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Skew-monoidal categories and bialgebroids☆

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Abstract

Skew-monoidal categories arise when the associator and the left and right units of a monoidal category are, in a specific way, not invertible. We prove that the closed skew-monoidal structures on the category of right R-modules are precisely the right bialgebroids over the ring R. These skew-monoidal structures induce quotient skew-monoidal structures on the category of R–R-bimodules and this leads to the following generalization: Opmonoidal monads on a monoidal category correspond to skew-monoidal structures with the same unit object which are compatible with the ordinary monoidal structure by means of a natural distributive law. Pursuing a Theorem of Day and Street we also discuss monoidal lax comonads to describe the comodule categories of bialgebroids beyond the flat case.

MSC: 16T10; 18D10

Keywords: Bialgebroids; Monoidal categories; Lax comonads

1. Introduction

Bialgebroids [26,16,27,14] are generalizations of bialgebras to non-commutative base ring. By replacing the commutative base ring k of a bialgebra with a non-commutative ring R the symmetric role of the monoid and comonoid structure is lost: A bialgebroid H over R is a comonoid $H \xrightarrow{\Delta} H \underset{R}{\otimes} H$ in the category ${}_R\mathsf{Ab}_R$ of R-bimodules but a monoid $H \otimes_{R^e} H \xrightarrow{m} H$ in the category of $R^e := R^{\mathrm{op}} \otimes R$-bimodules. The compatibility condition between the R^e-ring

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and the R-coring structure is too complicated to witness about something fundamental which may motivate to search for other generalizations of bialgebras [18]. However, if we look at the functor $\_\otimes_{R^e} H$ on the monoidal category $\mathsf{Ab}_{R^e} = {}_R\mathsf{Ab}_R$ instead of the object $H \in {}_{R^e}\mathsf{Ab}_{R^e}$ itself, the condition becomes amazingly simple. As it was observed in [25] a bimodule H is a bialgebroid precisely if $\_\otimes_{R^e} H$ is an opmonoidal monad [19,17].

The language of monads tells us that the modules over the bialgebroid H have to be the objects of the Eilenberg–Moore category of the monad $\_\otimes_{R^e} H$. Opmonoidality is then precisely the structure that makes the category of modules monoidal and the Eilenberg–Moore forgetful functor strict monoidal. This gives nothing new with respect to the 'classical' algebraic formalism: The Eilenberg–Moore category is the category of H-modules (H as an R^e-ring). But what are the comodules of an opmonoidal monad? The monadic language gives no hint. Classically one knows that there is the category of comodules over the R-coring H and several authors argued [22,6,3] that this category becomes monoidal with a strict monoidal forgetful functor to ${}_R\mathsf{Ab}_R$. This comodule category, however, is not the Eilenberg–Moore category of a monoidal comonad (unless H is flat as left R-module) which is a further asymmetry between modules and comodules of bialgebroids. Instead of monoidal comonad there is a lax monoidal structure given by Takeuchi's $\times_R$-product with respect to which bialgebroids can be seen as comonoids [12] and therefore have comodules in a natural way.

In this paper, we propose to consider a fragment of the structure of bialgebroids which lets their modules and comodules seen symmetrically or, better to say, dually. This fragment, called a skew-monoidal category, has left and right versions just like bialgebroids have [14]. A right-monoidal category consists of a category $\mathcal{M}$, a functor $\mathcal{M}\times\mathcal{M} \xrightarrow{*} \mathcal{M}$, an object $R\in\mathcal{M}$ and comparison natural transformations

$$L*(M*N) \xrightarrow{\gamma} (L*M)*N, \qquad M \xrightarrow{\eta} R*M, \qquad M*R \xrightarrow{\varepsilon} M$$

satisfying the usual pentagon and triangle equations of a monoidal category without assuming, however, invertibility of either γ, η or ε. In left-monoidal categories all comparisons go in the opposite way and the names η and ε are interchanged. For a right bialgebroid H over R the category $\mathcal{M}$ is the category Ab_R of right R-modules, R is the regular right R-module, ε and η are essentially the counit and the source map of H, respectively, while the skew-associator γ is the Galois map or canonical map $H \underset{R}{\otimes} H \to H \underset{R}{\bar{\otimes}} H$ built of the multiplication and comultiplication of H. What is not so simple to explain is the skew-monoidal product $*$.

The advantage of looking at the skew-monoidal category $\mathcal{M}$ instead of the bialgebroid H is that it encodes all information on the categories of right H-modules and of right H-comodules as simply as the Eilenberg–Moore categories of the canonical monad $T = R*\_$ and of the canonical comonad $Q = \_*R$ on $\mathcal{M}$. The disadvantage is that their monoidal structure is not seen. It is hidden in the properties of the category $\mathcal{M}$ together with all asymmetries between modules and comodules encoded in exactness properties of $\mathcal{M}$ and $*$.

Generalizations of monoidal categories or bicategories by relaxing invertibility of the comparison cells are not unknown in the literature. Burroni's *pseudocategory* [9] has comparison cells $(L*M)*N \to L*(M*N)$, $M \to R*M$, $M \to M*R$ and Grandis' *d-lax 2-category* [13] has $L*(M*N) \to (L*M)*N$, $R*M \to M$ and $M \to M*R$ therefore they are neither the left- nor the right-monoidal structures of the present paper. Blute et al. introduced the notion of *context category* [4] which contains, as part of the structure, precisely what we call right-monoidal comparison cells and the 5 axioms of a right-monoidal category can also be found among their axioms. Lax monoidal categories [15] provide another 'unbiased' way to generalize

monoidal categories which also have non-invertible comparison cells but no associator in the ordinary 'biased' sense. Much closer in spirit to our approach is the *2-monoidal* and *duoidal categories* [1,5] of Aguiar and Mahajan in spite of that they use two ordinary monoidal structures instead of a 'skew' one. For example the tensor square $H = R * R$ of the skew-monoidal unit, which is both a T-algebra and a Q-coalgebra, is reminiscent to a bimonoid in a 2-monoidal category although the precise connection is not clear. A direct predecessor of our skew-monoidal product is the non-unital monoidal product $*$ Ross Street constructs in [24] on a braided monoidal category equipped with a *tricocycloid* $H \otimes H \xrightarrow{\sim} H \otimes H$. Our $\gamma_{R,R,R}$ corresponds to a non-invertible tricocycloid on the object $H \in {}_R\mathsf{Ab}_R$ in a situation where no braiding is present.

The main result of this paper is the following characterization of bialgebroids (Theorem 9.1): The closed right-monoidal structures on Ab_R with skew-monoidal unit R are precisely the right bialgebroids over R. Similar statement holds for left-monoidal structures on ${}_R\mathsf{Ab}$ and left bialgebroids. The proof of this Theorem has four ingredients: 1. By left closedness of $*$ and by the Eilenberg–Watts Theorem there is a natural isomorphism $M \underset{R}{\otimes} TN \xrightarrow{\sim} M * N$. 2. Right exactness of T leads to a lifting of $*$ to a skew-monoidal product $*_q$ on ${}_R\mathsf{Ab}_R$ which admits an isomorphism $w_{M,N} : M \underset{R}{\otimes} T_q N \xrightarrow{\sim} M *_q N$ in terms of the canonical monad T_q of the $*_q$-structure. 3. The $w_{M,N}$ satisfies two coherence conditions in the form of a heptagon and a tetragon equation which turns out to be equivalent, by our Representability Theorem (Theorem 8.6), to that T is opmonoidal, hence a bimonad on ${}_R\mathsf{Ab}_R$. 4. Finally, by right closedness of $*$ this bimonad is left adjoint, hence the bimonad of a bialgebroid by a Theorem of [25].

The Representability Theorem is valid for any category equipped with two monoidal structures, an ordinary one $\otimes$ and a skew one $*$, and says that $*$ can be expressed as $M * N \cong M \otimes TN$ with a bimonad T precisely if the two monoidal structures are related by a *tetrahedral isomorphism* $L \otimes (M * N) \to (L \otimes M) * N$. The skew-monoidal structures on a monoidal category that can be expressed by a bimonad as above are called *representable*. This notion was inspired by the fusion operator formalism of [7] since a *fusion operator* $T(M \otimes TN) \to TM \otimes TN$ is the essential part of a skew-associator $\gamma_{L,M,N}$. As a matter of fact, for a bimonad T the expression $M * N := M \otimes TN$ always defines a skew-monoidal product (Proposition 7.2).

Although the Representability Theorem can be dualized and skew-monoidal structures can be constructed from monoidal comonads, this Corepresentability Theorem is not applicable to the monoidal (lax) comonad of a bialgebroid because of the different exactness properties we encounter. It could be applicable, however, to quantum categories [12] or to bicoalgebroids [8,2]. In order to complete the picture with the comodules of bialgebroids we use a lax version of the notion of comonad in Section 6, called *cohypomonad* in [11], and show in Theorem 10.2 that at least in case of the skew-monoidal category of a bialgebroid this lax comonad is monoidal. These results are not really new but a reformulation in a minimalistic language of what has been called in [12] a comonoid in a lax monoidal category provided by the iterated Takeuchi product.

2. Skew-monoidal categories

Definition 2.1. A right-monoidal category $\langle \mathcal{M}, *, R, \gamma, \eta, \varepsilon \rangle$ consists of a category $\mathcal{M}$, a functor $\_ * \_ : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, an object R of $\mathcal{M}$ and natural transformations

$$\gamma_{L,M,N} : L * (M * N) \to (L * M) * N$$
$$\eta_M : M \to R * M$$
$$\varepsilon_M : M * R \to M$$

subject to the following axioms: For all objects K, L, M, N

$$(\gamma_{K,L,M} * N) \circ \gamma_{K,L*M,N} \circ (K * \gamma_{L,M,N}) = \gamma_{K*L,M,N} \circ \gamma_{K,L,M*N} \tag{1}$$

$$\gamma_{R,M,N} \circ \eta_{M*N} = \eta_M * N \tag{2}$$

$$\varepsilon_{M*N} \circ \gamma_{M,N,R} = M * \varepsilon_N \tag{3}$$

$$(\varepsilon_M * N) \circ \gamma_{M,R,N} \circ (M * \eta_N) = M * N \tag{4}$$

$$\varepsilon_R \circ \eta_R = R. \tag{5}$$

If we replace $\mathcal{M}$ with $\mathcal{M}^{\mathrm{op,rev}}$, the category with opposite composition and with right-monoidal product of reversed order, we obtain again a right-monoidal category, with roles of η and ε interchanged. But replacing $\mathcal{M}$ with either $\mathcal{M}^{\mathrm{op}}$ or $\mathcal{M}^{\mathrm{rev}}$ what we obtain is different from the above structure. We call it a left-monoidal category.

If γ, η, ε are isomorphisms we recover the notion of a monoidal category with somewhat strange names for the associator and left and right units.

Definition 2.2. If $\mathcal{M}$ and $\mathcal{N}$ are right-monoidal categories (with structures denoted by $*$, R, γ, η, ε in both cases) then a right-monoidal functor $\mathcal{M} \to \mathcal{N}$ is a triple $\langle F, F_2, F_0 \rangle$ where F is a functor $\mathcal{M} \to \mathcal{N}$ of the underlying categories, F_0 is an arrow $R \to FR$ and F_2 is a natural transformation $F_{X,Y} : FX * FY \to F(X * Y)$ satisfying

$$F\gamma_{X,Y,Z} \circ F_{X,Y*Z} \circ (FX * F_{Y,Z}) = F_{X*Y,Z} \circ (F_{X,Y} * FZ) \circ \gamma_{FX,FY,FZ} \tag{6}$$

$$F_{R,X} \circ (F_0 * FX) \circ \eta_{FX} = F\eta_X \tag{7}$$

$$F\varepsilon_X \circ F_{X,R} \circ (FX * F_0) = \varepsilon_{FX} \tag{8}$$

for all $X, Y, Z \in \mathcal{M}$. Left-monoidal functors are similar functors between left-monoidal categories. They together will be referred to as skew-monoidal functors.

A skew-opmonoidal functor $\mathcal{M} \to \mathcal{N}$ is a triple $\langle F, F^2, F^0 \rangle$ where F is a functor $\mathcal{M} \to \mathcal{N}$, F^0 is an arrow $FR \to R$ and F^2 is a natural transformation $F^{X,Y} : F(X * Y) \to FX * FY$ such that F, $F_0 := F^0$ and $F_{X,Y} := F^{Y,X}$ define a skew-monoidal functor $\mathcal{M}^{\mathrm{op,rev}} \to \mathcal{N}^{\mathrm{op,rev}}$.

Example 2.3. Every right-monoidal category $\mathcal{M}$ has a canonical right-monoidal functor into the strict monoidal category $\mathrm{End}\,\mathcal{M}$ of endofunctors of $\mathcal{M}$. Define $\mathbf{L} : \mathcal{M} \to \mathrm{End}\,\mathcal{M}$ by $\mathbf{L}(M)N = M * N$. Then the natural transformation

$$\mathbf{L}(M)\mathbf{L}(N) \xrightarrow{\gamma_{M,N,-}} \mathbf{L}(M * N)$$

together with the arrow $\mathrm{id}_{\mathcal{M}} \xrightarrow{\eta} \mathbf{L}(R)$ is a right-monoidal structure on $\mathbf{L}$. Unlike for monoidal categories when this functor is a strong monoidal embedding, for general $\mathcal{M}$ the functor $\mathbf{L}$ is not even strong right-monoidal.

Similarly, the functor $\mathbf{R}(M)N = N * M$ has a right-opmonoidal structure as a functor $\mathcal{M} \to \mathrm{End}^{\mathrm{op}}\,\mathcal{M}$, to the category $\mathrm{End}\,\mathcal{M}$ equipped with opposite composition as (strict) monoidal structure.

Obviously, if both $\mathcal{M}$ and $\mathcal{N}$ are monoidal then the notions of left- and right-(op)monoidal functors coincide and they are precisely the usual (op)monoidal functors.

Definition 2.4. If $*$ and $*'$ are two right-monoidal structures on the same category $\mathcal{M}$ with the same unit object R then a twist from the $*$ structure to the $*'$-structure is a natural isomorphism

$w_{M,N} : M * N \xrightarrow{\sim} M *' N$ such that $\langle \mathrm{id}_{\mathcal{M}}, w, 1_R \rangle$ is a right-monoidal functor from $\mathcal{M}$ with $*'$ to $\mathcal{M}$ with $*$ structure.

One can define skew-(op)monoidal natural transformations although there is nothing 'skew' in them, so we drop the adjective:

Definition 2.5. Let $F, G : \mathcal{M} \to \mathcal{N}$ be skew-monoidal functors. A monoidal natural transformation $\nu : F \to G$ is a natural transformation of the underlying functors which satisfies

$$\nu_{X*Y} \circ F_{X,Y} = G_{X,Y} \circ (\nu_X * \nu_Y) \tag{9}$$

$$\nu_R \circ F_0 = G_0. \tag{10}$$

Opmonoidal transformations are similar transformations between skew-opmonoidal functors.

The right-monoidal categories together with the right-(op)monoidal functors and (op)monoidal natural transformations form the 2-category r-MonCat (r-OpmonCat). Similar 2-categories can be defined for left-monoidal categories.

In ordinary monoidal categories tensoring with the unit object defines rather trivial monads and/or comonads. In the skew-monoidal setting they are more interesting.

Lemma 2.6. *Let $\langle \mathcal{M}, *, R, \gamma, \eta, \varepsilon \rangle$ be a right-monoidal category and define $\mu_M := (\varepsilon_R * M) \circ \gamma_{R,R,M}$ and $\delta_M := \gamma_{M,R,R} \circ (M * \eta_R)$. Then*

$$T = \langle R * \_, \mu, \eta \rangle$$
$$Q = \langle \_ * R, \delta, \varepsilon \rangle$$

are a monad and a comonad on $\mathcal{M}$, respectively, and $\chi_M := \gamma_{R,M,R}$ is a (mixed) distributive law $\chi : TQ \to QT$.

Proof. Inserting $M = N = R$ in (2), composing with η_R and using naturality of η we obtain

$$\gamma_{R,R,R} \circ (R * \eta_R) \circ \eta_R = (\eta_R * R) \circ \eta_R. \tag{11}$$

In a similar fashion we obtain

$$\varepsilon_R \circ (\varepsilon_R * R) \circ \gamma_{R,R,R} = \varepsilon_R \circ (R * \varepsilon_R) \tag{12}$$

using (3). Now we can verify associativity of μ,

$$\begin{aligned}
\mu_M \circ (R * \mu_M) &= (\varepsilon_R * M) \circ \gamma_{R,R,M} \circ (R * (\varepsilon_R * M)) \circ (R * \gamma_{R,R,M}) \\
&= (\varepsilon_R * M) \circ ((R * \varepsilon_R) * M) \circ \gamma_{R,R*R,M} \circ (R * \gamma_{R,R,M}) \\
&\overset{(12)}{=} (\varepsilon_R * M) \circ ((\varepsilon_R * R) * M) \circ (\gamma_{R,R,R} * M) \\
&\quad \circ \gamma_{R,R*R,M} \circ (R * \gamma_{R,R,M}) \\
&\overset{(1)}{=} (\varepsilon_R * M) \circ ((\varepsilon_R * R) * M) \circ \gamma_{R*R,R,M} \circ \gamma_{R,R,R*M} \\
&= (\varepsilon_R * M) \circ \gamma_{R,R,M} \circ (\varepsilon_R * (R * M)) \circ \gamma_{R,R,R*M} \\
&= \mu_M \circ \mu_{R*M}
\end{aligned}$$

and coassociativity of δ,

$$\begin{aligned}(\delta_M * R)\circ\delta_M &= (\gamma_{M,R,R} * R)\circ((M * \eta_R) * R)\circ\gamma_{M,R,R}\circ(M * \eta_R)\\ &= (\gamma_{M,R,R} * R)\circ\gamma_{M,R*R,R}\circ(M * (\eta_R * R))\circ(M * \eta_R)\\ &\overset{(11)}{=} (\gamma_{M,R,R} * R)\circ\gamma_{M,R*R,R}\circ(M * \gamma_{R,R,R})\\ &\quad\circ(M * (R * \eta_R))\circ(M * \eta_R)\\ &\overset{(1)}{=} \gamma_{M*R,R,R}\circ\gamma_{M,R,R*R}\circ(M * (R * \eta_R))\circ(M * \eta_R)\\ &= \gamma_{M*R,R,R}\circ((M * R) * \eta_R)\circ\gamma_{M,R,R}\circ(M * \eta_R)\\ &= \delta_{M*R}\circ\delta_M.\end{aligned}$$

As for the left and right unit and counit equations

$$\mu_N\circ\eta_{R*N} = R * N \tag{13}$$
$$\mu_N\circ(R * \eta_N) = R * N \tag{14}$$
$$\varepsilon_{M*R}\circ\delta_M = M * R \tag{15}$$
$$(\varepsilon_M * R)\circ\delta_M = M * R \tag{16}$$

notice that inserting $M = R$ in (4) we obtain (14), inserting $N = R$ in (4) we obtain (16), inserting $M = R$ in (2) and composing with $\varepsilon_R * N$ we obtain (13) and inserting $N = R$ in (3) and composing with $M * \eta_R$ we obtain (15).

