{\{|\partial_i w| = \epsilon\}} \frac{\operatorname{tr}(FG(dQ)^{\wedge n})}{\partial_1 w \partial_2 w \cdots \partial_n w},$$

where Q is the twisted differential associated with Y. The first attempt to put the pairing into a mathematical context was outlined in [23].

In [21] the second author gave a mathematical derivation of this formula and proved its non-degeneracy, as a special case of a general statement about Serre duality in the singularity category of an arbitrary isolated Gorenstein singularity. In this work we give an alternative and more direct derivation of the pairing for hypersurfaces, using the techniques developed in *ibid*. Fundamentally the pairing is obtained from local duality applied to the mapping complexes in the category T, and the explicit form of the duality isomorphism is obtained by employing the basic perturbation lemma as well as the theory of residue symbols. In this context the formula naturally takes the form (Theorem 3.4)

$$(F, G) \mapsto (-1)^{\binom{n+1}{2}} \frac{1}{n!} \operatorname{Res} \begin{bmatrix} \operatorname{tr}(FG(dQ)^{\wedge n}) \\ \partial_1 w, \partial_2 w, \ldots, \partial_n w \end{bmatrix}.$$

As a second main result, we show in Section 5 that the pairing given by the Kapustin–Li formula is part of a Calabi–Yau structure on the dg category $\mathrm{MF}(R, w)$. This means that the pairing factors canonically over the cyclic complex of $\mathrm{MF}(R, w)$. The importance of this structure lies in the fact that it allows us to define 2-dimensional topological quantum field theories in the sense of [6,20]. This result is based on a variant of the calculation of the boundary–bulk map in [22] which we perform in Section 4. The representative

$$\operatorname{Hom}(E, E) \to \Omega_w[n], \qquad F \mapsto (-1)^{\binom{n+1}{2}} \frac{1}{n!} \operatorname{tr}(F(dQ)^{\wedge n})$$

of the map which we provide in Theorem 4.11 is adapted to the Kapustin–Li formula on the chain level. Our argument involves another application of the basic perturbation lemma where we use an explicit homotopy which contracts a Koszul complex onto its cohomology. The construction of this canonical contracting homotopy in Section 4.2 may be considered as a result of independent interest.

As a well-known application of the field theory formalism, we illustrate in Section 5.3 how a Riemann–Roch formula, which presumably agrees with the one given in [22], can be pictorially deduced from the existence of a field theory.

We will now outline our derivation of the Kapustin–Li formula. To exhibit the relation to classical local duality, we sketch the argument in a $\mathbb{Z}$-graded context, the detailed argumentation will be given in a purely $\mathbb{Z}/2$-graded setting. Fixing two objects X, Y in the category $T = \mathrm{MF}(R, w)$, we think of the mapping complex $T(X, Y)$ as a 2-periodic $\mathbb{Z}$-graded complex. Local duality provides an isomorphism

$$\mathrm{R}\Gamma_{\mathfrak{m}}(T(X, Y)) \xrightarrow{\simeq} \operatorname{Hom}_R(\operatorname{Hom}_R(T(X, Y), R), \mathrm{R}\Gamma_{\mathfrak{m}}(R)) \tag{1}$$

in the derived category of R-modules. Analyzing the right hand side, observe that the graded trace pairing yields an isomorphism between the complex $\mathrm{Hom}_R(T(X,Y),R)$ and $T(Y,X)$. Since R is a regular local ring, thus Gorenstein, we have an isomorphism $\mathrm{R}\Gamma_{\mathfrak{m}}(R) \cong \mathrm{H}^n_{\mathfrak{m}}(R)[-n]$, and $\mathrm{H}^n_{\mathfrak{m}}(R)$ is an injective hull of the residue field. Composing these maps, we obtain an isomorphism

$$\mathrm{R}\Gamma_{\mathfrak{m}}(T(X,Y)) \xrightarrow{\simeq} \mathrm{Hom}_R(T(Y,X[n]),\mathrm{H}^n_{\mathfrak{m}}(R))$$

which we denote by g. Furthermore, there exists a natural map

$$\mathrm{Res} : \mathrm{H}^n_{\mathfrak{m}}(R) \longrightarrow k$$

given by the Grothendieck residue symbol. Since the singularity is assumed to be isolated, the cohomology modules of the complex $T(Y,X)$ have finite length which in turn implies that the map

$$\mathrm{Hom}_R(T(Y,X),\mathrm{H}^n_{\mathfrak{m}}(R)) \xrightarrow{\mathrm{Res}_*} \mathrm{Hom}_k(T(Y,X),k)$$

is a quasi-isomorphism. On the left hand side of the duality isomorphism (1), we deduce that the natural map

$$\mathrm{R}\Gamma_{\mathfrak{m}}(T(X,Y)) \xrightarrow{f} T(X,Y)$$

is a quasi-isomorphism since the restriction of $T(X,Y)$ to the complement of $\mathfrak{m}$ in $\mathrm{Spec}(R)$ is contractible. Combining the above observations, we obtain the diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathfrak{m}}(T(X,Y)) \arrow[r, "g"] \arrow[d, "f"] & \mathrm{Hom}(T(Y,X[n]),\mathrm{H}^n_{\mathfrak{m}}(R)) \arrow[d, "\mathrm{Res}_*"] \\
T(X,Y) \arrow[r, dotted] & \mathrm{Hom}(T(Y,X[n]),k)
\end{tikzcd}$$

which exhibits the duality pairing of the category T. We subdivide the problem of finding an explicit formula for this pairing into

(I) Find a model of $\mathrm{R}\Gamma_m\, T(X,Y)$ in which the maps f and g become explicit (*Koszul model*)
(II) Invert the map f up to homotopy (*Basic Perturbation Lemma*)
(III) Describe the map Res_* explicitly (*Grothendieck residues*),

where we indicated the techniques which we will use in brackets. We refer to Remark 3.5 for a brief comparison to the approach of [21].

In conclusion, the outline of the paper is as follows. After collecting the necessary preliminaries in Section 2, the new derivation of the Kapustin–Li pairing is detailed in Section 3, following steps (I) through (III). Our calculation of the boundary–bulk map is given in Section 4 and then used in Section 5 to construct topological quantum field theories and deduce a Riemann–Roch formula.

Conventions. We use the symbol $\cong$ to denote an isomorphism of complexes and the symbol $\simeq$ to denote a quasi-isomorphism. Duals in various contexts will be referred to by the symbol $-^\vee$. Applied to an R-linear complex Z, the complex $Z^\vee$ will be the mapping complex $\mathrm{Hom}(Z,R)$ with the usual Koszul signs. Applied to an object X in $\mathrm{MF}(R,w)$, we obtain an object $X^\vee$ in $\mathrm{MF}(R,-w)$ by forming the mapping complex $\mathrm{Hom}(X,R)$ and ignoring the fact that the differential on X does not square to 0. Similarly, $\mathrm{Hom}(-,-)$ always refers to a mapping complex where in the context of matrix factorizations the differential does not necessarily square to 0.

The 2-periodic $\mathbb{Z}$-graded mapping complexes in the category $T = \mathrm{MF}(R, w)$ will be denoted by $T(X, Y)$ whereas we use $\mathrm{Hom}(X, Y)$ (or $\mathrm{Hom}_R(X, Y)$) for their $\mathbb{Z}/2$-graded counterpart. Occasionally, we will form $\mathrm{Hom}_R(X, Y)$ where X and Y are objects of $\mathrm{MF}(R, w)$ and $\mathrm{MF}(R, w')$, respectively. In this case, we interpret $\mathrm{Hom}_R(X, Y)$ as a matrix factorization of $w' - w$.

2. Preliminaries

2.1. The basic perturbation lemma

Homological perturbation theory is concerned with the transport of algebraic structures along homotopy equivalences of complexes. A typical example of a structure which admits such a transport feature is the structure of an A_∞ algebra as introduced by Stasheff. Roughly, the basic perturbation lemma is concerned with transporting an additional differential δ on a complex (B, d) along a homotopy equivalence of complexes $f : (A, d) \to (B, d)$. One thinks of $d + \delta$ as a small perturbation of (B, d) and attempts to perturb f and (A, d) to obtain a new, perturbed homotopy equivalence. In comparison with spectral sequence techniques, the lemma has the advantage of producing explicit formulas.

We recall the variant of the basic perturbation lemma from [7] which we will apply. Let R be a commutative ring with unit. A *deformation retract datum* consists of

$$\left[\ (A, d) \underset{p}{\overset{\iota}{\rightleftarrows}} (B, d),\ h\ \right], \tag{2}$$

where (A, d) and (B, d) are complexes of R-modules, ι and p are maps of complexes, and h is a homotopy on B such that

(1) $p\iota = \mathrm{id}_A$
(2) $\iota p = \mathrm{id}_B + dh + hd$.

Given a perturbation of the differential on B, the lemma produces a new deformation retract.

Lemma 2.1 (*Basic Perturbation Lemma*)**.** *Suppose we are given a deformation retract datum (2) and bounded below increasing filtrations on A and B which are preserved by ι, p and h. Let δ be a degree one map on B which lowers the filtration and suppose that $(d + \delta)^2 = 0$. Then the operator $\psi = \sum_{j \geq 0} (\delta h)^j \delta$ is well-defined and*

- $\iota_\infty = \iota + h\psi\iota$,
- $p_\infty = p + p\psi h$,
- *and* $h_\infty = h + h\psi h$

define a new perturbed *deformation retract datum*

$$\left[\ (A, d + p\psi\iota) \underset{p_\infty}{\overset{\iota_\infty}{\rightleftarrows}} (B, d + \delta),\ h_\infty\ \right]. \tag{3}$$

Proof. See [7, Theorem 2.3]. □

2.2. The Koszul model for local cohomology

We present a quick derivation of some aspects of local cohomology which will be relevant for us. Details can be found in [13,14,2,18].

