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Global existence of solutions of the liquid crystal flow for the Oseen–Frank model in $\mathbb{R}^2$

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Abstract

In the first part of this paper, we establish the global existence of solutions of the liquid crystal (gradient) flow for the well-known Oseen–Frank model. The liquid crystal flow is a prototype of equations from the Ericksen–Leslie system in the hydrodynamic theory and generalizes the heat flow for harmonic maps into the 2-sphere. The Ericksen–Leslie system is a system of the Navier–Stokes equations coupled with the liquid crystal flow. In the second part of this paper, we also prove the global existence of solutions of the Ericksen–Leslie system for a general Oseen–Frank model in $\mathbb{R}^2$.

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1. Introduction

A *liquid crystal* is a state of matter intermediate between a crystalline solid and a normal isotropic liquid. Research into liquid crystals is an area of a very successful synergy between mathematics and physics. There are a lot of analytical and computational issues, which arise in the attempt to study static equilibrium configurations. Numerical and experimental analysis has

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shown that equilibrium configurations are expected to have point and line singularities [21]. Mathematically, Hardt et al. in their fundamental papers [15,16] proved the existence of an energy minimizer u of the liquid crystal functional and showed that a minimizer u is smooth away from a closed set Σ of Ω. Moreover, Σ has Hausdorff dimension strictly less than one. In [1], Almgren and Lieb did some related analysis indicating that the phenomenon is of wider interest. In physical theory, an equilibrium configuration corresponds to a critical point, not necessarily an energy minimizer, of the liquid crystal energy. Critical points are much harder to understand mathematically than minima. From the above result of Hardt et al., minimizers cannot have line singularities. Following the work of Bethuel–Brezis–Coron on harmonic maps in [4], Giaquinta et al. [12] found a relaxed energy for the liquid crystal systems, whose minimizers are also equilibrium configurations. On the other hand, Giaquinta et al. [11] also proved that minimizers of the relaxed energy for harmonic maps are smooth away from a 1-dimensional singular set. Further developments on the regularity results on harmonic maps were surveyed in [13]. There is an interesting open problem to prove that minimizers of the relaxed liquid crystal energy have line singularities. The first author in [17] proved the partial regularity of minimizers of the modified relaxed energy of the liquid crystal energy. However, the partial regularity of minimizers of the relaxed energy for liquid crystals is still mysterious. In some related studies of liquid crystals, Bauman et al. [3] studied the Landau–de Gennes free energy used to describe the transition between chiral nematic and the smectic liquid crystal phase, Lin and Pan [29] used the Landau–de Gennes models to investigate the magnetic field induced instabilities in liquid crystals, and the existence of infinitely many liquid crystal equilibrium configurations prescribing the same boundary was obtained in [18].

A general description of the static theory of liquid crystals is given by Ericksen in [9]. A liquid crystal is composed of rod like molecules which display orientational order, unlike a liquid, but lacking the lattice structure of a solid. The kinematic variable in the nematic and the cholesteric phase may be taken to the optic axis, which is a unit vector field u in a region $\Omega \subset \mathbb{R}^3$ occupied by the materials. The liquid crystal energy for a configuration $u \in H^1(\Omega; S^2)$ is given by

$$E(u; \Omega) = \int_\Omega W(u, \nabla u)\, dx, \tag{1.1}$$

where the Oseen–Frank density $W(u, \nabla u)$, depending on positive material constants k_1, k_2, k_3 and k_4, is given by

$$W(u, \nabla u) = k_1(\operatorname{div} u)^2 + k_2(u \cdot \operatorname{curl} u)^2 + k_3|u \times \operatorname{curl} u|^2 + k_4[\operatorname{tr}(\nabla u)^2 - (\operatorname{div} u)^2].$$

Without loss of generality, as in [15] or [13], we rewrite the density

$$W(u, \nabla u) = a|\nabla u|^2 + V(u, \nabla u), \quad a = \min\{k_1, k_2, k_3\} > 0, \tag{1.2}$$

where

$$V(u, \nabla u) = (k_1 - a)(\operatorname{div} u)^2 + (k_2 - a)(u \cdot \operatorname{curl} u)^2 + (k_3 - a)|u \times \operatorname{curl} u|^2.$$

A static equilibrium configuration corresponds to an extremal (critical point) of the energy functional E in $H^1(\Omega, S^2)$. The Euler–Lagrange system for the general Oseen–Frank functional (1.1) (see details in the Appendix of Section 5) is

$$\begin{aligned} \nabla_\alpha \Big[W_{p^i_\alpha}(u, \nabla u) - u^l u^i V_{p^l_\alpha}(u, \nabla u) \Big] - W_{u^i}(u, \nabla u) + W_{u^l}(u, \nabla u) u^l u^i \\ + W_{p^l_\alpha}(u, \nabla u) \nabla_\alpha u^l u^i + V_{p^l_\alpha}(u, \nabla u) u^l \nabla_\alpha u^i = 0 \quad \text{in } \Omega \end{aligned} \tag{1.3}$$

for $i = 1, 2, 3$, where we adopt the standard summation convention. In a special case of $k_1 = k_2 = k_3$, the system (1.3) becomes the harmonic map equations into S^2. However, the equilibrium system associated to the energy functional (1.1) is not elliptic for every choice of the constants k_1, k_2 and k_3.

In the first part of this paper, we investigate the liquid crystal flow for a model with the Oseen–Frank density (1.2). For a domain Ω in $\mathbb{R}^3$ or in $\mathbb{R}^2$, a map $u(x,t) : \Omega \times [0,\infty) \to S^2$ is a solution of the liquid crystal flow if u satisfies

$$\begin{aligned}\frac{\partial u^i}{\partial t} &= \nabla_\alpha \left[W_{p^i_\alpha}(u, \nabla u) - u^l u^i V_{p^l_\alpha}(u, \nabla u) \right] - W_{u^i}(u, \nabla u) \\ &\quad + W_{u^l}(u, \nabla u) u^l u^i + W_{p^l_\alpha}(u, \nabla u) \nabla_\alpha u^l u^i + V_{p^l_\alpha}(u, \nabla u) u^l \nabla_\alpha u^i \end{aligned} \tag{1.4}$$

in $\Omega \times [0,\infty)$ for $i = 1, 2, 3$.

The flow equation (1.4) is a prototype of equations from the Ericksen–Leslie system in the hydrodynamic theory (cf. [9]). The liquid crystal flow (1.4) also generalizes the heat flow for harmonic maps into the 2-sphere. Since the seminal work of Eells and Sampson [7], many studies on the heat flow for harmonic maps have been carried out. In 2-dimensional case, Struwe [33] established the global existence of the weak solution of the harmonic maps flow with initial data, where the solution is smooth except for a finite number of singularities. In higher dimensional cases, Chen and Struwe [6] proved the global existence of partially regular solutions to the harmonic map flow. Since (1.4) is not parabolic, the system of the liquid crystal flow is complicated, so the question on global existence for the liquid crystal flow (1.4) for the Oseen–Frank model remains unresolved. In this paper, we prove the global existence of solutions of the liquid crystal flow in 2D.

We set

$$H^1_b(\mathbb{R}^2; S^2) := \left\{ u : u - b \in H^1(\mathbb{R}^2; \mathbb{R}^3), |u| = 1 \text{ a.e. in } \mathbb{R}^2 \right\}$$

for a constant vector $b \in S^2$.

Then, one of our main results in this paper is the following global existence for this flow in 2D (i.e. u is a constant along a direction in $\mathbb{R}^3$):

Theorem 1. *Let $u_0 \in H^1_b(\mathbb{R}^2; S^2)$ be a given map. Then there exists a global weak solution $u(x,t) : \mathbb{R}^2 \times [0,+\infty) \to S^2$ of (1.4) with initial value $u(0) = u_0$ such that u is smooth in $\Omega \times [0,+\infty)$ except for a finite number of singularities $\{(x^l_i, T_l)\}_{l=1}^K \in \mathbb{R}^2 \times [0,+\infty)$ with an integer $K > 0$ depending on u_0. Moreover, there are two constants $\varepsilon_0 > 0$ and $R_0 > 0$ such that each singular point x^l_i at the time T_l is characterized by the condition*

$$\limsup_{t \nearrow T_l} E\left(u(t), B_R\left(x^l_i\right)\right) \geq \varepsilon_0$$

for any $R > 0$ with $R \leq R_0$.

This result can be regarded as an extension of the well-known result of Struwe in [33] on the heat flow for harmonic maps in dimension two. Since the liquid crystal flow is not a parabolic system, the flow (1.4) is more complicated than the harmonic map flow. In particular, we cannot apply the well-known theory of partial differential equations directly to prove the local existence for the liquid crystal flow. Instead, we consider a family of Ginzburg–Landau approximation flows to prove the local existence of solutions to (1.4). To prove Theorem 1, we need to get a

L^2-estimate of $\nabla^2 u$ similarly to one in [33]. However, the flow (1.4) is not a parabolic system, so we overcome the difficulties due to the term $\nabla_\alpha[u^l u^i V_{p^l_\alpha}(u, \nabla u)]$ by using the fact that $|u| = 1$ as observed in [17].

In the second part of this paper, we investigate the Ericksen–Leslie system with the Oseen–Frank density $W(u, \nabla u)$ in (1.2). In the 1960s, Ericksen [9] and Leslie [23] established the hydrodynamic theory of liquid crystals independently. The Ericksen–Leslie theory describes the dynamic flow of liquid crystals, including the velocity vector v and direction vector u of the fluid. Let $v = (v^1, v^2, v^3)$ be the velocity vector of the fluid and $u = (u^1, u^2, u^3)$ the unit direction vector. The Ericksen–Leslie system in $\Omega \times [0, \infty)$ is given by (e.g. [24,27])

$$v^i_t + (v \cdot \nabla)v^i - \nu \triangle v^i + \nabla_{x_i} P = -\lambda \nabla_{x_j}(\nabla_{x_i} u^k W_{p^k_j}(u, \nabla u)), \tag{1.5}$$

$$\nabla \cdot v = 0, \tag{1.6}$$

$$\begin{aligned} u^i_t + (v \cdot \nabla)u^i \ = \ & \nabla_\alpha \left[W_{p^i_\alpha}(u, \nabla u) - u^k u^i V_{p^k_\alpha}(u, \nabla u) \right] - W_{u^i}(u, \nabla u) \\ & + W_{u^k}(u, \nabla u) u^k u^i + W_{p^l_\alpha}(u, \nabla u) \nabla_\alpha u^l u^i + V_{p^k_\alpha}(u, \nabla u) u^k \nabla_\alpha u^i \end{aligned} \tag{1.7}$$

for $i = 1, 2, 3$, prescribing the boundary condition

$$v(x, t) = 0, \qquad u(x, t) = u_0(x), \quad \forall (x, t) \in \partial\Omega \times (0, \infty) \tag{1.8}$$

and with initial data

$$v(x, 0) = v_0(x), \qquad u(x, 0) = u_0(x), \qquad \operatorname{div} v_0 = 0 \quad \forall x \in \Omega. \tag{1.9}$$

Here ν, λ are given positive constants, and P is the pressure.

The system (1.5)–(1.7) is a system of the Navier–Stokes equations coupled with the liquid crystal flow (1.4). The study of the Navier–Stokes equations is of great interest. Tremendous results on the existence and the partial regularity for the Navier–Stokes equations have been established (e.g. [31,5,25,35]). In this paper, we are only concentrating on the existence of solutions of the Ericksen–Leslie system. Since the functional $E(u; \Omega)$ in (1.1) with the constraint $|u| = 1$ is complicated, one considers Ginzburg–Landau functionals

$$E_\varepsilon(u; \Omega) = \int_\Omega \left[W(u, \nabla u) + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \right] dx$$

for any function $u \in H^1(\Omega; \mathbb{R}^3)$. Then, the approximating Ericksen–Leslie system is given by

$$v^i_t + (v \cdot \nabla)v^i - \nu \triangle v^i + \nabla_{x_i} P = -\lambda \nabla_{x_j}(\nabla_{x_i} u^k W_{p^k_j}(u, \nabla u)), \tag{1.10}$$

$$\nabla \cdot v = 0, \tag{1.11}$$

$$u^i_t + (v \cdot \nabla)u^i = \nabla_\alpha \left[W_{p^i_\alpha}(u, \nabla u) \right] - W_{u^i}(u, \nabla u) + \frac{1}{\varepsilon^2} u^i (1 - |u|^2) \tag{1.12}$$

for $i = 1, 2, 3$, prescribing the boundary condition (1.8) and initial condition (1.9).

In the case of $k_1 = k_2 = k_3$, Lin and Liu [27] proved the global existence of the classical solution of (1.10)–(1.12) with (1.8) and (1.9) in dimension two and the weak solution of the same system in dimension three. Lin and Liu in [28] also analyzed the limit of solutions $(v_\varepsilon, u_\varepsilon)$ of (1.10)–(1.12) as $\varepsilon \to 0$, but it is not clear that the limiting solution satisfies the original Ericksen–Leslie system (1.5)–(1.7) with $|u| = 1$. Therefore, there is an interesting question to establish the global existence of solutions of (1.5)–(1.7) with (1.8) and (1.9). The question for the case of $k_1 = k_2 = k_3$ has been answered by the first author in [19] in $\mathbb{R}^2$ and Lin et al. [26] in a general case for a domain of $\mathbb{R}^2$ independently. The system (1.5)–(1.7) or (1.10)–(1.12) for the general Oseen–Frank model is more complicated than the system for the case of $k_1 = k_2 = k_3$ since there is no maximum principle for the parabolic system (1.12) in the case $k_1 \neq k_2$ (see [2]) and the term $W_{u^i}(u, \nabla u)$ in (1.12) will cause a trouble to prove the global existence for the system.

In this paper, we will prove the global existence of weak solutions to the Ericksen–Leslie system (1.5)–(1.7) for a general Oseen–Frank model in $\mathbb{R}^2$. More precisely, we have the following.

Theorem 2. *Let $(u_0, v_0) \in H_b^1(\mathbb{R}^2; S^2) \times L^2(\mathbb{R}^2, \mathbb{R}^2)$ be given initial data with dvi $v_0 = 0$. Then, there exists a global weak solution $(u, v) : \mathbb{R}^2 \times [0, +\infty) \to S^2 \times \mathbb{R}^2$ of (1.5)–(1.7) with initial values (1.9), where the solution (u, v) is smooth in $\mathbb{R}^2 \times ((0, +\infty) \setminus \{T_l\}_{l=1}^L)$ for a finite number of times $\{T_l\}_{l=1}^L$. Moreover, there are two constants $\varepsilon_0 > 0$ and $R_0 > 0$ such that each singular point $(x_i^l, T_l) \in \Sigma \times \{T_l\}$ is characterized by the condition*

$$\limsup_{t \nearrow T_l} \int_{B_R(x_i^l)} |\nabla u(\cdot, t)|^2 + |v(\cdot, t)|^2 \, dx \geq \varepsilon_0$$

for any $R > 0$ with $R \leq R_0$.

The main idea to prove Theorem 2 is to combine the idea in [19] with the proofs of Theorem 1. The first key step is to prove the local existence of solutions of the system (1.5)–(1.7) by considering the approximation system (1.10)–(1.12). To prove the global existence of solutions to (1.5)–(1.7), one of the key steps is to get a L^2-estimate of $\nabla^2 u$ and ∇v in $\mathbb{R}^2 \times [0, T]$ under a small energy condition as in [33]. To show the regularity of the weak solution (u, v) of (1.5)–(1.7) in $\mathbb{R}^2 \times (0, T)$, we establish a local energy inequality under the small energy condition, which was first used by Struwe in [34] for the H-system flow. Finally, we prove the regularity of solutions by controlling L^2-estimate of $\nabla^2 u$ and ∇v in $\mathbb{R}^2$ for $t \in (0, T)$. Since (1.7) is not a parabolic system, the proof of Theorem 2 is more difficult than one for the case of $k_1 = k_2 = k_3$ in [19]. We overcome a number of difficulties on the regularity and the uniqueness for the systems by employing the invariance of the density (1.2) after a rotation.

Remark 1.1. The referee kindly pointed out to us that Lin and Wang [30] proved recently the uniqueness of the weak solutions in Theorem 2 for the case of $k_1 = k_2 = k_3$. However, it does not seem easy for us to adapt their analysis to prove the uniqueness of the weak solutions in Theorems 1 and 2, due to the difficulty in handling the term $\nabla_\alpha[W_{p_\alpha^i}(u, \nabla u) - u^l u^i V_{p_\alpha^l}(u, \nabla u)]$ in Eq. (1.4) or (1.7). We hope to investigate this issue elsewhere.

The rest of the paper is organized as follows. In Section 2, we prove the global existence for the liquid crystal flow (1.4) in 2D. Some global estimates for (1.5)–(1.7) are established in Section 3. Then, we complete a proof of Theorem 2 in Section 4. Finally, the regularity issue for the systems is dealt with in Section 5.

2. Existence of partial regular solutions of the liquid crystal flow

First, it is noted that due to (1.2), the density function $W(z, p)$ satisfies

$$a|p|^2 \leq W(z, p) \leq C|p|^2, \quad \forall z \in \mathbb{R}^3, \; p \in \mathbb{M}^{3\times 3}$$

for some $0 < a \le C < +\infty$ and since $W(z,p)$ is quadratic and convex in p, it holds that

$$\nabla^2_{p^k_i p^l_j} W(z,p)\xi^k_i \xi^l_j \ge a|\xi|^2$$

for any $\xi \in \mathbb{M}^{3\times 3}$, any $z \in \mathbb{R}^3$ and $p \in \mathbb{M}^{3\times 3}$.

In this section, we consider the flow (1.4) in $\mathbb{R}^2$. For simplicity of notations, u is assumed to be a constant along x_3-direction in $\mathbb{R}^3$; i.e. $\frac{\partial u}{\partial x_3} = 0$.

