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Free products, cyclic homology, and the Gauss–Manin connection ☆

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Abstract

We use the techniques of Cuntz and Quillen to present a new approach to periodic cyclic homology. Our construction is based on $((\Omega^\bullet A)[t], \mathsf{d} + t \cdot \imath_\Delta)$, a *noncommutative equivariant de Rham complex* of an associative algebra A. Here d is the Karoubi–de Rham differential and $\imath_\Delta$ is an operation analogous to contraction with a vector field. As a byproduct, we give a simple explicit construction of the Gauss–Manin connection, introduced earlier by E. Getzler, on the relative periodic cyclic homology of a flat family of associative algebras over a central base ring.

We introduce and study *free-product deformations* of an associative algebra, a new type of deformation over a not necessarily commutative base ring. Natural examples of free-product deformations arise from preprojective algebras and group algebras for compact surface groups.

Keywords: Free products; Cyclic homology; Gauss–Manin connection

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☆ With an Appendix by Boris Tsygan.

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1. Introduction

Throughout, we fix a field $\Bbbk$ of characteristic 0 and write $\otimes = \otimes_{\Bbbk}$ (although, up to passing to reduced versions of homology theories, everything generalizes to the case that $\Bbbk$ is an arbitrary commutative ring containing $\mathbb{Q}$, provided the algebra A has the property that $\Bbbk \to A$ is a $\Bbbk$-split central injection). By an algebra we always mean an associative unital $\Bbbk$-algebra, unless explicitly stated otherwise. Given an algebra A, we view the space $A \otimes A$ as an A-bimodule with respect to the *outer* bimodule structure, which is defined by the formula $b(a' \otimes a'')c := (ba') \otimes (a''c)$, for any $a', a'', b, c \in A$. By an A-bimodule, we will always mean an $(A \otimes A^{\mathrm{op}})$-module, i.e., an A-bimodule on which the left and the right action of $\Bbbk$ coincide.

1.1. *Double derivations*

It is well-known that a regular vector field on a smooth affine algebraic variety X is the same thing as a derivation $\Bbbk[X] \to \Bbbk[X]$ of the coordinate ring of X. Thus, derivations of a commutative algebra play the role of vector fields.

It has been commonly accepted until recently that this point of view applies to noncommutative algebras as well. A first indication towards a different point of view was a discovery by Crawley-Boevey [4] that, for a smooth affine curve X with coordinate ring $A = \Bbbk[X]$, the algebra of differential operators on X can be constructed by means of *double derivations* $A \to A \otimes A$, rather than ordinary derivations $A \to A$. Since then, the significance of double derivations in noncommutative geometry was explored further in [30,5].

To explain the role of double derivations in more detail we first recall some basic definitions.

1.2. *(Double) derivations as infinitesimal automorphisms*

Recall that a free product of two algebras A and B, is an associative algebra $A * B$ that contains A and B as subalgebras and whose elements are formal $\Bbbk$-linear combinations of words $a_1b_1a_2b_2\ldots a_nb_n$, for any $n \geq 1$ and $a_1, \ldots, a_n \in A,\ b_1, \ldots, b_n \in B$. These words are taken up to the equivalence imposed by the relation $1_A = 1_B$; for instance, $\cdots b1_Ab' \cdots = \cdots b1_Bb' \cdots = \cdots (b \cdot b') \cdots$, for any $b, b' \in B$.

Let N be an A-*bimodule*. A $\Bbbk$-linear map $f : A \to N$ is said to be a derivation of A with coefficients in N if $f(a_1a_2) = f(a_1)a_2 + a_1f(a_2),\ \forall a_1, a_2 \in A$. Given a subalgebra $R \subset A$, we let $\mathrm{Der}_R(A, N)$ denote the space of R-linear derivations of A with respect to the subalgebra R, that is, of derivations $A \to N$ that annihilate the subalgebra R.

Derivations of an algebra A may be viewed as 'infinitesimal automorphisms'. Specifically, let $A[t] = A \otimes \Bbbk[t]$ be the polynomial ring in one variable with coefficients in A. The natural algebra embedding $A \hookrightarrow A[t]$ makes $A[t]$ an A-bimodule.

A well-known elementary calculation yields the following.

Lemma 1.2.1. *The following properties of a $\Bbbk$-linear map $\theta : A \to A$ are equivalent:*

- *the map θ is a derivation of the algebra A;*
- *the map $A \to t \cdot A[t],\ a \mapsto t \cdot \theta(a)$ is a derivation of the algebra A with coefficients in $t \cdot A[t]$;*
- *the map $a \mapsto a + t \cdot \theta(a)$ gives an algebra homomorphism $A \to A[t]/t^2 \cdot A[t]$.*

All the above holds true, of course, no matter whether the algebra A is commutative or not. Yet, the element t, the formal parameter, is by definition a *central* element of the algebra $A[t]$. In noncommutative geometry, the assumption that the formal parameter t be central is not quite natural, however. Put another way, while the tensor product is a coproduct in the category of commutative associative algebras, the free product is a coproduct in the category of not necessarily commutative associative algebras.

We see that, in noncommutative geometry, the algebra $A_t = A * \Bbbk[t]$, freely generated by A and an indeterminate t, should play the role of the polynomial algebra $A[t]$. We are going to argue that, once the polynomial algebra $A[t]$ is replaced by the algebra A_t, it becomes more natural to replace derivations $A \to A$ by *double derivations*, i.e., by derivations $A \to A \otimes A$ where $A \otimes A$ is viewed as an A-bimodule with respect to the outer bimodule structure. To see this, write $A_t^+ = A_t \cdot t \cdot A_t$, a two-sided ideal generated by the element t. Then, there are natural A-bimodule isomorphisms

$$\begin{aligned} A_t/A_t^+ &\xrightarrow{\sim} A, \quad \text{and} \quad A_t/(A_t^+)^2 \xrightarrow{\sim} A \oplus (A \otimes A), \\ a + a'\, t\, a'' &\longmapsto a \oplus (a' \otimes a''). \end{aligned} \tag{1.2.2}$$

Given a $\Bbbk$-linear map $\Theta : A \to A \otimes A$ we will use symbolic Sweedler notation to write this map as $a \mapsto \Theta'(a) \otimes \Theta''(a)$, where we systematically suppress the summation symbol.

Now, a free product analogue of Lemma 1.2.1 reads as follows.

Lemma 1.2.3. *The following properties of a $\Bbbk$-linear map $\Theta : A \to A \otimes A$ are equivalent:*

- *the map Θ is a double derivation;*
- *the map $A \to A_t^+$, $a \mapsto \Theta'(a)\, t\, \Theta''(a)$, is a derivation of the algebra A with coefficients in the A-bimodule A_t^+;*
- *the map $a \mapsto a + \Theta'(a)\, t\, \Theta''(a)$ gives an algebra homomorphism $A \to A_t/(A_t^+)^2$.*

1.3. Layout of the paper

In Section 3, we recall the definition of the DG algebra of noncommutative differential forms [2,3], following [6], and that of the Karoubi–de Rham complex [18]. We also introduce an *extended Karoubi–de Rham complex*, that will play a crucial role later. In Section 2, we develop the basics of noncommutative calculus involving the action of double derivations on the extended Karoubi–de Rham complex, via Lie derivative and contraction operations.

In Section 4, we state three main results of the paper. The first two, Theorems 4.1.1 and 4.3.2, provide a description, in terms of the Karoubi–de Rham complex, of the Hochschild homology of an algebra A and of the periodic cyclic homology of A, respectively. The third result, Theorem 4.4.1, gives a formula for the Gauss–Manin connection on periodic cyclic homology of a family of algebras, [13], in a way that avoids complicated formulas and resembles equivariant cohomology. These results are proved in Section 5, using properties of the Karoubi operator and the harmonic decomposition of noncommutative differential forms introduced by Cuntz and Quillen, [6,7].

In Section 6, we establish a connection between cyclic homology and equivariant cohomology via the representation functor. More precisely, we give a homomorphism from our noncommutative equivariant de Rham complex (which extends the complex used to compute cyclic homology) to the equivariant de Rham complex computing equivariant cohomology of the representation variety.

In Section 7, we introduce a new notion of free product deformation over a not necessarily commutative base. We extend classic results of Gerstenhaber concerning deformations of an associative algebra A to our new setting of free product deformations. To this end, we consider a double-graded Hochschild complex $\big(\oplus_{p,k\geq 2}\ C^p(A, A^{\otimes k}),\ \mathsf{b}\big)$. We define a new associative product $f, g \mapsto f \vee g$ on that complex, and study Maurer–Cartan equations of the form $\mathsf{b}(\beta) + \frac{1}{2}\beta \vee \beta = 0$.

2. Noncommutative calculus

2.1. The commutator quotient

Let $B = \oplus_{k\in\mathbb{Z}} B^k$ be a $\mathbb{Z}$-graded algebra and $M = \oplus_{k\in\mathbb{Z}} M^k$ a $\mathbb{Z}$-graded B-bimodule. Given a homogeneous element $u \in B^k$ or $u \in M^k$, we put $|u| := k$. A linear map $f : B^\bullet \to M^{\bullet+n}$ is said to be a *degree n graded* derivation if, for any homogeneous $u, v \in B$, $f(uv) = f(u)\cdot v + (-1)^{n|u|}u\cdot f(v)$. Given a graded subalgebra $R \subset B$, we let $\mathrm{Der}^n_R(B, M)$ denote the vector space of degree n graded R-linear derivations. The direct sum $\mathrm{Der}^\bullet_R B := \bigoplus_{n\in\mathbb{Z}} \mathrm{Der}^n_R(B, B)$ has a natural Lie *super*-algebra structure given by the *super*-commutator.

Let $[B, B]$ be the $\Bbbk$-linear span of the set $\{b_1b_2 - (-1)^{pq}b_2b_1, b_1 \in B^p, b_2 \in B^q,\ p, q \in \mathbb{Z}\}$. We put $B_{\mathrm{cyc}} := B/[B, B]$. Any (ungraded) algebra may be regarded as a graded algebra concentrated in degree zero. Thus, for an algebra B without grading, we have the subspace $[B, B] \subset B$ spanned by ordinary commutators and the corresponding commutator quotient space $B_{\mathrm{cyc}} = B/[B, B]$.

We write $T_B M = \oplus_{n\geq 0} T^n_B M$ for the tensor algebra of a B-bimodule M. Thus, $T^\bullet_B M$ is a graded associative algebra with $T^0_B M = B$.

2.2. Noncommutative differential forms

Fix an algebra A and a subalgebra $R \subset A$. (After this section, we will only take R to be either $\Bbbk$ or $\Bbbk[t]$, but other interesting examples include when $R = \Bbbk^I$ and A is a quotient of the path algebra of a quiver with vertex set I.) Let $\Omega^1_R A := \mathrm{Ker}(m)$ be the kernel of the multiplication map $m : A \otimes_R A \to A$, and write $i_\Delta : \Omega^1_R A \hookrightarrow A \otimes_R A$ for the tautological embedding. Thus, $\Omega^1_R A$ is a A-bimodule, called the bimodule of *noncommutative one-forms* on the algebra A relative to the subalgebra R. One has a short exact sequence of A-bimodules, see [6, Section 2],

$$0 \longrightarrow \Omega^1_R A \xrightarrow{i_\Delta} A \otimes_R A \xrightarrow{m} A \longrightarrow 0. \tag{2.2.1}$$

The assignment $a \mapsto \mathsf{d}a := 1\otimes a - a \otimes 1$ gives a canonical derivation $\mathsf{d} : A \to \Omega^1_R A$. This derivation is 'universal' in the sense that, for any A-bimodule M, there is a canonical bijection

$$\mathrm{Der}_R(A, M) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-bimod}}(\Omega^1_R A, M), \quad \theta \mapsto i_\theta, \tag{2.2.2}$$

where $i_\theta : \Omega^1_R A \to M$ stands for a A-bimodule map defined by the formula $i_\theta(u\,\mathsf{d}v) := u\cdot\theta(v)$.

The map d extends uniquely to a degree 1 derivation of $\Omega_R A := T^\bullet_A(\Omega^1_R A)$, the tensor algebra of the A-bimodule $\Omega^1_R A$. Thus, $(\Omega^\bullet_R A, \mathsf{d})$ is a DG algebra called the algebra of noncommutative differential forms on A relative to the subalgebra R (we will interchangeably use the notation $\Omega_R A$ or $\Omega^\bullet_R A$ depending on whether we want to emphasize the grading or not). For each $n \geq 1$, there is a standard isomorphism of left A-modules, see [6], $\Omega^n_R A = A \otimes_R T^n_R(A/R)$; usually, one writes $a_0\,\mathsf{d}a_1\,\mathsf{d}a_2 \ldots \mathsf{d}a_n \in \Omega^n_R A$ for the n-form corresponding to an element $a_0 \otimes (a_1 \otimes \cdots \otimes a_n) \in A \otimes_R T^n_R(A/R)$ under this isomorphism. The de Rham differential $\mathsf{d} : \Omega^\bullet_R A \to \Omega^{\bullet+1}_R A$ is given by the formula $\mathsf{d} : a_0\,\mathsf{d}a_1\,\mathsf{d}a_2 \ldots \mathsf{d}a_n \mapsto \mathsf{d}a_0\,\mathsf{d}a_1\,\mathsf{d}a_2 \ldots \mathsf{d}a_n$.

Following Karoubi [18], we define the (relative) noncommutative de Rham complex of A as

$$\mathrm{DR}_R A := (\Omega_R A)_{\mathrm{cyc}} = \Omega_R A/[\Omega_R A, \Omega_R A],$$

the super-commutator quotient of the *graded* algebra $\Omega^\bullet_R A$. The space $\mathrm{DR}_R A$ comes equipped with a natural grading and with the de Rham differential $\mathsf{d} : \mathrm{DR}^\bullet_R A \to \mathrm{DR}^{\bullet+1}_R A$, induced from the one on $\Omega^\bullet_R A$. In degrees 0 and 1, $\mathrm{DR}^0_R A = A/[A, A]$ and $\mathrm{DR}^1_R A = \Omega^1_R A/[A, \Omega^1_R A]$, respectively.

In the 'absolute' case $R = \Bbbk$ we will use unadorned notation $\Omega^n A := \Omega^n_{\Bbbk} A$, $\mathrm{DR} A := \mathrm{DR}_{\Bbbk} A$, etc.

The DG algebra $(\Omega^\bullet_R A, \mathsf{d})$ can be characterized by a universal property saying that it is the universal DG R-algebra generated by A (see [6, Corollary 2.2]). The universal property implies, in particular, for any algebras A and B, a canonical isomorphism

$$\Omega(A * B) \cong \Omega A * \Omega B. \tag{2.2.3}$$

2.3. Lie derivative and contraction for noncommutative differential forms

In this section, we introduce operations of Lie derivative and contraction on noncommutative differential forms.

Fix an algebra A and a subalgebra $R \subset A$. Any derivation $\theta \in \mathrm{Der}_R A$ gives rise, naturally, to a pair of graded derivations of the graded algebra $\Omega^\bullet_R A$, a contraction operation $i_\theta \in \mathrm{Der}^{\bullet-1}_A(\Omega^\bullet_R A)$ and a Lie derivative operation $L_\theta \in \mathrm{Der}^\bullet_R(\Omega^\bullet_R A)$, respectively. To define these operations it is convenient to use the following construction.

Let K be a graded $\Omega^\bullet_R A$-bimodule and $f : \Omega^1_R A \to K$ an A-bimodule map. One shows, by adapting the proof of [6, Proposition 2.6] to a graded setting, that the assignment $a \mapsto 0$, $\mathsf{d}a \mapsto f(\mathsf{d}a)$ can be extended uniquely to an A-linear derivation $T^\bullet_A(\Omega^1_R A) \to K$. Hence, one has the following chain of canonical isomorphisms

$$\mathrm{Der}^\bullet_R(A, K) \underset{(2.2.2)}{\xrightarrow{\sim}} \mathrm{Hom}^\bullet_{A\text{-bimod}}(\Omega^1_R A, K) \xrightarrow{\sim} \mathrm{Der}^{\bullet-1}_A(T^\bullet_A(\Omega^1_R A), K)$$
$$= \mathrm{Der}^{\bullet-1}_A(\Omega_R A, K). \tag{2.3.1}$$

Let $\theta \in \mathrm{Der}^\bullet_R(A, K)$ be a graded derivation. Applying to θ the composite isomorphism above one obtains a graded derivation $i_\theta \in \mathrm{Der}^{\bullet-1}_A(\Omega_R A, K)$, called *contraction with* θ.

To define contraction on differential forms, we put $K = \Omega^\bullet_R A$. For any $\theta \in \mathrm{Der}_R A$, the image of θ in $\mathrm{Der}^{\bullet-1}_A(T^\bullet_A(\Omega^1_R A), K)$ above produces a graded derivation, which we also call i_θ, on $\Omega^\bullet_R A = T^\bullet_A(\Omega^1_R A)$ (in fact, it is an A-derivation, not merely an R-derivation). For a one-form $u\,\mathsf{d}v$, we have $i_\theta(u\,\mathsf{d}v) = u \cdot \theta(v)$.