Let R be a regular local augmented k-algebra of Krull dimension n with maximal ideal $\mathfrak{m}$. For a finitely generated R-module M we define the functor

$$\Gamma_{\mathfrak{m}} M = \{x \in M : \mathfrak{m}^k x = 0 \text{ for some } k \geq 0\}$$

of global sections with support in $\{\mathfrak{m}\}$. Recall that the right derived functors of $\Gamma_{\mathfrak{m}}$ are the local cohomology functors with respect to $\mathfrak{m}$ which we denote by $\mathrm{H}^i_{\mathfrak{m}}(-)$. We can calculate local cohomology by using the fact that there is a triangle

$$\mathrm{R}\Gamma_{\mathfrak{m}} M \longrightarrow M \longrightarrow \mathrm{R}\Gamma(U, \widetilde{M}_{|U}) \longrightarrow \mathrm{R}\Gamma_{\mathfrak{m}} M[1] \tag{4}$$

expressing $\mathrm{R}\Gamma_{\mathfrak{m}} M[1]$ as the cone of the restriction map to the open subscheme $U := \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ of $\mathrm{Spec}(R)$. Indeed, assume that $\underline{t} = \{t_1, \ldots, t_n\}$ is a system of parameters for R. This system defines a covering $\mathfrak{U}$ of the punctured spectrum U. The corresponding normalized Čech complex has graded pieces

$$\widetilde{C}^p(\mathfrak{U}, \widetilde{M}) = \bigoplus_{i_0 < \cdots < i_p} M_{t_{i_0} \cdots t_{i_p}} \tag{5}$$

with the usual differential given by the alternating sum over restriction maps. It is well-known that the complex $\widetilde{C}^\bullet(\mathfrak{U}, \widetilde{M})$ is quasi-isomorphic to $\mathrm{R}\Gamma(U, \widetilde{M}_{|U})$. The restriction map $M \to \widetilde{C}^\bullet(\mathfrak{U}, \widetilde{M})$ is given by the sum over the restriction maps $M \to M_{t_i}$. Using the above triangle (4), we can now calculate $\mathrm{R}\Gamma_{\mathfrak{m}} M[1]$ and obtain

$$\mathrm{R}\Gamma_{\mathfrak{m}} M \simeq K^\infty(\underline{t}; M) := K^\infty(\underline{t}; R) \otimes_R M$$

where

$$K^\infty(\underline{t}; R) := \bigotimes_{i=1}^{n} \left(R \longrightarrow R_{t_i}\right)$$

is called the stable cohomological Koszul complex corresponding to the sequence $\underline{t}$. We remark that this argumentation generalizes in a straightforward way, when we replace the module M by a complex of R-modules.

2.3. Generalized fractions

As in the previous section, let R be a regular local augmented k-algebra of Krull dimension n with maximal ideal $\mathfrak{m}$. Let M be a finite R-module. From the triangle (4), we deduce the existence of a surjective map

$$\mathrm{H}^{n-1}(U, \widetilde{M}) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{H}^n_{\mathfrak{m}}(M).$$

Using the normalized Čech model for $\mathrm{H}^\bullet(U, \widetilde{M})$ described in the previous section, this allows us to represent local cohomology classes by Čech cocycles. More precisely, after choosing a system of parameters $\underline{t}$, we obtain a surjective map

$$M_{t_1 \cdots t_n} \xrightarrow{\sigma_{\underline{t}}} \mathrm{H}^n_{\mathfrak{m}}(M)$$

and introduce the notation

$$\begin{bmatrix} m \\ t_1, t_2, \ldots, t_n \end{bmatrix} := \sigma_{\underline{t}}\left(\frac{m}{t_1 \cdots t_n}\right).$$

The expression on the left is called a generalized fraction. Note, that if the denominator of a generalized fraction is changed, then the map $\sigma_{\underline{t}}$ changes accordingly. Assume that $\underline{t}'$ is another system of parameters, such that

$$t_i' = \sum_{j=1}^{n} C_{ij} t_j$$

for a matrix C with coefficients in R. Then we have the transformation rule

$$\begin{bmatrix} m \\ t_1, t_2, \ \ldots, t_n \end{bmatrix} = \begin{bmatrix} \det(C)m \\ t_1', t_2', \ \ldots, t_n' \end{bmatrix}. \tag{6}$$

Detailed proofs of these statements can be found in [19,18].

2.4. Dualizing functors and Grothendieck residues

We recall the dualizing theory developed in [14, Section 4]. Let R be as in the previous section and consider the category $\mathcal{T}$ of R-modules of finite length. A functor $D : \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups is called dualizing, if

(1) D is exact,
(2) $D(k) \cong k$.

To a dualizing functor D one associates the injective R-module

$$I = \operatorname{colim}_i D(R/\mathfrak{m}^i)$$

and proves that there exists a natural equivalence of functors

$$D(-) \xrightarrow{\cong} \operatorname{Hom}_R(-, I).$$

One verifies that I is an injective hull of the residue field k, and thus dualizing functors are unique (up to non-canonical equivalence). In our situation, where R is regular local, there are two natural dualizing functors which we can consider.

(1) The functor $\operatorname{Hom}_k(-, k)$ obviously defines a dualizing functor.
(2) The functor $\operatorname{Ext}^n(-, R)$ is a dualizing functor. Indeed, $\operatorname{Ext}^\bullet(k, R)$ is concentrated in degree n and $\operatorname{Ext}^n(k, R) \cong k$, as one easily calculates via a Koszul resolution of k. But every module M of finite length can be obtained via finitely many extensions by the module k. The long exact sequence for Ext tells us that $\operatorname{Ext}^\bullet(M, R)$ is concentrated in degree n, thus $\operatorname{Ext}^n(-, R)$ is exact. The injective module corresponding to this dualizing functor is given by

$$I = \operatorname{colim}_i \operatorname{Ext}^n(R/\mathfrak{m}^i, R).$$

Directly from the definition of local cohomology, we deduce that I is isomorphic to the top local cohomology module $\mathrm{H}^n_{\mathfrak{m}}(R)$. In conclusion, we obtain a natural equivalence

$$\operatorname{Ext}^n(-, R) \cong \operatorname{Hom}_R(-, \mathrm{H}^n_{\mathfrak{m}}(R))$$

of dualizing functors.

By the uniqueness of dualizing functors, there must exist a possibly non-canonical equivalence between both functors. After identifying R with the rank 1 free R-module of top differential forms Ω^n (more precisely, we should use universally finite differentials as explained in [17]), one can actually construct a canonical identification via residues. Namely, there exists a natural map

$$\mathrm{Res}: \ \mathrm{H}^n_{\mathfrak{m}}(\Omega^n) \longrightarrow k$$

which is called the Grothendieck residue symbol. It induces an equivalence of dualizing functors

$$\mathrm{Res}_*: \ \mathrm{Hom}_R(-, \mathrm{H}^n_{\mathfrak{m}}(\Omega^n)) \xrightarrow{\cong} \mathrm{Hom}_k(-, k). \tag{7}$$

Details on the construction of the residue symbol and its natural properties can be found in [19]. We will only recall how to calculate it in terms of generalized fractions. Let us choose a regular sequence $\underline{x}$ of generators of the maximal ideal $\mathfrak{m}$ in R. This yields a trivialization of the module Ω^n by choice of the generator $d\underline{x} = dx_1 \wedge \cdots \wedge dx_n$, where $x_1, \ldots, x_n$ is a regular system of parameters for $(R, \mathfrak{m})$. Now let

$$\begin{bmatrix} \omega \\ t_1, t_2, \ldots, t_n \end{bmatrix}$$

be a generalized fraction representing an element of $\mathrm{H}^n_{\mathfrak{m}}(\Omega^n)$ in the sense of the previous section. Since $\underline{t}$ is a system of parameters, there exists i such that $\mathfrak{m}^i \subset (t_1, \ldots, t_n)$. Using the transformation rule for generalized fractions, we find

$$\begin{bmatrix} \omega \\ t_1, t_2, \ldots, t_n \end{bmatrix} = \begin{bmatrix} r\,d\underline{x} \\ x_1^i, x_2^i, \ldots, x_n^i \end{bmatrix}$$

for some $r \in R$ which can be calculated by formula (6). We embed $R \subset k[[x_1, \ldots, x_n]]$ and represent r as a power series. Expanding $\frac{r}{x_1^i \cdots x_n^i}$ as a Laurent series, the residue is given by the coefficient corresponding to $(x_1 \cdots x_n)^{-1}$.

Finally, we mention that for $k = \mathbb{C}$ we can apply analytic methods to calculate the residue symbol. In this case, we have

$$\mathrm{Res}\begin{bmatrix} \omega \\ t_1, t_2, \ldots, t_n \end{bmatrix} = \frac{1}{(2\pi i)^n} \oint_{|t_i| = \varepsilon} \frac{\omega}{t_1 \ldots t_n}$$

and we refer to [12] for a detailed treatment of duality based on this analytic definition.

3. The Kapustin–Li formula

As above let R be a regular local augmented k-algebra of finite Krull dimension n with maximal ideal $\mathfrak{m}$ and consider $w \in \mathfrak{m}$ with isolated critical locus. We can therefore obtain a system of parameters of the form

$$\underline{t} = \{\partial_1 w, \partial_2 w, \ldots, \partial_n w\}.$$

As previously, we abbreviate $T = \mathrm{MF}(R, w)$. Let X, Y be matrix factorizations and consider the morphism complex $T(X, Y)$ as a 2-periodic $\mathbb{Z}$-graded complex of free R-modules. In the introduction, we defined the diagram

$$\begin{tikzcd} \mathrm{R}\Gamma_{\mathfrak{m}}(T(X,Y)) \arrow[r, "g"] \arrow[d, "f"] & \mathrm{Hom}_R(T(Y, X[n]), \mathrm{H}^n_{\mathfrak{m}}(R)) \arrow[d, "\mathrm{Res}_*"] \\ T(X,Y) \arrow[r, dotted] & \mathrm{Hom}_k(T(Y, X[n]), k) \end{tikzcd} \tag{8}$$

and subdivided the explicit derivation of the duality pairing into three steps. In this section we will provide the details. We choose to formulate the argument in a purely $\mathbb{Z}/2$-graded setting replacing the 2-periodic mapping complexes in $T(X, Y)$ by the $\mathbb{Z}/2$-graded ones which we denote by $\mathrm{Hom}(X, Y)$. The comparison between the $\mathbb{Z}/2$-graded and the $\mathbb{Z}$-graded context is given as follows. Define the 2-periodification

$$P : C^{\mathbb{Z}/2}(R) \to C^{\mathbb{Z}}(R)$$

which extends a $\mathbb{Z}/2$-graded complex 2-periodically and the $\mathbb{Z}/2$-folding

$$F : C^{\mathbb{Z}}(R) \to C^{\mathbb{Z}/2}(R), \qquad Z \mapsto \left(\bigoplus_{k \text{ even}} Z^k\right) \oplus \left(\bigoplus_{k \text{ odd}} Z^k\right).$$

Then observe that the following holds.