For any two positive constants τ and T with $\tau < T$, we define

$$V(\tau,T) := \Big\{ u : \mathbb{R}^2 \times [\tau,T] \to S^2, \mid u \text{ is measurable and satisfies}$$
$$\operatorname*{ess\,sup}_{\tau\le t\le T} \int_{\mathbb{R}^2} |\nabla u(\cdot,t)|^2\,dx + \int_\tau^T \int_{\mathbb{R}^2} |\nabla^2 u|^2 + |\partial_t u|^2\,dx\,dt < \infty \Big\}.$$

Lemma 2.1. *Let $u \in V(0,T)$ be a solution of the system (1.4) with initial value $u_0 \in H^1(\mathbb{R}^2, S^2)$. Then, for any $t_1 \in [0,T]$*

$$\int_{\mathbb{R}^2\times(0,t_1)} |\partial_t u|^2\,dx\,dt + E(u(t_1)) \le E(u_0). \tag{2.1}$$

Moreover, for all $t \in [0,T]$, $x_0 \in \mathbb{R}^2$ and $R > 0$, it holds that

$$\int_{B_R(x_0)} W(u(x,t),\nabla u(x,t))\,dx \le \int_{B_{2R}(x_0)} W(u_0(x),\nabla u_0(x))\,dx + C\frac{t}{R^2}\int_{\mathbb{R}^2} |\nabla u_0|^2\,dx, \tag{2.2}$$

where C is a constant.

Proof. Multiplying (1.4) by $\frac{\partial u^i}{\partial t}$ yields

$$\int_{\mathbb{R}^2} \left|\frac{\partial u}{\partial t}\right|^2 dx = -\int_{\mathbb{R}^2} W_{p^i_\alpha}(u,\nabla u)\frac{d}{dt}\nabla_\alpha u^i\,dx - \int_{\mathbb{R}^2} W_{u^i}(u,\nabla u)\frac{\partial u^i}{\partial t}\,dx.$$

This implies

$$\int_{\mathbb{R}^2} \left|\frac{\partial u}{\partial t}\right|^2 dx + \frac{d}{dt}\int_{\mathbb{R}^2} W(u,\nabla u)\,dx = 0.$$

(2.1) follows from integrating the above identity.

Let $\phi \in C_0^\infty(B_{2R}(x_0))$ be a cut-off function satisfying $0 \le \phi \le 1$, $|\nabla\phi| \le C/R$ and $\phi \equiv 1$ on $B_R(x_0)$. Multiplying (1.4) by $\frac{\partial u^i}{\partial t}\phi^2$ and then using Young's inequality yields

$$\int_{\mathbb{R}^2} \left|\frac{\partial u}{\partial t}\right|^2 \phi^2\,dx + \frac{d}{dt}\int_{\mathbb{R}^2} W(u(x,t),\nabla u(x,t))\phi^2\,dx \le C\int_{\mathbb{R}^2} \left|\frac{\partial u}{\partial t}\right| |\nabla u||\phi\nabla\phi|\,dx$$
$$\le \frac{1}{2}\int_{\mathbb{R}^2} \left|\frac{\partial u}{\partial t}\right|^2 \phi^2\,dx + C\int_{\mathbb{R}^2} W(u,\nabla u)|\nabla\phi|^2\,dx.$$

Then, (2.2) follows from using (2.1) and integrating the above inequality. □

From [33] we obtain the following.

Lemma 2.2. *There are constants C and R_0 such that for any $u \in V(0,T)$ and any $R \in (0, R_0]$, we have*

$$\int_{\mathbb{R}^2\times[0,T]} |\nabla u|^4\,dx\,dt \le C \operatorname*{ess\,sup}_{0\le t\le T, x\in\mathbb{R}^2} \int_{B_R(x)} |\nabla u(\cdot,t)|^2\,dx \cdot \left(\int_{\mathbb{R}^2\times[0,T]} |\nabla^2 u|^2\,dx\,dt + R^{-2}\int_{\mathbb{R}^2\times[0,T]} |\nabla u|^2\,dx\,dt\right).$$

Lemma 2.3. *Let $u \in V(0,T)$ be a solution of (1.4) with initial value $u_0 \in H^1$. Then there are constants ε_1 and $R_0 > 0$ such that if*

$$\operatorname*{ess\,sup}_{0\le t\le T, x\in\mathbb{R}^2} \int_{B_R(x)} |\nabla u(\cdot,t)|^2\,dx < \varepsilon_1$$

for any $R \in (0, R_0]$, then

$$\int_{\mathbb{R}^2\times[0,T]} |\nabla^2 u|^2\,dx\,dt \le CE(u_0)\,(1+TR^{-2}), \tag{2.3}$$

$$\int_{\mathbb{R}^2\times[0,T]} |\nabla u|^4\,dx\,dt \le C\varepsilon_1 E(u_0)\,(1+TR^{-2}). \tag{2.4}$$

Proof. Multiplying (1.4) by $\triangle u^i$ yields

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{\partial u^i}{\partial t} \triangle u^i\,dx &= \int_{\mathbb{R}^2} \nabla_\alpha \Big[W_{p^i_\alpha}(u,\nabla u) - u^k u^i V_{p^k_\alpha}(u,\nabla u)\Big] \triangle u^i\,dx \\ &\quad - \int_{\mathbb{R}^2} W_{u^i}(u,\nabla u)(\triangle u^i - u^k u^i \triangle u^k)\,dx \\ &\quad + \int_{\mathbb{R}^2} W_{p^k_\alpha}(u,\nabla u)\nabla_\alpha u^k u^i \triangle u^i\,dx \\ &\quad + \int_{\mathbb{R}^2} V_{p^k_\alpha}(u,\nabla u) u^k \nabla_\alpha u^i \triangle u^i\,dx := I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Note that the terms I_2 and I_3 of the above identity can be controlled by $C|\nabla u|^2|\triangle u|$. It suffices to estimate terms I_1 and I_4. Since $|u|^2 = 1$, $-u^i \triangle u^i = |\nabla u|^2$. We note

$$\nabla_\alpha[u^k u^i V_{p^k_\alpha}(u,\nabla u)] = \nabla_\alpha u^k u^i V_{p^k_\alpha}(u,\nabla u) + u^k \nabla_\alpha u^i V_{p^k_\alpha}(u,\nabla u) + u^k u^i \nabla_\alpha V_{p^k_\alpha}(u,\nabla u).$$

Integration by parts twice yields

$$\begin{aligned} I_1 + I_4 &= \int_{\mathbb{R}^2} \nabla_\beta \Big[W_{p^i_\alpha}(u,\nabla u)\Big] \nabla^2_{\alpha\beta} u^i\,dx + \int_{\mathbb{R}^2} \nabla_\alpha u^k u^i V_{p^k_\alpha}(u,\nabla u) \triangle u^i\,dx \\ &\quad - \int_{\mathbb{R}^2} u^k \nabla_\alpha V_{p^k_\alpha}(u,\nabla u)|\nabla u|^2\,dx.\end{aligned}$$

Note

$$\nabla_\alpha V_{p^k_\alpha}(u,\nabla u) = V_{p^k_\alpha p^l_\gamma}(u,\nabla u)\nabla^2_{\gamma\alpha} u^l + V_{p^k_\alpha u^l}(u,\nabla u)\nabla_\alpha u^l$$

and

$$\nabla_\beta W_{p^i_\alpha}(u,\nabla u) = W_{p^i_\alpha p^j_\gamma}(u,\nabla u)\nabla^2_{\gamma\beta}u^j + W_{p^i_\alpha u^j}(u,\nabla u)\nabla_\beta u^j.$$

This implies

$$\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla u|^2\,dx + \int_{\mathbb{R}^2} W_{p^i_\alpha p^j_\gamma}(u,\nabla u)\nabla^2_{\alpha\beta}u^i\nabla^2_{\gamma\beta}u^j\,dx \le C\int_{\mathbb{R}^2}|\nabla u|^2(|\nabla u|^2+|\nabla^2 u|)\,dx. \tag{2.5}$$

As pointed out at the beginning of this section, we have

$$W_{p^i_\alpha p^j_\gamma}(u,\nabla u)\nabla^2_{\alpha\beta}u^i\nabla^2_{\gamma\beta}u^j \ge a|\nabla^2 u|^2$$

for the constant $a > 0$. Then, choosing $\varepsilon_1 > 0$ to be sufficiently small and applying Lemma 2.2 lead to (2.3) and (2.4). □

Lemma 2.4. *Let $u \in V(0,T)$ be a solution of (1.4) with initial value $u_0 \in H^1$. Assume that*

$$\operatorname*{ess\,sup}_{0\le t\le T, x\in\mathbb{R}^2}\int_{B_R(x)}|\nabla u(\cdot,t)|^2\,dx < \varepsilon_1$$

for any $R \in (0, R_0]$. Let $\tau \in (0,T]$ be any constant. Then it holds for all $t \in [\tau, T]$,

$$\int_{\mathbb{R}^2}|\nabla^2 u(x,t)|^2\,dx \le C_0, \tag{2.6}$$

with a uniform constant depending only on τ, T, R_0, and $E(u_0)$.

Proof. The proof is similar to Lemma 3.10 of [33]. Using a proper cut-off function if necessary, we assume in the following proof that $\int|\partial_t\nabla u|^2(\cdot,t),dx$ is finite.

Differentiate (1.4) with respect to t, multiply the resulting identity by $\partial_t u^i$, and then integrate to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\partial_t u|^2\,dx + \int_{\mathbb{R}^2}W_{p^i_\alpha p^j_\beta}(u,\nabla u)\nabla_\alpha\,\partial_t\,u^i\,\nabla_\beta\,\partial_t\,u^j\,dx \le C\int_{\mathbb{R}^2}(|\partial_t\,u|^2\,|\nabla u|^2 + |\nabla u|\,|\partial_t\,u|\,|\nabla\partial_t\,u|]\,dx + \int_{\mathbb{R}^2}\partial_t\,u^i\,\nabla_\alpha\left[u^i\,u^k\,V_{p^k_\alpha}(u,\nabla\partial_t\,u)\right]dx.$$

Note that

$$\int_{\mathbb{R}^2}W_{p^i_\alpha p^j_\beta}(u,\nabla u)\nabla_\alpha\,\partial_t\,u^i\,\nabla_\beta\,\partial_t\,u^j\,dx \ge a\int_{\mathbb{R}^2}|\nabla\partial_t\,u(x,t)|^2\,dx.$$

Since $|u| = 1$, so $\sum_i\ \partial_t\,u^i\,u^i = 0$. And hence,

$$\int_{\mathbb{R}^2}\partial_t u^i\nabla_\alpha\left(u^i\,u^k\,V_{p^k_\alpha}(u,\nabla\partial_t\,u)\right)dx \le C\int_{\mathbb{R}^2}|\partial_t\,u|\,|\nabla u|\,|\nabla\partial_t\,u|\,dx.$$

It follows from these and Cauchy's inequality that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\partial_t\,u(x,t)|^2\,dx + \frac{a}{2}\int_{\mathbb{R}^2}|\nabla\partial_t\,u(x,t)|^2\,dx \le C\int_{\mathbb{R}^2}|\partial_t\,u|^2\,|\nabla u|^2\,dx. \tag{2.7}$$

Note that

$$\begin{aligned} C\int_{\mathbb{R}^2} |\partial_t u|^2\,|\nabla u|^2\,dx &\le C\left(\int_{\mathbb{R}^2} |\partial_t u|^4\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^2} |\nabla u|^4\,dx\right)^{\frac12} \\ &\le C\left(\int_{\mathbb{R}^2} |\partial_t u|^2\,(x,t)\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^2} |\partial_t \nabla u|^2\,(x,t)\right)^{\frac12}\left(\int_{\mathbb{R}^2} |\nabla u|^4\,dx\right)^{\frac12} \\ &\le \frac{a}{4}\int_{\mathbb{R}^2} |\nabla\partial_t u(x,t)|^2\,dx + \left(C\int_{\mathbb{R}^2} |\nabla u(x,t)|^4\,dx\right)\int_{\mathbb{R}^2} |\partial_t u(x,t)|^2\,dx. \end{aligned}$$

This, together with (2.7), yields that for all $t \in (0, T]$,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^2} |\partial_t u(\cdot,t)|^2\,dx + \frac{a}{2}\int_{\mathbb{R}^2} |\nabla\partial_t u(\cdot,t)|^2\,dx \\ &\quad\le \left(C\int_{\mathbb{R}^2} |\nabla u(\cdot,t)|^4\,dx\right)\int_{\mathbb{R}^2} |\partial_t u(\cdot,t)|^2\,dt. \end{aligned} \tag{2.8}$$

It follows from (2.8), (2.4), Lemma 2.1, and Gronwall's inequality that for any $0 < s \le t \le T$,

$$\begin{aligned} \int_{\mathbb{R}^2} |\partial_t u(\cdot,t)|^2\,dt &\le \left(e^{C\int_s^t\int_{\mathbb{R}^2} |\nabla u(\cdot,l)|^4\,dx\,dl}\right)\int_{\mathbb{R}^2} |\partial_t u(\cdot,s)|^2\,dx \\ &\le e^{C\varepsilon_1 E(u_0)(1+TR^{-2})}\cdot\int_{\mathbb{R}^2} |\partial_t u(\cdot,s)|^2\,dx. \end{aligned}$$

Combining this with (2.1) shows that for any fixed $0 < \tau < T$, there exists a constant C such that

$$\operatorname*{ess\,sup}_{\tau\le t\le T}\int_{\mathbb{R}^2} |\partial_t u(\cdot,t)|^2\,dt \le C\tau^{-1}\,E(u_0)\,e^{C\varepsilon_1 E(u_0)(1+TR^{-2})}, \tag{2.9}$$

with a uniform constant C. On the other hand, using (2.5), integration by parts yields that for any $t \in [\tau, T]$,

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla^2 u(\cdot,t)|^2\,dx &\le C\int_{\mathbb{R}^2} |\nabla u(\cdot,t)|^4\,dx + C\int_{\mathbb{R}^2} |\partial_t u(\cdot,t)|^2\,dx \\ &\le C\varepsilon_1\int_{\mathbb{R}^2} |\nabla^2 u(\cdot,t)|^2\,dx + \frac{C\varepsilon_1}{R_0^2}E(u_0) + C\int_{\mathbb{R}^2} |\partial_t u(\cdot,t)|^2\,dx. \end{aligned}$$

Combining this with (2.9) shows that for suitably small ε_1, the desired estimate (2.6) holds with

$$C_0 \equiv CE(u_0)\left(\frac{\varepsilon_1}{R^2} + \tau^{-1}\,e^{C\varepsilon_1 E(u_0)(1+TR^{-2})}\right). \tag{2.10}$$

By the well-known Gagliardo–Nirenberg–Sobolev inequality, we have for any $x \in \mathbb{R}^2$

$$\begin{aligned} |u(x,t_1) - u(x,t_2)| &\le C\|u(x,t_1) - u(x,t_2)\|_{H^2(B_1(x))}^{3/4}\,\|u(x,t_1) - u(x,t_2)\|_{L^2(B_1(x))}^{1/4} \\ &\le C\left(\sup_{\tau\le t\le T}\|\nabla^2 u(\cdot,t)\|_{L^2(\mathbb{R}^2)}^{3/4} + 1\right)|t_1 - t_2|^{1/8} \\ &\quad\times\left(\int_0^T\int_{\mathbb{R}^2} |\partial_t u|^2\,dx\,dt\right)^{1/8} \\ &\le C|t_1 - t_2|^{1/8}. \end{aligned}$$

It follows from (2.6) and the Sobolev embedding theorem that $u(x,t)$ is Hölder continuous in x uniformly for $t \in [\tau, T]$. Then we get that u is Hölder continuous in $C^{1/8}(\mathbb{R}^2 \times [\tau, T])$ for any $T < T_1$. Due to Proposition 5.1 in the Appendix, u is in $C^{1,\frac{1}{8}}$. Hence, u is regular in $(0, T_1)$. □

Remark 2.1. Let $u \in V(0,T)$ be a solution of (1.4) with initial value $u_0 \in H_b^2$. Assume that there are constants ε_1 and $R_0 > 0$ such that

$$\operatorname*{ess\,sup}_{0\le t\le T, x\in\mathbb{R}^2} \int_{B_R(x)} |\nabla u(\cdot,t)|^2\,dx < \varepsilon_1$$

for any $R \in (0, R_0]$. Then, for any $t \in [0,T]$ and $R \le R_0$, we have

$$\int_{\mathbb{R}^2} |\nabla^2 u(x,t)|^2\,dx \le C_1 \equiv C_1(\|u_0\|_{H_b^2}, C_0).$$

Theorem 3 (*Local Existence*). *For a map $u_0 \in H_b^1(\mathbb{R}^2, S^2)$, there is a solution $u \in V(0,t_1)$ of (1.4) with initial value u_0 for some $t_1 > 0$.*

Proof. For any map $u_0 \in H_b^1(\mathbb{R}^2, S^2)$, it can be approximated by a sequence of smooth maps in $H_b^2(\mathbb{R}^2, S^2)$. Without loss of generality, we assume that $u_0 \in H_b^2(\mathbb{R}^2, S^2)$ is smooth. The liquid crystal flow is not a parabolic system, so one cannot apply the well-known local existence theory. Instead, we prove the local existence by an approximation of the Ginzburg–Landau flow in the following:

$$\frac{\partial u_\varepsilon^i}{\partial t} = \nabla_\alpha \left[W_{p_\alpha^i}(u_\varepsilon, \nabla u_\varepsilon)\right] - W_{u^i}(u_\varepsilon, \nabla u_\varepsilon) + \frac{1}{\varepsilon^2} u_\varepsilon^i (1 - |u_\varepsilon|^2) \tag{2.11}$$

with initial value $u_0 \in H_b^2(\mathbb{R}^2, S^2)$ and $u_0 \in C^\infty$. Applying the standard local existence theory of quasi-linear parabolic systems (cf. [8] or [2]), there is a local regular solution u_ε of (2.11) with initial value $u_\varepsilon(0) = u_0$.

