Monads in double categories

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ABSTRACT

We extend the basic concepts of Street's formal theory of monads from the setting of 2-categories to that of double categories. In particular, we introduce the double category $\mathrm{Mnd}(\mathbb{C})$ of monads in a double category $\mathbb{C}$ and define what it means for a double category to admit the construction of free monads. Our main theorem shows that, under some mild conditions, a double category that is a framed bicategory admits the construction of free monads if its horizontal 2-category does. We apply this result to obtain double adjunctions which extend the adjunction between graphs and categories and the adjunction between polynomial endofunctors and polynomial monads.

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Introduction

The development of the formal theory of monads, begun in [23] and continued in [15], shows that much of the theory of monads [1] can be generalized from the setting of the 2-category **Cat** of small categories, functors and natural transformations to that of a general 2-category. The generalization, which involves defining the 2-category $\mathrm{Mnd}(\mathcal{K})$ of monads, monad maps and monad 2-cells in a 2-category $\mathcal{K}$, is useful for studying homogeneously a variety of important mathematical structures. For example, as explained in [17], categories, operads, multicategories and T-multicategories can all be seen as monads in appropriate bicategories. However, the most natural notions of a morphism between these mathematical structures do not appear as instances of the notion of a monad map. For example, it is well-known that, while categories can be viewed as monads in the bicategory of spans [2], functors are not monad maps therein.

To address this issue, we define the double category $\mathrm{Mnd}(\mathbb{C})$ of monads, horizontal monad maps, vertical monad maps and monad squares in a double category $\mathbb{C}$. Monads and horizontal monad maps in $\mathbb{C}$ are exactly monads and monad maps in the horizontal 2-category of $\mathbb{C}$, while the definitions of vertical monad maps and monad squares in $\mathbb{C}$ involve vertical arrows of $\mathbb{C}$ that are not necessarily identities. This combination of horizontal and vertical arrows of $\mathbb{C}$ in the definition of $\mathrm{Mnd}(\mathbb{C})$ allows us to describe mathematical structures and morphisms between them as monads and vertical monad maps in appropriate double categories. For example, small categories and functors can be viewed as monads and vertical monad maps in the double category of spans.

For a double category $\mathbb{C}$, we define also the double category $\mathrm{End}(\mathbb{C})$ of endomorphisms, horizontal endomorphism maps, vertical endomorphism maps and endomorphism squares. The double categories $\mathrm{Mnd}(\mathbb{C})$ and $\mathrm{End}(\mathbb{C})$ are related by a forgetful double functor $U : \mathrm{Mnd}(\mathbb{C}) \to \mathrm{End}(\mathbb{C})$, mapping a monad to its underlying endomorphism. By definition, a double category $\mathbb{C}$ is said to admit the construction of free monads if U has a vertical left adjoint. In view of our applications, we consider the construction of free monads in double categories that satisfy the additional assumption of being framed

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bicategories, in the sense of [21]. Our main result shows that a framed bicategory satisfying some mild assumptions admits the construction of free monads if its horizontal 2-category does. Here, the notion of a 2-category admitting the construction of free monads is obtained by generalizing the characterization of the free monads in the 2-category **Cat** obtained in [22, Section 6.1].

We apply the general theory to the study of two free constructions. First, we consider the construction of the free category on a graph (relatively to a category with finite limits), which plays an important role in Joyal's abstract treatment of Gödel's incompleteness theorems [18]. We show that if $\mathcal{E}$ is a pretopos with parametrized list objects, then the double category of spans in $\mathcal{E}$ admits the construction of free monads. Secondly, we consider the construction of the free monad on a polynomial endofunctor (relatively to a locally cartesian closed category, which is always assumed here to have a terminal object), which contributes to the category-theoretic analysis of Martin-Löf's types of well-founded trees, begun in [19] and continued in [7,8]. We show that if $\mathcal{E}$ is a locally cartesian closed category with finite disjoint coproducts and W-types, then the double category of polynomials in $\mathcal{E}$ admits the construction of free monads. Both of these results are obtained by application of our main result, which is possible since the double categories of interest are framed bicategories. Examples of categories $\mathcal{E}$ satisfying the hypotheses above abound: for example, every elementary topos with a natural numbers object is both a pretopos with parametrized list objects and a locally cartesian closed category with finite disjoint coproducts and W-types [19]. Thus, our theory applies in particular to the category **Set** of sets and functions and to categories of sheaves.

The double categories of spans and of polynomials are defined such that if we consider the vertical part of the free monad double adjunction, we recover exactly the adjunction between graphs and categories [16, Section II.7] and the adjunction between polynomial endofunctors and polynomial monads [8, Section 4.6]. Hence, we both strengthen these adjointness results and put them in a general context. Indeed, one of the original motivations for the research presented here was to make precise the analogy between the two constructions. In both cases, the application of our main theorem simplifies a problem regarding double categories by reducing it to a question on 2-categories. Note, however, that the combination of horizontal and vertical arrows is exploited essentially to recover the existing results, since the free monad construction acts on endomorphisms (which are defined using horizontal arrows) but its universal property is expressed with respect to vertical endomorphism maps.

Some double-categorical aspects of monads have also been investigated within the theory of fc-multicategories in [17, Chapter 5] and within the theory of framed bicategories in [21, Section 11]. However, the notion of a horizontal monad map considered there generalizes the ring-theoretic notion of a bimodule, whereas our horizontal monad maps are essentially the monad maps of Street [23].

Plan of the paper. Section 1 discusses monads in a 2-category, recalling some basic notions from [23] and giving a characterization of the free monads in a 2-category. Section 2 introduces the double category Mnd($\mathbb{C}$) of monads in a double category $\mathbb{C}$ and illustrates its definition with examples. Section 3 establishes some special properties of Mnd($\mathbb{C}$) under the assumption that $\mathbb{C}$ is a framed bicategory. In particular, we state our main result, Theorem 3.7, and apply it to our examples. Finally, Section 4 contains the proof of Theorem 3.7.

1. Monads in a 2-category

Preliminaries. We recall some definitions concerning endomorphisms, monads and their algebras in a 2-category. Let $\mathcal{K}$ be a 2-category. An *endomorphism* in $\mathcal{K}$ is a pair (X, P) consisting of an object X and a 1-cell $P : X \rightarrow X$. An *endomorphism map* $(F, \phi) : (X, P) \rightarrow (Y, Q)$ consists of a 1-cell $F : X \rightarrow Y$ and a 2-cell $\phi : QF \rightarrow FP$, which is not required to satisfy any condition. An *endomorphism* 2-*cell* $\alpha : (F, \phi) \rightarrow (F', \phi')$ is a 2-cell $\alpha : F \rightarrow F'$ making the following diagram commute:

$$
\begin{array}{ccc}
QF & \xrightarrow{\ \phi\ } & FP \\
{\scriptstyle Q\alpha}\downarrow & & \downarrow{\scriptstyle \alpha P} \\
QF' & \xrightarrow[\ \phi'\ ]{} & F'P.
\end{array}
$$

We write End($\mathcal{K}$) or End$_{\mathcal{K}}$ for the 2-category of endomorphisms, endomorphism maps and endomorphism 2-cells in $\mathcal{K}$. There is a 2-functor Inc : $\mathcal{K} \rightarrow$ End($\mathcal{K}$) which sends an object $X \in \mathcal{K}$ to the identity endomorphism $(X, 1_X)$ on X. Let us now consider a fixed endomorphism (Y, Q) in $\mathcal{K}$. For $X \in \mathcal{K}$, the category Q-alg$_X$ of X-indexed Q-algebras, in the sense of Lambek, is defined by letting

$$Q\text{-alg}_X =_{\text{def}} \text{End}_{\mathcal{K}}((X, 1_X), (Y, Q)).$$

Explicitly, an X-indexed Q-algebra consists of a 1-cell $F : X \rightarrow Y$, called the underlying 1-cell of the algebra, and a 2-cell $f : QF \rightarrow F$, called the structure map of the algebra. Note that the structure map is not required to satisfy any conditions. These definitions extend to a 2-functor

$$Q\text{-alg}_{(-)} : \mathcal{K} \rightarrow \textbf{Cat}.$$

We write $U_{(-)} : Q\text{-alg}_{(-)} \rightarrow \mathcal{K}(-, Y)$ for the 2-natural transformations whose components are the forgetful functors $U_X : Q\text{-alg}_X \rightarrow \mathcal{K}(X, Y)$ mapping an X-indexed Q-algebra to its underlying 1-cell.

We write $\mathrm{Mnd}(\mathcal{K})$ or $\mathrm{Mnd}_{\mathcal{K}}$ for the 2-category of monads, monad maps and monad 2-cells in $\mathcal{K}$, as defined in [23]. As usual, we refer to a monad by mentioning only its underlying endomorphism, leaving implicit its multiplication and unit. With a minor abuse of notation, we write $\mathrm{Inc} : \mathcal{K} \to \mathrm{Mnd}(\mathcal{K})$ for the 2-functor mapping an object X to the monad $(X, 1_X)$. If (Y, Q) is a monad, for every $X \in \mathcal{K}$ we may consider not only the category $Q\text{-alg}_X$ of Lambek algebras for its underlying endomorphism, but also the category $Q\text{-Alg}_X$ of X-indexed Eilenberg–Moore Q-algebras, which is defined by letting

$$Q\text{-Alg}_X =_{\mathrm{def}} \mathrm{Mnd}_{\mathcal{K}}((X, 1_X), (Y, Q)).$$

Note that we write $Q\text{-alg}_X$ for the category of algebras for the endomorphism and $Q\text{-Alg}_X$ for the category of Eilenberg–Moore algebras for the monad. Explicitly, an X-indexed Eilenberg–Moore Q-algebra consists of a 1-cell $F : X \to Y$ and a 2-cell $f : QF \to F$ satisfying the axioms

Again, these definitions extend to a 2-functor $Q\text{-Alg}_{(-)} : \mathcal{K} \to \mathbf{Cat}$ and there is a 2-natural transformation $U_{(-)} : Q\text{-Alg}_{(-)} \to \mathcal{K}(-, Y)$, with components given by the evident forgetful functors. Since (Y, Q) is assumed to be a monad, for every $X \in \mathcal{K}$ the forgetful functor $U_X : Q\text{-Alg}_X \to \mathcal{K}(X, Y)$ has a left adjoint, defined by composition with $Q : Y \to Y$.

A characterization of free monads. We generalize the characterization of the free monad on an endomorphism given by Staton in [22, Theorem 6.1.5] from the 2-category **Cat** to an arbitrary 2-category $\mathcal{K}$. The generalization is essentially straightforward, but we indicate the main steps of the proof. See [1, Section 9.4] for background material on free monads and [13] for a general account of several examples of the free monad construction.