Proposition 3.1. *Let A be a $\mathbb{Z}/2$-graded complex and B a bounded $\mathbb{Z}$-graded complex. Then*

(a) *we have*

$$\mathrm{Hom}_R^{\mathbb{Z}}(P(A), B) \cong P\,\mathrm{Hom}_R^{\mathbb{Z}/2}(A, F(B)),$$

in particular, after passing to homotopy classes of maps, we obtain

$$[P(A), B] \cong [A, F(B)].$$

(b) *Further we have*

$$P(A) \otimes_R^{\mathbb{Z}} B \cong P(A \otimes_R^{\mathbb{Z}/2} F(B)).$$

Thus we can translate diagram (8) into the $\mathbb{Z}/2$-graded setting in virtue of the functors P and F.

3.1. (I) Koszul model

We will use the Koszul model for the complex $\mathrm{R}\Gamma_{\mathfrak{m}}(T(X, Y))$ in which both maps f and g become explicit. As explained in Section 2.2, this model is obtained as the tensor product

$$T(X, Y) \otimes_R K^{\infty}(t_1, t_2, \ldots, t_n; R), \tag{9}$$

where $K^{\infty}(t_1, t_2, \ldots, t_n; R)$ denotes the stable Koszul complex of a system of parameters $t_1, \ldots, t_n$. As already pointed out above, since the singularity of $\mathrm{Spec}(R/w)$ is isolated, the sequence of partial derivatives of w

$$\underline{t} = \{\partial_1 w, \partial_2 w, \ldots, \partial_n w\}$$

forms a system of parameters. We fix this system for the remainder of the section. In view of Proposition 3.1(b) we replace the complex (9) by the $\mathbb{Z}/2$-graded tensor product

$$Z \otimes_R K,$$

where $Z = \mathrm{Hom}_R(X, Y)$ denotes the $\mathbb{Z}/2$-graded mapping complex and K denotes the $\mathbb{Z}/2$-folding of the stable Koszul complex. Explicitly, we denote by (K, δ) the $\mathbb{Z}/2$-graded complex

$$\bigotimes_{i=1}^{n} (R \to R_{t_i}\theta_i)$$

where θ_i are odd bookkeeping variables. In the obvious way, we introduce a graded-commutative multiplication on K. Then, the differential will simply be given by the left-multiplication

$$\delta = \sum_i \theta_i .$$

We think of the variables θ_i in K as 1-forms.

In this $\mathbb{Z}/2$-graded context we will now make diagram (8) explicit. Even though local duality is of course the underlying motivation, we will not apply any particular duality theorem, but rather reprove it explicitly in our specific situation. The complex $Z \otimes_R K$ is a $\mathbb{Z}/2$-graded model for $\mathrm{R}\Gamma_{\mathfrak{m}}(T(X, Y))$, with respect to which we can give an explicit description of f and g.

We begin with the map

$$g : \operatorname{Hom}(Y, X) \otimes_R K \longrightarrow \operatorname{Hom}(\operatorname{Hom}(Y, X[n]), \mathrm{H}^n_{\mathfrak{m}}(R))$$

which is obtained as a composition of various natural maps. The tensor evaluation isomorphism is defined as

$$\xi : Z \otimes_R K \longrightarrow \operatorname{Hom}(Z^\vee, K), \qquad F \otimes \omega \mapsto \Big[g \mapsto (-1)^{|g|(|\omega|+|F|)} g(F)\omega \Big]. \tag{10}$$

The cohomology of K is concentrated in the n-form component, and we have $\mathrm{H}^n(K) \cong \mathrm{H}^n_{\mathfrak{m}}(R)[-n]$, so there is a quasi-isomorphism $\nu : K \to \mathrm{H}^n_{\mathfrak{m}}(R)[-n]$. The induced map

$$\nu_* : \operatorname{Hom}(Z^\vee, K) \longrightarrow \operatorname{Hom}(Z^\vee, \mathrm{H}^n_{\mathfrak{m}}(R)[-n])$$

is a quasi-isomorphism as the complex $\operatorname{cone}(\nu_*)$ is acyclic. To see this, observe that the cone of ν is the $\mathbb{Z}/2$-folding of a bounded acyclic complex C. Thus, any map from a $\mathbb{Z}$-graded complex P into C factors through a brutal truncation of P from above, and is therefore null-homotopic. The statement in the $\mathbb{Z}/2$-graded category now follows from Proposition 3.1(a). Finally, there is a natural isomorphism of complexes

$$\tau : \operatorname{Hom}(Y, X) \longrightarrow \operatorname{Hom}(X, Y)^\vee, \qquad G \mapsto \operatorname{tr}(G \circ -).$$

Here tr is the graded trace map, given by

$$\operatorname{tr} : \operatorname{Hom}(X, X) \to R, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \operatorname{trace}(A) - \operatorname{trace}(D).$$

With this terminology, we have

$$g = \tau^* \circ \nu_* \circ \xi .$$

We now move on to study the map f.

3.2. (II) Homotopy inverse of f

As above, we use the notation $(Z, d) = \operatorname{Hom}_R(X, Y)$ for the $\mathbb{Z}/2$-graded mapping complex in the category $\mathrm{MF}(R, w)$ and denote the $\mathbb{Z}/2$-folded stable Koszul complex by (K, δ). Thinking of the variables θ_i in K as 1-forms, the map

$$f : Z \otimes_R K \longrightarrow Z$$

is given by projection onto the 0-form component. The following lemma is well-known.

Lemma 3.2. *Multiplication by $t_i = \partial_i w$ is null-homotopic on the complex $Z = \mathrm{Hom}_R(X, Y)$. If*

$$Q = \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix}$$

represents the twisted differential of the matrix factorization Y, then postcomposition by $\partial_i Q$ provides a homotopy of t_i with zero. In particular, the restriction of the complex Z to $\mathrm{Spec}(R_{t_i})$ is contractible with contracting homotopy given by

$$h_i = \frac{\partial_i Q}{t_i} \circ -.$$

Proof. The relation $Q^2 = w$ implies by the Leibniz rule

$$\frac{\partial Q}{\partial x_i} Q + Q \frac{\partial Q}{\partial x_i} = \frac{\partial w}{\partial x_i} \mathrm{id}_X$$

for every $1 \leq i \leq n$. This implies all assertions. $\square$

Note that, ignoring the Koszul differential δ for now, we can split

$$(Z \otimes_R K, d \otimes 1) \cong (Z, d) \oplus \left(Z \otimes \bigoplus_{\substack{i_1 < \cdots < i_l \\ l > 0}} R_{t_{i_1} \cdots t_{i_l}} \theta_{i_1} \cdots \theta_{i_l}, d \otimes 1 \right)$$

where the right-hand side summand is contractible. Indeed, by combining the homotopies from Lemma 3.2 we can form the contracting homotopy

$$h = \bigoplus_{i_1 < \cdots < i_l} \frac{1}{l} \sum_{k=1}^{l} h_{i_k}. \tag{11}$$

In other words, we obtain a deformation retract datum

$$\left[(Z, d) \underset{f}{\overset{\iota}{\rightleftarrows}} (Z \otimes_R K, d), \; -h \right]$$

where ι is the canonical inclusion. Considering the differential $d + \delta$ on the complex $Z \otimes K$ as a perturbation of d, we apply Lemma 2.1 to obtain the perturbed deformation retract

$$\left[(Z, d) \underset{f}{\overset{\iota_\infty}{\rightleftarrows}} (Z \otimes_R K, d + \delta), h_\infty \right]$$

where

$$\iota_\infty = \iota + \sum_{j \geq 0} (-h)(\delta(-h))^j \delta \iota = \sum_{j \geq 0} (-h\delta)^j \iota.$$

In particular, ι_∞ represents the desired inverse to f in the homotopy category of R-modules.

With this calculation of the inverse in hand, we now pursue a concrete description of the composite $g \circ \iota_\infty$, which is the quasi-isomorphism

$$Z \xrightarrow{\iota_\infty} Z \otimes K \xrightarrow{\xi} \mathrm{Hom}(Z^\vee, K) \xrightarrow{\tau^* \circ \nu_*} \mathrm{Hom}(T(Y, X[n]), H^n_{\mathfrak{m}}(R)) \; . \tag{12}$$

Since this involves the projection ν the only relevant summand of ι_∞ is the one mapping to the n-form component of $Z \otimes_R K$, given by $(-1)^n (h\delta)^n \iota$. More precisely, there is a commutative diagram

$$\begin{tikzcd} Z \otimes R_{t_1 \cdots t_n} \theta_1 \cdots \theta_n \arrow[r] & Z \otimes H^n_{\mathfrak{m}}(R)[-n] \arrow[d, "\cong"] \\ Z \arrow[u, "(-1)^n (h\delta)^n \iota"] \arrow[r, "g \circ \iota_\infty"] & \mathrm{Hom}(T(Y, X[n]), H^n_{\mathfrak{m}}(R)) \end{tikzcd} \quad . \tag{13}$$

To evaluate $(h\delta)^n \iota$, recall that the differential δ on the stable Koszul complex is given by the left-multiplication $\delta = \sum \theta_i$. Thus, we calculate

$$\begin{aligned} (-1)^n (h\delta)^n \iota(F) &= (-1)^n \frac{1}{n!} \sum_{\sigma \in S^n} \frac{\partial_{\sigma(1)} Q \theta_{\sigma(1)} \partial_{\sigma(2)} Q \theta_{\sigma(2)} \cdots \partial_{\sigma(n)} Q \theta_{\sigma(n)} F}{\partial_1 w \cdots \partial_n w} + \rho \\ &= (-1)^{n|F| + \binom{n+1}{2}} \frac{1}{n!} \sum_{\sigma \in S^n} \mathrm{sign}(\sigma) \frac{\partial_{\sigma(1)} Q \partial_{\sigma(2)} Q \cdots \partial_{\sigma(n)} Q F}{\partial_1 w \cdots \partial_n w} \\ &\quad \times \theta_1 \cdots \theta_n + \rho. \end{aligned} \tag{14}$$

Here the remainder ρ consists of terms whose denominator is not divisible by $\partial_1 w \cdots \partial_n w$. Therefore, ρ will be annihilated by the residue map Res_* and can thus be neglected.