It remains to show that χ is a distributive law in the sense of the equations

$$(\mu_M * R)\circ\chi_{R*M}\circ(R * \chi_M) = \chi_M\circ\mu_{M*R} \tag{17}$$
$$(\chi_M * R)\circ\chi_{M*R}\circ(R * \delta_M) = \delta_{R*M}\circ\chi_M \tag{18}$$
$$\chi_M\circ\eta_{M*R} = \eta_M * R \tag{19}$$
$$\varepsilon_{R*M}\circ\chi_M = R * \varepsilon_M. \tag{20}$$

Eqs. (17) and (18) are simple consequences of the pentagon (1) while (19) and (20) follow trivially from (2) and (3), respectively. □

The monad T and the comonad Q on the right-monoidal category $\mathcal{M}$ will be called the *canonical monad* and the *canonical comonad* of $\mathcal{M}$. For left monoidal categories they are $T = \_ * R$ and $Q = R * \_$.

Lemma 2.7. *If $\langle F, F_2, F_0\rangle$ is a right-monoidal functor $\mathcal{M}\to\mathcal{N}$ then the pair $\langle F,\varphi\rangle$, where $\varphi_M := F_{R,M}\circ(F_0 * FM)$, is a monad morphism from the canonical monad T of $\mathcal{M}$ to the canonical monad T on $\mathcal{N}$, i.e.,*

$$F\mu\circ\varphi T\circ T\varphi = \varphi\circ\mu F \tag{21}$$
$$F\eta = \varphi\circ\eta F. \tag{22}$$

Dually, if $\langle F, F^2, F^0\rangle$ is a right-opmonoidal functor $\mathcal{M}\to\mathcal{N}$ then the pair $\langle F,\psi\rangle$, where $\psi_M := (FM * F^0)\circ F^{M,R}$, is a comonad morphism from the canonical comonad Q of $\mathcal{M}$ to the canonical comonad Q of $\mathcal{N}$.

Proof. The statement for the monad morphism can be easily shown using the definition of μ and the right-monoidal functor axioms (6)–(8). The statement for the comonad morphism is then obtained by passing to the dual right-monoidal category $\mathcal{M}^{\mathrm{op,rev}}$. □

Remark 2.8. If we want to formulate a bialgebra-like compatibility condition between μ and δ then here is a commutative diagram

$$\begin{array}{ccccc} R*(R*R) & \xrightarrow{\mu_R} & R*R & \xrightarrow{\delta_R} & (R*R)*R \\ {\scriptstyle R*\delta_R}\downarrow & & & & \uparrow{\scriptstyle \mu_R*R} \\ R*((R*R)*R) & & & & (R*(R*R))*R \\ {\scriptstyle \delta_{R,Q^2R}}\downarrow & & & & \uparrow{\scriptstyle \mu_{T^2R,R}} \\ (R*R)*((R*R)*R) & & \xrightarrow{\sigma_{R,R*R,R}} & & (R*(R*R))*(R*R) \end{array} \tag{23}$$

where

$$\sigma_{L,M,N} \coloneqq ((L*M)*\eta_N)\circ\gamma_{L,M,N}\circ(\varepsilon_L*(M*N)) \;:\; (L*R)*(M*N) \to (L*M)*(R*N)$$

and where the 2-argument δ and μ are defined by

$$\delta_{K,L} \coloneqq \gamma_{K,R,L}\circ(K*\eta_L) \;:\; K*L \to QK*L \tag{24}$$

$$\mu_{K,L} \coloneqq (\varepsilon_K*L)\circ\gamma_{K,R,L} \;:\; K*TL \to K*L. \tag{25}$$

They obey the relations

$$\delta_{QK,L}\circ\delta_{K,L} = (\delta_K*L)\circ\delta_{K,L} \qquad (\varepsilon_K*L)\circ\delta_{K,L} = K*L \tag{26}$$

$$\mu_{K,L}\circ\mu_{K,TL} = \mu_{K,L}\circ(K*\mu_L) \qquad \mu_{K,L}\circ(K*\eta_L) = K*L. \tag{27}$$

Although diagram (23) is reminiscent to the compatibility condition between multiplication and comultiplication of a bialgebroid, in order to confirm this interpretation one should investigate in which sense σ is a generalized braiding, if at all.

Remark 2.9. The composite $\delta_R\circ\mu_R$ is built from γ, η, δ and identity arrows and has the same source and target as $\gamma_{R,R,R}$. But there is no sign that they would be equal. Instead,

$$\delta_R\circ\mu_R = (\mu_R*R)\circ\gamma_{R,R*R,R}\circ(R*\delta_R),$$

that is to say χ_{R*R} fits into diagram (23) as a second row. So coherence for skew-monoidal categories is expected to fail in its naive form.

Remark 2.10. Using the notations (24), (25) there is an identity in any right-monoidal category:

$$\mu_{R*R,R}\circ\delta_{R,R*R} = \gamma_{R,R,R}.$$

More generally, we have

$$\mu_{QM,N}\circ\delta_{M,TN} = \gamma_{M,R,N}, \quad M,N\in\mathcal{M}.$$

This result suggests that we should think of the skew-associator γ as the Galois map of the 'underlying' quantum groupoid of $\mathcal{M}$ even if there is no such a quantum groupoid in general.

3. The motivating example: bialgebroids

Let Ab_R denote the category of right R-modules over the ring R. This category has no (obvious) monoidal structure. But every R-bialgebroid defines a right-monoidal structure on Ab_R as we shall see below.

Let H be a right R-bialgebroid with $R^{\mathrm{op}}\otimes R$-ring and R-coring structure

$$t^H \otimes s^H : R^{\mathrm{op}} \otimes R \to H \tag{28}$$

$$\Delta^H : H \to H \underset{R_1}{\otimes} H. \tag{29}$$

The unit element of H is denoted by 1^H and the counit $H \to R$ by ε^H. Then H carries two left and two right actions of R defined by

$$\lambda_1(r)(h) := ht^H(r) \qquad \rho_1(r)(h) := t^H(r)h$$
$$\lambda_2(r)(h) := s^H(r)h \qquad \rho_2(r)(h) := hs^H(r)$$

for $r \in R$, $h \in H$. The codomain $H \underset{R_1}{\otimes} H$ of the comultiplication Δ^H is the tensor square w.r.t. ρ_2 and λ_1.

For right R-modules M and N we introduce

$$M * N := M \underset{R_1}{\otimes} (N \underset{R_2}{\otimes} H) \tag{30}$$

where $L \underset{R_i}{\otimes} \_$ refers to tensoring over R with respect to the λ_i left action on H. The result $M * N$ is considered as a right R-module w.r.t. the ρ_2 right action on H. Elements of $M * N$ are denoted by $[m, n, h]$ instead of $m \otimes (n \otimes h)$. They therefore obey the relations

$$[m \cdot r, n, h] = [m, n, ht^H(r)]$$
$$[m, n \cdot r, h] = [m, n, s^H(r)h]$$
$$[m, n, h] \cdot r = [m, n, hs^H(r)]$$

so the following natural transformations are well-defined:

$$\eta_M : M \to R * M, \qquad \eta_M(m) = [1^R, m, 1^H]$$
$$\varepsilon_M : M * R \to M, \qquad \varepsilon([m, r, h]) = m \cdot \varepsilon^H(s^H(r)h)$$
$$\gamma_{L,M,N} : L * (M * N) \to (L * M) * N,$$
$$\gamma_{L,M,N}([l, [m, n, g], h]) = [[l, m, h^{(1)}], n, gh^{(2)}].$$

It is easy to verify, using the bialgebroid axioms, that $\langle \mathsf{Ab}_R, *, R_R, \gamma, \eta, \varepsilon \rangle$ is a right-monoidal category.

One can notice that the skew-associator γ, which is uniquely determined by $\gamma_{R,R,R}$, is, up to isomorphisms $R*(R*R) \cong H \underset{R_2}{\otimes} H$ and $(R*R)*R \cong H \underset{R_1}{\otimes} H$, the canonical map or Galois map

$$H \underset{R_2}{\otimes} H \to H \underset{R_1}{\otimes} H, \qquad g \otimes h \mapsto h^{(1)} \otimes gh^{(2)}$$

of H as a left H-comodule algebra. Therefore the bialgebroid is a Hopf algebroid (or $\times_R$-Hopf algebra) in the sense of [23] precisely when the skew-associator γ is invertible.

4. E-objects

Let $E = \operatorname{End} R$ be the endomorphism monoid of the right-monoidal unit R. An E-object in $\mathcal{M}$ is an object M together with a morphism $\lambda_M : E \to \mathcal{M}(M, M)$ of monoids. The category $\mathcal{E}$ of E-objects in $\mathcal{M}$ has arrows $M \to N$ the arrows $t \in \mathcal{M}(M, N)$ which satisfy $t \circ \lambda_M(r) = \lambda_N(r) \circ t$ for all $r \in E$.

Since the category of E-objects in Ab_R is the category of bimodules, ${}_R\mathsf{Ab}_R$, hence monoidal, we would like to see if this category inherits a skew-monoidal structure from the one given

on Ab_R. This is the first step on the path going from skew-monoidal structures on Ab_R to bialgebroids.

One can define the category of $E^{\otimes m}$-$E^{\otimes n}$-bimodules in $\mathcal{M}$ as the category of objects equipped with m left E-actions and n right E-actions that pairwise commute with each other. Such objects will be called (m,n)-type E-objects.

Lemma 4.1. *If K and L are left E-objects (i.e., they are $(1,0)$-type) then $K * L$ is a $(2,1)$-type E-object with*

$$\begin{aligned}\lambda_1(r) &= \lambda_K(r) * L\\ \lambda_2(r) &= K * \lambda_L(r)\\ \rho_1(r) &= (\varepsilon_K * L)\circ\gamma_{K,R,L}\circ(K*(r*L))\circ(K*\eta_L).\end{aligned}$$

More generally, if K is an E-object of (m_1,n_1)-type and L is of (m_2,n_2)-type then $K*L$ is an E-object of (m_1+m_2, n_1+1+n_2)-type.

Proof. λ_1 and λ_2 are obviously left actions and commute with each other. ρ_1 is natural in $K\in\mathcal{M}$ and $L\in\mathcal{M}$ therefore it commutes with both λ_1 and λ_2 and also with any other left or right actions the objects K or L may possess. Therefore the statement follows immediately if we prove that the formula for ρ_1 defines a right action. Unitalness $\rho(R)=K*L$ follows directly from (4). As for multiplicativity

$$\begin{aligned}\rho_1(r_1)\circ\rho_1(r_2) &= (\varepsilon_K*L)\circ\gamma_{K,R,L}\circ(K*(r_1*L))\circ(\varepsilon_K*\eta_L)\\ &\quad\circ\gamma_{K,R,L}\circ(K*(r_2*L))\circ(K*\eta_L)\\ &= (\varepsilon_K*L)\circ((\varepsilon_K*R)*L)\circ\gamma_{K*R,R,L}\circ\gamma_{K,R,R*L}\\ &\quad\circ(K*(r_2*(r_1*L)))\circ(K*(R*\eta_L))\circ(K*\eta_L)\\ &\overset{(1)}{=} (\varepsilon_K*L)\circ(\varepsilon_{K*R}*L)\circ(\gamma_{K,R,R}*L)\circ\gamma_{K,R*R,L}\\ &\quad\circ(K*\gamma_{R,R,L})\circ(K*(r_2*(r_1*L)))\circ(K*\eta_{R*L})\circ(K*\eta_L)\\ &= (\varepsilon_K*L)\circ(\varepsilon_{K*R}*L)\circ(\gamma_{K,R,R}*L)\circ((K*(r_2*R))*L)\\ &\quad\circ\gamma_{K,R*R,L}\circ(K*((R*r_1)*L))\circ(K*\gamma_{R,R,L})\\ &\quad\circ(K*\eta_{R*L})\circ(K*\eta_L)\\ &\overset{(3)}{=} (\varepsilon_K*L)\circ((K*\varepsilon_R)*L)\circ((K*(r_2*R))*L)\circ\gamma_{K,R*R,L}\\ &\quad\circ(K*((R*r_1)*L))\circ(K*\gamma_{R,R,L})\circ(K*\eta_{R*L})\circ(K*\eta_L)\\ &\overset{(2)}{=} (\varepsilon_K*L)\circ((K*\varepsilon_R)*L)\circ((K*(r_2*R))*L)\circ\gamma_{K,R*R,L}\\ &\quad\circ(K*((R*r_1)*L))\circ(K*(\eta_R*L))\circ(K*\eta_L)\\ &= (\varepsilon_K*L)\circ((K*r_2)*L)\circ((K*\varepsilon_R)*L)\circ\gamma_{K,R*R,L}\\ &\quad\circ(K*(\eta_R*L))\circ(K*(r_1*L))\circ(K*\eta_L)\\ &\overset{(5)}{=} (\varepsilon_K*L)\circ\gamma_{K,R,L}\circ(K*((r_2\circ r_1)*L))\circ(K*\eta_L)\\ &= \rho_1(r_2\circ r_1).\end{aligned}$$

This completes the proof. □

If we have n left E-objects and we $*$ them in any order, so the parenthesizing is arbitrary, then the resulting object will have n left actions of the obvious $1*\ldots 1*\lambda(r)*1*\ldots*1$ type and

less obvious right actions, $n-1$ in number, each corresponding to one $*$ sign. These actions will be numbered from left to right as shown:

$$\underset{\lambda_1}{A} \overset{\rho_1}{*} \underset{\lambda_2}{B} \overset{\rho_2}{*} \cdots \overset{\rho_{n-1}}{*} \underset{\lambda_n}{Z}$$

The simplest left E-object is R. Its left action is the identity morphism $E \to \mathcal{M}(R,R)$. By the above Lemma the object $R * R$ is equipped with two left actions λ_1, λ_2 and one right action ρ_1. As such a $(2,1)$-type object $R * R$ is denoted by H. It is to be interpreted as the underlying object of a quantum groupoid, at least for $\mathcal{M} = \mathsf{Ab}_R$.

In the next Lemma we summarize how the structure maps $\gamma, \eta, \varepsilon$ and their derivatives μ and δ behave with respect to the λ and ρ actions.

Lemma 4.2. *For E-objects L, M, N and for all $r \in E$*

$$\lambda_i(r) \circ \gamma_{L,M,N} = \gamma_{L,M,N} \circ \lambda_i(r) \quad i = 1,2,3 \tag{31}$$
$$\lambda_2(r) \circ \eta_N = \eta_N \circ \lambda_1(r) \tag{32}$$
$$\lambda_1(r) \circ \varepsilon_L = \varepsilon_L \circ \lambda_1(r) \tag{33}$$
$$\lambda_1(r) \circ \mu_N = \mu_N \circ \lambda_1(r) \tag{34}$$
$$\lambda_2(r) \circ \mu_N = \mu_N \circ \lambda_3(r) \tag{35}$$
$$\lambda_3(r) \circ \delta_L = \delta_L \circ \lambda_2(r) \tag{36}$$
$$\lambda_1(r) \circ \delta_L = \delta_L \circ \lambda_1(r). \tag{37}$$

For arbitrary L, M, N of $\mathcal{M}$ and for all $r \in E$

$$\rho_i(r) \circ \gamma_{L,M,N} = \gamma_{L,M,N} \circ \rho_i(r) \quad i = 1,2 \tag{38}$$
$$\rho_1(r) \circ \eta_N = \lambda_1(r) \circ \eta_N \tag{39}$$
$$\varepsilon_L \circ \rho_1(r) = \varepsilon_L \circ \lambda_2(r) \tag{40}$$
$$\rho_1(r) \circ \mu_N = \mu_N \circ \rho_2(r) \tag{41}$$
$$\rho_1(r) \circ \delta_L = \delta_L \circ \rho_1(r) \tag{42}$$
$$\mu_N \circ \rho_1(r) = \mu_N \circ \lambda_2(r) \tag{43}$$
$$\rho_2(r) \circ \delta_L = \lambda_2(r) \circ \delta_L. \tag{44}$$

Proof. Relations involving λ-s only are just naturalities of the structure maps. Those involving ρ-s require some computations which, however, are left to the reader. □

Among the various multiple E-objects there are distinguished ones that behave nicely under the $*$-product. For each $n > 0$ let $\mathcal{M}^{(n)}$ denote the category of $(n, n-1)$-type of E-objects in $\mathcal{M}$. Then $\mathcal{M}^{(m)} * \mathcal{M}^{(n)} \subset \mathcal{M}^{(m+n)}$ by Lemma 4.1. Clearly, $\mathcal{M}^{(1)} = \mathcal{E}$ and $R \in \mathcal{M}^{(1)}$, $H \in \mathcal{M}^{(2)}$. The coproduct $\mathcal{M}^{(\bullet)} = \bigsqcup_{n>0} \mathcal{M}^{(n)}$ is then closed under $*$ but has no unit object.

Now assume that the category $\mathcal{M}$ has limits and colimits. For two left E-objects L and M we can make new E-objects from the $(2,1)$-type object $L * M$ either by forming the λ_1-ρ_1 center or by forming the ρ_1-λ_2 quotient:

$$\int_{\lambda_1 \rho_1} L * M \overset{z_{L,M}}{\rightarrowtail} L * M \underset{\rho_1}{\overset{\lambda_1}{\rightrightarrows}} \prod_{r \in E} L * M \tag{45}$$

$$\coprod_{r \in E} L * M \underset{\lambda_2}{\overset{\rho_1}{\rightrightarrows}} L * M \overset{q_{L,M}}{\twoheadrightarrow} \int^{\rho_1 \lambda_2} L * M. \tag{46}$$

Then the λ_2 action on $L * M$ inherits to $\int_{\lambda_1\rho_1} L * M$ a left E-object structure and λ_1 inherits one to $\int^{\rho_1\lambda_2} L * M$. In this way, the above end and coend define functors $\mathcal{E} \times \mathcal{E} \to \mathcal{E}$. The identity arrow on $L * M$ restricts–corestricts to a natural transformation

$$\theta_{L,M} := q_{L.M} \circ z_{L,M} \ : \quad \int_{\lambda_1\rho_1} L * M \to \int^{\rho_1\lambda_2} L * M. \tag{47}$$

Indeed, for $r \in E$

$$\begin{aligned}\theta_{L,M} \circ \lambda_2(r) &= q_{L,M} \circ \lambda_2(r) \circ z_{L,M} = q_{L,M} \circ \rho_1(r) \circ z_{L,M} \\ &= q_{L,M} \circ \lambda_1(r) \circ z_{L,M} = \lambda_1(r) \circ \theta_{L,M}\end{aligned}$$

shows that $\theta_{L,M}$ belongs to $\mathcal{E}$. Its naturality follows from that z and q are natural.