Next, one defines a Lie derivative map $L_\theta : \Omega^\bullet_R A \to \Omega^\bullet_R A$ by the Cartan formula $L_\theta := i_\theta\,\mathsf{d} + \mathsf{d}\,i_\theta$. Here, the expression on the right hand side is the super-commutator of i_θ, a graded derivation of degree -1, with d, a graded derivation of degree $+1$. It follows that L_θ is a degree 0 derivation of the algebra $\Omega^\bullet_R A$. For a one-form $u\,\mathsf{d}v$ we have $L_\theta(u\,\mathsf{d}v) = (\theta(u))\,\mathsf{d}v + u\,\mathsf{d}(\theta(v))$.

Each of the maps L_θ and i_θ descends to a well-defined operation on the de Rham complex $\mathrm{DR}^\bullet_R A = (\Omega_R A^\bullet)_{\mathrm{cyc}}$.

2.4. Lie derivative and contraction for double derivations

Write $\mathbb{D}\mathrm{er}_R A := \mathrm{Der}_R(A, A \otimes A)$ for the vector space of R-derivations $A \to A \otimes A$, which we refer to as "double derivations". When $R = \Bbbk$ we write $\mathbb{D}\mathrm{er} A$. Double derivations do *not* give rise to natural operations on the DG algebra $\Omega^\bullet_R A$ itself. Instead, for any double derivation

$\Theta : A \to A \otimes A$, we will define associated contraction and Lie derivative operations, which are double derivations $\Omega_R A \to \Omega_R A \otimes \Omega_R A$.

To do so, we apply the general construction based on (2.3.1) in the special case where $K = \Omega^\bullet_R A \otimes \Omega^\bullet_R A$, an $\Omega^\bullet_R A$-bimodule with respect to the outer action. For $\Theta \in \mathbb{D}\mathrm{er}_R A$, we consider the composition $A \to A \otimes A = \Omega^0_R A \otimes \Omega^0_R A \hookrightarrow \Omega^\bullet_R A \otimes \Omega^\bullet_R A = K$. This is a degree zero derivation $A \to K$ so, using (2.3.1), we obtain a contraction map $\Omega_R A \to \Omega_R A \otimes \Omega_R A$. This map, to be denoted $\mathbf{i}_\Theta$, is an A-linear graded derivation of degree (-1).

In the special case $n = 1$, the contraction $\mathbf{i}_\Theta$ reduces to the map $i_\Theta : \ \Omega^1_R A \to A \otimes A$, $\alpha \mapsto (i'_\Theta \alpha) \otimes (i''_\Theta \alpha)$, that corresponds to the derivation $\Theta \in \mathbb{D}\mathrm{er}_R A$ via the canonical bijection (2.2.2). More generally, for $n \geq 1$, separating individual homogeneous components, the contraction gives maps

$$\mathbf{i}_\Theta : \ \Omega^n_R A \longmapsto \bigoplus_{1\leq k\leq n} \Omega^{k-1}_R A \otimes \Omega^{n-k}_R A.$$

Explicitly, for any $\alpha_1, \ldots, \alpha_n \in \Omega^1_R A$, one finds

$$\mathbf{i}_\Theta(\alpha_1\alpha_2 \ldots \alpha_n) = \sum_{1\leq k\leq n} (-1)^{k-1} \cdot \alpha_1 \ldots \alpha_{k-1} \, (i'_\Theta \alpha_k) \otimes (i''_\Theta \alpha_k)\, \alpha_{k+1} \ldots \alpha_n. \tag{2.4.1}$$

Next, we define the Lie derivative. To this end, one first extends the de Rham differential on $\Omega_R A$ to a degree one map $\mathsf{d} : \ \Omega^\bullet_R A \otimes \Omega^\bullet_R A \to \Omega^\bullet_R A \otimes \Omega^\bullet_R A$ defined, for any $\alpha \in \Omega^p_R A$ and $\beta \in \Omega^q_R A$, by the formula $\mathsf{d}(\alpha \otimes \beta) := (\mathsf{d}\alpha) \otimes \beta + (-1)^p \alpha \otimes (\mathsf{d}\beta)$. Now, given $\Theta \in \mathbb{D}\mathrm{er}_R A$, we use Cartan's formula as a definition and put $\mathbf{L}_\Theta := \mathbf{i}_\Theta\, \mathsf{d} + \mathsf{d}\, \mathbf{i}_\Theta$. This gives a graded double derivation $\mathbf{L}_\Theta : \ \Omega_R A \to \Omega_R A \otimes \Omega_R A$ of degree 0.

3. The extended Karoubi–de Rham complex

3.1. Cyclic quotients for free products

Given a graded algebra $B = \bigoplus_q \ B^q$, we equip $B_t = B * \Bbbk[t]$, a free product algebra, with a *bi*grading $B_t = \bigoplus_{p,q} \ B_t^{p,q}$ such that the homogeneous component B^q, of the subalgebra $B \subset B_t$ is assigned bidegree $(0, q)$ and the variable t is assigned bidegree $(2, 0)$. Thus, the p-grading counts twice the number of occurrences of the variable t.

The bigrading on B_t descends to a bigrading $(B_t)_{\mathrm{cyc}} = \bigoplus_{p,q} \ (B_t)^{p,q}_{\mathrm{cyc}}$ on the super-commutator quotient. In particular, $(B_t)^{0,\bullet}_{\mathrm{cyc}} = B^\bullet_{\mathrm{cyc}}$. Further, it is clear that the assignment $u_1\, t\, u_2 \mapsto (-1)^{|u_1|\cdot|u_2|} u_2\, u_1$ yields an isomorphism $(B_t)^{2,\bullet}_{\mathrm{cyc}} \cong B$. More generally, for any $n \geq 1$, the space $(B_t)^{2n,\bullet}_{\mathrm{cyc}}$ is spanned by *c*yclic words $u_1\, t\, u_2\, t \ldots t\, u_n\, t$, where cyclic means that, for instance, one has $u_1\, t\, u_2\, t\, u_3\, t = (-1)^{|u_1|(|u_2|+|u_3|)} t\, u_3\, t\, u_1\, t\, u_2$ (modulo commutators).

Given a graded vector space V, we let the group $\mathbb{Z}/n\mathbb{Z}$ act on $V^{\otimes n} = T^n_\Bbbk V$ by cyclic permutations tensored by the sign character and write

$$V^{\otimes n}_{\mathrm{cyc}} := V^{\otimes n}/(\mathbb{Z}/n\mathbb{Z}).$$

Thus, the assignments $u_1\, t\, u_2\, t \ldots t\, u_n \mapsto u_1 \otimes u_2 \otimes \cdots \otimes u_n$ and $u_1\, t\, u_2\, t \ldots t\, u_n\, t \mapsto u_1 \otimes u_2 \otimes \cdots \otimes u_n$ yield natural vector space isomorphisms

$$B_t^{2n-2,\bullet} \ \xrightarrow{\sim}\ (B^\bullet)^{\otimes n} \quad \text{and} \quad (B_t)^{2n,\bullet}_{\mathrm{cyc}} \cong (B^\bullet)^{\otimes n}_{\mathrm{cyc}}, \quad \forall n = 1, 2, \ldots. \tag{3.1.1}$$

Any graded derivation $\theta : B \to B$ has a natural extension to a graded derivation $\theta_t : B_t \to B_t$ that restricts to θ on the subalgebra $B \subset B_t$ and sends t to 0. Further, any double derivation $\Theta : B \to B \otimes B$ gives rise to a unique derivation $\Theta_t : B_t \to B_t$ such that $\Theta_t(b) = \Theta'(b)\, t\, \Theta''(b)$, for any $b \in B$, and $\Theta_t(t) = 0$.

Now fix an ungraded algebra B. We view it as a graded algebra concentrated in degree zero. Then, the second grading on B_t becomes trivial, so the bigrading effectively reduces to a grading $B_t = \bigoplus_p B_t^{2p}$, by even integers. Then, we immediately deduce the following.

Lemma 3.1.2. $\mathrm{Der}^p_{\Bbbk[t]}(B_t, B_t) = 0$ *for any* $p < 0$*. The assignments* $\theta \mapsto \theta_t$ *and* $\Theta \mapsto \Theta_t$*, yield isomorphisms* $\mathrm{Der}(B, B) \xrightarrow{\sim} \mathrm{Der}^0_{\Bbbk[t]}(B_t, B_t)$ *and* $\mathbb{D}\mathrm{er}(B, B) \xrightarrow{\sim} \mathrm{Der}^2_{\Bbbk[t]}(B_t, B_t)$*, respectively.*

3.2. The extended de Rham complex

We are going to introduce an enlargement of the noncommutative de Rham complex ΩA. This enlargement is a DG algebra that has three equivalent definitions according to the following lemma.

Lemma 3.2.1. *There are natural algebra isomorphisms*

$$\Omega_{\Bbbk[t]}(A_t) \cong (\Omega A) * \Bbbk[t] \cong T_A(A^{\otimes 2} \oplus \Omega^1 A). \tag{3.2.2}$$

The differential on $T_A(A^{\otimes 2} \oplus \Omega^1 A)$ *obtained by transporting the de Rham differential on* $\Omega_{\Bbbk[t]}(A_t)$ *via the isomorphisms in (3.2.2) is the derivation induced by the composite* $i_\Delta \circ \mathsf{d} : A \to \Omega^1 A \hookrightarrow A \otimes A$ *(using (2.2.1)).*

Proof. The algebra $\Omega_R(A * R)$ is the quotient of the algebra $\Omega(A * R)$ by the two-sided ideal generated by the space $\mathsf{d}R \subset \Omega^1 R \subset \Omega^1(A * R)$. Since $\Omega(A * R) \cong (\Omega A) * (\Omega R)$, by (2.2.3), we deduce a DG algebra isomorphism

$$\Omega_R(A * R) \cong (\Omega A) * R. \tag{3.2.3}$$

Applying this in the special case $R = \Bbbk[t]$ yields the first isomorphism in (3.2.2).

To prove the second isomorphism in (3.2.2), fix an A-bimodule M. View $A^{\otimes 2} \oplus M$ as an A-bimodule. The assignment $(a' \otimes a'') \oplus m \mapsto a'ta'' + m$ clearly gives an A-bimodule map $A^{\otimes 2} \oplus M \to (T_A M)_t$. This map can be extended, by the universal property of the tensor algebra, to an algebra morphism $T_A(A^{\otimes 2} \oplus M) \to (T_A M)_t$. To show that this morphism is an isomorphism, we explicitly construct an inverse map as follows.

We start with a natural algebra embedding $f : T_A M \hookrightarrow T_A(A^{\otimes 2} \oplus M)$, induced by the A-bimodule embedding $M = 0 \oplus M \hookrightarrow A^{\otimes 2} \oplus M$. Then, by the universal property of free products, we can (uniquely) extend the map f to an algebra homomorphism $(T_A M)_t = (T_A M) * \Bbbk[t] \to T_A(A^{\otimes 2} \oplus M)$ by sending $t \mapsto 1_A \otimes 1_A \in A^{\otimes 2} \subset T_A^1(A^{\otimes 2} \oplus M)$. It is straightforward to check that the resulting homomorphism is indeed an inverse of the homomorphism constructed earlier.

Applying the above in the special case $M = \Omega^1 A$ yields the second isomorphism of the lemma.

The statement of the lemma concerning differentials is left to the reader. □

It is convenient to introduce a special notation $\Omega_t A := \Omega_{\Bbbk[t]}(A_t)$. This algebra comes equipped with a natural *bi*-grading $\Omega_t A = \oplus_{p,q \geq 0}\, \Omega_t^{2p,q} A$, where the even p-grading is induced from the one on A_t, and the q-component corresponds to the grading induced by the natural one on $\Omega^\bullet A$. It is easy to see that the p-grading corresponds, under the isomorphism (3.2.2) to the grading on $(\Omega A) * \Bbbk[t]$ that counts twice the number of occurrences of the variable t.

The *extended de Rham complex* of A is defined as a super-commutator quotient

$$\mathrm{DR}_t A := \mathrm{DR}_{\Bbbk[t]}(A_t) = (\Omega_{\Bbbk[t]}(A_t))_{\mathrm{cyc}} \cong ((\Omega A)_t)_{\mathrm{cyc}}.$$

The bigrading on $\Omega_t A$ clearly descends to a bigrading on the extended de Rham complex of A. The de Rham differential has bidegree $(0, 1)$:

$$\mathrm{DR}_t A = \oplus_{p,q}\ \mathrm{DR}_t^{2p,q} A, \qquad \mathsf{d}:\ \mathrm{DR}_t^{2p,q} A \to \mathrm{DR}_t^{2p,q+1} A.$$

Next, we use the identification (3.1.1) for $B := \Omega A$, and equip $(\Omega^\bullet A)^{\otimes p}$ with the tensor product grading that counts the total degree of differential forms involved, e.g., given $\alpha_i \in \Omega^{k_i} A,\ i = 1, \dots, p$, for $\alpha := \alpha_1 \otimes \dots \otimes \alpha_p \in (\Omega A)^{\otimes p}_{\mathrm{cyc}}$, we put $\deg \alpha := k_1 + \dots + k_p$. Then, we get

$$\mathrm{DR}_t^{2p,q} A = \begin{cases} \mathrm{DR}^q A & \text{if } p = 0; \\ \text{degree } q \text{ component of } (\Omega^\bullet A)^{\otimes p}_{\mathrm{cyc}} & \text{if } p \geq 1. \end{cases} \tag{3.2.4}$$

3.3. *Operations on the extended de Rham complex*

Let $\theta \in \mathrm{Der} A$. The Lie derivative L_θ is a derivation of the algebra $\Omega^\bullet A$. Hence, there is an associated derivation $(L_\theta)_t : (\Omega A)_t \to (\Omega A)_t$; see Lemma 3.1.2(ii). On the other hand, we may first extend θ to a derivation $\theta_t : A_t \to A_t$, and then consider the Lie derivative L_{θ_t}, a derivation of the algebra $\Omega_{\Bbbk[t]}(A_t) = \Omega_t A$, of bidegree $(0, 0)$. Very similarly, we also have graded derivations, $(i_\theta)_t$ and i_{θ_t}, of bidegree $(0, -1)$.

It is immediate to see that the two procedures above agree with each other in the sense that, under the identification $\Omega_{\Bbbk[t]}(A_t) \cong (\Omega A)_t$ provided by (3.2.2),

$$L_{\theta_t} = (L_\theta)_t, \quad \text{and} \quad i_{\theta_t} = (i_\theta)_t. \tag{3.3.1}$$

Next, we consider operations induced by double derivations.

Proposition 3.3.2. *Any double derivation $\Theta \in \mathbb{D}\mathrm{er} A$ gives a canonical Lie derivative operation, a graded derivation $L_\Theta \in \mathrm{Der}_{\Bbbk[t]}(\Omega_t A)$ of bidegree $(2, 0)$, and a contraction operation, a graded derivation $i_\Theta \in \mathrm{Der}_{\Bbbk[t]}(\Omega_t A)$ of bidegree $(2, -1)$.*

Proof. Given $\Theta \in \mathbb{D}\mathrm{er} A$, we first extend it to a free product derivation $\Theta_t : A_t \to A_t$, as in Section 3.1. Hence, there are associated Lie derivative L_{Θ_t}, and contraction i_{Θ_t}, operations on the complex $\Omega_{\Bbbk[t]}(A_t)$, of relative differential forms on the algebra A_t. Thus, we may use (3.2.2) to interpret L_{Θ_t} and i_{Θ_t} as operations on $\Omega_t A$, to be denoted by L_Θ and i_Θ, respectively. □

Remark 3.3.3. One can use the above defined actions of double derivations on $\Omega_t A$ to obtain an alternative construction of the operations introduced in Section 2.4. Specifically, we observe that the operations L_{Θ_t} and i_{Θ_t} may be viewed, thanks to the first isomorphism in (3.2.2), as derivations of the algebra $(\Omega A)_t$ of degree 2 with respect to the p-grading, that counts (twice) the number of occurrences of t. Hence, applying Lemma 3.1.2, we conclude that there exists a unique double derivation $\mathbf{L}_\Theta : \Omega A \to \Omega A \otimes \Omega A$ such that, for the corresponding map

$(\Omega A)_t \to (\Omega A)_t$, we have $L_{\Theta_t} = (\mathbf{L}_\Theta)_t$. A similar argument yields a double derivation $\mathbf{i}_\Theta : \Omega A \to \Omega A \otimes \Omega A$ such that $i_{\Theta_t} = (\mathbf{i}_\Theta)_t$. It is easy to check that the derivations $\mathbf{L}_\Theta$ and $\mathbf{i}_\Theta$ thus defined are the same as those introduced in Section 2.4 in a different way. $\square$

Both the Lie derivative and contraction operations on $\Omega_t A$ descend to the commutator quotient. This way, we obtain the Lie derivative L_Θ and the contraction i_Θ on $\mathrm{DR}_t A$, the extended de Rham complex. Explicitly, using isomorphisms (3.2.4), we can write the Lie derivative L_Θ and contraction i_Θ operations as chains of maps of the form

$$\mathrm{DR}A \longrightarrow \Omega A \longrightarrow (\Omega A)^{\otimes 2}_{\mathrm{cyc}} \longrightarrow (\Omega A)^{\otimes 3}_{\mathrm{cyc}} \longrightarrow \cdots. \tag{3.3.4}$$

There are some standard identities involving the Lie derivative and contraction operations on ΩA associated with ordinary derivations. Similarly, the Lie derivative and contraction operations on $\Omega_t A$ resulting from Proposition 3.3.2 satisfy the following identities:

$$\begin{aligned} &L_\Theta = \mathsf{d} \circ i_\Theta + i_\Theta \circ \mathsf{d}, \qquad i_\Theta \circ i_\Phi + i_\Phi \circ i_\Theta = 0,\\ &i_\xi \circ i_\Theta + i_\Theta \circ i_\xi = 0, \quad \forall \Theta, \Phi \in \mathbb{D}\mathrm{er} A,\ \xi \in \mathrm{Der} A. \end{aligned} \tag{3.3.5}$$

It follows, in particular, that the Lie derivative L_Θ commutes with the de Rham differential d.