3.3. (III) Grothendieck residues

The final step of the argument will make use of Grothendieck's residue symbol.

Lemma 3.3. *The cohomology modules of the mapping complex* $\mathrm{Hom}_R(X, Y)$ *are* R*-modules of finite length.*

Proof. By Lemma 3.2 the partial derivatives $\partial_i w$ act trivially on the cohomology of the complex $\mathrm{Hom}(X, Y)$. Therefore, the cohomology modules are modules over the Milnor algebra

$$\Omega_w \cong R/(\partial_1 w, \ldots, \partial_n w).$$

However, the Milnor algebra is finite dimensional over k, since we assume that the singularity of $\mathrm{Spec}(R/w)$ is isolated. Because the cohomology modules are finitely generated R-modules, and thus finitely generated Ω_w-modules, the claim follows. □

In combination with (7) from Section 2.4, the lemma implies that the map Res_* in the diagram (8) is a quasi-isomorphism. In view of (14) this leads us to the Kapustin–Li formula: the composition $\mathrm{Res}_* \circ g \circ \iota_\infty$ provides an explicit quasi-isomorphism

$$\mathrm{Hom}(X, Y) \xrightarrow{\simeq} \mathrm{Hom}_k(\mathrm{Hom}(Y, X[n]), k).$$

We reformulate this statement in terms of the corresponding pairing. We use the notation

$$(dQ)^{\wedge n} = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \partial_{\sigma(1)} Q \cdots \partial_{\sigma(n)} Q,$$

where Q is the twisted differential corresponding to X.

Theorem 3.4. *The pairing*

$$\operatorname{Hom}(X, Y) \otimes_R \operatorname{Hom}(Y, X[n]) \to k, \ (F, G) \mapsto (-1)^{\binom{n+1}{2}} \frac{1}{n!} \operatorname{Res}\begin{bmatrix} \operatorname{tr}(FG(dQ)^{\wedge n}) \\ \partial_1 w, \partial_2 w, \ldots, \partial_n w \end{bmatrix}$$

provides a homologically non-degenerate pairing on the morphism complexes of the category of matrix factorizations $\operatorname{MF}(R, w)$ *associated to the local germ of an isolated hypersurface singularity.*

Proof. We simply have to evaluate the composition

$$\operatorname{Res}_* \circ g \circ \iota_\infty(F)(G)$$

keeping track of the Koszul signs. Careful sign bookkeeping yields

$$\begin{aligned} \operatorname{Res}_* \circ g \circ \iota_\infty(F)(G) &= (-1)^{n|F| + \binom{n+1}{2} + 2n|G| + |F||G|} \frac{1}{n!} \operatorname{Res}\begin{bmatrix} \operatorname{tr}(G(dQ)^{\wedge n} F) \\ \partial_1 w, \partial_2 w, \ldots, \partial_n w \end{bmatrix} \\ &= (-1)^{\binom{n+1}{2}} \frac{1}{n!} \operatorname{Res}\begin{bmatrix} \operatorname{tr}(FG(dQ)^{\wedge n}) \\ \partial_1 w, \partial_2 w, \ldots, \partial_n w \end{bmatrix}. \quad \square \end{aligned}$$

Remark 3.5. We conclude this section with a comparison to the approach in [21]. For an isolated Gorenstein singularity A the analogue of the category $\operatorname{MF}(R, w)$ is the stable category of maximal Cohen–Macaulay (CM) modules $\underline{\operatorname{CM}}(A)$, and following Buchweitz [3] we identify this category with the homotopy category of acyclic complexes of finitely generated free A-modules. The equivalence sends a CM module to its complete free resolution which, viewed as an infinite sequence of matrices, generalizes the notion of a matrix factorization (which may be viewed as a two-periodic complete free resolution). In [21] the perturbation lemma is applied to these complete free resolutions to obtain explicit complete injective resolutions, which give rise to a duality isomorphism in $\underline{\operatorname{CM}}(A)$ specializing to the Kapustin–Li formula when $A = R/w$.

In the present article we exploit the fact that in the hypersurface case we can apply local duality and the perturbation lemma directly to the morphism complexes $T(X, Y)$, which allows us to avoid the introduction of complete injective resolutions.

4. The boundary–bulk map

The work in this section should be seen as an addendum to the results in [22]. We establish an explicit formula for the boundary–bulk map which is adapted to the Kapustin–Li formula in Theorem 3.4 on the chain level. Again, R denotes a regular local augmented k-algebra of Krull dimension n with maximal ideal $\mathfrak{m}$, where k is a field of characteristic 0. However, in order for the tensor product $R \otimes_k R$ to be well-behaved, we also assume that R is essentially of finite type over k.

Remark 4.1. A little caution is required, since the ring $R \otimes_k R$ is almost never local. However, for $w \in \mathfrak{m}$ with isolated critical locus, the critical locus of the element $\widetilde{w} = 1 \otimes w - w \otimes 1$ is supported over a unique maximal ideal $\widetilde{\mathfrak{m}}$ of $R \otimes_k R$. Thus, as shown in [10, Section 4.6], the matrix factorization category of $(R \otimes_k R, \widetilde{w})$ is naturally quasi-equivalent to the one of the localization $((R \otimes_k R)_{\widetilde{\mathfrak{m}}}, \widetilde{w})$.

Remark 4.2. The additional finiteness condition on R excludes for example power series algebras $k[[x_1, \dots, x_n]]$. However, in view of the quasi-equivalence

$$\mathrm{MF}^{\infty}(R, w) \simeq \mathrm{MF}^{\infty}(\widehat{R}, w)$$

of [10, Section 5.4] and the finite determinancy result [8, Theorem 2.1], no generality is lost. Alternatively, one could reformulate all proofs using power series rings and completed tensor products.

4.1. Morita-theoretic construction

We recall the definition of Hochschild homology in the context of Toën's derived Morita theory for dg categories ([26,25]). More precisely, we will use the 2-periodic variant defined in [10, Section 5.1]. Let T be a 2-periodic dg category which we may consider as a module over $T^{\mathrm{op}} \otimes T$ via

$$T^{\mathrm{op}} \otimes T \to C(k[u, u^{-1}]), (x, y) \mapsto T(x, y)$$

for objects x, y in T. By [26, Theorem 7.2], this map has a continuous extension

$$\underline{\mathrm{tr}} : \widehat{T^{\mathrm{op}} \otimes T} \to C(k[u, u^{-1}])$$

which is unique up to homotopy. The induced map on homotopy categories yields

$$[\underline{\mathrm{tr}}] : \mathrm{D}(T^{\mathrm{op}} \otimes T) \to \mathrm{D}(k[u, u^{-1}])$$

and the Hochschild homology of the category T is defined to be $[\underline{\mathrm{tr}}](T)$. Thus, the Hochschild homology is the trace of the identity functor which we may choose to think of as the dimension of T.

Now, every object e in T, defines an object (e, e) of $T^{\mathrm{op}} \otimes T$ which induces a representable functor

$$h^{(e,e)} : T^{\mathrm{op}} \otimes T \to C(k[u, u^{-1}]), (x, y) \mapsto T(x, e) \otimes T(e, y).$$

The composition law in T provides us with a natural map

$$\pi_e : \; h^{(e,e)} \to T \tag{15}$$

in $\mathrm{D}(T^{\mathrm{op}} \otimes T)$ and thus we obtain an induced map

$$[\underline{\mathrm{tr}}](\pi_e) : \; T(e, e) \simeq [\underline{\mathrm{tr}}](h^{(e,e)}) \to [\underline{\mathrm{tr}}](T) \simeq \mathrm{HH}(T) \tag{16}$$

in $\mathrm{D}(k[u, u^{-1}])$, which we call the *boundary–bulk map*.

The derived Morita theory developed by Toën can be used in the context of matrix factorization categories to calculate the Hochschild chain complex $\mathrm{HH}(T)$ (see [10]). In recent work of Polishchuk and Vaintrob [22], an explicit formula for the boundary–bulk map is calculated. We will provide a (homotopic) variant of the formula which is better adapted to the form of the Kapustin–Li pairing from Theorem 3.4. Following the method in *ibid*, we will describe the map in the context of derived Morita theory. The compatibility between the Kapustin–Li pairing and the boundary–bulk map will lead to the existence of an oriented 2-dimensional topological quantum field theory as discussed in Section 5.

As in the previous sections, we fix the notation $T = \mathrm{MF}(R, w)$. We also introduce $\widetilde{R} = R \otimes_k R$ and $\widetilde{w} = 1 \otimes w - w \otimes 1$. Using the results of [10], we have an equivalence

$$\mathrm{D}(T^{\mathrm{op}} \otimes T) \simeq \mathrm{MF}^{\infty}(\widetilde{R}, \widetilde{w}). \tag{17}$$

Given a matrix factorization E in the category $\mathrm{MF}(R, w)$, the representable module which corresponds under the above equivalence to $h^{(E,E)}$ is $E^{\vee} \otimes_k E$. By [10, Proposition 6.3], the identity functor is represented by the stabilized diagonal Δ^{stab}. We have to identify the natural map

$$\varphi_E : E^{\vee} \otimes_k E \to \Delta^{\mathrm{stab}}$$

in $\mathrm{MF}^{\infty}(\widetilde{R}, \widetilde{w})$ which corresponds under the equivalence (17) to the map π_E. By [10, Lemma 4.2], there is a quasi-isomorphism

$$\mathrm{Hom}_{\widetilde{R}}(E^{\vee} \otimes_k E, \Delta^{stab}) \simeq \mathrm{Hom}_{\widetilde{R}}(E^{\vee} \otimes_k E, R) \tag{18}$$

where R is the diagonal $\widetilde{R}/\widetilde{w}$-module. The symbol $\mathrm{Hom}_{\widetilde{R}}$ refers to the $\mathbb{Z}/2$-graded $\widetilde{R}$-linear mapping complex and we keep this convention throughout this section. The right-hand side complex is in turn quasi-isomorphic to the R-linear mapping complex $\mathrm{Hom}_R(E, E)$.

