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Advances in Mathematics 231 (2012) 1417–1438
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Model category structures arising from Drinfeld vector bundles

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Received 27 June 2009; accepted 26 June 2012
Available online 9 August 2012

Communicated by the Managing Editors of AIM

Abstract

We present a general construction of model category structures on the category $\mathbb{C}(\mathfrak{Qco}(X))$ of unbounded chain complexes of quasi-coherent sheaves on a semi-separated scheme X. The construction is based on making compatible the filtrations of individual modules of sections at open affine subsets of X. It does not require closure under direct limits as previous methods. We apply it to describe the derived category $\mathbb{D}(\mathfrak{Qco}(X))$ via various model structures on $\mathbb{C}(\mathfrak{Qco}(X))$. As particular instances, we recover recent results on the flat model structure for quasi-coherent sheaves. Our approach also includes the case of (infinite-dimensional) vector bundles, and restricted Drinfeld vector bundles. Finally, we prove that the unrestricted case does not induce a model category structure as above in general.

MSC: primary 14F05; 55U35; secondary 18G35; 18E15; 18E30

Keywords: Drinfeld vector bundle; Model structure; Flat Mittag-Leffler module

1. Introduction

Let $\mathcal{A}$ be a Grothendieck category. A convenient way of approaching the derived category $\mathbb{D}(\mathcal{A})$ consists in considering Quillen's notion of model category (cf. [30]) on $\mathbb{C}(\mathcal{A})$, the category

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0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.06.011

of unbounded chain complexes on $\mathcal{A}$. In particular, one can compute morphisms between two objects A and B of $\mathbb{D}(\mathcal{A})$ as the $\mathbb{C}(\mathcal{A})$-morphisms between cofibrant and fibrant replacements of A and B, respectively, modulo chain homotopy.

Recently, Hovey has shown that model category structures naturally arise from small cotorsion pairs over $\mathbb{C}(\mathcal{A})$, [27]. Since $\mathfrak{Qco}(X)$, the category of quasi-coherent sheaves on a scheme X, is a Grothendieck category [11], there is a canonical injective model category structure on $\mathbb{C}(\mathfrak{Qco}(X))$. However, this structure is not monoidal, that is, compatible with the tensor product on $\mathfrak{Qco}(X)$, [25, pp. 111-2]. Another natural, but not monoidal, model structure on $\mathbb{C}(\mathfrak{Qco}(X))$ was constructed in [26] under the assumption of X being a noetherian separated scheme with enough locally frees.

The lack of compatibility with the tensor product was partially solved in [1,2,18,29] by using flat quasi-coherent sheaves. The main result of [18] shows that in case X is quasi-compact and semi-separated, it is possible to construct a monoidal flat model structure on $\mathbb{C}(\mathfrak{Qco}(X))$. The weak equivalences of this model structure are the same as the ones for the injective model structure, hence they induce the same cohomology functors (see [1] for a different approach). However, the structure of flat quasi-coherent sheaves is rather complex, and it is difficult to compute the associated fibrant and cofibrant replacements. Moreover, the methods of the main application of [18] (see [18, Section 6.4]) depend heavily on the fact that the class of all flat modules is closed under direct limits.

A different approach has recently been suggested in [13] for the particular case of quasi-coherent sheaves on the projective line $\mathbf{P^1}(k)$. In that paper it was shown that the class of infinite-dimensional vector bundles (i.e., those quasi-coherent sheaves whose modules of sections in all open affine sets are projective, cf. [6]) imposes a monoidal model category structure on $\mathbb{C}(\mathfrak{Qco}(\mathbf{P^1}(k)))$. The proofs and techniques in [13] are strongly based on the Grothendieck decomposition theorem for vector bundles over the projective line [22], hence they cannot be extended to more general situations.

In the present paper, we show that the main results of [13,18] are particular instances of the following general theorem. It provides a variety of model category structures on $\mathbb{C}(\mathfrak{Qco}(X))$, and hence a variety of ways to represent $\mathbb{D}(\mathcal{A})$, parametrized by sets S_v $(v \in V)$ of modules of sections (see Notation 4.2 and Section 4 for unexplained terminology).

Theorem 1.1. *Let X be a semi-separated scheme. There is a model category structure on $\mathbb{C}(\mathfrak{Qco}(X))$ in which the weak equivalences are the homology isomorphisms, the cofibrations (resp. trivial cofibrations) are the monomorphisms with cokernels in $dg\,\widetilde{\mathcal{C}}$ (resp. $\widetilde{\mathcal{C}}$), and the fibrations (resp. trivial fibrations) are the epimorphisms whose kernels are in $dg\,\widetilde{\mathcal{C}^\perp}$ (resp. $\widetilde{\mathcal{C}^\perp}$). Moreover, if every $M \in S_v$ is a flat $\mathcal{R}(v)$-module, and $M \otimes_{\mathcal{R}(v)} N \in S_v$ for all $M, N \in S_v$, then the model category structure is monoidal.*

The proof of Theorem 1.1 is based on new tools for handling filtrations of quasi-coherent sheaves developed in this paper. Thus it avoids the usual assumption of closure under direct limits (see [18, Theorem 4.12]).

Now, different choices of sets S_v in Theorem 1.1 provide the applications given below (Corollaries 1.2 and 1.3). The first one is a generalization of [13, Theorem 6.1].

Corollary 1.2. *Let X be a scheme having enough infinite-dimensional vector bundles (for example, a quasi-compact and quasi-separated scheme that admits an ample family of invertible sheaves, or a noetherian, integral, separated, and locally factorial scheme). Let $\mathcal{C}$ be the class of all vector bundles on X.*

Then there is a monoidal model category structure on $\mathbb{C}(\mathfrak{Qco}(X))$ where weak equivalences are homology isomorphisms, the cofibrations (resp. trivial cofibrations) are the monomorphisms whose cokernels are dg-complexes of vector bundles (resp. exact complexes of vector bundles whose every quasi-coherent sheaf of cycles is a vector bundle), and the fibrations (resp. trivial fibrations) are the epimorphisms whose kernels are in $dg\,\widetilde{\mathcal{C}^{\perp}}$ (resp. in $\widetilde{\mathcal{C}^{\perp}}$).

For vector bundles, we also prove an analogue of the Kaplansky theorem for projective modules in Corollary 3.16.

In another application of Theorem 1.1, we immediately recover [18, Theorem 6.7].

Corollary 1.3. *Let X be a scheme with enough flat quasi-coherent sheaves (for instance, let X be quasi-compact and semi-separated, see [1, (1.2)]). Then there is a monoidal model category structure on $\mathbb{C}(\mathfrak{Qco}(X))$ where weak equivalences are homology isomorphisms, the cofibrations (resp. trivial cofibrations) are the monomorphisms whose cokernels are dg-flat complexes (resp. flat complexes). The fibrations (resp. trivial fibrations) are the epimorphisms whose kernels are dg-cotorsion complexes (resp. cotorsion complexes).*

However, there are further interesting applications of Theorem 1.1. Drinfeld has proposed quasi-coherent sheaves whose sections at affine open sets are flat and Mittag-Leffler modules (in the sense of Raynaud and Gruson [31]) as the appropriate objects defining infinite-dimensional vector bundles on a scheme, see [6, p. 266]. Here we call such quasi-coherent sheaves the Drinfeld vector bundles, and show that the restricted ones, that is, those defined by the class $\mathcal{C}$ in the next corollary, fit into another instance of Theorem 1.1.

Corollary 1.4. *Let X be a scheme possessing a generating set of Drinfeld vector bundles (for instance if X has enough vector bundles, see Corollary 3.8). Let κ be an infinite cardinal such that $\kappa \geq |V|$. For each $v \in V$, let S_v denote the class of all $\leq \kappa$-generated flat Mittag-Leffler modules. Denote by $\mathcal{C}$ the class of all Drinfeld vector bundles $\mathcal{M}$ such that $\mathcal{M}(v)$ has a S_v-filtration for each $v \in V$.*

Then there is a monoidal model category structure on $\mathbb{C}(\mathfrak{Qco}(X))$ where weak equivalences are homology isomorphisms, the cofibrations (resp. trivial cofibrations) are monomorphisms with cokernels in $dg\,\widetilde{\mathcal{C}}$ (resp. in $\widetilde{\mathcal{C}}$), and the fibrations (resp. trivial fibrations) are epimorphisms whose kernels are in $dg\,\widetilde{\mathcal{C}^{\perp}}$ (resp. in $\widetilde{\mathcal{C}^{\perp}}$).

Note that if V is countable, then the property of X having a generating set of Drinfeld vector bundles is equivalent to X having a generating set of vector bundles (see Corollary 3.8 below).

The reader may wonder whether it is possible to apply Theorem 1.1 to the entire class of Drinfeld vector bundles and impose thus a (monoidal) model category structure on $\mathbb{C}(\mathfrak{Qco}(X))$. Our final theorem shows that this is not the case in general. We adapt a recent consistency result of Eklof and Shelah [9] concerning Whitehead groups to this setting, and prove (in ZFC):

Theorem 1.5. *The class $\mathcal{D}$ of all flat Mittag-Leffler abelian groups is not precovering. Thus $\mathcal{D}$ cannot induce a cofibrantly generated model category structure on $\mathfrak{Qco}(\mathrm{Spec}(\mathbb{Z})) \cong \mathrm{Mod}\text{-}\mathbb{Z}$ compatible with its abelian structure.*

2. Notation and preliminaries

Let $\mathcal{A}$ be a Grothendieck category. A well-ordered direct system of objects of $\mathcal{A}$, $(A_\alpha \mid \alpha \leq \lambda)$, is said to be *continuous* if $A_0 = 0$ and, for each limit ordinal $\beta \leq \lambda$, we have $A_\beta = \lim_{\rightarrow} A_\alpha$:

where the limit is taken over all ordinals $\alpha < \beta$. A continuous direct system $(A_\alpha \mid \alpha \leq \lambda)$ is called a *continuous directed union* if all morphisms in the system are monomorphisms.

Definition 2.1. Let $\mathcal{L}$ be a class of objects of $\mathcal{A}$. An object A of $\mathcal{A}$ is *$\mathcal{L}$-filtered* if $A = \lim_{\rightarrow} A_\alpha$ for a continuous directed union $(A_\alpha \mid \alpha \leq \lambda)$ satisfying that, for each $\alpha + 1 \leq \lambda$, $\operatorname{Coker}(A_\alpha \to A_{\alpha+1})$ is isomorphic to an element of $\mathcal{L}$.

We denote by $Filt(\mathcal{L})$ the class of all $\mathcal{L}$-filtered objects in $\mathcal{A}$. A class $\mathcal{L}$ is said to be *closed under $\mathcal{L}$-filtrations* in case $Filt(\mathcal{L}) = \mathcal{L}$.

Note that if $\mathcal{L}$ has a representative set of objects, $\mathcal{S}$, up to isomorphism, then $Filt(\mathcal{L}) = Filt(\mathcal{S})$. This happens in case there is a cardinal κ such that each object in $\mathcal{L}$ is $< \kappa$-presented.

Definition 2.2. Let $\mathcal{D}$ be a class of objects of $\mathcal{A}$. We will denote by $\mathcal{D}^\perp$ the subclass of $\mathcal{A}$ defined by

$$\mathcal{D}^\perp = \operatorname{KerExt}^1_{\mathcal{A}}(\mathcal{D}, -) = \{Y \in \mathcal{O}b(\mathcal{A}) \mid \operatorname{Ext}^1_{\mathcal{A}}(D, Y) = 0, \text{ for all } D \in \mathcal{D}\}.$$

Similarly,

$$^\perp\mathcal{D} = \operatorname{KerExt}^1_{\mathcal{A}}(-, \mathcal{D}) = \{Z \in \mathcal{O}b(\mathcal{A}) \mid \operatorname{Ext}^1_{\mathcal{A}}(Z, D) = 0, \text{ for all } D \in \mathcal{D}\}.$$

Analogously, we will define

$$\mathcal{D}^{\perp_\infty} = \{Y \in \mathcal{O}b(\mathcal{A}) \mid \operatorname{Ext}^i_{\mathcal{A}}(D, Y) = 0, \text{ for all } D \in \mathcal{D} \text{ and } i \geq 1\}$$

and

$$^{\perp_\infty}\mathcal{D} = \{Z \in \mathcal{O}b(\mathcal{A}) \mid \operatorname{Ext}^i_{\mathcal{A}}(Z, D) = 0, \text{ for all } D \in \mathcal{D} \text{ and } i \geq 1\}.$$

Let us recall the following definitions from [20].

Definition 2.3. A pair $(\mathcal{F}, \mathcal{C})$ of classes of objects of $\mathcal{A}$ is called a *cotorsion pair* if $\mathcal{F}^\perp = \mathcal{C}$ and $^\perp\mathcal{C} = \mathcal{F}$. The cotorsion pair is said to have *enough injectives* (resp. *enough projectives*) if for each object Y of $\mathcal{A}$ there exists an exact sequence $0 \to Y \to C \to F \to 0$ (resp. for each object Z of $\mathcal{A}$ there exists an exact sequence $0 \to C' \to F' \to Z \to 0$) such that $F, F' \in \mathcal{F}$ and $C, C' \in \mathcal{C}$. A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is *complete* provided it has enough injectives and enough projectives.

The proof of the following lemma is the same as for module categories (see [20, Lemma 2.2.10]).

Lemma 2.4. *Let $\mathcal{A}$ be a Grothendieck category with enough projectives and let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair on $\mathcal{A}$. The following conditions are equivalent.*

(a) *If $0 \to F' \to F \to F'' \to 0$ is exact with $F, F'' \in \mathcal{F}$, then $F' \in \mathcal{F}$.*
(b) *If $0 \to C' \to C \to C'' \to 0$ is exact with $C', C \in \mathcal{C}$, then $C'' \in \mathcal{C}$.*
(c) $\operatorname{Ext}^2(F, C) = 0$ *for all $F \in \mathcal{F}$ and $C \in \mathcal{C}$.*
(d) $\operatorname{Ext}^n(F, C) = 0$ *for all $n \geq 1$ and all $F \in \mathcal{F}$ and $C \in \mathcal{C}$.*

A cotorsion pair satisfying the equivalent conditions above is called *hereditary*. So $(\mathcal{F}, \mathcal{C})$ is a hereditary cotorsion pair, if and only if $\mathcal{F} = {}^{\perp_\infty}\mathcal{C}$ and $\mathcal{C} = \mathcal{F}^{\perp_\infty}$.