Proposition 4.3. *Let $\langle \mathcal{M}, *, R, \gamma, \eta, \varepsilon\rangle$ be a right-monoidal category in which the category $\mathcal{M}$ has colimits and $L * \_ : \mathcal{M} \to \mathcal{M}$ preserves finite colimits for each $L \in \mathcal{M}$. Choosing a coequalizer (46) for each pair of E-objects $\langle L, M\rangle$ and making the quotient*

$$L *_q M := \int^{\rho_1\lambda_2} L * M$$

an E-object by means of λ_1 there is a unique right-monoidal structure $\langle \mathcal{E}, *_q, R, \gamma^q, \eta^q, \varepsilon^q\rangle$ on the category of E-objects such that the forgetful functor $\phi : \mathcal{E} \to \mathcal{M}$ together with $q_{L,M} : L * M \to L *_q M$ and the identity arrow 1_R becomes a right-monoidal functor $\mathcal{E} \to \mathcal{M}$.

Proof. For $\langle \phi, q, 1_R\rangle$ to be a right-monoidal functor the γ^q, η^q and ε^q must obey to commutativity of the diagrams

$$\begin{array}{ccccc} L * (M * N) & \xrightarrow{L * q_{M,N}} & L * (M *_q N) & \xrightarrow{q_{L,M *_q N}} & L *_q (M *_q N) \\ {\scriptstyle \gamma_{L,M,N}}\downarrow & & & & \downarrow{\scriptstyle \gamma^q_{L,M,N}} \\ (L * M) * N & \xrightarrow{q_{L,M} * N} & (L *_q M) * N & \xrightarrow{q_{L *_q M,N}} & (L *_q M) *_q N \end{array} \tag{48}$$

$$\begin{array}{ccccc} M & \xrightarrow{\eta_M} & R * M & \qquad M * R & \xrightarrow{\varepsilon_M} & M \\ \| & & \downarrow{\scriptstyle q_{R,M}} & {\scriptstyle q_{M,R}}\downarrow & & \| \\ M & \xrightarrow{\eta^q_M} & R *_q M & \qquad M *_q R & \xrightarrow{\varepsilon^q_M} & M \end{array} \tag{49}$$

The existence and uniqueness of γ^q follow from that the composite $\xi := q_{L *_q M,N} \circ (q_{L,M} * N) \circ \gamma_{L,M,N}$ satisfies both $\xi \circ \rho_1 = \xi \circ \lambda_2$ and $\xi \circ \rho_2 = \xi \circ \lambda_3$ as a consequence of (31), (38). By the latter there is a unique factorization $\xi = \xi' \circ (L * q_{M,N})$ in which $\xi' \circ \rho_1 = \xi' \circ \lambda_2$. Then γ^q is obtained as the unique factorization $\xi' = \gamma^q_{L,M,N} \circ q_{L,M *_q N}$. ε^q is obtained in a similar way while η^q is readily defined by the diagram as it stands.

The verification of the right-monoidal category axioms is now a routine computation. □

The dual of Proposition 4.3 is the following.

Proposition 4.4. *Let $\langle \mathcal{M}, *, R, \gamma, \eta, \varepsilon\rangle$ be a right-monoidal category in which the category $\mathcal{M}$ has limits and $\_ * M : \mathcal{M} \to \mathcal{M}$ preserves finite limits for each $M \in \mathcal{M}$. Choosing an*

equalizer (45) for each pair of E-objects $\langle L, M\rangle$ and making the center

$$L *_z M := \int_{\lambda_1 \rho_1} L * M$$

an E-object by means of λ_2 there is a unique right-monoidal structure $\langle \mathcal{E}, *_z, R, \gamma^z, \eta^z, \varepsilon^z\rangle$ on the category of E-objects such that $\langle \phi, z, 1_R\rangle$ is a right-opmonoidal functor, i.e.,

$$\begin{array}{ccccc} L *_z (M *_z N) & \xrightarrow{z_{L,M*_zN}} & L * (M *_z N) & \xrightarrow{L*z_{M,N}} & L * (M * N) \\ \gamma^z_{L,M,N}\downarrow & & & & \downarrow \gamma_{L,M,N} \\ (L *_z M) *_z N & \xrightarrow{z_{L*_zM,N}} & (L *_z M) * N & \xrightarrow{z_{L,M}*N} & (L * M) * N \end{array} \tag{50}$$

$$\begin{array}{ccccc} M & \xrightarrow{\eta^z_M} & R *_z M & \qquad M *_z R & \xrightarrow{\varepsilon^z_M} & M \\ \| & & \downarrow z_{R,M} & z_{M,R}\downarrow & & \| \\ M & \xrightarrow{\eta_M} & R * M & M * R & \xrightarrow{\varepsilon_M} & M \end{array} \tag{51}$$

are commutative for each $L, M, N \in \mathcal{E}$.

Applying Lemma 2.7 to the skew-(op)monoidal functor of Propositions 4.3 and 4.4, respectively, we obtain the following.

Corollary 4.5. *Let $\langle \mathcal{M}, *, R, \gamma, \eta, \varepsilon\rangle$ be a right-monoidal category with canonical monad T and canonical comonad Q.*

(i) *If $\mathcal{M}$ has colimits and for all $L \in \mathcal{M}$ the endofunctor $L * \_$ preserves finite colimits then*
  (a) *$\mathcal{E}$ has a right-monoidal structure with canonical monad $T_q = \int^{\rho_1 \lambda_2} R * \_$*
  (b) *and $\kappa_M := q_{R,M}$ defines a monad morphism $\langle \phi, \kappa\rangle$ from T_q to T.*

(ii) *If $\mathcal{M}$ has limits and for all $M \in \mathcal{M}$ the endofunctor $\_ * M$ preserves finite limits then*
  (a) *$\mathcal{E}$ has a right-monoidal structure with canonical comonad $Q^z = \int_{\lambda_1 \rho_1} \_ * R$*
  (b) *and $\zeta_L := z_{L,R}$ defines a comonad morphism $\langle \phi, \zeta\rangle$ from Q^z to Q.*

As we shall see in the next section some results of this Corollary hold under weaker hypotheses.

5. Comodules and modules

If right-monoidal categories are to be interpreted as quantum groupoids then it must have associated categories of modules and comodules. The Eilenberg–Moore categories of the canonical monad T and comonad Q are the obvious candidates, albeit apparently without monoidal structures.

Let $\mathcal{M}^Q$ denote the Eilenberg–Moore category of Q-comodules, also called Q-coalgebras, for the comonad $Q = \langle \_ * R, \delta, \varepsilon\rangle$. Its objects are pairs $\langle M, \Delta_M\rangle$ where M is an object of M and $\Delta_M : M \to M * R$ satisfies

$$(\Delta_M * R) \circ \Delta_M = \delta_M \circ \Delta_M \tag{52}$$

$$\varepsilon_M \circ \Delta_M = M. \tag{53}$$

The arrows $M \to N$ in $\mathcal{M}^Q$ are defined to be the arrows $t \in \mathcal{M}(M, N)$ such that

$$\Delta_N \circ t = (t * R) \circ \Delta_M. \tag{54}$$

Dually, in the category $\mathcal{M}_T$ of T-modules the objects $\nabla_M : R * M \to M$ are defined by the equations

$$\nabla_M \circ (R * \nabla_M) = \nabla_M \circ \mu_M \tag{55}$$

$$\nabla_M \circ \eta_M = M \tag{56}$$

and its arrows $t : M \to N$ by

$$t \circ \nabla_M = \nabla_N \circ (R * t). \tag{57}$$

Entwined modules of a skew-monoidal category can be defined as the category of triples $\langle M, \nabla, \Delta\rangle$ such that $\langle M, \nabla\rangle$ is a T-module and $\langle M, \Delta\rangle$ is a Q-comodule which satisfy the compatibility condition

$$\begin{array}{ccccc} TM & \xrightarrow{\nabla} & M & \xrightarrow{\Delta} & QM \\ {\scriptstyle T\Delta}\downarrow & & & & \uparrow{\scriptstyle Q\nabla} \\ TQM & & \xrightarrow{\chi_M} & & QTM \end{array}$$

The arrows $\langle M, \Delta, \nabla\rangle \to \langle M', \Delta', \nabla'\rangle$ are the arrows $t \in \mathcal{M}(M, M')$ which are both T-module and Q-comodule morphisms. The basic example of an entwined module is the object $R * R$ with action μ_R and coaction δ_R.

Lemma 5.1. *If L is a Q-comodule and N is a T-module then both L and N are left E-objects via*

$$\lambda_L(r) = \varepsilon_L \circ (L * r) \circ \Delta_L \tag{58}$$

$$\lambda_N(r) = \nabla_N \circ (r * N) \circ \eta_N, \tag{59}$$

respectively. With respect to these actions every arrow in $\mathcal{M}^Q$ and every arrow in $\mathcal{M}_T$ are morphisms of left E-objects. This defines the faithful functors

$$\mathcal{F}_z : \mathcal{M}^Q \to \mathcal{E}, \qquad \mathcal{F}_q : \mathcal{M}_T \to \mathcal{E}.$$

Proof. Since T-modules in $\mathcal{M}$ are the Q-comodules of the opposite-reversed right-monoidal category $\mathcal{M}^{\mathrm{op,rev}}$, it suffices to show that λ_L is a monoid morphism and that every $t \in \mathcal{M}^Q$ is a morphism of E-objects.

$$\begin{aligned} \lambda_L(R) = \varepsilon_L \circ \Delta_L &\overset{(53)}{=} L \\ \lambda_L(r_1) \circ \lambda_L(r_2) &= \varepsilon_L \circ \varepsilon_{L*R} \circ ((L * r_1) * R) \circ ((L * R) * r_2) \circ (\Delta_L * R) \circ \Delta_L \\ &\overset{(52)}{=} \varepsilon_L \circ \varepsilon_{L*R} \circ ((L * r_1) * r_2) \circ \delta_L \circ \Delta_L \\ &= \varepsilon_L \circ \varepsilon_{L*R} \circ \gamma_{L,R,R} \circ (L * (r_1 * r_2)) \circ (L * \eta_R) \circ \Delta_L \\ &\overset{(3)}{=} \varepsilon_L \circ (L * \varepsilon_R) \circ (L * (r_1 * r_2)) \circ (L * \eta_R) \circ \Delta_L \\ &= \varepsilon_L \circ (L * r_1) \circ (L * \varepsilon_R) \circ (L * \eta_R) \circ (L * r_2) \circ \Delta_L \\ &\overset{(5)}{=} \varepsilon_L \circ (L * (r_1 \circ r_2)) \circ \Delta_L \\ &= \lambda_L(r_1 \circ r_2). \end{aligned}$$

If $t : K \to L$ is a Q-comodule morphism then

$$t \circ \lambda_K(r) = \varepsilon_L \circ (t * r) \circ \Delta_K = \varepsilon_L \circ (L * r) \circ \Delta_L \circ t = \lambda_L \circ t. \quad \square$$

We note that for the free Q-comodules $N * R \xrightarrow{\delta_N} (N * R) * R$, where N is an arbitrary object in $\mathcal{M}$, the above left E-action λ_{N*R} reduces to the canonical $N * r$ left action λ_2 of the right-monoidal product $N * R$ of a $(0,0)$-type object with a $(1,0)$-type object. Dually, for free T-modules $\lambda_{R*N}(r) = r * N$. However, if L is a Q-comodule and M is a T-module then $L * R$ and $R * M$ are type $(2,1)$ and the question arises how the coaction and action behave with respect to the extra two E-actions.

Lemma 5.2. *Assume $\mathcal{M}$ is complete. For every Q-comodule L the coaction Δ_L is a morphism of left E-objects and factorizes uniquely through the center of the $(2,1)$-type E-object $L * R$ as*

$$L \xrightarrow{\Delta_L^z} \int_{\lambda_1 \rho_1} L * R \xrightarrow{z_{L,R}} L * R$$

in $\mathcal{E}$. Dually, assume $\mathcal{M}$ is cocomplete. Then the action ∇_M of every T-module M belongs to $\mathcal{E}$ and has a unique factorization

$$R * M \xrightarrow{q_{R,M}} \int^{\rho_1 \lambda_2} R * M \xrightarrow{\nabla_M^q} M$$

in $\mathcal{E}$ through the quotient of the $(2,1)$-type E-object $R * M$.

Proof. We prove the statement for Q-coactions. Since every comodule L is an equalizer

$$L \xrightarrow{\Delta_L} L * R \underset{\Delta_L * R}{\overset{\delta_L}{\rightrightarrows}} (L * R) * R$$

in $\mathcal{M}$ (it is split by $L \xleftarrow{\varepsilon_L} L * R \xleftarrow{\varepsilon_{L*R}} (L * R) * R$), the coaction Δ_L is a morphism of Q-comodules from L to the free Q-comodule $L * R$. Therefore by Lemma 5.1 it is also a morphism of E-objects with respect to the λ_2 action on $L * R$, i.e.,

$$\Delta_L \circ \lambda_L(r) = (L * r) \circ \Delta_L, \quad r \in E.$$

As for the remaining two actions we can compute, using the expressions in Lemma 4.1 for ρ_1, λ_1, that

$$\begin{aligned} \rho_1(r) \circ \Delta_L &= (\varepsilon_L * R) \circ ((L * r) * R) \circ \delta_L \circ \Delta_L \\ &= (\varepsilon_L * R) \circ ((L * r) * R) \circ (\Delta_L \circ R) \circ \Delta_L = (\lambda_L(r) * R) \circ \Delta_L \\ &= \lambda_1(r) \circ \Delta_L \end{aligned}$$

from which the unique factorization through $z_{L,R} \in \mathcal{E}$ follows. $\square$

Note that in the above Lemma we avoided to use the notation $*_q$ and $*_z$ because under the given conditions they need not be skew-monoidal products.

Theorem 5.3. *If $\mathcal{M}$ has colimits and the endofunctor $R * \_$ preserves coequalizers then*

(i) *the endofunctor $M \mapsto T_q M := \int^{\rho_1 \lambda_2} R * M$ on $\mathcal{E}$ carries a unique monad structure such that the forgetful functor $\phi : \mathcal{E} \to \mathcal{M}$ together with the coequalizer $T\phi M \overset{\kappa_M}{\twoheadrightarrow} \phi T_q M$ of ρ_1 and λ_2 is a monad morphism $\langle \phi, \kappa \rangle$ from T_q to T;*

(ii) *the functor ϕ_q induced by the monad morphism $\langle\phi,\kappa\rangle$ is an equivalence of the Eilenberg–Moore categories such that*

$$\begin{array}{ccc} \mathcal{E}_{T_q} & \xrightarrow{\phi_q} & \mathcal{M}_T \\ {\scriptstyle \mathcal{F}_{T_q}}\downarrow & & \downarrow{\scriptstyle \mathcal{F}_T} \\ \mathcal{E} & \xrightarrow{\phi} & \mathcal{M} \end{array} \tag{60}$$

and the functor $\mathcal{F}_q : \mathcal{M}_T \to \mathcal{E}$ of Lemma 5.1 is monadic and satisfies

$$\mathcal{F}_q\phi_q = \mathcal{F}_{T_q}, \qquad \phi\mathcal{F}_q = \mathcal{F}_T. \tag{61}$$

Proof. This Theorem follows by dualizing the next Theorem 5.4. □

Theorem 5.4. *If $\mathcal{M}$ has limits and the endofunctor $\_ * R$ preserves equalizers then*

(i) *the endofunctor $M \mapsto Q^z M := \int_{\lambda_1\rho_1} M * R$ on $\mathcal{E}$ carries a unique comonad structure such that the forgetful functor $\phi : \mathcal{E} \to \mathcal{M}$ together with the equalizer $\phi Q^z M \overset{\zeta_M}{\rightarrowtail} Q\phi M$ of λ_1 and ρ_1 is a comonad morphism $\langle\phi,\zeta\rangle$ from Q^z to Q;*

(ii) *the functor ϕ_z induced by the comonad morphism $\langle\phi,\zeta\rangle$ is an equivalence of the Eilenberg–Moore categories such that $\mathcal{F}^Q\phi_z = \phi\mathcal{F}^{Q^z}$ and the functor $\mathcal{F}_z : \mathcal{M}^Q \to \mathcal{E}$ of Lemma 5.1 is comonadic and satisfies*

$$\mathcal{F}_z\phi_z = \mathcal{F}^{Q^z}, \qquad \phi\mathcal{F}_z = \mathcal{F}^Q.$$

Proof. This Theorem is the special case of the lax version proven in the next section. Part (i) follows from Proposition 6.2 and part (ii) from Theorem 6.3 after noticing that left exactness of Q implies the possibility to choose the equalizers ζ^n in such a way that $\mathbf{Q}_n = (Q^z)^n$ for each $n \geq 0$. □

Example 5.5. For a right R-bialgebroid H as in Section 3 the monad T is $\_ \underset{R}{\otimes} H$ associated to the R-ring $R \xrightarrow{s^H} H$ and T_q is $\_ \underset{R^e}{\otimes} H$ associated to the R^e-ring $R^{\mathrm{op}} \otimes R \xrightarrow{t^H \otimes s^H} H$. The monad morphism κ_M is the canonical projection $M \underset{R}{\otimes} H \twoheadrightarrow M \underset{R^e}{\otimes} H$ and the fact that it induces an equivalence between the corresponding right H-module categories can be considered as a well-known fact in the bialgebroid literature and it is a consequence of the fact that T is right exact. However, the dual statement Theorem 5.4 presents a warning that the category $(\mathsf{Ab}_R)^H$ of right comodules over the R-coring H may not be equivalent to the Eilenberg–Moore category of the comonad Q^z on ${}_R\mathsf{Ab}_R$ unless ${}_R H$ is flat, i.e., Q is left exact. This equivalence is crucial in Tannaka duality where we want Q^z a monoidal comonad on the bimodule category ${}_R\mathsf{Ab}_R$. Without left exactness the Q^z will not even be a comonad. What replaces Q^z in the general case is a lax comonad discussed in the next section.

6. The lax comonad Q

In [12, Proposition 4.2] Day and Street have characterized (left) R-bialgebroids as comonoids in the lax monoidal category of monads on R^e where the lax monoidal structure is given by n-fold Takeuchi products $M_1 \times_R \cdots \times_R M_n$. Here we shall concentrate on the closely related

but simpler structure of monoidal lax comonads on the category $\mathcal{E}$ of E-objects but ignore monoidality altogether, as we did so far for T and T_q, and be content with proving equivalence of $\mathcal{M}^Q$ with the category $\mathcal{E}^{\mathbf{Q}}$ of comodules for the lax comonad $\mathbf{Q}$ with the hope in mind that if $\mathcal{E}$ is provided a 'good' monoidal structure then $\mathcal{E}^{\mathbf{Q}}$ will become monoidal, too.