Lemma 4.3 (*[22]*). *The composition map π_E corresponds to the unique class φ_E in $\mathrm{Hom}_{\widetilde{R}}(E^{\vee} \otimes_k E, \Delta^{stab})$ which maps to id_E under the quasi-isomorphism (18).*

Proof. The composition functor

$$T(-, E) \otimes T(E, -) \to T(-, -)$$

is uniquely characterized by the property that it maps $\mathrm{id} \otimes \mathrm{id}$ in $T(E, E) \otimes T(E, E)$ to the identity in $T(E, E)$. We interpret this statement in the category $\mathrm{MF}^{\infty}(\widetilde{R}, \widetilde{w})$: the map

$$\varphi_E : E^{\vee} \otimes_k E \to \Delta^{\mathrm{stab}}$$

is characterized by the property that the induced map

$$\begin{aligned}(\varphi_E)_* : \ &\mathrm{Hom}_{\widetilde{R}}(E^{\vee} \otimes_k E, E^{\vee} \otimes_k E) \to \mathrm{Hom}_{\widetilde{R}}(E^{\vee} \otimes_k E, \Delta^{\mathrm{stab}}) \\ &\cong \mathrm{Hom}_R(E, E), \quad f \mapsto \varphi_E \circ f\end{aligned}$$

maps $\mathrm{id}_{E^{\vee} \otimes_k E}$ to id_E. But this proves the claim, since $(\varphi_E)_*(\mathrm{id}) = \varphi_E$. □

To find φ_E we therefore have to find an explicit homotopy inverse of the quasi-isomorphism (18). Again, we will use the basic perturbation lemma to provide a solution to this problem. For a regular system of parameters $\{x_i\}$ of $\mathfrak{m}$ in R, the sequence $\{\Delta_i = x_i - y_i\}$ forms a regular system for the diagonal ideal in $\widetilde{R} = R \otimes_k R$. By [10, Section 2.3], we obtain the explicit description

$$\Delta^{\mathrm{stab}} = (\widetilde{R}\langle \theta_1, \ldots, \theta_n \rangle, \iota_{\Delta} + \epsilon_{\lambda})$$

where $\widetilde{w} = 1 \otimes w - w \otimes 1 = w_1 \Delta_1 + \cdots + w_n \Delta_n \in \widetilde{R}$, ι_{Δ} is contraction with the element $\Delta_1 \theta_1^{\vee} + \cdots + \Delta_n \theta_n^{\vee}$ and ϵ_{λ} denotes exterior left multiplication with the element

$$\lambda = w_1 \theta_1 + \cdots + w_n \theta_n. \tag{19}$$

Note that the coefficients $\{w_i\}$ are not unique, different choices of $\{w_i\}$ will lead to different (but isomorphic) models of Δ^{stab}. As explained below (Section 4.3) we will make a specific canonical choice for these coefficients during the calculation. We reformulate

$$\mathrm{Hom}_{\widetilde{R}}(E^\vee \otimes_k E, \Delta^{\mathrm{stab}}) \cong \mathrm{Hom}_{\widetilde{R}}(E_y, E_x) \otimes_{\widetilde{R}} \widetilde{R}\langle \theta_1, \ldots, \theta_n \rangle \tag{20}$$

where we set $E_x := E \otimes_R \widetilde{R}$ and $E_y := E \otimes_R \widetilde{R}$ using the embeddings of R into the first and the second component of $\widetilde{R} = R \otimes_k R$, respectively. We denote the twisted differentials on E_x and E_y by Q_x and Q_y, respectively, which satisfy $Q_x^2 = w \otimes 1$ and $Q_y^2 = 1 \otimes w$. Via (20), we can thus think of elements in $\mathrm{Hom}_{\widetilde{R}}(E^\vee \otimes_k E, \Delta^{\mathrm{stab}})$ as supermatrix-valued differential forms. For a homogeneous element

$$A \otimes \eta \in \mathrm{Hom}_{\widetilde{R}}(E_y, E_x) \otimes_{\widetilde{R}} \widetilde{R}\langle \theta_1, \ldots, \theta_n \rangle$$

we define

$$\begin{aligned} d_Q(A \otimes \eta) &= (Q_x A - (-1)^{|A|} A Q_y) \otimes \eta \\ \iota_\Delta(A \otimes \eta) &= (-1)^{|A|} A \otimes \iota_\Delta(\eta) \\ \epsilon_\lambda(A \otimes \eta) &= (-1)^{|A|} A \otimes \epsilon_\lambda(\eta). \end{aligned}$$

The differential on the complex

$$\mathrm{Hom}_{\widetilde{R}}(E_y, E_x) \otimes_{\widetilde{R}} \widetilde{R}\langle \theta_1, \ldots, \theta_n \rangle$$

is then given by $d_Q + \epsilon_\lambda + \iota_\Delta$. Note that we may interpret this differential as a perturbation of the Koszul differential ι_Δ by $\delta = d_Q + \epsilon_\lambda$. This allows us to apply the basic perturbation lemma. Indeed, the Koszul complex

$$(\widetilde{R}\langle \theta_1, \ldots, \theta_n \rangle, \iota_\Delta) \tag{21}$$

has cohomology R concentrated in degree 0 which allows us to define the deformation retract

$$\left[\, (R, 0) \underset{p}{\overset{\iota}{\rightleftarrows}} (\widetilde{R}\langle \theta_1, \ldots, \theta_n \rangle, \iota_\Delta), \; -H \, \right].$$

Here, p is the projection onto the cohomology, ι is the inclusion of R into the first component of $\widetilde{R}$ and H is a k-linear homotopy which contracts the complex (21) onto its cohomology. Observe that we have an isomorphism of graded vector spaces

$$\mathrm{Hom}_{\widetilde{R}}(E_y, E_x) \otimes_{\widetilde{R}} \widetilde{R}\langle \theta_1, \ldots, \theta_n \rangle \cong \mathrm{Mat}_n(k) \otimes_k \widetilde{R}\langle \theta_1, \ldots, \theta_n \rangle$$

where $r = \mathrm{rank}(E)$ and $\mathrm{Mat}_r(k)$ denotes the $\mathbb{Z}/2$-graded vector space of r-by-r supermatrices. This allows us to extend the above deformation retract linearly to obtain the retract

$$\left[\, (\mathrm{Hom}_{\widetilde{R}}(E_y, E_x) \otimes_{\widetilde{R}} R, 0) \underset{p}{\overset{\iota}{\rightleftarrows}} (\mathrm{Hom}_{\widetilde{R}}(E_y, E_x) \otimes_{\widetilde{R}} \widetilde{R}\langle \theta_1, \ldots, \theta_n \rangle, \iota_\Delta), \; -H \, \right].$$

The basic perturbation lemma allows us to perturb this retract by $\delta = d_Q + \epsilon_\lambda$ to obtain the following result.

Proposition 4.4. *A homotopy inverse of the projection map*

$$(\mathrm{Hom}_{\widetilde{R}}(E_y, E_x) \otimes_{\widetilde{R}} \widetilde{R}\langle \theta_1, \ldots, \theta_n \rangle, d_Q + \iota_\Delta + \epsilon_\lambda) \to (\mathrm{Hom}_R(E, E), d_Q)$$

is given by the map

$$\iota_\infty = \sum_{k=0}^{n} (-H\delta)^k \iota.$$

To find an explicit formula for ι_∞, we thus have to construct an explicit contracting homotopy H of the Koszul complex (21). We will do this in the next section, but before that, let us conclude how to obtain the boundary–bulk map from $\iota_\infty(\mathrm{id})$.

Lemma 4.5. *Let $\eta = \iota_\infty(\mathrm{id})$ with $\eta_n = B\theta_1 \ldots \theta_n$. Then the boundary–bulk map is given explicitly by*

$$[\underline{\mathrm{tr}}](\pi_E) : \mathrm{Hom}_R(E, E) \to \Omega_w[n], \ F \mapsto (-1)^{n|F|} \operatorname{tr}(BF).$$

Proof. The map

$$\varphi_E : \mathrm{Hom}_{\widetilde{R}}(E_x, E_y) \to \Delta^{\mathrm{stab}}$$

corresponding to η, is given by

$$F \mapsto (-1)^{n|F|} \operatorname{tr}(BF)\theta_1 \cdots \theta_n + l. \tag{22}$$

Here the sign is contributed by the tensor evaluation map (10) and l consists of terms involving k-forms with $k < n$ which are, as we will see, irrelevant. From Lemma 6.8 in [10], we deduce that we have

$$\underline{\mathrm{tr}} : \mathrm{MF}(\widetilde{R}, \widetilde{w}) \to C(k[u, u^{-1}]), \ X \mapsto \mathrm{Hom}_{\widetilde{R}}(\Delta^{\mathrm{stab}}_{\widetilde{w}}[n], X) \cong (\Delta^{\mathrm{stab}}_{\widetilde{w}}[n])^\vee \otimes_{\widetilde{R}} X.$$

Now, directly from the definition of Δ^{stab}, it is easy to see that $(\Delta^{\mathrm{stab}}_{\widetilde{w}}[n])^\vee$ is isomorphic to $\Delta^{\mathrm{stab}}_{-\widetilde{w}}$. By the perturbation argument in the proof of Proposition 4.4 we deduce in complete analogy that the projection map

$$\Delta^{\mathrm{stab}}_{-\widetilde{w}} \otimes_{\widetilde{R}} X \longrightarrow R \otimes_{\widetilde{R}} X$$

has a homotopy inverse, in particular it is a quasi-isomorphism. Thus, we have an equivalence $[\underline{\mathrm{tr}}] \simeq R \otimes_{\widetilde{R}} -$. The cohomology of the Hochschild chain complex $R \otimes_{\widetilde{R}} \Delta^{\mathrm{stab}}$ is concentrated in degree n (i.e. the parity of n due to the $\mathbb{Z}/2$-grading) where it is isomorphic to the Milnor algebra Ω_w. More precisely the cohomology is concentrated in the n-form component. Therefore, projection from the Hochschild complex onto $\Omega_w\theta_1 \ldots \theta_n$ is a quasi-isomorphism. In view of (22), postcomposing $[\underline{\mathrm{tr}}](\varphi_E)$ with this projection results in the asserted formula. $\square$

Note that this lemma corresponds to [22, 3.1.1].