For simplicity of notations, we define

$$\tilde V(\tau, T) = \Big\{ u : \mathbb{R}^2 \times [\tau, T] \to \mathbb{R}^3 | u \text{ is measurable and satisfies}$$
$$\operatorname*{ess\,sup}_{\tau\le t\le T} \int_{\mathbb{R}^2} |\nabla u(\cdot,t)|^2\,dx + \int_\tau^T \int_{\mathbb{R}^2} (|\nabla^2 u|^2 + |\partial_t u|^2)\,dx\,dt < \infty \Big\}$$

for $0 \le \tau < T < +\infty$ and set

$$e_\varepsilon(u) = W(u, \nabla u) + \frac{1}{2\varepsilon^2}(1-|u|^2)^2, \qquad E_\varepsilon(u) = \int_{\mathbb{R}^2} e_\varepsilon(u)\,dx.$$

Taking the inner product of (2.11) with $\partial_t u_\varepsilon$, one can obtain that for any $s > 0$ in the maximal interval of existence,

$$\int_{\mathbb{R}^2\times(0,s)} |\partial_t u_\varepsilon|^2\,dx\,dt + E_\varepsilon(u_\varepsilon(s)) \le E(u_0). \tag{2.12}$$

Moreover, repeating similar arguments in Lemma 2.3 (see (2.29) below) yields that the solution u_ε belongs to $\tilde V(0, T_\varepsilon)$ for a maximum time T_ε and hence is regular $\mathbb{R}^2 \times [0, T_\varepsilon)$. The maximum time T_ε is characterized in the following: For a singular point x_0 at T_ε, there are ε_0 and $R_0 > 0$ such that

$$\limsup_{t\to T_\varepsilon} \int_{B_R(x_0)} |\nabla u_\varepsilon(\cdot, t)|^2\, dx \geq \varepsilon_0 > 0$$

for any positive $R \leq R_0$.

Next, we will show that there is a uniform lower bound time $t_1 > 0$ such that $T_\varepsilon \geq t_1$ and u_ε is bounded in $\tilde{V}(0, t_1)$ uniformly in ε.

A similar argument as in Lemma 2.1 shows

$$\int_{B_R(x_0)} e_\varepsilon(u_\varepsilon(x, t))\, dx \leq \int_{B_{2R}(x_0)} e_\varepsilon(u_0(x))\, dx + C\frac{t}{R^2}\int_{\mathbb{R}^2} |\nabla u_0|^2\, dx$$

for $t \leq T_\varepsilon$.

It follows from this inequality that for suitably small ε_1 and R_0, there is a time t_1 uniform in ε with $t_1 \leq T_\varepsilon$ such that

$$\sup_{0\leq t\leq t_1} \int_{B_R(x_0)} e_\varepsilon(u_\varepsilon(x, t))\, dx < \varepsilon_1 \tag{2.13}$$

for $R \leq R_0$ and thus u_ε is smooth for $[0, t_1]$ for all $\varepsilon > 0$. Next, we claim that for $0 \leq t \leq t_1$

$$\frac{1}{2} \leq |u_\varepsilon(x, t)| \leq \frac{3}{2} \quad \text{for all } x \in \mathbb{R}^2. \tag{2.14}$$

To verify this claim, we re-scale the solution by $\tilde{u}(x, t) = u_\varepsilon(\varepsilon x, \varepsilon^2 t)$. Then $\tilde{u}$ satisfies

$$\frac{\partial \tilde{u}^i}{\partial t} = \nabla_\alpha \left[W_{p^i_\alpha}(\tilde{u}, \nabla\tilde{u})\right] - W_{u^i}(\tilde{u}, \nabla\tilde{u}) + \tilde{u}^i(1 - |\tilde{u}|^2) \tag{2.15}$$

with initial value $u_0(\varepsilon x)$. Let τ be the maximal time in $\left[0, \frac{t_1}{\varepsilon^2}\right]$ such that (2.14) holds, i.e.,

$$\frac{1}{2} \leq |\tilde{u}(x, t)| \leq \frac{3}{2}$$

for any $(x, t) \in \mathbb{R}^2 \times [0, \tau]$. Note that in this case, the basic energy inequality (2.12) becomes

$$\int_{\mathbb{R}^2\times[0,s]} |\partial_t \tilde{u}|^2\, dx\, dt + \int_{\mathbb{R}^2} \left(W(\tilde{u}, \nabla\tilde{u})(s) + \frac{1}{2}(1 - |\tilde{u}(s)|^2)^2\right) dx \leq E(u_0)$$
$$\text{for all } s \in \left[0, \frac{t_1}{\varepsilon^2}\right], \tag{2.16}$$

and the condition (2.13) turns into

$$\operatorname*{ess\,sup}_{0\leq s\leq \frac{t_1}{\varepsilon^2}, x\in\mathbb{R}^2} \int_{B_{\frac{R}{\varepsilon}}(x)} \left(|\nabla\tilde{u}(\cdot, s)|^2 + \frac{1}{2}(1 - |\tilde{u}|^2)^2\right) dx < \varepsilon_1 \tag{2.17}$$

for $R \leq R_0$.

Multiplying (2.15) by $\triangle\tilde{u}$ and integrating over $\mathbb{R}^2$ lead to

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla\tilde{u}|^2\, dx + \int_{\mathbb{R}^2} \nabla_\alpha \left[W_{p^i_\alpha}(\tilde{u}, \nabla\tilde{u})\right] \triangle\, \tilde{u}^i\, dx$$
$$- \int_{\mathbb{R}^2} W_{\tilde{u}^i}(\tilde{u}, \nabla\tilde{u}) \triangle\, \tilde{u}^i\, dx + \int_{\mathbb{R}^2} \tilde{u}^i(1 - |\tilde{u}|^2) \triangle\, \tilde{u}^i\, dx = 0. \tag{2.18}$$

Note that

$$\int_{\mathbb{R}^2} \tilde{u}^i (1-|\tilde{u}|^2) \triangle \tilde{u}^i \, dx = -\int_{\mathbb{R}^2} |\nabla \tilde{u}|^2 (1-|\tilde{u}|^2)\, dx + 2\int_{\mathbb{R}^2} |\nabla |\tilde{u}|^2|^2 \, dx.$$

Then, combining the above identity with (2.18) yields that for any $s, t \in [0, \tau]$ with $s \le t$,

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla \tilde{u}(\cdot, t)|^2 \, dx + \int_s^t \int_{\mathbb{R}^2} W_{p^i_\alpha p^j_\gamma}(\tilde{u}, \nabla \tilde{u}) \nabla^2_{\alpha\beta} \tilde{u}^i \nabla^2_{\gamma\beta} \tilde{u}^j \, dx\, dt$$
$$\le C\int_{\mathbb{R}^2} |\nabla u(\cdot, s)|^2 \, dx + C\int_s^t \int_{\mathbb{R}^2} |\nabla \tilde{u}|^4 + \eta[(1-|\tilde{u}|^2)^2 + |\nabla^2 \tilde{u}|^2]\, dx\, dt \tag{2.19}$$

for a sufficiently small $\eta > 0$ to be chosen.

On the other hand, it follows from (2.15) that for $s, t \in [0, \tau]$

$$\int_{\mathbb{R}^2 \times [s,t]} (1-|\tilde{u}|^2)^2 \, dx\, dt \le C\int_{\mathbb{R}^2 \times [s,t]} (|\partial_t \tilde{u}|^2 + |\nabla \tilde{u}|^4 + |\nabla^2 \tilde{u}|^2) dx\, dt. \tag{2.20}$$

Combining Lemma 2.2 with (2.17) shows that

$$\int_{\mathbb{R}^2 \times [s,t]} |\nabla \tilde{u}|^4 \, dx\, dt \le C_1 \varepsilon_1 \left(\int_{\mathbb{R}^2 \times [s,t]} |\nabla^2 \tilde{u}|^2 \, dx\, dt + \frac{\varepsilon^2}{R_0^2} \int_{\mathbb{R}^2 \times [s,t]} |\nabla \tilde{u}|^2 \, dx\, dt \right). \tag{2.21}$$

As a consequence of (2.19)–(2.21), (2.16), and suitable choices of η and ε_1, one can get that $\tilde{u} \in \tilde{V}_\tau$, and for any $0 \le s \le t \le \tau$,

$$\int_{\mathbb{R}^2 \times [s,t]} \left[|\nabla^2 \tilde{u}|^2 + (1-|\tilde{u}|^2)^2 \right] dx\, dt \le C E(u_0)(1+\varepsilon^2 (t-s) R_0^{-2}), \tag{2.22}$$

$$\int_{\mathbb{R}^2 \times [s,t]} |\nabla \tilde{u}(x,t)|^4 \, dx\, dt \le C\varepsilon_1 E(u_0)(1+\varepsilon^2 (t-s) R_0^{-2}). \tag{2.23}$$

By an argument similar to the proof of Lemma 2.4, one can derive from (2.15) that there exists a positive uniform constant a such that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\partial_t \tilde{u}(x,t)|^2 \, dx + a\int_{\mathbb{R}^2} |\partial_t \nabla \tilde{u}(x,t)|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^2} |\partial_t (|\tilde{u}(x,t)|^2)|^2 \, dx$$
$$\le C\int_{\mathbb{R}^2} |\partial_t \tilde{u}(x,t)|^2 \, |\nabla \tilde{u}(x,t)|^2 dx + \int_{\mathbb{R}^2} |\partial_t \tilde{u}(x,t)|^2 \, (1-|\tilde{u}(x,t)|^2) dx. \tag{2.24}$$

Note that

$$C\int_{\mathbb{R}^2} |\partial_t \tilde{u}(x,t)|^2 \, |\nabla \tilde{u}(x,t)|^2 dx$$
$$\le C\left(\int_{\mathbb{R}^2} |\partial_t \tilde{u}(x,t)|^4 \, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\nabla \tilde{u}(x,t)|^4 \, dx\right)^{\frac{1}{2}}$$
$$\le C\left(\int_{\mathbb{R}^2} |\partial_t \tilde{u}(x,t)|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\partial_t \nabla \tilde{u}(x,t)|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\nabla \tilde{u}(x,t)|^4 \, dx\right)^{\frac{1}{2}}$$
$$\le \frac{a}{2}\int_{\mathbb{R}^2} |\partial_t \nabla \tilde{u}(x,t)|^2 \, dx + C\left(\int_{\mathbb{R}^2} |\nabla \tilde{u}(x,t)|^4 \, dx\right) \int_{\mathbb{R}^2} |\partial_t \tilde{u}(x,t)|^2 \, dx.$$

Hence,

$$\frac{d}{dt}\int_{\mathbb{R}^2}|\partial_t\,\tilde u(x,t)|^2\,dx + a\int_{\mathbb{R}^2}|\partial_t\,\nabla\tilde u(x,t)|^2\,dx + \int_{\mathbb{R}^2}|\partial_t(|\tilde u(x,t)|^2)|^2\,dx$$
$$\le \left(C\int_{\mathbb{R}^2}|\nabla\tilde u(x,t)|^4\,dx\right)\int_{\mathbb{R}^2}|\partial_t\,\tilde u(x,t)|^2\,dx$$
$$+2\int_{\mathbb{R}^2}|\partial_t\,\tilde u(x,t)|^2(1-|\tilde u(x,t)|^2)dx,$$

which yields immediately that for any $0\le t\le\tau\in\left(0,\frac{t_1}{\varepsilon^2}\right]$,

$$\int_{\mathbb{R}^2}|\partial_t\,\tilde u(x,t)|^2\,dx \le e^{C\int_0^t\int_{\mathbb{R}^2}|\nabla\tilde u(x,t)|^4\,dx\,dt}\int_{\mathbb{R}^2}|\partial_t\,\tilde u(x,0)|^2\,dx$$
$$+\int_0^t\left(e^{C\int_0^t\int_{\mathbb{R}^2}|\nabla\tilde u(x,l)|^4\,dx\,dl}\int_{\mathbb{R}^2}|\partial_t\,\tilde u(x,s)|^2(1-|\tilde u(x,s)|^2)dx\right)ds.$$

It follows from this, (2.23), $u_0\in H_b^2$, and the definition of τ that

$$\int_{\mathbb{R}^2}|\partial_t\,\tilde u(x,t)|^2\,dx\le C_1\equiv C_1(E(u_0),\varepsilon_1,t_1,\|u_0\|_{H_b^2},R_0) \tag{2.25}$$

with a positive constant C_1 independent of $\tau\in\left(0,\frac{t_1}{\varepsilon^2}\right)$ given by

$$C_1 = C(\|u_0\|_{H_b^2})\,e^{\varepsilon_1\,E(u_0)\left(1+\frac{t_1}{R_0^2}\right)}. \tag{2.26}$$

Using (2.18), an integration by parts implies that for all $t\in(0,\tau]$,

$$\int_{\mathbb{R}^2}|\nabla^2\tilde u(\cdot,t)|^2\,dx\le C\int_{\mathbb{R}^2}|\nabla\tilde u(x,t)|^4\,dx + C\int_{\mathbb{R}^2}(1-|\tilde u|^2)^2\,dx$$
$$+C\int_{\mathbb{R}^2}|\partial_t\,\tilde u(x,t)|^2\,dx$$

Due to Lemma 2.2, and (2.17), one has

$$C\int_{\mathbb{R}^2}|\nabla\tilde u(x,t)|^4\,dx\le\varepsilon_1\int_{\mathbb{R}^2}|\nabla^2\,\tilde u(x,t)|^2\,dx+\frac{C\varepsilon_1\,\varepsilon^2}{R_0^2}E(u_0).$$

Thus one can get that for all $t\in(0,\tau)$

$$\int_{\mathbb{R}^2}|\nabla^2\,u(x,t)|^2\,dx\le CE(u_0)\left(1+\frac{\varepsilon_1\,\varepsilon^2}{R_0^2}\right)+CC_1. \tag{2.27}$$

By the Sobolev embedding theorem, $\tilde u$ is β-Hölder continuous in x uniformly in all $t\in[0,\tau]$ with $\beta<1$. Repeating the similar analysis as in the proof of Lemma 2.4 and using Proposition 5.1 in the Appendix, we get $\tilde u\in C^{1,\frac{1}{8}}$ on $\mathbb{R}^2\times(0,\tau)$. If there is a $x_1\in\mathbb{R}^2$ such that either $|\tilde u(x_1,t)|<\frac{1}{2}$ or $|\tilde u(x_1,t)|>\frac{3}{2}$. By the uniform Hölder continuity of $\tilde u$, there exists a constant C_2 with the property that $\frac{1}{4C_2}<\frac{R_0}{\varepsilon}$, and

$$(1-|\tilde u(x,t)|^2)^2\ge\frac{1}{4},\quad x\in B_{\frac{1}{4C_2}}(x_1).$$

Hence,

$$\int_{B_{\frac{1}{4C_2}}(x_1)} (1 - |\tilde{u}(x,t)|^2)^2\, dx \geq \frac{1}{4}|B_{\frac{1}{4C_2}}(0)| > 2\varepsilon_1, \tag{2.28}$$

which contradicts to (2.17) for suitably small ε_1. Here we have used the fact that C_2 depends only on the upper bound of C_1, which may be chosen to be independent of ε_1 by the choice of t_1. This implies that $\frac{1}{2} \leq |\tilde{u}(x,t)| \leq \frac{3}{2}$ for all $t \in [0, \tau]$. By the continuity of u at τ and the maximal choice of τ, τ must be the value $\frac{t_1}{\varepsilon^2}$. This shows that (2.14) holds for all $t \in [0, t_1]$.

Next, it follows from (2.12), (2.22) and (2.23) that u_ε are uniformly bounded in $\tilde{V}(0, t_1)$ for all ε and

$$\int_0^{t_1} \int_{\mathbb{R}^2} \left[|\nabla^2 u_\varepsilon(x,t)|^2 + \frac{1}{\varepsilon^2}(1 - |u_\varepsilon(x,t)|^2)^2 \right] dx\, dt \leq C E(u_0) \left(1 + \frac{t_1}{R_0^2}\right), \tag{2.29}$$

$$\int_0^{t_1} \int_{\mathbb{R}^2} |\nabla u_\varepsilon(x,t)|^4\, dx\, dt \leq C\varepsilon_1\, E(u_0) \left(1 + \frac{t_1}{R_0^2}\right). \tag{2.30}$$

Letting $\varepsilon \to 0$, we can prove the local existence of a solution of (1.4) in $V(0, t_1)$. □

Now we complete the proof of Theorem 1.

Proof. By Theorem 3, there is a local solution u on $[0, t_1)$ for some $t_1 > 0$. By Lemmas 2.3 and 2.4, the solution can be extended to $[0, T_1)$ for a maximal time $T_1 > 0$ such that there is a singular set Σ at T_1. Each singularity $x_i^1 \in \Sigma$ at T_1 is characterized by the condition

$$\limsup_{t \nearrow T_1} E\left(u(t), B_R\left(x_i^1\right)\right) \geq \varepsilon_0$$

for any $R > 0$ with $R \leq R_0$. It is easily shown that the solution $u \in V$ is regular for all $t \in (0, T_1)$. By Lemma 2.1, we can show that the singular set Σ and the singular times are finite (See [33]). Theorem 1 is thus proved. □

Remark 2.2. Although we cannot prove the uniqueness of the weak solutions in Theorem 1, we will prove the uniqueness of smooth solutions (see Lemma 3.5 below in Section 3).

3. Global existence for the Ericksen–Leslie system

In this section, we derive a priori estimates for solutions to the Ericksen–Leslie system (1.5)–(1.7). Without loss of generality, we assume that $\nu = \lambda = 1$ in (1.5).

For the case $\Omega = \mathbb{R}^2$, we still consider (1.5)–(1.7) in $\mathbb{R}^3$ by taking $\frac{\partial v}{\partial x_3} = 0$, $\frac{\partial u}{\partial x_3} = 0$. In this case, $\nabla \cdot v = \frac{\partial v^1}{\partial x_1} + \frac{\partial v^2}{\partial x_2} = 0$ in (1.6) is well-defined.