To prove (3.3.5), one first verifies these identities on the generators of the algebra $\Omega_t A = (\Omega A)_t$, that is, on differential forms of degrees 0 and 1, which is a simple computation. The general case then follows by observing that any commutation relation between (graded)-derivations that holds on generators of the algebra holds true for all elements of the algebra.

It is immediate that the induced operations on $\mathrm{DR}_t A$ also satisfy (3.3.5).

3.4. *Reduced Lie derivative and contraction*

The second component (the q-component) of the bigrading on $\mathrm{DR}_t A$ induces a grading on each of the spaces $(\Omega A)^{\otimes k}_{\mathrm{cyc}}$, $k = 1, 2, \ldots,$ appearing in (3.3.4). Each of the maps in (3.3.4) corresponding to the Lie derivative induced by a double derivation $\Theta \in \mathbb{D}\mathrm{er} A$ preserves the q-grading. In the case of contraction with Θ, all maps in the corresponding chain (3.3.4) decrease the q-grading by one.

The leftmost map in (3.3.4), to be denoted $\imath_\Theta$ in the contraction case and $\mathscr{L}_\Theta$ in the Lie derivative case, will be especially important for us. These maps, which we will call the *reduced contraction* and *reduced Lie derivative*, respectively, have the form

$$\imath_\Theta : \mathrm{DR}^\bullet A \longrightarrow \Omega^{\bullet -1} A, \quad \text{and} \quad \mathscr{L}_\Theta : \mathrm{DR}^\bullet A \longrightarrow \Omega^\bullet A. \tag{3.4.1}$$

Explicitly, we see from (2.4.1) that the operation $\imath_\Theta$, for instance, is given, for any $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Omega^1 A$, by the following formula:

$$\imath_\Theta(\alpha_1 \alpha_2 \ldots \alpha_n) = \sum_{k=1}^{n} (-1)^{(k-1)(n-1)} \cdot (i''_\Theta \alpha_k) \cdot \alpha_{k+1} \ldots \alpha_n\, \alpha_1 \ldots \alpha_{k-1} \cdot (i'_\Theta \alpha_k). \tag{3.4.2}$$

An *ad hoc* definition of the maps in (3.4.1) via explicit formulas like (3.4.2) was first given in [5]. Proving the commutation relations (3.3.5) using explicit formulas is, however, very painful; this was carried out in [5] by rather long brute-force computations. Our present approach based on the free product construction yields the commutation relations for free.

3.5. The derivation Δ

There is a distinguished double derivation

$$\Delta : A \to A \otimes A, \qquad a \mapsto 1 \otimes a - a \otimes 1.$$

The corresponding contraction map $i_\Delta : \Omega^1 A \to A \otimes A$ is the tautological embedding (2.2.1). Furthermore, the derivation $\Delta_t : A_t \to A_t$ associated with Δ, cf Section 3.1, equals $\operatorname{ad} t : u \mapsto t \cdot u - u \cdot t$. Hence, the Lie derivative map $L_\Delta : \Omega_t A \to \Omega_t A$ reads $\omega \mapsto \operatorname{ad} t(\omega) := t \cdot \omega - \omega \cdot t$.

Lemma 3.5.1. (i) *For any $a_0, a_1, \ldots, a_n \in A$,*

$$\imath_\Delta(a_0\, \mathsf{d}a_1 \ldots \mathsf{d}a_n) = \sum_{1 \le k \le n} (-1)^{(k-1)(n-1)+1} \cdot [a_k,\ \mathsf{d}a_{k+1} \ldots \mathsf{d}a_n\, a_0\, \mathsf{d}a_1 \ldots \mathsf{d}a_{k-1}].$$

(ii) *In $\Omega^\bullet A$, we have $\imath_\Delta \circ \mathsf{d} + \mathsf{d} \circ \imath_\Delta = 0$ and $\mathsf{d}^2 = (\imath_\Delta)^2 = 0$; similar equations also hold in $DR^\bullet_t A$, with i_Δ in place of $\imath_\Delta$.*

Proof. Part (i) is verified by a straightforward computation based on formula (2.4.1). We claim next that, in $\Omega_t A$, we have $L_\Delta = \operatorname{ad} t$. Indeed, it suffices to check this equality on the generators of the algebra $\Omega_t A$. It is clear that $L_\Delta(t) = 0 = \operatorname{ad} t(t)$, and it is easy to see that both derivations agree on zero-forms and on one-forms. This proves the claim.

The equation $\imath_\Delta \circ \mathsf{d} + \mathsf{d} \circ \imath_\Delta = 0$, of part (ii) of the lemma, now follows by the Cartan formula on the left of (3.3.5), since the equation $L_\Delta = \operatorname{ad} t$ clearly implies that the map $L_\Delta : \mathrm{DR}_t A \to \mathrm{DR}_t A$, as well as the map $\mathscr{L}_\Delta$, vanishes. Finally, the formula of part (i) shows that the image of the map $\imath_\Delta$ is contained in $[A, \Omega A]$. Hence, we deduce $(\imath_\Delta)^2(\Omega A) \subset \imath_\Delta([A, \Omega A]) = 0$, since the map $\imath_\Delta$ vanishes on commutators. $\square$

Let $A_\tau := A * \Bbbk[\tau]$ be the graded algebra such that A is placed in degree zero and τ is an *odd* variable placed in degree 1. Let $\frac{d}{d\tau}$ be the degree -1 derivation of the algebra A_τ that annihilates A and satisfies $\frac{d}{d\tau}(\tau) = 1$. Similarly, let $\tau^2 \frac{d}{d\tau}$ be the degree $+1$ graded derivation of the algebra A_τ that annihilates A and satisfies $\tau^2 \frac{d}{d\tau}(\tau) = \tau^2$. For any homogeneous element $x \in A_\tau$, put $\operatorname{ad} \tau(x) := \tau x - (-1)^{|x|}\tau$; in particular, one finds that $\operatorname{ad}\tau(\tau) = 2\tau^2$.

It is easy to check that each of the derivations $\frac{d}{d\tau}$, $\tau^2 \frac{d}{d\tau}$, and $\operatorname{ad}\tau - \tau^2 \frac{d}{d\tau}$ squares to zero.

Claim 3.5.2. (i) *The following assignment gives a graded algebra embedding:*

$$j :\ \Omega_t A \hookrightarrow A * \Bbbk[\tau], \quad t \mapsto \tau^2, \qquad a_0\, \mathsf{d}a_1 \ldots \mathsf{d}a_n \mapsto a_0 \cdot [\tau, a_1] \cdot \cdots \cdot [\tau, a_n].$$

Moreover, the above map intertwines the contraction operation $\mathbf{i}_\Delta$ with the differential $\tau^2 \frac{d}{d\tau}$, and the Karoubi–de Rham differential d with the differential $\operatorname{ad}\tau - \tau^2 \frac{d}{d\tau}$.

(ii) *The image of the map j is annihilated by the derivation $\frac{d}{d\tau}$.*

(iii) *The complex $\left((A_\tau)_{cyc},\ \frac{d}{d\tau}\right)$ computes cyclic homology of the algebra A.* $\square$

We will neither use nor prove this result; cf. [6, Proposition 1.4] and [21, Section 4.1 and Lemma 4.2.1].

4. Applications to Hochschild homology and cyclic homology

4.1. Hochschild homology

Given an algebra A and an A-bimodule M, we let $H_k(A, M)$ denote the k-th Hochschild homology group of A with coefficients in M. Also, write $[A, M] \subset M$ for the $\Bbbk$-linear span of the set $\{am - ma \mid a \in A, m \in M\}$. Thus, $H_0(A, M) = M/[A, M]$.

We extend some ideas of Cuntz and Quillen [7] to obtain our first important result.

Theorem 4.1.1. *For any unital $\Bbbk$-algebra A, there is a natural graded space isomorphism*

$$H_\bullet(A, A) \cong \mathrm{Ker}[\imath_\Delta : DR^\bullet A \to \Omega^{\bullet-1} A].$$

To put Theorem 4.1.1 in context, recall that Cuntz and Quillen used noncommutative differential forms to compute Hochschild homology. Specifically, following [6,7], consider a complex $\ldots \xrightarrow{\mathsf{b}} \Omega^2 A \xrightarrow{\mathsf{b}} \Omega^1 A \xrightarrow{\mathsf{b}} \Omega^0 A \longrightarrow 0$. Here, b is the *Hochschild differential* given by the formula

$$\mathsf{b} : \ \alpha\, \mathsf{d}a \longmapsto (-1)^n \cdot [\alpha, a], \quad \forall a \in A/\Bbbk,\ \alpha \in \Omega^n A,\ n > 0. \tag{4.1.2}$$

It was shown in [7] that the complex $(\Omega^\bullet A, \mathsf{b})$ can be identified with the standard Hochschild chain complex. It follows that $H_\bullet(\Omega A, \mathsf{b}) = H_\bullet(A, A)$ are the Hochschild homology groups of A.

Theorem 4.1.1 will follow directly from Proposition 5.1.1 below (see the discussion after this proposition). Proposition 5.1.1 itself will be proved in Section 5.2.

Remark 4.1.3. A somewhat more geometric interpretation of Theorem 4.1.1, from the point of view of representation functors, is provided by the map (6.2.6): see Theorem 6.2.5 of Section 6 below.

4.2. *An application*

The algebra A is said to be *connected* if the following sequence is exact:

$$0 \longrightarrow \Bbbk \longrightarrow \mathrm{DR}^0 A \xrightarrow{\mathsf{d}} \mathrm{DR}^1 A. \tag{4.2.1}$$

Proposition 4.2.2. *Let A be a connected algebra such that $H_2(A, A) = 0$. Then,*

- $H_1(A, A) = (DR^1 A)_{\mathrm{closed}}$ *and* $(DR^2 A)_{\mathrm{exact}} = (DR^2 A)_{\mathrm{closed}}$.
- *There is a natural vector space isomorphism* $(DR^2 A)_{\mathrm{closed}} \xrightarrow{\sim} [A, A]$.

Proof. We will freely use the notation of [5, Section 4.1]. According to [5, Proposition 4.1.4], for any connected algebra A, there is a map $\widetilde{\mu_{\mathrm{nc}}}$, a lift of the *noncommutative moment map*, that fits into the following commutative diagram:

$$\begin{tikzcd} \mathrm{DR}^1 A \arrow[r, "\mathsf{d}"] \arrow[d, "\imath_\Delta"'] & (\mathrm{DR}^2 A)_{\mathrm{closed}} \arrow[d, "\imath_\Delta"] \arrow[dl, dotted, "\widetilde{\mu_{\mathrm{nc}}}"'] \\ {[A, A]} \arrow[r, "\mathsf{d}"] & {[A, \Omega^1 A].} \end{tikzcd} \tag{4.2.3}$$

Since $H_2(A, A) = 0$, we deduce from the short exact sequence of Theorem 4.1.1 for $n = 2$ that the right vertical map $\imath_\Delta$ in diagram (4.2.3) is injective. It follows, by commutativity of the lower right triangle in (4.2.3), that the map $\widetilde{\mu_{\mathrm{nc}}}$ must be injective. Hence, by the commutativity of the upper left triangle in the diagram, the kernel of the map d in the top row equals the kernel of the left vertical map $\imath_\Delta$. Thus, from Theorem 4.1.1 for $n = 1$, we deduce $H_1(A, A) = \ker(\imath_\Delta : \mathrm{DR}^1 A \to [A, A]) = (\mathrm{DR}^1 A)_{\mathrm{closed}}$.

Next, the left vertical map $\imath_\Delta$ in the diagram is given by the formula $a\,\mathsf{d}b \mapsto [a,b]$. Hence, this map is surjective. Since the map $\widetilde{\mu_{\mathrm{nc}}}$ is injective, it follows from the commutativity of the upper left triangle in (4.2.3) that $\widetilde{\mu_{\mathrm{nc}}}$, as well as the map d in the top row of diagram (4.2.3), must be surjective. We conclude that the map $\widetilde{\mu_{\mathrm{nc}}}$ yields an isomorphism $(\mathrm{DR}^2A)_{\mathrm{closed}} \xrightarrow{\sim} [A,A]$ and also that $(\mathrm{DR}^2A)_{\mathrm{exact}} = (\mathrm{DR}^2A)_{\mathrm{closed}}$. $\square$

Proposition 4.2.2 can be easily extended to a relative setting where the algebra A contains a subalgebra of the form $R = \Bbbk^r$, for some $r \geq 1$. Then, the correct relative counterpart of the commutator space $[A,A]$ turns out to be the subspace $[A,A]^R \subset [A,A]$, formed by the elements which commute with R. The corresponding formalism has been worked out in [5]. The relative version of Proposition 4.2.2 reads as follows.

Corollary 4.2.4. *Let $R = \Bbbk^r$. Let A be an algebra containing R and such that the sequence $0 \to R \to A \to DR^1_R A$ is exact and $H_2(A,A) = 0$. Then, there is a natural vector space isomorphism $(DR^2_R A)_{\mathrm{closed}} \xrightarrow{\sim} [A,A]^R$.* $\square$

An important example where the above corollary applies is the case where A is the path algebra of a quiver with r vertices.

Remark 4.2.5. The isomorphism of Proposition 4.2.2 and Corollary 4.2.4 plays a role in the theory of Calabi–Yau algebras; see [14, Claim 3.9.11].

4.3. Cyclic homology

We recall some standard definitions, following [22, Chapter 2 and p. 162]. For any graded vector space $M = \oplus_{i\geq 0} M^i$, we introduce a $\mathbb{Z}$-graded $\Bbbk[t,t^{-1}]$-module

$$M\hat{\otimes}\Bbbk[t,t^{-1}] := \bigoplus_{n\in\mathbb{Z}} \left(\prod_{i\in\mathbb{Z}} t^i\, M^{n-2i} \right), \tag{4.3.1}$$

where the grading is such that the space $M \subset M\hat{\otimes}\Bbbk[t,t^{-1}]$ has the natural grading, and $|t| := 2$.

Below, we will use a complex of *reduced* differential forms, defined by setting $\overline{\Omega}^0 := \Omega^0 A/\Bbbk = A/\Bbbk$ and $\overline{\Omega}^k := \Omega^k A$ for all $k > 0$. Let $\overline{\Omega}^\bullet := \bigoplus_{k\geq 0} \overline{\Omega}^k$. The Hochschild differential induces a $\Bbbk[t,t^{-1}]$-linear differential $\mathsf{b} : \overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}] \to \overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]$ of degree -1.

We also have the Connes differential $\mathsf{B} : \overline{\Omega}^\bullet \to \overline{\Omega}^{\bullet+1}$ [2]. Following Loday and Quillen [23], we extend it to a $\Bbbk[t,t^{-1}]$-linear differential on $\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]$ of degree $+1$. It is known that $\mathsf{B}^2 = \mathsf{b}^2 = 0$ and $\mathsf{B}\circ\mathsf{b} + \mathsf{b}\circ\mathsf{B} = 0$. Thus, the map $\mathsf{B} + t\cdot\mathsf{b} : \overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}] \to \overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]$ gives a degree $+1$ differential on $\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]$.

Write $HP_{-\bullet}(A)$, where, '$-\bullet$' denotes *inverting* the degrees, for the *reduced periodic cyclic homology* of A as defined in [23] or [22, Section 5.1], using a complex with differential of degree -1. According to [7], the groups $HP_{-\bullet}(A)$ turn out to be isomorphic to homology groups of the complex $(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}], \mathsf{B} + t\cdot\mathsf{b})$, with differential of degree $+1$ (which is why we must invert the degrees). It is known that the action of multiplication by t yields periodicity isomorphisms $HP_\bullet(A) \cong HP_{\bullet+2}(A)$. Thus, up to isomorphism, there are only two groups, $HP_{\mathrm{even}}(A) := HP_0(A)$, and $HP_{\mathrm{odd}}(A) := HP_1(A)$.

Next, we compose the map $\imath_\Delta : \mathrm{DR}^\bullet A \to \Omega^{\bullet-1}A$ with the natural projection $\Omega^\bullet A \twoheadrightarrow \mathrm{DR}^\bullet A$ to obtain a map $\Omega^\bullet A \to \Omega^{\bullet-1}A$. The latter map descends to a map $\overline{\Omega}^\bullet \to \overline{\Omega}^{\bullet-1}$. Furthermore, we may extend this last map, as well as the de Rham differential $\mathsf{d} : \overline{\Omega}^\bullet \to \overline{\Omega}^{\bullet+1}$, to $\Bbbk[t, t^{-1}]$-linear maps $\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}] \to \overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}]$, of degrees -1 and $+1$, respectively.

The resulting maps d and $\imath_\Delta$ satisfy $\mathsf{d}^2 = (\imath_\Delta)^2 = 0$ and $\mathsf{d}\circ \imath_\Delta + \imath_\Delta \circ d = 0$, by Lemma 3.5.1(ii). Thus, the map $\mathsf{d} + t \cdot \imath_\Delta$ gives a degree $+1$ differential on $\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}]$. This differential may be thought of as some sort of equivariant differential for the 'vector field' Δ.