We finish this section by recalling some notions from module theory.

Let κ be an infinite cardinal. A module M is *$\leq \kappa$-generated* provided there is an epimorphism $f : F \to M$ where F is free of rank $\leq \kappa$. If moreover $\operatorname{Ker}(f)$ is $\leq \kappa$-generated, then M is called

$\leq \kappa$*-presented*. M is called *strongly* κ*-presented* provided that M has a projective resolution consisting of $\leq \kappa$-generated projective modules.

A ring R is *left* κ*-noetherian* if each left ideal of R is $\leq \kappa$-generated, or, equivalently, all $\leq \kappa$-generated modules are $\leq \kappa$-presented. For instance, if $|R| \leq \kappa$ or R is left noetherian, then R is left κ-noetherian.

Notice that over a κ-noetherian ring, the notions of a $\leq \kappa$-generated, $\leq \kappa$-presented, and strongly $\leq \kappa$-presented module coincide.

We also recall the notion of a Mittag-Leffler module from [31].

Definition 2.5. Let R be a ring and M a right R-module. Then M is *Mittag-Leffler* provided that the canonical map $M \otimes_R \prod_{i \in I} M_i \to \prod_{i \in I} M \otimes_R M_i$ is monic for each family of left R-modules $(M_i \mid i \in I)$.

For example, all finitely presented modules, and all projective modules, are Mittag-Leffler. Any countably generated flat Mittag-Leffler module is projective. In fact, projectivity of a module M is equivalent to M being flat Mittag-Leffler and a direct sum of countably generated submodules (see [31] and [6, Theorem 2.2]).

The basic characterization of flat Mittag-Leffler modules is due to Raynaud and Gruson (see [31, Seconde partie, Section 2.2] and [3, Proposition 6]).

Theorem 2.6. *Let R be a ring and $M \in$ Mod-R. Then the following are equivalent.*

(1) *M is a flat Mittag-Leffler module.*
(2) *Every finitely (or countably) generated submodule of M is contained in a countably generated projective submodule which is pure in M.*

The following properties of flat Mittag-Leffler modules will be needed in the sequel.

Lemma 2.7. *Let R be a ring and κ an infinite cardinal.*

(1) *Let M be a $\leq \kappa$-generated flat Mittag-Leffler module. Then M is strongly $\leq \kappa$-presented.*
(2) *Let M be a flat Mittag-Leffler module, and N be a $\leq \kappa$-generated submodule of M. Then N is contained in a $\leq \kappa$-generated pure submodule P of M.*

Proof. (1) By Definition 2.5, the class of all flat Mittag-Leffler modules is closed under pure submodules. So it suffices to prove that M is $\leq \kappa$-presented.

Let $0 \to K \to R^{(I)} \xrightarrow{\pi} M \to 0$ be a presentation of M with $|I| \leq \kappa$. Let G be a generating subset of M of cardinality $|I|$. By induction on $|F|$, we define for each finite subset F of G a countably generated projective and pure submodule P_F of M and a countable subset $J_F \subseteq I$ such that $\pi(R^{(J_F)}) = P_F$, and if $F' \subseteq F$ then $J_{F'} \subseteq J_F$ and $P_{F'} \subseteq P_F$.

If $F = \emptyset$ then $J = \emptyset$ and $P_F = 0$. If $|F| = n$, let C be a countably generated submodule of M containing $P_{F'}$ for all $F' \subsetneq F$. Iterated use of condition (2) of Theorem 2.6 yields a projective countably generated and pure submodule P_F of M containing C, and a countable subset $J_F \subseteq I$ containing $\bigcup_{F' \subsetneq F} J_{F'}$, such that $\pi(R^{(J_F)}) = P_F$.

Let $\pi_F = \pi \restriction R^{(J_F)}$. Then $\{\pi_F \mid F$ a finite subset of $G\}$ is a direct system of epimorphisms with $\varinjlim_F \pi_F : R^{(J)} \twoheadrightarrow M$ where $J = \bigcup_F J_F$ has cardinality $\leq \kappa$. Since P_F is projective, π_F splits, so its kernel is a $\leq \kappa$-generated projective module. Then $\operatorname{Ker}(\varinjlim_F \pi_F) \cong \varinjlim_F \operatorname{Ker}(\pi_F)$ is also $\leq \kappa$-generated. This proves that M is $\leq \kappa$-presented.

(2) We prove the assertion by induction on κ. The case of $\kappa = \aleph_0$ follows by Theorem 2.6. Let $\kappa > \aleph_0$. We have $\kappa = \sup_{\gamma < \mathrm{cf}(\kappa)} \lambda_\gamma$ for an increasing continuous chain of ordinals,

$(\lambda_\gamma \mid \gamma < \mathrm{cf}(\kappa))$, where $\mathrm{cf}(\kappa)$ denotes the cofinality of κ. Let $\{n_\alpha \mid \alpha < \kappa\}$ be an R-generating subset of N.

By induction on $\gamma < \mathrm{cf}(\kappa)$, we define a continuous chain of $< \kappa$-generated pure submodules P_γ of M so that $\{n_\beta \mid \beta < \lambda_\gamma\} \subseteq P_\gamma$ for each $\gamma < \mathrm{cf}(\kappa)$. First, $P_0 = 0$, and $P_{\gamma+1}$ is defined as a $< \kappa$-generated pure submodule of M containing $P_\gamma \cup \{x_\beta \mid \lambda_\gamma \leq \beta < \lambda_{\gamma+1}\}$ (such $P_{\gamma+1}$ exists by the inductive premise). If $\gamma < \mathrm{cf}(\kappa)$ is the limit, we let $P_\gamma = \bigcup_{\delta<\gamma} P_\delta$. Then P_γ is pure in M, and it is $< \kappa$-generated because $\gamma < \mathrm{cf}(\kappa)$. Now, $P = \bigcup_{\gamma<\mathrm{cf}(\kappa)} P_\gamma$ is a $\leq \kappa$-generated pure submodule of M containing N. $\square$

We refer the reader to [8,20,23,25] for unexplained terminology used in this paper.

3. Filtrations of quasi-coherent sheaves

Let X be a scheme. Let $Q_X = (V, E)$ be the quiver whose set, V, of vertices is a subfamily of the family of all open affine sets of X such that V covers both X and all intersections $O \cap O'$ of open affine sets O, O' of V. The set of edges, E, consists of the reversed arrows $v \to u$ corresponding to the inclusions $u \subseteq v$ where u and v are in V. We say that Q_X is a *quiver associated to the scheme* X. Note that different choices of the set of vertices V may give rise to non-isomorphic quivers associated to the same scheme X. From now on we fix a quiver Q_X on X.

As explained in [11, Section 2], there is an equivalence between the category of quasi-coherent sheaves on X and the category of quasi-coherent $\mathcal{R}$-modules where $\mathcal{R}$ is the representation of the quiver Q_X by the sections of the structure sheaf $\mathcal{O}_X$. A quasi-coherent sheaf $\mathcal{F}$ on X corresponds to a *quasi-coherent* $\mathcal{R}$-*m*odule $\mathcal{M}$ defined by the following data:

(1) an $\mathcal{R}$-*module* on X, that is, an $\mathcal{R}(u)$-module $\mathcal{M}(u)$, for each $u \in V$ and a $\mathcal{R}(u)$-morphism $\rho_{uv} : \mathcal{M}(u) \to \mathcal{M}(v)$ for each edge $u \to v$ in E;
(2) the *quasi-coherence* condition, saying that the induced morphism

$$id_{\mathcal{R}(v)} \otimes \rho_{uv} : \mathcal{R}(v) \otimes_{\mathcal{R}(u)} \mathcal{M}(u) \to \mathcal{R}(v) \otimes_{\mathcal{R}(u)} \mathcal{M}(v) \cong \mathcal{M}(v)$$

is an $\mathcal{R}(v)$-isomorphism, for each arrow $u \to v$ in E;
(3) the *compatibility condition*, saying that if $w \subseteq v \subseteq u$, with $w, v, u \in V$, then $\rho_{uw} = \rho_{vw} \circ \rho_{uv}$.

Note that quasi-coherent subsheaves $\mathcal{F}'$ of $\mathcal{F}$ correspond to quasi-coherent $\mathcal{R}$-submodules $\mathcal{M}'$ of $\mathcal{M}$ (where the latter means that $\mathcal{M}'(v)$ is an $\mathcal{R}(v)$-submodule of $\mathcal{M}(v)$ for each $v \in V$, and the map ρ'_{uv} is a restriction of ρ_{uv} for each edge $u \to v$ in E). If $(\mathcal{F}_i)_{i\in I}$ are quasi-coherent subsheaves of $\mathcal{F}$ then $\mathcal{F}' = \sum_{i\in I} \mathcal{F}_i$ (resp. $\mathcal{F}' = \mathcal{F}_1 \cap \mathcal{F}_2$) corresponds to the quasi-coherent submodule $\mathcal{M}'$ such that $\mathcal{M}'(v) = \sum_{i\in I} \mathcal{M}_i(v)$ (resp. such that $\mathcal{M}'(v) = \mathcal{M}_1(v) \cap \mathcal{M}_2(v)$) and the maps ρ'_{uv} are restrictions of ρ_{uv}.

Recall that $\mathfrak{Qco}(X)$ denotes the category of all quasi-coherent sheaves on X. This is a Grothendieck category by [11, p.290]. Note that in our setting $\mathcal{R}(v)$ is commutative for each $v \in V$, and if $u \subseteq v$ are affine open subsets in V, then $\mathcal{R}(u)$ is a flat $\mathcal{R}(v)$-module, see [23, III.9].

Recall that a quasi-coherent sheaf $\mathcal{M}$ on X is a (classical algebraic) vector bundle if $\mathcal{M}(u)$ is a finitely generated projective $\mathcal{R}(u)$-module for every open affine set u. In this paper we adopt the following more general definition: $\mathcal{M}$ is a *vector bundle* if $\mathcal{M}(u)$ is a (not necessarily finitely generated) projective $\mathcal{R}(u)$-module for each open affine set u (see [6, Section 2. Definition]).

In [6, Section 2.Remarks], Drinfeld proposed to consider the following more general notion of a vector bundle (see also [5, Appendices 5 and 6]). Thus, we call a quasi-coherent sheaf $\mathcal{M}$

a *Drinfeld vector bundle* provided that $\mathcal{M}(u)$ is a flat Mittag-Leffler $\mathcal{R}(u)$-module for each open affine set u (cf. [6, p. 266]). Finally we call a quasi-coherent sheaf $\mathcal{M}$ a *κ-restricted Drinfeld vector bundle*, for κ an infinite cardinal, provided that $\mathcal{M}(u)$ has a filtration by $\leq \kappa$-presented flat Mittag-Leffler modules for each open affine set u.

The properties of being vector bundle and Drinfeld vector bundle are local by 3.1.4.(3) and 2.5.2 in [31, Seconde partie, 2.5.2]. The property of being a κ-restricted Drinfeld vector bundle is local for each infinite cardinal κ, cf. [16]. In our situation this means that the construction of vector bundles or (κ-restricted) Drinfeld vector bundles is independent of the choice of the quiver associated to the scheme X.

One of the main goals of this paper is to construct monoidal model category structures associated to these generalized notions of vector bundles. In order to achieve this aim we will need to characterize these classes as closures under filtrations of certain of their subsets.

The following tools will play a central role in our study of these filtrations, both in the case of modules over a ring, and of quasi-coherent sheaves on a scheme.

The first tool is known as Eklof's Lemma (see [7, Theorem 1.2]).

Lemma 3.1. *Let R be a ring and $\mathcal{C}$ be a class of modules. Let M be a module possessing a ${}^{\perp}\mathcal{C}$-filtration. Then $M \in {}^{\perp}\mathcal{C}$.*

Remark 3.2. The proof of Lemma 3.1 given in [20, Lemma 3.1.2] needs only embeddability of each module into an injective one, so the lemma holds in $\mathfrak{Qco}(X)$, and in fact in any Grothendieck category.

Our second tool is known as Hill's Lemma (see [20, Theorem 4.2.6], [34, Lemma 1.4], or [35, Theorem 6]). It will allow us to extend a given filtration of a module M to a complete lattice of its submodules having similar properties.

Lemma 3.3. *Let R be a ring, λ a regular infinite cardinal, and $\mathcal{J}$ a class of $< \lambda$-presented modules. Let M be a module with a $\mathcal{J}$-filtration $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$. Then there is a family $\mathcal{H}$ consisting of submodules of M such that*

(1) $\mathcal{M} \subseteq \mathcal{H}$,
(2) *$\mathcal{H}$ is closed under arbitrary sums and intersections,*
(3) *P/N has a $\mathcal{J}$-filtration for all $N, P \in \mathcal{H}$ such that $N \subseteq P$, and*
(4) *If $N \in \mathcal{H}$ and T is a subset of M of cardinality $< \lambda$, then there exists $P \in \mathcal{H}$ such that $N \cup T \subseteq P$ and P/N is $< \lambda$-presented.*

We will also need the following application of Lemma 3.3 (see [20, Theorem 4.2.11] and [35, Theorem 10]).

Lemma 3.4. *Let R be a ring, λ a regular uncountable cardinal, and $\mathcal{J}$ a class of $< \lambda$-presented modules. Let $\mathcal{A} = {}^{\perp}(\mathcal{J}^{\perp})$, and let $\mathcal{A}^{<\lambda}$ denote the class of all $< \lambda$-presented modules from $\mathcal{A}$. Then every module in $\mathcal{A}$ is $\mathcal{A}^{<\lambda}$-filtered.*

If κ is a cardinal and $\mathcal{M}$ a quasi-coherent sheaf, then $\mathcal{M}$ is called *locally $\leq \kappa$-presented* (*locally $\leq \kappa$-generated*) if for each $v \in V$, the $\mathcal{R}(v)$-module $\mathcal{M}(v)$ is $\leq \kappa$-presented ($\leq \kappa$-generated).

Notice that if $\kappa \geq |V|$ and $\kappa \geq |\mathcal{R}(v)|$ for each $v \in V$, then both these notions are equivalent to saying that $\mathcal{M}$ is κ^+-presentable in the sense of [18, Lemma 6.1], and also to $|\bigoplus_{v \in V} \mathcal{M}(v)| \leq \kappa$.