For two positive constants τ and T with $\tau < T$, we denote

$$V(\tau, T) := \Big\{ u : \mathbb{R}^2 \times [\tau, T] \to S^2 \,|\, u \text{ is measurable and satisfies}$$

$$\operatorname*{ess\,sup}_{\tau \leq t \leq T} \int_{\mathbb{R}^2} |\nabla u(\cdot,t)|^2\, dx + \int_\tau^T \int_{\mathbb{R}^2} |\nabla^2 u|^2 + |\partial_t u|^2\, dx\, dt < \infty \Big\}$$

and

$$H(\tau,T) := \Big\{ v : \mathbb{R}^2 \times [\tau,T] \to \mathbb{R}^2 \,|\, v \text{ is measurable and satisfies}$$
$$\operatorname*{ess\,sup}_{\tau\le t\le T} \int_{\mathbb{R}^2} |v(\cdot,t)|^2\,dx + \int_\tau^T \int_{\mathbb{R}^2} |\nabla v|^2\,dx\,dt < \infty \Big\}.$$

For each pair (u,v), define

$$e(u,v) = W(u,\nabla u) + \frac{1}{2}|v|^2, \qquad E(u,v) = \int_{\mathbb{R}^2} e(u,v)dx.$$

Lemma 3.1. *Let $(u,v) \in V(0,T) \times H(0,T)$ be a solution of (1.5)–(1.7) with initial values $u_0 \in H^1(\mathbb{R}^2;S^2)$ and $v_0 \in L^2(\mathbb{R}^2;\mathbb{R}^3)$. Then for $t \in (0,T]$,*

$$\int_{\mathbb{R}^2} e(u(\cdot,t),v(\cdot,t))\,dx + \int_0^t \int_{\mathbb{R}^2} (|u_t + (v\cdot\nabla)u|^2 + |\nabla v|^2)\,dx\,dt$$
$$= \int_{\mathbb{R}^2} e(u_0,v_0)\,dx. \tag{3.1}$$

Proof. Multiplying (1.5) by v and using (1.6), one gets

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |v|^2\,dx + \int_{\mathbb{R}^2} |\nabla v|^2\,dx = \int_{\mathbb{R}^2} \nabla_j v^i \nabla_i u^k W_{p_j^k}(u,\nabla u)\,dx. \tag{3.2}$$

Multiplying (1.7) by $u_t + (v\cdot\nabla)u$ yields

$$\int_{\mathbb{R}^2} (u_t^i + (v\cdot\nabla)u^i)\Big(\nabla_\alpha\Big[W_{p_\alpha^i}(u,\nabla u) - u^k u^i V_{p_\alpha^k}(u,\nabla u)\Big]\Big)\,dx$$
$$+ \int_{\mathbb{R}^2} (u_t^i + (v\cdot\nabla)u^i)(-W_{u^i}(u,\nabla u) + W_{u^k}(u,\nabla u)u^k u^i)\,dx$$
$$+ \int_{\mathbb{R}^2} (u_t^i + (v\cdot\nabla)u^i)\big(+W_{p_\alpha^k}(u,\nabla u)\nabla_\alpha u^k u^i + V_{p_\alpha^k}(u,\nabla u)u^k\nabla_\alpha u^i\big)\,dx$$
$$= \int_{\mathbb{R}^2} |u_t + (v\cdot\nabla)u|^2\,dx. \tag{3.3}$$

Note that $|u|^2 = 1$ implies

$$u^i\partial_t u^i = 0, \qquad u^i \nabla_{x_\alpha} u^i = 0.$$

Integration by parts yields

$$\int_{\mathbb{R}^2} u_t^i\Big(\nabla_\alpha\Big[W_{p_\alpha^i}(u,\nabla u) - u^k u^i V_{p_\alpha^k}(u,\nabla u)\Big]$$
$$- W_{u^i}(u,\nabla u) + V_{p_\alpha^k}(u,\nabla u)u^k\nabla_\alpha u^i\Big)\,dx$$
$$= -\int_{\mathbb{R}^2} \nabla_\alpha u_t^i\Big[W_{p_\alpha^i}(u,\nabla u)\Big] - u_t^i W_{u^i}(u,\nabla u)\,dx = -\frac{d}{dt}\int_{\mathbb{R}^2} W(u,\nabla u)\,dx. \tag{3.4}$$

Using (1.6) and integrating by parts, we get

$$\begin{aligned}
&\int_{\mathbb{R}^2}(v\cdot\nabla)u^i\left(\nabla_\alpha\left[W_{p^i_\alpha}(u,\nabla u)-u^k u^i V_{p^k_\alpha}(u,\nabla u)\right]\right.\\
&\qquad\left.-W_{u^i}(u,\nabla u)+V_{p^k_\alpha}(u,\nabla u)u^k\nabla_\alpha u^i\right)dx\\
&\quad=-\int_{\mathbb{R}^2}\nabla_\alpha v^k\nabla_k u^i W_{p^i_\alpha}(u,\nabla u)+v^k[\nabla_k\nabla_\alpha u^i W_{p^i_\alpha}(u,\nabla u)+\nabla_k u^i W_{u^i}(u,\nabla u)]\,dx\\
&\quad=-\int_{\mathbb{R}^2}\nabla_\alpha v^k\nabla_k u^i W_{p^i_\alpha}(u,\nabla u)\,dx.
\end{aligned}\tag{3.5}$$

It follows from (3.3)–(3.5) that

$$\frac{d}{dt}\int_{\mathbb{R}^2}W(u,\nabla u)\,dx+\int_{\mathbb{R}^2}|u_t+(v\cdot\nabla)u|^2\,dx=-\int_{\mathbb{R}^2}\nabla_\alpha v^k\nabla_k u^i W_{p^i_\alpha}(u,\nabla u)\,dx.\tag{3.6}$$

Therefore, (3.1) follows from integrating (3.2) and (3.6) in t. □

By the same proof as in Lemma 3.1 of [33] (see also [20]), there exists a constant C_1 such that for any $f\in H(0,T)$ and any $R>0$, it holds that

$$\begin{aligned}
\int_{\mathbb{R}^2\times[0,T]}|f|^4\,dx\,dt\le C_1\operatorname*{ess\,sup}_{0\le t\le T,x\in\mathbb{R}^2}\int_{B_R(x)}|f(\cdot,t)|^2\,dx\\
\cdot\left(\int_{\mathbb{R}^2\times[0,T]}|\nabla f|^2\,dx\,dt+R^{-2}\int_{\mathbb{R}^2\times[0,T]}|f|^2\,dx\,dt\right).
\end{aligned}\tag{3.7}$$

Then, we have the following.

Lemma 3.2. *Let $(u,v)\in V(0,T)\times H(0,T)$ be a solution of (1.5)–(1.7) with initial values $u_0\in H^1$ and $v_0\in L^2$. Then there are constants ε_1 and $R_0>0$ such that if*

$$\operatorname*{ess\,sup}_{0\le t\le T,x\in\mathbb{R}^2}\int_{B_R(x)}e(u(\cdot,t),v(\cdot,t))\,dx<\varepsilon_1$$

for any $R\in(0,R_0]$, then

$$\int_{\mathbb{R}^2\times[0,T]}|\nabla^2u|^2+|\nabla v|^2\,dx\,dt\le C(1+TR^{-2})\int_{\mathbb{R}^2}e(u_0,v_0)\,dx,\tag{3.8}$$

$$\int_{\mathbb{R}^2\times[0,T]}(|\nabla u|^4+|v|^4)dx\,dt\le C\varepsilon_1(1+TR^{-2})\int_{\mathbb{R}^2}e(u_0,v_0)dx.\tag{3.9}$$

Proof. Multiplying $\triangle u^i$ with (1.7) yields

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left(\frac{\partial u^i}{\partial t}+(v\cdot\nabla)u^i\right)\triangle u^i\,dx\\
&\quad=\int_{\mathbb{R}^2}\nabla_\alpha\left[W_{p^i_\alpha}(u,\nabla u)-u^k u^i V_{p^k_\alpha}(u,\nabla u)\right]\triangle u^i\,dx\\
&\qquad-\int_{\mathbb{R}^2}W_{u^i}(u,\nabla u)\triangle u^i dx+\int_{\mathbb{R}^2}u^k u^i\,W_{u^k}(u,\nabla u)\triangle u^i\,dx\\
&\qquad+\int_{\mathbb{R}^2}W_{p^k_\alpha}(u,\nabla u)\nabla_\alpha u^k u^i\triangle u^i\,dx+\int_{\mathbb{R}^2}V_{p^k_\alpha}(u,\nabla u)u^k\nabla_\alpha u^i\triangle u^i\,dx.
\end{aligned}$$

As in the proof of Lemma 2.3, one can derive

$$\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla u|^2\,dx+\int_{\mathbb{R}^2}W_{p^i_\alpha p^j_\gamma}(u,\nabla u)\nabla^2_{\alpha\beta}u^i\nabla^2_{\gamma\beta}u^j\,dx$$
$$\le C\int_{\mathbb{R}^2}(|\nabla u|^2+|v|^2)(|\nabla u|^2+|\nabla^2 u|)\,dx$$
$$\le \frac{b}{4}\int_{\mathbb{R}^2}|\nabla^2 u|^2\,dx+C\int_{\mathbb{R}^2}(|\nabla u|^4+|v|^4)\,dx.$$

Applying (3.7) and Lemma 2.2 again shows

$$\int_{\mathbb{R}^2\times[0,T]}|\nabla u|^4+|v|^4\,dx\,dt\le C_1\varepsilon_1\int_{\mathbb{R}^2\times[0,T]}|\nabla^2 u|^2+|\nabla v|^2\,dx\,dt$$
$$+C_1\,\varepsilon_1\,R^{-2}\int_{\mathbb{R}^2\times[0,T]}|\nabla u|^2+|v|^2\,dx\,dt.$$

Then (3.8) and (3.9) follow by choosing $\varepsilon_1=\frac{b}{4C_1}$. □

Lemma 3.3. *Let $(u,v)\in V(0,T)\times H(0,T)$ be a solution of (1.5)–(1.7) with initial values (u_0,v_0). Assume that there exist constants $\varepsilon_1>0$ and $R_0>0$ such that*

$$\sup_{x\in\mathbb{R}^2,0\le t\le T}\int_{B_{R_0}(x)}|\nabla u(x,t)|^2+|v(\cdot,t)|^2\,dx<\varepsilon_1.$$

Then for all $t\in[0,T]$, $x_0\in\mathbb{R}$ and $R\le R_0$, it holds that

$$\int_{B_R(x_0)}e(u(\cdot,t),v(\cdot,t))\,dx+\int_0^t\int_{B_R(x_0)}\left(|\nabla v|^2+\frac{1}{2}|\partial_t u+v\cdot\nabla u|^2\right)dx\,dt$$
$$\le\int_{B_{2R}(x_0)}e(u_0,v_0)\,dx+C_2\frac{t^{\frac{1}{2}}}{R}\left(1+\frac{t}{R^2}\right)^{\frac{1}{2}}\int_{\mathbb{R}^2}e(u_0,v_0)dx, \tag{3.10}$$

where C_2 is a uniform positive constant.

Proof. Let $\phi\in C_0^\infty(B_{2R}(x_0))$ be a cut-off function with $\phi\equiv 1$ on $B_R(x_0)$ and $|\nabla\phi|\le\frac{C}{R}$, $|\nabla^2\phi|\le\frac{C}{R^2}$ for all $R\le R_0$.

Multiplying (1.5) by $v\phi^2$ and integrating over $\mathbb{R}^2$ show

$$\int_{\mathbb{R}^2}v_t\cdot v\phi^2+(v\cdot\nabla)v\cdot v\phi^2-\triangle v\cdot v\phi^2+\nabla P\cdot v\phi^2\,dx$$
$$=\int_{\mathbb{R}^2}\nabla_{x_i}u^kW_{p^k_j}(u,\nabla u)\nabla_{x_j}v^i\phi^2\,dx+\int_{\mathbb{R}^2}\nabla_{x_i}u^kW_{p^k_j}(u,\nabla u)v^i\nabla_{x_j}\phi^2\,dx.$$

Integration by parts yields

$$\int_{\mathbb{R}^2}(v_t\cdot v\phi^2-\triangle v\cdot v\phi^2)\,dx=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|v|^2\phi^2\,dx+\int_{\mathbb{R}^2}\nabla v\cdot\nabla(v\phi^2)\,dx$$
$$=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|v|^2\phi^2\,dx+\int_{\mathbb{R}^2}|\nabla v|^2\phi^2\,dx-\int_{\mathbb{R}^2}|v|^2(|\nabla\phi|^2+\phi\triangle\phi)\,dx.$$

Integrating by parts and using (1.6) give

$$\int_{\mathbb{R}^2}\nabla_{x_i}Pv^i\phi^2\,dx=-2\int_{\mathbb{R}^2}Pv^i\phi\nabla_{x_i}\phi\,dx$$

and

$$\int_{\mathbb{R}^2} v^k \nabla_{x_k} v^i v^i \phi^2 = \frac{1}{2} \int_{\mathbb{R}^2} v^k \nabla_{x_k} (|v|^2) \phi^2 = -\int_{\mathbb{R}^2} v^k |v|^2 \phi \nabla_{x_k} \phi \, dx.$$

Hence,

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |v|^2 \phi^2 \, dx + \int |\nabla v|^2 \phi^2 \, dx \\
&\quad = \int_{\mathbb{R}^2} (|v|^2 + 2P + |\nabla u|^2) v^i \phi \nabla_{x_i} \phi \, dx + \int_{\mathbb{R}^2} |v|^2 (|\nabla \phi|^2 + \phi \triangle \phi) \, dx \\
&\qquad + \int_{\mathbb{R}^2} \nabla_{x_i} u^k W_{p_j^k}(u, \nabla u) \nabla_{x_j} v^i \phi^2 \, dx + \int_{\mathbb{R}^2} \nabla_{x_i} u^k W_{p_j^k}(u, \nabla u) v^i \nabla_{x_j} \phi^2 \, dx. \qquad (3.11)
\end{aligned}$$

Multiplying (1.7) by $(u_t^i + (v \cdot \nabla) u^i) \phi^2$ and using $|u| = 1$ lead to

$$\begin{aligned}
&\int_{\mathbb{R}^2} |u_t + (v \cdot \nabla) u|^2 \phi^2 \, dx = \int_{\mathbb{R}^2} (u_t^i + v^l \nabla_l u^i) \nabla_\alpha [W_{p_\alpha^i}(u, \nabla u) - u^k u^i V_{p_\alpha^k}(u, \nabla u)] \varphi^2 \, dx \\
&\quad + \int_{\mathbb{R}^2} (u_t^i + v^l \nabla_l u^i)(-W_{u^i}(u, \nabla u) + V_{p_\alpha^k}(u, \nabla u) u^k \nabla_\alpha u^i) \phi^2 \, dx.
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
&\int_{\mathbb{R}^2} u_t^i \nabla_\alpha [W_{p_\alpha^i}(u, \nabla u) - u^k u^i V_{p_\alpha^k}(u, \nabla u)] \phi^2 \, dx \\
&\qquad + \int_{\mathbb{R}^2} u_t^i (-W_{u^i}(u, \nabla u) + V_{p_\alpha^k}(u, \nabla u) u^k \nabla_\alpha u^i) \phi^2 \, dx \\
&\quad = -\int_{\mathbb{R}^2} [\nabla_\alpha u_t^i W_{p_\alpha^i}(u, \nabla u) + u_t^i W_{u^i}(u, \nabla u)] \phi^2 \, dx - \int_{\mathbb{R}^2} u_t^i W_{p_\alpha^i}(u, \nabla u) \nabla_\alpha \phi^2 \, dx \\
&\quad = -\frac{d}{dt} \int_{\mathbb{R}^2} W(u, \nabla u) \phi^2 \, dx - 2 \int_{\mathbb{R}^2} u_t^i W_{p_\alpha^i}(u, \nabla u) \phi \nabla_\alpha \phi \, dx.
\end{aligned}$$

Integrating by parts twice and using (1.6), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} v^l \nabla_l u^i \nabla_\alpha [W_{p_\alpha^i}(u, \nabla u) - u^k u^i V_{p_\alpha^k}(u, \nabla u)] \phi^2 \, dx \\
&\qquad + \int_{\mathbb{R}^2} v^l \nabla_l u^i (-W_{u^i}(u, \nabla u) + V_{p_\alpha^k}(u, \nabla u) u^k \nabla_\alpha u^i) \phi^2 \, dx \\
&\quad = -\int_{\mathbb{R}^2} (\nabla_\alpha v^l \nabla_l u^i + v^l \nabla_l \nabla_\alpha u^i) W_{p_\alpha^i}(u, \nabla u) \phi^2 \, dx \\
&\qquad - \int_{\mathbb{R}^2} (v^l \nabla_l u^i) W_{p_\alpha^i}(u, \nabla u) \nabla_\alpha \phi^2 \, dx - \int_{\mathbb{R}^2} v^l \nabla_l u^i W_{u^i}(u, \nabla u) \phi^2 \, dx \\
&\quad = -\int_{\mathbb{R}^2} \nabla_\alpha v^l \nabla_l u^i W_{p_\alpha^i}(u, \nabla u) \phi^2 \, dx - \int_{\mathbb{R}^2} (v^l \nabla_l u^i) W_{p_\alpha^i}(u, \nabla u) \nabla_\alpha \phi^2 \, dx \\
&\qquad + 2 \int_{\mathbb{R}^2} v^l W(u, \nabla u) \phi \nabla_l \phi \, dx.
\end{aligned}$$