The following theorem, to be proved in Section 5.4 below, is one of the main results of the paper. It shows the importance of the reduced contraction map $\imath_\Delta$ for cyclic homology.

Theorem 4.3.2. *The homology of the complex* $(\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}], \mathsf{d} + t \cdot \imath_\Delta)$ *is isomorphic to* $\overline{HP}_{-\bullet}(A)$*, the reduced periodic cyclic homology of* A *(with inverted degrees).*

Remark 4.3.3 (*Hodge Filtration*)**.** In [19, Section 1.17], Kontsevich considers a 'Hodge filtration' on periodic cyclic homology. In terms of Theorem 4.3.2, the Hodge filtration $\mathsf{F}^\bullet_{\text{Hodge}}$ may be defined as follows:

- $\mathsf{F}^n_{\text{Hodge}} HP_{\text{even}}$ consists of those classes representable by sums $\sum_{i \geq n} t^{-i}\gamma_{2i}$, $\gamma_{2i} \in \overline{\Omega}^{2i}$;
- $\mathsf{F}^{n+\frac{1}{2}}_{\text{Hodge}} HP_{\text{odd}}$ consists of those classes representable by sums $\sum_{i \geq n} t^{-i}\gamma_{2i+1}$, $\gamma_{2i+1} \in \overline{\Omega}^{2i+1}$.

4.4. The Gauss–Manin connection

It is well-known that, given a smooth family $p : \mathscr{X} \to S$ of complex schemes over a smooth base S, there is a canonical flat connection on the relative algebraic de Rham cohomology groups $H^\bullet_{DR}(\mathscr{X}/S)$, called the *Gauss–Manin connection*. More algebraically, let A be a commutative $\Bbbk$-algebra which is smooth over a regular subalgebra $B \subset A$. In such a case, the relative algebraic de Rham cohomology may be identified with $HP^B_\bullet(A)$, the relative periodic cyclic homology; see, e.g., [12]. The Gauss–Manin connection therefore provides a flat connection on the relative periodic cyclic homology.

In [13], Getzler extended the definition of the Gauss–Manin connection to a noncommutative setting. Specifically, let A be a (not necessarily commutative) associative algebra equipped with a *central* algebra embedding $B = \Bbbk[x_1, \ldots, x_n] \hookrightarrow A$. Assuming that A is free as a B-module, Getzler has defined a flat connection on $HP^B_\bullet(A)$. Unfortunately, Getzler's definition of the connection involves quite complicated calculations in the Hochschild complex that make it difficult to relate his definition with the classical geometric construction of the Gauss–Manin connection on de Rham cohomology. Alternative approaches to the definition of Getzler's connection, also based on homological algebra, were suggested more recently by Kaledin [17] and by Tsygan [29].

Below, we propose a new, geometrically more transparent (we believe) approach for the Gauss–Manin connection using the construction of cyclic homology from the previous subsection. Unlike earlier constructions, our formula for the connection on periodic cyclic homology is identical, essentially, to the classic formula for the Gauss–Manin connection in de Rham cohomology, though the objects involved have different meanings.

Our version of Getzler's result reads as follows.

Theorem 4.4.1. *Let B be a commutative algebra. Let A be an associative algebra equipped with a central algebra embedding $B \hookrightarrow A$ such that the quotient A/B is a free B-module.*

Then, there is a canonical flat connection ∇_{GM} on $HP^B_\bullet(A)$.

Notation 4.4.2. (i) Given an algebra R and a subset $J \subset R$, let (J) denote the two-sided ideal in R generated by the set J.

(ii) For a commutative algebra B, we set $\Omega^\bullet_{\mathrm{comm}} B := \Lambda^\bullet_B(\Omega^1_{\mathrm{comm}} B)$, the super-commutative DG algebra of differential forms, generated by the B-module $\Omega^1_{\mathrm{comm}} B$ of Kähler differentials.

Construction of the Gauss–Manin connection. Given a *central* algebra embedding $B \hookrightarrow A$, we may realize the relative periodic cyclic homology of A over B as follows. First, we define the following quotient DG algebras of $(\Omega^\bullet A, \mathsf{d})$:

$$\Omega^B A := \Omega^\bullet A/([\Omega^\bullet A, \Omega^\bullet B]), \qquad \Omega(A; B) := \Omega^B A/(\mathsf{d}B).$$

Thus, we have a super-central DG algebra embedding $\Omega^\bullet_{\mathrm{comm}} B \hookrightarrow \Omega^B A$ induced by the natural embedding $\Omega^\bullet B \hookrightarrow \Omega^\bullet A$. We introduce the descending filtration $F^\bullet(\Omega^B A)$ by powers of the ideal $(\mathsf{d}B)$. For the corresponding associated graded algebra, there is a natural surjection

$$\Omega^\bullet(A; B) \otimes_B \Omega^i_{\mathrm{comm}} B \twoheadrightarrow \mathrm{gr}^i_F \Omega^B A, \qquad \alpha \otimes \beta \mapsto \alpha\beta,$$
$$\forall \alpha \in \Omega^B A,\ \beta \in \Omega^i_{\mathrm{comm}} B. \tag{4.4.3}$$

Below, we will also make use of the objects $\overline{\Omega}^B A$ and $\overline{\Omega}(A; B)$, obtained by killing $\Bbbk \subset A = \Omega^0 A$. Thus, $\overline{\Omega}(A; B)\hat{\otimes}\Bbbk[t, t^{-1}]$ and $\overline{\Omega}^B A\hat{\otimes}\Bbbk[t, t^{-1}]$ are modules over $\Bbbk[t, t^{-1}]$. There is a natural descending filtration $F^\bullet$ on $\overline{\Omega}^B A\hat{\otimes}\Bbbk[t, t^{-1}]$ induced by $F^\bullet(\Omega^B A)$ and such that $\Bbbk[t, t^{-1}]$ is placed in filtration degree zero. This filtration is obviously stable under the differential d. It is also stable under the differential $t \cdot \imath_\Delta$ since the commutators that appear in $t \cdot \imath_\Delta(\omega)$ (see Lemma 3.5.1(i)) vanish, by definition of $\Omega^B A$. Therefore, the map (4.4.3) induces a morphism of double complexes, equipped with the differentials $d \otimes_B \mathrm{Id}$ and $t \cdot \imath_\Delta \otimes_B \mathrm{Id}$,

$$\overline{\Omega}^\bullet(A; B)\hat{\otimes}\Bbbk[t, t^{-1}] \otimes_B \Omega^i_{\mathrm{comm}} B \twoheadrightarrow \mathrm{gr}^i_F \overline{\Omega}^B A\hat{\otimes}\Bbbk[t, t^{-1}]. \tag{4.4.4}$$

We will show in Section 5.6 below that the assumptions of Theorem 4.4.1 ensure that the map (4.4.3) is an isomorphism.

Assume this for the moment and consider the standard spectral sequence associated with the filtration $F^\bullet(\overline{\Omega}^B A\hat{\otimes}\Bbbk[t, t^{-1}])$. The first page of this sequence consists of terms $\mathrm{gr}_F(\overline{\Omega}^B A\hat{\otimes}\Bbbk[t, t^{-1}])$. Under the above assumption, the LHS of (4.4.4), summed over all i, composes the first page of the spectral sequence of $\big(F^\bullet(\overline{\Omega}^B A\hat{\otimes}\Bbbk[t, t^{-1}]),\ \mathsf{d} + t \cdot \imath_\Delta\big)$. Then, for the second page of the spectral sequence we get

$$E^2 = H^\bullet\big(\overline{\Omega}(A; B)\hat{\otimes}\Bbbk[t, t^{-1}],\ \mathsf{d} + t \cdot \imath_\Delta\big) \otimes_B \Omega^\bullet_{\mathrm{comm}} B.$$

We now describe the differential ∇ on the second page. Let

$$\nabla_{GM} :\ H^\bullet(\overline{\Omega}(A; B)\hat{\otimes}\Bbbk[t, t^{-1}]) \longrightarrow H^\bullet\big(\overline{\Omega}(A; B)\hat{\otimes}\Bbbk[t, t^{-1}]\big) \otimes_B \Omega^1_{\mathrm{comm}} B \tag{4.4.5}$$

be the restriction of ∇ to degree zero. Then we immediately see that

$$\nabla(\alpha \otimes \beta) = \nabla_{GM}(\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \otimes (d_{\mathrm{DR}}\beta),$$
$$\nabla_{GM}(b\alpha) = b\nabla_{GM}(\alpha) + (-1)^{|\alpha|}\alpha \otimes (d_{\mathrm{DR}} b), \quad \forall b \in B,$$

where now d_{DR} is the usual de Rham differential. From these equations, we deduce that the map ∇_{GM}, from (4.4.5), gives a flat connection on $H^i\big(\overline{\Omega}(A; B)\hat{\otimes}\Bbbk[t, t^{-1}]\big)$ for all i.

Explicitly, we may describe the connection ∇_{GM} as follows. Suppose that $\bar{\alpha} \in \overline{\Omega}(A; B)$ has the property that $(\mathsf{d} + t \cdot \imath_\Delta)(\bar{\alpha}) = 0$. Let $\alpha \in \overline{\Omega}^B A$ be any lift, and consider $(\mathsf{d} + t \cdot \imath_\Delta)(\alpha)$. This must lie in $(\mathsf{d}B)$, and its image in $\Omega(A; B) \otimes_B \Omega^1_{\mathrm{comm}} B$ is the desired element. $\square$

Remark 4.4.6. In [13], Getzler takes $B = \Bbbk[[x_1, \ldots, x_n]]$, and takes A to be a *formal* deformation over B of an associative algebra A_0. Although such a setting is not formally covered by Theorem 4.4.1, our construction of the Gauss–Manin connection still applies.

To explain this, write $\mathfrak{m} \subset B = \Bbbk[[x_1, \ldots, x_n]]$ for the augmentation ideal of the formal power series without constant term. Let A_0 be a $\Bbbk$-vector space with a fixed nonzero element 1_A, and let $A = A_0[[x_1, \ldots, x_n]]$ be the B-module of formal power series with coefficients in A_0. We equip B and A with the $\mathfrak{m}$-adic topology, and view B as a B-submodule in A via the embedding $b \mapsto b \cdot 1_A$.

Corollary 4.4.7. *Let $\star : A \times A \to A$ be a B-bilinear, continuous associative (not necessarily commutative) product that makes 1_A the unit element. Then, the conclusion of Theorem 4.4.1 holds for $HP^B_\bullet(A)$.*

5. Proofs

5.1. The Karoubi operator

Throughout this section, we fix an algebra A and abbreviate $\Omega^n := \Omega^n A$ and $\Omega := \oplus_n \Omega^n$.

Given an A-bimodule M, put $M_\natural := M/[A, M] = H_0(A, M)$. In particular, an algebra homomorphism $A \to B$ makes B an A-bimodule. In such a case, one has a canonical projection $B_\natural = B/[A, B] \twoheadrightarrow B_{\mathrm{cyc}} = B/[B, B]$. This applies for $B = \Omega^\bullet$. So, we get a natural projection $\Omega^\bullet_\natural \to \mathrm{DR}^\bullet A$, which is not an isomorphism, in general.

Following Cuntz and Quillen [7], we consider a diagram

$$\Omega^0 \underset{\mathsf{b}}{\overset{\mathsf{d}}{\rightleftarrows}} \Omega^1 \underset{\mathsf{b}}{\overset{\mathsf{d}}{\rightleftarrows}} \Omega^2 \underset{\mathsf{b}}{\overset{\mathsf{d}}{\rightleftarrows}} \cdots.$$

Here, the de Rham differential d and the Hochschild differential b, defined in (4.1.2), are related via an important *Karoubi operator* $\kappa : \Omega^\bullet \to \Omega^\bullet$ [18]. The latter is defined by the formula $\kappa : \alpha\, \mathsf{d}a \mapsto (-1)^{\deg \alpha}\, \mathsf{d}a\, \alpha$ if $\deg \alpha > 0$, and $\kappa(\alpha) = \alpha$ if $\alpha \in \Omega^0$. By [18,6], one has

$$\mathsf{b} \circ \mathsf{d} + \mathsf{d} \circ \mathsf{b} = \mathrm{Id} - \kappa.$$

It follows that κ commutes with both d and b. Furthermore, it is easy to verify (see [6] and the proof of Lemma 5.2.1 below) that the Karoubi operator descends to a well-defined map $\kappa : \Omega^n_\natural \to \Omega^n_\natural$, which is essentially a cyclic permutation; specifically, in $\Omega^n_\natural$, we have

$$\kappa(\alpha_1 \alpha_2 \ldots \alpha_{n-1} \alpha_n) = (-1)^{n-1} \alpha_n \alpha_1 \alpha_2 \ldots \alpha_{n-1}, \quad \forall \alpha_1, \ldots, \alpha_n \in \Omega^1.$$

Let $(-)^\kappa$ denote taking κ-invariants. In particular, write $(\Omega^\bullet)^\kappa_\natural := [(\Omega^\bullet)_\natural]^\kappa \subset (\Omega^\bullet)_\natural$.

Proposition 5.1.1. *For any $n \geq 1$, we have an equality*

$$\imath_\Delta = (1 + \kappa + \kappa^2 + \cdots + \kappa^{n-1}) \circ \mathsf{b} \quad \text{as maps} \quad \Omega^n \to \Omega^{n-1}.$$

Furthermore, the map $\imath_\Delta$ fits into a canonical short exact sequence

$$0 \longrightarrow H^n(\Omega A, \mathsf{b}) \longrightarrow DR^n A \xrightarrow{\imath_\Delta} [A, \Omega^{n-1} A]^\kappa \longrightarrow 0.$$

We recall that the cohomology group $H^n(\Omega A, \mathsf{b})$ that occurs in the above displayed short exact sequence is isomorphic, as has been mentioned in Section 4.1, to the Hochschild homology $H_n(A, A)$. Thus, Theorem 4.1.1 is an immediate consequence of the short exact sequence of the proposition.

Special case: $H_1(A, A)$. For one-forms, the formula of Proposition 5.1.1 gives $\imath_\Delta = \mathsf{b}$. Thus, using the identification $H_1(A, A) = H^1(\Omega^\bullet, \mathsf{b})$, the short exact sequence of Proposition 5.1.1 reads

$$0 \longrightarrow H_1(A, A) \longrightarrow \mathrm{DR}^1 A \xrightarrow{\mathsf{b}=\imath_\Delta} [A, A] \longrightarrow 0. \tag{5.1.2}$$

The short exact sequence (5.1.2) may be obtained in an alternate way as follows. We apply the right exact functor $(-)_\natural$ to (2.2.1). The corresponding long exact sequence of Tor-groups reads

$$\cdots \to H_1(A, A \otimes A) \to H_1(A, A) \to (\Omega^1)_\natural \to (A \otimes A)_\natural \xrightarrow{c} A_\natural \to 0.$$

Now, by the definition of Tor, $H_k(A, A \otimes A) = 0$ for all $k > 0$. Also, we have natural identifications $(\Omega^1)_\natural = \mathrm{DR}^1 A$ and $(A \otimes A)_\natural \cong A$. This way, the map c on the right of the displayed formula above may be identified with the natural projection $A \twoheadrightarrow A/[A, A]$. Thus, $\mathrm{Ker}(c) = [A, A]$, and the long exact sequence above reduces to the short exact sequence (5.1.2).

It is immediate from definitions that the map $\mathsf{b} = \imath_\Delta$ in (5.1.2) is given by Quillen's formula $u\,\mathsf{d}v \mapsto [u, v]$ [6]. In particular, we deduce that $(\mathrm{DR}^1 A)_{\mathrm{exact}} \subset \mathrm{Ker}(\imath_\Delta) = H_1(A, A)$.

5.2. *Proof of Proposition 5.1.1*

We first state and prove a lemma which was implicit in [7] and [22, Section 2.6] and which will play an important role in Section 5 below.

Lemma 5.2.1. (i) *The projection $(\Omega^\bullet)_\natural \to DR^\bullet A$ restricts to a* ***bijection*** $(\Omega^\bullet)_\natural^\kappa \xrightarrow{\sim} DR^\bullet A$.

(ii) *The map* b *descends to a map* $\mathsf{b}_\natural : (\Omega^\bullet)_\natural \to \Omega^{\bullet-1}$.

(iii) *The kernel of the map* $\mathsf{b}_\natural : (\Omega^\bullet)_\natural^\kappa \to \Omega^{\bullet-1}$, *the restriction of* $\mathsf{b}_\natural$ *to the space of* κ*-invariants, is isomorphic to* $H^n(\Omega A, \mathsf{b})$.

Proof of Lemma. The argument below follows the proof of [22, Lemma 2.6.8].

From definitions, we get $[A, \Omega] = \mathsf{b}\Omega$ and $[\mathsf{d}A, \Omega] = (\mathrm{Id} - \kappa)\Omega$. Hence, we obtain, cf. [6]:

$$[\Omega, \Omega] = [A, \Omega] + [\mathsf{d}A, \Omega] = \mathsf{b}\Omega + (\mathrm{Id} - \kappa)\Omega.$$

We deduce that $\Omega_\natural = \Omega/\mathsf{b}\Omega$ and $\mathrm{DR}^\bullet A = \Omega/[\Omega, \Omega] = \Omega_\natural/(\mathrm{Id} - \kappa)\Omega_\natural$. In particular, since $\mathsf{b}^2 = 0$, the map b descends to a well defined map $\mathsf{b}_\natural : \Omega_\natural = \Omega/\mathsf{b}\Omega \to \Omega$.