Theorem 1.1. *Let (Y, Q) be an endomorphism in a 2-category $\mathcal{K}$. For a monad (Y, Q^*) and a 2-cell $\iota_Q : Q \to Q^*$, the following conditions are equivalent.*

(i) *The endomorphism map $(1_Y, \iota_Q) : (Y, Q^*) \to (Y, Q)$ is universal, in the sense that for every monad (X, P), composition with $(1_Y, \iota_Q)$ induces an isomorphism fitting in the diagram*

$$(1)$$

where the downward arrows are the evident forgetful functors.

(ii) *The 2-cell $\nu_{Q^*} : QQ^* \to Q^*$, defined as the composite*

$$QQ^* \xrightarrow{\;\iota_Q Q^*\;} Q^*Q^* \xrightarrow{\;\mu_{Q^*}\;} Q^*, \tag{2}$$

equips Q^* with a universal Q-algebra structure, in the sense that for every $X \in \mathcal{K}$, the functor $\mathcal{K}(X, Y) \to Q\text{-alg}_X$ defined by mapping $F : X \to Y$ to the Q-algebra with underlying 1-cell Q^*F and structure map the 2-cell $\nu_{Q^*}F : QQ^*F \to Q^*F$ is left adjoint to the forgetful functor $U_X : Q\text{-alg}_X \to \mathcal{K}(X, Y)$.

Proof. To see that (i) implies (ii), consider the following diagram:

where the bottom triangular diagram is an instance of the diagram in (1). The functor defined in (ii) is left adjoint to the forgetful functor $U_X : Q\text{-alg}_X \to \mathcal{K}(X, Y)$ since it is exactly the composite of the left adjoint $\mathcal{K}(X, Y) \to Q^*\text{-Alg}_X$, which is given by composition with Q^* (since Q^* is a monad), with the isomorphism $Q^*\text{-Alg}_X \to Q\text{-alg}_X$, which is defined by composition with ι_Q.

For the proof that (ii) implies (i), we need to define an isomorphism as in (1). Given an endomorphism map $(F, \phi) : (X, P) \to (Y, Q)$, where $\phi : QF \to FP$, we need to define a monad map $(F, \phi^{\sharp}) : (X, P) \to (Y, Q^*)$, where $\phi^{\sharp} : Q^*F \to FP$.

For this, we exploit the adjointness in (ii). Note that the left adjoint to $Q\text{-alg}_X \to \mathcal{K}(X, Y)$ sends F to the Q-algebra with underlying 1-cell Q^*F and structure map $\nu_{Q*}F : QQ^*F \to Q^*F$. Now, observe that the map

$$QFP \xrightarrow{\phi P} FPP \xrightarrow{F\mu_P} FP$$

equips FP with a Q-algebra structure. By adjointness, the map $\phi^\sharp : Q^*F \to FP$ is defined as the unique Q-algebra morphism such that the following diagram commutes:

$$
\begin{array}{ccc}
F & \xrightarrow{\eta_{Q*}F} & Q^*F \\
& \searrow{\scriptstyle F\eta_P} & \downarrow{\scriptstyle \phi^\sharp} \\
& & FP.
\end{array}
$$

Note that saying that $\phi^\sharp$ is a Q-algebra morphism amounts to saying that the following diagram commutes:

$$
\begin{array}{ccc}
QQ^*F & \xrightarrow{Q\phi^\sharp} & QFP \\
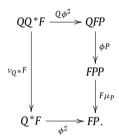\downarrow{\scriptstyle \nu_{Q*}F} & & \downarrow{\scriptstyle \phi P} \\
& & FPP \\
& & \downarrow{\scriptstyle F\mu_P} \\
Q^*F & \xrightarrow[\phi^\sharp]{} & FP.
\end{array}
$$

The isomorphism is defined as the identity on 2-cells. It remains to check that what we have defined is indeed an inverse to the functor defined by composition with $(1_Y, \iota_Q)$, but the verification is essentially identical to the one given in detail in the proof of [22, Theorem 6.1.5] and hence we omit it. $\square$

Definition 1.2. A 2-category $\mathcal{K}$ is said to *admit the construction of free monads* if for every endomorphism (Y, Q) there exists a monad (Y, Q^*) and a 2-cell $\iota_Q : Q \to Q^*$ satisfying the equivalent conditions of Theorem 1.1.

Remark 1.3. Let us point out that the universal property of the free monad (Y, Q^*) on an endomorphism (Y, Q) stated in item (i) of Theorem 1.1 includes the assertion that for every monad (X, P) and every endomorphism map $(F, \phi) : (X, P) \to (Y, Q)$, there exists a unique 2-cell $\phi^\sharp : Q^*F \to FP$ such that $(F, \phi^\sharp) : (X, P) \to (Y, Q^*)$ is a monad map and the diagram

$$
\begin{array}{ccc}
QF & \xrightarrow{\iota_Q F} & Q^*F \\
& \searrow{\scriptstyle \phi} & \downarrow{\scriptstyle \phi^\sharp} \\
& & FP
\end{array}
$$

commutes. From the statement in item (ii) of Theorem 1.1, it also follows that if $\mathcal{K}$ is a 2-category that admits the construction of free monads and has local coproducts, *i.e.* coproducts in its hom-categories, then for every $F : X \to Y$, the initial algebra for the endofunctor

$$
\begin{array}{ccc}
\mathcal{K}(X, Y) & \to & \mathcal{K}(X, Y) \\
(-) & \mapsto & F + Q\,(-)
\end{array}
$$

has Q^*F as its underlying object and the copair of the 2-cells $\eta_{Q*}F : F \to Q^*F$ and $\nu_{Q*}F : QQ^*F \to Q^*F$ as its structure map.

By the bicategorical Yoneda lemma [24], every bicategory is biequivalent to a 2-category [9, Theorem 1.4]. Hence, the remarks and the results above can be applied also to bicategories. We now introduce our two main classes of examples: bicategories of spans and bicategories of polynomials.

Example 1.4. Let $\mathcal{E}$ be a category with finite limits. Recall that a span in $\mathcal{E}$ is a diagram of the form

$$
\begin{array}{ccc}
& F & \\
{\scriptstyle \sigma}\swarrow & & \searrow{\scriptstyle \tau} \\
X & & Y,
\end{array}
\tag{3}
$$

and that a span morphism is a commutative diagram of the form

$$
\begin{array}{ccc}
X & \xleftarrow{\ \sigma\ } F \xrightarrow{\ \tau\ } & Y \\
\| & \big\downarrow{\phi} & \| \\
X & \xleftarrow[\ \sigma'\ ]{} F' \xrightarrow[\ \tau'\ ]{} & Y.
\end{array}
\tag{4}
$$

We write $\mathbf{Span}_{\mathcal{E}}$ for the bicategory of spans in $\mathcal{E}$, originally defined in [2], which has the objects of $\mathcal{E}$ as 0-cells, spans as 1-cells and span morphisms as 2-cells. It is well-known that graphs and categories in $\mathcal{E}$ can be identified with endomorphisms and monads in $\mathbf{Span}_{\mathcal{E}}$ [2,3]. For our purposes, it is convenient to recall the definition of the 2-category of linear functors over $\mathcal{E}$, which is biequivalent to the bicategory $\mathbf{Span}_{\mathcal{E}}$. Given a span as in (3), we define its associated linear functor to be the composite

$$
\mathcal{E}/X \xrightarrow{\ \sigma^*\ } \mathcal{E}/F \xrightarrow{\ \tau_!\ } \mathcal{E}/Y \ ,
$$

where σ^* acts by pullback along σ and $\tau_!$ acts by composition with τ. In general, a functor between slices of $\mathcal{E}$ is said to be linear if it is naturally isomorphic to a functor of this form. Now, recall from [8, Section 1.3] that slice categories of $\mathcal{E}$ are tensored over $\mathcal{E}$ and that linear functors have a canonical strength. The 2-category of linear functors is then defined as the sub-2-category of $\mathbf{Cat}$ having slice categories of $\mathcal{E}$ as 0-cells, linear functors between them as 1-cells, and strong natural transformations as 2-cells, *i.e.* natural transformations compatible with the canonical strength on linear functors. Let us also recall that a strong natural transformation between linear functors is cartesian, *i.e.* its naturality squares are pullbacks. By the biequivalence, graphs in $\mathcal{E}$ can be thought of as linear endofunctors and categories in $\mathcal{E}$ can be thought of as linear monads, *i.e.* monads whose underlying functor is linear and whose multiplication and unit are strong natural transformations.

Example 1.5. Let $\mathcal{E}$ be a locally cartesian closed category. Recall from [8, Section 1.4] that a polynomial over $\mathcal{E}$ is a diagram of the form

$$
\begin{array}{ccc}
 & \bar{F} \xrightarrow{\ \theta\ } F & \\
{}^{\sigma}\swarrow & & \searrow^{\tau} \\
X & & Y
\end{array}
\tag{5}
$$

and a cartesian morphism of polynomials is a diagram of the form

$$
\begin{array}{ccccc}
X & \longleftarrow \bar{F} & \longrightarrow F & \longrightarrow & Y \\
\| & \big\downarrow \quad \lrcorner & \| & & \| \\
X & \longleftarrow \bar{F}' & \longrightarrow F' & \longrightarrow & Y,
\end{array}
$$

where the central square is a pullback. We write $\mathbf{Poly}_{\mathcal{E}}$ for the bicategory of polynomials over $\mathcal{E}$, as defined in [8, Section 1.16], which has the objects of $\mathcal{E}$ as 0-cells, polynomials as 1-cells, and cartesian morphisms of polynomials as 2-cells. Working in the internal logic of $\mathcal{E}$, for a polynomial as in (5) we may represent an element $f \in F$ as an arrow

$$
f : (x_i \mid i \in I) \to y,
$$

where $I =_{\mathrm{def}} \theta^{-1}(f)$, the family $(x_i \mid i \in I)$ is defined by letting $x_i =_{\mathrm{def}} \sigma(i)$, for $i \in I$, and $y =_{\mathrm{def}} \tau(f)$. Thus, we think of the set I as the arity of the arrow f. The biequivalence between the bicategory of spans and the 2-category of linear functors extends to a biequivalence between the bicategory of polynomials and the 2-category of polynomial functors [8, Theorem 2.17], as we now proceed to recall. For a polynomial as in (5), the polynomial functor associated with it is defined as the composite

$$
\mathcal{E}/X \xrightarrow{\ \sigma^*\ } \mathcal{E}/\bar{F} \xrightarrow{\ \theta_*\ } \mathcal{E}/F \xrightarrow{\ \tau_!\ } \mathcal{E}/Y \ ,
$$

where θ_* is the right adjoint to the pullback functor θ^*. A functor between slices of $\mathcal{E}$ is said to be polynomial if it is naturally isomorphic to a functor of this form. Like linear functors, polynomial functors have a canonical strength and so we can define the 2-category of polynomial functors as the sub-2-category of $\mathbf{Cat}$ having slices of $\mathcal{E}$ as 0-cells, polynomial functors as 1-cells and cartesian strong natural transformations as 2-cells. The biequivalence between $\mathbf{Poly}_{\mathcal{E}}$ and the 2-category of polynomial functors allows us to identify endomorphisms and monads in $\mathbf{Poly}_{\mathcal{E}}$ with polynomial endofunctors and polynomial monads on slices of $\mathcal{E}$, respectively, where by a polynomial monad we mean a monad whose underlying endofunctor is polynomial and whose multiplication and unit are cartesian strong natural transformations.