Combining the above three identities yields

$$\frac{d}{dt}\int_{\mathbb{R}^2} W(u,\nabla u)\phi^2\,dx + \int_{\mathbb{R}^2}|u_t+(v\cdot\nabla)u|^2\phi^2\,dx$$
$$= -\int_{\mathbb{R}^2}\nabla_\alpha v^k\nabla_k u^i W_{p^i_\alpha}(u,\nabla u)\phi^2\,dx - \int_{\mathbb{R}^2}(u^i_t+v^k\nabla_k u^i)W_{p^i_\alpha}(u,\nabla u)\nabla_\alpha\phi^2\,dx$$
$$+2\int_{\mathbb{R}^2} v^k W(u,\nabla u)\phi\nabla_k\phi\,dx$$
$$\le -\int_{\mathbb{R}^2}\nabla_\alpha v^k\nabla_k u^i W_{p^i_\alpha}(u,\nabla u)\phi^2\,dx + \frac{1}{2}\int_{\mathbb{R}^2}|u_t+(v\cdot\nabla)u|^2\phi^2\,dx$$
$$+C\int_{\mathbb{R}^2}|\nabla u|^2|\nabla\phi|^2\,dx + 2\int_{\mathbb{R}^2} v^k\,W(u,\nabla u)\phi\,\nabla_k\,\phi\,dx. \tag{3.12}$$

Integrating (3.11) and (3.12) in t on $[0,s]$ leads to

$$\int_{\mathbb{R}^2} e(u(\cdot,s),v(\cdot,s))\phi^2\,dx + \int_0^s\int_{\mathbb{R}^2}\left(|\nabla v|^2+\frac{1}{2}|u_t+(v\cdot\nabla)u|^2\right)\phi^2\,dxdt$$
$$\le \int_{\mathbb{R}^2} e(u_0,v_0)\phi^2\,dx + \int_0^s\int_{\mathbb{R}^2}(|v|^2+|\nabla u|^2+2P)v^i\phi\nabla_{x_i}\phi\,dxdt$$
$$+2\int_0^s\int_{\mathbb{R}^2}(v\cdot\nabla)u^k W_{p^k_i}(u,\nabla u)\phi\nabla_{x_i}\phi\,dxdt + 2\int_0^s\int_{\mathbb{R}^2} v^l W(u,\nabla u)\phi\nabla_l\phi\,dx$$
$$+C\int_0^s\int_{\mathbb{R}^2}(|v|^2+|\nabla u|^2)(|\nabla\phi|^2+|\phi|\,|\triangle\phi|)\,dxdt. \tag{3.13}$$

This, together with (3.1), shows immediately that

$$\int_{B_R(x_0)}(|v(\cdot,s)|^2+|\nabla u(\cdot,s)|^2)\,dx + \int_0^s\int_{\mathbb{R}^2}\left(|\nabla v|^2+\frac{1}{2}|u_t+(v\cdot\nabla)u|^2\right)\phi^2\,dxdt$$
$$\le \int_{B_{2R}(x_0)}(|v_0|^2+|\nabla u_0|^2)\,dx + C\int_0^s\int_{\mathbb{R}^2}(|v|^2+|\nabla u|^2+|P|)|v||\phi||\nabla\phi|\,dxdt$$
$$+C\frac{s}{R^2}\int_{\mathbb{R}^2}(|v_0|^2+|\nabla u_0|^2)\,dx. \tag{3.14}$$

It follows from Hölder inequality, (3.1) and (3.9) that

$$\int_0^s\int_{\mathbb{R}^2}(|v|^2+|\nabla u|^2)|v||\phi||\nabla\phi|\,dxdt \le\ C\int_0^s\int_{\mathbb{R}^2}(|v|^2+|\nabla u|^2)\frac{|v|}{R}dx\,dt$$
$$\le C\left(\int_0^s\int_{\mathbb{R}^2}(|v|^4+|\nabla u|^4)dx\,dt\right)^{\frac{1}{2}}\left(\int_0^s\int_{\mathbb{R}^2}\frac{|v|^2}{R^2}dx\,dt\right)$$
$$\le C\varepsilon^{\frac{1}{2}}\frac{s^{\frac{1}{2}}}{R}\left(1+\frac{s}{R^2}\right)^{\frac{1}{2}}\int_{\mathbb{R}^2} e(u_0,v_0)dx. \tag{3.15}$$

Similarly,

$$\int_0^s\int_{\mathbb{R}^2}|P||v||\phi||\nabla\phi|\,dxdt \le C\int_0^s\int_{\mathbb{R}^2}|P|\frac{|v|}{R}dx\,dt$$

$$\leq C\left(\int_0^s\int_{\mathbb{R}^2}|P|^2\,dx\,dt\right)^{\frac{1}{2}}\left(\int_0^s\int_{\mathbb{R}^2}\frac{|v|^2}{R^2}dx\,dt\right)^{\frac{1}{2}}$$
$$\leq C\frac{s^{\frac{1}{2}}}{R}\left(\int_{\mathbb{R}^2}e(u_0,v_0)dx\right)^{\frac{1}{2}}\left(\int_0^s\int_{\mathbb{R}^2}|P|^2\,dxdt\right)^{\frac{1}{2}} \tag{3.16}$$

for $R \leq R_0$.

On the other hand, it follows from the relation that

$$\triangle P = -\nabla_{x_i x_j}\left[\nabla_{x_i}u^k W_{p_j^k}(u,\nabla u) + v^j v^i\right] \quad \text{on } \mathbb{R}^2\times(0,T],$$

due to (1.5), and the Calderon–Zygmund estimate (cf. [5]) that

$$\int_0^s\int_{\mathbb{R}^2}|P|^2\,dx\,dt \leq C\int_0^s\int_{\mathbb{R}^2}(|\nabla u|^4+|v|^4)\,dx\,dt \leq C\,\varepsilon_1\left(1+\frac{s}{R^2}\right)\int_{\mathbb{R}^2}e(u_0,v_0)dx.$$

This, together with (3.16), yields

$$\int_0^s\int_{\mathbb{R}^2}|P|\,|v|\,|\varphi|\,|\nabla\phi|\,dx\,dt \leq C\frac{s^{\frac{1}{2}}}{R}\left(1+\frac{s}{R^2}\right)^{\frac{1}{2}}\varepsilon_1^{\frac{1}{2}}\int_{\mathbb{R}^2}e(u_0,v_0)dx. \tag{3.17}$$

The desired estimate (3.10) now follows from (3.14), (3.15) and (3.17). □

Lemma 3.4. *Let $(u,v)\in V(0,T)\times H(0,T)$ be a solution of (1.5)–(1.7) with initial value $(u_0,v_0)\in H_b^1(\mathbb{R}^2,S^2)\times L^2(\mathbb{R}^2,\mathbb{R}^3)$ and div $v_0=0$. Assume that there are constants ε_1 and $R_0>0$ such that*

$$\operatorname*{ess\,sup}_{0\leq t\leq T,x\in\mathbb{R}^2}\int_{B_R(x)}|\nabla u(\cdot,t)|^2+|v\,(\cdot,t)\,|^2\,dx < \varepsilon_1$$

for any $R\in(0,R_0]$. Let τ be any positive constant. Then, for $t\in[\tau,T]$, it holds that

$$\int_{\mathbb{R}^2}|\nabla^2u(x,t)|^2+|\nabla v(x,t)|^2\,dx \leq C\,\tau^{-1}(1+TR^{-2}). \tag{3.18}$$

Moreover, u and v are regular for all $t\in(0,T)$.

Proof. Note that, in a-priority, $\int_{\mathbb{R}^2}|\triangle v|^2$and $\int_{\mathbb{R}^2}|\nabla^3u|^2$ might not be finite. However, by a standard cut-off argument, we can assume that $\int_{\mathbb{R}^2}|\triangle v|^2$ and $\int_{\mathbb{R}^2}|\nabla^3u|^2$ are finite without loss of generality in the following proof.

Multiplying (1.5) by $\triangle v^i$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla v|^2\,dx+\int_{\mathbb{R}^2}|\triangle v|^2\,dx$$
$$=\int_{\mathbb{R}^2}(v\cdot\nabla v^i)\triangle v^i\,dx+\int_{\mathbb{R}^2}\nabla_j[\nabla_i u^k W_{p_j^k}(u,\nabla u)]\triangle v^i\,dx$$
$$\leq\frac{1}{4}\int_{\mathbb{R}^2}|\triangle v|^2\,dx+C\int_{\mathbb{R}^2}|v\cdot\nabla v|^2\,dx+C\int_{\mathbb{R}^2}(|\nabla^2u|^2+|\nabla u|^4)|\nabla u|^2\,dx. \tag{3.19}$$

Differentiating (1.7) in x_β, multiplying the above equation by $\nabla_\beta\triangle u^i$ and then integrating by parts, one can obtain

$$-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\triangle u|^2\,dx+\int_{\mathbb{R}^2}\left[(\nabla_\beta v\cdot\nabla)u^i+(v\cdot\nabla)\nabla_\beta u^i\right]\nabla_\beta\triangle u^i\,dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} \left[\nabla_\beta \nabla_\alpha \left[W_{p^i_\alpha}(u, \nabla u) - u^k u^i V_{p^k_\alpha}(u, \nabla u) \right] - \nabla_\beta W_{u^i}(u, \nabla u) \right] \nabla_\beta \triangle u^i \, dx \\
&\quad + \int_{\mathbb{R}^2} \nabla_\beta [W_{u^k}(u, \nabla u) u^k u^i + W_{p^k_\alpha}(u, \nabla u) \nabla_\alpha u^k u^i] \nabla_\beta \triangle u^i \, dx \\
&\quad + \int_{\mathbb{R}^2} \nabla_\beta [V_{p^k_\alpha}(u, \nabla u) u^k \nabla_\alpha u^i] \nabla_\beta \triangle u^i \, dx. \qquad (3.20)
\end{aligned}
$$

The first term on the right hand side of (3.20) is a bit more complicated. Since $W(u, p)$ is quadratic in p, we have

$$
\begin{aligned}
\nabla^2_{\gamma\beta} W_{p^i_\alpha}(u, \nabla u) &= \nabla_\gamma [W_{u^j p^i_\alpha}(u, \nabla u) \nabla_\beta u^j + W_{p^i_\alpha}(u, \nabla \nabla_\beta u)] \\
&= W_{u^j p^i_\alpha}(u, \nabla u) \nabla^2_{\gamma\beta} u^j + W_{u^j u^k p^i_\alpha}(u, \nabla u) \nabla_\gamma u^k \nabla_\beta u^j \\
&\quad + W_{p^j_l p^i_\alpha}(u, \nabla \nabla_\beta u) \nabla^3_{\beta\gamma l} u^j + W_{u^j p^i_\alpha}(u, \nabla \nabla_\beta u) \nabla_\gamma u^j. \qquad (3.21)
\end{aligned}
$$

Then, integrating by parts and using Young's inequality, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \nabla_\beta \nabla_\alpha W_{p^i_\alpha}(u, \nabla u) \nabla_\beta \triangle u^i \, dx = \int_{\mathbb{R}^2} \nabla^2_{\gamma\beta} W_{p^i_\alpha}(u, \nabla u) \nabla^3_{\gamma\beta\alpha} u^i \, dx \\
&\quad \geq \frac{a}{4} \int_{\mathbb{R}^2} |\nabla^3 u|^2 \, dx - C \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla u|^4 + |\nabla^2 u|^2) \, dx. \qquad (3.22)
\end{aligned}
$$

Note that $|u|^2 = 1$ implies

$$
u^i \nabla_\beta \triangle u^i + \nabla_\beta u^i \triangle u^i = -\nabla_\beta |\nabla u|^2.
$$

By this identity, one can estimate the second term and the last term on the right hand side of (3.20) as follows:

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \nabla_\beta \left[\nabla_\alpha \left(u^k u^i V_{p^k_\alpha}(u, \nabla u) \right) - V_{p^k_\alpha}(u, \nabla u) u^k \nabla_\alpha u^i \right] \nabla_\beta \triangle u^i \, dx \\
&\quad = \int_{\mathbb{R}^2} \nabla^2_{\alpha\beta} \left(u^k V_{p^k_\alpha}(u, \nabla u) \right) u^i \, \nabla_\beta \triangle u^i \, dx \\
&\qquad + \int_{\mathbb{R}^2} \nabla_\beta u^i \, \nabla_\alpha \left(u^k V_{p^k_\alpha}(u, \nabla u) \right) \nabla_\beta \triangle u^i \, dx \\
&\quad \leq \frac{a}{4} \int_{\mathbb{R}^2} |\nabla^3 u|^2 \, dx + C \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla^2 u|^2 + |\nabla u|^4) \, dx. \qquad (3.23)
\end{aligned}
$$

The other terms can be estimated easily in (3.20). Then it follows from (3.20)–(3.23) that

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} | \triangle u|^2 dx + \frac{a}{4} \int_{\mathbb{R}^2} |\nabla^3 u|^2 \, dx \\
&\quad \leq C \int_{\mathbb{R}^2} |\nabla v|^2 |\nabla u|^2 + |v|^2 |\nabla^2 u|^2 + (|\nabla^2 u|^2 + |\nabla u|^4) |\nabla u|^2 \, dx. \qquad (3.24)
\end{aligned}
$$

It follows from $-u \cdot \triangle u = |\nabla u|^2$, (3.18) and (3.24) that

$$
\begin{aligned}
&\frac{d}{dt} \left(\int_{\mathbb{R}^2} |\nabla^2 u|^2 + |\nabla v|^2 \right) + \frac{a}{4} \int_{\mathbb{R}^2} \left(|\nabla^3 u|^2 + |\nabla^2 v|^2 \right) dx \\
&\quad \leq C \int_{\mathbb{R}^2} (|v|^2 + |\nabla u|^2) \left(|\nabla v|^2 + |\nabla^2 u|^2 \right) dx. \qquad (3.25)
\end{aligned}
$$

By the Gagliardo–Nirenberg–Sobolev inequality, one has

$$
\begin{aligned}
&C\int_{\mathbb{R}^2}(|v|^2+|\nabla u|^2)(|\nabla v|^2+|\nabla^2 u|^2)dx\\
&\quad\le C\left(\int_{\mathbb{R}^2}(|v|^4+|\nabla u|^4)dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}(|\nabla v|^4+|\nabla^2 u|^4)dx\right)^{\frac12}\\
&\quad\le C\left(\int_{\mathbb{R}^2}(|\nabla v|^2+|\nabla^2 u|^2)dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}(|\nabla^2 v|^2+|\nabla^3 u|^2)dx\right)^{\frac12}\\
&\qquad\cdot\left(\int_{\mathbb{R}^2}(|v|^4+|\nabla u|^4)dx\right)^{\frac12}\\
&\quad\le \frac{a}{8}\int_{\mathbb{R}^2}(|\nabla^2 v|^2+|\nabla^3 u|^2)dx+\left(C\int_{\mathbb{R}^4}(|v|^4+|\nabla u|^4)dx\right)\\
&\qquad\cdot\left(\int_{\mathbb{R}^2}(|\nabla v|^2+|\nabla^2 u|^2)dx\right).
\end{aligned}
$$

This, together with (3.25), shows that for $t\in(0,T)$,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}(|\nabla v(x,t)|^2+|\nabla^2 u(x,t)|^2)dx+\frac{a}{8}\int_{\mathbb{R}^2}(|\nabla^2 v(x,t)|^2+|\nabla^3 u(x,t)|^2)dx\\
&\quad\le\left(C\int_{\mathbb{R}^2}(|\nabla u|^4+|v|^4)dx\right)\int_{\mathbb{R}^2}(|\nabla v(x,t)|^2+|\nabla^2 u(x,t)|^2)dx. \qquad (3.26)
\end{aligned}
$$

It follows from (3.9), (3.26), and Gronwall's inequality that for any s and t with $\tau\le s<t\le T$,

$$
\begin{aligned}
&\int_{\mathbb{R}^2}(|\nabla v|^2+|\nabla^2 u|^2)(x,t)\,dx\\
&\quad\le\left(e^{C\int_s^t\int_{\mathbb{R}^2}(|\nabla u|^4+|v|^4)(x,t)\,dx\,dl}\right)\left(\int_{\mathbb{R}^2}(|\nabla v|^2+|\nabla^2 u|^2)(x,s)\,dx\right)\\
&\quad\le\left(e^{C\varepsilon_1(1+TR^{-2})}\int_{\mathbb{R}^2}e(u_0,v_0)dx\right)\int_{\mathbb{R}^2}(|\nabla v|^2+|\nabla^2 u|^2)(x,s)dx. \qquad (3.27)
\end{aligned}
$$

Thanks to (3.8), (3.27), and the mean value theorem, we conclude that

$$
\begin{aligned}
&\sup_{\tau\le t\le T}\int_{\mathbb{R}^2}(|\nabla^2 u|^2+|\nabla v|^2)(\cdot,t)dx\\
&\quad\le C\tau^{-1}(1+TR^{-2})\,E(u_0,v_0)\,e^{C\varepsilon_1(1+TR^{-2})E(u_0,v_0)} \qquad (3.28)
\end{aligned}
$$

for any $\tau>0$. Then, by a proof similar to the one in Lemma 2.4, we can show that u belongs to $C^{1/8}(\mathbb{R}^2\times[\tau,T])$ for any $\tau>0$. In the Appendix below (Section 5), we can show that (u,v) is regular for all $t\in(0,T]$. □

Remark 3.1. Let $(u,v)\in V(0,T)\times H(0,T)$ be a solution of (1.5)–(1.7) with initial values $(u_0,v_0)\in H_b^2(\mathbb{R}^2;S^2)\times H^1(\mathbb{R}^2;\mathbb{R}^2)$ and $\operatorname{div} v_0=0$. Assume that there are constants ε_1 and

$R_0 > 0$ such that

$$\operatorname*{ess\,sup}_{0\le t\le T, x\in\mathbb{R}^2} \int_{B_R(x)} |\nabla u(\cdot,t)|^2 + |v(\cdot,t)|^2\,dx < \varepsilon_1$$

for any $R \in (0, R_0]$. Then, for $t \in [0,T]$, we have

$$\sup_{0\le t\le T} \int_{\mathbb{R}^2} (|\nabla^2 u(x,t)|^2 + |\nabla v(x,t)|^2)\,dx$$
$$\le C_3(1+TR^{-2})\,E(u_0,v_0)\,e^{C\varepsilon_1(1+TR^{-2})E(u_0,v_0)}, \tag{3.29}$$

with $C_3 = C(\|u_0\|_{H_b^2} + \|v_0\|_{H^1})$.