Let Δ be the category of finite ordinals and order preserving maps equipped with the strict monoidal structure of ordinal addition $+$. By a lax comonad on a category $\mathcal{E}$ we mean a monoidal functor $G : \Delta^{\mathrm{op}} \to \mathrm{End}\,\mathcal{E}$ to the strict monoidal category of endofunctors on $\mathcal{E}$ with composition of functors as monoidal product. The monoidal structure of G is given by an 'arrow' $\iota : \mathrm{id}_{\mathcal{E}} \to G_0$ of $\mathrm{End}\,\mathcal{E}$ and a natural transformation $\nu_{m,n} : G_m G_n \to G_{m+n}$ satisfying 3 axioms, as usual. If the functor G happens to be strict monoidal then the object map of G is $G_n = (G_1)^n$ and we recover an ordinary comonad $\langle G_1, G_{2\to 1}, G_{0\to 1}\rangle$ on $\mathcal{E}$.

The generalization of the Eilenberg–Moore category for the lax situation goes as follows. A comodule over a lax comonad $\langle \mathcal{E}, G\rangle$ consists of an object M of $\mathcal{E}$ and arrows $\alpha_n : M \to G_n M$ for each $n \geq 0$ such that

$$
\begin{aligned}
&G_f \circ \alpha_n = \alpha_m \quad \forall f : m \to n\\
&\alpha_{m+n} = \nu_{m,n} M \circ G_m \alpha_n \circ \alpha_m \quad \forall m, n \geq 0\\
&\alpha_0 = \iota_M .
\end{aligned}
$$

A comodule map $\langle M, \alpha\rangle \xrightarrow{t} \langle N, \beta\rangle$ is an arrow $M \xrightarrow{t} N$ in $\mathcal{E}$ such that

$$
\begin{array}{ccc}
M & \xrightarrow{t} & N\\
{\scriptstyle \alpha_n}\downarrow & & \downarrow{\scriptstyle \beta_n} \quad \forall n \geq 0.\\
G_n M & \xrightarrow{G_n t} & G_n N
\end{array}
$$

The category of G-comodules and their comodule maps is denoted by $\mathcal{E}^G$. The forgetful functor $\mathcal{E}^G \to \mathcal{E}$, $\langle M, \alpha\rangle \mapsto M$ is faithful, reflects isomorphisms but not left adjoint in general.

In order to justify the above definition of $\mathcal{E}^G$ it is worth looking at its 2-categorical interpretation. For lax comonads F on $\mathcal{D}$ and G on $\mathcal{E}$ a morphism of lax comonads $\langle \mathcal{D}, F\rangle \to \langle \mathcal{E}, G\rangle$ can be defined to consist of a functor $U : \mathcal{D} \to \mathcal{E}$ and natural transformations

$$
\xi_n : U F_n \to G_n U : \mathcal{D} \to \mathcal{E} \quad \text{natural in } n \in \Delta^{\mathrm{op}}
$$

and obeying the following monoidality conditions

$$
\begin{array}{ccccc}
U F_m F_n & \xrightarrow{\xi_m F_n} & G_m U F_n & \xrightarrow{G_m \xi_n} & G_m G_n U\\
\downarrow & & & & \downarrow\\
U F_{m+n} & & \xrightarrow{\xi_{m+n}} & & G_{m+n} U
\end{array}
\qquad
\begin{array}{ccc}
U & = & U\\
\downarrow & & \downarrow\\
U F_0 & \xrightarrow{\xi_0} & G_0 U
\end{array}
$$

A modification $\tau : \langle U, \xi\rangle \to \langle V, \upsilon\rangle : \langle \mathcal{D}, F\rangle \to \langle \mathcal{E}, G\rangle$ is a natural transformation $\tau : U \to V$ satisfying

$$
\begin{array}{ccc}
U F_m & \xrightarrow{\tau F_m} & V F_m\\
{\scriptstyle \xi_m}\downarrow & & \downarrow{\scriptstyle \upsilon_m} \quad \forall m \geq 0.\\
G_m U & \xrightarrow{G_m \tau} & G_m V
\end{array}
$$

With the obvious horizontal and vertical compositions the lax comonads, their morphisms and modifications form a 2-category Lax-Cmd.

Lemma 6.1. *Let* $\mathbf{1}$ *be the identity comonad on the terminal category 1. Then for any lax comonad* $\langle \mathcal{E}, G\rangle$ *the Eilenberg–Moore category* $\mathcal{E}^G$ *of* G*-comodules can be identified with the hom-category* Lax-Cmd$(\langle 1, \mathbf{1}\rangle, \langle \mathcal{E}, G\rangle)$.

Proof. A morphism of lax comonads from $\mathbf{1}$ to G is an object M of $\mathcal{E}$ equipped with $\alpha_n : M \to G_n M$, $n \geq 0$, satisfying precisely the defining relations of a G-comodule. A modification $\langle M, \alpha\rangle \xrightarrow{t} \langle N, \beta\rangle$ in turn is an arrow $M \xrightarrow{t} N$ satisfying $\beta_n \circ t = G_n t \circ \alpha_n$, $n \geq 0$, i.e., a comodule map. □

By extending Lemma 6.1 notice that a morphism $\langle U, \xi\rangle : \langle \mathcal{D}, F\rangle \to \langle \mathcal{E}, G\rangle$ of lax comonads induces a functor

$$\text{Lax-Cmd}(\langle 1, \mathbf{1}\rangle, \langle U, \xi\rangle) \ : \ \mathcal{D}^F \to \mathcal{E}^G$$

between the Eilenberg–Moore categories the object map of which is

$$\langle D, \alpha\rangle \mapsto \langle UD, (UD \xrightarrow{U\alpha_n} UF_nD \xrightarrow{\xi_n D} G_nUD)_{n\geq 0}\rangle.$$

After this preparation we can introduce the *canonical lax comonad* $\mathbf{Q}$ of a right-monoidal category $\langle \mathcal{M}, *, R, \gamma, \eta, \varepsilon\rangle$. For an E-object M we define $\mathbf{Q}_n M$ by delaying the action of the ends in $(Q^z)^n M$, i.e., by the formula

$$\mathbf{Q}_n M := \int_{\lambda_1 \rho_1} \dots \int_{\lambda_n \rho_n} (\dots (M * R) * \dots * R) * R$$

where the number of R-s is n and the left and right E-actions λ_i, ρ_i are labeled according to what we said in Section 4. The result $\mathbf{Q}_n M$ becomes a left E-object via λ_{n+1} which is the action on the last R factor.

$\mathbf{Q}_0$ is the identity functor and $\mathbf{Q}_1$ is the endofunctor Q^z of Corollary 4.5(ii). But now, without the assumption that $\_ * R$ preserves equalizers, Q^z does not inherit a comonad structure from that of Q. Although $\varepsilon^z : \mathbf{Q}_1 \to \mathbf{Q}_0$ exists we cannot define comultiplication $\mathbf{Q}_1 \to \mathbf{Q}_1^2$. Instead we can define a natural transformation $\delta_0^1 : \mathbf{Q}_1 \to \mathbf{Q}_2$.

Proposition 6.2. *Let* $\mathcal{M}$ *be a right-monoidal category whose underlying category* $\mathcal{M}$ *is complete. Let* $\mathcal{E}$ *be the category of* E*-objects in* $\mathcal{M}$*,* ϕ *the forgetful functor* $\mathcal{E} \to \mathcal{M}$ *and define the endofunctors* $\mathbf{Q}_n$ *on* $\mathcal{E}$ *for* $n \geq 0$ *by the equalizers*

$$\phi\mathbf{Q}_n M \xrightarrowtail{\zeta^n_M} Q^n \phi M \underset{\langle \rho_1, \dots, \rho_n\rangle}{\overset{\langle \lambda_1, \dots, \lambda_n\rangle}{\rightrightarrows}} \{E^{\otimes n}, Q^n \phi M\}$$

where $\{\ ,\ \}$ *denotes cotensor (=power) in* $\mathcal{M}$*. Then* $n \mapsto \mathbf{Q}_n$ *is the object map of a unique lax comonad* $\mathbf{Q}$ *on* $\mathcal{E}$ *such that* ϕ *together with* $\{\zeta^n | n \geq 0\}$ *is a morphism of lax comonads* $\mathbf{Q} \to Q$.

Proof. In order to extend $\mathbf{Q}$ to a functor $\Delta^{\mathrm{op}} \to \mathrm{End}\,\mathcal{E}$ it suffices to define it on the elementary monotone functions $i + (2 \to 1) + j$ and $i + (0 \to 1) + j$. Naturality of ζ^n determines them to be the unique $\delta_i^n : \mathbf{Q}_n \to \mathbf{Q}_{n+1}$ and $\varepsilon_i^n : \mathbf{Q}_n \to \mathbf{Q}_{n-1}$, respectively, such that

$$\zeta^{n+1} \circ \delta_i^n = Q^i \delta Q^{n-i-1} \circ \zeta^n \quad i = 0, 1, \dots, n-1, \ n \geq 0 \tag{62}$$

$$\zeta^{n-1} \circ \varepsilon_i^n = Q^i \varepsilon Q^{n-i-1} \circ \zeta^n \quad i = 0, 1, \dots, n-1, \ n > 0. \tag{63}$$

For their existence the reader should check that the RHS satisfies the equalizing conditions of the ζ on the LHS as a consequence of the properties of δ and ε given in Lemma 4.2. The form of the RHS of these equations makes it obvious that they satisfy the usual relations that a simplicial object in $\mathrm{End}\,\mathcal{E}$ should have. This proves that $\mathbf{Q}$ is a functor.

As for the monoidal structure $\nu^{m,n} : \mathbf{Q}_m\mathbf{Q}_n \to \mathbf{Q}_{m+n}$ the requirement that ζ be monoidal leaves only one possibility,

$$\zeta^{m+n} \circ \nu^{m,n} = Q^m\zeta^n \circ \zeta^m\mathbf{Q}_n \equiv \zeta^m Q^n \circ \mathbf{Q}_m\zeta^n\,, \tag{64}$$

which exists by the equalizing properties of the RHS. Since $\mathbf{Q}_0 = \mathbf{1}_{\mathcal{E}}$, we can take ι to be the identity natural transformation $\mathbf{1}_{\mathcal{E}} \to \mathbf{1}_{\mathcal{E}}$, provided we also choose ζ^0 to be the identity. Then the monoidality conditions on ζ are built in the definition of ν and ι and the monoidality constraints on ν and ι boil down to

$$\begin{aligned}
&\nu^{l+m,n} \circ \nu^{l,m}\mathbf{Q}_n = \nu^{l,m+n} \circ \mathbf{Q}_l\nu^{m,n}\\
&\nu^{0,n} = \mathbf{Q}_n\\
&\nu^{m,0} = \mathbf{Q}_m.
\end{aligned}$$

The last two follow from uniqueness of ν and the first can be shown by multiplying it with ζ^{l+m+n} and using (64).

Finally, we have to show naturality of ν (and of ι). That is to say, we need a proof of

$$\begin{aligned}
&\mathbf{Q}_{f+g} \circ \nu^{m,n} = \nu^{m',n'} \circ \mathbf{Q}_f\mathbf{Q}_g\\
&\forall f : m' \to m, \quad g : n' \to n \text{ in } \Delta.
\end{aligned}$$

It suffices to prove this for f and g being elementary functions, that is to say, to prove

$$\delta_i^{m+n} \circ \nu^{m,n} = \begin{cases} \nu^{m+1,n} \circ \delta_i^m\mathbf{Q}_n & \text{if } i < m\\ \nu^{m,n+1} \circ \mathbf{Q}_m\delta_{i-m}^n & \text{if } i \geq m \end{cases}$$

and

$$\varepsilon_i^{m+n} \circ \nu^{m,n} = \begin{cases} \nu^{m-1,n} \circ \varepsilon_i^m\mathbf{Q}_n & \text{if } i < m\\ \nu^{m,n-1} \circ \mathbf{Q}_m\varepsilon_{i-m}^n & \text{if } i \geq m. \end{cases}$$

Multiplying the first with ζ^{m+n+1} and the second with ζ^{m+n-1} they can be easily verified using the defining relations (62) and (63). □

Theorem 6.3. *The functor $\hat{\phi} : \mathcal{E}^{\mathbf{Q}} \to \mathcal{M}^{Q}$ induced by the lax comonad morphism $\langle\phi, \zeta\rangle : \langle\mathcal{E}, \mathbf{Q}\rangle \to \langle\mathcal{M}, Q\rangle$ of the above Proposition is an equivalence of categories.*

Proof. $\hat{\phi}$ is the lift of the faithful ϕ along the Eilenberg–Moore forgetful functors,

$$\begin{array}{ccc} \mathcal{E}^{\mathbf{Q}} & \xrightarrow{\hat{\phi}} & \mathcal{M}^{Q} \\ \downarrow & & \downarrow \\ \mathcal{E} & \xrightarrow{\phi} & \mathcal{M} \end{array}$$

therefore it is faithful, too. For an arrow $t \in \mathcal{M}^Q(\hat{\phi}\langle M,\alpha\rangle, \hat{\phi}\langle N,\beta\rangle)$ we have

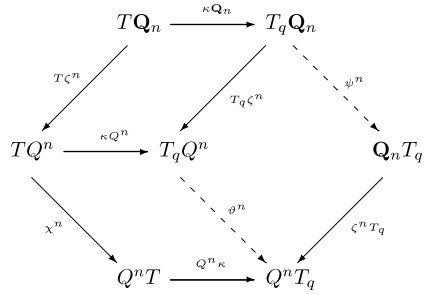
$$\begin{array}{ccccc} \phi M & \xrightarrow{\phi\alpha_n} & \phi\mathbf{Q}_n M & \xrightarrow{\zeta^n_M} & Q^n\phi M \\ {\scriptstyle t}\downarrow & & & & \downarrow{\scriptstyle Q^n t} \\ \phi N & \xrightarrow{\phi\beta_n} & \phi\mathbf{Q}_n N & \xrightarrow{\zeta^n_N} & Q^n\phi N \end{array}$$

therefore by Lemma 5.1 $t = \phi\tau$ for a unique $\tau \in \mathcal{E}(M,N)$. This allows to insert the arrow $\phi\mathbf{Q}_n\tau$ in the middle of the diagram so that the right square is commutative. But ζ^n_N being monic implies commutativity of the left square, so τ lifts to an arrow in $\mathcal{E}^{\mathbf{Q}}(\langle M,\alpha\rangle, \langle N,\beta\rangle)$. This proves that $\hat{\phi}$ is full. Finally we show that $\hat{\phi}$ is eso, in fact surjective on objects. Let $\langle M,\alpha\rangle \in \mathcal{M}^Q$. Then by Lemma 5.2 there is an $\langle\hat{M},\hat{\alpha}\rangle \in \mathcal{E}^{\mathbf{Q}}$ such that

$$\left(M \xrightarrow{\phi\hat{\alpha}_n} \phi\mathbf{Q}_n\hat{M} \xrightarrow{\zeta^n_M} Q^n M\right) = \alpha_n \equiv Q^{n-1}\alpha \circ \ldots Q\alpha \circ \alpha$$

i.e., such that $\hat{\phi}\langle\hat{M},\hat{\alpha}\rangle = \langle M,\alpha\rangle$. Thus $\hat{\phi}$ is eso. □

Remark 6.4. There is a lift of the distributive law $\chi : TQ \to QT$ of Lemma 2.6 to a *lax distributive law* $\psi_n : T_q\mathbf{Q}_n \to \mathbf{Q}_n T_q$ provided we consider T, Q, T_q and $\mathbf{Q}_n$ as endofunctors on the category $\mathcal{M}^{(2)}$ of (2,1)-type E-objects, which is the category of $R^{\mathrm{op}}\otimes R$-$R^{\mathrm{op}}\otimes R$-bimodules in case $\mathcal{M} = \mathsf{Ab}_R$. Of course, $\mathcal{M}$ has to have limits and colimits and $R * \_$ has to preserve coequalizers in order for T_q to be a monad and κ a monad morphism. More precisely, T_q on $\mathcal{M}^{(2)}$ is defined as T_q on $\mathcal{M}^{(1)} = \mathcal{E}$ by considering $M \in \mathcal{M}^{(2)}$ as a (1, 1)-type E-object in $\mathcal{E}$ via ρ_1 and λ_2 and $\mathbf{Q}_n$ on $\mathcal{M}^{(2)}$ is defined as $\mathbf{Q}_n$ on $\mathcal{M}^{(1)}$ by considering $M \in \mathcal{M}^{(2)}$ as a (1, 1)-type E-object in $\mathcal{E}$ via λ_1 and ρ_1. Let $\chi^n := Q^{n-1}\chi \circ \cdots \circ Q\chi Q^{n-2} \circ \chi Q^{n-1}$ and consider the diagram



in which κQ^n is a coequalizer which defines ϑ^n and $\zeta^n T_q$ is an equalizer which defines ψ^n. Then one obtains the distributive laws

$$\begin{aligned} &\mathbf{Q}_n\mu^q \circ \psi^n T_q \circ T_q\psi^n = \psi^n \circ \mu^q\mathbf{Q}_n \\ &\psi^n \circ \eta^q\mathbf{Q}_n = \mathbf{Q}_n\eta^q \\ &\psi^{n+1} \circ T_q\delta^n_i = \delta^n_i T_q \circ \psi^n \\ &\psi^{n-1} \circ T_q\varepsilon^n_i = \varepsilon^n_i T_q \circ \psi^n \end{aligned}$$

as a consequence of (17), (19), (18) and (20), respectively. While the last two express only naturality of ψ, the first two contain the monad data $\langle T_q, \mu^q, \eta^q\rangle$. The difference disappears, however, if we introduce the lax monad $\mathbf{T}$ as a cosimplicial object $\Delta \to \mathrm{End}\,\mathcal{M}^{(2)}$ by

$$\mathbf{T}_m := T_q^m, \quad m \geq 0$$
$$\mathbf{T}_{i+(2\to 1)+j} := T_q^i \mu^q T_q^j, \quad i, j \geq 0$$
$$\mathbf{T}_{i+(1\to 0)+j} := T_q^i \varepsilon^q T_q^j, \quad i, j \geq 0.$$

Then the lax distributive law becomes deceptively simple, just a natural transformation

$$\mathbf{TQ} \xrightarrow{\ \psi\ } \mathbf{QT} \ : \ \Delta \times \Delta^{\mathrm{op}} \to \mathrm{End}\,\mathcal{M}^{(2)}.$$

Note that $\mathbf{TQ}$ and $\mathbf{QT}$ are not the composite of two functors as in $TQ \xrightarrow{\chi} QT$, rather the monoidal product on their common target category: $\Delta \times \Delta^{\mathrm{op}} \xrightarrow{\mathbf{T}\times\mathbf{Q}} \mathrm{End}\,\mathcal{M}^{(2)} \times \mathrm{End}\,\mathcal{M}^{(2)} \to \mathrm{End}\,\mathcal{M}^{(2)}$. All information on the compatibility of $\psi^{m,n}$ with μ^q, η^q, δ_i^n, ε_i^n seems to be comprised in the naturality of $\psi^{m,n}$ in $m \in \Delta$ and $n \in \Delta^{\mathrm{op}}$. However, $\psi^{m,n}$ also satisfies some 'monoidality' relations in m and n separately which are automatic in this example and which ought to belong to the axioms of a lax distributive law for general lax monad $\mathbf{T}$ and lax comonad $\mathbf{Q}$.