Let us also recall from [19] that a locally cartesian closed category $\mathcal{E}$ is said to have W-types if every polynomial endofunctor $P : \mathcal{E} \to \mathcal{E}$ has an initial algebra, called the W-type of the functor. Note that a polynomial functor $P : \mathcal{E} \to \mathcal{E}$ has to be represented by a diagram as in (5) in which both X and Y are the terminal object of $\mathcal{E}$, and hence is completely

determined by the map θ. The category-theoretic notion of a W-type is a counterpart of the notion of a type of well-founded trees, originally introduced by Martin-Löf within his dependent type theory [20]. As shown in [7, Theorem 12], if $\mathcal{E}$ has disjoint coproducts, the assumption of W-types is sufficient to show that, for all $X \in \mathcal{E}$, every polynomial endofunctor $P : \mathcal{E}/X \to \mathcal{E}/X$ has an initial algebra. For further material and references on polynomial functors, see [8] and its bibliography.

Proposition 1.6 provides the horizontal part of Proposition 3.8. Item (i) in its statement refers to the notion of a pretopos with parametrized list objects, for which we invite the reader to refer to [18].

Proposition 1.6.

(i) *If $\mathcal{E}$ is a pretopos with parametrized list objects, the bicategory* **Span**$_\mathcal{E}$ *admits the construction of free monads.*

(ii) *If $\mathcal{E}$ is a locally cartesian closed category with disjoint coproducts and W-types, the bicategory* **Poly**$_\mathcal{E}$ *admits the construction of free monads.*

Proof. We begin by proving (ii). We exploit the biequivalence between the bicategory of polynomials and the 2-category of polynomial functors. Let $Q : \mathcal{E}/Y \to \mathcal{E}/Y$ be a polynomial endofunctor. We show that there is a polynomial monad $Q^* : \mathcal{E}/Y \to \mathcal{E}/Y$ and a cartesian strong natural transformation $\iota : Q \to Q^*$ that satisfy the universal property in item (ii) of Theorem 1.1. By [7, Theorem 12], the assumption that $\mathcal{E}$ has W-types implies that the forgetful functor $U : Q\text{-alg} \to \mathcal{E}/Y$ has a left adjoint. We let $Q^* : \mathcal{E}/Y \to \mathcal{E}/Y$ be the monad resulting from the adjunction. The monad $Q^* : \mathcal{E}/Y \to \mathcal{E}/Y$ is polynomial by [8, Theorem 4.5]. If $Q : \mathcal{E}/Y \to \mathcal{E}/Y$ is represented by the polynomial

$$
\begin{array}{ccc}
\bar{Q} & \xrightarrow{\theta_Q} & Q \\
\swarrow & & \searrow \\
Y & & Y
\end{array}
\tag{6}
$$

then $Q^* : \mathcal{E}/Y \to \mathcal{E}/Y$ is represented by the polynomial

$$
\begin{array}{ccc}
\bar{Q}^* & \xrightarrow{\theta_{Q*}} & Q^* \\
\swarrow & & \searrow \\
Y & & Y,
\end{array}
\tag{7}
$$

where the object Q^* in (7) is described in the internal logic of $\mathcal{E}$ as the set of well-founded trees of profile Q, *i.e.* trees built up from identities and formal composites of the arrows in Q. The map θ_{Q*} in (7) describes the arities of the arrows in Q^* in the evident way. The inclusion of the arrows in Q into those in Q^* is part of a diagram

$$
\begin{array}{ccccccc}
Y & \longleftarrow & \bar{Q} & \xrightarrow{\theta_Q} & Q & \longrightarrow & Y \\
\| & & \downarrow & \ulcorner & \downarrow & & \| \\
Y & \longleftarrow & \bar{Q}^* & \xrightarrow[\theta_{Q*}]{} & Q^* & \longrightarrow & Y,
\end{array}
\tag{8}
$$

which represents the required cartesian strong natural transformation $\iota : Q \to Q^*$. A direct verification shows that the left adjoint to $U : Q\text{-alg} \to \mathcal{E}/Y$ maps an object A to the Q-algebra with underlying object Q^*A and structure map $\nu_A : QQ^*A \to Q^*A$, where $\nu_{Q*} : QQ^* \to Q^*$ is defined as in (2). To conclude the proof of item (ii) it is sufficient to observe that, for $X \in \mathcal{E}$, the category $Q\text{-alg}_X$ is equivalent to the category of polynomial functors $F : \mathcal{E}/X \to \mathcal{E}/Y$ equipped with a cartesian strong natural transformation $\phi : QF \to F$.

The proof of item (i) is similar, except that polynomial functors are replaced by linear functors. In this case, the assumption of W-types can be replaced by that of parametrized list objects, which suffice to prove the existence of the left adjoint to the forgetful functor $U : Q\text{-alg} \to \mathcal{E}/Y$ and that the resulting monad $Q^* : \mathcal{E}/Y \to \mathcal{E}/Y$ is linear. This is because linear endofunctors (respectively, linear monads) are just graphs (respectively, categories) internal to $\mathcal{E}$, and, as shown in [18, Proposition 7.3], the assumption of parametrized list objects guarantees the existence of the free category on a graph in $\mathcal{E}$. $\square$

If $\mathcal{E}$ is a locally cartesian closed pretopos with W-types, then it has list objects and these are parametrized since we are in a cartesian closed category. Hence, such a category satisfies the hypotheses of both item (i) and item (ii) of Proposition 1.6. In this case, the construction of the free monad for polynomial endofunctors generalizes the construction of the free monad for linear endofunctors.

2. Monads in a double category

Notation and preliminaries. We assume readers to be familiar with the basic concepts of the theory of double categories (see [4] for the original reference and [6,10,11] for modern accounts) and limit ourselves to introducing some notation and recalling some basic notions. For a double category $\mathbb{C}$, we write $\mathrm{Obj}_{\mathbb{C}}$ for its class of objects, $\mathrm{Hor}_{\mathbb{C}}$ for its class of horizontal arrows, $\mathrm{Ver}_{\mathbb{C}}$ for its class of vertical arrows and $\mathrm{Sq}_{\mathbb{C}}$ for its class of squares. We write $\mathbb{C}_0$ for the category of objects and vertical arrows and $\mathbb{C}_1$ for the category of horizontal arrows and squares. We allow horizontal composition to be associative and unital up to coherent invertible squares rather than strictly. For the sake of readability, however, we shall work as if horizontal composition were strict, as allowed by [10, Theorem 7.5]. Typically, a square will be written as follows:

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ u\downarrow & \alpha & \downarrow v \\ X' & \xrightarrow{F'} & Y'. \end{array} \qquad (9)$$

Identity squares will be written without a label, as follows:

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \| & & \| \\ X & \xrightarrow{F} & Y, \end{array} \qquad \begin{array}{ccc} X & =\!\!=\!\!= & X \\ u\downarrow & & \downarrow u \\ X' & =\!\!=\!\!= & X'. \end{array}$$

For a double category $\mathbb{C}$, its horizontal 2-category $\mathcal{H}_{\mathbb{C}}$ is defined as follows: the 0-cells are the objects of $\mathbb{C}$, the 1-cells are the horizontal arrows of $\mathbb{C}$ and the 2-cells are the squares of the form

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \| & \alpha & \| \\ X & \xrightarrow{F'} & Y. \end{array}$$

The notions of horizontal adjunction and vertical adjunction between double categories can be defined using the general notion of an adjunction in a 2-category [14]. A horizontal adjunction is an adjunction in the 2-category of double categories, double functors and horizontal natural transformations; vertical adjunctions are defined analogously, replacing horizontal natural transformations with vertical ones [10].

Example 2.1. Let $\mathcal{E}$ be a category with finite limits. With a minor abuse of notation, we write **Span**$_{\mathcal{E}}$ also for the double category of spans in $\mathcal{E}$, which has objects of $\mathcal{E}$ as objects, spans as horizontal arrows, maps of $\mathcal{E}$ as vertical arrows and diagrams of the form

$$\begin{array}{ccccc} X & \xleftarrow{\sigma} & F & \xrightarrow{\tau} & Y \\ u\downarrow & & \downarrow \phi & & \downarrow v \\ X' & \xleftarrow{\sigma'} & F' & \xrightarrow{\tau'} & Y' \end{array}$$

as squares. Note that the horizontal bicategory of this double category is exactly the bicategory of spans in $\mathcal{E}$ defined in Example 1.4.

Example 2.2. Let $\mathcal{E}$ be a locally cartesian closed category. With another abuse of notation, we write **Poly**$_{\mathcal{E}}$ also for the double category of polynomials over $\mathcal{E}$, which has the objects of $\mathcal{E}$ as objects, polynomials as horizontal arrows, maps of $\mathcal{E}$ as vertical arrows and diagrams of the form

$$\begin{array}{ccccccc} X & \xleftarrow{\sigma} & \bar{F} & \xrightarrow{\theta} & F & \xrightarrow{\tau} & Y \\ u\downarrow & & \downarrow & \lrcorner & \downarrow \phi & & \downarrow v \\ X' & \xleftarrow{\sigma'} & \bar{F}' & \xrightarrow{\theta'} & F' & \xrightarrow{\tau'} & Y', \end{array}$$

where the central square is a pullback, as squares. The bicategory of polynomials defined in Example 1.5 is the horizontal bicategory of this double category.

The double categories of endomorphisms and monads. Below, we define the double category Mnd($\mathbb{C}$) of monads in a double category $\mathbb{C}$. After giving the definition, we explain how it generalizes the definition of the 2-category Mnd($\mathcal{K}$) of monads in a 2-category $\mathcal{K}$. In view of our applications, we begin by introducing the double category End($\mathbb{C}$) of endomorphisms in a double category $\mathbb{C}$.

Definition 2.3. Let $\mathbb{C}$ be a double category.

 (i) A *horizontal endomorphism* is a pair (X, P) consisting of an object X and a horizontal arrow $P : X \to X$. Since we consider only horizontal endomorphisms, we refer to them simply as endomorphisms.