We are not able to prove the uniqueness of solutions to (1.5)–(1.7) for initial value in $H^1 \times L^2$ as one in Lemma 3.12 of [33]. However, we obtain

Lemma 3.5. *Let $(u_1, v_1), (u_2, v_2) \in V(0,T) \times H(0,T)$ be two smooth solutions of (1.5)–(1.7) with smooth initial values $(u_0, v_0) \in H_b^2(\mathbb{R}^2; S^2) \times H^1(\mathbb{R}^2; \mathbb{R}^2)$ and div $v_0 = 0$. Then $(u_1, v_1) = (u_2, v_2)$.*

Proof. Following the proof of Proposition 5.2 in the Appendix, we can assume that

$$|\nabla u_1| + |\nabla u_2| + |v_1| + |v_2| \le C$$

for a constant $C > 0$. For simplicity, we set in (1.7)

$$B(u,\nabla u) := -W_{u^i}(u,\nabla u) + W_{u^k}(u,\nabla u)u^k u^i + W_{p_\alpha^k}(u,\nabla u)\nabla_\alpha u^k u^i$$
$$+ V_{p_\alpha^k}(u,\nabla u)u^k\nabla_\alpha u^i$$

It follows from (1.7) that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |(u_1-u_2)|^2\,dx + \int_{\mathbb{R}^2} (W_{p_\alpha^i}(u_1,\nabla u_1) - W_{p_\alpha^i}(u_2,\nabla u_2))\nabla_\alpha(u_1^i - u_2^i)\,dx$$
$$= \int_{\mathbb{R}^2} (u_1^j u_1^i V_{p_\alpha^j}(u_1,\nabla u_1) - u_2^j u_2^i V_{p_\alpha^j}(u_2,\nabla u_2))\nabla_\alpha(u_1^i - u_2^i)\,dx$$
$$- \int_{\mathbb{R}^2} [-B(u_1,\nabla u_1) + B(u_2,\nabla u_2) + (v_1\cdot\nabla u_1) - (v_2\cdot\nabla)u_2]\cdot(u_1-u_2)\,dx$$
$$:= I_5 + I_6. \tag{3.30}$$

By Young's inequality, the last term on the right hand side of the above identity can be estimated as

$$I_6 = -\int_{\mathbb{R}^2} [-B(u_1,\nabla u_1) + B(u_2,\nabla u_2) + (v_1\cdot\nabla u_1) - (v_2\cdot\nabla)u_2]\cdot(u_1-u_2)\,dx$$
$$\le C\int_{\mathbb{R}^2} |u_1-u_2|^2 + |v_1-v_2|^2\,dx + \frac{a}{4}\int_{\mathbb{R}^2} |\nabla(u_1-u_2)|^2\,dx.$$

The difficult part is to estimate I_5. Using a uniform open ball covering of $\mathbb{R}^2$, we can estimate only the local integral

$$\int_{B_{r_0}(x_0)} (u_1^j u_1^i V_{p_\alpha^j}(u_1,\nabla u_1) - u_2^j u_2^i V_{p_\alpha^j}(u_2,\nabla u_2))\nabla(u_1-u_2)\,dx.$$

Now we can think about Eq. (1.7) with $\frac{\partial u}{\partial x_3} = 0$ in a domain of $\mathbb{R}^3$. After a rotation $\mathcal{R} \in O(3)$, the integrand (1.2) has the following invariant property:

$$W(\mathcal{R}u, \mathcal{R}\nabla u \mathcal{R}^T) = W(u, \nabla u).$$

Therefore, the system (1.5)–(1.7) is invariant for a rotation. Without loss of generality, we can assume that $u_0(x_0) = (0, 0, 1)$. Since u_1 and u_2 are uniformly continuous in $(x, t) \in \mathbb{R}^2 \times [0, \tau]$ for some $\tau > 0$, there exists a constant $r_0 > 0$ such that for any $(x, t) \in B_{r_0}(x_0) \times [0, \tau]$

$$|u_1(x, t) - u_0(x_0)| \le \varepsilon, \qquad |u_2(x, t) - u_0(x_0)| \le \varepsilon.$$

Then

$$\begin{aligned}
&\int_{B_{r_0}(x_0)} u_1^j u_1^i V_{p_\alpha^j}(u_1, \nabla u_1 - \nabla u_2) \nabla_\alpha (u_1^i - u_2^i)\, dx \\
&\quad \le C\varepsilon \int_{B_{r_0}(x_0)} |\nabla(u_1 - u_2)|^2\, dx + C \int_{B_{r_0}(x_0)} |\nabla(u_1^3 - u_2^3)|^2\, dx.
\end{aligned}$$

It follows from $|u| = 1$ that

$$u^3 \nabla_l u^3 = -u^1 \nabla_l u^1 - u^2 \nabla_l u^2.$$

Then an elementary calculation shows that

$$|\nabla(u_1^3 - u_2^3)| \le C|u_1 - u_2| + C\varepsilon |\nabla(u_1 - u_2)|.$$

By a covering argument, we apply all above estimates to obtain

$$I_5 \le C\varepsilon \int_{\mathbb{R}^2} |\nabla(u_1 - u_2)|^2\, dx + C \int_{\mathbb{R}^2} |(u_1 - u_2)|^2\, dx.$$

Therefore, choosing ε sufficiently small yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^2} |(u_1 - u_2)|^2\, dx + \frac{a}{2} \int_{\mathbb{R}^2} |\nabla(u_1 - u_2)|^2\, dx \\
&\quad \le C \int_{\mathbb{R}^2} |(u_1 - u_2)|^2 + |v_1 - v_2|^2\, dx. \qquad (3.31)
\end{aligned}$$

Using (1.5) and (1.6), one can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^2} |v_1 - v_2|^2\, dx + \int_{\mathbb{R}^2} |\nabla(v_1 - v_2)|^2\, dx \\
&\quad \le \tilde{C} \int_{\mathbb{R}^2} (|v_1 - v_2|^2 + |u_1 - u_2|^2 + |\nabla(u_1 - u_2)|^2)\, dx \\
&\qquad + \frac{1}{2} \int_{\mathbb{R}^2} |\nabla(v_1 - v_2)|^2\, dx. \qquad (3.32)
\end{aligned}$$

Combining (3.31) with (3.32) gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^2} \left(\tilde{C}|u_1 - u_2|^2 + \frac{a}{4}|v_1 - v_2|^2\right) dx \\
&\quad \le C \int_{\mathbb{R}^2} \left(\tilde{C}|u_1 - u_2|^2 + \frac{a}{4}|v_1 - v_2|^2\right) dx. \qquad (3.33)
\end{aligned}$$

Integrating (3.33) in t and applying Gronwall's inequality, we conclude

$$\int_{\mathbb{R}^2}\left(\tilde{C}|u_1-u_2|^2+\frac{a}{4}|v_1-v_2|^2\right)(\cdot,t)\,dt \leq C\int_{\mathbb{R}^2}\left(\tilde{C}|u_1-u_2|^2+\frac{a}{4}|v_1-v_2|^2\right)(\cdot,0)\,dt=0.$$

This proves our claim. □

4. Local existence and proof of Theorem 2

In this section, we prove the local existence of solutions of (1.5)–(1.7) and complete the proof of Theorem 2. Recall the notation that $\tilde{V}(\tau,t)$ denotes the space $V(\tau,t)$ where S^2 is replaced by $\mathbb{R}^3$.

Lemma 4.1. *For a pair $(u_0,v_0)\in H^1_b(\mathbb{R}^2,S^2)\times L^2(\mathbb{R}^2,\mathbb{R}^2)$ with div $v_0=0$ in $\mathbb{R}^2$ in the sense of distribution, there is a local regular solution $(u_\varepsilon,v_\varepsilon)\in\tilde{V}(0,T)\times H(0,T)$ of (1.10)–(1.12) with initial data (1.9) for some $T>0$.*

Proof. Although Lin–Liu proved only the global existence of the solution to (1.10)–(1.12) with initial data (1.9) for the case of $k_1=k_2=k_3$, their proofs still work for the local existence for the system (1.10)–(1.12). Thus we omit the details and refer readers to [27,28]. □

Theorem 4 (*Local Existence*)**.** *For a pair $(u_0,v_0)\in H^1_b(\mathbb{R}^2,S^2)\times L^2(\mathbb{R}^2,\mathbb{R}^3)$ with div $v_0=0$ in $\mathbb{R}^2$ in the sense of distribution, there is a local solution $(u,v)\in V(0,t_1)\times H(0,t_1)$ of (1.5)–(1.7) with initial value (u_0,v_0) for some $t_1>0$.*

Proof. For any map $u_0\in H^1_b(\mathbb{R}^2,S^2)$, one can approximate it by a sequence of smooth maps in $H^1_b(\mathbb{R}^2,S^2)$. Without loss of generality, we assume that $u_0\in H^2_b(\mathbb{R}^2,S^2)$ and $v_0\in H^1(\mathbb{R}^2,\mathbb{R}^3)$ with div $v_0=0$ in $\mathbb{R}^2$ are smooth. Then thanks to Lemma 4.1, there is a local regular solution $(u_\varepsilon,v_\varepsilon)\in\tilde{V}\times H$ of (1.10)–(1.12) with initial data (1.9).

For each pair (u,v), set

$$e_\varepsilon(u,v)=W(u,\nabla u)+\frac{1}{2\varepsilon^2}(1-|u|^2)^2+|v|^2,\qquad E(u,v)=\int_{\mathbb{R}^2}e_\varepsilon(u,v)dx.$$

Then same calculations as for (3.1) give

$$E(u_\varepsilon(\cdot,t),v_\varepsilon(\cdot,t)) +2\int_0^t\int_{\mathbb{R}^2}(|\partial_t u_\varepsilon+(v_\varepsilon\cdot\nabla u_\varepsilon)u_\varepsilon|^2+|\nabla v_\varepsilon|^2)^2\,dx\,dt=E(u_0,v_0). \tag{4.1}$$

By an analysis similar to the proof of Lemmas 3.2 and 3.1, one can show that there exist uniform positive constants R_0 and ε_1, and a positive time $T_\varepsilon=T(\varepsilon,R_0,\varepsilon_1)$ such that the problem (1.10)–(1.12) with initial data (1.9) has a regular solution $(u_\varepsilon,v_\varepsilon)\in\tilde{V}(0,T_\varepsilon)\times H(0,T_\varepsilon)$ for each fixed $\varepsilon>0$, and furthermore, it holds that

$$\sup_{0\le t\le T_\varepsilon}\int_{B_R(x_0)}|\nabla u_\varepsilon(\cdot,t)|^2+|v_\varepsilon(\cdot,t)|^2+\frac{1}{2\varepsilon^2}(1-|u_\varepsilon(\cdot,t)|^2)^2\,dx<\varepsilon_1 \tag{4.2}$$

for any positive $R\le R_0$.

Next, we will show that there is a constant $t_1 > 0$, independent of ε, such that $T_\varepsilon \geq t_1$ and the solution $(u_\varepsilon, v_\varepsilon)$ is bounded in $\tilde{V}(0, t_1) \times H(0, t_1)$ uniformly in ε.

First, we claim that for all $t \in [0, \min\{1, T_\varepsilon\}]$

$$\frac{1}{2} \leq |u_\varepsilon(x, t)| \leq \frac{3}{2}. \tag{4.3}$$

To verify (4.3), we re-scale the solution by

$$\tilde{u}(x, t) = u_\varepsilon(\varepsilon x, \varepsilon^2 t), \qquad \tilde{v}(x, t) = \varepsilon v_\varepsilon(\varepsilon x, \varepsilon^2 t), \qquad \tilde{P}(x, t) = \varepsilon^2 P_\varepsilon(\varepsilon x, \varepsilon^2 t).$$

Then $(\tilde{u}, \tilde{v})$ solves the following approximate Ericksen–Leslie system

$$\tilde{v}^i_t + (\tilde{v} \cdot \nabla)\tilde{v}^i - \triangle\tilde{v}^i + \nabla_{x_i}\tilde{P} = -\nabla_{x_j}(\nabla_{x_i}\tilde{u}^k\, W_{p^k_j}(\tilde{u}, \nabla\tilde{u})), \tag{4.4}$$

$$\nabla \cdot \tilde{v} = 0, \tag{4.5}$$

$$\tilde{u}^i_t + (\tilde{v} \cdot \nabla)\tilde{u}^i = \nabla_\alpha\left[W_{p^i_\alpha}(\tilde{u}, \nabla\tilde{u})\right] - W_{u^i}(\tilde{u}, \nabla\tilde{u}) + \tilde{u}^i(1 - |\tilde{u}|^2) \tag{4.6}$$

for $i = 1, 2, 3$, with initial data

$$\tilde{v}(x, 0) = \tilde{v}_0(x), \qquad \tilde{u}(x, 0) = \tilde{u}_0(x), \quad \forall x \in \mathbb{R}^2, \tag{4.7}$$

where $\tilde{u}_0(x) = u_0(\varepsilon x)$ and $\tilde{v}_0(x) = \varepsilon v_0(\varepsilon x)$ satisfy

$$\int_{\mathbb{R}^2} e(\tilde{u}_0(x), \tilde{v}_0(x))\, dx = \int_{\mathbb{R}^2} e(u_0(x), v_0(x))\, dx.$$

The condition (4.2) becomes

$$\operatorname*{ess\,sup}_{0 \leq t \leq \frac{T_\varepsilon}{\varepsilon^2}, x \in \mathbb{R}^2} \int_{B_{\frac{R}{\varepsilon}}(x)} |\nabla\tilde{u}(\cdot, t)|^2 + \frac{1}{2}(|1 - |\tilde{u}(\cdot, t)|^2)^2 + |\tilde{v}|^2\, dx < \varepsilon_1 \tag{4.8}$$

for any $R \in (0, R_0]$. While the basic energy identity (4.1) becomes

$$\int_{\mathbb{R}^2} \left[W(\tilde{u}, \nabla\tilde{u}) + \frac{1}{2}(1 - |\tilde{u}|^2)^2\right](\cdot, t)dx + \int_0^t \int_{\mathbb{R}^2} (|\partial_t\tilde{u}|^2 + |\partial_t\tilde{u} + (\tilde{v} \cdot \nabla)\tilde{u}|^2)(\cdot, t)dx\, dt = E(u_0, v_0) \tag{4.9}$$

for all $t \in \left(0, \frac{T_\varepsilon}{\varepsilon^2}\right)$.

Without loss of generality, we assume $T_\varepsilon \leq 1$. Let τ be the maximal time in $\left[0, \frac{T_\varepsilon}{\varepsilon^2}\right]$ such that

$$\frac{1}{4} \leq |\tilde{u}(x, t)| \leq 2. \tag{4.10}$$

By (4.1) and similar arguments as for Lemma 3.2, one can derive from (4.4)–(4.6) that there exists a uniform constant C_0 such that

$$\begin{aligned} C_0 \int_0^\tau \int_{\mathbb{R}^2} \left(|\nabla^2\tilde{u}|^2 + |\nabla\tilde{v}|^2\right) dx\, dt &\leq -\int_0^\tau \int_{\mathbb{R}^2} \triangle\tilde{u} \cdot \tilde{u}(1 - |\tilde{u}|^2)\, dx\, dt + C\int_{\mathbb{R}^2} |\nabla u_0|^2 + |v_0|^2\, dx \\ &\quad + C\int_0^\tau \int_{\mathbb{R}^2} |\nabla\tilde{u}|^4 + |\tilde{v}|^4\, dx\, dt. \end{aligned} \tag{4.11}$$

Integration by parts yields

$$-\int_0^\tau\int_{\mathbb{R}^2}\triangle\tilde u\ \cdot\tilde u(1-|\tilde u|^2)\,dx\,dt$$
$$=\int_0^\tau\int_{\mathbb{R}^2}|\nabla\tilde u|^2(1-|\tilde u|^2)\,dx\,dt-\frac{1}{2}\int_0^\tau\int_{\mathbb{R}^2}|\nabla|\tilde u|^2|^2\,dxdt \tag{4.12}$$

By Young's inequality and using (4.1), (4.6) and (4.10), one can obtain

$$\int_0^\tau\int_{\mathbb{R}^2}|\nabla\tilde u|^2(1-|\tilde u|^2)\,dx\,dt\le\eta\int_0^\tau\int_{\mathbb{R}^2}(1-|\tilde u|^2)^2\,dx\,dt+C\int_0^\tau\int_{\mathbb{R}^2}|\nabla\tilde u|^4\,dx\,dt$$
$$\le\eta\int_0^\tau\int_{\mathbb{R}^2}|\nabla^2u|^2\,dx\,dt+C\int_0^\tau\int_{\mathbb{R}^2}|\nabla\tilde u|^4\,dx\,dt+C\int_{\mathbb{R}^2}(|\nabla u_0|^2+|v_0|^2)\,dx \tag{4.13}$$

for a small constant η.

Combining (4.11)–(4.13) and choosing ε_1 sufficiently small in (4.2) with Lemma 2.2, we conclude that

$$\int_0^\tau\int_{\mathbb{R}^2}\left(|\nabla^2\tilde u|^2+|\nabla\tilde v|^2\right)dx\,dt+\int_0^\tau\int_{\mathbb{R}^2}(1-|\tilde u|^2)^2\,dx\,dt$$
$$\le C\left(1+\frac{\tau\varepsilon^2}{R^2}\right)\int_{\mathbb{R}^2}e(u_0,v_0)\,dx \tag{4.14}$$

for any $R\le R_0$.

It follows also from Lemma 2.2, (4.8), (4.9) and (4.14) that

$$\int_0^\tau\int_{\mathbb{R}^2}(|\nabla\tilde u|^4+|\tilde v|^4)\,dx\,dt\le C\,\varepsilon_1\left(1+\frac{\tau\varepsilon^2}{R^2}\right)E(u_0,v_0) \tag{4.15}$$

for any $R\le R_0$.