4.2. Canonical contracting homotopy of the Koszul complex

In this section, we construct an explicit canonical homotopy which contracts the Koszul complex

$$K(\Delta_1, \Delta_2, \ldots, \Delta_n) = \bigotimes_{i=1}^{n} (k[x_i, y_i] \xrightarrow{\Delta_i} k[x_i, y_i])$$

onto its cohomology. This will serve the purpose of finding an explicit expression for the homotopy inverse ι_∞ from Proposition 4.4. Aside from that the result may be considered interesting in its own right. For a more general and more conceptual perspective on the homotopy constructed in this section, we refer the reader to Section 8.1 in [9].

Remark 4.6. To simplify the notation, we will construct the homotopy over the polynomial ring $k[x_1, y_1, \dots, x_n, y_n]$. The argumentation carries over verbatim to the corresponding power series ring which we are really interested in. Indeed, one simply has to replace all tensor products over k by completed tensor products and extend the maps continuously.

For each $1 \le i \le n$, consider the augmented Koszul complex of the (length 1) sequence Δ_i in $k[x_i, y_i]$

$$k[x_i, y_i]\theta_i \xrightarrow{d_i} k[x_i, y_i] \xrightarrow{d_i} k[x_i]\xi_i.$$

So we have

$$d_i(f\theta_i) = f\Delta_i$$

and

$$d_i(f) = f \pmod{\Delta_i}.$$

We have canonical contracting homotopies h_i which are defined as follows. An element $f \in k[x_i, y_i]$ can be uniquely written as

$$f = f_0 + \Delta_i f_1 \quad \text{with } f_0 \in k[x_i],$$

and we define

$$h_i : k[x_i, y_i] \to k[x_i, y_i]\theta_i, \ f \mapsto f_1\theta_i \tag{23}$$

which we may think of as division by Δ_i without remainder. Furthermore, we let

$$h_i : k[x_i]\xi_i \to k[x_i, y_i], \ f\xi_i \mapsto f$$

be the inclusion. The variables ξ_i and θ_i are graded commutative bookkeeping variables of degree 1 and -1 respectively. We define the *extended Koszul complex* EK of the sequence $\{\Delta_1, \dots, \Delta_n\}$ to be the tensor product over k of the augmented Koszul complexes. Using the Koszul sign rule one can easily check that the map

$$h = \frac{1}{n}(h_1 + h_2 + \cdots + h_n)$$

defines a contracting homotopy on the extended Koszul complex. Note that as graded vector spaces, we have

$$EK = E \oplus K$$

where K is the graded space underlying the usual Koszul complex of $\{\Delta_1, \dots, \Delta_n\}$. In terms of the bookkeeping variables, K consists of those elements which do not have any ξ_i terms. We call K the *interior*, E the *exterior* of EK. The picture we have in mind is a half-open hypercube whose faces constitute E. The i-th face in E is given by those elements which are multiples of a ξ_i. Note that each face is a Koszul complex of one variables less.

We would like to use the contracting homotopy on EK to contract K onto its cohomology. The issue is, however, that even though K is stable under h, it is not stable under the differential d

on EK. Nevertheless, we can construct a canonical perturbation of h which provides an explicit contracting homotopy of K. To this end, we introduce one last bit of notation. We define the maps

$$\mathrm{pr}_i : K(\Delta_1, \Delta_2, \dots, \Delta_n) \to k[x_i] \otimes_k K(\Delta_1, \dots, \widehat{\Delta_i}, \dots, \Delta_n)$$

where $\mathrm{pr}_i\, \omega$ is obtained from ω via substituting y_i by x_i and removing all terms which are multiples of θ_i. Note that we can naturally think of the right-hand side Koszul complex as a subcomplex of $K(\Delta_1, \Delta_2, \dots, \Delta_n)$. This allows us to abuse notation and consider the element $\mathrm{pr}_i\, \omega$ as an element of $K(\Delta_1, \Delta_2, \dots, \Delta_n)$. One easily verifies that pr_i is a map of complexes.

Lemma 4.7. *There exists a unique family $\{H^{(n)} \mid n \geq 1\}$ of homotopies $H^{(n)}$ of the Koszul complexes $K(\Delta_1, \Delta_2, \dots, \Delta_n)$ satisfying the following.*

1. *The homotopy $H^{(1)}$ agrees with the one defined in (23).*
2. *For $n > 1$ we have the recursive formula*

$$H^{(n)} = h + \frac{1}{n} \sum_i H^{(n-1)} \circ \mathrm{pr}_i\,. \tag{24}$$

Each homotopy $H^{(n)}$ contracts the corresponding Koszul complex onto its cohomology.

Proof. We argue by induction on n. We decompose the differential d on EK into $d = d_{\mathrm{ext}} + d_K$, where $d_{\mathrm{ext}} = \mathrm{pr}_E \circ d$ and $d_K = \mathrm{pr}_K \circ d$. Similarly, the homotopy h does not preserve E and we have $h = h_E + h_{\mathrm{int}}$ with $h_E = \mathrm{pr}_E \circ h$ and $h_{\mathrm{int}} = \mathrm{pr}_K \circ h$. First, let ω be an element of negative degree in K. Then we have

$$\begin{aligned} \omega &= [d, h]\omega \\ &= d_{\mathrm{ext}} h\omega + d_K h\omega + h d_K \omega + h d_{\mathrm{ext}}\omega \\ &= [d_K, h]\omega + h_{\mathrm{int}} d_{\mathrm{ext}}\omega \end{aligned} \tag{25}$$

where the last equality follows since ω lies in K and so all exterior components must cancel out. Directly from the definitions we calculate

$$h_{\mathrm{int}} d_{\mathrm{ext}}\omega = \frac{1}{n}(\mathrm{pr}_1\, \omega + \mathrm{pr}_2\, \omega + \cdots + \mathrm{pr}_n\, \omega). \tag{26}$$

Assume, $H^{(n-1)}$ is a contracting homotopy for the Koszul complex in $n - 1$ variables. Then, we calculate

$$\begin{aligned} [d_K, H^{(n)}]\omega &= [d_K, h]\omega + \left[d_K, \frac{1}{n} \sum_i H^{(n-1)} \circ \mathrm{pr}_i \right] \omega \\ &= [d_K, h]\omega + \frac{1}{n}(\mathrm{pr}_1\, \omega + \mathrm{pr}_2\, \omega + \cdots + \mathrm{pr}_n\, \omega) \\ &\qquad \text{(since } d_K \text{ commutes with } \mathrm{pr}_i) \\ &= \omega \quad \text{(use (25) and (26))}. \end{aligned}$$

Now let f be an element of degree 0 in K. Define the augmentation maps

$$p^{(n)} : \widetilde{R} \to \widetilde{R}/(\Delta_1, \dots, \Delta_n) \cong k[x_1, \dots, x_n] \subset \widetilde{R}.$$

We have to show that

$$d_K H^{(n)} f + p^{(n)} f = f.$$

Again, we argue inductively and calculate

$$\begin{aligned} d_K H^{(n)} f + p^{(n)} f &= d_K hf + \left(\frac{1}{n} \sum_j d_K H^{(n-1)} \operatorname{pr}_j f \right) + p^{(n)} f \\ &= d_K hf + \frac{1}{n} \sum_j (d_K H^{(n-1)} \operatorname{pr}_j f + p^{(n-1)} \operatorname{pr}_j f) \\ &= \frac{1}{n} \left(\sum_j (d_j h_j f + \operatorname{pr}_j f) \right) \\ &= \frac{1}{n} \left(\sum_j f \right) = f. \end{aligned}$$

This proves all assertions. $\square$

We give an explicit formula for $H^{(n)}$.

Corollary 4.8. *Explicitly, we have*

$$H^{(n)} = (h_1 + h_2 + \cdots + h_n) \circ P^{(n)}$$

where

$$\begin{aligned} P^{(n)} &= \sum_{l=0}^{n-1} a(l) \sum_{j_1 < j_2 < \cdots < j_l} \operatorname{pr}_{j_1} \circ \operatorname{pr}_{j_2} \circ \cdots \circ \operatorname{pr}_{j_l} \\ &= \frac{1}{n} \operatorname{id} + \frac{1}{n(n-1)} \sum_j \operatorname{pr}_j + \cdots \end{aligned} \tag{27}$$

and

$$a(l) = \frac{1}{n-l} \binom{n}{l}^{-1}.$$

Proof. This can be easily deduced from the recursive properties of $H^{(n)}$. $\square$

We will never actually make use of this explicit formula in our calculation. In fact, we will only need various simple properties of $H^{(n)}$ and its components which are collected in the following proposition. For a k-form ω in $K(\Delta_1, \Delta_2, \ldots, \Delta_n)$, we let $\omega(y_i)$ be the k-form obtained from ω via substituting x_i by y_i. Analogously, we define $\omega(x_i)$. Note the difference between $\operatorname{pr}_i \omega$ and $\omega(x_i)$.

Proposition 4.9. (1) *The homotopies h_i and h_j anticommute for all i, j.*
(2) *For ω in $K(\Delta_1, \Delta_2, \ldots, \Delta_n)$ we have*

$$h_i(\omega(y_i)) \equiv \theta_i \frac{\partial}{\partial x_i} \omega(x_i) \pmod{\Delta_i}.$$

(3) *We have*

$$h_i \circ P^{(n)} = P^{(n)} \circ h_i$$

for the iterated projection map from Corollary 4.8.

(4) *Let $k > 0$ and let ω be a k-form in $K(\Delta_1, \Delta_2, \dots, \Delta_n)$. Then we have*

$$P^{(n)}(\omega) \equiv \frac{1}{k}\omega \pmod{\Delta_1, \dots, \Delta_n}.$$

Proof. (1) This is immediate from the construction.

(2) Expanding the coefficients of $\omega(y_i)$ into its Taylor series with respect to the variable x_i, we obtain

$$\omega(y_i) = \omega(x_i) + \Delta_i \frac{\partial}{\partial x_i}\omega(x_i) + \frac{\Delta_i^2}{2}\left(\frac{\partial}{\partial x_i}\right)^2 \omega(x_i) + \cdots,$$

which implies the assertion.