Our third tool is a version of [11, Proposition 3.3] in which we do not restrict the size of the rings $\mathcal{R}(v)$, because we do not need $\mathcal{M}'(v)$ to be a pure submodule of $\mathcal{M}(v)$. This tool will be applied to form filtrations of quasi-coherent sheaves by connecting the individual $\mathcal{R}(v)$-module filtrations for all $v \in V$.

Lemma 3.5. *Let $\mathcal{M} \in \mathfrak{Qco}(X)$ and let κ be an infinite cardinal such that $\kappa \geq |V|$. Let $X_v \subseteq \mathcal{M}(v)$ be subsets with $|X_v| \leq \kappa$ for all $v \in V$. Then there is a locally $\leq \kappa$-generated quasi-coherent subsheaf $\mathcal{M}' \subseteq \mathcal{M}$ such that $X_v \subseteq \mathcal{M}'(v)$ for all $v \in V$.*

Proof. By induction on n, we define subsets $X_{v,n} \subseteq \mathcal{M}(v)$ such that $X_{v,n} \subseteq X_{v,n+1}$ for all $v \in V$ and $n < \omega$ as follows.

(1) $X_{v,0} = X_v$ for all $v \in V$.
(2) Assume $X_{v,n}$ is defined for all $v \in V$ and n is even. Let $Y^n_{v,0} = X_{v,n}$, $Y^n_{v,i+1} = Y^n_{v,i} \cup \{\rho_{uv}(Y^n_{u,i}) \mid u \to v \text{ in } E\}$, and $X_{v,n+1} = \bigcup_{i<\omega} Y^n_{v,i}$.
(3) Assume $X_{v,n}$ is defined for all $v \in V$ and n is odd. Let $Z^n_{v,0} = X_{v,n}$, $Z^n_{v,i+1} = Z^n_{v,i} \cup \{\overleftarrow{Z^n_{u,i}} \mid v \to u \text{ in } E\}$, where $\overleftarrow{Z^n_{u,i}}$ is a subset of $\mathcal{M}(v)$ of cardinality at most κ such that $(id_{\mathcal{R}(u)} \otimes \rho_{vu})(\mathcal{R}(u) \otimes_{\mathcal{R}(v)} \langle \overleftarrow{Z^n_{u,i}} \rangle) \supseteq Z^n_{u,i}$. Such a subset exists because the map $id_{\mathcal{R}(u)} \otimes \rho_{vu}$ is an isomorphism. Finally, $X_{v,n+1} = \bigcup_{i<\omega} Z^n_{v,i}$.

We claim that $\mathcal{M}'(v) = \langle \bigcup_{n<\omega} X_{v,n} \rangle$ for $v \in V$ yield the desired subsheaf of $\mathcal{M}$. Clearly $X_v \subseteq \mathcal{M}'(v)$ for all $v \in V$ by step (1) above, and $\mathcal{M}'(v)$ is $\leq \kappa$-generated because $|E| \leq |V^2| \leq \kappa$. It remains to prove the quasi-coherence of $\mathcal{M}'$. That is, we prove that $id_{\mathcal{R}(v)} \otimes \rho_{uv}$ restricts to an isomorphism $\tau_{uv} : \mathcal{R}(v) \otimes_{\mathcal{R}(u)} \mathcal{M}'(u) \to \mathcal{M}'(v)$ for each edge $u \to v$ in E.

Let $x \in \mathcal{M}'(u)$. Then $x \in \langle X_{u,n+1} \rangle$ for some even $n < \omega$, so $x \in \langle Y^n_{u,i} \rangle$ for some $i < \omega$. If $u \to v$ in E, then $\rho_{uv}(x) \in \langle Y^n_{v,i+1} \rangle$ by step (2). So τ_{uv} maps $\mathcal{R}(v) \otimes_{\mathcal{R}(u)} \mathcal{M}'(u)$ into $\mathcal{M}'(v)$, and it is monic.

In order to prove that τ_{uv} is surjective for $u \to v$ in E, we consider $y \in \mathcal{M}'(v)$. Then $y \in \langle X_{v,n+1} \rangle$ for some odd $n < \omega$, so $y \in \langle Z^n_{v,i} \rangle$ for some $i < \omega$. By step (3), there exist $x_1, \dots, x_k \in \langle Z^n_{u,i+1} \rangle$ and $r_1, \dots, r_k \in \mathcal{R}(v)$ such that

$$\tau_{uv}\left(\sum_{i=1}^{k} r_i \otimes x_i\right) = (id_{\mathcal{R}(v)} \otimes \rho_{uv})\left(\sum_{i=1}^{k} r_i \otimes x_i\right) = y.$$

This proves that τ_{uv} is surjective. So $\mathcal{M}'$ is a quasi-coherent subsheaf of $\mathcal{M}$. $\square$

We can do better in the case of Drinfeld vector bundles.

Proposition 3.6. *Let $\mathcal{M} \in \mathfrak{Qco}(X)$ be a Drinfeld vector bundle and let κ be an infinite cardinal such that $\kappa \geq |V|$. Let $X_v \subseteq \mathcal{M}(v)$ be subsets with $|X_v| \leq \kappa$ for all $v \in V$. Then there is a locally $\leq \kappa$-generated Drinfeld vector bundle $\mathcal{N} \subseteq \mathcal{M}$ such that $X_v \subseteq \mathcal{N}(v)$ for all $v \in V$.*

Proof. Let $\mathcal{R} = (\mathcal{R}(v) \mid v \in V)$ be the representation of the quiver Q_X by the sections of the structure sheaf $\mathcal{O}_X$. By induction on n, we define $\mathcal{R}(v)$-submodules $M_{v,n} \subseteq \mathcal{M}(v)$ such that $M_{v,n} \subseteq M_{v,n+1}$ for all $v \in V$ and $n < \omega$ as follows.

(1) $M_{v,0} = \langle X_v \rangle$ for all $v \in V$.
(2) Assume $M_{v,n}$ is defined for all $v \in V$ and n is even. We use Lemma 2.7(2) to define $M_{v,n+1}$ as a pure $\leq \kappa$-generated $\mathcal{R}(v)$-submodule of $M(v)$ containing $M_{v,n}$.

(3) Assume $M_{v,n}$ is defined for all $v \in V$ and n is odd. Let X_v be an $\mathcal{R}(v)$-generating subset of $M_{v,n}$ of cardinality $\leq \kappa$. An application of Lemma 3.5 yields a quasi-coherent sheaf $\mathcal{M}'$ as in 3.5 and we define $M_{v,n+1} = \mathcal{M}'(v)$.

Finally, we let $\mathcal{N}(v) = \bigcup_{n<\omega} M_{v,n}$. This is a pure submodule of $\mathcal{M}(v)$, so $\mathcal{N} = (\mathcal{N}(v) \mid v \in V)$ has the desired properties. □

Corollary 3.7. *Let X be a scheme having enough Drinfeld vector bundles, and $Q_X = (V, E)$ a quiver associated to X. Let $\kappa \geq |V|$. Then X has enough locally $\leq \kappa$-presented Drinfeld vector bundles.*

Proof. By Proposition 3.6, there is a generating set of locally $\leq \kappa$-generated Drinfeld vector bundles, and these are $\leq \kappa$-presented by Lemma 2.7(1).

As locally $\leq \aleph_0$-generated Drinfeld vector bundles are necessarily vector bundles in the sense of [6, Section 2. Definition], we have the following.

Corollary 3.8. *Let X be a scheme with enough Drinfeld vector bundles. Suppose we can choose a quiver $Q_X = (V, E)$ associated to X so that $|V| \leq \aleph_0$. Then X has enough vector bundles.*

Remark 3.9. (1) The existence of enough vector bundles follows from, and can be seen as a natural generalization of, the so-called resolution property: a noetherian scheme X is said to satisfy the *resolution property* if every coherent sheaf is a quotient of a locally free sheaf of finite rank. Since for such X each quasi-coherent sheaf is a filtered union of its coherent subsheaves, the resolution property for X implies that $\mathfrak{Qco}(X)$ has enough locally frees of finite rank (so, in particular, enough classical vector bundles). It is open whether the resolution property holds for each noetherian separated scheme, cf. [36, Question 1].

(2) In view of Corollary 3.8, possible examples of schemes having enough Drinfeld vector bundles, but not enough vector bundles, will require schemes for which each covering by open affine sets is uncountable. Corollary 3.8 could potentially be useful to establish the existence of enough vector bundles by proving the weaker condition of existence of enough Drinfeld vector bundles.

For future use in Section 4 we now present a version of Hill's Lemma for the category $\mathfrak{Qco}(X)$. For this version, we fix the following notation.

Notation 3.10. We assume that λ is a regular infinite cardinal such that $\lambda > |V|$, and $\mathcal{J}$ a class of locally $< \lambda$-presented objects of $\mathfrak{Qco}(X)$. Further, let $\mathcal{M}$ be a quasi-coherent sheaf possessing a $\mathcal{J}$-filtration $\mathcal{O} = (\mathcal{M}_\alpha \mid \alpha \leq \sigma)$.

By Lemma 3.5 there exist locally $< \lambda$-generated quasi-coherent sheaves $\mathcal{A}_\alpha \subseteq \mathcal{M}_{\alpha+1}$ such that $\mathcal{M}_{\alpha+1} = \mathcal{M}_\alpha + \mathcal{A}_\alpha$ for each $\alpha < \sigma$.

A set $S \subseteq \sigma$ is called *closed* provided that $\mathcal{M}_\alpha \cap \mathcal{A}_\alpha \subseteq \sum_{\beta<\alpha,\beta\in S} \mathcal{A}_\beta$ for each $\alpha \in S$.

Lemma 3.11. *Let $\mathcal{H} = \{\sum_{\alpha\in S} \mathcal{A}_\alpha \mid S \text{ closed}\}$. Then $\mathcal{H}$ satisfies the following conditions:*

(1) *$\mathcal{O} \subseteq \mathcal{H}$,*
(2) *$\mathcal{H}$ is closed under arbitrary sums,*
(3) *$\mathcal{P}/\mathcal{N}$ has a $\mathcal{J}$-filtration whenever $\mathcal{N}, \mathcal{P} \in \mathcal{H}$ are such that $\mathcal{N} \subseteq \mathcal{P}$.*
(4) *If $\mathcal{N} \in \mathcal{H}$ and $\mathcal{X}$ is a locally $< \lambda$-generated quasi-coherent subsheaf of $\mathcal{M}$, then there exists $\mathcal{P} \in \mathcal{H}$ such that $\mathcal{N} + \mathcal{X} \subseteq \mathcal{P}$ and $\mathcal{P}/\mathcal{N}$ is locally $< \lambda$-presented.*

Proof. Note that for each ordinal $\alpha \leq \sigma$, we have $\mathcal{M}_\alpha = \sum_{\beta<\alpha} \mathcal{A}_\beta$, hence α is a closed subset of σ. This proves condition (1). Since any union of closed subsets is closed, condition (2) holds.

Our proof of condition (3) follows from the proof of condition (iv) in [35], and condition (4) is proved similarly as condition $(H4)$ in [20, Theorem 4.26]. However, the proofs in [20,35] are restricted to module categories, so we prefer to present a detailed proof here for the setting of $\mathfrak{Qco}(X)$. In order to prove condition (3), we consider closed subsets S, T of σ such that $\mathcal{N} = \sum_{\alpha\in S} \mathcal{A}_\alpha$ and $\mathcal{P} = \sum_{\alpha\in T} \mathcal{A}_\alpha$. Since $S \cup T$ is closed, we will w.l.o.g. assume that $S \subseteq T$. We define a $\mathcal{J}$-filtration of $\mathcal{P}/\mathcal{N}$ as follows. For each $\beta \leq \sigma$, let $\mathcal{F}_\beta = (\sum_{\alpha\in T\setminus S,\alpha<\beta} \mathcal{A}_\alpha + \mathcal{N})/\mathcal{N}$. Then $\mathcal{F}_{\beta+1} = \mathcal{F}_\beta + (\mathcal{A}_\beta + \mathcal{N})/\mathcal{N}$ for $\beta \in T \setminus S$ and $\mathcal{F}_{\beta+1} = \mathcal{F}_\beta$ otherwise.

Let $\beta \in T \setminus S$. Then $\mathcal{F}_{\beta+1}/\mathcal{F}_\beta \cong \mathcal{A}_\beta/(\mathcal{A}_\beta \cap (\sum_{\alpha\in T\setminus S,\alpha<\beta} \mathcal{A}_\alpha + \mathcal{N}))$, and since $\beta \in T \setminus S$ and T is closed, we have

$$\begin{aligned}\mathcal{A}_\beta \cap \left(\sum_{\alpha\in T\setminus S,\alpha<\beta} \mathcal{A}_\alpha + \mathcal{N}\right) &= \mathcal{A}_\beta \cap \left(\sum_{\alpha\in S,\alpha>\beta} \mathcal{A}_\alpha + \sum_{\alpha\in T,\alpha<\beta} \mathcal{A}_\alpha\right)\\ &\supseteq \mathcal{A}_\beta \cap \left(\sum_{\alpha\in S,\alpha>\beta} \mathcal{A}_\alpha + (\mathcal{M}_\beta \cap \mathcal{A}_\beta)\right) \supseteq \mathcal{M}_\beta \cap \mathcal{A}_\beta.\end{aligned}$$

Let $\mathcal{B}_\beta = \sum_{\alpha\in S,\alpha>\beta} \mathcal{A}_\alpha + \sum_{\alpha\in T,\alpha<\beta} \mathcal{A}_\alpha$. We will prove that $\mathcal{A}_\beta \cap \mathcal{B}_\beta = \mathcal{M}_\beta \cap \mathcal{A}_\beta$. We have only to show that for each $v \in V$, $\mathcal{A}_\beta(v) \cap \mathcal{B}_\beta(v) \subseteq \mathcal{A}_\beta(v) \cap \mathcal{M}_\beta(v)$. Let $a \in \mathcal{A}_\beta(v) \cap \mathcal{B}_\beta(v)$. Then $a = c + a_{\alpha_0} + \cdots + a_{\alpha_k}$ where $c \in \sum_{\alpha\in T,\alpha<\beta} \mathcal{A}_\alpha(v) \subseteq \mathcal{M}_\beta(v)$, $\alpha_i \in S$ and $a_{\alpha_i} \in \mathcal{A}_{\alpha_i}(v)$ for all $i \leq k$ and $\alpha_i > \alpha_{i+1}$ for all $i < k$. W.l.o.g., we can assume that α_0 is minimal possible. If $\alpha_0 > \beta$, then $a_{\alpha_0} = a - c - a_{\alpha_1} + \cdots - a_{\alpha_k} \in \mathcal{M}_{\alpha_0}(v) \cap \mathcal{A}_{\alpha_0}(v) \subseteq \sum_{\alpha\in S,\alpha<\alpha_0} \mathcal{A}_\alpha(v)$ (since $\alpha_0 \in S$), in contradiction with the minimality of α_0. Since $\beta \notin S$, we infer that $\alpha_0 < \beta$, $a \in \mathcal{M}_\beta(v)$, and $\mathcal{A}_\beta \cap \mathcal{B}_\beta = \mathcal{A}_\beta \cap \mathcal{M}_\beta$.