Now following the calculation for (3.25), one can derive that for any $t\in(0,\tau)$,

$$\frac{d}{dt}\left(\int_{\mathbb{R}^2}(|\nabla^2\tilde u|^2+|\nabla\tilde v|)(\cdot,t)dx\right)+\frac{a}{4}\int_{\mathbb{R}^2}(|\nabla^3\tilde u|^2+|\nabla^2\tilde v|)(\cdot,t)dx$$
$$\le\ C\int_{\mathbb{R}^2}(|\tilde v|^2+|\nabla\tilde u|^2)(|\nabla\tilde v|^2+|\nabla^2\tilde u|^2)(\cdot,t)dx$$
$$+C\left|\int_{\mathbb{R}^2}\nabla_\beta(\tilde u^i(1-|\tilde u|^2))\cdot\nabla_\beta\ \triangle\tilde u^i\,dx\right|+C\int_{\mathbb{R}^2}|\nabla\tilde u|^6\,(\cdot,t)dx. \tag{4.16}$$

Note that

$$C\left|\int_{\mathbb{R}^2}\nabla_\beta(\tilde u^i(1-|\tilde u|^2))\cdot\nabla_\beta\ \triangle\tilde u^i\,dx\right|=C\left|\int_{\mathbb{R}^2}\triangle(\tilde u^i(1-|\tilde u|^2))\cdot\triangle\tilde u^i\,dx\right|$$
$$\le C\int_{\mathbb{R}^2}|\nabla^2\tilde u(\cdot,t)|^2dx+C\int_{\mathbb{R}^2}|\nabla\tilde u(\cdot,t)|^4\,dx \tag{4.17}$$

and

$$C\int_{\mathbb{R}^2}|\nabla\tilde u(\cdot,t)|^6\,dx\ =-C\int_{\mathbb{R}^2}|\nabla\tilde u(\cdot,t)|^4\,\tilde u\cdot\triangle\tilde u\,dx-C\int\nabla_\alpha(|\nabla\tilde u|^4)\tilde u\cdot\nabla_\alpha\,\tilde u\,dx$$
$$\le\ \frac{C}{2}\int_{\mathbb{R}^2}|\nabla\tilde u(\cdot,t)|^6\,dx+C\int_{\mathbb{R}^2}|\nabla\tilde u(\cdot,t)|^2\,|\nabla^2\tilde u(\cdot,t)|dx. \tag{4.18}$$

It follows from (4.16)–(4.18) that

$$\frac{d}{dt}\left(\int_{\mathbb{R}^2}(|\nabla^2\tilde{u}|^2+|\nabla\tilde{v}|^2)(\cdot,t)dx\right)+\frac{a}{4}\int_{\mathbb{R}^2}(|\nabla^3\tilde{u}|^2+|\nabla^2\tilde{v}|^2)(\cdot,t)dx$$
$$\le C\int_{\mathbb{R}^2}(|\tilde{v}|^2+|\nabla\tilde{u}|^2)(|\nabla\tilde{v}|^2+|\nabla^2\tilde{u}|^2)dx$$
$$+C\left(\int_{\mathbb{R}^2}(|\nabla^2\tilde{u}(\cdot,t)|^2+|\nabla\tilde{u}(\cdot,t)|^4)dx\right).$$

Using the Gagliardo–Nirenberg–Sobolev inequality, one can get

$$C\int_{\mathbb{R}^2}(|\tilde{v}|^2+|\nabla\tilde{u}|^2)(|\nabla\tilde{v}|^2+|\nabla^2\tilde{u}|^2)dx$$
$$\le C\left(\int_{\mathbb{R}^2}(|\tilde{v}|^4+|\nabla\tilde{u}|^4)dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}(|\nabla\tilde{v}|^4+|\nabla^2\tilde{u}|^4)dx\right)^{\frac{1}{2}}$$
$$\le C\left(\int_{\mathbb{R}^2}(|\nabla\tilde{v}|^2+|\nabla^2\tilde{u}|^2)dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}(|\nabla^2\tilde{v}|^2+|\nabla^3\tilde{u}|^2)dx\right)^{\frac{1}{2}}$$
$$\cdot\left(\int_{\mathbb{R}^2}(|\tilde{v}|^4+|\nabla\tilde{u}|^4)dx\right)^{\frac{1}{2}}$$
$$\le\frac{a}{8}\int_{\mathbb{R}^2}(|\nabla^2\tilde{v}|^2+|\nabla^3\tilde{u}|^2)dx+\left(C\int_{\mathbb{R}^2}(|\tilde{v}|^4+|\nabla\tilde{u}|^4)dx\right)$$
$$\cdot\int_{\mathbb{R}^2}(|\nabla\tilde{v}|^2+|\nabla^2\tilde{u}|^2)dx.$$

This, together with (4.18), shows that

$$\frac{d}{dt}\int_{\mathbb{R}^2}(|\nabla^2\tilde{u}|^2+|\nabla\tilde{v}|^2)(\cdot,t)dx+\frac{a}{8}\int_{\mathbb{R}^2}(|\nabla^3\tilde{u}|^2+|\nabla^2\tilde{v}|)(\cdot,t)dx$$
$$\le\left(C\int_{\mathbb{R}^2}(|\tilde{v}|^4+|\nabla\tilde{u}|^4)(\cdot,t)dx\right)\int_{\mathbb{R}^2}(|\nabla^2\tilde{u}|^2+|\nabla\tilde{v}|^2)(\cdot,t)dx+h(t) \tag{4.19}$$

with

$$h(t)=C\int_{\mathbb{R}^2}(|\nabla^2\tilde{u}|^2+|\nabla\tilde{u}|^4)(\cdot,t)dx. \tag{4.20}$$

It then follows from (4.14), (4.15), (4.19)–(4.20), and Gronwall's inequality that for all $t\in(0,\tau)$,

$$\int_{\mathbb{R}^2}(|\nabla^2\tilde{u}|^2+|\nabla\tilde{v}|^2)(\cdot,t)dx\le e^{C\int_0^t\int_{\mathbb{R}^2}(|\tilde{v}|^4+|\nabla\tilde{u}|^4)(\cdot,l)dx\,dl}\cdot\int_{\mathbb{R}^2}(|\nabla^2u_0|^2+|\nabla v_0|^2)dx$$
$$+\int_0^t e^{C\int_s^t\int_{\mathbb{R}^2}(|\nabla\tilde{u}|^4+|\tilde{v}|^4)(\cdot,l)dx\,dl}\cdot h(s)\,ds$$
$$\le e^{C\varepsilon_1\left(1+\frac{\tau\varepsilon^2}{R^2}\right)E(u_0,v_0)}\left(\|u_0\|_{H^2}^2+\|v_0\|_{H^1}^2+\int_0^t h(s)ds\right)$$
$$\le\left(\|u_0\|_{H^2}^2+\|v_0\|_{H^1}^2+C\left(1+\frac{\tau\varepsilon^2}{R^2}\right)E(u_0,v_0)\right)e^{C\varepsilon_1\left(1+\frac{\tau\varepsilon^2}{R^2}\right)E(u_0,v_0)}. \tag{4.21}$$

Suppose that there is a $x_1 \in \mathbb{R}^2$ such that $|\tilde{u}(x_1,t)| < 1/2$ $\left(\text{or } |\tilde{u}(x_1,t)| > \frac{3}{2}\right)$ with some $t \in [0,\tau]$. It follows from (4.21) that $\tilde{u}$ is $C^{\frac{1}{8}}$-continuous uniformly in (x,t). Then, there is a constant C_4 so that for $x \in B_{1/4C_4}(x_1)$ with $\frac{1}{4C_4} < \frac{R_0}{\varepsilon}$, we have

$$(1-|\tilde{u}(x,t)|^2)^2 \geq \frac{1}{2}.$$

Then

$$\frac{1}{2}\int_{B_{\frac{1}{4C_4}}(x_1)} (1-|\tilde{u}(x,t)|^2)^2\,dx \geq \frac{1}{8}|B_{\frac{1}{4C_4}}(0)| > \varepsilon_1$$

which contradicts (4.8) for a sufficiently small ε_1. This shows that our claim (4.3) holds for all $t \in \left[0, \frac{T_\varepsilon}{\varepsilon}\right]$.

Finally, we show that $(u_\varepsilon, v_\varepsilon)$ is bounded in $\tilde{V}(0,\min\{T_\varepsilon,1\}) \times H(0,\min\{T_\varepsilon,1\})$ uniformly for any positive $\varepsilon < 4C_4R_0$.

For any $t \leq \min(1,T_\varepsilon)$, it follows from (4.14) and (4.15) that

$$\int_0^t\int_{\mathbb{R}^2}(|\nabla^2 u_\varepsilon|^2 + |\nabla v^\varepsilon|^2)(x,t)dx\,dt + \frac{1}{4\varepsilon^4}\int_0^t\int_{\mathbb{R}^2}(1-|u_\varepsilon|^2)^2\,dx\,dt$$
$$\leq C\left(1+\frac{t}{R^2}\right)E(u_0,v_0), \tag{4.22}$$

$$\int_0^t\int_{\mathbb{R}^2}(|\nabla u_\varepsilon|^4 + |v_\varepsilon|^4)(x,t)dx\,dt \leq C\,\varepsilon_1\left(1+\frac{t}{R^2}\right)E(u_0,v_0). \tag{4.23}$$

Let φ be the cut-off function as in the proof of Lemma 3.3. Then by an analysis similar to the ones in (3.14)–(3.17) and using (4.22) and (4.23), one can get

$$\int_{\mathbb{R}^2} e(u_\varepsilon(\cdot,t),v_\varepsilon(\cdot,t))\varphi^2\,dx + \int_0^t\int_{\mathbb{R}^2}\left(|\nabla v_\varepsilon|^2 + \frac{1}{2}|\partial_t u_\varepsilon + v_\varepsilon\cdot\nabla u_\varepsilon|^2\right)\varphi^2\,dx\,dt$$
$$\leq \int_{\mathbb{R}^2} e(u_0,v_0)\varphi^2\,dx + C\frac{t^{\frac{1}{2}}}{R}\left(1+\frac{t}{R^2}\right)^{\frac{1}{2}}E(u_0,v_0)$$
$$+\int_0^t\int_{\mathbb{R}^2}\frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\,|v_\varepsilon\cdot\nabla(\phi^2)|dx\,dt. \tag{4.24}$$

On the other hand,

$$\int_0^t\int_{\mathbb{R}^2}\frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\,|v_\varepsilon\cdot\nabla(\varphi^2)|\,dx\,dt$$
$$\leq C\left(\int_0^t\int_{\mathbb{R}^2}\frac{1}{4\varepsilon^4}(1-|u_\varepsilon|^2)^4\,dx\,dt\right)^{\frac{1}{2}}\left(\int_0^t\int_{\mathbb{R}^2}\frac{|v_\varepsilon|^2}{R^2}dx\,dt\right)^{\frac{1}{2}}$$
$$\leq C\frac{t^{\frac{1}{2}}}{R}\left(\int_{\mathbb{R}^2}\frac{1}{4\varepsilon^4}(1-|u_\varepsilon|^2)^2\,dx\,dt\right)^{\frac{1}{2}}(E(u_0,v_0))^{\frac{1}{2}}$$
$$\leq C\frac{t^{\frac{1}{2}}}{R}\left(1+\frac{t}{R^2}\right)^{\frac{1}{2}}E(u_0,v_0) \tag{4.25}$$

where one has used (4.1) and (4.22).

Hence,

$$\int_{B_R(x_0)} e_\varepsilon(u_\varepsilon(x,t), v_\varepsilon(x,t))\,dx \le \int_{B_{RR}(x_0)} e(u_0, v_0)dx + C\frac{t^{\frac{1}{2}}}{R}\left(1+\frac{t}{R^2}\right)^{\frac{1}{2}} E(u_0, v_0) \tag{4.26}$$

for any $R \le R_0$. First, choose $R_1 > 0$ so that

$$\int_{B_{2R_1}(x_0)} e(u_0, v_0)dx < \frac{\varepsilon_1}{2} \tag{4.27}$$

for all $x_0 \in \mathbb{R}^2$. Then, set

$$t_1 = \min\left\{R_1, \frac{\varepsilon_1 R_1}{4C(\varepsilon_1 + E(u_0, v_0))}.\right\} \tag{4.28}$$

Then for $t \le t_1$,

$$\int_{B_R(x_0)} e_\varepsilon(u_\varepsilon(x,t), v_\varepsilon(x,t))dx < \varepsilon_1$$

for all $x_0 \in \mathbb{R}^2$ and $R \le R_1$. Consequently, we have shown that there is a uniform $t_1 \le \min\{T_\varepsilon, 1\}$ such that $(u_\varepsilon, v_\varepsilon)$ is bounded in $\tilde{V}(0, t_1) \times H(0, t_1)$ with t_1 independent of ε. Letting $\varepsilon \to 0$, we can prove the local existence of solution $(u, v) \in V(0, t_1) \times H(0, t_1)$ with initial data (1.9). □

Now we complete the proof of Theorem 2.

Proof. By Theorem 4, there is a local solution $(u, v) \in V(0, t_1) \times H(0, t_1)$ of (1.5)–(1.7) in $\mathbb{R}^2 \times [0, t_1]$ with initial conditions (1.9) for some $t_1 > 0$. By Lemmas 3.4 and 3.5, the solution can be extended in $[0, T_1)$ for a maximal times T_1 such that at T_1, there is at least a singular point $x_i^1 \in \mathbb{R}^2$ such that

$$\limsup_{t \nearrow T_1} \int_{B_R(x_i^1)} e(u, v)(\cdot, t)\,dx \ge \varepsilon_0$$

for any $R \le R_0$ for some $R_0 > 0$ and $\varepsilon_0 > 0$. It is easy to see the solution $(u, v) \in V \times H$ is regular for all $t \in (0, T_1)$. Then there exists a sequence of $\{t_n\}$ such that the sequence $(u(t_n), v(t_n))$ converges weakly to $(u(T_1), v(T_1))$ in $H^1(\mathbb{R}^2; S^2) \times L^2(\mathbb{R}^2; \mathbb{R}^3)$ satisfying

$$\int_{\mathbb{R}^2} e(u(T_1), v(T_1))\,dx \le \int_{\mathbb{R}^2} e(u_0, v_0)\,dx - \varepsilon_0, \quad \operatorname{div} v(T_1) = 0.$$

Using the energy identity, there is a finite number of singular times $\{T_l\}_{l=1}^L$ in Theorem 2. □

5. Appendix: The liquid crystal flow and regularity issue

In this section, we formulate the liquid crystal heat flow and discuss $C^{1,\alpha}$-regularity issues for solutions of the liquid crystal flow (1.4) and the system (1.5)–(1.7).

The liquid crystal equilibrium system in a form of vectors and tensors was derived by Hardt et al. in [15] using the Lagrange multiplier method, but we need a precise form of (1.3) in coordinates.

Let ϕ be a smooth functional in $C_0^\infty(\Omega, \mathbb{R}^3)$. We consider a variation

$$u_t(x) = \frac{u + t\phi}{|u(x) + t\phi(x)|} = \frac{u + t\phi}{(1 + 2tu \cdot \phi + t^2\phi^2)^{1/2}}$$

and compute

$$\frac{du_t}{dt} = \phi - \frac{(u + t\phi)(u \cdot \phi + t|\phi|^2)}{(1 + 2tu \cdot \phi + t^2\phi^2)^{1/2}}.$$

To derive the Euler–Lagrange equations, we compute

$$\frac{d}{dt} \int_\Omega W(u_t, \nabla u_t)\, dx \bigg|_{t=0} = 0.$$

This implies

$$\int_\Omega \left(W_{u^j} \frac{du_t^j}{dt} + W_{p_\alpha^i} \frac{d\nabla_\alpha u_t^i}{dt} \right) \bigg|_{t=0} dx = 0,$$

where $W_{p_\alpha^i}(u, p) = \frac{\partial W}{\partial p_\alpha^i}$ and $W_{u^i} = \frac{\partial W}{\partial u^i}$. Note

$$\frac{du_t^i}{dt}\bigg|_{t=0} = \phi^i - u^i(u \cdot \phi), \qquad \frac{d\nabla_\alpha u_t^i}{dt}\bigg|_{t=0} = \nabla_\alpha \phi^i - \nabla_\alpha u^i(u \cdot \phi) - u^i \nabla_\alpha(u \cdot \phi).$$

We conclude that

$$\int_\Omega W_{u^j}(u, \nabla u) \left[\phi^j - u^j(u \cdot \phi)\right] + W_{p_\alpha^i}(u, \nabla u) \left[\nabla_\alpha \phi^i - \nabla_\alpha u^i(u \cdot \phi) - u^i \nabla_\alpha(u \cdot \phi)\right] dx = 0 \tag{5.1}$$

for any $\phi \in C_0^\infty(\Omega, \mathbb{R}^3)$. Therefore, $u \in H^1(\Omega, S^2)$ is said to be a weak solution to the liquid crystal system if u satisfies

$$\begin{aligned} -\nabla_\alpha &\left[W_{p_\alpha^i}(u, \nabla u) - u^k u^i W_{p_\alpha^k}(u, \nabla u)\right] + W_{u^i}(u, \nabla u) \\ &- W_{u^k}(u, \nabla u) u^k u^i - W_{p_\alpha^k}(u, \nabla u) \nabla_\alpha u^k u^i - W_{p_\alpha^k}(u, \nabla u) u^k \nabla_\alpha u^i = 0 \end{aligned}$$

in the sense of distribution. Note $|u|^2 = 1$, then $u^i \nabla u^i = 0$. This system is the exact form of (1.3).