 (ii) A *horizontal endomorphism map* $(F, \phi) : (X, P) \to (Y, Q)$ consists of a horizontal arrow $F : X \to Y$ and a square

$$
\begin{array}{ccccc}
X & \xrightarrow{F} & Y & \xrightarrow{Q} & Y \\
\| & & \phi & & \| \\
X & \xrightarrow{P} & X & \xrightarrow{F} & Y.
\end{array}
$$

(iii) A *vertical endomorphism map* $(u, \bar{u}) : (X, P) \to (X', P')$ consists of a vertical arrow $u : X \to X'$ and a square

$$
\begin{array}{ccc}
X & \xrightarrow{P} & X \\
u\downarrow & \bar{u} & \downarrow u \\
X' & \xrightarrow{P'} & X'.
\end{array}
$$

(iv) An *endomorphism square*

$$
\begin{array}{ccc}
(X, P) & \xrightarrow{(F,\phi)} & (Y, Q) \\
(u,\bar{u})\downarrow & \alpha & \downarrow (v,\bar{v}) \\
(X', P') & \xrightarrow{(F',\phi')} & (Y', Q')
\end{array}
$$

is a square

$$
\begin{array}{ccc}
X & \xrightarrow{F} & Y \\
u\downarrow & \alpha & \downarrow v \\
X' & \xrightarrow{F'} & Y'
\end{array}
$$

satisfying the condition

$$
\begin{array}{ccccccccc}
X & \longrightarrow & Y & \longrightarrow & Y & & X & \longrightarrow & Y & \longrightarrow & Y \\
\| & & \phi & & \| & & \downarrow & & \alpha & \downarrow & \bar{v} & \downarrow \\
X & \longrightarrow & X & \longrightarrow & Y & = & X' & \longrightarrow & Y' & \longrightarrow & Y' \\
\downarrow & \bar{u} & \downarrow & \alpha & \downarrow & & \| & & \phi' & & \| \\
X' & \longrightarrow & X' & \longrightarrow & Y' & & X' & \longrightarrow & X' & \longrightarrow & Y'.
\end{array}
$$

We write End($\mathbb{C}$) for the double category of endomorphisms, horizontal endomorphism maps, vertical endomorphism maps and endomorphism squares. We omit the straightforward verification that End($\mathbb{C}$) is indeed a double category.

Definition 2.4. Let $\mathbb{C}$ be a double category.

 (i) A *monad* is an endomorphism (X, P) equipped with squares

$$
\begin{array}{ccccc}
X & \xrightarrow{P} & X & \xrightarrow{P} & X \\
\| & & \mu_P & & \| \\
X & & \xrightarrow{P} & & X
\end{array}
\qquad
\begin{array}{ccc}
X & = & X \\
\| & \eta_P & \| \\
X & \xrightarrow{P} & X
\end{array}
$$

satisfying the associativity law

$$
\begin{array}{ccc}
X \longrightarrow X \longrightarrow X \longrightarrow X & & X \longrightarrow X \longrightarrow X \longrightarrow X \\
\big\| \quad\; \mu_P \quad \big\| \quad\;\; \big\| & & \big\| \quad\;\; \big\| \quad\; \mu_P \quad \big\| \\
X \longrightarrow X \longrightarrow X & = & X \longrightarrow X \longrightarrow X \\
\big\| \quad\; \mu_P \quad\;\; \big\| & & \big\| \quad\; \mu_P \quad\;\; \big\| \\
X \longrightarrow X & & X \longrightarrow X
\end{array}
$$

and the unit laws

$$
\begin{array}{ccccc}
X \longrightarrow X = X & & & & X = X \longrightarrow X \\
\big\| \quad\;\; \big\| \;\eta_P\; \big\| & & X \longrightarrow X & & \big\| \;\eta_P\; \big\| \quad\;\; \big\| \\
X \longrightarrow X \longrightarrow X & = & \big\| \qquad \big\| & = & X \longrightarrow X \longrightarrow X \\
\big\| \quad\; \mu_P \quad\;\; \big\| & & X \longrightarrow X & & \big\| \quad\; \mu_P \quad\;\; \big\| \\
X \longrightarrow X & & & & X \longrightarrow X.
\end{array}
$$

As before, we refer to a monad as above by mentioning only its underlying endomorphism (X, P).

(ii) A *horizontal monad map* $(F, \phi) : (X, P) \to (Y, Q)$ is a horizontal endomorphism map between the underlying endomorphisms satisfying the following conditions:

$$
\begin{array}{ccc}
\begin{array}{c}
X \longrightarrow Y \longrightarrow Y \longrightarrow Y \\
\big\| \quad\;\; \big\| \quad\; \mu_Q \quad \big\| \\
X \longrightarrow Y \longrightarrow Y \\
\big\| \quad\; \phi \quad\;\; \big\| \\
X \longrightarrow X \longrightarrow Y
\end{array}
& = &
\begin{array}{c}
X \longrightarrow Y \longrightarrow Y \longrightarrow Y \\
\big\| \quad\; \phi \quad\;\; \big\| \quad\;\; \big\| \\
X \longrightarrow X \longrightarrow Y \longrightarrow Y \\
\big\| \quad\;\; \big\| \quad\; \phi \quad\;\; \big\| \\
X \longrightarrow X \longrightarrow X \longrightarrow Y \\
\big\| \quad\; \mu_P \quad\;\; \big\| \quad\;\; \big\| \\
X \longrightarrow X \longrightarrow Y
\end{array}
\end{array}
$$

$$
\begin{array}{ccc}
\begin{array}{c}
X \longrightarrow Y = Y \\
\big\| \quad\;\; \big\| \;\eta_Q\; \big\| \\
X \longrightarrow Y \longrightarrow Y \\
\big\| \quad\; \phi \quad\;\; \big\| \\
X \longrightarrow X \longrightarrow Y
\end{array}
& = &
\begin{array}{c}
X = X \longrightarrow Y \\
\big\| \;\eta_P\; \big\| \quad\;\; \big\| \\
X \longrightarrow X \longrightarrow Y.
\end{array}
\end{array}
$$

(iii) A *vertical monad map* $(u, \bar{u}) : (X, P) \to (X', P')$ is a vertical endomorphism map between the underlying endomorphisms satisfying the following conditions:

$$
\begin{array}{ccc}
\begin{array}{c}
X \longrightarrow X \longrightarrow X \\
\big\| \quad\; \mu_P \quad\;\; \big\| \\
X \longrightarrow X \\
\big\downarrow \quad\; \bar{u} \quad\;\; \big\downarrow \\
X' \longrightarrow X'
\end{array}
& = &
\begin{array}{c}
X \longrightarrow X \longrightarrow X \\
\big\downarrow \;\bar{u}\; \big\downarrow \;\bar{u}\; \big\downarrow \\
X' \longrightarrow X' \longrightarrow X' \\
\big\| \quad\; \mu_{P'} \quad\;\; \big\| \\
X' \longrightarrow X'
\end{array}
\end{array}
$$

$$
\begin{array}{ccc}
X \rule[0.5ex]{4em}{0.4pt} X & & X \rule[0.5ex]{4em}{0.4pt} X \\
\parallel \quad \eta_P \quad \parallel & & \downarrow \qquad \downarrow \\
X \longrightarrow X & = & X' \rule[0.5ex]{4em}{0.4pt} X' \\
\downarrow \quad \bar{u} \quad \downarrow & & \parallel \quad \eta_{P'} \quad \parallel \\
X' \longrightarrow X' & & X' \longrightarrow X'.
\end{array}
$$

(iv) A *monad square* is an endomorphism square between the underlying endomorphism maps.

We write Mnd($\mathbb{C}$) for the double category of monads, horizontal monad maps, vertical monad maps and monad squares; again, it is straightforward to check that Mnd($\mathbb{C}$) is a double category. Before giving examples, we clarify the relationship between our definitions and those in [23].

Remark 2.5. Let $\mathcal{K}$ be a 2-category and consider the double category $\mathbb{H}(\mathcal{K})$ that has $\mathcal{K}$ as its horizontal 2-category and only identity 1-cells as vertical arrows. Monads in $\mathcal{K}$ are the same as monads in $\mathbb{H}(\mathcal{K})$ and monad maps in $\mathcal{K}$ are the same as horizontal monad maps in $\mathbb{H}(\mathcal{K})$. Finally, monad 2-cells in $\mathcal{K}$ are the same as monad squares in $\mathbb{H}(\mathcal{K})$ of the special form

$$
\begin{array}{ccc}
(X, P) & \xrightarrow{(F, \phi)} & (Y, Q) \\
\parallel & \quad \alpha \quad & \parallel \\
(X, P) & \xrightarrow[(F', \phi')]{} & (Y, Q).
\end{array}
$$

In particular, the horizontal 2-category of Mnd($\mathbb{H}(\mathcal{K})$) is the 2-category Mnd($\mathcal{K}$) of [23]. As we explain in the following examples, the presence of non-trivial vertical arrows in a double category allows us to describe important mathematical structures as vertical monad maps.

Example 2.6. Let $\mathcal{E}$ be a category with finite limits. The category **Grph**$_{\mathcal{E}}$ of graphs and graph morphisms internal to $\mathcal{E}$ can be identified with the category of endomorphisms and vertical endomorphism maps in the double category **Span**$_{\mathcal{E}}$, while the category **Cat**$_{\mathcal{E}}$ of categories and functors internal to $\mathcal{E}$ can be identified with the category of monads and vertical monad maps in **Span**$_{\mathcal{E}}$. We see here an example of the benefits of considering monads in a double category rather than in a 2-category: while categories can be seen as monads in the bicategory of spans in $\mathcal{E}$, functors between categories are not the same as monad maps in that bicategory.

Example 2.7. Let $\mathcal{E}$ be a locally cartesian closed category with finite disjoint coproducts and W-types. We write **PolyEnd**$_{\mathcal{E}}$ for the category of endomorphisms and vertical endomorphism maps in the double category **Poly**$_{\mathcal{E}}$ and write **PolyMnd**$_{\mathcal{E}}$ for the category of monads and vertical monad maps in **Poly**$_{\mathcal{E}}$. If $M : \mathcal{E} \to \mathcal{E}$ is the free monoid monad in $\mathcal{E}$ (which exists by the assumptions on $\mathcal{E}$), then there is a double category **PolyEnd**$_{\mathcal{E}}/M$ whose objects are endomorphisms with a vertical endomorphism map to M. This is the double category of M-spans in the sense of [3,17], while **PolyMnd**$_{\mathcal{E}}$ is the double category of multicategories. The free monad on an endofunctor over M is the free multicategory on an M-span. Furthermore, the vertical maps in **PolyMnd**$_{\mathcal{E}}$ are the multifunctors, and hence we see again the benefits of considering monads in the double categories rather than just in 2-categories. Further variations are possible: with a polynomial monad T in the place of M we get the same result for T-spans and T-multicategories, and in the particular case where T is the identity monad, we are back to just plain categories in $\mathcal{E}$.

The function sending a monad (X, P) to its underlying object X extends to a double functor Und : Mnd($\mathbb{C}$) $\to \mathbb{C}$ and the function mapping an object $X \in \mathbb{C}$ to the identity monad $(X, 1_X)$ extends to a double functor Inc : $\mathbb{C} \to$ Mnd($\mathbb{C}$). It is easy to check that Inc is a horizontal right adjoint to Und, essentially as in the 2-categorical formal theory of monads [23, Theorem 1]. The question of when $\mathbb{C}$ admits the construction of Eilenberg–Moore objects, that is, of when the double functor Inc has a horizontal right adjoint, will be treated in a sequel to this paper. Here, instead, we focus on the construction of free monads.