In the rest of the paper we study the problem of how and when (ordinary) monoidal structures on the category $\mathcal{E}$ of E-objects will lead to monoidality of the Eilenberg–Moore categories $\mathcal{E}^{\mathbf{Q}}$ or $\mathcal{E}_{T_q}$ with a strong monoidal forgetful functor to $\mathcal{E}$.

7. Bi(co)monad induced structures

In Section 3 we have seen how right R-bialgebroids induce right-monoidal structures on the category Ab_R of right R-modules. Since bialgebroids correspond to bimonads, i.e., opmonoidal monads, on $\mathcal{E} = {}_R\mathsf{Ab}_R$ [25], it is natural to look for generalizations that produce right-monoidal categories from bimonads.

Let $\langle \mathcal{E}, \otimes, R, \mathbf{a}, \mathbf{l}^{-1}, \mathbf{r}\rangle$ be a monoidal category. Then a bimonad, more precisely a $\otimes$-bimonad, $\langle O, \omega, \iota\rangle$ consists of an endofunctor O on $\mathcal{E}$ together with an opmonoidal structure $O^{M,N} : O(M \otimes N) \to OM \otimes ON$, $O^0 : OR \to R$ and natural transformations $\omega : OO \to O$ and $\iota : \mathcal{E} \to O$ satisfying the monad axioms (not involving the $\otimes$-structure) and the opmonoidality axioms

$$\mathbf{a}_{OL,OM,ON} \circ (OL \otimes O^{M,N}) \circ O^{L,M\otimes N} = (O^{L,M} \otimes ON) \circ O^{L\otimes M,N} \circ O\mathbf{a}_{L,M,N} \quad (65)$$
$$(O^0 \otimes ON) \circ O^{R,N} \circ O\mathbf{l}_N^{-1} = \mathbf{l}_{ON}^{-1} \quad (66)$$
$$\mathbf{r}_{OM} \circ (OM \otimes O^0) \circ O^{M,R} = O\mathbf{r}_M \quad (67)$$
$$(\omega_M \otimes \omega_N) \circ O^{OM,ON} \circ OO^{M,N} = O^{M,N} \circ \omega_{M\otimes N} \quad (68)$$
$$O^0 \circ \omega_R = O^0 \circ OO^0 \quad (69)$$
$$O^{M,N} \circ \iota_{M\otimes N} = \iota_M \otimes \iota_N \quad (70)$$
$$O^0 \circ \iota_R = R. \quad (71)$$

We have written them using only $\mathbf{a}$, $\mathbf{l}^{-1}$ and $\mathbf{r}$ but never their inverses. This admits to speak about opmonoidal monads in right-monoidal categories. Such right-opmonoidal monads are not really new, they are just the monads in the 2-category r-OpmonCat. Indeed, relations (65)–(67) say exactly that O is a 1-cell and relations (68)–(71) say that ω and ι are 2-cells of this 2-category.

The so-called fusion operator [7] associated to a bimonad $\langle O, \omega, \iota\rangle$ is the natural transformation

$$h_{M,N} \coloneqq (OM \otimes \omega_N) \circ O^{M,ON} \;:\; O(M \otimes ON) \to OM \otimes ON. \tag{72}$$

Given a fusion operator we can recover the opmonoidal structure by

$$O^{M,N} = h_{M,N} \circ O(M \otimes \iota_N). \tag{73}$$

The next result is essentially [7, Proposition 2.6] of Bruguieres et al. although some of the output is turned into input. But the main difference is the observation that the statement is valid also when $\otimes$ is a skew-monoidal product.

Proposition 7.1. *Let $\langle \mathcal{E}, \otimes, R, \mathbf{a}, \mathbf{l}^{-1}, \mathbf{r}\rangle$ be a right-monoidal category and $\langle O, \omega, \iota\rangle$ be a monad on $\mathcal{E}$. Then opmonoidal structures on O, i.e., $O^{M,N}$, O^0 satisfying (65)–(71), are in bijection with data consisting of a natural transformation $h_{M,N} : O(M \otimes ON) \to OM \otimes ON$ and the same O^0 satisfying the following relations:*

$$(OM \otimes \omega_N) \circ h_{M,ON} = h_{M,N} \circ O(M \otimes \omega_N) \tag{74}$$

$$\begin{aligned}&(h_{L,M} \otimes ON) \circ h_{L\otimes OM,N} \circ O\mathbf{a}_{L,OM,ON} \circ O(L \otimes h_{M,N})\\ &\quad = \mathbf{a}_{OL,OM,ON} \circ (OL \otimes h_{M,N}) \circ h_{L,M\otimes ON}\end{aligned} \tag{75}$$

$$h_{M,N} \circ \iota_{M\otimes ON} = \iota_M \otimes ON \tag{76}$$

$$(O^0 \otimes ON) \circ h_{R,N} \circ O\mathbf{l}^{-1}_{ON} = \mathbf{l}^{-1}_{ON} \circ \omega_N \tag{77}$$

$$\mathbf{r}_{OM} \circ (OM \otimes O^0) \circ h_{M,R} = O\mathbf{r}_M \circ O(M \otimes O^0) \tag{78}$$

$$(\omega_M \otimes ON) \circ h_{OM,N} \circ Oh_{M,N} = h_{M,N} \circ \omega_{M\otimes ON} \tag{79}$$

$$O^0 \circ \iota_R = R. \tag{80}$$

The bijection is given by Eqs. (72) and (73).

Proof. Assume that an opmonoidal structure $O^{M,N}$, O^0 is given and h is defined by (72). Then (74) can be shown using associativity of the monad multiplication ω,

$$\begin{aligned}(OM \otimes \omega_N) \circ h_{M,ON} &= (OM \otimes \omega_N) \circ (OM \otimes \omega_{ON}) \circ O^{M,O^2N}\\ &= (OM \otimes \omega_N) \circ (OM \otimes O\omega_N) \circ O^{M,O^2N}\\ &= (OM \otimes \omega_N) \circ O^{M,ON} \circ O(M \otimes \omega_N)\\ &= h_{M,N} \circ O(M \otimes \omega_N).\end{aligned}$$

The proof of the associativity law (75) is a bit longer:

$$
\begin{aligned}
&(h_{L,M}\otimes ON)\circ h_{L\otimes OM,N}\circ O\mathbf{a}_{L,OM,ON}\circ O(L\otimes h_{M,N})\\
&\quad=((OL\otimes\omega_M)\otimes ON)\circ(O^{L,OM}\otimes ON)\circ(O(L\otimes OM)\otimes\omega_N)\\
&\qquad\circ O^{L\otimes OM,ON}\circ O\mathbf{a}_{L,OM,ON}\circ O(L\otimes(OM\otimes\omega_N))\circ O(L\otimes O^{M,ON})\\
&\quad=((OL\otimes\omega_M)\otimes\omega_N)\circ((OL\otimes O^2M)\otimes O\omega_N)\circ(O^{L,OM}\otimes O^3N)\\
&\qquad\circ O^{L\otimes OM,O^2N}\circ O\mathbf{a}_{L,OM,O^2N}\circ O(L\otimes O^{M,ON})\\
&\quad\overset{(65)}{=}((OL\otimes\omega_M)\otimes\omega_N)\circ((OL\otimes O^2M)\otimes O\omega_N)\circ\mathbf{a}_{OL,O^2M,O^3N}\\
&\qquad\circ(OL\otimes O^{OM,O^2N})\circ O^{L,OM\otimes O^2N}\circ O(L\otimes O^{M,ON})\\
&\quad=\mathbf{a}_{OL,OM,ON}\circ(OL\otimes(\omega_M\otimes\omega_N))\circ(OL\otimes(O^2M\otimes O\omega_N))\\
&\qquad\circ(OL\otimes O^{OM,O^2N})\circ(OL\otimes OO^{M,ON})\circ O^{L,O(M\otimes ON)}\\
&\quad=\mathbf{a}_{OL,OM,ON}\circ(OL\otimes(OM\otimes\omega_N))\\
&\qquad\circ(OL\otimes[(\omega_M\otimes\omega_{ON})\circ O^{OM,O^2N}\circ OO^{M,ON}])\circ O^{L,O(M\otimes ON)}\\
&\quad\overset{(68)}{=}\mathbf{a}_{OL,OM,ON}\circ(OL\otimes h_{M,N})\circ h_{L,M\otimes ON}.
\end{aligned}
$$

As for the remaining relations we proceed as follows:

$$
\begin{aligned}
h_{M,N}\circ\iota_{M\otimes ON} &= (OM\otimes\omega_N)\circ O^{M,ON}\circ\iota_{M\otimes ON}\\
&\overset{(70)}{=}(OM\otimes\omega_N)\circ(\iota_M\otimes\iota_{ON})=\iota_M\otimes ON,\\
(O^0\otimes ON)\circ h_{R,N}\circ O\mathbf{l}^{-1}_{ON} &= (R\otimes\omega_N)\circ(O^0\otimes O^2N)\circ O^{R,ON}\circ O\mathbf{l}^{-1}_{ON}\\
&\overset{(66)}{=}(R\otimes\omega_N)\circ\mathbf{l}^{-1}_{O^2N}=\mathbf{l}^{-1}_{ON}\circ\omega_N,\\
\mathbf{r}_{OM}\circ(OM\otimes O^0)\circ h_{M,R} &= \mathbf{r}_{OM}\circ(OM\otimes O^0)\circ(OM\otimes\omega_R)\circ O^{M,OR}\\
&\overset{(69)}{=}\mathbf{r}_{OM}\circ(OM\otimes O^0)\circ(OM\otimes OO^0)\circ O^{M,OR}\\
&=\mathbf{r}_{OM}\circ(OM\otimes O^0)\circ O^{M,R}\circ O(M\otimes O^0)\\
&\overset{(67)}{=}O\mathbf{r}_M\circ O(M\otimes O^0)
\end{aligned}
$$

and finally (79) follows from (68) easily.

Now assume that a fusion operator h is given, together with O^0, and define $O^{M,N}$ by (73). First, (70) follows easily from (76). Then associativity relation (65) can be shown by means of (75) and (70):

$$
\begin{aligned}
&\mathbf{a}_{OL,OM,ON}\circ(OL\otimes O^{M,N})\circ O^{L,M\otimes N}\\
&\quad=\mathbf{a}_{OL,OM,ON}\circ(OL\otimes h_{M,N})\circ(OL\otimes O(M\otimes\iota_N))\circ h_{L,M\otimes N}\circ O(L\otimes\iota_{M\otimes N})\\
&\quad=\mathbf{a}_{OL,OM,ON}\circ(OL\otimes h_{M,N})\circ h_{L,M\otimes ON}\circ O(L\otimes O(M\otimes\iota_N))\circ O(L\otimes\iota_{M\otimes N})\\
&\quad\overset{(75)}{=}(h_{L,M}\otimes ON)\circ h_{L\otimes OM,N}\circ O\mathbf{a}_{L,OM,ON}\\
&\qquad\circ O(L\otimes h_{M,N})\circ O(L\otimes O(M\otimes\iota_N))\circ O(L\otimes\iota_{M\otimes N})\\
&\quad\overset{(70)}{=}(h_{L,M}\otimes ON)\circ h_{L\otimes OM,N}\circ O\mathbf{a}_{L,OM,ON}\circ O(L\otimes(\iota_M\otimes\iota_N))\\
&\quad=(h_{L,M}\otimes ON)\circ(O(L\otimes\iota_M)\otimes ON)\circ h_{L\otimes M,N}\circ O((L\otimes M)\otimes\iota_N)\circ O\mathbf{a}_{L,M,N}\\
&\quad=(O^{L,M}\otimes ON)\circ O^{L\otimes M,N}\circ O\mathbf{a}_{L,M,N}.
\end{aligned}
$$

Eq. (66) is a simple consequence of (77) if we compose the latter with $O\iota_N$. Similarly, (67) follows from (78) and (80). For proving (68) we need relation (79) and the calculation

$$\begin{aligned}
&(\omega_M \otimes \omega_N) \circ h_{OM,ON} \circ O(OM \otimes \iota_{ON}) \circ Oh_{M,N} \circ O^2(M \otimes \iota_N) \\
&\quad = (\omega_M \otimes ON) \circ h_{OM,N} \circ Oh_{M,N} \circ O^2(M \otimes \iota_N) \\
&\quad \overset{(79)}{=} h_{M,N} \circ \omega_{M\otimes ON} \circ O^2(M \otimes \iota_N) \\
&\quad = h_{M,N} \circ O(M \otimes \iota_N) \circ \iota_{M\otimes N} = O^{M,N} \circ \iota_{M\otimes N}.
\end{aligned}$$

Finally, (69) is the consequence of (77) and (78),

$$\begin{aligned}
O^0 \circ \omega_R = \ & \mathbf{r}_R \circ \mathbf{l}_R^{-1} \circ O^0 \circ \omega_R = \mathbf{r}_R \circ (R \otimes O^0) \circ \mathbf{l}_{OR}^{-1} \circ \omega_R \\
&\overset{(77)}{=} \mathbf{r}_R \circ (O^0 \otimes O^0) \circ h_{R,R} \circ O\mathbf{l}_{OR}^{-1} = O^0 \circ \mathbf{r}_{OR} \circ (OR \otimes O^0) \circ h_{R,R} \circ O\mathbf{l}_{OR}^{-1} \\
&\overset{(78)}{=} O^0 \circ O\mathbf{r}_R \circ O(R \otimes O^0) \circ O\mathbf{l}_{OR}^{-1} = O^0 \circ O\mathbf{r}_R \circ O\mathbf{l}_{OR}^{-1} \circ OO^0 \\
&= O^0 \circ OO^0.
\end{aligned}$$

This finishes the proof that O is opmonoidal.

It remains to verify that (72) and (73) define a bijection between fusion operators and opmonoidal structures. While the composite mapping $O^{M,N} \mapsto h_{M,N} \mapsto O^{M,N}$ is the identity for whatever $O^{M,N}$, the composite $h_{M,N} \mapsto O^{M,N} \mapsto h_{M,N}$ becomes the identity after using (74). □

Proposition 7.2. *Let $\langle O, \omega, \iota\rangle$ be a bimonad on the (right-)monoidal category $\langle \mathcal{E}, \otimes, R, \mathbf{a}, \mathbf{l}^{-1}, \mathbf{r}\rangle$. Then there is a right-monoidal structure on $\mathcal{E}$ given by*

$$M \odot N := M \otimes ON \tag{81}$$

$$\dot\gamma_{L,M,N} := \mathbf{a}_{L,OM,ON} \circ (L \otimes (OM \otimes \omega_N)) \circ (L \otimes O^{M,ON}) \tag{82}$$

$$\dot\eta_M := \mathbf{l}_{OM}^{-1} \circ \iota_M \tag{83}$$

$$\dot\varepsilon_M := \mathbf{r}_M \circ (M \otimes O^0). \tag{84}$$

The unit $\mathbf{l}^{-1}$ of the $\otimes$-structure gives rise to a monad morphism $\mathbf{l}_{ON}^{-1} : ON \to \dot TN$ from O to the canonical monad $\dot T = R \odot \_$ of the $\odot$-structure.

Proof. By Proposition 7.1 the monad O is supplied with a fusion operator h. Since the associator $\dot\gamma$ is essentially given by the fusion operator, the pentagon equation (1) for the $\odot$ product is a consequence of (75) and of the pentagon equation for $\otimes$,

$$\begin{aligned}
&(\dot\gamma_{K,L,M} \odot N) \circ \dot\gamma_{K,L\odot M,N} \circ (K \odot \dot\gamma_{L,M,N}) \\
&\quad = (\mathbf{a}_{K,OL,OM} \otimes ON) \circ ((K \otimes h_{L,M}) \otimes ON) \circ \mathbf{a}_{K,O(L\otimes OM),ON} \circ (K \otimes h_{L\otimes OM,N}) \\
&\qquad \circ (K \otimes O\mathbf{a}_{L,OM,ON}) \circ (K \otimes O(L \otimes h_{M,N})) \\
&\quad = (\mathbf{a}_{K,OL,OM} \otimes ON) \circ \mathbf{a}_{K,OL\otimes OM,ON} \\
&\qquad \circ \big(K \otimes \big[(h_{L,M} \otimes ON) \circ h_{L\otimes OM,N} \circ O\mathbf{a}_{L,OM,ON} \circ O(L \otimes h_{M,N})\big]\big) \\
&\quad \overset{(75)}{=} (\mathbf{a}_{K,OL,OM} \otimes ON) \circ \mathbf{a}_{K,OL\otimes OM,ON} \circ (K \otimes \mathbf{a}_{OL,OM,ON}) \\
&\qquad \circ (K \otimes (OL \otimes h_{M,N})) \circ (K \otimes h_{L,M\otimes ON}) \\
&\quad = \mathbf{a}_{K\otimes OL,OM,ON} \circ \mathbf{a}_{K,OL,OM\otimes ON} \circ (K \otimes (OL \otimes h_{M,N})) \circ (K \otimes h_{L,M\otimes ON}) \\
&\quad = \dot\gamma_{K\odot L,M,N} \circ \dot\gamma_{K,L,M\odot N}.
\end{aligned}$$

The unit-triangle (2) for $\odot$ follows from (76) and from the unit triangle for $\otimes$,

$$\begin{aligned}\dot\gamma_{R,M,N}\circ\dot\eta_{M\odot N} &= \mathbf{a}_{R,OM,ON}\circ(R\otimes h_{M,N})\circ(R\otimes\iota_{M\otimes ON})\circ\mathbf{l}^{-1}_{M\otimes ON}\\ &\overset{(76)}{=} \mathbf{a}_{R,OM,ON}\circ(R\otimes(\iota_M\otimes ON))\circ\mathbf{l}^{-1}_{M\otimes ON}\\ &= ((R\otimes\iota_M)\otimes ON)\circ(\mathbf{l}^{-1}_M\otimes ON) = \dot\eta_M\odot N.\end{aligned}$$

The counit-triangle (3) for $\odot$ follows from the counit triangle for $\otimes$ and from (78),

$$\begin{aligned}\varepsilon_{M\odot N}\circ\dot\gamma_{M,N,R} &= \mathbf{r}_{M\otimes ON}\circ((M\otimes ON)\otimes O^0)\circ\mathbf{a}_{M,ON,OR}\circ(M\otimes h_{N,R})\\ &= (M\otimes\mathbf{r}_{ON})\circ(M\otimes(ON\otimes O^0))\circ(M\otimes h_{N,R})\\ &\overset{(78)}{=} (M\otimes O\mathbf{r}_N)\circ(M\otimes O(N\otimes O^0)) = M\odot\dot\varepsilon_N.\end{aligned}$$