Then, the liquid crystal flow can be formulated as in (1.4), i.e.,

$$\begin{aligned} \frac{\partial u^i}{\partial t} = \nabla_\alpha &\left[W_{p_\alpha^i}(u, \nabla u) - u^k u^i V_{p_\alpha^k}(u, \nabla u)\right] - W_{u^i}(u, \nabla u) \\ &+ W_{u^k}(u, \nabla u) u^k u^i + W_{p_\alpha^k}(u, \nabla u) \nabla_\alpha u^k u^i + V_{p_\alpha^k}(u, \nabla u) u^k \nabla_\alpha u^i. \end{aligned}$$

Next, we will prove that a Hölder continuous solution of (1.4) belongs to $C^{1,\alpha}$ for some α with $0 < \alpha < 1$. For any point $z_0 = (x_0, t_0) \in \Omega \times [0, \mathbb{R})$ and any number $R > 0$, we use standard notations:

$$B(x_0, R) = \{x \in \mathbb{R}^3 : |x - x_0| < R\}, \qquad Q(z_0, R) = B(x_0, R) \times (t_0 - R^2, t_0),$$
$$S_R(z_0) = B(x_0, R) \times \{t_0 - R^2\} \cap \partial B(x_0, R) \times (t_0 - R^2, t_0).$$

Proposition 5.1. *Let Ω be a domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Let u be a weak solution of (1.4) and Hölder continuous in $\Omega \times [0, T)$. Then, ∇u is (locally) Hölder continuous with the same exponent in $\Omega \times [0, T)$.*

Proof. Assume that $u(x, t)$ is Hölder continuous with exponent $\beta, 0 < \beta < 1$. Let $(x_0, t_0) \in \Omega \times (0, T)$ with $Q_{4R_0}(z_0) \subset \Omega \times (0, T)$ for some $R_0 > 0$. Note $u(x_0, t_0) = e \in S^2$. After a rotation, we can assume that $e = (0, 0, 1)$.

It follows from $|u| = 1$ and Cauchy's inequality that

$$|u^3|^2\,|\nabla u^3|^2 \le (1 - |u^3|^2)|\nabla u|^2 \le 2|u - u(x_0, t_0)||\nabla u|^2. \tag{5.2}$$

Denote

$$\tilde{p} = (p^j_\alpha)_{3\times 2}.$$

Using the structure of $W(u, p)$, we can write

$$W_{\tilde{p}}(u, \nabla u) = \tilde{W}_{\tilde{p}}(u, \nabla u^1, \nabla u^2) + f(u, \nabla u^3),$$

where $|f(u, \nabla u^3)| \le C|\nabla u^3|$.

Let $\tilde{v} = (v^1, v^2)$ be the solution of the Cauchy–Dirichlet problem

$$\begin{aligned} &v^i_t = \nabla_\alpha \left[W_{\tilde{p}^i_\alpha}(e, \nabla v^1, \nabla v^2)\right] \quad \text{in } Q_R(z_0) \\ &v^i = u^i \quad \text{on } S_R(z_0). \end{aligned} \tag{5.3}$$

for $i = 1, 2$. Since (5.3) is a parabolic system with constant coefficients, it follows from Proposition 1.2 of [14] that for all $\rho \le R \le R_0$

$$\int_{Q_\rho} |\nabla\tilde{v}|^2\,dz \le C\left(\frac{\rho}{R}\right)^5 \int_{Q_R} |\nabla\tilde{v}|^2\,dz$$

and

$$\int_{Q_\rho} |\nabla\tilde{v} - (\nabla\tilde{v})_\rho|^2\,dz \le C\left(\frac{\rho}{R}\right)^7 \int_{Q_R} |\nabla\tilde{v} - (\nabla\tilde{v})_R|^2\,dz.$$

Set $\tilde{w} = \tilde{u} - \tilde{v}$. Then for all $\rho < R$, we have

$$\int_{Q_\rho} |\nabla u|^2\,dz \le C\left(\frac{\rho}{R}\right)^5 \int_{Q_R} |\nabla u|^2\,dz + C\int_{Q_R} |\nabla\tilde{w}|^2\,dz + C\int_{Q_R} |\nabla u^3|^2\,dz \tag{5.4}$$

and

$$\begin{aligned} \int_{Q_\rho} |\nabla u - (\nabla u)_\rho|^2\,dz &\le C\left(\frac{\rho}{R}\right)^7 \int_{Q_R} |\nabla u - (\nabla u)_R|^2\,dz \\ &\quad + C\int_{Q_R} |\nabla\tilde{w}|^2\,dz + C\int_{Q_R} |\nabla u^3|^2\,dz. \end{aligned} \tag{5.5}$$

Note that u is β-Hölder continuous in $\Omega \times [0, T)$ and $u(x_0, t_0) = (0, 0, 1)$.

Although there is no maximum principle for the parabolic system (5.3) with constant coefficients, Giaquinta–Struwe in [14, p. 445] obtained that

$$\sup_{Q_R} |v - u(x_0, t_0)| \le C \sup_{Q_R} |u - u(x_0, t_0)|$$

with a constant C independent of R and u. This implies

$$|\tilde{w}| \le |u - u(x_0, t_0)| + |v - u(x_0, t_0)| \le CR^{\beta}. \tag{5.6}$$

Multiplying the difference between (5.3) and (1.4) by $\tilde{w}^i$ $(i = 1, 2)$ and integrating over Q_R lead to

$$\begin{aligned}
&\int_{B_R} |\tilde{w}|^2(\cdot, t_0)\,dx + \int_{Q_R} \sum_{i=1}^{2} \nabla_\alpha \tilde{w}^i W_{\tilde{p}^i_\alpha}(e, \nabla\tilde{w})\,dx \\
&\quad\le \int_{Q_R} \sum_{i=1}^{2} |\nabla_\alpha \tilde{w}^i|\,|\tilde{W}_{\tilde{p}^i_\alpha}(e, \nabla\tilde{v}) - \tilde{W}_{\tilde{p}^i_\alpha}(u, \nabla\tilde{u})|\,dx + C\int_{Q_R} |\nabla u^3 \parallel \nabla\tilde{w}|\,dx \\
&\qquad + \int_{Q_R} \sum_{i=1}^{2}\sum_{k=1}^{3} \nabla_\alpha \tilde{w}^i u^i u^k V_{p^k_\alpha}(u, \nabla u) + C\int_{Q_R} |\tilde{w}||\nabla u|^2\,dx.
\end{aligned} \tag{5.7}$$

Since u is β-Hölder continuous and $u(x_0, t_0) = (0, 0, 1)$, we have $|u^i| \le CR^{\beta}$ for $i = 1, 2$. Applying Young's inequality and (5.2) yields

$$\int_{Q_R} |\nabla\tilde{w}|^2\,dz \le CR^{\beta} \int_{Q_R} |\nabla u|^2\,dz. \tag{5.8}$$

It follows that for all $\rho < R$,

$$\int_{Q_\rho} |\nabla u|^2\,dz \le C\left(\frac{\rho}{R}\right)^5 \int_{Q_R} |\nabla u|^2\,dz + CR^{\beta}\int_{Q_R} |\nabla u|^2\,dz. \tag{5.9}$$

We claim the following Cacciopoli's inequality

$$\int_{Q(z_0,R)} |\nabla u|^2\,dz \le C\frac{1}{R^2}\int_{Q(z_0,2R)} |u - u_{2R}|^2\,dz \le CR^{3+2\beta}. \tag{5.10}$$

for any $z_0 \in \Omega \times (0, \infty)$ and $R \le R_0$, where u_{2R} is the average of u in $Q_{2R}(x_0, t_0)$.

Next, we prove this claim. Let ξ be a cut-off function in $C_0^\infty(B_{2R}(x_0))$ with $0 \le \xi \le 1$, $\xi \equiv 1$ in $B_R(x_0)$ and $|\nabla\xi| \le \frac{C}{R}$. Let $\tau \in C^\infty(\mathbb{R}, \mathbb{R})$ be a function that depends only on t with $0 \le \tau \le 1$, $\tau \equiv 1$ on $[t_0 - R^2, t_0]$ and $\tau \equiv 0$ on $(-\infty, t_0 - 4R^2)$ and $|\partial_t\tau| \le C/R^2$.

Testing (1.4) with $\phi = (u^i - u^i_{2R})\xi^2\tau^2 I_{(-\infty,t_0)}$ for $i = 1, 2$, where $I_{(-\infty,t_0)}$ is the characteristic function of $(-\infty, t_0)$, we have

$$\begin{aligned}
&\int_{B_{2R}(x_0)} |u(\cdot, t_0) - u_{2R}|^2\xi^2\tau^2(t_0)\,dx + \int_{Q_{2R}(z_0)} \sum_{i,j=1}^{2} W_{p^i_\alpha p^j_\beta} \nabla_\alpha u^i \nabla_\alpha u^j \xi^2\tau^2\,dz \\
&\quad\le 2\int_{Q_{2R}(z_0)} \left[W_{p^i_\alpha}(u, \nabla u) - u^k u^i V_{p^k_\alpha}(u, \nabla u)\right] \nabla_\alpha\xi(u^i - u^i_{2R})\xi\tau^2\,dz \\
&\qquad + C\int_{Q_{2R}(z_0)} |\nabla u|^2|u - u_{2R}|\xi^2\tau^2\,dz + 2\int_{Q_{2R}(z_0)} |u - u_{2R}|^2\xi^2\tau\partial_t\tau\,dz \\
&\qquad + \int_{Q_{2R}(z_0)} u^k u^i V_{p^k_\alpha}(u, \nabla u)\,\nabla_\alpha(u^i - u^i_{2R})\xi^2\tau^2\,dz + C\int_{Q_{2R}(z_0)} |\nabla u^3|^2\xi^2\tau^2\,dz.
\end{aligned}$$

Since u is β-Hölder continuous and $u(x_0, t_0) = (0, 0, 1)$, $u(x, t) - u_{2R}$ can be chosen sufficiently small when R_0 is small and $|u^1| + |u^2|$ is also small. We need to deal with the last term above.

However, due to (5.2), the term $|\nabla u^3|^2$ can be handled easily. Thus, by Young's inequality, the claim (5.10) follows.

Using (5.9) and (5.10), a standard iteration (cf. [10, Chapter III, Lemma 2.1]) yields that for all $\rho \le R_0$, one has

$$\int_{Q_\rho} |\nabla u|^2 \, dz \le C\rho^{3+3\beta}, \tag{5.11}$$

where C depends on R_0. An iteration by (5.9) and (5.10) yields that for any $\sigma < 1$,

$$\int_{Q_\rho} |\nabla u|^2 \, dz \le C\rho^{3+2\sigma}.$$

Using (5.2) and (5.8) yields

$$\begin{aligned}\int_{Q_\rho} |\nabla u - (\nabla u)_\rho|^2 \, dz &\le C\left(\frac{\rho}{R}\right)^7 \int_{Q_R} |\nabla u - (\nabla u)_R|^2 \, dz + CR^\beta \int_{Q_R} |\nabla u|^2 \, dz \\ &\le C\left(\frac{\rho}{R}\right)^7 \int_{Q_R} |\nabla u - (\nabla u)_R|^2 \, dz + CR^{3+2\sigma+\beta}.\end{aligned}$$

Choose σ sufficiently close to 1 so that $2\sigma + \beta > 2$. Then, for all $\rho \le \frac{R}{2}$, we have

$$\int_{Q_\rho} |\nabla u - (\nabla u)_\rho|^2 \, dz \le C\rho^{5+2\sigma_1}$$

for some σ_1 with $0 < \sigma_1 < 1$. This implies $\nabla u \in C^{1,\sigma_1}_{loc}$ and then $\nabla u \in C^{1,\beta}$ (cf [14]). □

Proposition 5.2. *Let (u, v) be a weak solution of (1.5)–(1.7) in $\mathbb{R}^2 \times [0, T]$ and assume that u is Hölder continuous in $\mathbb{R}^2 \times [0, T)$. Let τ be any positive constant. For $t \in [\tau, T]$, we have*

$$\int_{\mathbb{R}^2} |\nabla^2 u(x,t)|^2 + |\nabla v(x,t)|^2 \, dx \le C\tau^{-1}(1 + TR^{-2}).$$

Then, (u, v) is smooth in $\mathbb{R}^2 \times (0, T)$.

Proof. By the Sobolev embedding theorem, we have

$$\int_{B_1(x_0)} |\nabla u(x,t)|^p + |v(x,t)|^p \, dx \le C$$

for any $p > 1$ and for $x_0 \in \mathbb{R}^2$ and $t > \tau$. By an analysis similar to the one in Lemma 2.4, we can show that u is Hölder continuous in $\mathbb{R}^2 \times [\tau, T]$.

To get the higher order regularity, we rewrite (1.7) as

$$u_t^i - \nabla_\alpha \left[W_{p_\alpha^i}(u, \nabla u)\right] = -u^k u^i \nabla_\alpha \left[V_{p_\alpha^k}(u, \nabla u)\right] - (v \cdot \nabla) u^i + \tilde{B}(u, \nabla u), \tag{5.12}$$

where $\tilde{B}(u, \nabla u)$ is given by

$$\begin{aligned}\tilde{B}(u, \nabla u) = {} & -W_{u^i}(u, \nabla u) + W_{u^k}(u, \nabla u) u^k u^i + W_{p_\alpha^k}(u, \nabla u) \nabla_\alpha u^k u^i \\ & + V_{p_\alpha^k}(u, \nabla u) u^k \nabla_\alpha u^i - \nabla_\alpha \left[u^k u^i\right] V_{p_\alpha^k}(u, \nabla u).\end{aligned}$$

Since $W(u, p)$ is quadratic and convex in p, we can write

$$W_{p_\alpha^i}(u, \nabla u) = a_{\alpha\beta}^{ij}(u) \nabla_\alpha u^j.$$

Since u is uniformly Hölder continuous, the left-hand term of (5.12) is a parabolic operator. Let $\xi(x)$ be a cut-off function in $B_R(x_0)$ and let $\tau \in C^\infty(\mathbb{R}, \mathbb{R})$ be a function that depends only on t with $0 \le \tau \le 1$, $\tau \equiv 1$ on $\left[t_0 - \frac{1}{4}R^2, t_0\right]$ and $\tau \equiv 0$ on $(-\infty, t_0 - R^2)$ and $|\partial_t \tau| \le C/R^2$. Set $\phi = \tau\xi$. Multiplying (5.12) by ϕ, we have

$$
\begin{aligned}
&(u\phi)_t^i - \nabla_\alpha\left[a_{\alpha\beta}^{ij}(u)\nabla_\alpha(u^j\phi)\right] - u^i\phi_t \\
&\quad = -u^k u^i \nabla_\alpha\left[V_{p_\alpha^k}(u, \nabla u)\right]\phi - [(v\cdot\nabla)u^i + \tilde{B}(u, \nabla u)]\phi. \qquad (5.13)
\end{aligned}
$$

By the assumption, we have

$$(v\cdot\nabla)u \in L^p(Q_R(x_0)), \qquad |\nabla u|^2 \in L^p(Q_R(x_0)) \quad \forall p > 1$$

But the first term on the right hand side of (5.13) which is not a 'good' term, needs more analysis. Using the fact that $|u| = 1$, we have

$$u^3\nabla_{\alpha\beta}^2 u^3 = -(\nabla_\beta u\cdot\nabla u + u^1\nabla_{\alpha\beta}^2 u^1 + u^2\nabla_{\alpha\beta}^2 u^2), \qquad u^3u_t^3 = -(u^1u_t^1 + u^2u_t^2).$$

Without loss of generality, we regard the solution in $\mathbb{R}^3$. By a rotation, we assume

$$u(x_0, t_0) = (0, 0, 1).$$

Since u is Hölder continuous, there exists a small R such that

$$|u(x,t) - u(x_0, t_0)| \le \varepsilon$$

for a sufficiently small constant $\varepsilon > 0$. Therefore

$$|\nabla^2 u^3| \le C|\nabla u|^2 + 2\varepsilon(|\nabla^2 u^1| + |\nabla^2 u^2|)$$

Apply the classical L^p-estimate of parabolic systems (c.f. [8,22]) to (5.13) for $i = 1, 2$, we have

$$
\begin{aligned}
\|\tilde{u}_t\phi\|_{L^p(Q_R(x_0))} + \|\nabla^2(\tilde{u}\phi)\|_{L^p(Q_R(x_0))} &\le C\|\phi\nabla^2u^3\|_{L^p} + C\varepsilon\|\phi\nabla^2 u\|_{L^p(Q_R(x_0))} \\
&\quad + C(\|u\|_{L^{2p}(Q_R(x_0))} + \|v\|_{L^{2p}(Q_R(x_0))} + 1),
\end{aligned}
$$

where $\tilde{u} = (u^1, u^2)$. Choosing ε sufficiently small, we obtain the following.

$$\|u_t\phi\|_{L^p(Q_R(x_0))} + \|\nabla^2(u\phi)\|_{L^p(Q_R(x_0))} \le C.$$

To estimate v in (1.5), it follows from Hölder's inequality that

$$\int_{\mathbb{R}^2\times[\tau,T]} |(v\cdot\nabla)v|^p\,dx \le \left(\int_{\mathbb{R}^2\times[\tau,T]} |\nabla v|^4\,dx\,dt\right)^{p/4}\left(\int_{\mathbb{R}^2\times[\tau,T]} |v|^{\frac{4}{4-p}}\,dx\,dt\right)^{\frac{4-p}{4}}$$

for any p with $3 < p < 4$. By the L^p-estimate of Stoke's operator (e.g. [32]), v_t and $\nabla^2 v$ are in L^p for $3 < p < 4$. This implies that v is Hölder continuous.

Differentiating in x_l in (5.12), we have

$$
\begin{aligned}
(\nabla_{x_l}u^i)_t - \nabla_\alpha\left[a_{\alpha\beta}^{ij}(u)\nabla_\alpha(\nabla_{x_l}u^j)\right] &= -u^k u^i\nabla_\alpha\left[V_{p_\alpha^k}(u, \nabla\nabla_{x_l}u)\right] \\
&\quad + v\#\nabla^2 u + \nabla v\#\nabla u + \nabla u\#\nabla^2 u.
\end{aligned}
$$

By applying the L^p-theory, a similar argument yields that ∇u is uniformly continuous. Then, a standard bootstrap method implies that (u, v) are smooth. $\square$

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