Free monads in a double category. We write U : Mnd($\mathbb{C}$) $\to$ End($\mathbb{C}$) for the forgetful double functor mapping a monad to its underlying endomorphism.

Definition 2.8. A double category $\mathbb{C}$ is said to *admit the construction of free monads* if U : Mnd($\mathbb{C}$) $\to$ End($\mathbb{C}$) has a vertical left adjoint.

Remark 2.9. We now make explicit what it means for a double category $\mathbb{C}$ to admit the construction of free monads. By an analogue of the characterization of ordinary adjunctions in terms of universal arrows [16, Theorem IV.2], giving a vertical left adjoint to U amounts to giving the following data in (i)–(iv) satisfying the functoriality condition in (∗).

(i) For every endomorphism (X, P), a monad (X^*, P^*).

(ii) For every endomorphism (X, P), a universal vertical endomorphism map

   $(\iota_X, \iota_P) : (X, P) \to (X^*, P^*)$.

Universality means that for each vertical endomorphism map $(u, \bar{u}): (X, P) \to (X', P')$, where (X', P') is a monad, there exists a unique vertical monad map $(u^\sharp, \bar{u}^\sharp) : (X^*, P^*) \to (X', P')$ such that

$$
\begin{array}{ccc}
X \xrightarrow{\;P\;} X \\
u \downarrow \quad \bar{u} \quad \downarrow u \qquad = \\
X' \xrightarrow{\;P'\;} X'
\end{array}
\qquad
\begin{array}{ccc}
X \xrightarrow{\;P\;} X \\
\iota_X \downarrow \quad \iota_P \quad \downarrow \iota_X \\
X^* \longrightarrow X^* \\
u^\sharp \downarrow \quad \bar{u}^\sharp \quad \downarrow u^\sharp \\
X' \xrightarrow{\;P'\;} X'.
\end{array}
$$

(iii) For every horizontal endomorphism map $(F, \phi): (X, P) \to (Y, Q)$, a horizontal monad map $(F^*, \phi^*): (X^*, P^*) \to (Y^*, Q^*)$.

(iv) For every horizontal endomorphism map $(F, \phi) : (X, P) \to (Y, Q)$, a universal endomorphism square

$$
\begin{array}{ccc}
(X, P) & \xrightarrow{(F, \phi)} & (Y, Q) \\
(\iota_X, \iota_P) \downarrow & {}^{\iota_{(F,\phi)}} & \downarrow (\iota_Y, \iota_Q) \\
(X^*, P^*) & \xrightarrow[(F^*, \phi^*)]{} & (Y^*, Q^*).
\end{array}
$$

Universality means that for every endomorphism square

$$
\begin{array}{ccc}
(X, P) & \xrightarrow{(F, \phi)} & (Y, Q) \\
(u, \bar{u}) \downarrow & \alpha & \downarrow (v, \bar{v}) \\
(X', P') & \xrightarrow[(F', \phi')]{} & (Y', Q'),
\end{array}
\qquad (10)
$$

where (X', P'), (Y', Q') are monads and $(F, \phi') : (X', P') \to (Y', Q')$ is a horizontal monad map, there exists a unique monad square

$$
\begin{array}{ccc}
(X^*, P^*) & \xrightarrow{(F^*, \phi^*)} & (Y^*, Q^*) \\
(u^\sharp, \bar{u}^\sharp) \downarrow & \alpha^\sharp & \downarrow (v^\sharp, \bar{v}^\sharp) \\
(X', P') & \xrightarrow[(F', \phi')]{} & (Y', Q')
\end{array}
$$

such that

$$
\begin{array}{ccc}
(X, P) & \xrightarrow{(F, \phi)} & (Y, Q) \\
(u, \bar{u}) \downarrow & \alpha & \downarrow (v, \bar{v}) \\
(X', P') & \xrightarrow[(F', \phi')]{} & (Y', Q')
\end{array}
\quad = \quad
\begin{array}{ccc}
(X, P) & \xrightarrow{(F, \phi)} & (Y, Q) \\
(\iota_X, \iota_P) \downarrow & {}^{\iota_{(F,\phi)}} & \downarrow (\iota_Y, \iota_Q) \\
(X^*, P^*) & \longrightarrow & (Y^*, Q^*) \\
(u^\sharp, \bar{u}^\sharp) \downarrow & \alpha^\sharp & \downarrow (v^\sharp, \bar{v}^\sharp) \\
(X', P') & \xrightarrow[(F', \phi')]{} & (Y', Q').
\end{array}
$$

($*$) The assignments in (i) and (iii) give a functor

   $(-)^* : \big(\mathrm{Obj}_{\mathrm{End}(\mathbb{C})}, \mathrm{Hor}_{\mathrm{End}(\mathbb{C})}\big) \to \big(\mathrm{Obj}_{\mathrm{Mnd}(\mathbb{C})}, \mathrm{Hor}_{\mathrm{Mnd}(\mathbb{C})}\big)$

and the assignments in (ii) and (iv) give a functor

   $\iota : \big(\mathrm{Obj}_{\mathrm{End}(\mathbb{C})}, \mathrm{Hor}_{\mathrm{End}(\mathbb{C})}\big) \to \big(\mathrm{Ver}_{\mathrm{End}(\mathbb{C})}, \mathrm{Sq}_{\mathrm{End}(\mathbb{C})}\big)$.

Note that the data and the universality in (ii) actually follow from the data and the universality in (iv) by taking (F, ϕ) to be the horizontal identity on an endomorphism (X, P).

A necessary condition for $U : \mathrm{Mnd}(\mathbb{C}) \to \mathrm{End}(\mathbb{C})$ to have a vertical left adjoint is that its vertical part

$$U_0 : \mathrm{Mnd}(\mathbb{C})_0 \to \mathrm{End}(\mathbb{C})_0 \tag{11}$$

has a left adjoint. Indeed, this is precisely what items (i) and (ii) of Remark 2.9 amount to. Here, $\mathrm{End}(\mathbb{C})_0$ denotes the category of endomorphisms and vertical endomorphism maps and $\mathrm{Mnd}(\mathbb{C})_0$ denotes the category of monads and vertical monad maps.

Example 2.10. Let $\mathcal{E}$ be a category with finite limits. The functor in (11) for the double category **Span**$_{\mathcal{E}}$ is the forgetful functor $U_0 : \mathbf{Cat}_{\mathcal{E}} \to \mathbf{Grph}_{\mathcal{E}}$ mapping a category in $\mathcal{E}$ to its underlying graph.

Example 2.11. Let $\mathcal{E}$ be a locally cartesian closed category. The functor in (11) for the double category **Poly**$_{\mathcal{E}}$ is the forgetful functor $U_0 : \mathbf{PolyMnd}_{\mathcal{E}} \to \mathbf{PolyEnd}_{\mathcal{E}}$ mapping a polynomial monad to its underlying endofunctor.

3. Monads in a framed bicategory

We now proceed to establish some properties of the double category $\mathrm{Mnd}(\mathbb{C})$ under the assumption that $\mathbb{C}$ is a framed bicategory, leading to our main theorem (Theorem 3.7 below), which provides conditions for $\mathbb{C}$ to admit the construction of free monads. We begin by recalling from [21] the definition of a framed bicategory and some useful facts.

Framed bicategories. For a double category $\mathbb{C}$, the functor

$$(\partial_0, \partial_1) : \mathbb{C}_1 \to \mathbb{C}_0 \times \mathbb{C}_0,$$

mapping a horizontal arrow $F : X \to Y$ to (X, Y) and a square as in (9) to $(u, v) : (X, Y) \to (X', Y')$, is a Grothendieck fibration if and only if it is a Grothendieck opfibration [21, Theorem 4.1]. When these conditions hold, the double category $\mathbb{C}$ is said to be a framed bicategory [21, Definition 4.2]. As explained in [21, Example 4.4] and [8, Proposition 3.6], the double categories **Span**$_{\mathcal{E}}$ and **Poly**$_{\mathcal{E}}$ are framed bicategories.

Lemma 3.1 (*Shulman*). *If $\mathbb{C}$ is a framed bicategory, for every vertical arrow $u : X \to X'$ there exist horizontal arrows $u_! : X \to X'$ and $u^* : X' \to X$ together with squares*

satisfying the equalities

and

Proof. See [21, Theorem 4.1]. $\square$

Lemma 3.1 can be expressed equivalently by saying that every vertical arrow u in $\mathbb{C}$ has an orthogonal companion $u_!$ and an orthogonal adjoint u^* in the terminology of [11].

Lemma 3.2 (*Shulman*). *Let $\mathbb{C}$ be a framed bicategory. Let $u : X \to X'$ be a vertical arrow in $\mathbb{C}$. If we define*

and

then the following versions of the triangle identities hold:

Proof. See [21, Proposition 5.3]. □

Monads in framed bicategories. Let $\mathbb{C}$ be a double category. We have the diagram

$$\mathrm{Mnd}(\mathbb{C})_0 \xrightarrow{\ U_0\ } \mathrm{End}(\mathbb{C})_0$$
$$\partial_M \searrow \quad \swarrow \partial_E$$
$$\mathbb{C}_0, \tag{12}$$

where ∂_E and ∂_M send an endomorphism and a monad, respectively, to their underlying object and U_0 is the vertical part of the forgetful double functor U of Definition 2.8.

Proposition 3.3. *If $\mathbb{C}$ is a framed bicategory, the functors*

$$\partial_E : \mathrm{End}(\mathbb{C})_0 \to \mathbb{C}_0, \qquad \partial_M : \mathrm{Mnd}(\mathbb{C})_0 \to \mathbb{C}_0$$

are Grothendieck fibrations and the functor $U_0 : \mathrm{Mnd}(\mathbb{C})_0 \to \mathrm{End}(\mathbb{C})_0$ is a fibered functor relatively to these fibrations.

Proof. Writing $\Delta : \mathbb{C}_0 \to \mathbb{C}_0 \times \mathbb{C}_0$ for the diagonal functor, the functor ∂_E fits into the pullback diagram

$$\mathrm{End}(\mathbb{C})_0 \longrightarrow \mathbb{C}_1$$
$$\partial_E \downarrow \qquad\qquad \downarrow (\partial_0, \partial_1)$$
$$\mathbb{C}_0 \xrightarrow{\ \Delta\ } \mathbb{C}_0 \times \mathbb{C}_0.$$

We then have that ∂_E is a Grothendieck fibration since it is a pullback of (∂_0, ∂_1), which is a Grothendieck fibration by the hypothesis that $\mathbb{C}$ is a framed bicategory. Using Lemmas 3.1 and 3.2, we can define explicitly a base change operation for the

Grothendieck fibration ∂_E, as follows. Let $u : X \to X'$ be a map in $\mathbb{C}_0$ and (X', P') an endomorphism in $\mathbb{C}$. The base change of (X', P') along u is defined to be the endomorphism (X, P), where $P : X \to X$ is the composite

$$X \xrightarrow{u_!} X' \xrightarrow{P'} X' \xrightarrow{u^*} X \ .$$

The required cartesian morphism from (X, P) to (X', P') in $\mathrm{End}(\mathbb{C})_0$ (*i.e.* the cartesian lift of u) is given by the vertical endomorphism map $(u, \bar{u}) : (X, P) \to (X', P')$, where $\bar{u}$ is the square

$$
\begin{array}{ccccccc}
X & \xrightarrow{u_!} & X' & \xrightarrow{P'} & X' & \xrightarrow{u^*} & X \\
{\scriptstyle u}\downarrow & {\scriptstyle \alpha_u} & \| & & \| & {\scriptstyle \beta_u} & \downarrow {\scriptstyle u} \\
X' & = & X' & \xrightarrow{P'} & X' & = & X'.
\end{array}
$$

The verification of the required universal property is straightforward.