(3) The projection maps pr_i commute with h_j for all i, j. Note that $\mathrm{pr}_i\, h_i = h_i\, \mathrm{pr}_i = 0$.

(4) For $k = n$, we have

$$P^{(n)}\omega = \frac{1}{n}\omega$$

since all maps pr_j annihilate ω. We proceed by induction on $n - k$. The iterative projection map satisfies the recursion formula

$$P^{(n)} = \frac{1}{n}\left(\mathrm{id} + \sum_j P^{(n-1)} \circ \mathrm{pr}_j\right)$$

which allows us to proceed by induction on $n - k$. It suffices to prove the statement for $\omega = f\theta_{i_1}\theta_{i_2}\cdots\theta_{i_k}$. In this case, we have

$$\begin{aligned}
P^{(n)}\omega &= \frac{1}{n}\left(\omega + \sum_j P^{(n-1)} \circ \mathrm{pr}_j\right) \\
&= \frac{1}{n}\left(\omega + \sum_{j \neq i_l} P^{(n-1)} f(x_j)\theta_{i_1}\theta_{i_2}\cdots\theta_{i_k}\right) \\
&= \frac{1}{n}\left(\omega + (n-k)\frac{1}{k}\omega\right) \pmod{\Delta_1, \dots, \Delta_n} \\
&= \frac{1}{k}\omega \pmod{\Delta_1, \dots, \Delta_n}
\end{aligned}$$

which proves our claim. □

4.3. An explicit formula

In this section, we implicitly replace the algebra R from Section 4.1 by its completion $\widehat{R}_{\mathfrak{m}} \cong k[[x_1, \dots, x_n]]$. This is justified by the natural quasi-equivalences

$$\begin{aligned}
&\mathrm{MF}^{\infty}(R, w) \simeq \mathrm{MF}^{\infty}(\widehat{R}_{\mathfrak{m}}, w) \\
&\mathrm{MF}^{\infty}(R \otimes_k R, \widetilde{w}) \simeq \mathrm{MF}^{\infty}(\widehat{R \otimes_k R}_{\widetilde{\mathfrak{m}}}, \widetilde{w})
\end{aligned} \tag{28}$$

from Theorem 5.7 in [10]. All constructions of Section 4.2 hold verbatim over

$$\widehat{R \otimes_k R}_{\widetilde{\mathfrak{m}}} \cong k[[x_1, \ldots, x_n, y_1, \ldots, y_n]]$$

instead of $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by using completed tensor products (cf. Remark 4.6). We will now use the canonical Koszul homotopy to calculate $\eta = \iota_\infty(\mathrm{id})$ with ι_∞ from Proposition 4.4. In fact, in view of Lemma 4.5, we only need an explicit formula for η_n modulo $(\Delta_1, \Delta_2, \ldots, \Delta_n)$. We fix n and denote the canonical contracting homotopy $H^{(n)}$ and the iterative projection map $P^{(n)}$ of the previous section by H and P respectively. At this point, we choose the 1-form $\lambda = w_1\theta_1 + w_2\theta_2 + \cdots + w_n\theta_n$ from (19) to be given by $H(\widetilde{w})$, where $\widetilde{w} = 1 \otimes w - w \otimes 1 \in \widehat{R \otimes_k R}$. As already mentioned above, this leads to a canonical choice of a model for Δ^{stab}.

Theorem 4.10. *With the above choices we have*

$$\eta_n = (-1)^{\binom{n+1}{2}} \frac{1}{n!} (dQ)^n \pmod{\Delta_1, \ldots, \Delta_n}.$$

Proof. Let us introduce $H_i = h_i \circ P$. By Proposition 4.4, we have to calculate

$$(\iota_\infty(\mathrm{id}))_n = (-1)^n (H\delta)^n(\mathrm{id}).$$

In the following calculation, we use the convention to sum over all indices which appear (Einstein's sum convention). In addition to the usual Koszul sign rule, we will use the following key facts.

(i) All terms involving the composition $h_i h_j$ vanish after summing over all indices (4.9 (1)).
(ii) The operators h_i satisfy the Leibniz rule modulo Δ_i (4.9 (2)).
(iii) The projection operator P commutes with h_i (4.9 (3)).
(iv) By choice, we have $\lambda = h_i P(\widetilde{w})$.

We start by calculating

$$\begin{aligned}(H\delta)(\mathrm{id}) &= h_i P(d_Q(\mathrm{id}) + \lambda) = P(h_i(Q_x - Q_y) + h_i(h_j P(\widetilde{w}))\\ &= -P h_i(Q_y).\end{aligned}$$

Proceeding, we find

$$\begin{aligned}(H\delta)^2(\mathrm{id}) &= -h_j P[Q_x P(h_i(Q_y)) - P(h_i(Q_y))Q_y + \lambda P(h_i Q_y)]\\ &= -P[Q_x P(h_j(h_i(Q_y))) - P(h_j(h_i(Q_y)))Q_y - P(h_i(Q_y))h_j(Q_y)\\ &\quad + h_j(\lambda)(h_i Q_y) - \lambda(h_j(h_i Q_y))] \pmod{\Delta_j}\\ &= P(P(h_i(Q_y))h_j(Q_y)) \pmod{\Delta_j},\end{aligned}$$

where most of the terms vanish thanks to (i). An iteration of this argument leads to the formula

$$\begin{aligned}(H\delta)^n(\mathrm{id}) &= (-1)^n P(P(\ldots P(h_{i_1}(Q_y))h_{i_2}(Q_y)\ldots)h_{i_n}(Q_y)) \pmod{\Delta_1, \ldots, \Delta_n}\\ &= (-1)^n(-1)^{\binom{n+1}{2}} \frac{1}{n!} \partial_{i_1} Q \partial_{i_2} Q \cdots \partial_{i_n} Q \theta_{i_1}\theta_{i_2}\cdots\theta_{i_n} \pmod{\Delta_1, \ldots, \Delta_n},\end{aligned}$$

where the last equality follows from Proposition 4.9 (4) and the Koszul sign interaction of θ_i with Q. Incorporating the additional sign $(-1)^n$ leads to the claimed formula. □

Combining this result with Lemma 4.5, we obtain an explicit formula for the boundary–bulk map.

Theorem 4.11. *The boundary–bulk map admits the explicit formula*

$$[\underline{\mathrm{tr}}](\varphi_E): \mathrm{Hom}(E,E) \to \Omega_w[n], \qquad F \mapsto (-1)^{\binom{n+1}{2}} \frac{1}{n!} \mathrm{tr}(F(dQ)^{\wedge n}) \tag{29}$$

where Q is the twisted differential corresponding to E.

Proof. Note that the sign contribution $n|F|$ cancels by using the cyclic symmetry of the graded trace map. □

Note that on cohomology, our formula and the one given in [22] produce (up to sign) the same map. The difference is that our map is well adapted to the Kapustin–Li pairing on the chain level.

5. The Calabi–Yau structure and topological quantum field theories

5.1. Topological field theories

In this section, we explain the relevance of the category $\mathrm{MF}(R,w)$ as a category of boundary conditions in the context of topological quantum field theories of various flavors.

In light of the results in [10] we expect that there exists a 2-dimensional framed extended topological field theory in the sense of [20] which maps the trivially-framed point to the category $\mathrm{MF}(R,w)$. Roughly, the argumentation for this would go as follows. We consider $\mathrm{MF}(R,w)$ as an object of an appropriately defined $(\infty,2)$-category $\mathcal{C}$ of 2-periodic dg categories. In analogy to the nonperiodic situation, the smoothness and properness of $\mathrm{MF}(R,w)$ established in [10] will imply that this category is a fully dualizable object in $\mathcal{C}$. The assertion then follows from [20, 2.4.6].

As first established by Auslander, the triangulated category $[\mathrm{MF}(R,w)]$ admits a Calabi–Yau structure. This suggests that the category $\mathrm{MF}(R,w)$ will be a Calabi–Yau object in $\mathcal{C}$ in the sense of [20, 4.2.6]. The results of this work, will allow us to explicitly construct a Calabi–Yau structure on the dg category $\mathrm{MF}(R,w)$. In view of [20, 4.2.11], this implies the existence of a 2-dimensional oriented extended topological field theory.

Alternatively, using a theorem of Kontsevich and Soibelman [16, 10.2.2] (also cf. [5] for more details), the results of Section 5.2 imply the existence of a minimal A_∞ model on which the Calabi–Yau pairing has strictly cyclic symmetry. This implies the existence of an open-closed field theory in the sense of [6, Theorem A], where this notion of a strict Calabi–Yau A_∞ algebra is used (see [6, 7.2]). Using Costello's framework we will explain how to deduce a Riemann–Roch formula from the existence of the field theory. The formula presumably agrees with the one recently established in [22] (building on the work of [24]).

5.2. Calabi–Yau dg algebras

To put us into context, recall that a Frobenius algebra is a unital k-algebra A together with a non-degenerate pairing

$$A \otimes_k A \longrightarrow k, \qquad a \otimes b \mapsto \langle a, b\rangle$$

which satisfies

$$\langle ab, c\rangle = \langle bc, a\rangle = \langle ca, b\rangle \tag{30}$$

for all elements a, b, c in A. Equivalently, we could formulate the definition in terms of the trace map

$$\mathrm{tr}: \ A \to k, \qquad a \mapsto \langle a, 1 \rangle ,$$

the corresponding pairing is then recovered as $\langle a, b\rangle = \mathrm{tr}(ab)$. The cyclic symmetry can be reformulated by saying that there exists a commutative diagram

$$\begin{tikzcd} A \otimes_k A \arrow[r, "\langle -,- \rangle"] \arrow[d] & k \\ A \otimes_{A \otimes_k A^{\mathrm{op}}} A \arrow[ur, dotted] & \end{tikzcd} \tag{31}$$

or, in terms of the trace map,

$$\begin{tikzcd} A \arrow[r, "\mathrm{tr}"] \arrow[d] & k \\ A \otimes_{A \otimes_k A^{\mathrm{op}}} A. \arrow[ur, dotted] & \end{tikzcd} \tag{32}$$

Trying to generalize this notion to the context of differential graded algebras, we would certainly start by requiring the existence of a pairing

$$A \otimes_k A \longrightarrow k, \qquad a \otimes b \mapsto \langle a, b\rangle$$

which is homologically non-degenerate. Whatever the notion of cyclicity should be, we would like it to be invariant under weak equivalences. To achieve this desideratum, we require the existence of a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{tr}"] \arrow[d, "\beta"] & k \\ A \otimes^L_{A \otimes_k A^{\mathrm{op}}} A \arrow[ur, dotted, "\mathrm{tr}^\infty"'] & \end{tikzcd} \tag{33}$$

where β is the boundary–bulk map from (16) in the case where the dg category T has a single object with endomorphism dga A. Observe, that the existence of the lift of the map tr in diagram (32) is a property of the pairing. In contrast, specifying the map tr^∞ in diagram (33) requires the specification of additional structure, corresponding to a coherent system of homotopies between the expressions appearing in (30). Indeed, these higher homotopies become explicit by using the cyclic bar construction $\mathrm{C}(A)$ as a model for the (Hochschild) complex $A \otimes^L_{A \otimes_k A^{\mathrm{op}}} A$. Within this model, the map β is simply the inclusion of A as a subcomplex of $\mathrm{C}(A)$. Constructing a map tr^∞ such that (33) commutes thus amounts to providing an extension of the trace map on A to one on $\mathrm{C}(A)$.