Further, one has the following standard identities [7, Section 2], on Ω^n for all $n \geq 1$:

$$\kappa^n - \mathrm{Id} = \mathsf{b} \circ \kappa^n \circ \mathsf{d}, \qquad \kappa^{n+1} \circ \mathsf{d} = \mathsf{d}. \tag{5.2.2}$$

The Karoubi operator κ commutes with b, and hence induces a well-defined endomorphism of the vector space $\Omega^n/\mathsf{b}\Omega^n$, $n = 1, 2, \ldots$. Furthermore, from the first identity in (5.2.2) we see that $\kappa^n = \mathrm{Id}$ on $\Omega^n/\mathsf{b}\Omega^n$. Hence, we have a direct sum decomposition $\Omega_\natural = (\Omega_\natural)^\kappa \oplus (\mathrm{Id} - \kappa)\Omega_\natural$. It follows that the natural projection $\Omega_\natural = \Omega/\mathsf{b}\Omega \twoheadrightarrow \mathrm{DR}^\bullet A = \Omega_\natural/(\mathrm{Id} - \kappa)\Omega_\natural$ restricts to an isomorphism $(\Omega_\natural)^\kappa \xrightarrow{\sim} \mathrm{DR}^\bullet A$. Parts (ii)–(iii) of Lemma 5.2.1 are clear from the proof of [22, Lemma 2.6.8]. $\square$

Proof of Proposition 5.1.1. The first statement of the proposition is immediate from the formula of Lemma 3.5.1(i). To prove the second statement, we exploit the first identity in (5.2.2). Using the formula for $\imath_\Delta$ and the fact that b commutes with κ, we compute that

$$\begin{aligned}(\kappa-1)\circ\ \imath_\Delta &= \mathsf{b}\circ(\kappa-1)\circ(1+\kappa+\kappa^2+\cdots+\kappa^{n-1})\\ &= \mathsf{b}\circ(\kappa^n-1)=\mathsf{b}^2\circ\kappa^n\circ\mathsf{d}=0.\end{aligned}\tag{5.2.3}$$

Hence, we deduce that the image of $\imath_\Delta$ is contained in $(\mathsf{b}\Omega)^\kappa$. Conversely, given any element $\alpha=\mathsf{b}(\beta)\in(\mathsf{b}\Omega)^\kappa$, we find that

$$\imath_\Delta(\beta)=(1+\kappa+\kappa^2+\cdots+\kappa^{n-1})\circ\mathsf{b}\beta=n\cdot\mathsf{b}\beta=n\cdot\alpha.$$

Thus, $\mathrm{Im}(\imath_\Delta)=(\mathsf{b}\Omega)^\kappa=[A,\Omega]^\kappa$, since $\mathsf{b}\Omega=[A,\Omega]$. Furthermore, since $(1+\kappa+\kappa^2+\cdots+\kappa^{n-1})\circ\mathsf{b}=n\mathsf{b}$ on $(\Omega^\bullet)^\kappa_\natural$, the two maps have the same kernel. The exact sequence of the proposition now follows from Lemma 5.2.1. □

5.3. *Harmonic decomposition*

Our proof of Theorem 4.3.2 is an adaptation of the strategy used in [7, Section 2], based on the *harmonic decomposition*

$$\overline{\Omega}=P\overline{\Omega}\oplus P^\perp\overline{\Omega},\quad \text{where } P\overline{\Omega}:=\mathrm{Ker}(\mathrm{Id}-\kappa)^2,\quad P^\perp\overline{\Omega}:=\mathrm{Im}(\mathrm{Id}-\kappa)^2.\tag{5.3.1}$$

The differentials B, b, and d commute with κ, hence preserve the harmonic decomposition.

It will be convenient to introduce two degree preserving linear maps $\mathsf{N},\ \mathsf{N}!:\ \overline{\Omega}\to\overline{\Omega}$, such that, for any $n\geq 0$,

$$\mathsf{N}|_{\overline{\Omega}^n}\ \text{is multiplication by } n,\quad \text{and}\quad \mathsf{N}!|_{\overline{\Omega}^n}\ \text{is multiplication by } n!.\tag{5.3.2}$$

Then,

$$\text{(i) } \mathsf{B}=\mathsf{N}\,\mathsf{d}\,P,\quad \text{and}\quad \text{(ii) } \imath_\Delta=\mathsf{b}\,\mathsf{N}\,P.\tag{5.3.3}$$

Here, equation (i) is proved by Cuntz and Quillen [7, Section 2, formula (11)] using the second identity in (5.2.2).

Proof of 5.3.3(ii). First, we use that b commutes with κ. Therefore, applying the formula of Proposition 5.1.1, we find $i_\Delta\circ(\mathrm{Id}-\kappa)^2=(\kappa-1)\circ i_\Delta\circ(\kappa-1)=0$. We conclude that the operation $\imath_\Delta$ annihilates $P^\perp\overline{\Omega}$.

It remains to show that, on $P\overline{\Omega}^n$, one has $\imath_\Delta=(n-1)\cdot\mathsf{b}$. To this end, let $\alpha\in\overline{\Omega}^n$. From the first identity in (5.2.2), $\alpha-\kappa^n(\alpha)\in\mathsf{b}\overline{\Omega}$. Hence, $\mathsf{b}\alpha-\kappa^n(\mathsf{b}\alpha)\in\mathsf{b}^2\overline{\Omega}=0$, since $\mathsf{b}^2=0$. Thus, the operator κ has finite order on $\mathsf{b}\overline{\Omega}$, and hence on $\mathsf{b}(P\overline{\Omega})$. But, for any operator T of finite order, $\mathrm{Ker}(\mathrm{Id}-T)=\mathrm{Ker}((\mathrm{Id}-T)^2)$. It follows that, if $\alpha\in P\overline{\Omega}^n$, then $\mathsf{b}\alpha\in\mathrm{Ker}((\mathrm{Id}-\kappa)^2)=\mathrm{Ker}(\mathrm{Id}-\kappa)$. We conclude that the element $\mathsf{b}\alpha$ is fixed by κ. Hence, $(1+\kappa+\kappa^2+\cdots+\kappa^{n-1})\circ\mathsf{b}\alpha=n\cdot\mathsf{b}\alpha$. Therefore, by Proposition 5.1.1, $\imath_\Delta(\alpha)=n\cdot\mathsf{b}\alpha$, and (5.3.3)(ii) is proved. □

5.4. *Proof of Theorem 4.3.2*

Since the harmonic decomposition is stable under all four differentials $\mathsf{B},\mathsf{b},\mathsf{d}$, and $\imath_\Delta$, we may analyze the homology of each of the direct summands, $P\overline{\Omega}$ and $P^\perp\overline{\Omega}$, separately.

First of all, we know that $\mathsf{B} = 0$ on $P^{\perp}\Omega$, by (5.3.3)(i), and moreover it has been shown by Cuntz and Quillen [7, Proposition 4.1(1)] that $(P^{\perp}\Omega, \mathsf{b})$ is acyclic. Furthermore, since the complex $(\overline{\Omega}, \mathsf{d})$ is acyclic (see [7, Section 1] or [5, formula (2.5.1)]), we deduce the following.

$$\text{Each of the complexes } (P\overline{\Omega}, \mathsf{d}) \quad \text{and} \quad (P^{\perp}\overline{\Omega}, \mathsf{d}) \text{ is acyclic.} \tag{5.4.1}$$

Now, the map $\imath_\Delta$ vanishes on $P^{\perp}\overline{\Omega}$ by (5.3.3)(ii). Hence, on $P^{\perp}\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}]$, we have $\mathsf{d} + t \cdot \imath_\Delta = \mathsf{d}$. Therefore, we conclude using (5.4.1) that $(P^{\perp}\overline{\Omega}[t], \mathsf{d})$, and hence also $(P^{\perp}\overline{\Omega}[t], \mathsf{d} + t \cdot \imath_\Delta)$, are acyclic complexes.

Thus, to complete the proof of the theorem, we must compare cohomology of the complexes $(P\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}], \mathsf{d}+t\cdot\imath_\Delta)$ and $(P\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}], \mathsf{B}+t\cdot\mathsf{b})$. We have $\mathsf{N}\cdot\mathsf{d}+(\mathsf{N}+1)^{-1}\cdot t\cdot\imath_\Delta = \mathsf{B}+t\mathsf{b}$. Post-composing this by $\mathsf{N}!$ (see (5.3.2)), we obtain $(\mathsf{N}!) \cdot (\mathsf{d} + t \cdot \imath_\Delta) = (\mathsf{B} + t \cdot \mathsf{b}) \cdot (\mathsf{N}!)$. We deduce the following isomorphism of complexes which completes the proof of the theorem:

$$\mathsf{N}! : \big(P\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}],\ \mathsf{d} + t \cdot \imath_\Delta\big) \xrightarrow{\sim} \big(P\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}],\ \mathsf{B} + t \cdot \mathsf{b}\big). \quad \square$$

5.5. Negative cyclic homology and ordinary cyclic homology

It is possible to extend Theorem 4.3.2 to the case of (nonperiodic) cyclic homology and negative cyclic homology using harmonic decomposition. To explain this, put

$$(\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}])_+ := \bigoplus_{m\ge 0}\left(\prod_{i<m} t^i\,\Omega^{m-2i}\right),$$
$$(\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}])_{\ge 0} := \bigoplus_{m\ge 0}\left(\prod_{i\le m} t^i\,\Omega^{m-2i}\right).$$

It follows from definitions that cyclic homology and negative cyclic homology, respectively, may be defined in terms of the following complexes (see, e.g., [22, Chapters 2–3]):

$$\begin{aligned} HC_\bullet &= H^{-\bullet}\big(\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}]/(\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}])_+,\ \mathsf{B} + t \cdot \mathsf{b}\big), \\ HC_\bullet^- &= H^{-\bullet}\big((\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}])_{\ge 0},\ \mathsf{B} + t \cdot \mathsf{b}\big). \end{aligned} \tag{5.5.1}$$

Furthermore, we introduce two other homology theories, ${}^{\heartsuit}HC$, ${}^{\heartsuit}HC^-$. The corresponding homology groups are defined as the homology groups of complexes similar to (5.5.1), but where the differential $\mathsf{B} + t \cdot \mathsf{b}$ is replaced by $\mathsf{d} + t \cdot \imath_\Delta$.

Now, in terms of the projection to the harmonic part (recall (5.3.1)), we have the following.

Proposition 5.5.2. *There are natural graded $\Bbbk[t]$-module isomorphisms*

$$P({}^{\heartsuit}HC_\bullet) \cong HC_\bullet \quad \textit{and} \quad P({}^{\heartsuit}HC_\bullet^-) \cong HC_\bullet^-.$$

Proof. There are natural splittings

$$\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}] = \frac{\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}]}{(\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}])_+} \bigoplus (\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}])_+,$$
$$\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}] = \frac{\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}]}{(\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}])_{\ge 0}} \bigoplus (\overline{\Omega}\hat{\otimes}\Bbbk[t, t^{-1}])_{\ge 0}.$$

These splittings are stable under the differential $t \cdot \mathsf{b}$. It follows that $P^{\perp}HC = 0 = P^{\perp}HC_{-}$, since $\mathsf{B} = 0$ on $P^{\perp}\overline{\Omega}$, and the differential $t \cdot \mathsf{b}$ is acyclic here.

Observe further that, while ι_{Δ} is zero on $P^{\perp}\overline{\Omega}$, the above splittings do *not* stabilize d. So, we can pick up some nonzero groups $P^{\perp}({}^{\heartsuit}HC)$, or $P^{\perp}({}^{\heartsuit}HC^{-})$. However, restricting to the harmonic part, the proof of Theorem 4.3.2 yields an isomorphism of complexes

$$\begin{aligned}&\big(P[\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]/(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}])_{+}],\ \mathsf{B}+t\cdot\mathsf{b}\big)\\&\quad\cong\big(P[\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]/(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}])_{+}],\ \mathsf{d}+t\cdot\iota_{\Delta}\big),\end{aligned}$$

and also a similar isomorphism involving $(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}])_{\geq 0}$. □

Remark 5.5.3. In view of these results, it is reasonable to ask why our construction of the Gauss–Manin connection does not extend to ordinary cyclic homology and negative cyclic homology. The point is that, in these cases, one must replace $\Omega^{i}_{\mathrm{comm}}B$ in (4.4.3) by $t^{-\lfloor i/2\rfloor}\Omega^{2i}_{\mathrm{comm}}B$, and so one does not obtain a connection over B. (Note also that the results of this subsection require passing to the harmonic part, which does not seem to fit well into our construction of the Gauss–Manin connection.)

5.6. *Proof of Theorem 4.4.1*

Let $B \subset A$ and assume that A/B is a free B-module. The main step of the proof is the following.

Lemma 5.6.1. *Let $\{a_s,\ s \in \mathscr{S}\}$ be a basis of A/B as a free B-module. Then, the elements below form a basis for $\Omega^{B}(A)$ as a free $\Omega^{\bullet}(B)$-module:*

$$a_{s_0}\mathsf{d}a_{s_1}\mathsf{d}a_{s_2}\dots\mathsf{d}a_{s_m},\qquad \mathsf{d}a_{s_1}\mathsf{d}a_{s_2}\dots\mathsf{d}a_{s_m},\quad s_j \in \mathscr{S}. \tag{5.6.2}$$

Proof. Observe that the short exact sequence $B \to A \to A/B$ splits as a sequence of B-modules, since A/B is a free B-module. By abuse of notation we will view A/B as a subspace of A via a fixed choice of such splitting. Hence, the elements 1 and $\{a_s,\ s \in \mathscr{S}\}$ give a B-module basis of A.

Now, $\Omega^{m}A \cong A \otimes (A/\Bbbk)^{\otimes m}$ is spanned over $\Bbbk$ by elements of the form

$$f_0\mathsf{d}f_1\cdots\mathsf{d}f_m, \tag{5.6.3}$$

where each f_i is of the form ba_s for some $b \in B$ and $s \in \mathscr{S}$, except for f_0 which can also be an element of B itself.

Now, let $\Omega_{\mathscr{S}} \subseteq \Omega A$ be the subspace spanned by elements of the form (5.6.2). Let $\Omega' := \Omega B \otimes_{\Bbbk} \Omega_{\mathscr{S}}$ be the free (ΩB)-module it generates. We have a projection $\pi : \Omega A \twoheadrightarrow \Omega'$, given on elements (5.6.3) by rewriting $\mathsf{d}(ba_s) = (\mathsf{d}b)a_s + b(\mathsf{d}a_s)$, and then moving all b and $\mathsf{d}b$ terms to the front of the expression (via supercommutators). It is easy to see that this is well-defined. Thus, Lemma 5.6.1 reduces to the next claim, which we prove below. □

Claim 5.6.4. *The map π descends to an isomorphism $\pi' : \Omega^{B}A \xrightarrow{\sim} \Omega'$.*

Proof. Consider the multiplication map $\mu : \Omega' = \Omega B \otimes_{\Bbbk} \Omega_{\mathscr{S}} \to \Omega A$. It is easy to see that μ is injective and that the composition $\pi \circ \mu$ is the identity. This realizes Ω' as a direct summand of ΩA. Let $\mathrm{pr} : \Omega A \twoheadrightarrow \Omega^B A$ be the projection. Note that the composition

$$\Omega A \xrightarrow{\pi} \Omega' \xrightarrow{\mu} \Omega A \xrightarrow{\mathrm{pr}} \Omega^B A$$

coincides with pr, by the definition of π.

We will show that π factors through pr, i.e., $\pi = \pi' \circ \mathrm{pr}$ for $\pi' : \Omega^B A \to \Omega'$. From this and the above, it follows that $(\mathrm{pr} \circ \mu) \circ \pi' = \mathrm{Id}$. Since π' is surjective, $(\mathrm{pr} \circ \mu)$ and π' are therefore inverse isomorphisms.

Equivalently, we need to show that π annihilates the kernel of $\Omega A \twoheadrightarrow \Omega^B A$, i.e., the two-sided ideal generated by $[B, \Omega A]$ and $[\mathsf{d}B, \Omega A]$. We first prove the following multiplicative property of π:

$$\pi(\alpha\beta) = \pi\big((\mu \circ \pi)(\alpha)(\mu \circ \pi)(\beta)\big). \tag{5.6.5}$$

To prove this, it is enough to assume that α and β are elements of the form (5.6.3). Say $\beta = f\mathsf{d}\beta'$ where β' is also of this form. Then, we need to show that $\pi(\alpha \cdot f \cdot \mathsf{d}\beta') = \pi(\pi(\alpha) \cdot f \cdot \pi(\mathsf{d}\beta'))$. Since, in either expansion, each term $\mathsf{d}(ba)$ in $\mathsf{d}\beta'$ is replaced by $(\mathsf{d}b)a + b(\mathsf{d}a)$, and then after everything else is so expanded, the $\mathsf{d}b$ and b are moved to the front, it follows that one can ignore the $\mathsf{d}\beta'$ and merely prove that $\pi(\alpha \cdot f) = \pi(\alpha) \cdot f$. Now, to expand $\pi(\alpha \cdot f)$, we need to use

$$\begin{aligned} \mathsf{d}f_1 \cdots \mathsf{d}f_m \cdot f_{m+1} &= (-1)^m f_1 \mathsf{d}f_2 \cdots \mathsf{d}f_{m+2} \\ &\quad + \sum_{i=1}^{m} (-1)^{m-i} \mathsf{d}f_1 \cdots \mathsf{d}f_{i-1} \mathsf{d}(f_i f_{i+1}) \mathsf{d}f_{i+2} \cdots \mathsf{d}f_{m+1}. \end{aligned} \tag{5.6.6}$$

So, we have to show that applying (5.6.6) and then applying π is the same as applying π to α, multiplying by f, and applying (5.6.6). This is a straightforward verification.