So if $\beta \in T \setminus S$ then $\mathcal{F}_{\beta+1}/\mathcal{F}_\beta \cong \mathcal{A}_\beta/(\mathcal{M}_\beta \cap \mathcal{A}_\beta) \cong \mathcal{M}_{\beta+1}/\mathcal{M}_\beta$, and the latter is isomorphic to an element of $\mathcal{J}$ because $\mathcal{O}$ is a $\mathcal{J}$-filtration of $\mathcal{M}$. This finishes the proof of condition (3).

For condition (4) we first claim that each subset of σ of cardinality $< \lambda$ is contained in a closed subset of cardinality $< \lambda$. Since λ is regular and unions of closed sets are closed, it suffices to prove the claim only for one-element subsets of σ. By induction on β we prove that each $\beta < \sigma$ is contained in a closed set C of cardinality $< \lambda$. If $\beta < \lambda$ we take $C = \beta + 1$.

Otherwise, consider the short exact sequence $0 \to \mathcal{M}_\beta \cap \mathcal{A}_\beta \to \mathcal{A}_\beta \to \mathcal{M}_{\beta+1}/\mathcal{M}_\beta \to 0$. So for each $v \in V$ we have an exact sequence $0 \to \mathcal{M}_\beta(v) \cap \mathcal{A}_\beta(v) \to \mathcal{A}_\beta(v) \to \mathcal{M}_{\beta+1}(v)/\mathcal{M}_\beta(v) \to 0$. By our assumption, $\mathcal{M}_{\beta+1}(v)/\mathcal{M}_\beta(v)$ is $< \lambda$-presented and $\mathcal{A}_\beta(v)$ is $< \lambda$-generated, so $\mathcal{M}_\beta(v) \cap \mathcal{A}_\beta(v)$ is $< \lambda$-generated. Hence for each $v \in V$, $\mathcal{M}_\beta(v) \cap \mathcal{A}_\beta(v) \subseteq \sum_{\alpha\in C_v} \mathcal{A}_\alpha(v)$ for a subset $C_v \subseteq \beta$ of cardinality $< \lambda$. Since $|V| < \lambda$, our inductive premise yields that the set $\bigcup_{v\in V} C_v$ is contained in a closed subset C' of cardinality $< \lambda$. Let $C = C' \cup \{\beta\}$. Then C is closed because C' is closed, and $\mathcal{M}_\beta \cap \mathcal{A}_\beta \subseteq \sum_{\alpha\in C'} \mathcal{A}_\alpha$.

Finally if $\mathcal{N} = \sum_{\alpha\in C} \mathcal{A}_\alpha$ and $\mathcal{X}$ is a locally $< \lambda$-generated quasi-coherent subsheaf of $\mathcal{M}$, then $\mathcal{X} \subseteq \sum_{\alpha\in T} \mathcal{A}_\alpha$ for a subset D of σ of cardinality $< \lambda$ (because $|V| < \lambda$). By the above we can assume that D is closed and put $\mathcal{P} = \sum_{\alpha\in C\cup D} \mathcal{A}_\alpha$. By (the proof of) condition (3) $\mathcal{P}/\mathcal{N}$ is $\mathcal{J}$-filtered, and the length of the filtration can be taken $\leq |D \setminus C| < \lambda$. This implies that $\mathcal{P}/\mathcal{N}$ is locally $< \lambda$-presented. $\square$

Now we fix our notation.

Notation 3.12. Let $Q_X = (V, E)$ be a quiver associated to a scheme X, and κ be an infinite cardinal such that $\kappa \geq |V|$. For each $v \in V$, let S_v be a class of $\leq \kappa$-presented $\mathcal{R}(v)$-modules,

$\mathcal{F}_v = {}^{\perp}(S_v^{\perp})$, $\mathcal{L}$ be the class of all locally $\leq \kappa$-presented quasi-coherent sheaves $\mathcal{N}$ such that $\mathcal{N}(v) \in \mathcal{F}_v$ for each $v \in V$, and $\mathcal{C}$ be the class of all quasi-coherent sheaves $\mathcal{M}$ such that $\mathcal{M}(v) \in \mathcal{F}_v$ for each $v \in V$.

The following theorem is a sheaffified (quasi-coherent) version of Lemma 3.4 in the light of Lemma 3.5.

Theorem 3.13. *Each quasi-coherent sheaf $\mathcal{M} \in \mathcal{C}$ has an $\mathcal{L}$-filtration.*

Proof. Let $v \in V$ and put $\lambda = \kappa^+$. Denote by $\mathcal{F}_v^{\leq\kappa}$ the subclass of $\mathcal{F}_v$ consisting of all $\leq \kappa$-presented modules. By Lemma 3.4, $\mathcal{M}(v)$ has a $\mathcal{F}_v^{\leq\kappa}$-filtration $\mathcal{M}_v$. Denote by $\mathcal{H}_v$ the family associated to $\mathcal{M}_v$ in Lemma 3.3. And let $\{m_{v,\alpha} \mid \alpha < \tau_v\}$ be an $\mathcal{R}(v)$-generating set of the $\mathcal{R}(v)$-module $\mathcal{M}(v)$. W.l.o.g., we can assume that $\tau = \tau_v$ for all $v \in V$.

We will construct an $\mathcal{L}$-filtration $(\mathcal{M}_\alpha \mid \alpha \leq \tau)$ of $\mathcal{M}$ by induction on α. Let $\mathcal{M}_0 = 0$. Assume that $\mathcal{M}_\alpha$ is defined for some $\alpha < \tau$ so that $\mathcal{M}_\alpha(v) \in \mathcal{H}_v$ and $m_{v,\beta} \in \mathcal{M}_\alpha(v)$ for all $\beta < \alpha$ and all $v \in V$. Set $N_{v,0} = \mathcal{M}_\alpha(v)$. By Lemma 3.3(4), there is a module $N_{v,1} \in \mathcal{H}_v$ such that $N_{v,0} \subseteq N_{v,1}$, $m_{v,\alpha} \in N_{v,1}$ and $N_{v,1}/N_{v,0}$ is $\leq \kappa$-presented.

By Lemma 3.5 (with $\mathcal{M}$ replaced by $\mathcal{M}/\mathcal{M}_\alpha$, and $X_v = N_{v,1}/\mathcal{M}_\alpha(v)$) there is a quasi-coherent subsheaf $\mathcal{M}'_1$ of $\mathcal{M}$ such that $\mathcal{M}_\alpha \subseteq \mathcal{M}'_1$ and $\mathcal{M}'_1/\mathcal{M}_\alpha$ is locally $\leq \kappa$-generated. Then $\mathcal{M}'_1(v) = N_{v,1} + \langle T_v \rangle$ for a subset $T_v \subseteq \mathcal{M}'_1(v)$ of cardinality $\leq \kappa$, for each $v \in V$.

By Lemma 3.3(4) there is a module $N_{v,2} \in \mathcal{H}_v$ such that $\mathcal{M}'_1(v) = N_{v,1} + \langle T_v \rangle \subseteq N_{v,2}$ and $N_{v,2}/N_{v,1}$ is $\leq \kappa$-presented.

Proceeding similarly, we obtain a countable chain $(\mathcal{M}'_n \mid n < \aleph_0)$ of quasi-coherent subsheaves of $\mathcal{M}$, as well as a countable chain $(N_{v,n} \mid n < \aleph_0)$ of $\mathcal{R}(v)$-submodules of $\mathcal{M}(v)$, for each $v \in V$. Let $\mathcal{M}_{\alpha+1} = \bigcup_{n<\aleph_0} \mathcal{M}'_n$. Then $\mathcal{M}_{\alpha+1}$ is a quasi-coherent subsheaf of $\mathcal{M}$ satisfying $\mathcal{M}_{\alpha+1}(v) = \bigcup_{n<\aleph_0} N_{v,n}$ for each $v \in V$. By Lemma 3.3(2) and (3) we deduce that $\mathcal{M}_{\alpha+1}(v) \in \mathcal{H}_v$ and $\mathcal{M}_{\alpha+1}(v)/\mathcal{M}_\alpha(v) \in \mathcal{F}_v^{\leq\kappa}$. Therefore $\mathcal{M}_{\alpha+1}/\mathcal{M}_\alpha \in \mathcal{L}$.

Assume $\mathcal{M}_\beta$ has been defined for all $\beta < \alpha$ where α is a limit ordinal $\leq \tau$. Then we define $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$. Since $m_{v,\alpha} \in \mathcal{M}_{\alpha+1}(v)$ for all $v \in V$ and $\alpha < \tau$, we have $\mathcal{M}_\tau(v) = \mathcal{M}(v)$, so $(\mathcal{M}_\alpha \mid \alpha \leq \tau)$ is an $\mathcal{L}$-filtration of $\mathcal{M}$. $\square$

It is clear that the class $\mathcal{C}$ is closed under extensions, retractions and direct sums. As a consequence of Theorem 3.13 we get the following two corollaries.

Corollary 3.14. *Let $\mathcal{C}$ and $\mathcal{L}$ be the subclasses of $\mathfrak{Qco}(X)$ defined above. Then $\mathcal{C} = Filt(\mathcal{L})$.*

Proof. The inclusion $\mathcal{C} \subseteq Filt(\mathcal{L})$ follows by Theorem 3.13, and $Filt(\mathcal{L}) \subseteq \mathcal{C}$ by Lemma 3.1. $\square$

Corollary 3.15. *Let $\mathcal{C}$ and $\mathcal{L}$ be the subclasses of $\mathfrak{Qco}(X)$ defined above. Suppose that $\mathcal{C}$ contains a generator of $\mathfrak{Qco}(X)$. Then $(\mathcal{C}, \mathcal{L}^{\perp})$ is a complete cotorsion pair.*

Proof. Since $\mathcal{L} \subseteq {}^{\perp}(\mathcal{L}^{\perp})$, we have $Filt(\mathcal{L}) \subseteq Filt({}^{\perp}(\mathcal{L}^{\perp}))$. By Lemma 3.1, $Filt({}^{\perp}(\mathcal{L}^{\perp})) = {}^{\perp}(\mathcal{L}^{\perp})$. So by Corollary 3.14, $\mathcal{C} \subseteq {}^{\perp}(\mathcal{L}^{\perp})$.

In order to prove that $(\mathcal{C}, \mathcal{L}^{\perp})$ is a complete cotorsion pair, we first show that ${}^{\perp}(\mathcal{L}^{\perp}) \subseteq \mathcal{C}$. By [14, Lemma 2.4, Theorem 2.5], for all $\mathcal{Q} \in \mathfrak{Qco}(X)$ there exists a short exact sequence

$$0 \to \mathcal{Q} \to \mathcal{P} \to \mathcal{Z} \to 0 \tag{1}$$

where $\mathcal{P} \in \mathcal{L}^{\perp}$ and $\mathcal{Z}$ has an $\mathcal{L}$-filtration. Given any $\mathcal{M} \in \mathfrak{Qco}(X)$, since the generator $\mathcal{G}$ of $\mathfrak{Qco}(X)$ is in $\mathcal{C}$, there exists a short exact sequence

$$0 \to \mathcal{U} \to \mathcal{G}' \to \mathcal{M} \to 0$$

where $\mathcal{G}'$ is a direct sum of copies of $\mathcal{G}$. Now let

$$0 \to \mathcal{U} \to \mathcal{N} \to \mathcal{Z} \to 0$$

be exact with $\mathcal{N} \in \mathcal{L}^{\perp}$ and $\mathcal{Z}$ admitting an $\mathcal{L}$-filtration. Form a pushout and get

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & \mathcal{U} \arrow[r] \arrow[d] & \mathcal{G}' \arrow[r] \arrow[d] & \mathcal{M} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{W} \arrow[r] \arrow[d] & \mathcal{M} \arrow[r] & 0 \\ & \mathcal{Z} \arrow[r, equal] \arrow[d] & \mathcal{Z} \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd}$$

Then since $\mathcal{G}'$ is a direct sum of copies of $\mathcal{G} \in \mathcal{C}$ and $\mathcal{Z}$ has an $\mathcal{L}$-filtration (so $\mathcal{Z} \in \mathcal{C}$ by Corollary 3.14), we see that $\mathcal{W} \in \mathcal{C}$. Also $\mathcal{N} \in \mathcal{L}^{\perp}$. Hence if $\mathcal{M} \in {}^{\perp}(\mathcal{L}^{\perp})$ we get that $0 \to \mathcal{N} \to \mathcal{W} \to \mathcal{M} \to 0$ splits and so $\mathcal{M}$ is a direct summand of $\mathcal{W} \in \mathcal{C}$. But then $\mathcal{M} \in \mathcal{C}$ because $\mathcal{C}$ is closed under direct summands.

This proves that $\mathcal{C} = {}^{\perp}(\mathcal{L}^{\perp})$. Moreover (1) shows that the cotorsion pair $(\mathcal{C}, \mathcal{L}^{\perp})$ has enough injectives, and the second line of the diagram above that it has enough projectives. □

Focusing on particular classes of modules, we obtain several interesting corollaries of Theorem 3.13.