The mixed triangle (4) can be shown using (77) and then the analogous triangle for $\otimes$:

$$\begin{aligned}&(\dot\varepsilon_M\odot N)\circ\dot\gamma_{M,R,N}\circ(M\odot\dot\eta_N)\\ &\quad= ((\mathbf{r}_M\circ(M\otimes O^0))\otimes ON)\circ\mathbf{a}_{M,OR,ON}\circ(M\otimes h_{R,N})\circ(M\otimes O\mathbf{l}^{-1}_{ON}\circ O\iota_N)\\ &\quad= (\mathbf{r}_M\otimes ON)\circ\mathbf{a}_{M,R,ON}\circ(M\otimes(O^0\otimes ON))\\ &\qquad\circ(M\otimes h_{R,N})\circ(M\otimes O\mathbf{l}^{-1}_{ON})\circ(M\otimes O\iota_N)\\ &\quad\overset{(77)}{=} (\mathbf{r}_M\otimes ON)\circ\mathbf{a}_{M,R,ON}\circ(M\otimes\mathbf{l}^{-1}_{ON})\circ(M\otimes\omega_N)\circ(M\otimes O\iota_N)\\ &\quad= (\mathbf{r}_M\otimes ON)\circ\mathbf{a}_{M,R,ON}\circ(M\otimes\mathbf{l}^{-1}_{ON}) = M\odot N.\end{aligned}$$

Finally, (5) for $\odot$ follows from (71) and from the analogous axiom for $\otimes$,

$$\dot\varepsilon_R\circ\dot\eta_R = \mathbf{r}_R\circ(R\otimes O^0)\circ(R\otimes\iota_R)\circ\mathbf{l}^{-1}_R = \mathbf{r}_R\circ\mathbf{l}^{-1}_R = R.$$

This finishes the proof that $\odot$ is a right-monoidal structure. The natural transformation $\mathbf{l}^{-1}_{ON}$ (together with the identity functor on $\mathcal{E}$) is a monad morphism $O\to\dot T$ if it satisfies the following two conditions:

$$\dot\mu_N\circ\mathbf{l}^{-1}_{O\dot TN}\circ O\mathbf{l}^{-1}_{ON} = \mathbf{l}^{-1}_{ON}\circ\omega_N \tag{85}$$

$$\dot\eta_M \coloneqq \mathbf{l}^{-1}_{OM}\circ\iota_M. \tag{86}$$

The LHS of the first can be written as

$$\begin{aligned}&(\dot\varepsilon_R\otimes ON)\circ\dot\gamma_{R,R,N}\circ\mathbf{l}^{-1}_{O(R\otimes ON)}\circ O\mathbf{l}^{-1}_{ON}\\ &\quad= (\mathbf{r}_R\otimes ON)\circ((R\otimes O^0)\otimes ON)\circ\mathbf{a}_{R,OR,ON}\circ(R\otimes h_{R,N})\\ &\qquad\circ(R\otimes O\mathbf{l}^{-1}_{ON})\circ\mathbf{l}^{-1}_{O^2N}\\ &\quad\overset{(77)}{=} (\mathbf{r}_R\otimes ON)\circ\mathbf{a}_{R,R,ON}\circ(R\otimes\mathbf{l}^{-1}_{ON})\circ(R\otimes\omega_N)\circ\mathbf{l}^{-1}_{O^2N}\\ &\quad= \mathbf{l}^{-1}_{ON}\circ\omega_N\end{aligned}$$

which is the RHS. The second condition is just the definition (83) of $\dot\eta$, so $\mathbf{l}^{-1}_{ON}$ is a monad morphism as claimed. □

Definition 7.3. The right-monoidal structures twist isomorphic (see Definition 2.4) to ones arising from a bimonad w.r.t. some ordinary monoidal structure $\otimes$ as in Proposition 7.2 are called $\otimes$-representable or representable by a $\otimes$-bimonad.

Passing to the reversed right-monoidal structures one obtains the notion of representability of left-monoidal categories by opmonoidal monads. Up to twist-isomorphism they are given by

$$M \odot N := OM \otimes N$$
$$\dot{\gamma}_{L,M,N} := \mathbf{a}^{-1}_{OL,OM,N} \circ ((\omega_L \otimes OM) \otimes N) \circ (O^{OL,M} \otimes N)$$
$$\dot{\eta}_M := \mathbf{r}^{-1}_{OM} \circ \iota_M$$
$$\dot{\varepsilon}_M := \mathbf{l}_M \circ (O^0 \otimes M).$$

Passing to the opposite category opmonoidal monads become monoidal comonads and we obtain the notion of *corepresentability*.

Definition 7.4. A right-monoidal category $\langle \mathcal{M}, *, R, \gamma, \eta, \varepsilon \rangle$ is corepresentable by a monoidal comonad $\langle C, C_2, C_0, \Delta, \epsilon \rangle$ in a (left-) monoidal structure $\langle \mathcal{M}, \otimes, R, \mathbf{a}^{-1}, \mathbf{r}^{-1}, \mathbf{l} \rangle$ when it is twist-isomorphic to the following right-monoidal structure:

$$M \odot N := N \otimes CM$$
$$\dot{\gamma}_{L,M,N} := (N \otimes C_{M,CL}) \circ (N \otimes (CM \otimes \Delta_L)) \circ \mathbf{a}^{-1}_{N,CM,CL}$$
$$\dot{\eta}_M := (M \otimes C_0) \circ \mathbf{r}^{-1}_M$$
$$\dot{\varepsilon}_M := \epsilon_M \circ \mathbf{l}_{CM}.$$

It is left to the reader to write up what corepresentability means for left-monoidal categories.

8. The representability theorem

We wish to study the situation of a category $\mathcal{E}$ endowed with two right-monoidal structures $\langle \mathcal{E}, *, R, \gamma, \eta, \varepsilon \rangle$ and $\langle \mathcal{E}, \otimes, R, \mathbf{a}, \mathbf{l}^{-1}, \mathbf{r} \rangle$ with a common unit object R. Later the second structure will be assumed to be an ordinary monoidal structure, this explains the notation, but for a good while the unit $\mathbf{l}^{-1}_M : M \to R \otimes M$ is not assumed to be invertible, neither are $\mathbf{a}_{L,M,N}$ and $\mathbf{r}_M$. We shall briefly refer to them as the $*$-structure and the $\otimes$-structure.

In order to relate this situation to that of earlier sections one may think $\mathcal{E}$ as the category of left E-objects in Ab_R, i.e., $\mathcal{E}$ is the bimodule category ${}_R\mathsf{Ab}_R$ with $\otimes$ the tensor product $\underset{R}{\otimes}$. Then $*$ is the quotient $*_q$ of a right-monoidal structure on Ab_R as it was described in Proposition 4.3.

Definition 8.1. A tetrahedral homomorphism from the $*$-structure to the $\otimes$-structure is a natural transformation

$$t_{L,M,N} : L \otimes (M * N) \to (L \otimes M) * N$$

satisfying the following axioms:

$$(\mathbf{a}_{K,L,M} * N) \circ t_{K,L\otimes M,N} \circ (K \otimes t_{L,M,N}) = t_{K\otimes L,M,N} \circ \mathbf{a}_{K,L,M*N} \tag{87}$$
$$(t_{K,L,M} * N) \circ t_{K,L*M,N} \circ (K \otimes \gamma_{L,M,N}) = \gamma_{K\otimes L,M,N} \circ t_{K,L,M*N} \tag{88}$$
$$t_{R,M,N} \circ \mathbf{l}^{-1}_{M*N} = \mathbf{l}^{-1}_M * N \tag{89}$$
$$\varepsilon_{M\otimes N} \circ t_{M,N,R} = M \otimes \varepsilon_N. \tag{90}$$

A tetrahedral isomorphism is a tetrahedral homomorphism t for which

$$w_{M,N} := (\mathbf{r}_M * N) \circ t_{M,R,N} : M \otimes TN \to M * N \tag{91}$$

is a natural isomorphism where $T = R * \_$.

Axioms (87) and (88) are pentagons on the string of symbols $K\otimes L\otimes M* N$ and $K\otimes L* M* N$, respectively. Axioms (89) and (90) are analogous to the unit and counit axioms (2) and (3). The analogue of (4) is void since we have no distinguished arrow $M\otimes N\to M* N$ to put on the right hand side, except the one on the left hand side.

The above axioms for t can be recognized to be a fragment of the Cockett–Seely axioms for 'linearly distributive categories' [10] although we do not assume either $*$ or $\otimes$ to be monoidal structures. Our terminology "tetrahedral" refers to the early 90s when Ocneanu used a tetrahedral calculus to formulate his 'double-triangle algebras' [20,21].

Lemma 8.2. *For t a tetrahedral isomorphism from a $*$-structure to a $\otimes$-structure we have the following results.*

$$w_{R,N}\circ \mathbf{l}^{-1}_{TN} = TN \tag{92}$$

$$t_{L,M,N} = w_{L\otimes M,N}\circ \mathbf{a}_{L,M,TN}\circ (L\otimes w^{-1}_{M,N}) \tag{93}$$

$$\begin{aligned}&(w_{L,M} * N)\circ w_{L\otimes TM,N}\circ \mathbf{a}_{L,TM,TN}\circ (L\otimes w^{-1}_{TM,N})\circ (L\otimes \gamma_{R,M,N})\\ &\quad = \gamma_{L,M,N}\circ w_{L,M*N}\end{aligned} \tag{94}$$

$$\varepsilon_M\circ w_{M,R} = \mathbf{r}_M\circ (M\otimes \varepsilon_R). \tag{95}$$

Proof. Setting $M = R$ in (89) and multiplying it with $\mathbf{r}_R * N$ we obtain $w_{R,N}\circ \mathbf{l}^{-1}_{TN} = (\mathbf{r}_R * N)\circ (\mathbf{l}^{-1}_R * N)$ the RHS of which is the identity by axiom (5) for the $\otimes$-structure. This proves (92).

Set $(K, L, M, N) = (L, M, R, N)$ in the pentagon (87), multiply it with $\mathbf{r}_{L\otimes M} * N$ and use (3) for the $\otimes$. Then we obtain

$$((L\otimes \mathbf{r}_M) * N)\circ t_{L,M\otimes R,N}\circ (L\otimes t_{M,R,N}) = w_{L\otimes M,N}\circ \mathbf{a}_{L,M,TN}.$$

Using naturality of t the LHS becomes $t_{L,M,N}\circ (L\otimes w_{M,N})$ from which (93) follows immediately.

Setting $(K, L, M, N) = (L, R, M, N)$ in (88) and then multiplying it with $(\mathbf{r}_L * M) * N$ we obtain

$$(w_{L,M} * N)\circ t_{L,TM,N}\circ (L\otimes \gamma_{R,M,N}) = \gamma_{L,M,N}\circ w_{L,M*N}.$$

Inserting here the expression (93) we obtain the heptagon (94).

Setting $N = R$ in (90), multiplying it with $\mathbf{r}_M$ and then using naturality of ε on the LHS leads to (95). □

Proposition 8.3. *Given right-monoidal structures $\otimes$ and $*$ on the same category and with same unit object R Eqs. (91) and (93) provide a bijection between*

tetrahedral isomorphisms $t_{L,M,N} : L\otimes (M * N)\to (L\otimes M) * N$

and natural isomorphisms $w_{M,N} : M\otimes TN\xrightarrow{\sim} M * N$ satisfying (94) and (95).

Proof. Given a tetrahedral isomorphism t the natural isomorphism w defined by (91) satisfies (94) and (95) by Lemma 8.2.

Assume w is a natural isomorphism satisfying (94) and (95) and define the natural transformation t by (93). Then the pentagon (87) is a simple consequence of the pentagon for $\mathbf{a}$ (and invertibility of w). But in order to prove the other pentagon (88) we need its special case (94). The LHS of (88) can be written as

$$\begin{aligned}
\text{LHS} = {} & (w_{K\otimes L,M} * N) \circ (\mathbf{a}_{K,L,TM} * N) \circ ((K \otimes w^{-1}_{L,M}) * N) \\
& \circ w_{K\otimes(L*M),N} \circ \mathbf{a}_{K,L*M,TN} \circ (K \otimes w^{-1}_{L*M,N}) \circ (K \otimes \gamma_{L,M,N}) \\
= {} & (w_{K\otimes L,M} * N) \circ w_{(K\otimes L)\otimes TM,N} \circ (\mathbf{a}_{K,L,TM} \otimes TN) \circ \mathbf{a}_{K,L\otimes TM,TN} \\
& \circ (K \otimes w^{-1}_{L\otimes TM,N}) \circ (K \otimes (w^{-1}_{L,M} * N)) \circ (K \otimes \gamma_{L,M,N}) \\
\overset{(94)}{=} {} & (w_{K\otimes L,M} * N) \circ w_{(K\otimes L)\otimes TM,N} \circ (\mathbf{a}_{K,L,TM} \otimes TN) \circ \mathbf{a}_{K,L\otimes TM,TN} \\
& (K \otimes \mathbf{a}_{L,TM,TN}) \circ (K \otimes (L \otimes w^{-1}_{TM,N})) \circ (K \otimes (L \otimes \gamma_{R,M,N})) \circ (K \otimes w^{-1}_{L,M*N}) \\
= {} & (w_{K\otimes L,M} * N) \circ w_{(K\otimes L)\otimes TM,N} \circ \mathbf{a}_{K\otimes L,TM,TN} \circ \mathbf{a}_{K,L,TM\otimes TN} \\
& \circ (K \otimes (L \otimes w^{-1}_{TM,N})) \circ (K \otimes (L \otimes \gamma_{R,M,N})) \circ (K \otimes w^{-1}_{L,M*N}) \\
= {} & (w_{K\otimes L,M} * N) \circ w_{(K\otimes L)\otimes TM,N} \circ \mathbf{a}_{K\otimes L,TM,TN} \circ ((K \otimes L) \otimes w^{-1}_{TM,N}) \\
& \circ ((K \otimes L) \otimes \gamma_{R,M,N}) \circ \mathbf{a}_{K,L,T(M*N)} \circ (K \otimes w^{-1}_{L,M*N}) \\
\overset{(94)}{=} {} & \gamma_{K\otimes L,M,N} \circ w_{K\otimes L,M*N} \circ \mathbf{a}_{K,L,T(M*N)} \circ (K \otimes w^{-1}_{L,M*N})
\end{aligned}$$

which is exactly the RHS. In order to prove (89) insert $L = R$ in the definition (93) of t and multiply it with $\mathbf{l}^{-1}_{M*N}$.

$$\begin{aligned}
t_{R,M,N} \circ \mathbf{l}^{-1}_{M*N} &= w_{R\otimes M,N} \circ \mathbf{a}_{R,M,TN} \circ \mathbf{l}^{-1}_{M\otimes TN} \circ w^{-1}_{M,N} \\
&= w_{R\otimes M,N} \circ (\mathbf{l}^{-1}_{M} \otimes TN) \circ w^{-1}_{M,N} = \mathbf{l}^{-1}_{M} * N
\end{aligned}$$

where we used (2) for $\otimes$. Axiom (90) in turn can be proven by using (95) and (3) for $\otimes$:

$$\begin{aligned}
\varepsilon_{M\otimes N} \circ t_{M,N,R} = {} & \varepsilon_{M\otimes N} \circ w_{M\otimes N,R} \circ \mathbf{a}_{M,N,TR} \circ (M \otimes w^{-1}_{N,R}) \\
\overset{(95)}{=} {} & \mathbf{r}_{M\otimes N} \circ ((M \otimes N) \otimes \varepsilon_R) \circ \mathbf{a}_{M,N,TR} \circ (M \otimes w^{-1}_{N,R}) \\
= {} & \mathbf{r}_{M\otimes N} \circ \mathbf{a}_{M,N,R} \circ (M \otimes (N \otimes \varepsilon_R) \circ w^{-1}_{N,R}) \\
\overset{(3)}{=} {} & M \otimes \left[\mathbf{r}_N \circ (N \otimes \varepsilon_R) \circ w^{-1}_{N,R}\right] \overset{(95)}{=} M \otimes \varepsilon_N.
\end{aligned}$$

This finishes the proof that t is a tetrahedral homomorphism. That it is also a tetrahedral isomorphism will be a consequence of that the composite map $w \mapsto t \mapsto w$ is the identity. Indeed, it maps w to

$$\begin{aligned}
& (\mathbf{r}_M * N) \circ w_{M\otimes R,N} \circ \mathbf{a}_{M,R,TN} \circ (M \otimes w^{-1}_{R,N}) \\
& = w_{M,N} \circ (\mathbf{r}_M \otimes TN) \circ \mathbf{a}_{M,R,TN} \circ (M \otimes \mathbf{l}^{-1}_{TN}) \\
& = w_{M,N}
\end{aligned}$$

by (92) and by the (4) axiom for $\otimes$. That $t \mapsto w \mapsto t$ is also the identity has been already proven in Lemma 8.2 when we verified (93). $\square$

Note that in case of tetrahedral isomorphisms axiom (89) is redundant, it follows from (87) alone. Indeed, in Lemma 8.2 (93) was a consequence of only (87) and in the proof of Proposition 8.3 we derived axiom (89) using only (93).

Having a natural isomorphism w as in Proposition 8.3 we can define what looks like an opmonoidal structure for the canonical monad T, namely

$$T^{M,N} := w^{-1}_{TM,N} \circ \gamma_{R,M,N} \circ Tw_{M,N} \circ T(M \otimes \eta_N) \ : \ T(M \otimes N) \to TM \otimes TN \tag{96}$$

$$T^0 := \varepsilon_R \ : \ TR \to R. \tag{97}$$

In order to prove that they make the monad $\langle T, \mu, \eta\rangle$ opmonoidal, we use the technology of fusion operators. In contrast to Section 7, however, we need h to be expressed in terms of w. Comparing (96) with (72) the conjecture is that

$$h_{M,N} := w^{-1}_{TM,N} \circ \gamma_{R,M,N} \circ Tw_{M,N} \ : \ T(M \otimes TN) \to TM \otimes TN \tag{98}$$

is a fusion operator.