Now, using (5.6.5), the claim reduces to showing that $\pi([B, \mathsf{d}A]) = \pi([\mathsf{d}B, A]) = \pi([\mathsf{d}B, \mathsf{d}A]) = 0$. The first two identities are equivalent, since $\mathsf{d}[B, A] = 0$, and are straightforward to verify. It is immediate that $\pi([\mathsf{d}B, \mathsf{d}A]) = 0$. So this reduces the claim to (5.6.5). $\square$

It is instructive, for the proof of Theorem 4.4.1 presented below, to have in mind the situation of Remark 4.4.6, where $B = \Bbbk[\![x_1, \ldots, x_n]\!]$ and $A = (A_0[\![x_1, \ldots, x_n]\!], \star)$. Then, $\{a_s\}$ is a $\Bbbk$-basis of A_0, and the (topologically-free version of) Lemma 5.6.1 becomes more obvious.

Remark 5.6.7. More generally, given an arbitrary *regular* commutative algebra B and a maximal ideal $\mathfrak{m} \subset B$, taking the $\mathfrak{m}$-adic completions $\widehat{A}_{\mathfrak{m}}$ and $\widehat{B}_{\mathfrak{m}}$ reduces to the above situation. $\square$

Completion of the proof of Theorem 4.4.1. We take B as a ground ring and apply Theorem 4.3.2 (which applies over any commutative base ring containing $\Bbbk$). We deduce that

$$H^{-i}\big(\overline{\Omega}(A; B) \hat{\otimes} \Bbbk[t, t^{-1}], \mathsf{d} + t\,\imath_\Delta\big) \cong HP^B_\bullet(A), \quad \forall i \in \mathbb{Z}.$$

Now Lemma 5.6.1 implies that the map in (4.4.3) is an isomorphism, since the basis for $\Omega^B A$ as a free $\Omega^\bullet(B)$-module is also a basis for the associated graded $\mathrm{gr}^i_F \Omega^B A$, and a B-module basis for $\Omega^\bullet(A; B) = \mathrm{gr}^0_F \Omega^B A$. The construction of the Gauss–Manin connection given in Section 4.4 completes the proof of the theorem. $\square$

Proof of Corollary 4.4.7. We only need to show that, in the present setting, the map in (4.4.3) is an isomorphism. For that, we observe that the argument used in the proof goes through provided that A is only *topologically* free over B, and our claim follows. $\square$

6. The representation functor

6.1. The evaluation map

We fix a finite-dimensional $\Bbbk$-vector space V. Set $\mathrm{End} := \mathrm{End}_{\Bbbk}(V)$. Given affine schemes X and S, let $X(S) = \mathrm{Hom}(S, X) = \mathrm{Hom}_{\Bbbk\text{-alg}}(\Bbbk[X], \Bbbk[S])$ denote the set of S-points of X.

Given an algebra A, we may consider the set $\mathrm{Hom}_{\Bbbk\text{-alg}}(A, \mathrm{End})$ of all algebra maps $\rho : A \to \mathrm{End}$. More precisely, to any *finitely presented* associative $\Bbbk$-algebra A we associate an affine scheme of finite type over $\Bbbk$, to be denoted $\mathrm{Rep}(A, V)$, such that $\mathrm{Rep}(A, V)(B) \cong \mathrm{Hom}_{\Bbbk\text{-alg}}(A, B \otimes \mathrm{End})$. That is, the B-points of $\mathrm{Rep}(A, V)$ correspond to families of representations of A parametrized by $\mathrm{Spec} B$. Write $\Bbbk[\mathrm{Rep}(A, V)]$ for the coordinate ring of the affine scheme $\mathrm{Rep}(A, V)$, which will be always assumed to be *nonempty*.

The tensor product $\mathrm{End} \otimes \Bbbk[\mathrm{Rep}(A, V)]$ is an associative algebra of polynomial maps $\mathrm{Rep}(A, V) \to \mathrm{End}$. To each element $a \in A$, we associate the element $\widehat{a} \in \mathrm{End} \otimes \Bbbk[\mathrm{Rep}(A, V)]$, which on the level of points, has the form $\widehat{a} : \mathrm{Rep}(A, V)(B) \to \mathrm{End} \otimes B$, $\widehat{a}(\rho) = \rho(a)$. This yields an algebra homomorphism $\mathrm{ev} : A \longrightarrow \mathrm{End} \otimes \Bbbk[\mathrm{Rep}(A, V)]$, $a \mapsto \widehat{a}$, called the *evaluation map*.

6.2. The extended de Rham complex and equivariant cohomology

Let X be an affine scheme with coordinate ring $\Bbbk[X]$, tangent module $\mathscr{T}(X) := \mathrm{Der}(\Bbbk[X],\ \Bbbk[X])$, and module of Kähler differentials $\Omega^1(X)$. We write $\Omega^\bullet(X) = \Lambda^\bullet_{\Bbbk[X]}\Omega^1(X)$ for the DG algebra of differential forms, equipped with the de Rham differential d_X.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let $\mathfrak{g}$ act on $\Bbbk[\mathfrak{g}]$, the polynomial algebra on the vector space $\mathfrak{g}$, by the adjoint action. We view $\Bbbk[\mathfrak{g}]$ as an even-graded algebra such that the vector space of linear functions on $\mathfrak{g}$ is assigned degree 2.

Given a Lie algebra map $\mathfrak{g} \to \mathscr{T}(X)$, $e \mapsto \vec{e}$, we get a $\mathfrak{g}$-action $\omega \mapsto L_{\vec{e}}\omega$ on $\Omega^\bullet(X)$, by the Lie derivative. This makes the tensor product $\Omega^\bullet(X, \mathfrak{g}) := \Omega^\bullet(X) \otimes \Bbbk[\mathfrak{g}]$ a graded algebra, equipped with the total grading and with the $\mathfrak{g}$-diagonal action. For $e \in \mathfrak{g}$, let $i_{\vec{e}}$ denote the contraction. Then, set

$$d_{\mathfrak{g}} :\ \Omega^\bullet(X, \mathfrak{g}) \longrightarrow \Omega^{\bullet+1}(X,\ \mathfrak{g}), \qquad \omega \otimes f \longmapsto \sum_{r=1}^{\dim \mathfrak{g}} (i_{\vec{e}_r}\omega) \otimes (e_r^* \cdot f), \tag{6.2.1}$$

where $\{e_r\}$ and $\{e_r^*\}$ stand for dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively. This map restricts to a differential $d_{\mathfrak{g}}$ on $\Omega^\bullet(X,\ \mathfrak{g})^{\mathfrak{g}}$, the graded subalgebra of $\mathfrak{g}$-diagonal invariants.

Definition 6.2.2. A differential form $\omega \in \Omega^\bullet(X)$ is called *basic* if, for any $e \in \mathfrak{g}$, both $L_{\vec{e}}\omega = 0$ and $i_{\vec{e}}\omega = 0$. Basic forms form a subcomplex $\Omega^\bullet{}_{\mathrm{basic}}(X) \subset \Omega^\bullet(X)$ of the de Rham complex.

Furthermore, define the $\mathfrak{g}$-*equivariant algebraic de Rham complex* of X to be the complex

$$\big(\Omega^\bullet(X,\ \mathfrak{g})^{\mathfrak{g}},\ d_{\mathrm{DR}} + d_{\mathfrak{g}}\big), \quad d_{\mathrm{DR}} := d_X \otimes \mathrm{id}_{\Bbbk[\mathfrak{g}]}. \tag{6.2.3}$$

We now return to the setup of Section 6.1. Thus we fix a finitely-presented algebra A, a finite-dimensional vector space V, and consider the scheme $\mathrm{Rep}(A, V)$.

Let $G = \mathrm{GL}(V)$. This is an algebraic group over $\Bbbk$ that acts naturally on the algebra End by inner automorphisms, via conjugation. Hence, given an algebra homomorphism $\rho : A \to \mathrm{End}$ and $g \in G(\Bbbk)$, one has a conjugate homomorphism $g(\rho) : a \mapsto g \cdot \rho(a) \cdot g^{-1}$. Then, the action $\rho \mapsto g(\rho)$ makes $\mathrm{Rep}(A, V)$ a G-scheme (extending in the obvious way to B-valued representations for any B).

Let $\mathfrak{g} := \mathrm{Lie}\, G$ be the Lie algebra of G. The action of G on $\mathrm{Rep}(A, V)$ induces a Lie algebra map

$$\mathrm{act}_A : \mathfrak{g} \longrightarrow \mathscr{T}(\mathrm{Rep}(A, V)), \qquad x \longmapsto \vec{x} = \mathrm{act}_A(x). \tag{6.2.4}$$

Thus, we may consider $\Omega^\bullet(\mathrm{Rep}(A, V), \mathfrak{g})^{\mathfrak{g}}$, the corresponding $\mathfrak{g}$-equivariant algebraic de Rham complex.

Now, thanks to Lemma 3.5.1(ii), the map $\mathsf{d}+i_\Delta : \mathrm{DR}_t A \to \mathrm{DR}_t A$ squares to zero. We call the resulting complex $(\mathrm{DR}_t A,\ \mathsf{d} + i_\Delta)$ the **noncommutative equivariant de Rham complex**. The second isomorphism of the following theorem, which is the main result of this section, shows that this complex is indeed a noncommutative analogue of the equivariant de Rham complex (6.2.3).

Recall the operator N from (5.3.2).

Theorem 6.2.5. *The evaluation map induces the following canonical morphisms of complexes:*

$$\big(H_\bullet(A, A),\ \mathsf{B}\big) \xrightarrow{\mathrm{ev}} \big(\Omega^\bullet{}_{\mathrm{basic}}(\mathrm{Rep}(A, V)),\ (\mathsf{N}+1)\circ d_{DR}\big), \quad \textit{and} \tag{6.2.6}$$

$$\big(DR^\bullet_t A,\ \mathsf{d} + i_\Delta\big) \xrightarrow{\mathrm{ev}} \big(\Omega^\bullet(\mathrm{Rep}(A, V), \mathfrak{g})^{\mathfrak{g}},\ d_{DR} + d_{\mathfrak{g}}\big). \tag{6.2.7}$$

We begin the proof with some general constructions.

6.3. The evaluation map on differential forms

Observe that giving an algebra homomorphism $\rho : \Bbbk[t] \to B \otimes \mathrm{End}$ amounts to specifying an arbitrary element $x = \rho(t) \in B \otimes \mathrm{End}$. Thus, $\mathrm{Rep}(\Bbbk[t], V)(B) = B \otimes \mathrm{End}$.

Similarly, for any algebra A, giving an algebra morphism $\rho : A * \Bbbk[t] \to B \otimes \mathrm{End}$ amounts to giving a homomorphism $A \to B \otimes \mathrm{End}$ and an arbitrary additional element $x = \rho(t) \in B \otimes \mathrm{End}$. We see that $\mathrm{Rep}(A_t, V) \cong \mathrm{Rep}(A, V) \times \mathrm{End}$. Let π denote the second projection, which is G-equivariant. We will use shorthand notation

$$\mathrm{Rep}_t := \mathrm{Rep}(A_t, V) = \mathrm{Rep}(A, V) \times \mathrm{End}, \quad \text{and} \quad \mathrm{Rep} := \mathrm{Rep}(A, V).$$

Let $\Omega^\bullet_\pi(\mathrm{Rep}_t)$ be the DG algebra of relative (with respect to π) algebraic differential forms on the scheme Rep_t (in the ordinary sense of commutative algebraic geometry). By definition,

$$\Omega^\bullet_\pi(\mathrm{Rep}_t) := \Lambda^\bullet_{\Bbbk[\mathrm{Rep}_t]}\, \Omega^1_\pi(\mathrm{Rep}_t) \cong \Omega^\bullet(\mathrm{Rep}) \otimes \Bbbk[\mathrm{End}]. \tag{6.3.1}$$

Generalizing the construction of Section 6.1, we now introduce an evaluation map on relative differential forms. In more detail, given $n = 0, 1, 2, \ldots,$ write $m : \mathrm{End}^{\otimes(n+1)} \to \mathrm{End}$ for the n-fold multiplication map. We define a map ev_Ω as the following composite:

$$\begin{aligned}
\Omega^n_{\Bbbk[t]}(A_t) = A_t \otimes (A_t/\Bbbk[t])^{\otimes n} &\xrightarrow{\mathrm{ev}} \ (\mathrm{End} \otimes \Bbbk[\mathrm{Rep}_t]) \otimes \big(\mathrm{End} \otimes \Omega^1_\pi(\mathrm{Rep}_t)\big)^{\otimes n} \\
&\to \ \mathrm{End}^{\otimes n+1} \otimes \bigotimes\big(\Lambda^n_{\Bbbk[\mathrm{Rep}_t]}\, \Omega^1_\pi(\mathrm{Rep}_t)\big) \\
&\xrightarrow{m \otimes \mathrm{Id}} \mathrm{End} \otimes \Omega^n_\pi(\mathrm{Rep}_t).
\end{aligned}$$

Any element in the image of this composite is easily seen to be G-invariant with respect to the G-diagonal action on $\mathrm{End} \otimes \Omega^n_\pi(\mathrm{Rep}_t)$. Thus, the composite above yields a well-defined, canonical DG algebra map

$$\mathrm{ev}_{\Omega} : \Omega_t A \to \left(\mathrm{End} \otimes \Omega^\bullet_\pi(\mathrm{Rep}_t)\right)^G, \qquad \alpha = a_0\, \mathsf{d}a_1 \ldots \mathsf{d}a_n \mapsto \widehat{\alpha} = \widehat{a}_0\, d_{\mathrm{DR}}\widehat{a}_1 \ldots d_{\mathrm{DR}}\widehat{a}_n.$$

Furthermore, we have the linear function $\mathrm{Tr} : \mathrm{End} \to \Bbbk,\ x \mapsto \mathrm{Tr}(x)$. We form the composite

$$\Omega_t A \xrightarrow{\mathrm{ev}_\Omega} \left(\mathrm{End} \otimes \Omega^\bullet_\pi(\mathrm{Rep}_t)\right)^G \xrightarrow{\mathrm{Tr}\otimes\mathrm{Id}} \left(\Bbbk \otimes \Omega^\bullet_\pi(\mathrm{Rep}_t)\right)^G = \Omega^\bullet_\pi(\mathrm{Rep}_t)^G,$$
$$\alpha \mapsto \mathrm{Tr}\widehat{\alpha}. \tag{6.3.2}$$

The above composite vanishes on the (graded) commutator space $[\Omega_t A, \Omega_t] \subset \Omega_t A$, due to symmetry of the trace function. Therefore, the map in (6.3.2) descends to $\mathrm{DR}^\bullet(\Omega_t A)$.

We remark next that the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ is nothing but the associative algebra End viewed as a Lie algebra. Hence, using the isomorphisms in (6.3.1), we can write

$$\Omega^\bullet_\pi(\mathrm{Rep}_t) = \Omega^\bullet(\mathrm{Rep}) \otimes \Bbbk[\mathrm{End}] = \Omega^\bullet(\mathrm{Rep}) \otimes \Bbbk[\mathfrak{g}] = \Omega^\bullet(\mathrm{Rep},\ \mathfrak{g}).$$

Thus, by the definition of the extended de Rham complex, $\mathrm{DR}^\bullet_t A$, the composite in (6.3.2) gives a map

$$(\mathrm{Id} \otimes \mathrm{Tr}) \circ \mathrm{ev}_\Omega :\ \mathrm{DR}^\bullet_t A \longrightarrow \Omega^\bullet_\pi(\mathrm{Rep}_t)^G = \Omega^\bullet(\mathrm{Rep},\ \mathfrak{g})^{\mathfrak{g}}. \tag{6.3.3}$$

6.4. *Proof of Theorem 6.2.5*

It is clear that d is taken to d_{DR} under (6.3.3). Hence, proving (6.2.7), where the map 'ev' stands for $(\mathrm{Id} \otimes \mathrm{Tr}) \circ \mathrm{ev}_\Omega$, amounts to showing commutativity of the diagram

$$\begin{tikzcd} \mathrm{DR}_t A \arrow[r, "(\mathrm{Id}\otimes\mathrm{Tr})\,\circ\,\mathrm{ev}_\Omega"] \arrow[d, "i_\Delta"'] & \Omega(\mathrm{Rep},\ \mathfrak{g})^{\mathfrak{g}} \arrow[d, "d_{\mathfrak{g}}"'] \\ \mathrm{DR}_t A \arrow[r, "(\mathrm{Id}\otimes\mathrm{Tr})\,\circ\,\mathrm{ev}_\Omega"] & \Omega(\mathrm{Rep},\ \mathfrak{g})^{\mathfrak{g}}. \end{tikzcd} \tag{6.4.1}$$

To see this, we note that, for any $a_0, \ldots, a_n \in A_t$, one has

$$\mathrm{ev}_\Omega \circ i_\Delta[a_0\, \mathsf{d}a_1\, \mathsf{d}a_2 \ldots \mathsf{d}a_n]$$
$$= (\mathrm{Id} \otimes \mathrm{Tr}) \left(\sum_{i=1}^{n} \widehat{a}_0\, d_{\mathrm{DR}}\widehat{a}_1 \ldots d_{\mathrm{DR}}\widehat{a}_{i-1}\, [\widehat{t}, \widehat{a}_i]\, d_{\mathrm{DR}}\widehat{a}_{i+1} \ldots d_{\mathrm{DR}}\widehat{a}_n \right). \tag{6.4.2}$$

Next, note that $\hat{t}$ may be identified with the element $\mathrm{Id} \in \mathfrak{g} \otimes \mathfrak{g}^* \cong \mathrm{End} \otimes \mathfrak{g}^* \subset \mathrm{End} \otimes \Bbbk[\mathfrak{g}]$. Furthermore, for any element $e \in \mathfrak{g}$ and any $s \in (\Bbbk[\mathrm{Rep}(A, V)] \otimes \mathrm{End})^{\mathfrak{g}}$, we evidently have $i_{\vec{e}}(ds) = \mathrm{ad}\, e(s) = [(1 \otimes e), s]$. As a consequence, by (6.2.1), we obtain that the RHS of (6.4.2) may be identified with $d_{\mathfrak{g}}((\mathrm{Id} \otimes \mathrm{Tr})(\widehat{a}_0 \mathsf{d}\widehat{a}_1 \ldots \mathsf{d}\widehat{a}_n))$, as desired.