To show that ∂_M is a Grothendieck fibration, we first observe that if (X', P') is a monad, then (X, P) inherits a monad structure: its multiplication is the square

$$
\begin{array}{ccccccccccccc}
X & \xrightarrow{u_!} & X' & \xrightarrow{P'} & X' & \xrightarrow{u^*} & X & \xrightarrow{u_!} & X' & \xrightarrow{P'} & X' & \xrightarrow{u^*} & X \\
\| & & \| & & \| & & & {\scriptstyle \varepsilon_u} & & \| & & \| & \| \\
X & \to & X' & \xrightarrow{P'} & X' & = & & & X' & \xrightarrow{P'} & X' & \to & X \\
\| & & \| & & & & {\scriptstyle \mu} & & & & \| & & \| \\
X & \xrightarrow{u_!} & X' & & & & \xrightarrow{P'} & & & & X' & \xrightarrow{u^*} & X
\end{array}
$$

and its unit is the square

$$
\begin{array}{ccccccc}
X & = & X & = & X & = & X \\
\| & & & {\scriptstyle \eta_u} & & & \| \\
X & \xrightarrow{u_!} & X' & = & X' & \xrightarrow{u^*} & X \\
\| & & \| & {\scriptstyle \eta} & \| & & \| \\
X & \xrightarrow{u_!} & X' & \xrightarrow{P'} & X' & \xrightarrow{u^*} & X.
\end{array}
$$

The monad axioms are easily verified. Now it only remains to verify that the cartesian lift $(u, \bar{u})$ is a vertical monad map and that it is cartesian for ∂_M. This verification is straightforward, using Lemmas 3.1 and 3.2. This also shows that U_0 is fibered as claimed. $\square$

Lemma 3.4. *Let (X, P) and (X', P') be endomorphisms in a framed bicategory $\mathbb{C}$. There is a bijection between vertical endomorphism maps $(u, \bar{u}) : (X, P) \to (X', P')$ and horizontal endomorphism maps of the form $(u^*, \phi) : (X', P') \to (X, P)$, which restricts to a bijection between vertical monad maps and horizontal monad maps when (X, P) and (X', P') are monads.*

Proof. For a vertical endomorphism map $(u, \bar{u}) : (X, P) \to (X', P')$, define the horizontal endomorphism map $(u^*, \phi_u) : (X', P') \to (X, P)$ by letting ϕ_u be the square

$$
\begin{array}{ccccc}
X' & \xrightarrow{u^*} & X & \xrightarrow{P} & X \\
\| & & {\scriptstyle \phi_u} & & \| \\
X' & \xrightarrow{P'} & X' & \xrightarrow{u^*} & X
\end{array}
\quad =_{\mathrm{def}} \quad
\begin{array}{ccccccc}
X' & \xrightarrow{u^*} & X & \xrightarrow{P} & X & = & X \\
\| & {\scriptstyle \beta_u} & \downarrow & {\scriptstyle \bar{u}} & \downarrow & {\scriptstyle \gamma_u} & \| \\
X' & = & X' & \xrightarrow{P'} & X' & \xrightarrow{u^*} & X.
\end{array}
$$

In the other direction, given a horizontal endomorphism map $(u^*, \phi)\colon (X', P') \to (X, P)$, define the vertical endomorphism map $(u, \bar{u}_\phi)\colon (X, P) \to (X', P')$ by letting $\bar{u}_\phi$ be the square

$$
\begin{array}{ccc}
X \xrightarrow{\ P\ } X \\
u \downarrow \quad \bar{u}_\phi \quad \downarrow u \quad =_{\text{def}} \\
X' \xrightarrow[P']{} X'
\end{array}
\qquad
\begin{array}{ccccc}
X & =\!=\!= & X & \xrightarrow{\ P\ } & X \\
u\downarrow & \gamma_u & \| & & \| \\
X' & \longrightarrow & X & \longrightarrow & X \\
\| & & \phi & & \| \\
X' & \longrightarrow & X' & \longrightarrow & X \\
\| & & \| & \beta_u & \downarrow u \\
X' & \xrightarrow[P']{} & X' & =\!=\!= & X'.
\end{array}
$$

Using Lemma 3.1, it is possible to show that these functions are mutually inverse, that (u^*, ϕ_u) is a horizontal monad map if $(u, \bar{u})$ is a vertical monad map, and that $(u, \bar{u}_\phi)$ is a vertical monad map if (u^*, ϕ) is a horizontal monad map. $\quad\square$

Let us point out that the bijection defined in the proof of Lemma 3.4 is an example of a cofolding in the sense of [5, Definition 3.16].

Free monads in a framed bicategory. We now consider the construction of free monads in a framed bicategory $\mathbb{C}$. Since the functor U_0 in (12) is fibered, a sufficient condition for it to have a left adjoint is that each of its fibers has a left adjoint. In this case, the free monad on an endomorphism (X, P) has the form (X, P^*) and the component of the unit $(\iota_X, \iota_P)\colon (X, P) \to (X, P^*)$ is a vertical endomorphism map of the form $(1_X, \iota_P)\colon (X, P) \to (X, P^*)$, where ι_P is a square of the form

$$
\begin{array}{ccc}
X & \xrightarrow{\ P\ } & X \\
\| & \iota_P & \| \\
X & \xrightarrow[P^*]{} & X.
\end{array}
$$

The universal property in the fiber asserts that for every endomorphism square of the form

$$
\begin{array}{ccc}
X & \xrightarrow{\ P\ } & X \\
\| & \alpha & \| \\
X & \xrightarrow[P']{} & X,
\end{array}
$$

where (X, P') is a monad, there exists a unique monad square of the form

$$
\begin{array}{ccc}
X & \xrightarrow{\ P^*\ } & X \\
\| & \alpha^\sharp & \| \\
X & \xrightarrow[P']{} & X
\end{array}
$$

such that

$$
\begin{array}{ccc}
X & \xrightarrow{\ P\ } & X \\
\| & \alpha & \| \\
X & \xrightarrow[P']{} & X
\end{array}
\quad = \quad
\begin{array}{ccc}
X & \xrightarrow{\ P\ } & X \\
\| & \iota_P & \| \\
X & \longrightarrow & X \\
\| & \alpha^\sharp & \| \\
X & \xrightarrow[P']{} & X.
\end{array}
$$

The universal property in the fiber implies a more general universal property, with respect to general monads (and not just monads with X as underlying object) and general vertical endomorphism maps (and not just the special ones considered above), as in item (ii) of Remark 2.9. Note, however, that the left adjoint to U_0 constructed from the left adjoints to its fibers need not be a fibered left adjoint, since the so-called Beck–Chevalley conditions are not necessarily satisfied [12, Section 1.8].

Example 3.5. Let us consider the framed bicategory **Span**$_\mathcal{E}$ associated with a category $\mathcal{E}$ with finite limits. The diagram in (12) becomes

$$
\begin{array}{ccc}
\mathbf{Cat}_\mathcal{E} & \xrightarrow{\ U_0\ } & \mathbf{Grph}_\mathcal{E} \\
& \searrow \partial_M \qquad \swarrow \partial_E & \\
& \mathcal{E}, &
\end{array}
$$

where ∂_E sends a graph to its object of vertices and ∂_M sends a small category to its object of objects. Since **Span**$_\mathcal{E}$ is a framed bicategory, the preceding remarks reduce to the familiar fact that the free category on a graph has the object of vertices of the graph as its object of objects.

Example 3.6. Let us consider the framed bicategory **Poly**$_\mathcal{E}$ associated with a locally cartesian closed category $\mathcal{E}$ with finite disjoint coproducts. For **Poly**$_\mathcal{E}$, the diagram in (12) amounts to

$$
\begin{array}{ccc}
\mathbf{PolyMnd}_\mathcal{E} & \xrightarrow{\ U_0\ } & \mathbf{PolyEnd}_\mathcal{E} \\
& \searrow \partial_M \qquad \swarrow \partial_E & \\
& \mathcal{E}. &
\end{array}
$$

In this case, the remarks above amount to the fact, exploited in the proof of [8, Corollary 4.7], that to prove the universal property of the free monad on a polynomial endofunctor with respect to maps in **PolyEnd**$_\mathcal{E}$, it is sufficient to check it with respect to a special class of them.

Theorem 3.7, which is our main result, gives sufficient conditions for a framed bicategory to admit the construction of free monads, facilitating the verification of this property in our examples. Recall that, for a double category $\mathbb{C}$, we write $\mathbb{C}_0$ for its category of objects and vertical arrows and $\mathbb{C}_1$ for its category of horizontal arrows and squares.

Theorem 3.7. *Let $\mathbb{C}$ be a framed bicategory such that the category $\mathbb{C}_1$ has equalizers and the source and target functors $\partial_0, \partial_1 : \mathbb{C}_1 \to \mathbb{C}_0$ preserve them. If the horizontal 2-category of $\mathbb{C}$ has local coproducts and admits the construction of free monads, then $\mathbb{C}$ admits the construction of free monads.*

The proof of Theorem 3.7 is given in Section 4. Here, instead, we apply it to our two running examples.

Proposition 3.8.