Corollary 3.16 (*Kaplansky Theorem for Vector Bundles*). *Let X be a scheme and κ an infinite cardinal such that $\kappa \geq |V|$. Then every vector bundle on X has an $\mathcal{L}$-filtration where $\mathcal{L}$ is the class of all locally $\leq \kappa$-presented vector bundles.*

In particular, if X is a scheme, $Q_X = (V, E)$ is a quiver associated to X, and V is countable (for instance, there is such a choice of Q_X when X has a countable basis of affine open sets), then every vector bundle on X has a filtration by locally countably generated vector bundles.

Proof. This follows by taking $S_v = \{\mathcal{R}(v)\}$ (so $\mathcal{F}_v$ is the class of all projective $\mathcal{R}(v)$-modules) for all $v \in V$, and then applying Theorem 3.13. □

Note that in the next corollary, $\mathcal{C}$ is the class of κ-restricted Drinfeld vector bundles, that is, the quasi-coherent sheaves $\mathcal{M}$ such that $\mathcal{M}(v)$ has a filtration by $\leq \kappa$-presented flat Mittag-Leffler modules.

Corollary 3.17. *Let X be a scheme, κ an infinite cardinal such that $\kappa \geq |V|$. For each $v \in V$, let S_v denote the class of all $\leq \kappa$-presented flat Mittag-Leffler modules. Let $\mathcal{F}_v$, $\mathcal{L}$ and $\mathcal{C}$ be defined as in 3.12. Then $\mathcal{L}$ is the class of all locally $\leq \kappa$-presented Drinfeld vector bundles and $Filt(\mathcal{L}) = \mathcal{C}$.*

In 5.7, we will see that in general Corollary 3.17 fails for arbitrary Drinfeld vector bundles. Our final application goes back to [11, Section 4].

Corollary 3.18. *Let X be a scheme. Let κ be an infinite cardinal such that $\kappa \geq |V|$ and $\kappa \geq |\mathcal{R}(v)|$ for all $v \in V$.*

Then every flat quasi-coherent sheaf on X has an $\mathcal{L}$-filtration where $\mathcal{L}$ is the class of all locally $\leq \kappa$-presented flat quasi-coherent sheaves.

Proof. For each vertex $v \in V$, we take a set S_v of representatives of isoclasses of flat $\mathcal{R}(v)$-modules of cardinality $\leq \kappa$. Then by Lemma 3.1 and [4, Lemma 1] it follows that $\mathcal{F}_v = {}^{\perp}(S_v^{\perp})$ is the class of all flat $\mathcal{R}(v)$-modules. Finally, we apply Theorem 3.13. $\square$

4. Quillen model category structures on $\mathbb{C}(\mathfrak{Qco}(X))$

In this section we develop a method for constructing a model structure on $\mathbb{C}(\mathfrak{Qco}(X))$ starting from *a priori* given sets of modules over sections of the structure sheaf associated to X. Our main tool will be Hovey's Theorem relating cotorsion pairs to model category structures (see [27, Theorem 2.2]).

We recall some standard definitions concerning complexes of objects in a Grothendieck category $\mathcal{A}$. Let (M, δ) (or just M, for simplicity) denote a chain complex in $\mathcal{A}$.

$$\cdots \to M^{-1} \xrightarrow{\delta^{-1}} M^0 \xrightarrow{\delta^0} M^1 \xrightarrow{\delta^1} \cdots.$$

We write $Z(M) = \cdots \to Z_n M \to Z_{n+1} M \to \cdots$ and $B(M) = \cdots \to B_n M \to B_{n+1} M \to \cdots$ for the subcomplexes consisting of the cycles and the boundaries of M.

Given an M in $\mathcal{A}$, let $S^n(M)$ denote the complex which has M in the $(-n)$th position and 0 elsewhere ($n \in \mathbb{Z}$). We denote by $D^n(M)$ the complex $\cdots \to 0 \to M \xrightarrow{id} M \to 0 \to \cdots$ where M is in the $-(n+1)$th and $(-n)$th positions ($n \in \mathbb{Z}$).

If (M, δ_M) and (N, δ_N) are two chain complexes, we define $Hom(M, N)$ as the complex

$$\cdots \to \prod_{k \in \mathbb{Z}} \mathrm{Hom}(M^k, N^{k+n}) \xrightarrow{\delta^n} \prod_{k \in \mathbb{Z}} \mathrm{Hom}(M^k, N^{k+n+1}) \to \cdots,$$

where $(\delta^n f)^k = \delta_N^{k+n} f^k - (-1)^n f^{k+1} \delta_M^k$. Write $\mathrm{Ext}_{\mathbb{C}(\mathcal{A})}(M, N)$ for the group of equivalence classes of short exact sequences of complexes $0 \to N \to L \to M \to 0$. Let us note that $\mathbb{C}(\mathcal{A})$ is a Grothendieck category having the set $\{S^n(G) : n \in \mathbb{Z}\}$ (or $\{D^n(G) : n \in \mathbb{Z}\}$) as a family of generators (where G is a generator for $\mathcal{A}$). So the functors $\mathrm{Ext}^i_{\mathbb{C}(\mathcal{A})}$, $i \in \mathbb{Z}$, can be computed using injective resolutions.

Let $(\mathcal{C}, \mathcal{C}^{\perp})$ be a cotorsion pair in $\mathcal{A}$. Following [17, Definition 3.3] we define the classes $\widetilde{\mathcal{C}^{\perp}}$, $dg\,\widetilde{\mathcal{C}}$, $\widetilde{\mathcal{C}}$ and $dg\,\widetilde{\mathcal{C}^{\perp}}$ of complexes of objects in $\mathcal{A}$. So an exact complex $E \in \widetilde{\mathcal{C}^{\perp}}$ (resp. $E \in \widetilde{\mathcal{C}}$) if $Z_n E \in \mathcal{C}^{\perp}$ (resp. $Z_n E \in \mathcal{C}$), for each $n \in \mathbb{Z}$. Then a complex $M = (M^n) \in dg\,\widetilde{\mathcal{C}}$ (resp. $M \in dg\,\widetilde{\mathcal{C}^{\perp}}$) if $Hom(M, E)$ (resp. $Hom(E, M)$) is an exact complex of abelian groups for any complex $E \in \widetilde{\mathcal{C}^{\perp}}$ (resp. $E \in \widetilde{\mathcal{C}}$) and $M^n \in \mathcal{C}$ (resp. $M^n \in \mathcal{C}^{\perp}$), for each $n \in \mathbb{Z}$.

We will need the following lemma.

Lemma 4.1. *Let X be a scheme and κ be a regular infinite cardinal such that $\kappa \geq |V|$ and $\mathcal{R}(v)$ is κ-noetherian for all $v \in V$. Let $\mathcal{N} = (\mathcal{N}^n)$, $\mathcal{M} = (\mathcal{M}^n)$ be exact complexes of quasi-coherent sheaves on X such that $\mathcal{N} \subseteq \mathcal{M}$. For each $n \in \mathbb{Z}$, let $\mathcal{X}_n$ be a locally $\leq \kappa$-presented quasi-coherent subsheaf of $\mathcal{M}^n$. Then there exists an exact complex of quasi-coherent sheaves $\mathcal{T} = (\mathcal{T}^n)$ such that $\mathcal{N} \subseteq \mathcal{T} \subseteq \mathcal{M}$, and for each $n \in \mathbb{Z}$, $\mathcal{T}^n \supseteq \mathcal{N}^n + \mathcal{X}_n$, and the quasi-coherent sheaf $\mathcal{T}^n/\mathcal{N}^n$ is locally $\leq \kappa$-presented.*

Proof. (I) First, consider the particular case of $\mathcal{N} = 0$. Let $\mathcal{Y}_0^n = \mathcal{X}_n + \delta^{n-1}(\mathcal{X}_{n-1})$. Then $(\mathcal{Y}_0^n)$ is a subcomplex of $\mathcal{M}$.

If $i < \omega$ and $\mathcal{Y}_i^n$ is a locally $\leq \kappa$-presented quasi-coherent subsheaf of $\mathcal{M}^n$, put $\mathcal{Y}_{i+1}^n = \mathcal{Y}_i^n + \mathcal{D}_i^n + \delta^{n-1}(\mathcal{D}_i^{n-1})$ where $\mathcal{D}_i^n$ is a locally $\leq \kappa$-presented quasi-coherent subsheaf of $\mathcal{M}^n$ such that $\delta^n(\mathcal{D}_i^n) \supseteq Z_{n+1}\mathcal{M} \cap \mathcal{Y}_i^{n+1}$. (Such $\mathcal{D}_i^n$ exists by our assumption on κ, since $Z_{n+1}\mathcal{M} \cap \mathcal{Y}_i^{n+1} \subseteq \operatorname{Ker}(\delta^{n+1}) = \operatorname{Im}(\delta^n)$.) Let $\mathcal{T}^n = \bigcup_{i<\omega} \mathcal{Y}_i^n$. Then $Z_{n+1}\mathcal{M} \cap \mathcal{T}^{n+1} = \bigcup_{i<\omega}(Z_{n+1}\mathcal{M} \cap \mathcal{Y}_i^{n+1}) \subseteq \bigcup_{i<\omega} \delta^n(\mathcal{Y}_{i+1}^n) \subseteq \delta^n(\mathcal{T}^n)$. It follows that $\mathcal{T} = (\mathcal{T}^n)$ is an exact subcomplex of $\mathcal{M}$. By our assumption on κ, $\mathcal{T}^n$ is locally $\leq \kappa$-presented.

(II) In general, let $\bar{\mathcal{M}} = \mathcal{M}/\mathcal{N}$ and $\bar{\mathcal{X}}_n = (\mathcal{X}_n + \mathcal{N}^n)/\mathcal{N}^n$. By part (I), there is an exact complex of quasi-coherent sheaves $\bar{\mathcal{T}}$ such that $\bar{\mathcal{T}} \subseteq \bar{\mathcal{M}}$, and for each $n \in \mathbb{Z}$, $\bar{\mathcal{T}}^n \supseteq \bar{\mathcal{X}}_n$, and the quasi-coherent sheaf $\bar{\mathcal{T}}^n$ is locally $\leq \kappa$-presented. Then $\bar{\mathcal{T}} = \mathcal{T}/\mathcal{N}$ for an exact subcomplex $\mathcal{N} \subseteq \mathcal{T} \subseteq \mathcal{M}$, and $\mathcal{T}$ clearly has the required properties. $\square$

As mentioned above, we will apply [27, Theorem 2.2] to get a model structure on $\mathbb{C}(\mathfrak{Qco}(X))$. We point out that $\mathfrak{Qco}(X)$ is a closed symmetric monoidal category under the tensor product (in the sense of [25, Section 4.1]) and hence $\mathbb{C}(\mathfrak{Qco}(X))$ is also closed symmetric monoidal. We will therefore investigate when the model structure is compatible with the induced closed symmetric monoidal structure.

Recall that a scheme X is *semi-separated* provided that the intersection of any two affine open subsets of X is again affine.

We fix our notation for the rest of this section.

Notation and Assumptions 4.2. We will assume that X is a semi-separated scheme and adopt Notation 3.12. We let $\lambda = \kappa^+$.

We will moreover assume that

(1) $\mathcal{R}(v)$ is κ-noetherian for each $v \in V$,
(2) $\mathcal{C}$ contains a generator of $\mathfrak{Qco}(X)$, and
(3) $\mathcal{F}_v$ is closed under kernels of epimorphisms for each $v \in V$.

(2) implies that $(\mathcal{C}, \mathcal{L}^\perp)$ is a complete cotorsion pair by Corollary 3.15, and (3) just says that $\mathcal{S}_v^\perp = \mathcal{S}_v^{\perp\infty}$ for each $v \in V$.

Lemma 4.3. *$(\widetilde{\mathcal{C}}, dg\widetilde{\mathcal{L}^\perp})$ is a complete cotorsion pair in $\mathbb{C}(\mathfrak{Qco}(X))$.*

Proof. $(\widetilde{\mathcal{C}}, dg\widetilde{\mathcal{L}^\perp})$ is a cotorsion pair by [17, Corollary 3.8].

We will prove that each complex $\mathcal{C} \in \widetilde{\mathcal{C}}$ is $\widetilde{\mathcal{L}}$-filtered. Then the completeness of $(\widetilde{\mathcal{C}}, dg\widetilde{\mathcal{L}^\perp})$ follows as in the proof of Corollary 3.15 because $\widetilde{\mathcal{C}}$ contains a generating set of $\mathbb{C}(\mathfrak{Qco}(X))$ (for example $\{D^n(G) \mid n \in \mathbb{Z}\}$ where $G \in \mathcal{C}$ is a generator of $\mathfrak{Qco}(X)$).

Let $\mathcal{C} = (\mathcal{M}^n) \in \widetilde{\mathcal{C}}$. Then for each $n \in \mathbb{Z}$, $Z_n\mathcal{C} \in \mathcal{C}$ and therefore $Z_n\mathcal{C}$ has an $\mathcal{L}$-filtration $\mathcal{O}_n = (\mathcal{M}_\alpha^n \mid \alpha \leq \sigma_n)$. For each $n \in \mathbb{Z}$, $\alpha < \sigma_n$, consider a locally $\leq \kappa$-presented quasi-coherent sheaf $\mathcal{A}_\alpha^n$ such that $\mathcal{M}_{\alpha+1}^n = \mathcal{M}_\alpha^n + \mathcal{A}_\alpha^n$, and the corresponding family $\mathcal{H}_n$ as in Lemma 3.11. Since the complex $\mathcal{C}$ is exact, the $\mathcal{L}$-filtration $\mathcal{O}_{n+1}$ determines a prolongation of $\mathcal{O}_n$ into a filtration $\mathcal{O}'_n = (\mathcal{M}_\alpha^n \mid \alpha \leq \tau_n)$ of $\mathcal{M}^n$ where $\tau_n = \sigma_n + \sigma_{n+1}$ (the ordinal sum), and $\mathcal{M}_{\sigma_n+\beta}^n = \delta_n^{-1}(\mathcal{M}_\beta^{n+1})$ for each $\beta < \sigma_{n+1}$.