To prove (6.2.6), ev becomes the restriction of $(\mathrm{Id} \otimes \mathrm{Tr}) \circ \mathrm{ev}_\Omega$ to $\ker(i_\Delta) \subset \Omega^\bullet(\mathrm{Rep})^{\mathfrak{g}}$ (recall Theorem 4.1.1). Commutativity of (6.4.1) together with Theorem 4.1.1 immediately gives that this induces a morphism $H_\bullet(A, A) \to \Omega^\bullet_{basic}(\mathrm{Rep}(A, V))$. It remains only to show that B is carried to $(\mathsf{N} + 1)\mathsf{d}$. To see this, we use the harmonic decomposition (5.3.1). Under the quotient $\overline{\Omega}^\bullet A \twoheadrightarrow \mathrm{DR}^\bullet A/\Bbbk$ (considering $\Bbbk$ to be the span of the image of $1 \in A$), $P^\perp \overline{\Omega}$ is killed, so the differential B is carried to $(\mathsf{N} + 1)d_{\mathrm{DR}}$. Thus, on $\mathrm{DR}^\bullet A$, the differential B must reduce to the same as $(\mathsf{N} + 1)\mathsf{d}$ up to a scalar. However, since B has degree $+1$, the scalar must be zero. So $\mathsf{B} = (\mathsf{N} + 1)\mathsf{d}$ on $\mathrm{DR}^\bullet A$. Thus, the same is true after passing to $\ker(i_\Delta)$.

7. Free products and deformations

7.1. First order deformations based on free products

Recall that, given an associative algebra A, we let $A_t = A*\Bbbk[t]$ and write $I = A_t^+ = (t) \subset A_t$ for the augmentation ideal.

A *first order free product deformation* of an associative algebra A is the structure of an associative algebra on the vector space A_t/I^2 that makes the vector space $I/I^2 \subset A_t/I^2$ a two-sided ideal and that makes the natural bijection below an algebra isomorphism,

$$\big(A_t/I^2\big)/\big(I/I^2\big) = A_t/I \ \xrightarrow{\sim}\ A.$$

It is convenient to identify the vector space A_t/I^2 with $A \oplus (A \otimes A)$, using (1.2.2). Thus, we are interested in associative products on the vector space $A \oplus (A \otimes A)$ that have the following form:

$$\big(u \oplus (u' \otimes u'')\big) \times \big(v \oplus (v' \otimes v'')\big) \xmapsto{\star_\beta} uv \oplus \big(u' \otimes u''v + uv' \otimes v'' + \beta(u, v)\big), \quad (7.1.1)$$

where $\beta : A \times A \to A \otimes A$ is a certain $\Bbbk$-bilinear map.

These products are taken up to an equivalence. Specifically, for any $\Bbbk$-linear map $f : A \to A \otimes A$, we define a linear bijection

$$\tilde{f} :\ A \oplus (A \otimes A) \longrightarrow A \oplus (A \otimes A), \qquad u \oplus (u' \otimes u'') \longmapsto u \oplus (u' \otimes u'' + f(u)).$$

Given a product $\star_\beta$ and a map f, we define a new product by transporting the structure via $\tilde{f}$, that is, by the formula $x \star_\gamma y := \tilde{f}^{-1}(\tilde{f}(x) \star_\beta \tilde{f}(y))$. We say that the products $\star_\gamma$ and $\star_\beta$ are *equivalent*.

A routine calculation, completely analogous to the classical one due to Gerstenhaber, now yields the following.

- A product $\star_\beta$ as in (7.1.1) is associative $\iff \beta \in C^2(A, A \otimes A)$ is a Hochschild two-cocycle with coefficients in $A \otimes A$.
- The products $\star_\beta$ and $\star_\gamma$ corresponding to two-cochains β and γ are equivalent $\iff \beta - \gamma$ is a Hochschild coboundary.

Thus, similarly to the classical theory, we obtain a classification of first order star product deformations in terms of Hochschild cohomology as follows.

Proposition 7.1.2. *Equivalence classes of associative products, as in (7.1.1), are in one-to-one correspondence with the elements of $H^2(A, A \otimes A)$, the second Hochschild cohomology group of the A-bimodule $A \otimes A$.* $\square$

To study *higher order* free product deformations, we have to introduce first some new operations on Hochschild cohomology, to be defined below.

7.2. Operations on Hochschild cohomology of A valued in the tensor algebra of A

For any algebra A, the natural embedding $A \hookrightarrow A_t$ makes A_t a graded A-bimodule. Using the identification (3.1.1), we may write $A_t = \bigoplus_{k \geq 1} A^{\otimes k}$. Here, the summand $A^{\otimes k}$ is assigned degree $2k - 2$ and is equipped with the *outer* A-bimodule structure defined by the formula $b(a' \otimes u \otimes a'')c := (ba') \otimes u \otimes (a''c)$, for any $a', a'', b, c \in A$ and $u \in A^{\otimes(k-2)}$.

Let $C^\bullet(A, A_t) = \bigoplus_{p,k\geq 1} C^p(A, A^{\otimes k})$ be the Hochschild cochain complex with coefficients in the A-bimodule A_t. Multiplication in the algebra A_t induces, for any $p, q, k, m \geq 1$, a cup product

$$\cup : C^p(A, A^{\otimes k}) \times C^q(A, A^{\otimes m}) \longrightarrow C^{p+q}(A, A^{\otimes(k+m-1)}).$$

This way, $C^\bullet(A, A_t)$ acquires the structure of a bigraded associative algebra such that the direct summand $C^p(A, A^{\otimes k})$ is assigned bidegree $(p, 2k-2)$.

Next, on $C^\bullet(A, A_t)$, we introduce a pair of new binary operations, $\vdash$ and $\dashv$:

$$\begin{aligned}
&C^p(A, A^{\otimes k}) \times C^q(A, A^{\otimes m}) \longrightarrow C^{p+q-1}(A, A^{\otimes(k+m-1)}),\\
&(f, g) \mapsto f \vdash g := f^{[1,p]} \circ g^{[p,p+q-1]}, \quad \text{and}\\
&(f, g) \mapsto f \dashv g := g^{[k,k+q-1]} \circ f^{[1,p]},
\end{aligned}$$

where $f^{[i,j]}$ denotes applying f to the consecutive components $i, i+1, i+2, \ldots, j$, that is,

$$f^{[i,j]}(a_1 \otimes \cdots \otimes a_\ell) = a_1 \otimes \cdots \otimes a_{i-1} \otimes f(a_i \otimes \cdots \otimes a_j) \otimes a_{j+1} \otimes \cdots \otimes a_\ell.$$

Proposition 7.2.1. (i) *The operation $f \vee g := f \vdash g - f \dashv g$ and Hochschild differential* b *give the space $C(A, A_t)_{\geq 2} := \bigoplus_{p,k\geq 2} C^p(A, A^{\otimes k})$ the structure of an associative DG algebra, i.e.,*

$$f \vee (g \vee h) = (f \vee g) \vee h, \tag{7.2.2}$$

$$\mathsf{b}(f \vee g) = (\mathsf{b}f) \vee g + (-1)^{p-1} f \vee (\mathsf{b}g) \quad \forall f \in C^p(A, A^{\otimes k}). \tag{7.2.3}$$

(ii) *The cup product $\cup$ is associative and induces the zero map on cohomology:*

$$H^p(A, A^{\otimes k}) \otimes H^q(A, A^{\otimes m}) \xrightarrow{\cup=0} H^{p+q}(A, A^{\otimes k+m-1})$$
$$\forall p, q \geq 1 \text{ whenever } k \geq 2 \text{ or } m \geq 2.$$

(iii) *The following compatibility identities hold:*

$$(f \cup g) \vee h = f \cup (g \vee h), \qquad f \vee (g \cup h) = (f \vee g) \cup h. \tag{7.2.4}$$

Proof. We note the following identities (for any x, y, z):

$$\begin{aligned}
x \vdash (y \vdash z) &= (x \vdash y) \vdash z, & x \dashv (y \dashv z) &= (x \dashv y) \dashv z,\\
x \vdash (y \dashv z) &= (x \vdash y) \dashv z, & x \dashv (y \vdash z) &= (x \dashv y) \vdash z.
\end{aligned}$$

The first set of identities is fairly obvious from the definition, and the second follows because, since $y \in C^{p'}(A, A^{\otimes k'})$ for $p', k' \geq 2$, x (on the left) and z (on the right) are always applied to disjoint sets of consecutive components. This is all we need to prove the associativity (7.2.2). In fact, $\dashv$ and $\vdash$ are mutually associative (7.2.7).

To prove the DG property (7.2.3), we show the following two identities:

$$(\mathsf{b}f \vdash g) + (-1)^{p-1}(f \vdash \mathsf{b}g) = \mathsf{b}(f \vdash g) + (-1)^{p+1} f \cup g, \tag{7.2.5}$$

$$(\mathsf{b}f \dashv g) + (-1)^{p-1}(f \dashv \mathsf{b}g) = \mathsf{b}(f \dashv g) + (-1)^{p-1} f \cup g. \tag{7.2.6}$$

Actually, in (7.2.5), we only need $m \geq 2$, and in (7.2.6), we only need $k \geq 2$. These identities also prove that the cup product is zero on cohomology. This implies part (ii), since it is clear that the cup product is associative.

We show only (7.2.5), as the other identity is the same verification. Write $m : A \otimes A \to A$ for the multiplication map. We compute

$$\begin{aligned}((\mathsf{b}f \vdash g) + (-1)^{p-1}(f \vdash \mathsf{b}g))(a_1 \otimes \cdots \otimes a_{p+q}) &= a_1(f \vdash g)(a_2 \otimes \cdots \otimes a_{p+q})\\ &+ (-1)^{p+q+1}(f \vdash g)(a_1 \otimes \cdots \otimes a_{p+q-1})a_{p+q} + (-1)^{m+1} f \cup g\\ &+ \sum_{i=1}^{p+q-1} (-1)^i (f \vdash g) \circ m^{i,i+1}.\end{aligned}$$

Finally, part (iii) of the proposition follows from Proposition 7.2.8(ii) below. □

For each $n \geq 1$, we may introduce an operad, $As^{(n)}$, generated by n binary operations, $\star_i$, $i = 1, \ldots, n$, subject to the following relations of pairwise mutual associativity (considered also by Loday):

$$a \star_i (b \star_j c) = (a \star_i b) \star_j c, \quad \forall i, j = 1, \ldots, n. \tag{7.2.7}$$

Proposition 7.2.8. (i) *Each of the operads $As^{(2)}$ and $As^{(3)}$ is Koszul and self-dual, (see* [15] *for a definition).*

(ii) *The operations $(\vee, \cup)$ and $(\vdash, \dashv, \cup)$ make $C(A, A_t^+)_{\geq 2}$ an $As^{(2)}$- and an $As^{(3)}$-algebra, respectively.*

Sketch of Proof. It is easy to see, as in the associative case, that the quadratic dual of $As^{(2)}$ is itself and similarly for $As^{(3)}$. The fact that these two operads are Koszul is a straightforward consequence of Koszulity criteria due to Dotsenko and Khoroshkin [10] and Hoffbeck [16]. This can also be seen directly using the same argument as in the associative case [15,24]. Specifically, we show that the operadic homology of the free $As^{(2)}$ or $As^{(3)}$-algebra vanishes in degrees ≥ 2. To this end, we split up the operadic homology complexes for $As^{(2)}$ and $As^{(3)}$ into direct sums of pieces corresponding to particular sequences of operations, e.g., $(*, \star)$ would consist of terms that multiply out to a sum of terms of the form $a * b \star c$. Each such has the vanishing homology property by the same proof as in the usual case of Hochschild homology of a free associative algebra; see [22, Section 3.1].

Next, it is straightforward to verify the following identities:

$$\begin{aligned} x \vdash (y \cup z) &= (x \vdash y) \cup z, & x \dashv (y \cup z) &= (x \dashv y) \cup z,\\ (x \cup y) \vdash z &= x \cup (y \vdash z), & (x \cup y) \dashv z &= x \cup (y \dashv z).\end{aligned}$$

This yields part (ii), and also implies the identities in (7.2.4). □

7.3. Infinite order deformations

In the classical theory, an infinite order formal deformation of an algebra A with multiplication map $m : A \times A \to A$ is a formally associative star-product

$$\begin{aligned} &a \star a' = m(a, a') + t\beta^{(1)}(a, a') + t^2\beta^{(2)}(a, a') + \cdots \in A[\![t]\!],\\ &\qquad \beta^{(k)} \in C^2(A, A),\ k \geq 1. \end{aligned} \tag{7.3.1}$$

Given such a star-product, we extend the formal series $m + t\beta^{(1)} + t^2\beta^{(2)} + \cdots \in \sum t^k C^2(A, A) = C^2(A, A[\![t]\!])$, by $\Bbbk[\![t]\!]$-bilinearity, to obtain a continuous cochain $\beta := \beta^{(1)} + t\beta^{(2)} + \cdots \in C^2_{\Bbbk[\![t]\!]}(A[\![t]\!], A[\![t]\!])$.

With an appropriate equivalence relation on the set of associative star-products, one has the following well-known result.

Proposition 7.3.2. *Equivalence classes of associative star products (7.3.1) are in one-to-one correspondence with gauge equivalence classes in the set of solutions of the following Maurer–Cartan equation*

$$\{\beta \in C^2_{\Bbbk[[t]]}(A[[t]], A[[t]]) \mid \mathsf{b}_{A[[t]]}(\beta) + (1/2)t\{\beta, \beta\}_{A[[t]]} = 0\}. \quad \square$$

To consider free product deformations of an algebra A, let $\widehat{A_t} := \prod_{k\geq 0} A_t^{2k} \cong \prod_{m\geq 1} A^{\otimes m}$ be the completion of the free product algebra A_t in the t-adic topology, and write $\widehat{A}_t^+ := \prod_{k\geq 1} A_t^{2k}$ for the corresponding augmentation ideal.

An infinite order free product deformation of A is, by definition, a formally associative star-product of the form

$$a \star_\beta a' = aa' + \beta^{(1)}(a, a') + \beta^{(2)}(a, a') + \cdots, \quad \beta^{(k)} \in C^2(A, A_t^{2k}). \tag{7.3.3}$$

In more detail, given an arbitrary sequence $\beta^{(k)} \in C^2(A, A_t^{2k})$, $k = 1, 2, \ldots,$ of two-cochains, we first extend each map $\beta^{(k)}$ to a $\Bbbk[t]$-bilinear map $\beta^{(k)} : A_t \times A_t \to A_t$ given by the formula

$$\beta^{(k)} : (a_1 t a_2 t \ldots t a_m) \times (b_1 t \ldots t b_n) \longmapsto a_1 t \ldots t a_{m-1} t \beta(a_m \otimes b_1) t b_2 t \ldots t b_n.$$

For any $u, u' \in A_t$, the corresponding formal series $uu' + \beta_t^{(1)}(u, u') + \beta_t^{(2)}(u, u') + \cdots$ clearly converges in $\widehat{A}_t$. In this way, we obtain a well defined and continuous $\Bbbk[t]$-bilinear map $A_t \times A_t \to \widehat{A}_t$, that can be uniquely extended, by continuity, to a map $\beta : \widehat{A}_t \times \widehat{A}_t \to \widehat{A}_t$. We are interested in those star-products (7.3.3) which give rise to an *associative* product β, on $\widehat{A}_t$.