Lemma 8.4. *Let the natural isomorphism w satisfy (94) and (95). Then (98), together with $T^0 = \varepsilon_R$, is a fusion operator for the monad $\langle T, \mu, \eta\rangle$, i.e., it satisfies Eqs. (74)–(80) with O, ω, ι, O^0 replaced by T, μ, η, T^0, respectively.*

Proof. First we prove (75) by unpacking it by means of (98) and then using (94) twice:

$$\begin{aligned}
&(h_{L,M} \otimes TN) \circ h_{L\otimes TM,N} \circ T\mathbf{a}_{L,TM,TN} \circ T(L \otimes h_{M,N}) \\
&\quad = (w^{-1}_{TL,M} \otimes TN) \circ (\gamma_{R,L,M} \otimes TN) \circ (Tw_{L,M} \otimes TN) \circ w^{-1}_{T(L\otimes TM),N} \\
&\qquad \circ \gamma_{R,L\otimes TM,N} \circ Tw_{L\otimes TM,N} \circ T\mathbf{a}_{L,TM,TN} \circ T(L \otimes w^{-1}_{TM,N}) \\
&\qquad \circ T(L \otimes \gamma_{R,M,N}) \circ T(L \otimes Tw_{M,N}) \\
&\quad = (w^{-1}_{TL,M} \otimes TN) \circ w^{-1}_{TL*M,N} \circ (\gamma_{R,L,M} * N) \circ \gamma_{R,L*M,N} \\
&\qquad \circ T[(w_{L,M} * N) \circ w_{L\otimes TM,N} \circ \mathbf{a}_{L,TM,TN} \\
&\qquad \circ (L \otimes w^{-1}_{TM,N}) \circ (L \otimes \gamma_{R,M,N}) \circ (L \otimes Tw_{M,N})] \\
&\quad \overset{(94)}{=} (w^{-1}_{TL,M} \otimes TN) \circ w^{-1}_{TL*M,N} \circ (\gamma_{R,L,M} * N) \circ \gamma_{R,L*M,N} \circ T\gamma_{L,M,N} \\
&\qquad \circ Tw_{L,M*N} \circ T(L \otimes Tw_{M,N}) \\
&\quad \overset{(1)}{=} (w^{-1}_{TL,M} \otimes TN) \circ w^{-1}_{TL*M,N} \circ \gamma_{TL,M,N} \\
&\qquad \circ \gamma_{R,L,M*N} \circ Tw_{L,M*N} \circ T(L \otimes Tw_{M,N}) \\
&\quad = w^{-1}_{TL\otimes TM,N} \circ (w^{-1}_{TL,M} * N) \circ \gamma_{TL,M,N} \circ w_{TL,M*N} \circ h_{L,M*N} \circ T(L \otimes Tw_{M,N}) \\
&\quad \overset{(94)}{=} \mathbf{a}_{TL,TM,TN} \circ (TL \otimes w^{-1}_{TM,N}) \circ (TL \otimes \gamma_{R,M,N}) \circ h_{L,M*N} \circ T(L \otimes Tw_{M,N}) \\
&\quad = \mathbf{a}_{TL,TM,TN} \circ (TL \otimes h_{M,N}) \circ h_{L,M\otimes TN}.
\end{aligned}$$

Eqs. (76)–(78) can be shown as follows:

$$\begin{aligned} h_{M,N}\circ\eta_{M\otimes TN} &= w^{-1}_{TM,N}\circ\gamma_{R,M,N}\circ Tw_{M,N}\circ\eta_{M\otimes TN}\\ &= w^{-1}_{TM,N}\circ\gamma_{R,M,N}\circ\eta_{M*N}\circ w_{M,N}\\ &\overset{(2)}{=} w^{-1}_{TM,N}\circ(\eta_M * N)\circ w_{M,N} = \eta_M\otimes TN. \end{aligned}$$

$$\begin{aligned} (T^0\otimes TN)\circ h_{R,N}\circ T\mathbf{l}^{-1}_{TN} &= (\varepsilon_R\otimes TN)\circ w^{-1}_{TR,N}\circ\gamma_{R,R,N}\circ Tw_{R,N}\circ T\mathbf{l}^{-1}_{TN}\\ &\overset{(92)}{=} w^{-1}_{R,N}\circ(\varepsilon_R * N)\circ\gamma_{R,R,N}\\ &\overset{(92)}{=} \mathbf{l}^{-1}_{TN}\circ\mu_N. \end{aligned}$$

$$\begin{aligned} \mathbf{r}_{TM}\circ(TM\otimes T^0)\circ h_{M,R} &= \mathbf{r}_{TM}\circ(TM\otimes\varepsilon_R)\circ w^{-1}_{TM,R}\circ\gamma_{R,M,R}\circ Tw_{M,R}\\ &\overset{(95)}{=} \varepsilon_{TM}\circ\gamma_{R,M,R}\circ Tw_{M,R}\\ &\overset{(3)}{=} T\varepsilon_M\circ Tw_{M,R}\\ &\overset{(95)}{=} T\mathbf{r}_M\circ T(M\otimes T^0). \end{aligned}$$

In order to prove (74) we need some preparation.

$$\begin{aligned} w_{M,N}\circ(M\otimes\mu_N) = {} & w_{M,N}\circ(\mathbf{r}_M\otimes TN)\circ\mathbf{a}_{M,R,TN}\circ(M\otimes w^{-1}_{R,N})\\ & \circ(M\otimes(\varepsilon_R * N))\circ(M\otimes\gamma_{R,R,N})\\ = {} & w_{M,N}\circ(\mathbf{r}_M\otimes TN)\circ((M\otimes\varepsilon_R)\otimes TN)\circ\mathbf{a}_{M,TR,TN}\\ & \circ(M\otimes w^{-1}_{TR,N})\circ(M\otimes\gamma_{R,R,N})\\ \overset{(95)}{=} {} & w_{M,N}\circ(\varepsilon_M\otimes TN)\circ(w_{M,R}\otimes TN)\\ & \circ\mathbf{a}_{M,TR,TN}\circ(M\otimes w^{-1}_{TR,N})\circ(M\otimes\gamma_{R,R,N})\\ \overset{(94)}{=} {} & w_{M,N}\circ(\varepsilon_M\otimes TN)\circ w^{-1}_{M*R,N}\circ\gamma_{M,R,N}\circ w_{M,TN}\\ = {} & (\varepsilon_M * N)\circ\gamma_{M,R,N}\circ w_{M,TN}\\ = {} & \mu_{M,N}\circ w_{M,TN} \end{aligned} \tag{99}$$

where in the first line we inserted an identity arrow in the form of the $\otimes$-version of axiom (4), using also (92), and in the last line we used the notation of (25). It follows that

$$\begin{aligned} (TM\otimes\mu_N)\circ h_{M,TN} &= (TM\otimes\mu_N)\circ w^{-1}_{TM,TN}\circ\gamma_{R,M,TN}\circ Tw_{M,TN}\\ &\overset{(99)}{=} w^{-1}_{TM,N}\circ\mu_{TM,N}\circ\gamma_{R,M,TN}\circ Tw_{M,TN}\\ &\overset{(1)}{=} w^{-1}_{TM,N}\circ(\varepsilon_{TM} * N)\circ(\gamma_{R,M,R} * N)\\ &\quad\circ\gamma_{R,M*R,N}\circ T\gamma_{M,R,N}\circ Tw_{M,TN}\\ &\overset{(3)}{=} w^{-1}_{TM,N}\circ\gamma_{R,M,N}\circ T\mu_{M,N}\circ Tw_{M,TN}\\ &\overset{(99)}{=} w^{-1}_{TM,N}\circ\gamma_{R,M,N}\circ Tw_{M,N}\circ T(M\otimes\mu_N)\\ &= h_{M,N}\circ T(M\otimes\mu_N) \end{aligned} \tag{100}$$

which is relation (74). While (80) obviously follows from (5) the proof of (79) needs some work:

$$\begin{aligned}
&(\mu_M \otimes TN) \circ h_{TM,N} \circ Th_{M,N} \\
&\quad = (\mu_M \otimes TN) \circ w^{-1}_{T^2M,N} \circ \gamma_{R,TM,N} \circ Tw_{TM,N} \circ Tw^{-1}_{TM,N} \circ T\gamma_{R,M,N} \circ T^2 w_{M,N} \\
&\quad = w^{-1}_{TM,N} \circ (\mu_M * N) \circ \gamma_{R,TM,N} \circ T\gamma_{R,M,N} \circ T^2 w_{M,N} \\
&\quad \overset{(1)}{=} w^{-1}_{TM,N} \circ ((\varepsilon_R * M) * N) \circ \gamma_{R*R,M,N} \circ \gamma_{R,R,M*N} \circ T^2 w_{M,N} \\
&\quad = w^{-1}_{TM,N} \circ \gamma_{R,M,N} \circ \mu_{M*N} \circ T^2 w_{M,N} = h_{M,N} \circ \mu_{M\otimes TN}. \quad \square
\end{aligned}$$

Proposition 8.5. *Given a monoidal structure $\otimes$ and a right-monoidal structure $*$ on the same category and with the same unit object R the existence of a natural isomorphism $w_{M,N} : M \otimes (R * N) \to M * N$ satisfying Eqs. (94) and (95) implies that the formulas (96), (97) define a $\otimes$-opmonoidal structure for the canonical monad $T = \langle R * \_, \mu, \eta\rangle$ of the $*$-structure.*

Proof. This is an immediate consequence of Lemma 8.4 and Proposition 7.1. $\square$

Theorem 8.6. *Let $\langle \mathcal{E}, \otimes, R, \mathbf{a}, \mathbf{l}^{-1}, \mathbf{r}\rangle$ be a monoidal category. Then for a right-monoidal structure $*$ on $\mathcal{E}$ with unit object R the following conditions are equivalent:*

(i) *The $*$-structure is $\otimes$-representable (by a $\otimes$-bimonad) in the sense of Definition 7.3.*

(ii) *There exists a natural isomorphism $w_{M,N} : M \otimes (R * N) \to M * N$ satisfying the heptagon (94) and the tetragon (95).*

(iii) *There exists a tetrahedral isomorphism $t_{L,M,N} : L \otimes (M * N) \xrightarrow{\sim} (L \otimes M) * N$.*

Proof. Equivalence of (ii) and (iii) has been shown in Proposition 8.3. Assume (i). This means that there exist a bimonad $\langle O, \omega, \iota\rangle$ w.r.t. the $\otimes$-structure and a skew-twist $v_{M,N} : M \odot N \to M * N$ where $\odot$ is the skew-monoidal structure induced by O in the sense of Proposition 7.2. Therefore v satisfies the relations

$$v_{L*M,N} \circ (v_{L,M} \otimes ON) \circ \dot\gamma_{L,M,N} = \gamma_{L,M,N} \circ v_{L,M*N} \circ (L \otimes Ov_{M,N}) \tag{101}$$

$$v_{R,N} \circ \dot\eta_N = \eta_N \tag{102}$$

$$\dot\varepsilon_M = \varepsilon_M \circ v_{M,R} \tag{103}$$

where $\dot\gamma$, $\dot\eta$, $\dot\varepsilon$ are the expressions (82), (83), (84). We claim that the composite

$$w_{M,N} := \left(M \otimes TN \xrightarrow{M\otimes v^{-1}_{R,N}} M \otimes (R \odot N) \xrightarrow{M\otimes \mathbf{l}_{ON}} M \odot N \xrightarrow{v_{M,N}} M * N \right) \tag{104}$$

is a natural isomorphism satisfying (94) and (95). With the notation $u_N := \mathbf{l}_{ON} \circ v^{-1}_{R,N}$ the left hand side of (94) can be transformed to the right hand side as follows.

$$
\begin{aligned}
&v_{L*M,N}\circ(v_{L,M}\otimes ON)\circ((L\otimes u_M)\otimes ON)\circ((L\otimes TM)\otimes u_N)\circ \mathbf{a}_{L,TM,TN}\\
&\quad\circ(L\otimes(TM\otimes u_N^{-1}))\circ(L\otimes v_{TM,N}^{-1})\circ(L\otimes\gamma_{R,M,N})\\
&=v_{L*M,N}\circ(v_{L,M}\otimes ON)\circ\mathbf{a}_{L,OM,ON}\circ(L\otimes(u_M\otimes ON))\\
&\quad\circ(L\otimes v_{TM,N}^{-1})\circ(L\otimes\gamma_{R,M,N})\\
&\overset{(101)}{=}v_{L*M,N}\circ(v_{L,M}\otimes ON)\circ\mathbf{a}_{L,OM,ON}\circ(L\otimes(\mathbf{l}_{OM}\otimes ON))\circ(L\otimes\dot\gamma_{R,M,N})\\
&\quad\circ(L\otimes(R\otimes Ov_{M,N}^{-1}))\circ(L\otimes v_{R,M*N}^{-1})\\
&\overset{(82)}{=}v_{L*M,N}\circ(v_{L,M}\otimes ON)\circ\mathbf{a}_{L,OM,ON}\circ(L\otimes\mathbf{l}_{OM\otimes ON})\\
&\quad\circ(L\otimes(R\otimes(OM\otimes\omega_N)))\circ(L\otimes(R\otimes O^{M,ON}))\\
&\quad\circ(L\otimes(R\otimes Ov_{M,N}^{-1}))\circ(L\otimes v_{R,M*N}^{-1})\\
&=v_{L*M,N}\circ(v_{L,M}\otimes ON)\circ\mathbf{a}_{L,OM,ON}\circ(L\otimes(OM\otimes\omega_N))\circ(L\otimes O^{M,ON})\\
&\quad\circ(L\otimes Ov_{M,N}^{-1})\circ(L\otimes u_{M*N})\\
&=v_{L*M,N}\circ(v_{L,M}\otimes ON)\circ\dot\gamma_{L,M,N}\circ(L\otimes Ov_{M,N}^{-1})\circ(L\otimes u_{M*N})\\
&\overset{(101)}{=}\gamma_{L,M,N}\circ v_{L,M*N}\circ(L\otimes u_{M*N})\\
&=\gamma_{L,M,N}\circ w_{L,M*N}.
\end{aligned}
$$

In order to prove (95) we compute its left hand side

$$
\begin{aligned}
\varepsilon_M\circ w_{M,R}&\overset{(103)}{=}\dot\varepsilon_M\circ(M\otimes u_R)=\dot\varepsilon_M\circ(\mathbf{r}_M\otimes OR)\circ\mathbf{a}_{M,R,OR}\circ(M\otimes v_{R,R}^{-1})\\
&=\mathbf{r}_M\circ\dot\varepsilon_{M\otimes R}\circ\mathbf{a}_{M,R,OR}\circ(M\otimes v_{R,R}^{-1})\\
&\overset{(84)}{=}\mathbf{r}_M\circ\mathbf{r}_{M\otimes R}\circ((M\otimes R)\otimes O^0)\circ\mathbf{a}_{M,R,OR}\circ(M\otimes v_{R,R}^{-1})\\
&=\mathbf{r}_M\circ(M\otimes\mathbf{r}_R)\circ(M\otimes(R\otimes O^0))\circ(M\otimes v_{R,R}^{-1})\\
&\overset{(84)}{=}\mathbf{r}_M\circ(M\otimes\dot\varepsilon_R)\circ(M\otimes v_{R,R}^{-1})\\
&\overset{(103)}{=}\mathbf{r}_M\circ(M\otimes\varepsilon_R)
\end{aligned}
$$

and arrive to the expression on the right hand side. This proves the implication (i)⇒(ii).

Now assume (ii). Then we know by Proposition 8.5 that T is a bimonad, so by Proposition 7.2 that $M\odot N:=M\otimes TN$ is a right-monoidal product. Therefore $\otimes$-representability of the $*$-structure would follow immediately if we could show that $w_{M,N}:M\odot N\to M*N$ is a twist.

$$
\begin{aligned}
&w_{L*M,N}\circ(w_{L,M}\otimes TN)\circ\dot\gamma_{L,M,N}\\
&\quad=w_{L*M,N}\circ(w_{L,M}\otimes TN)\circ\mathbf{a}_{L,TM,TN}\circ(L\otimes(TM\otimes\mu_N))\circ(L\otimes T^{M,TN})\\
&\quad\overset{(98)}{=}(w_{L,M}*N)\circ w_{L\otimes TM,N}\circ\mathbf{a}_{L,TM,TN}\\
&\qquad\circ(L\otimes w_{TM,N}^{-1})\circ(L\otimes\gamma_{R,M,N})\circ(L\otimes Tw_{M,N})\\
&\quad\overset{(94)}{=}\gamma_{L,M,N}\circ w_{L,M*N}\circ(L\otimes Tw_{M,N})
\end{aligned}
$$

proves the hexagon relation (101) for w. The following simple computations yield the remaining relations:

$$
\begin{aligned}
&w_{R,N}\circ\dot\eta\overset{(83)}{=}w_{R,N}\circ\mathbf{l}_{TN}^{-1}\circ\eta_N\overset{(92)}{=}\eta_N\\
&\varepsilon_M\circ w_{M,R}=\mathbf{r}_M\circ(M\otimes\varepsilon_R)=\mathbf{r}_M\circ(M\otimes T^0)\overset{(84)}{=}\dot\varepsilon_M.
\end{aligned}
$$

So, w is indeed a twist and this finishes the proof of the implication (ii)⇒(i). □

9. Closed skew-monoidal categories

A skew-monoidal category $\langle \mathcal{M}, *, R, \gamma, \eta, \varepsilon\rangle$ is called left (right) closed if the endofunctor $\_ * N$ (resp. $N * \_$) has a right adjoint $\hom^l(N, \_)$ (resp. $\hom^r(N, \_)$) for all object $N \in \mathcal{M}$. It is called closed if it is both left closed and right closed.

Theorem 9.1. *Let R be a ring. Then closed right-monoidal structures $\langle \mathsf{Ab}_R, *, R, \gamma, \eta, \varepsilon\rangle$ on the category of right R-modules, with unit object being the right-regular R-module, are precisely the right bialgebroids over R.*

Proof. In Section 3 we have shown how bialgebroids over R give rise to right-monoidal structures on Ab_R. The definition of the right-monoidal product (30) makes it obvious that it is closed.

Let $*$ be a closed right-monoidal structure on Ab_R. Since Ab_R is cocomplete and $\_ * N$ is left adjoint, by the Eilenberg–Watts Theorem there is an isomorphism

$$v_{M,N} : M \underset{R}{\otimes} TN \xrightarrow{\sim} M * N$$

natural in M for each N where $\underset{R}{\otimes}$ stands for the action on the monoidal category ${}_R\mathsf{Ab}_R$ on Ab_R. (Note that the left R-module structure of $TN = R * N$ is defined by the endomorphism ring of the right-regular module R, i.e., by λ_1 in the notation of Section 4.) Without loss of generality we may assume that v also satisfies the normalization

$$v_{R,N} = \mathbf{l}_{TN} \tag{105}$$

for each N. (Otherwise compose it with $(M \otimes (\mathbf{l}_{TN} \circ v^{-1}_{R,N}))$.) Then considering $N \mapsto (\_ * N)$ as the object map of a functor $\mathsf{Ab}_R \to \mathrm{End}\,\mathsf{Ab}_R$ the $v_{M,N}$ becomes natural in N, too. Now substituting v for w in the heptagon (94) with $L = R$ we obtain an identity due to (105). Similarly, (95) with $w = v$ and $M = R$ is an identity. Therefore, using that R is a generator, it follows that both (94) and (95) are identities for all values of their arguments L, M and N.