By definition, for each $\alpha \leq \sigma_{n+1}$, δ^n maps $\mathcal{M}_{\sigma_n+\alpha}^n$ onto $\mathcal{M}_\alpha^{n+1}$. So for each $\alpha < \sigma_{n+1}$ there is a locally $\leq \kappa$-presented quasi-coherent subsheaf $\mathcal{A}_{\sigma_n+\alpha}^n$ of $\mathcal{M}_{\sigma_n+\alpha+1}^n$ such that $\delta^n(\mathcal{A}_{\sigma_n+\alpha}^n) = \mathcal{A}_\alpha^{n+1}$. Since for each $\sigma_n \leq \alpha < \tau_n$ we have $\operatorname{Ker}(\delta^n) \subseteq \mathcal{M}_\alpha^n$, it follows that $\mathcal{M}_{\alpha+1}^n = \mathcal{M}_\alpha^n + \mathcal{A}_\alpha^n$.

Let $\mathcal{H}'_n$ be the family corresponding to $\mathcal{A}^n_\alpha$ ($\alpha < \tau_n$) by Lemma 3.11. Since each closed subset of σ_n is also closed when considered as a subset of τ_n, we have $\mathcal{H}_n \subseteq \mathcal{H}'_n$. Note that $Filt(\mathcal{L}) \subseteq \mathcal{C}$, so $\mathcal{H}'_n \subseteq \mathcal{C}$ by condition (3) of Lemma 3.11.

Notice that $Z_n\mathcal{C} = \mathcal{M}^n_{\sigma_n} = \sum_{\alpha<\sigma_n} \mathcal{A}^n_\alpha$. We claim that for each closed subset $S \subseteq \tau_n$, we have $Z_n\mathcal{C} \cap \sum_{\alpha\in S} \mathcal{A}^n_\alpha = \sum_{\alpha\in S\cap\sigma_n} \mathcal{A}^n_\alpha \in \mathcal{H}_n$. To see this, we first show that $\sum_{\alpha<\sigma_n} \mathcal{A}^n_\alpha(v) \cap \sum_{\alpha\in S} \mathcal{A}^n_\alpha(v) = \sum_{\alpha\in S\cap\sigma_n} \mathcal{A}^n_\alpha(v)$ for each $v \in V$. The inclusion $\supseteq$ is clear, so consider $a \in (\sum_{\alpha<\sigma_n} \mathcal{A}^n_\alpha(v)) \cap \sum_{\alpha\in S} \mathcal{A}^n_\alpha(v)$. Then $a = a_{\alpha_0} + \cdots + a_{\alpha_k}$ where $\alpha_i \in S$, $a_{\alpha_i} \in \mathcal{A}^n_{\alpha_i}(v)$ for all $i \le k$, and $\alpha_i > \alpha_{i+1}$ for all $i < k$. W.l.o.g., we can assume that α_0 is minimal possible. If $\alpha_0 \ge \sigma_n$, then $a_{\alpha_0} = a - a_{\alpha_1} - \cdots - a_{\alpha_k} \in (\sum_{\alpha<\alpha_0} \mathcal{A}^n_\alpha(v)) \cap \mathcal{A}^n_{\alpha_0}(v) \subseteq \sum_{\alpha\in S,\alpha<\alpha_0} \mathcal{A}^n_\alpha(v)$ as $\alpha_0 \in S$ and S is closed, in contradiction with the minimality of α_0. Hence $\alpha_0 < \sigma_n$, and $a \in \sum_{\alpha\in S\cap\sigma_n} \mathcal{A}^n_\alpha(v)$. So $Z_n\mathcal{C} \cap \sum_{\alpha\in S} \mathcal{A}^n_\alpha = \sum_{\alpha\in S\cap\sigma_n} \mathcal{A}^n_\alpha$, and the latter quasi-coherent sheaf is in $\mathcal{H}_n$ because $S \cap \sigma_n$ is closed in σ_n. This proves our claim.

By induction on α, we will construct an $\widetilde{\mathcal{L}}$-filtration $(\mathcal{C}_\alpha \mid \alpha \le \sigma)$ of $\mathcal{C}$ such that $\mathcal{C}_\alpha = (\mathcal{N}^n_\alpha)$, $Z_n\mathcal{C}_\alpha \in \mathcal{H}_n$ and $\mathcal{N}^n_\alpha \in \mathcal{H}'_n$ for each $n \in \mathbb{Z}$.

First, $\mathcal{C}_0 = 0$, and if $\mathcal{C}_\alpha$ is defined and $\mathcal{C}_\alpha \ne \mathcal{C}$, then for each $n \in \mathbb{Z}$ we take a locally $\le \kappa$-presented quasi-coherent sheaf $\mathcal{X}_n$ such that $\mathcal{X}_n \nsubseteq \mathcal{N}^n_\alpha$ in case $\mathcal{N}^n_\alpha \subsetneq \mathcal{M}^n$ (this is possible by Lemma 3.5), or $\mathcal{X}_n = 0$ if $\mathcal{M}^n = \mathcal{N}^n_\alpha$. If $\mathcal{M}^n = \mathcal{N}^n_\alpha$ for all $n \in \mathbb{Z}$, we let $\sigma = \alpha$ and finish our construction.

By Lemma 4.1 there exists an exact subcomplex $\mathcal{T} = (\mathcal{T}^n)$ of $\mathcal{C}$ containing $\mathcal{C}_\alpha$ such that for each $n \in \mathbb{Z}$, $\mathcal{T}^n \supseteq \mathcal{N}^n_\alpha + \mathcal{X}_n$, and the quasi-coherent sheaf $\mathcal{T}^n/\mathcal{N}^n_\alpha$ is locally $\le \kappa$-presented. Then $\mathcal{Y}_n = \mathcal{T}^n = \mathcal{N}^n_\alpha + \mathcal{X}'_n$ for a locally $\le \kappa$-presented quasi-coherent subsheaf $\mathcal{X}'_n$ of $\mathcal{M}^n$. By condition (4) of Lemma 3.11 (for $\mathcal{N} = \mathcal{N}^n_\alpha$ and $\mathcal{X} = \mathcal{X}'_n$), there exists a quasi-coherent sheaf $\mathcal{Y}'_n = \mathcal{P}_n$ in $\mathcal{H}'_n$ such that $\mathcal{N}^n_\alpha + \mathcal{X}'_n = \mathcal{T}^n \subseteq \mathcal{P}_n$ and $\mathcal{P}_n/\mathcal{N}^n_\alpha$ is locally $\le \kappa$-presented. Iterating this process we obtain a countable chain $\mathcal{Y}_n \subseteq \mathcal{Y}'_n \subseteq \mathcal{Y}''_n \subseteq \dots$ whose union $\mathcal{N}^n_{\alpha+1} \in \mathcal{H}'_n$ by condition (2) of Lemma 3.11. Then $\mathcal{C}_{\alpha+1} = (\mathcal{N}^n_{\alpha+1})$ is an exact subcomplex of $\mathcal{C}$ containing $\mathcal{C}_\alpha$. Since $\mathcal{N}^n_{\alpha+1} \in \mathcal{H}'_n$, we have $Z_n\mathcal{C}_{\alpha+1} = Z_n\mathcal{C} \cap \mathcal{N}^n_{\alpha+1} \in \mathcal{H}_n$ by the claim above.

In order to prove that $\mathcal{C}_{\alpha+1}/\mathcal{C}_\alpha \in \widetilde{\mathcal{L}}$, it remains to show that for each $n \in \mathbb{Z}$, $Z_n(\mathcal{C}_{\alpha+1}/\mathcal{C}_\alpha) \in \mathcal{C}$. Since the complex $\mathcal{C}_{\alpha+1}/\mathcal{C}_\alpha$ is exact, it suffices to prove that $F = (\delta^n(\mathcal{N}^n_{\alpha+1}) + \mathcal{N}^{n+1}_\alpha)/\mathcal{N}^{n+1}_\alpha \in \mathcal{C}$.

We have $\mathcal{N}^n_{\alpha+1} = \sum_{\alpha\in S} \mathcal{A}^n_\alpha$ where w.l.o.g., S is a closed subset of τ_n containing σ_n. Let $S' = \{\alpha < \sigma_{n+1} \mid \sigma_n + \alpha \in S\}$. Then S' is a closed subset of $\tau_{n+1} = \sigma_{n+1} + \sigma_{n+2}$. Indeed, for each $\alpha \in S'$, we have

$$\begin{aligned}\sum_{\beta<\alpha} \mathcal{A}^{n+1}_\beta \cap \mathcal{A}^{n+1}_\alpha &= \delta^n\left(\sum_{\beta<\sigma_n+\alpha} \mathcal{A}^n_\beta\right) \cap \delta^n(\mathcal{A}^n_{\sigma_n+\alpha}) \\ &\subseteq \delta^n\left(\sum_{\beta<\sigma_n+\alpha,\beta\in S} \mathcal{A}^n_\beta\right) = \sum_{\beta<\alpha,\beta\in S'} \mathcal{A}^{n+1}_\alpha\end{aligned}$$

where the inclusion $\subseteq$ holds because S is closed in τ_n and $\mathrm{Ker}(\delta^n) \subseteq \sum_{\beta<\sigma_n+\alpha} \mathcal{A}^n_\beta$.

Since $\delta^n(\mathcal{N}^n_{\alpha+1}) = \sum_{\beta\in S'} \mathcal{A}^{n+1}_\beta$, and $\mathcal{N}^{n+1}_\alpha = \sum_{\beta\in T} \mathcal{A}^{n+1}_\beta$ for a closed subset T of τ_{n+1}, we have $F = \sum_{\beta\in S'\cup T} \mathcal{A}^{n+1}_\beta / \sum_{\beta\in T} \mathcal{A}^{n+1}_\beta$, so $F \in \mathcal{C}$ by condition (3) of Lemma 3.11 for $\mathcal{H}'_{n+1}$. This finishes the proof of $\mathcal{C}_{\alpha+1}/\mathcal{C}_\alpha \in \widetilde{\mathcal{L}}$.

If α is a limit ordinal we define $\mathcal{C}_\alpha = \bigcup_{\beta<\alpha} \mathcal{C}_\beta = (\mathcal{N}^n_\alpha)$. Then $\mathcal{N}^n_\alpha \in \mathcal{H}'_n$ by condition (2) of Lemma 3.11, and $Z_n\mathcal{C}_\alpha = Z_n\mathcal{C} \cap \mathcal{N}^n_\alpha \in \mathcal{H}_n$ by the claim above. This finishes the construction of the $\widetilde{\mathcal{L}}$-filtration of $\mathcal{C}$. □

Following [27, Definition 6.4], we call a cotorsion pair $(\mathcal{U}, \mathcal{V})$ in an abelian category $\mathcal{A}$ *small* provided that (A1) $\mathcal{U}$ contains a generator of $\mathcal{A}$, (A2) $\mathcal{V} = \mathcal{S}^{\perp}$ for a subset $\mathcal{S} \subseteq \mathcal{U}$, and (A3) for each $S \in \mathcal{S}$ there is a monomorphism i_S with cokernel S such that if $\mathcal{A}(i_S, X)$ is surjective for all $S \in \mathcal{S}$, then $X \in \mathcal{V}$.

We now show that condition (A3) above which is redundant in case $\mathcal{A}$ is a Grothendieck category.

Lemma 4.4. *Let* $(\mathcal{U}, \mathcal{V})$ *be a cotorsion pair in a Grothendieck category* $\mathcal{A}$ *satisfying conditions* (A1) *and* (A2) *above. Then* $(\mathcal{U}, \mathcal{V})$ *is small.*

Proof. We will show that $(\mathcal{U}, \mathcal{V})$ satisfies a slightly weaker version of condition (A3), namely that for each $L \in \mathcal{S}$ there is a set $\mathcal{E}_L$ of exact sequences $0 \to K \to U \to L \to 0$ such that $Y \in \mathcal{V}$ if and only if $\mathrm{Hom}(U, Y) \to \mathrm{Hom}(K, Y) \to 0$ is exact for each exact sequence in $\mathcal{E}_L$. For a given L, we define $\mathcal{E}_L$ as the set of all representatives of short exact sequences $0 \to K \to U \to L \to 0$ where U is $\leq \kappa$-presented and κ comes from [12, Corollary 2.3] for $Y = L$.

Suppose that G is an object of $\mathcal{A}$ such that $\mathrm{Hom}(U, G) \to \mathrm{Hom}(K, G) \to 0$ is exact for each exact sequence in $\mathcal{E}_L$. We will prove that $\mathrm{Ext}^1(L, G) = 0$ for all $L \in \mathcal{U}$. By condition (A2), it suffices to prove that $\mathrm{Ext}^1(L, G) = 0$ for all $L \in \mathcal{S}$. So let $0 \to G \to V \to L \to 0$ be exact with $L \in \mathcal{S}$. We want to show that this sequence splits. By our choice of κ, there is $U \subseteq V$ such that U is $\leq \kappa$-presented and $V = G + U$. Then the sequence $0 \to G \cap U \to U \to L \to 0$ is isomorphic to one in $\mathcal{E}_L$.

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G \cap U & \longrightarrow & U & \longrightarrow & L & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & G & \longrightarrow & G + U & \longrightarrow & L & \longrightarrow & 0. \end{array}$$

Our hypothesis now implies that the inclusion $G \cap U \to G$ can be extended to $U \to G$ so, since the left-hand square is a pushout, we see that the bottom row splits. This proves that $\mathrm{Ext}^1(L, G) = 0$. Now, replacing the set $\mathcal{S}$ by $\mathcal{S}' = \{L^{(card(\mathcal{E}_L))} \mid L \in \mathcal{S}\}$, we see that both conditions (A2) and (A3) hold for $\mathcal{S}'$, hence the cotorsion pair $(\mathcal{U}, \mathcal{V})$ is small. $\square$

Now we can prove the main theorem of our paper (see Notation and Assumptions 4.2).