The existence of the commutative diagram (33) does not suffice to obtain an oriented extended field theory. Indeed, assuming the existence of such a field theory, the Hochschild complex of A will be assigned to the circle. The symmetries of the circle will therefore act on the Hochschild complex and the trace map is seen to be equivariant with respect to this action. On the level of chain complexes, the action of the circle on the Hochschild complex, translates into the action of Connes' B-operator. The equivariance condition amounts to providing a lift of the trace map from the Hochschild complex to the cyclic complex.

$$\begin{tikzcd}
A \arrow[r, "\mathrm{tr}"] \arrow[d, "\beta"] & k \arrow[dd] \\
A \otimes^L_{A\otimes_k A^{\mathrm{op}}} A \arrow[ur, dotted, "\mathrm{tr}^\infty"'] \arrow[d] & \\
(A \otimes^L_{A\otimes_k A^{\mathrm{op}}} A)_{S^1} \arrow[r, dotted, "\mathrm{tr}^\infty_{S^1}"'] & k_{S^1}
\end{tikzcd} \tag{34}$$

Here, we may choose Connes' cyclic complex

$$(A \otimes^L_{A\otimes_k A^{\mathrm{op}}} A)_{S^1} \simeq \mathrm{CC}(A) := (\mathrm{C}(A)[u^{-1}], b + uB)$$

as an explicit model in which the complex $\mathrm{C}(A)$ appears as a subcomplex. A dg algebra A with a homologically non-degenerate pairing tr together with a lift $\mathrm{tr}^\infty_{S^1}$ to the cyclic complex is called a Calabi–Yau dg algebra. This structure was already studied by Kontsevich and Soibelman in [16]. Interestingly, as proved in [16], one can always strictify a Calabi–Yau structure by passing to an appropriate minimal A_∞-model of A on which the pairing becomes strictly cyclic. This strict notion of a Calabi–Yau A_∞ algebra is used in [6]. Costello proves that every strict Calabi–Yau A_∞-algebra defines an open topological conformal field theory (TCFT) which canonically extends to a universal open-closed TCFT.

We outline how to construct the Calabi–Yau dg algebra which will provide an open-closed TCFT associated to the dg category $T = \mathrm{MF}(R, w)$ of matrix factorizations.

First, we apply Theorem 5.2 in [10] which allows us to restrict our study to the endomorphism dg algebra $A = T(E, E)$ of a single matrix factorization E in $\mathrm{MF}(R, w)$. The Kapustin–Li formula provides us with a trace map

$$\mathrm{tr}_{KL} : A \longrightarrow k,\ \ F \mapsto (-1)^{\binom{n+1}{2}} \frac{1}{n!} \operatorname{Res}\begin{bmatrix} \mathrm{tr}(F(dQ)^{\wedge n}) \\ \partial_1 w, \partial_2 w, \ldots, \partial_n w \end{bmatrix} \tag{35}$$

which, by Theorem 3.4, induces a homologically non-degenerate pairing on A. We have to show that this trace map is part of a Calabi–Yau structure on A, in other words, we have to extend the trace map to a map $\mathrm{tr}^\infty_{S^1}$ on the cyclic complex. By Theorem 6.6 in [10], the Hochschild complex $\mathrm{C}(A)$ is quasi-isomorphic to the Milnor algebra Ω_w concentrated in degree given by the parity of n. By the degeneration of the Hochschild-to-cyclic spectral sequence (cf. [10]), we further know that the cyclic complex $\mathrm{CC}(A)$ is quasi-isomorphic to $\Omega_w[u^{-1}]$ concentrated in the same degree. Thus, the diagram (34) specializes to

$$\begin{tikzcd}
A \arrow[r, "\mathrm{tr}_{KL}"] \arrow[d, "\beta"] & k \arrow[dd] \\
\Omega_w \arrow[ur, dotted, "\mathrm{tr}^\infty"'] \arrow[d] & \\
\Omega_w[u^{-1}] \arrow[r, dotted, "\mathrm{tr}^\infty_{S^1}"'] & k[u^{-1}],
\end{tikzcd} \tag{36}$$

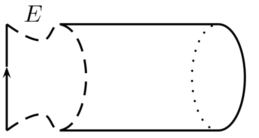


Fig. 1. The boundary–bulk map.

where we omitted the shifts by $[n]$. The map β coincides with the boundary–bulk map $[\underline{\mathrm{tr}}](\pi_E)$ studied in Section 4. By Theorem 4.11, we have the formula

$$\beta(F) = (-1)^{\binom{n+1}{2}} \frac{1}{n!} \operatorname{tr}(F(dQ)^{\wedge n}).$$

Therefore, we can complete diagram (36) by letting

$$\operatorname{tr}^{\infty}(\omega) := \operatorname{Res}\left[\begin{matrix}\omega \\ \partial_1 w, \partial_2 w, \ldots, \partial_n w\end{matrix}\right]$$

and defining $\operatorname{tr}^{\infty}_{S^1}$ by extending $k[u^{-1}]$-linearly. This provides A with a Calabi–Yau structure. In particular, the above mentioned result due to Kontsevich–Soibelman assures the existence of minimal strictly cyclic models of A. In [4], the author develops and implements an algorithm to explicitly calculate such cyclic minimal models.

5.3. *The Riemann–Roch formula*

Finally, we sketch how to deduce a Riemann–Roch formula from the existence of a field theory. Using [16, 10.2.2], we pass to a strictly cyclic minimal A_∞-model of $\mathrm{MF}(R, w)$. Here, we restrict our attention to a direct sum of finitely many objects in $\mathrm{MF}(R, w)$ such that the method explained in the previous section becomes applicable.

By a $\mathbb{Z}/2$-graded variant of Costello's Theorem A [6], we obtain the existence of a canonical open-closed field theory associated to $\mathrm{MF}(R, w)$. Within this field theory, the boundary–bulk map $[\underline{\mathrm{tr}}](\pi_E)$ is the map of chain complexes associated to the cobordism visualized in Fig. 1.

We define the Chern character of E to be

$$\operatorname{ch}(E) = [\underline{\mathrm{tr}}](\pi_E)(\mathrm{id}_E) \in \mathrm{HH}_0(\mathrm{MF}(R, w)).$$

For matrix factorizations E, F in $\mathrm{MF}(R, w)$ we define the $\mathbb{Z}/2$-graded Euler characteristic

$$\chi \operatorname{Hom}(E, F) = \dim \mathrm{H}^0(\operatorname{Hom}(E, F)) - \dim \mathrm{H}^1(\operatorname{Hom}(E, F)).$$

The field theory corresponding to $\mathrm{MF}(R, w)$ assigns a scalar $\lambda \in k$ to the cobordism drawn in Fig. 2 which is a sphere with two disks removed, where the dashed lines indicate free boundaries labeled by the objects E, F.

The field theory formalism allows us to calculate this number in two different ways, by decomposing the above punctured sphere. Consider first the decomposition illustrated in Fig. 3. Interpreting all components appropriately, this yields the formula

$$\lambda = \langle \operatorname{ch}(E), \operatorname{ch}(F) \rangle .$$

Second, we first flatten the punctured sphere into the plane and then decompose as illustrated in Fig. 3b. This yields the formula

$$\lambda = \operatorname{tr}(\mathrm{id}_{\mathrm{H}^*(\operatorname{Hom}(E,F))}) = \chi \operatorname{Hom}(E, F),$$

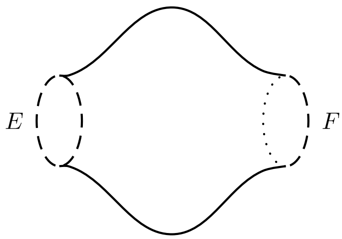
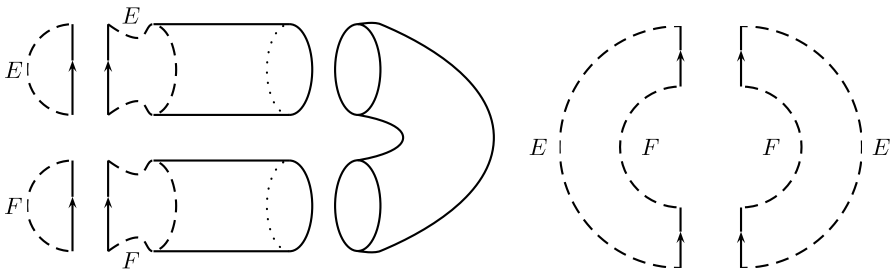


Fig. 2. Twice punctured sphere.



(a) First decomposition.

(b) Second decomposition.

Fig. 3. Two decompositions.

where tr denotes the graded trace map. Thus, we obtain the Hirzebruch–Riemann–Roch formula

$$\chi\,\mathrm{Hom}(E, F) = \langle \mathrm{ch}(E), \mathrm{ch}(F)\rangle\,.$$

To compare to the formula obtained in [22, 4.1.4], one had to calculate the pairing $\langle -, -\rangle$ on the Hochschild homology produced by the field theory (which depends on our choice of the Calabi–Yau structure $\mathrm{tr}^{\infty}_{S^1}$) and compare it to the canonical pairing calculated in [22].

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