 (i) *If $\mathcal{E}$ is a pretopos with parametrized list objects, the double category **Span**$_\mathcal{E}$ admits the construction of free monads.*
(ii) *If $\mathcal{E}$ is a locally cartesian closed category with disjoint coproducts and W-types, the double category **Poly**$_\mathcal{E}$ admits the construction of free monads.*

Proof. For both (i) and (ii), we apply Theorem 3.7. For (i), the hypotheses on equalizers are verified because in this case the category $\mathbb{C}_1$ is a category of internal presheaves, in which equalizers exist and are computed pointwise. For (ii), the hypotheses on equalizers are also satisfied, since pullbacks preserve equalizers. For both items, the existence of free monads in the horizontal 2-categories is established in Proposition 1.6. □

4. Proof of the main theorem

Let $\mathbb{C}$ be a double category satisfying the hypotheses of Theorem 3.7. We use the characterization of free monads in a 2-category given in Theorem 1.1 to exhibit the data listed in Remark 2.9. For items (i) and (ii) of Remark 2.9, let (X, P) be an endomorphism. By the existence of free monads in $\mathcal{H}_\mathbb{C}$, we have a monad (X, P^*) and a square

$$
\begin{array}{ccc}
X & \xrightarrow{\ P\ } & X \\
\| & \iota_P & \| \\
X & \xrightarrow{\ P^*\ } & X
\end{array}
$$

satisfying the equivalent conditions in items (i) and (ii) of Theorem 1.1 in $\mathcal{H}_\mathbb{C}$. We then obtain a vertical endomorphism map $(1_X, \iota_P) : (X, P) \to (X, P^*)$. We need to show that $(1_X, \iota_P)$ enjoys the required universal property. For this, let us consider a vertical endomorphism map $(u, \bar{u}) : (X, P) \to (X', P')$, where (X', P') is a monad. Here, $\bar{u}$ is a square of the form

$$
\begin{array}{ccc}
X & \xrightarrow{\ P\ } & X \\
u \downarrow & \bar{u} & \downarrow u \\
X' & \xrightarrow{\ P'\ } & X'.
\end{array}
$$

By the cofolding bijection defined in the proof of Lemma 3.4, we have a horizontal endomorphism map $(u^*, \phi_u) : (X', P') \to (X, P)$, where ϕ_u is a square of the form

$$
\begin{array}{ccccc}
X' & \xrightarrow{u^*} & X & \xrightarrow{P} & X \\
\| & & \phi_u & & \| \\
X' & \xrightarrow{P'} & X' & \xrightarrow{u^*} & X.
\end{array}
$$

By the universal property in item (i) of Theorem 1.1 for (X, P^*), there exists a unique square

$$
\begin{array}{ccccc}
X' & \xrightarrow{u^*} & X & \xrightarrow{P^*} & X \\
\| & & \phi_u^\sharp & & \| \\
X' & \xrightarrow{P'} & X' & \xrightarrow{u^*} & X
\end{array}
$$

such that $(u^*, \phi_u^\sharp) : (X', P') \to (X, P^*)$ is a horizontal monad map and

$$
\begin{array}{ccccc}
X' & \xrightarrow{u^*} & X & \xrightarrow{P} & X \\
\| & & \phi_u & & \| \\
X' & \xrightarrow{P'} & X' & \xrightarrow{u^*} & X
\end{array}
\quad = \quad
\begin{array}{ccccc}
X' & \xrightarrow{u^*} & X & \xrightarrow{P} & X \\
\| & & \| & \iota_P & \| \\
X' & \longrightarrow & X & \longrightarrow & X \\
\| & & \phi_u^\sharp & & \| \\
X' & \xrightarrow{P'} & X' & \xrightarrow{u^*} & X.
\end{array}
$$

Using again the cofolding bijection of Lemma 3.4, we obtain the vertical monad morphism $(u, \bar{u}^\sharp) : (X, P^*) \to (X', P')$ that factors $(u, \bar{u})$ through $(1_X, \iota_P)$, as required. By the definition of the bijection and Theorem 1.1, the square $\bar{u}^\sharp$ satisfies the equations

$$
\begin{array}{ccc}
X == X & & X == X \\
\| \;\eta_{P*}\; \| & & u \downarrow \quad \downarrow u \\
X \longrightarrow X & = & X' == X' \\
u \downarrow \;\bar{u}^\sharp\; \downarrow u & & \| \;\eta_{P'}\; \| \\
X' \xrightarrow{P'} X' & & X' \xrightarrow{P'} X'
\end{array}
\tag{13}
$$

and

$$
\begin{array}{ccc}
X \xrightarrow{P^*} X \xrightarrow{P} X & & X \xrightarrow{P^*} X \xrightarrow{P} X \\
\| \quad \nu_{P*} \quad \| & & \downarrow \;\bar{u}^\sharp\; \downarrow \;\bar{u}\; \downarrow \\
X \longrightarrow X & = & X' \longrightarrow X' \longrightarrow X' \\
\downarrow \;\bar{u}^\sharp\; \downarrow & & \| \quad \mu_{P'} \quad \| \\
X' \xrightarrow{P'} Y & & X' \xrightarrow{P'} X',
\end{array}
\tag{14}
$$

where the square ν_{P*} is defined by

$$
\begin{array}{ccc}
X \xrightarrow{P^*} X \xrightarrow{P} X & & X \xrightarrow{P^*} X \xrightarrow{P} X \\
\| \quad \nu_{P*} \quad \| & =_{\text{def}} & \| \quad \| \;\iota_P\; \| \\
X \xrightarrow{P^*} X & & X \longrightarrow X \longrightarrow X \\
& & \| \quad \mu_{P*} \quad \| \\
& & X \xrightarrow{P^*} X.
\end{array}
$$

For item (iii) of Remark 2.9, let $(F, \phi) : (X, P) \to (Y, Q)$ be a horizontal endomorphism map. Exploiting the universal property in item (i) of Theorem 1.1 for (Y, Q^*), we define

$$
\begin{array}{ccc}
X & \xrightarrow{F} Y & \xrightarrow{Q^*} Y \\
\| & \phi^* & \| \\
X & \xrightarrow{P^*} X & \xrightarrow{F} X
\end{array}
$$

to be the unique square such that $(F, \phi^*) : (X, P^*) \to (Y, Q^*)$ is a horizontal monad map and

$$
\begin{array}{ccc}
X \xrightarrow{F} Y \xrightarrow{Q} Y \\
\| \quad \phi \quad \| \\
X \longrightarrow X \longrightarrow Y \\
\| \quad \iota_P \quad \| \quad \| \\
X \xrightarrow{P^*} X \xrightarrow{F} Y
\end{array}
\;=\;
\begin{array}{ccc}
X \xrightarrow{F} Y \xrightarrow{Q} Y \\
\| \quad \| \quad \iota_Q \quad \| \\
X \longrightarrow Y \longrightarrow Y \\
\| \quad \phi^* \quad \| \\
X \xrightarrow{P^*} X \xrightarrow{F} X
\end{array}
\tag{15}
$$

Observe that, by the fact that (F, ϕ^*) is a horizontal monad map, we have that

$$
\begin{array}{ccc}
X \xrightarrow{F} Y = Y \\
\| \quad \| \quad \eta_{Q^*} \quad \| \\
X \longrightarrow Y \longrightarrow Y \\
\| \quad \phi^* \quad \| \\
X \xrightarrow{P^*} X \xrightarrow{F} Y
\end{array}
\;=\;
\begin{array}{ccc}
X = X \xrightarrow{F} Y \\
\| \quad \eta_{P^*} \quad \| \quad \| \\
X \xrightarrow{P^*} X \xrightarrow{F} Y
\end{array}
\tag{16}
$$

and

$$
\begin{array}{ccc}
X \xrightarrow{F} Y \xrightarrow{Q^*} Y \xrightarrow{Q} Y \\
\| \quad \| \quad \nu_{Q^*} \quad \| \\
X \longrightarrow Y \longrightarrow Y \\
\| \quad \phi^* \quad \| \\
X \xrightarrow{P^*} X \xrightarrow{F} Y
\end{array}
\;=\;
\begin{array}{ccc}
X \xrightarrow{F} Y \xrightarrow{Q^*} Y \xrightarrow{Q} Y \\
\| \quad \phi^* \quad \| \quad \| \\
X \longrightarrow X \longrightarrow Y \longrightarrow Y \\
\| \quad \| \quad \phi \quad \| \\
X \longrightarrow X \longrightarrow X \longrightarrow Y \\
\| \quad \nu_{P^*} \quad \| \quad \| \\
X \xrightarrow{P^*} X \xrightarrow{F} Y
\end{array}
\tag{17}
$$

In particular, (17) holds by the definitions of ν_{P^*} and ν_{Q^*}, the first axiom for a horizontal monad map and (15). For item (iv) of Remark 2.9, the required universal endomorphism square needs to have the form

$$
\begin{array}{ccc}
(X, P) & \xrightarrow{(F, \phi)} & (Y, Q) \\
(1_X, \iota_P) \downarrow & \iota_{(F, \phi)} & \downarrow (1_Y, \iota_Q) \\
(X, P^*) & \xrightarrow{(F, \phi^*)} & (Y, Q^*).
\end{array}
$$

Therefore, $\iota_{(F, \phi)}$ has to be a square in $\mathbb{C}$ of the form

$$
\begin{array}{ccc}
X & \xrightarrow{F} & Y \\
\| & \iota_{(F, \phi)} & \| \\
X & \xrightarrow{F} & Y
\end{array}
$$

and satisfy the equation

$$
\begin{array}{ccc}
\begin{array}{ccc}
X \xrightarrow{F} Y \xrightarrow{Q} Y \\
\parallel \quad \phi \quad \parallel \\
X \longrightarrow X \longrightarrow Y \\
\parallel \quad {}_{\iota_P} \quad {}_{\iota_{(F,\phi)}} \parallel \\
X \xrightarrow{P^*} X \xrightarrow{F} Y
\end{array}
& = &
\begin{array}{ccc}
X \xrightarrow{F} Y \xrightarrow{Q} Y \\
\parallel {}_{\iota_{(F,\phi)}} \parallel {}_{\iota_Q} \parallel \\
X \longrightarrow Y \longrightarrow Y \\
\parallel \quad \phi^* \quad \parallel \\
X \xrightarrow{P^*} X \xrightarrow{F} Y.
\end{array}
\end{array}
\tag{18}
$$

We define $\iota_{(F,\phi)}$ to be the identity square on F, so (18) above is verified by (15). To verify the universal property, we need to show that for an endomorphism square

$$
\begin{array}{ccc}
(X, P) & \xrightarrow{(F,\phi)} & (Y, Q) \\
{}_{(u,\bar{u})}\downarrow & \alpha & \downarrow{}_{(v,\bar{v})} \\
(X', P') & \xrightarrow[(F',\phi')]{} & (Y', Q'),
\end{array}
$$

there exists a unique monad square

$$
\begin{array}{ccc}
(X, P^*) & \xrightarrow{(F,\phi^*)} & (Y, Q^*) \\
{}_{(u,\bar{u}^\sharp)}\downarrow & \alpha^\sharp & \downarrow{}_{(v,\bar{v}^\sharp)} \\
(X', P') & \xrightarrow[(F',\phi')]{} & (Y', Q')
\end{array}
$$

satisfying

$$
\begin{array}{ccc}
(X, P) & \xrightarrow{(F,\phi)} & (Y, Q) \\
{}_{(u,\bar{u})}\downarrow & \alpha & \downarrow{}_{(v,\bar{v})} \\
(X', P') & \xrightarrow[(F',\phi')]{} & (Y', Q')
\end{array}
\quad = \quad
\begin{array}{ccc}
(X, P) & \xrightarrow{(F,\phi)} & (Y, Q) \\
{}_{(1_X,\iota_P)}\downarrow & {}_{\iota_{(F,\phi)}} & \downarrow{}_{(1_Y,\iota_Q)} \\
(X, P^*) & \longrightarrow & (Y, Q^*) \\
{}_{(u,\bar{u}^\sharp)}\downarrow & \alpha^\sharp & \downarrow{}_{(v,\bar{v}^\sharp)} \\
(X', P') & \xrightarrow[(F',\phi')]{} & (Y', Q').
\end{array}
\tag{19}
$$