Next we want to construct a w for the quotient right-monoidal structure $*_q$ (see Proposition 4.3) on the monoidal category ${}_R\mathsf{Ab}_R$. There is a unique w such that for all $M, N \in {}_R\mathsf{Ab}_R$

$$\begin{array}{ccc} M \underset{R}{\otimes} TN & \xrightarrow{v_{M,N}} & M * N \\ {\scriptstyle M \underset{R}{\otimes} q_{R,N}}\downarrow & & \downarrow{\scriptstyle q_{M,N}} \\ M \underset{R}{\otimes} T_q N & \xrightarrow{w_{M,N}} & M *_q N \end{array} \tag{106}$$

since $q_{M,N}$ is a coequalizer. $w_{M,N}$ is invertible since $M \underset{R}{\otimes} \_$ preserves coequalizers. Now use (48), (49) to show that the heptagon (94) and tetragon (95) for v and $*$ implies the heptagon and tetragon for w and $*_q$. Then by Theorem 8.6 T_q is a bimonad on ${}_R\mathsf{Ab}_R$. Thus we could conclude by [25, Theorem 4.5] that T_q is the bimonad of a bialgebroid if we knew that T_q is left adjoint. Using that $*$ is also right closed the Eilenberg–Watts Theorem provides an isomorphism $M * N \cong N \underset{R_2}{\otimes} (M * R)$; hence $TN \cong N \underset{R_2}{\otimes} H$ where $H = R * R$. The quotient

$$T_q N = \int^{\rho_1 \lambda_2} TN \;\cong\; \int^{\rho_1 \lambda_N} (N \underset{R_2}{\otimes} H) \;\cong\; N \underset{R^e}{\otimes} H$$

amalgamates the left R-action on N with the right R-action ρ_1 on H which, together with $\underset{R_2}{\otimes}$, amounts to taking tensor product over $R^e = R^{\mathrm{op}} \otimes R$ by considering N as right R^e-module and H as left R^e-module via $(r' \otimes r) \cdot h = \rho_1(r') \circ \lambda_2(r)(h)$. As such, T_q is left adjoint. □

Combining the above result with Mitchell's Theorem on the characterization of module categories we can obtain a characterization of skew-monoidal categories of bialgebroids without explicit reference to the base ring.

Corollary 9.2. *A right monoidal category $\langle \mathcal{M}, *, R, \gamma, \eta, \varepsilon \rangle$ is equivalent to the right-monoidal category of a right-bialgebroid iff*

(i) *$\mathcal{M}$ is cocomplete abelian,*
(ii) *$*$ preserves colimits in both arguments*
(iii) *and R is a small projective generator.*

10. Monoidal (lax) comonads

In this last section, we discuss two results that lead to monoidality of the canonical lax comonad of a skew-monoidal category.

10.1. The corepresentability theorem

We would like to characterize the skew-monoidal categories that can be "corepresented" in the sense of Definition 7.4 by a monoidal comonad. For that purpose we dualize the construction of Section 8.

Let $\langle \mathcal{E}, *, R \rangle$ be a right-monoidal category, the dual $\langle \mathcal{E}^{\mathrm{op}}, *^{\mathrm{op}}, R \rangle$ of which is representable by an opmonoidal monad in the right-monoidal category $\langle \mathcal{E}^{\mathrm{op}}, \otimes, R \rangle$. This means precisely that the original $*$-structure is corepresentable by a monoidal comonad w.r.t the left-monoidal structure $\otimes$. So we can speak about tetrahedral homomorphisms t as natural transformations

$$t_{L,M,N} \ : \ N * (L \otimes M) \ \to \ L \otimes (N * M)$$

satisfying the pentagons

$$(K \otimes t_{L,M,N}) \circ t_{K,L\otimes M,N} \circ (N * \mathbf{a}^{-1}_{K,L,M}) = \mathbf{a}^{-1}_{K,L,N*M} \circ t_{K\otimes L,M,N}$$
$$(K \otimes \gamma_{N,M,L}) \circ t_{K,M*L,N} \circ (N * t_{K,L,M}) = t_{K,L,N*M} \circ \gamma_{N,M,K\otimes L}$$

and the triangles

$$\mathbf{l}_{N*M} \circ t_{R,M,N} = N * \mathbf{l}_M$$
$$t_{M,N,R} \circ \eta_{M\otimes N} = M \otimes \eta_N .$$

(We have written t exactly for what it was in Section 8, without even permuting indices, now using the opposite composition and opposite skew-monoidal product.) Such a t is then a tetrahedral isomorphism if

$$w_{M,N} \coloneqq t_{M,R,N} \circ (N * \mathbf{r}^{-1}_M) \ : \ N * M \to M \otimes QN$$

is a natural isomorphism.

Dualizing Proposition 8.3 we obtain that t is a tetrahedral isomorphism if and only if w satisfies the following heptagon and tetragon equations:

$$w_{L,N*M} \circ \gamma_{N,M,L} = (L \otimes \gamma_{N,M,R}) \circ (L \otimes w^{-1}_{QM,N}) \circ \mathbf{a}^{-1}_{L,QM,QN} \circ w_{L\otimes QM,N} \circ (N * w_{L,M}) \tag{107}$$

$$w_{M,R} \circ \eta_M = (M \otimes \eta_R) \circ \mathbf{r}^{-1}_M. \tag{108}$$

The fusion operators can be defined as the composite natural transformation

$$h_{M,N} := Qw_{M,N} \circ \gamma_{N,M,R} \circ w^{-1}_{QM,N} : QM \otimes QN \to Q(M \otimes QN).$$

This allows to write up the would-be monoidal structure for the canonical comonad $Q = \langle \_ * R, \delta, \varepsilon \rangle$ as follows

$$Q_{M,N} := Q(M \otimes \varepsilon_N) \circ h_{M,N} : QM \otimes QN \to Q(M \otimes N) \tag{109}$$

$$Q_0 := \eta_R : R \to QR. \tag{110}$$

Then by dualizing Theorem 8.6 we obtain the following corepresentability theorem:

Theorem 10.1. *Let $\mathcal{E}$ be a category equipped with a right-monoidal structure $*$ and a monoidal structure $\otimes$ with a common unit object R. Then the following statements are equivalent.*

(i) *$*$ is $\otimes$-corepresentable, i.e., there is a $\otimes$-monoidal comonad C and a twist-isomorphism $M * N \xrightarrow{\sim} N \otimes CM$ of right monoidal structures.*

(ii) *There is a natural isomorphism $w_{M,N} : N * M \xrightarrow{\sim} M \otimes QN$ satisfying the heptagon and tetragon equations (107) and (108) where Q is the canonical comonad of the $*$-structure.*

(iii) *There is a tetrahedral isomorphism $t_{L,M,N} : N * (L \otimes M) \to L \otimes (N * M)$.*

One may try to apply this corepresentation theorem to a situation dual to that of Section 9, e.g., by considering categories of right comodules of a coalgebra and coclosed skew-monoidal structures on them. Unfortunately this dualization seems to require more than what is known, to the present author, about bicoalgebroids [8,2].

10.2. Monoidality of the lax comonad on ${}_R\mathsf{M}_R$

If $\mathcal{E}$ is a monoidal category then monoidality of the lax comonad $\mathbf{Q} : \Delta^{\mathrm{op}} \to \mathrm{End}\,\mathcal{E}$ means the structure on $\mathbf{Q}$ that allows its factorization through the faithful functor $\mathrm{End}^{\otimes}\,\mathcal{E} \hookrightarrow \mathrm{End}\,\mathcal{E}$ which forgets monoidality of monoidal endofunctors and their monoidal natural transformations. If $\mathcal{E}$ is the category of E-objects of a complete right-monoidal category $\mathcal{M}$ and $\mathbf{Q}$ is the lax comonad on $\mathcal{E}$ constructed in Section 6 then one would like to find conditions on a monoidal structure $\otimes$ on $\mathcal{E}$ which implies monoidality of $\mathbf{Q}$. For the monad T_q the existence of tetrahedral isomorphism between $\otimes$ and $*_q$ on $\mathcal{E}$ implied its opmonoidality. Unfortunately we do not know analogous conditions that would imply monoidality of $\mathbf{Q}$. However, if $\mathcal{E}$ is the category ${}_R\mathsf{Ab}_R$ of bimodules over a ring R and $\mathbf{Q}$ is the lax comonad of a right R-bialgebroid one expects that monoidality of $\mathbf{Q}$ follows without any additional conditions.

As the proof of [12, Proposition 4.2] indicates, in order to construct the monoidal structure of $\mathbf{Q}$, it is not sufficient to work within ${}_R\mathsf{Ab}_R$, it has to be embedded into a monoidal bicategory of bimodules. The basic idea of the proof of the next Theorem is that of the above mentioned construction of [12] although some differences in the conventions may disguise it.

Theorem 10.2. *For a commutative ring k and a k-algebra R let $\langle \mathsf{M}_R, *, R, \gamma, \eta, \varepsilon\rangle$ be a closed right-monoidal structure on the category of right R-modules. Then the lax comonad $\mathbf{Q}$ on ${}_R\mathsf{M}_R$ defined in Proposition 6.2 is monoidal and the Eilenberg–Moore category ${}_R\mathsf{M}_R^{\mathbf{Q}}$ has a unique monoidal structure such that the forgetful functor ${}_R\mathsf{M}_R^{\mathbf{Q}} \to {}_R\mathsf{M}_R$ is strict monoidal.*

Proof. Let $\mathcal{E}(m,n)$ be the category of R_n-R_m-bimodules where $R_n := R \otimes (R^{\mathrm{op}} \otimes R)^{\otimes(n-1)}$ and $\otimes$ denotes tensor product over k. Tensor product over R_n is denoted by $\square$ for any n.

Let H denote $R * R$ as an $R^{\mathrm{op}} \otimes R$-bimodule. Since H is a monoid in the category of $R^{\mathrm{op}} \otimes R$-bimodules, tensoring with H (n times) defines monoidal functors $\mathbf{H}^n : \mathcal{E}(1,l) \to \mathcal{E}(n+1, n+l)$ given recursively by $\mathbf{H}^0 M := M$ and $\mathbf{H}^n M := \mathbf{H}^{n-1} M \otimes H$ if $n > 0$.

Let $P \in \mathcal{E}(1,2)$ be the k-module $R \otimes R$ equipped with $(R \otimes R^{\mathrm{op}} \otimes R)$-$R$-bimodule structure

$$(r_1 \otimes r' \otimes r_2) \cdot (x \otimes y) \cdot r_3 := r_1 x r' \otimes r_2 y r_3.$$

We shall also need the n-th iterate of P

$$P_1 := P \quad \text{and} \quad P_n := (P \otimes R_{n-1}) \square P_{n-1} \in \mathcal{E}(1, n+1), \quad n > 1.$$

Since $\_ * N$ is left adjoint for each $N \in \mathsf{M}_R$, there is an isomorphism $M * N \xrightarrow{\sim} M \underset{R}{\otimes} (R * N)$, natural in M, where the left R-module structure of $R * N$ is given by λ_1. Setting $N = R$ we obtain $QM \xrightarrow{\sim} M \underset{R}{\otimes} H = \mathbf{H}M \square P$ and iterating $Q^n M \xrightarrow{\sim} \mathbf{H}^n M \square P_n$.

Using that $P_n \square \_ : \mathcal{E}(1,1) \to \mathcal{E}(1, n+1)$ has a right adjoint, the object map of the lax comonad $\mathbf{Q}$ can be given by the functors

$$M \mapsto \mathbf{Q}_n M = \mathrm{Hom}_{R_{n+1}}(P_n, Q^n M) \xrightarrow{\sim} \mathrm{Hom}_{R_{n+1}}(P_n, \mathbf{H}^n M \square P_n).$$

The counit of this hom–tensor adjunction, i.e., the evaluation $\mathrm{ev}^n : P_n \square \mathrm{Hom}_{R_{n+1}}(P_n, \_) \to \_$, allows us to define $(\mathbf{Q}_n)_{M,N}$ by the following commutative diagram (in which the associators for $\square$ are suppressed and ev^n_M is written instead of $\mathrm{ev}^n_{H^n M \square P_n}$ for brevity)

$$\begin{array}{ccc} P_n \square \mathbf{Q}_n M \square \mathbf{Q}_n N & \xrightarrow{(1 \square \mathrm{ev}^n_N) \circ (\mathrm{ev}^n_M \square 1)} & \mathbf{H}^n M \square \mathbf{H}^n N \square P_n \\ {\scriptstyle 1 \square (\mathbf{Q}_n)_{M,N}}\downarrow & & \downarrow{\scriptstyle (\mathbf{H}^n)_{M,N} \square 1} \\ P_n \square \mathbf{Q}_n (M \square N) & \xrightarrow{\mathrm{ev}^n_{M \square N}} & \mathbf{H}^n (M \square N) \square P_n \end{array} \tag{111}$$

The unit $(\mathbf{Q}_n)_0 : R \to \mathbf{Q}_n R$, in turn, is defined by the unit of $\mathbf{H}^n$ via the diagram

$$\begin{array}{ccc} P_n \square R & \xrightarrow{\sim} & R_{n+1} \square P_n \\ {\scriptstyle 1 \square (\mathbf{Q}_n)_0}\downarrow & & \downarrow{\scriptstyle (\mathbf{H}^n)_0 \square 1} \\ P_n \square \mathbf{Q}_n R & \xrightarrow{\mathrm{ev}^n_R} & \mathbf{H}^n R \square P_n \end{array} \tag{112}$$

That $(\mathbf{Q}_n)_{M,N}$ and $(\mathbf{Q}_n)_0$ make $\mathbf{Q}_n$ a monoidal functor is now a simple consequence of monoidality of the functor $\mathbf{H}^n$.

Next we have to show that $\mathbf{Q}_f$ is a monoidal natural transformation for all $f : m \to n$ in Δ. For $f = i + (2 \to 1) + (n-1-i)$ this means showing commutativity of the diagrams

$$\begin{array}{ccc} \mathbf{Q}_n M \,\square\, \mathbf{Q}_n N & \xrightarrow{(\mathbf{Q}_n)_{M,N}} & \mathbf{Q}_n(M \,\square\, N) \\ {\scriptstyle \delta^i_n M \,\square\, \delta^i_n N}\downarrow & & \downarrow{\scriptstyle \delta^i_n (M \,\square\, N)} \\ \mathbf{Q}_{n+1} M \,\square\, \mathbf{Q}_{n+1} N & \xrightarrow{(\mathbf{Q}_{n+1})_{M,N}} & \mathbf{Q}_{n+1}(M \,\square\, N) \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{(\mathbf{Q}_n)_0} & \mathbf{Q}_n R \\ \| & & \downarrow{\scriptstyle \delta^i_n} \\ R & \xrightarrow{(\mathbf{Q}_{n+1})_0} & \mathbf{Q}_{n+1} R \end{array}$$

To make a long story short, we already know by Theorem 9.1 that H is a right R-bialgebroid therefore the factorization of the comultiplication $\Delta^H : H \xrightarrow{\Delta^H_\times} H \times H \hookrightarrow H \underset{R}{\otimes} H$ through the Takeuchi product is an algebra map $\Delta^H_\times$. Commutativity of the above two diagrams follows precisely from multiplicativity and unitality of $\Delta^H_\times$. Similar observation for the counit leads to monoidality of ε^i_n. This defines the required factorization of the functor $\mathbf{Q} : \Delta^{\mathrm{op}} \to \mathrm{End}\,\mathcal{E}(1,1)$ through the category $\mathrm{End}^{\otimes}\,\mathcal{E}(1,1)$ of monoidal endofunctors and monoidal natural transformations.

It remains to show that the monoidal structure of $\mathbf{Q}$, namely ν and ι, consists also of monoidal natural transformations. For ι there is nothing to prove since it can be chosen to be the identity as we have seen in the proof of Proposition 6.2. For ν this is the commutativity of the diagrams

$$\begin{array}{ccc} \mathbf{Q}_m \mathbf{Q}_n M \,\square\, \mathbf{Q}_m \mathbf{Q}_n N & \longrightarrow & \mathbf{Q}_m \mathbf{Q}_n (M \,\square\, N) \\ {\scriptstyle \nu^{m,n} M \,\square\, \nu^{m,n} N}\downarrow & & \downarrow{\scriptstyle \nu^{m,n}(M \,\square\, N)} \\ \mathbf{Q}_{m+n} M \,\square\, \mathbf{Q}_{m+n} N & \longrightarrow & \mathbf{Q}_{m+n}(M \,\square\, N) \end{array} \qquad \begin{array}{ccc} R & \longrightarrow & \mathbf{Q}_m \mathbf{Q}_n R \\ \| & & \downarrow{\scriptstyle \nu^{m,n} R} \\ R & \longrightarrow & \mathbf{Q}_{m+n} R \end{array} \tag{113}$$

Since $P_{m+n} = (P_n \otimes (R^{\mathrm{op}} \otimes R)^m) \,\square\, P_m$, we obtain the following multiplicativity rule for the evaluation:

$$\begin{array}{ccc} P_{m+n} \,\square\, \mathbf{Q}_m \mathbf{Q}_n M & \xrightarrow{1 \,\square\, \mathrm{ev}^m_{\mathbf{Q}_n M}} & H^m(P_n \,\square\, \mathbf{Q}_n M) \,\square\, P_m \\ {\scriptstyle 1 \,\square\, \nu^{m,n}_M}\downarrow & & \downarrow{\scriptstyle H^m \mathrm{ev}^n_M \,\square\, 1} \\ P_{m+n} \,\square\, \mathbf{Q}_{m+n} M & \xrightarrow{\mathrm{ev}^{m+n}_M} & H^{m+n} M \,\square\, P_{m+n} \end{array} \tag{114}$$

Using (111) and (114) one can show that

$$\begin{aligned} &\mathrm{ev}^{m+n}_{M \,\square\, N} \circ (P_{m+n} \,\square\, \nu^{m,n}_{M \,\square\, N}) \circ (P_{m+n} \,\square\, (\mathbf{Q}_m \mathbf{Q}_n)_{M,N}) \\ &\quad = \mathrm{ev}^{m+n}_{M \,\square\, N} \circ (P_{m+n} \,\square\, (\mathbf{Q}_{m+n})_{M,N}) \circ (P_{m+n} \,\square\, \nu^{m,n}_M \,\square\, \nu^{m,n}_N) \end{aligned}$$

from which the first diagram in (113) follows by adjunction. As for the second diagram one utilizes the fact that $\mathbf{H}^{m+n} = \mathbf{H}^m \mathbf{H}^n$ in diagram (112) to obtain

$$\begin{aligned} \mathrm{ev}^{m+n}_R \circ (1 \,\square\, (\mathbf{Q}_{m+n})_0) = {} & (H^m \mathrm{ev}^n_R \,\square\, 1) \circ (H^m (1 \,\square\, (\mathbf{Q}_n)_0) \,\square\, 1) \\ & \circ (1 \,\square\, \mathrm{ev}^m_R) \circ (1 \,\square\, 1 \,\square\, (\mathbf{Q}_m)_0) \end{aligned}$$

from which the statement can be obtained by rewriting the RHS using (114). This finishes the proof of monoidality of the lax comonad. The way the Eilenberg–Moore forgetful functor becomes strict monoidal is standard and needs no explanation. □

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