Theorem 4.5. *Let* X *be a semi-separated scheme. There is a model category structure on* $\mathbb{C}(\mathfrak{Qco}(X))$ *where the weak equivalences are the homology isomorphisms, the cofibrations (resp. trivial cofibrations) are the monomorphisms with cokernels in* $dg\,\widetilde{\mathcal{C}}$ *(resp. in* $\widetilde{\mathcal{C}}$*), and the fibrations (resp. trivial fibrations) are the epimorphisms whose kernels are in* $dg\,\widetilde{\mathcal{L}^{\perp}}$ *(resp.* $\widetilde{\mathcal{L}^{\perp}}$*).*

Moreover, if every $M \in S_v$ *is a flat* $\mathcal{R}(v)$*-module, and* $M \otimes_{\mathcal{R}(v)} N \in S_v$ *whenever* $M, N \in S_v$*, then the model structure is monoidal with respect to the usual tensor product of complexes of quasi-coherent sheaves.*

Proof. We will apply Hovey's Theorem [27, Theorem 2.2]. First, the results of [27, Section 5] guarantee that the weak equivalences of our model structure are the homology isomorphisms. In our case $\mathcal{W}$ is the class of all exact complexes of quasi-coherent sheaves. It is easy to check that this is a thick subcategory of $\mathbb{C}(\mathfrak{Qco}(X))$. Now, according to Hovey's Theorem, we will have to show that the pairs $(dg\,\widetilde{\mathcal{C}}, dg\,\widetilde{\mathcal{L}^{\perp}} \cap \mathcal{W})$ and $(dg\,\widetilde{\mathcal{C}} \cap \mathcal{W}, dg\,\widetilde{\mathcal{L}^{\perp}})$ are complete cotorsion pairs (notice that our notion of completeness coincides with Hovey's notion of 'functorial completeness'). We will proceed in three steps, proving the following.

(1) The pairs $(\widetilde{\mathcal{C}}, dg\,\widetilde{\mathcal{L}^{\perp}})$ and $(dg\,\widetilde{\mathcal{C}}, \widetilde{\mathcal{L}^{\perp}})$ are cotorsion pairs.
(2) $dg\,\widetilde{\mathcal{C}} \cap \mathcal{W} = \widetilde{\mathcal{C}}$ and $dg\,\widetilde{\mathcal{L}^{\perp}} \cap \mathcal{W} = \widetilde{\mathcal{L}^{\perp}}$.
(3) The cotorsion pairs $(\widetilde{\mathcal{C}}, dg\,\widetilde{\mathcal{L}^{\perp}})$ and $(dg\,\widetilde{\mathcal{C}}, \widetilde{\mathcal{L}^{\perp}})$ are complete.

Condition (1) follows from [17, Corollary 3.8].

Let us check condition (2). By [18, Corollary 3.9] (4. $\Rightarrow$ 1.) it suffices to prove that $dg\,\widetilde{\mathcal{C}} \cap \mathcal{W} = \widetilde{\mathcal{C}}$. The inclusion $\widetilde{\mathcal{C}} \subseteq dg\,\widetilde{\mathcal{C}} \cap \mathcal{W}$ was proven in [17, Lemma 3.10]. Let us prove that $dg\,\widetilde{\mathcal{C}} \cap \mathcal{W} \subseteq \widetilde{\mathcal{C}}$. So let $\mathcal{Y}$ be a complex in $dg\,\widetilde{\mathcal{C}} \cap \mathcal{W}$ (so $\mathcal{Y}(v)$ is a complex of $\mathcal{R}(v)$-modules, for all $v \in V$). To see that $\mathcal{Y}$ is in $\widetilde{\mathcal{C}}$ we have to check that $Z_n\mathcal{Y} \in \mathcal{C}$, for all $n \in \mathbb{Z}$. But this means that the $\mathcal{R}(v)$-module $Z_n\mathcal{Y}(v)$ belongs to $\mathcal{F}_v$ for all $v \in V$. Since $\mathcal{F}_v$ is closed under kernels of epimorphisms by assumption, [18, Corollary 3.9] shows that if a complex of $\mathcal{R}(v)$-modules is exact and belongs to $dg\widetilde{\mathcal{F}_v}$ then it belongs to $\widetilde{\mathcal{F}_v}$ (so $Z_n\mathcal{Y}(v) \in \mathcal{F}_v$ for all $v \in V$). Therefore we will be done if we prove that $\mathcal{Y}(v)$ is exact and belongs to $dg\widetilde{\mathcal{F}_v}$. Since the complex $\mathcal{Y}$ is exact, for each affine open set $v \in V$, $\mathcal{Y}(v)$ is an exact complex of $\mathcal{R}(v)$-modules. Let us see that $\mathcal{Y}(v) \in dg\widetilde{\mathcal{F}_v}$, for all $v \in V$. So let E be a complex of $\mathcal{R}(v)$-modules in $\widetilde{\mathcal{S}_v{}^{\perp}}$ (so E is exact and $Z_nE \in \mathcal{S}_v{}^{\perp}$). We have to check that $Hom(\mathcal{Y}(v), E)$ is exact. Since X is semi-separated, by [23, Proposition 5.8] there exists a right adjoint $i_{*v} : \mathcal{R}(v)\text{-Mod} \to \mathfrak{Qco}(X)$ of the restriction functor $i_v^* : \mathfrak{Qco}(X) \to \mathcal{R}(v)\text{-Mod}$ (defined by $i_v^*(\mathcal{M}) = \mathcal{M}(v)$). The adjointness situation can be lifted up to $\mathbb{C}(\mathfrak{Qco}(X))$. Then there is an isomorphism

$$Hom_{\mathbb{C}(\mathcal{R}(v))}(\mathcal{Y}(v), E) = Hom_{\mathbb{C}(\mathcal{R}(v))}(i_v^*(\mathcal{M}), E) \cong Hom_{\mathbb{C}(\mathfrak{Qco}(X))}(\mathcal{Y}, i_{*v}(E))$$

and since the functor i_{*v} preserves exactness, $i_{*v}(E)$ will be an exact complex in $\mathbb{C}(\mathfrak{Qco}(X))$. Since $\mathcal{Y} \in dg\,\widetilde{\mathcal{C}}$, once we show that $i_{*v}(E) \in \widetilde{\mathcal{L}^{\perp}}$ we will finish by the comment above. But, $Z_n i_{*v}(E) = i_{*v}(Z_nE)$. Hence, for each $\mathcal{T} \in \mathcal{C}$,

$$\operatorname{Ext}^1_{\mathfrak{Qco}(X)}(\mathcal{T}, i_{*v}(Z_nE)) \cong \operatorname{Ext}^1_{\mathcal{R}(v)}(i_v^*(\mathcal{T}), Z_nE) = 0,$$

where the last equality follows because $i_v^*(\mathcal{T}) = \mathcal{T}(v) \in \mathcal{F}_v$ and $Z_nE \in \mathcal{S}_v{}^{\perp}$.

Now let us prove condition (3). By Lemma 4.3 the cotorsion pair $(\widetilde{\mathcal{C}}, dg\,\widetilde{\mathcal{L}^{\perp}})$ is complete. We claim that the cotorsion pair $(dg\,\widetilde{\mathcal{C}}, \widetilde{\mathcal{L}^{\perp}})$ is also complete. Let $\mathcal{I}'$ be a set of representatives of the quasi-coherent sheaves in $\mathcal{L}$. Then clearly $(\mathcal{I}')^{\perp} = \mathcal{L}^{\perp}$. We will prove that $\mathcal{I}^{\perp} = \widetilde{\mathcal{L}^{\perp}}$ where $\mathcal{I} = \{S^n(\mathcal{A}) \mid \mathcal{A} \in \mathcal{I}', n \in \mathbb{Z}\} \cup \{S^n(\mathcal{G}) \mid n \in \mathbb{Z}\}$ (and $\mathcal{G} \in \mathcal{C}$ is a generator of $\mathfrak{Qco}(X)$). Then the claim will follow by Lemma 4.4 and [27, Corollary 6.6]. First we will prove that $\widetilde{\mathcal{L}^{\perp}} \subseteq \mathcal{I}^{\perp}$. To do this we will check that $\mathcal{I} \subseteq dg\,\widetilde{\mathcal{C}}$. It is clear that $S^m(\mathcal{A})^l \in \mathcal{C}$ ($l \in \mathbb{Z}$, $\mathcal{A} \in \mathcal{I}' \cup \{\mathcal{G}\}$). Now, let us consider $\mathcal{M} \in \widetilde{\mathcal{L}^{\perp}}$. Then $Hom(S^m(\mathcal{A}), \mathcal{M})$ is the complex

$$\cdots \to \operatorname{Hom}(\mathcal{A}, \mathcal{M}^l) \to \operatorname{Hom}(\mathcal{A}, \mathcal{M}^{l+1}) \to \cdots.$$

This complex is obviously exact because $\mathcal{M}$ is exact and $Z_n\mathcal{M}, B_n\mathcal{M} \in \mathcal{L}^{\perp}$, and so $S^m(\mathcal{A}) \in dg\,\widetilde{\mathcal{C}}$ ($m \in \mathbb{Z}$, $\mathcal{A} \in \mathcal{I}' \cup \{\mathcal{G}\}$). Therefore $\mathcal{I}^{\perp} \supseteq (dg\,\widetilde{\mathcal{C}})^{\perp} = \widetilde{\mathcal{L}^{\perp}}$. We now prove the converse: let $\mathcal{N} \in \mathcal{I}^{\perp}$. We have to see that $\mathcal{N}$ is exact and that $Z_n\mathcal{N} \in \mathcal{L}^{\perp}$. First, we prove that $\mathcal{N}$ is exact. It is clear that this is equivalent to each morphism $S^n(\mathcal{G}) \to \mathcal{N}$ (for $\mathcal{G}$ a generator of $\mathfrak{Qco}(X)$) being extendable to $D^n(\mathcal{G}) \to \mathcal{N}$ for each $n \in \mathbb{Z}$. But this follows from the short exact sequence

$$0 \to S^n(\mathcal{G}) \to D^n(\mathcal{G}) \to S^{n+1}(\mathcal{G}) \to 0$$

since $\mathrm{Ext}^1(S^{n+1}(\mathcal{G}), \mathcal{N}) = 0$. Now we prove that $Z_n\mathcal{N} \in \mathcal{L}^{\perp}$. Since $\mathcal{I}^{\perp} = \mathcal{L}^{\perp}$ we only need to prove that $\mathrm{Ext}^1_{\mathfrak{Qco}(X)}(\mathcal{A}, Z_n\mathcal{N}) = 0$ for all $n \in \mathbb{Z}$ and $\mathcal{A} \in \mathcal{I}$. But there exists a monomorphism of abelian groups

$$0 \to \mathrm{Ext}^1_{\mathfrak{Qco}(X)}(\mathcal{A}, Z_n\mathcal{N}) \to \mathrm{Ext}^1_{\mathbb{C}(\mathfrak{Qco}(X))}(S^{-n}(\mathcal{A}), \mathcal{N})$$

(see e.g. [13, Lemma 5.1]) and since the latter group is 0, we get that $Z_n\mathcal{N} \in \mathcal{L}^{\perp}$. This proves our claim, and thus finishes the proof of condition (3).

Finally to get that the model structure is monoidal we apply [18, Theorem 5.1] (by noticing that the argument of the proof of [18, Theorem 5.1] carries over without the assumption of $\mathcal{F}$ being closed under direct limits). If S_v is contained in the class of all flat modules then every quasi-coherent sheaf in $\mathcal{C}$ is flat. So condition (1) of [18, Theorem 5.1] holds. Now if $M \otimes_{\mathcal{R}(v)} N \in S_v$, where M, N are $\mathcal{R}(v)$-modules in S_v, it follows that $L \otimes_{\mathcal{R}(v)} T \in \mathcal{F}_v$, where L, T are direct summands of S_v-filtered $\mathcal{R}(v)$-modules (because the tensor product commutes with direct limits, and S_v consists of flat modules). And so $\mathcal{L} \otimes_{\mathcal{R}} \mathcal{T} \in \mathcal{C}$, for any $\mathcal{L}, \mathcal{T} \in \mathcal{C}$. So condition (2) of [18, Theorem 5.1] also holds. Finally condition (3) of [18, Theorem 5.1] is immediate because, for all $v \in V$, $\mathcal{F}_v$ contains all projective $\mathcal{R}(v)$-modules, so in particular $\mathcal{R} \in \mathcal{C}$. □

The proof of Corollaries 1.2 and 1.3. In Theorem 4.5, we take $S_v = \{R(v)\}$, so $\mathcal{F}_v$ is the class of all projective $\mathcal{R}(v)$-modules, and S_v = a representative set of all flat modules of cardinality $\leq \mathrm{card}(R(v)) + \aleph_0$, so $\mathcal{F}_v$ is the class of all flat $\mathcal{R}(v)$-modules, respectively. Notice that in the first case, $\mathcal{C}$ is the class of all vector bundles, while in the second, $\mathcal{C}$ is the class of all flat quasi-coherent sheaves. □

If $X = \mathbf{P}^n(R)$ where R is any commutative noetherian ring, then every quasi-coherent sheaf on X is a filtered union of coherent subsheaves, and the family of so-called twisting sheaves $\{\mathcal{O}(n) \mid n \in \mathbb{Z}\}$ generates the category of coherent sheaves on X cf. [23, Corollary 5.18], so $\bigoplus_{i\in\mathbb{Z}} \mathcal{O}(n)$ is a (vector bundle) generator for $\mathfrak{Qco}(X)$. So Corollary 1.2 applies to this setting. In particular, we extend here [13, Theorem 6.1] which deals with the case of the projective line.

Finally, we consider the case of restricted Drinfeld vector bundles.

The proof of Corollary 1.4. First we notice that the class $\mathcal{C}$ in this case contains a generator of $\mathfrak{Qco}(X)$ as a consequence of Corollary 3.7, and that $\mathcal{S}_v^{\perp} = \mathcal{S}_v^{\perp\infty}$ because the class of all flat Mittag-Leffler modules is closed under pure submodules, and hence under syzygies. In view of Theorem 4.5, the proof will be complete once we show the following.

Lemma 4.6. *If R is a commutative ring and M and N are $\leq \kappa$-presented flat Mittag-Leffler modules, then so is $N \otimes_R M$.*

Proof. It is clear that $N \otimes_R M$ is $\leq \kappa$-presented. Let us check the Mittag-Leffler condition (see Definition 2.5). So let $(M_i \mid i \in I)$ be a family of R-modules. Since N is flat Mittag-Leffler the canonical map $N \otimes_R \prod_{i\in I} M_i \to \prod_{i\in I} N \otimes_R M_i$ is a monomorphism. Now since M is flat, we get a monomorphism

$$(M \otimes_R N) \otimes_R \prod_{i\in I} M_i \cong M \otimes_R \left(N \otimes_R \prod_{i\in I} M_i\right) \to M \otimes_R \left(\prod_{i\in I} N \otimes_R M_i\right).$$

Now we apply the fact that M is Mittag-Leffler to the family $(N \otimes_R M_i \mid i \in I)$ to get a monomorphism

$$M \otimes_R \left(\prod_{i \in I} N \otimes_R M_i \right) \to \prod_{i \in I} M \otimes_R (N \otimes_R M_i) \cong \prod_{i \in I} (M \otimes_R N) \otimes_R M_i .$$

So the claim follows by composing the previous monomorphisms. □

5. Flat Mittag-Leffler abelian groups

Let X be a scheme having a generating set consisting of Drinfeld vector bundles. We have already seen that restricted Drinfeld vector bundles impose monoidal model structures on $\mathbb{C}(\mathfrak{Qco}(X))$ whose weak equivalences are the homology isomorphisms (see Corollary 1.4). This result suggests that the entire class of all Drinfeld vector bundles could also impose a cofibrantly generated model structure in $\mathbb{C}(\mathfrak{Qco}(X))$. The aim of this section is to prove Theorem 1.5 and show thus that this is not the case in general.