First of all, observe that α is a square in $\mathbb{C}$ of the form

$$
\begin{array}{ccc}
X & \xrightarrow{F} & Y \\
{}_u\downarrow & \alpha & \downarrow{}_v \\
X' & \xrightarrow[F']{} & Y'
\end{array}
$$

which satisfies the compatibility condition

$$
\begin{array}{ccc}
\begin{array}{ccc}
X \xrightarrow{F} Y \xrightarrow{Q} Y \\
\parallel \quad \phi \quad \parallel \\
X \longrightarrow X \longrightarrow Y \\
\downarrow \quad {}_{\bar{u}} \quad \downarrow \quad \alpha \quad \downarrow \\
X' \xrightarrow{P'} X' \xrightarrow{F'} Y'
\end{array}
& = &
\begin{array}{ccc}
X \xrightarrow{F} Y \xrightarrow{Q} Y \\
\downarrow \quad \alpha \quad \downarrow \quad \bar{v} \quad \downarrow \\
X' \longrightarrow Y' \longrightarrow Y' \\
\parallel \quad \phi' \quad \parallel \\
X' \xrightarrow{P'} X' \xrightarrow{F'} Y'.
\end{array}
\end{array}
\tag{20}
$$

The required monad square $\alpha^\sharp$ has to be a square in $\mathbb{C}$ of the form

$$
\begin{array}{ccc}
X & \xrightarrow{F} & Y \\
u \downarrow & \alpha^\sharp & \downarrow v \\
X' & \xrightarrow[F']{} & Y'
\end{array}
$$

satisfying the compatibility condition

$$
\begin{array}{ccccc}
X & \xrightarrow{F} & Y & \xrightarrow{Q^*} & Y \\
\| & \phi^* & \| & & \\
X & \longrightarrow & X & \longrightarrow & Y \\
\downarrow & \bar{u}^\sharp & \alpha^\sharp & \downarrow & \\
X' & \xrightarrow[P']{} & X' & \xrightarrow[F']{} & Y'
\end{array}
\quad = \quad
\begin{array}{ccccc}
X & \xrightarrow{F} & Y & \xrightarrow{Q^*} & Y \\
\downarrow & \alpha^\sharp & \downarrow & \bar{v}^\sharp & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Y' \\
\| & & \phi' & & \| \\
X' & \xrightarrow[P']{} & X' & \xrightarrow[F']{} & Y'.
\end{array}
\tag{21}
$$

We define $\alpha^\sharp =_{\text{def}} \alpha$, so Eq. (19) holds trivially, since $\iota_{(F,\phi)}$ is the identity.

To complete the verification of the universal property of $\iota_{(F,\phi)}$, it only remains to show that Eq. (21) holds. The idea is to consider the sub-horizontal arrow E of Q^*F for which (21) and show that E must be isomorphic to Q^*F. More precisely, let us define the horizontal arrow $E : X \to Y$ via the following equalizer in the category $\mathbb{C}_1$ of horizontal arrows and squares:

$$
E \rightarrowtail^{\theta} Q^*F \underset{\phi'\,(\alpha,\bar{v}^\sharp)}{\overset{(\bar{u}^\sharp,\alpha)\,\phi^*}{\rightrightarrows}} F'P'.
\tag{22}
$$

Since the vertical boundaries of the squares in (21) are equal, the assumption that the source and target functors $\partial_0, \partial_1 : \mathbb{C}_1 \to \mathbb{C}_0$ preserve equalizers implies that E is indeed a horizontal arrow from X to Y and that θ has vertical boundaries given by identity morphisms. The commutativity of the equalizer diagram in (22) can be expressed as the equation

$$
\begin{array}{ccccc}
X & \xrightarrow{E} & Y & & \\
\| & \theta & \| & & \\
X & \longrightarrow & Y & \longrightarrow & Y \\
\| & \phi^* & \| & & \\
X & \longrightarrow & X & \longrightarrow & Y \\
\downarrow & \bar{u}^\sharp & \alpha & \downarrow & \\
X' & \xrightarrow[P']{} & X' & \xrightarrow[F']{} & Y'
\end{array}
\quad = \quad
\begin{array}{ccccc}
X & \xrightarrow{E} & Y & & \\
\| & \theta & \| & & \\
X & \longrightarrow & Y & \longrightarrow & Y \\
\downarrow & \alpha & \downarrow & \bar{v}^\sharp & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Y' \\
\| & & \phi' & & \| \\
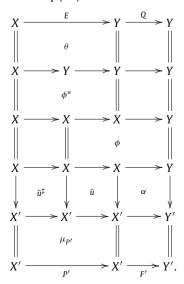X' & \xrightarrow[P']{} & X' & \xrightarrow[F']{} & Y'.
\end{array}
\tag{23}
$$

To prove Eq. (21) we show that $\theta : E \to Q^*F$ is an isomorphism. For this, we exploit the fact (observed in Remark 1.3) that $Q^*F : X \to Y$ is the initial algebra for the endofunctor

$$
\begin{aligned}
\mathcal{H}_\mathbb{C}(X, Y) &\longrightarrow \mathcal{H}_\mathbb{C}(X, Y) \\
(-) &\longmapsto F + Q(-),
\end{aligned}
\tag{24}
$$

where $\mathcal{H}_\mathbb{C}(X, Y)$ denotes the hom-category of horizontal arrows from X to Y of the horizontal 2-category $\mathcal{H}_\mathbb{C}$ of $\mathbb{C}$. Note that here we are using our assumption that $\mathcal{H}_\mathbb{C}$ has local coproducts. By the initiality of Q^*F, in order to show that $\theta : E \to Q^*F$ is an isomorphism, it is sufficient to show that E admits an algebra structure for the endofunctor in (24). The required algebra structure is given by the copair $(\lambda, \rho) : F + QE \to E$, where $\lambda : F \to E$ and $\rho : QE \to E$ are determined, via the universal property of the equalizer E, by the commutative diagrams

$$
F \xrightarrow{\eta_{Q^*F}} Q^*F \underset{\phi'\,(\alpha,v^\sharp)}{\overset{(\bar{u}^\sharp,\alpha)\,\phi^*}{\rightrightarrows}} F'P'
\tag{25}
$$

and

$$QE \xrightarrow{\ Q\,\theta\ } QQ^*F \xrightarrow{\ \nu_{Q*}\,F\ } Q^*F \xrightarrow[\phi'\,(\alpha,\nu^\sharp)]{\overset{(\bar{u}^\sharp,\alpha)\,\phi^*}{\Longrightarrow}} F'P', \tag{26}$$

respectively. It remains to show that the diagrams in (25) and (26) commute. The commutativity of (25) amounts to the equation

$$\tag{27}$$

Starting from the left-hand side of Eq. (27), we apply Eq. (16) in the top two rows and get

Then, Eq. (13) gives us

$$\tag{28}$$

Considering now the right-hand side of Eq. (27), an application of the analogue of Eq. (13) for $\bar{\nu}^\sharp$ gives us

An application of the second axiom for a horizontal monad map for (F', ϕ') then gives us exactly (28), as required. It remains to show the commutativity of the diagram in (26), which amounts to the equation

$$\tag{29}$$

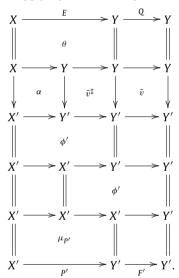Starting from the left-hand side of Eq. (29), we use Eq. (17) in the second and third row to get

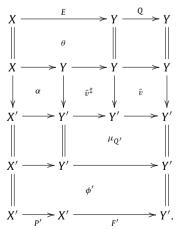We then use Eq. (14) in the bottom two rows and obtain

We apply Eq. (20), which is the assumption that α is an endomorphism square, in the third and the fourth row, so as to get

$$
\begin{array}{ccccccc}
X & \xrightarrow{\ E\ } & Y & \xrightarrow{\ Q\ } & Y \\
\| & \theta & \| & & \| \\
X & \longrightarrow & Y & \longrightarrow Y \longrightarrow & Y \\
\| & \phi^* & \| & & \| \\
X & \longrightarrow X \longrightarrow & Y & \longrightarrow & Y \\
\downarrow \bar{u}^\sharp & \alpha & \downarrow & \bar{v} & \downarrow \\
X' & \longrightarrow X' \longrightarrow & Y' & \longrightarrow & Y' \\
\| & & \phi' & & \| \\
X' & \longrightarrow X' \longrightarrow & X' & \longrightarrow & Y' \\
\| & \mu_{P'} & \| & & \| \\
X' & \xrightarrow{\ P'\ } & X' & \xrightarrow{\ F'\ } & Y'.
\end{array}
$$

We now apply Eq. (23) in the top three rows and we obtain

$$
\begin{array}{ccccccc}
X & \xrightarrow{\ E\ } & Y & \xrightarrow{\ Q\ } & Y \\
\| & \theta & \| & & \| \\
X & \longrightarrow Y & \longrightarrow Y \longrightarrow & Y \\
\downarrow & \alpha & \downarrow \bar{v}^\sharp & \bar{v} & \downarrow \\
X' & \longrightarrow Y' \longrightarrow & Y' & \longrightarrow & Y' \\
\| & \phi' & \| & & \| \\
X' & \longrightarrow X' \longrightarrow & Y' & \longrightarrow & Y' \\
\| & \phi' & \| & & \| \\
X' & \longrightarrow X' \longrightarrow & X' & \longrightarrow & Y' \\
\| & \mu_{P'} & \| & & \| \\
X' & \xrightarrow{\ P'\ } & Y' & \xrightarrow{\ F'\ } & Y'.
\end{array}
$$

We use the first axiom for a horizontal monad map (see item (ii) of Definition 2.4) for (F', ϕ') in the bottom three rows so as to get

$$
\begin{array}{ccccccc}
X & \xrightarrow{\ E\ } & Y & \xrightarrow{\ Q\ } & Y \\
\| & \theta & \| & & \| \\
X & \longrightarrow Y & \longrightarrow Y \longrightarrow & Y \\
\downarrow & \alpha & \downarrow \bar{v}^\sharp & \bar{v} & \downarrow \\
X' & \longrightarrow Y' & \longrightarrow Y' \longrightarrow & Y' \\
\| & & \mu_{Q'} & & \| \\
X' & \longrightarrow Y' & \longrightarrow & Y' \\
\| & \phi' & & & \| \\
X' & \xrightarrow{\ P'\ } X' & \longrightarrow & \xrightarrow{\ F'\ } & Y'.
\end{array}
$$

We obtain exactly this diagram also by applying the analogue of Eq. (14) for $\bar{v}^{\sharp}$ to the second and third row of the right-hand side of Eq. (29). This concludes the proof of Theorem 3.7.   □

Acknowledgements

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