We recall that, given a class of objects $\mathcal{F}$ in a Grothendieck category $\mathcal{A}$, an *$\mathcal{F}$-precover* of an object M is a morphism $\varphi : F \to M$ with $F \in \mathcal{F}$ such that $\mathrm{Hom}_{\mathcal{A}}(F', F) \to \mathrm{Hom}_{\mathcal{A}}(F', M)$ is an epimorphism for every $F' \in \mathcal{F}$. The class $\mathcal{F}$ is said to be precovering if every object of $\mathcal{A}$ admits an $\mathcal{F}$-precover (see [15, Chapters 5 and 6] for properties of such classes). For example the class of projective modules $\mathcal{P}$ is precovering. Similarly as $\mathcal{P}$ is used to define projective resolutions, one can employ a precovering class $\mathcal{F}$ to define $\mathcal{F}$-resolutions and a version of relative homological algebra can be developed (see [15]).

Let $\mathcal{D}$ denote the class of all flat Mittag-Leffler modules over a ring R. In the case when $R = \mathbb{Z}$, the class $\mathcal{D}$ was characterized by Azumaya and Facchini.

Lemma 5.1 *([3, Proposition 7]). A group A is flat and Mittag-Leffler, if and only if A is $\aleph_1$-free (i.e., all countably generated subgroups of A are free).*

Remark 5.2. Let R be an arbitrary ring. A module M is $\aleph_1$-*projective* provided there exists a system, $\mathcal{S}$, consisting of countably generated projective submodules of M such that for each countable subset C of M, there is $S \in \mathcal{S}$ with $C \subseteq S$, and $\mathcal{S}$ is closed under unions of countable chains. This notion is due to Shelah (cf. [8, Chap. IV]); notice that for abelian groups, $\aleph_1$-projective = $\aleph_1$-free.

As recently proved in [24], flat Mittag-Leffler modules coincide with the $\aleph_1$-projective ones. Particular instances of this result are much older: besides abelian groups [3], it is known for modules over von Neumann regular rings [21], and over Dedekind domains [32]. Moreover, [32,33] reveal yet another facet of Mittag-Leffler modules: they coincide with the positively atomic ones.

From now on, we will restrict ourselves to the particular case of (abelian) groups. Our aim is to show that the class of all $\aleph_1$-free groups is not precovering. We will prove this following an idea from [9] where the analogous result was proven consistent with (but independent of) ZFC + GCH for the subclass of $\mathcal{D}$ consisting of all Whitehead groups.

The reason why our result on $\mathcal{D}$ holds in ZFC rather than only in some of its forcing extensions rests in the following fact whose proof goes back to [28] (see also [19]): for each non-cotorsion group A, there is a *Baer–Specker group* (that is, the product $\mathbb{Z}^\kappa$ for some $\kappa \geq \aleph_0$) such that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^\kappa, A) \neq 0$; moreover, the Baer–Specker group can be taken small in the following sense.

Lemma 5.3. *Define a sequence of cardinals κ_α ($\alpha \geq 0$) as follows:*

- $\kappa_0 = \aleph_0$,
- $\kappa_{\alpha+1} = \sup_{i<\omega} \kappa_{\alpha,i}$ *where* $\kappa_{\alpha,0} = \kappa_\alpha$ *and* $\kappa_{\alpha,n+1} = 2^{\kappa_{\alpha,n}}$, *and*
- $\kappa_\alpha = \sup_{\beta<\alpha} \kappa_\beta$ *when α is a limit ordinal.*

Let α be an ordinal and A be a non-cotorsion group of cardinality $\leq 2^{\kappa_\alpha}$. Then $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^{\kappa_\alpha}, A) \neq 0$.

Proof. This is a consequence of [19, 1.2(4)] where the following stronger assertion is proven.

'If A is any group of cardinality $\leq 2^{\kappa_\alpha}$ such that $\mathrm{Ext}^1_{\mathbb{Z}}(D_{\kappa_\alpha}, A) = 0$ (where D_{κ_α} is a certain subgroup of $\mathbb{Z}^{\kappa_\alpha}$) then $\mathrm{Ext}^1_R(\mathbb{Q}, A) = 0$, that is, A is a cotorsion group.' $\square$

Remark 5.4. Under GCH, the definition of the κ_α's simplifies as follows: if $\kappa_\alpha = \aleph_\beta$, then $\kappa_{\alpha+1} = \aleph_{\beta+\omega}$.

It follows that though the class $\mathcal{D}$ of all $\aleph_1$-free groups is closed under $\mathcal{D}$-filtrations, it is not of the form ${}^\perp\mathcal{C}$ for any class of groups $\mathcal{C}$.

Theorem 5.5. *Let $R = \mathbb{Z}$. Then $\mathcal{D} \neq {}^\perp \mathcal{C}$, for each class $\mathcal{C} \subseteq$ Mod-$\mathbb{Z}$. In fact, ${}^\perp(\mathcal{D}^\perp)$ is the class of all flat (= torsion-free) groups.*

Proof. Since all the Baer–Specker groups are $\aleph_1$-free (see e.g. [8, IV.2.8]), Lemma 5.3 implies that $\mathcal{D}^\perp$ coincides with the class of all cotorsion groups, so ${}^\perp(\mathcal{D}^\perp)$ is the class of all flat groups. Since $\mathbb{Q} \notin \mathcal{D}$, we have $\mathcal{D} \neq {}^\perp \mathcal{C}$ for each class $\mathcal{C} \subseteq$ Mod-$\mathbb{Z}$. $\square$

Lemma 5.6. *Let α be an ordinal and A be an $\aleph_1$-free group of cardinality $\leq 2^{\kappa_\alpha}$ where κ_α is defined as in Lemma 5.3. Then* $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^{\kappa_\alpha}, A) \neq 0$.

Proof. In view of Lemma 5.3, it suffices to verify that no non-zero $\aleph_1$-free group is cotorsion. Indeed, each reduced torsion free cotorsion group A has a direct summand isomorphic to $\mathbb{J}_p$ (the group of all p-adic integers for some prime $p \in \mathbb{Z}$) by [8, V.2.7 and V.2.9(5),(6)]. However, if A is $\aleph_1$-free, then it is cotorsion-free by [8, V.2.10(ii)], so $\mathbb{J}_p$ does not embed into A. $\square$

Remark 5.7. Theorem 5.5 already implies that we cannot improve Corollary 1.4 by extending the claim to all Drinfeld vector bundles (that is, removing the κ-filtration restriction): consider the affine scheme $X = \mathrm{Spec}(\mathbb{Z})$. Then there is a category equivalence $\mathfrak{Qco}(X) \cong$ Mod-$\mathbb{Z}$ by [23, Corollary 5.5]. By Theorem 5.5, for each infinite cardinal κ, there is a Drinfeld vector bundle $\mathcal{M}$ which does not have a $\mathcal{C}$-filtration where $\mathcal{C}$ is the class of all locally $\leq \kappa$-presented Drinfeld vector bundles. (Recently, it has been shown in [24] that this extends to the affine schemes $X = \mathrm{Spec}(R)$ where R is commutative, but not perfect.)

In more detail, if $\kappa = \kappa_\alpha$ (see Lemma 5.3) and $\mathcal{D}^{\leq\kappa}$ denotes the class of all $\leq \kappa$-generated $\aleph_1$-free groups, then $\mathbb{Z}^{2^\kappa} \in \mathcal{D} \setminus {}^\perp((\mathcal{D}^{\leq\kappa})^\perp)$.

Indeed, denote by $\mathcal{E}$ a representative set of elements of the class $\mathcal{D}^{\leq\kappa}$. Then $|\mathcal{E}| \leq 2^\kappa$, so by [10, Theorem 2], there exists $A \in \mathcal{E}^\perp$ such that $|\mathcal{A}| = 2^{2^\kappa}$ and A has a $\mathcal{E}$-filtration. In particular, A is $\aleph_1$-free, and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^{2^\kappa}, A) \neq 0$ by Lemma 5.6. Hence $\mathbb{Z}^{2^\kappa} \notin {}^\perp(\mathcal{E}^\perp)$.

In view of Remark 5.7, the class of all $\aleph_1$-free groups cannot induce a cofibrantly generated model category structure on $\mathfrak{Qco}(\mathrm{Spec}(\mathbb{Z})) \cong$ Mod-$\mathbb{Z}$ compatible with its abelian structure. This is the second claim of Theorem 1.5. In order to prove the (stronger) first claim of Theorem 1.5, it remains to show the following.

Theorem 5.8. *The class of all $\aleph_1$-free groups is not precovering.*

Proof. Assume there exists a $\mathcal{D}$-precover of $\mathbb{Q}$, and denote it by $p : B \to \mathbb{Q}$. We will construct an $\aleph_1$-free group G of infinite rank such that there is no non-zero homomorphism from G to B. Since G has infinite rank and $\mathbb{Q}$ is injective, there is a non-zero (even surjective) homomorphism $g : G \to \mathbb{Q}$. Clearly g does not factorize through p, a contradiction.

First, we take an ordinal α such that $\mu = 2^{\kappa_\alpha} \geq \operatorname{card}(B)$ (see Lemma 5.3). The $\aleph_1$-free group G will be the last term of a continuous chain of $\aleph_1$-free groups of infinite rank, $(G_\nu \mid \nu \leq \tau)$, of length $\tau \leq \mu^+$. The chain will be constructed by induction on ν as follows: first, G_0 is any free group of infinite rank.

Assume G_ν is defined for some $\nu < \mu^+$ and consider the set I_ν of all non-zero homomorphisms from G_ν to B. If $I_\nu = \emptyset$, we put $\tau = \nu$ and finish the construction. Otherwise, we fix a free presentation $0 \to K \hookrightarrow F \to \mathbb{Z}^{\kappa_\alpha} \to 0$ of $\mathbb{Z}^{\kappa_\alpha}$, and denote by θ the inclusion of K into F.

For each $h \in I_\nu$, let A_h be the image of h. By Lemma 5.6, $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^{\kappa_\alpha}, A_h) \neq 0$, so there exists a homomorphism $\phi_h : K \to A_h$ which does not extend to F. Since K is free and h maps onto A_h, there is a homomorphism $\psi_h : K \to G_\nu$ such that $h\psi_h = \phi_h$.

Denote by Θ the inclusion of $K^{(I_\nu)}$ into $F^{(I_\nu)}$, and define $\Psi \in \operatorname{Hom}_{\mathbb{Z}}(K^{(I_\nu)}, G_\nu)$ so that the h-th component of Ψ is ψ_h, for each $h \in I_\nu$.

The group $G_{\nu+1}$ is defined by the pushout of Θ and Ψ:

$$\begin{tikzcd}
K^{(I_\nu)} \arrow[r, "\Theta"] \arrow[d, "\Psi"'] & F^{(I_\nu)} \arrow[d, "\Omega"'] \\
G_\nu \arrow[r, "\subseteq"] & G_{\nu+1}.
\end{tikzcd}$$

Note that $G_{\nu+1}/G_\nu \cong F^{(I_\nu)}/K^{(I_\nu)}$ is $\aleph_1$-free because $\mathbb{Z}^{\kappa_\alpha}$ is $\aleph_1$-free by [8, IV.2.8]. It follows that $G_{\nu+1}$ is an $\aleph_1$-free group of infinite rank.

If $\nu \leq \mu^+$ is a limit ordinal we put $G_\nu = \bigcup_{\sigma<\nu} G_\sigma$. Clearly G_ν has infinite rank, and since $G_{\sigma+1}/G_\sigma$ is $\aleph_1$-free for each $\sigma < \nu$ by construction, G_ν is also $\aleph_1$-free.

It remains to show that there exists $\nu \leq \mu^+$ such that $I_\nu = \emptyset$. Assume $I_\nu \neq \emptyset$ for all $\nu < \mu^+$ (hence G_ν is defined for all $\nu \leq \mu^+$); we will prove that $I_{\mu^+} = \emptyset$.

Assume there is a non-zero homomorphism $f : G_{\mu^+} \to B$ and let $\nu < \mu^+$ be such that $h := f \restriction G_\nu \neq 0$.

Using the notation introduced in the non-limit step of the construction, we will prove that A_h is a proper submodule of the image of $h' = f \restriction G_{\nu+1}$. If not, then $h'\Omega$ extends $h\Psi$ to a homomorphism $F^{(I_\nu)} \to A_h$. Denote by ι_h and ι'_h the h-th canonical embedding of K into $K^{(I_\nu)}$ and of F into $F^{(I_\nu)}$, respectively. Then $h'\Omega\iota'_h$ extends $h\Psi\iota_h = h\psi_h = \phi_h$ to a homomorphism $F \to A_h$, in contradiction with the definition of ϕ_h.

This proves that the image of $f \restriction G_\nu$ is a proper submodule of the image of $f \restriction G_{\nu+1}$ for each $\nu \in C$, where C is the set of all $\nu < \mu^+$ such that $f \restriction G_\nu \neq 0$. However, $f \neq 0$ implies that C has cardinality μ^+, in contradiction with $\operatorname{card}(B) < \mu^+$. This proves that $\operatorname{Hom}_{\mathbb{Z}}(G_{\mu^+}, B) = 0$, that is, $I_{\mu^+} = \emptyset$. $\square$

Acknowledgments

The first two authors were partially supported by the DGI and by the Fundación Seneca 04555/06. The first author was also partially supported by the Junta de Andalucía, Consejería de EIC and FEDER funds. The fourth author was supported by GAČR 201/09/0816 and MSM 0021620839.

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