We define a natural equivalence relation on such star-products as follows. First, for any map $\phi : A \to A_t$ we define a map $\phi_t : A_t \to A_t$ by a Leibniz-type formula:

$$\phi_t : a_1\, t\, a_2\, t \ldots t\, a_n \longmapsto \sum_{k=1}^{n} a_1\, t \ldots a_{k-1}\, t\, \phi(a_k)\, t\, a_{k+1}\, t \ldots t\, a_n, \tag{7.3.4}$$

for any $a_1, \ldots, a_n \in A$. Thus, given a sequence of one-cochains $f^{(k)} \in C^1(A, A_t^{2k})$, $k = 1, 2, \ldots,$ we have a continuous map $f = \mathrm{Id} + f_t^{(1)} + f_t^{(2)} + \cdots : \widehat{A}_t \to \widehat{A}_t$. Furthermore, given any star-product $\star_\beta$, we define a new star product by the formula $a \star_\gamma a' := f^{-1}(f(a) \star_\beta f(a'))$. We say that the star-products $\star_\gamma$ and $\star_\beta$ are *equivalent*.

Given a star-product (7.3.3), we form $\beta := \beta^{(1)} + \beta^{(2)} + \cdots \in C^2(A, \widehat{A}_t^+)$, an associated two-cochain. We further define gauge equivalence of chains to be infinitesimally generated by the following $C^1(A, \widehat{A}_t^+)$-action: $\phi :\ \beta \mapsto \phi \cdot \beta$, where

$$\begin{aligned}\phi \cdot \beta(a_1 \otimes a_2) = {} & \phi(a_1)a_2 + a_1\phi(a_2) - \phi(a_1a_2) + \beta'(\phi(a_1) \otimes a_2) \\ & + \beta'(a_1 \otimes \phi(a_2)) - \phi_t \circ \beta'(a_1 \otimes a_2).\end{aligned}$$

Here, ϕ_t is defined according to formula (7.3.4), and we put

$$\beta'(a_1 t \ldots t a_m \otimes c_1 t \ldots t c_n) = a_1 t \ldots t a_{m-1} t (\beta(a_m \otimes c_1)) t c_2 t \ldots t c_n.$$

The following result provides a cohomological description of free product deformations, similar to the one given in Proposition 7.3.2 (below, b stands for the Hochschild differential).

Theorem 7.3.5. (i) *Linear maps $\beta^{(i)}$ in (7.3.3) define an associative product on $\widehat{A}_t$ if and only if the Maurer–Cartan equation,*

$$\mathsf{b}(\beta) + \frac{1}{2}\,\beta \vee \beta = 0, \quad \textit{holds for } \beta := \beta^{(1)} + \beta^{(2)} + \cdots \in C^2(A, \widehat{A}_t^+). \tag{7.3.6}$$

(ii) *Star products are equivalent if and only if the solutions of the equation (7.3.6) are gauge-equivalent.*

Proof. In this proof (only) we temporarily change our notation and given $\beta^{(m)} \in C^2(A, A^{\otimes m+1})$, write $\beta_m^{12} \in C^3(A, A^{\otimes m+2})$ for the map $a \otimes b \otimes c \mapsto \beta^{(m)}(a,b) \otimes c$, etc.

It is easy to see that it suffices to check associativity on $A^{\otimes 3}$, namely that $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in A$. This is equivalent to the following (similar to (7.1.1)), for all $p \geq 2$:

$$\beta^{(p)}(a \otimes bc) + a\beta^{(p)}(b \otimes c) + \sum_{m+n=p} \beta_m^{12} \circ \beta_n^{23}(a \otimes b \otimes c)$$
$$= \beta^{(p)}(ab \otimes c) + \beta^{(p)}(a \otimes b)c + \sum_{m+n=p} \beta_n^{m,m+1} \circ \beta_m^{12}(a \otimes b \otimes c) \tag{7.3.7}$$

which is the Maurer–Cartan equation (7.3.6). Part (ii) is then not difficult to verify. □

Below, we summarize a few basic properties of free product deformations which are entirely analogous to the well-known properties of ordinary one-parameter formal deformations:

(1) First order free product deformations are classified by $H^2(A, A \otimes A)$.
(2) The obstruction to extending a first-order deformation to second order lives in $H^3(A, A^{\otimes 3})$.
(3) Let $\star_n$ be an associative product on $A/(A_t^+)^{n+1}$ of the form

$$a \star_n b = ab + \sum_{m=1}^{n} \beta^{(m)}(a \otimes b), \quad \beta^{(m)} \in C^2(A, A^{m+1}) = C^2(A, A_t^{2m}). \tag{7.3.8}$$

There is an obstruction in $H^3(A, A^{\otimes n+1})$ to the existence of $\beta^{(n+1)} \in C^2(A, A^{\otimes(n+2)})$, such that the formula $a \star_{n+1} b := a \star_n b + \beta^{(n+1)}(a \otimes b)$ gives an associative product on $A/(t)^{n+2}$.
Explicitly, the condition on $\beta^{(n+1)}$ reads

$$\mathsf{b}\beta^{(n+1)} = \sum_{i+j-1=n+1} \beta^{(i)} \vee \beta^{(j)}. \tag{7.3.9}$$

(4) If the obstruction in (3) vanishes then, the space of possible $\beta^{(n+1)}$ (up to equivalence of the resulting star product, $\star + \beta^{(n+1)}$ modulo $(t)^{n+2}$), is $H^2(A, A^{\otimes(n+2)})$.

Proof. In degrees $n = 1, 2$, the Maurer–Cartan equation (7.3.9) says that

$$\mathsf{b}\beta^{(1)} = 0, \quad \text{and} \quad \mathsf{b}\beta^{(2)} = \beta^{(1)} \vee \beta^{(1)}.$$

Using (7.2.3), we find $\mathsf{b}(\beta^{(1)} \vee \beta^{(1)}) = (\mathsf{b}\beta^{(1)}) \vee \beta^{(1)} + \beta^{(1)} \vee (\mathsf{b}\beta^{(1)}) = 0 + 0 = 0$. This yields (1)–(2).

In general, if $\beta^{(1)}, \ldots, \beta^{(n)}$ satisfy the Maurer–Cartan conditions up to $O(t^{n+1})$ (i.e., $\mathsf{b}\beta^{(m)} = \sum_{i+j-1=m} \beta^{(i)} \vee \beta^{(j)}$ for $m \leq n$), then we consider b of the RHS of (7.3.9):

$$\begin{aligned}\sum_{i+j-1=n+1} \mathsf{b}(\beta^{(i)} \vee \beta^{(j)}) &= \sum_{i+j-1=n+1} \big[(\mathsf{b}\beta^{(i)}) \vee \beta^{(j)} - \beta^{(i)} \vee (\mathsf{b}\beta^{(j)})\big] \\ &= \sum_{i+j+k-2=n+1} \big[(\beta^{(i)} \vee \beta^{(j)}) \vee \beta^{(k)} - \beta^{(i)} \vee (\beta^{(j)} \vee \beta^{(k)})\big] \\ &= 0,\end{aligned}$$

where we used both (7.2.3) and (7.2.2). Thus, the RHS is indeed a Hochschild three-cocycle. Thus, if this represents the zero class of $H^3(A, A^{\otimes n+1})$ (i.e., it is a Hochschild three-coboundary), then the space of choices of $\beta^{(n+1)}$ is the space of Hochschild two-cocycles. Furthermore, we have the freedom of conjugating by automorphisms $\phi : A \to A$ of the form $\phi = \mathrm{Id} + \phi'$ as follows:

$$\phi^{-1}(\phi(a) \star \phi(b)) \equiv a \star b + \phi'(a)b + a\phi'(b) - \phi'(ab) \pmod{(t)^{n+2}}.$$

We conclude that the space of $\beta^{(n+1)}$'s, taken up to equivalence of the obtained star product on $A/(t)^{n+2}$, is $H^2(A, A^{\otimes n+2})$. $\square$

7.4. Deformations of NCCI algebras

It will be convenient below to work in a slightly more general setting of deformations that are not necessarily written in the form of a star product.

To define such deformations, fix an augmented (not necessarily commutative) associative algebra R, and let $R^+ \subset R$ be the augmentation ideal. Given an algebra $\widehat{A}$ and an algebra embedding $R \hookrightarrow \widehat{A}$, write $(R^+) \subset \widehat{A}$ for the two-sided ideal in $\widehat{A}$ generated by R^+. We will view R and $\widehat{A}$ as filtered algebras equipped with the R^+-adic and (R^+)-adic descending filtrations, respectively, and let $\operatorname{gr} R$ and $\operatorname{gr} \widehat{A}$ denote the associated graded algebras. Thus, there is a canonical algebra map $\operatorname{gr} R \to \operatorname{gr} \widehat{A}$.

Given an algebra A and an algebra isomorphism $\phi : \widehat{A}/(R^+) \xrightarrow{\sim} A$, we say that $\widehat{A}$ is a deformation of A over R. We may view A as a graded algebra concentrated in degree zero.

Definition 7.4.1. The deformation $\widehat{A}$ of A over R is said to be a **flat** *free product formal deformation* if the algebra $\widehat{A}$ is complete in the (R^+)-adic topology, and the maps $\phi^{-1} : A \to \widehat{A}/(R^+)$ and $\operatorname{gr} R \to \operatorname{gr} \widehat{A}$ induce a graded algebra isomorphism

$$A * \operatorname{gr} R \xrightarrow{\sim} \operatorname{gr} \widehat{A}. \tag{7.4.2}$$

Now, fix V, a $\mathbb{Z}_+$-graded finite dimensional vector space, and let $F := TV$. Let $L \subset TV$ be a finite-dimensional vector subspace. Assuming certain favorable conditions, we can describe the equivalence classes of *all* infinite order free product deformations of an algebra of the form $A = F/(L)$ quite explicitly.

To explain this, write $\widehat{F_t}$ for the standard completion of the algebra $F_t = (TV) * \Bbbk[t]$ and $\widehat{F}_t^+ \subset \widehat{F_t}$ for the augmentation ideal. Given any linear map $\phi : L \to \widehat{F}_t^+$, we introduce a $\Bbbk[\![t]\!]$-algebra

$$A_\phi := \widehat{F_t}/(x - \phi(x))_{x \in L}. \tag{7.4.3}$$

It is clear that the projection $\widehat{F_t} \twoheadrightarrow \widehat{F_t}/\widehat{F}_t^+ = F$ induces an algebra isomorphism $A_\phi/(t) \xrightarrow{\sim} A$.

Thus, we may view the algebra A_ϕ as a one-parameter infinite order free product deformation of A. This deformation is *not* necessarily flat, in general, i.e., the corresponding map (7.4.2) for $\widehat{A} = A_\phi$ may fail to be an isomorphism.

To formulate a sufficient condition for flatness, we recall the notion of a *noncommutative complete intersection* (NCCI); see [11]. An algebra of the form $A = TV/(L)$ is said to be an NCCI if the two-sided ideal $J := (L)$ has the property that J/J^2 is *projective* as an $A \otimes A^{op}$-module. Moreover, such a linear subspace $L \subset TV$ is called *minimal* if $L \cap J^2 = 0$, or equivalently, L has minimal dimension (assuming J is finitely-generated).

An NCCI algebra A is known to have Hochschild dimension ≤ 2, so that $H^3(A, A \otimes A) = 0$ [11]. Thus, free product deformations of A are unobstructed. Moreover, we have the following.

Proposition 7.4.4. *Let $A = TV/(L)$ be an NCCI, with $L \subset TV$ minimal, and let $\phi : L \to \widehat{F}_t^+$ be a linear map. Then, we have the following.*

(i) *The deformation A_ϕ defined in (7.4.3) is flat.*
(ii) *Any flat one-parameter infinite order free product deformation of A is equivalent to a deformation of the form A_ϕ for an appropriate map ϕ.*
(iii) *Two deformations A_ϕ and A_ψ associated, to linear maps $\phi, \psi \in \mathrm{Hom}_{\Bbbk}(L, \widehat{F}_t^+)$, respectively, are equivalent if and only if there exists a linear map $f : V \to A_t^+$, such that*

$$\pi \circ (\phi - \psi) = \Theta_f|_L. \tag{7.4.5}$$

In the last formula, we used the notation $\pi : \widehat{F}_t \twoheadrightarrow A_t$ for the canonical quotient map and, given a linear map $f : V \to A_t^+$, write

$$\Theta_f(v_1 v_2 \cdots v_n) := \sum_{i=1}^{n} \pi(v_1 v_2 \cdots v_{i-1}) f(v_i) \pi(v_{i+1} \cdots v_n), \quad \forall v_1, \ldots, v_n \in V.$$

Remark 7.4.6. Our proof below shows that, in the case where the image of the map ϕ is contained in the subalgebra $F_t \subset \widehat{F}_t$, we may replace the algebra A_ϕ, in (7.4.3), by algebra $F_t/(x - \phi(x))_{x \in L}$, its non-completed counterpart. In this way, we obtain a *genuine*, rather than merely a 'formal', flat free product deformation of A.

Proof of Proposition 7.4.4. (i) By the inductive argument of [27, Proposition 4.2.1], one may show that the NCCI property is equivalent to the statement that the canonical surjection $\mathrm{gr}_{(L)} F \twoheadrightarrow A * TL$ is an isomorphism. Thus, we conclude that the surjection $A_t \twoheadrightarrow \mathrm{gr}_{(t)} A_\phi$ is an isomorphism. It follows that A_ϕ is a free product deformation of A over $\Bbbk[\![t]\!]$.

(ii) For an NCCI algebra, there is a standard Anick's free resolution of A as an A-bimodule [1]:

$$0 \to A \otimes L \otimes A \to A \otimes V \otimes A \to A \otimes A \twoheadrightarrow A \to 0, \tag{7.4.7}$$

where the first map is the restriction to L of the map

$$a \otimes (v_1 v_2 \cdots v_n) \otimes b \longmapsto \sum_{i=1}^{n} a v_1 \cdot \cdots \cdot v_{i-1} \otimes v_i \otimes v_{i+1} \cdots v_n b, \quad v_i \in V,$$

and the second map has the form $a \otimes v \otimes b \mapsto av \otimes b - a \otimes vb$.

One may use Anick's resolution to compute Hochschild cohomology. We see in particular that the group $H^2(A, A \otimes A)$ is a quotient of $\mathrm{Hom}(L, A \otimes A)$. Now let A_ϕ be the deformation associated with $\phi \in \mathrm{Hom}(L, \widehat{F}_t^+)$. Then, it is easy to check that the element of $H^2(A, A \otimes A)$ corresponding to the induced first-order deformation $A_\phi/(t)^2$ is represented by the composite

$$L \to \widehat{F}_t^+ \twoheadrightarrow \widehat{F}_t^+/(\widehat{F}_t^+)^2 = F \otimes F \twoheadrightarrow A \otimes A,$$

of the map ϕ followed by two natural projections (cf. [5, Lemma 10.2.1]).

Furthermore, by the inductive description of all possible star-products in Section 7.3, the deformations A_ϕ must exhaust all possible deformations (note that all possible classes of $H^2(A, A^{\otimes m})$ at every step of the way are attained, which is as it must be, since $H^3(A, A \otimes A) = 0$).

(iii) At the first-order stage, we see from Anick's resolution (7.4.7) that two elements of the space $\mathrm{Hom}_k(L, A\otimes A)$ yield the same cohomology class in $H^2(A, A\otimes A)$ if and only if they differ by Θ_f from condition (7.4.5) modulo $(t)^2$, for some f. The desired result now follows from the inductive construction of all free-product deformations given in Section 7.3. □

Remark 7.4.8. (i) In general, given an arbitrary algebra A such that $H^3(A, A \otimes A) = 0$, one can show that there exist 'versal' free product deformations of A. The base of such a versal deformation is a completed tensor algebra of the vector space H^*, where $H := H^2(A, A \otimes A)$.

(ii) Proposition 7.4.4 may be generalized easily to the case where the ground field $\Bbbk$ is replaced by a ground ring R, a finite dimensional semisimple $\Bbbk$-algebra, as in [11]. Such a generalized version of Proposition 7.4.4 applies to preprojective algebras of non-Dynkin quivers, in particular. Thus the proposition may be viewed as a generalization of [5, Theorem 10.1.3].

(iii) Let $\pi_1(X)$ be the fundamental group of a compact oriented Riemann surface X of genus ≥ 1. The group algebra $\Bbbk[\pi_1(X)]$ may be thought of as a multiplicative analogue of the preprojective algebra of a non-Dynkin quiver (being non-Dynkin corresponds to the condition that the Euler characteristic of X be nonpositive). Accordingly, there is a similar construction of free product deformations of the group algebra as follows.

Let g be the genus of X, and write $a_1, \ldots, a_g, b_1, \ldots, b_g$ for the standard loops around the handles, which generate $\pi_1(X)$. The group $\pi_1(X)$ is the quotient of Γ, the free group generated by the letters a_i, b_i, by the normal subgroup generated by the following element:

$$\boldsymbol{\gamma} := (a_1 b_1 a_1^{-1} b_1^{-1}) \cdot \cdots \cdot (a_g b_g a_g^{-1} b_g^{-1}) \in \Gamma.$$

To construct free product deformations of the group algebra $\Bbbk[\pi_1(X)]$, we put $F := \Bbbk[\Gamma]$. The algebra F is a 'multiplicative analogue' of a free algebra. To any element $u \in 1 + F_t^+$, we associate an algebra $A_u := F_t/(\boldsymbol{\gamma} - u)$.

There is a 'multiplicative analogue' of Proposition 7.4.4, saying that the algebra A_u gives a flat free-product deformation of the group algebra $\Bbbk[\pi_1(X)]$, and moreover that these are all such deformations up to equivalence. One may prove this result by using the fact that the prounipotent completion of $\Bbbk[\pi_1(X)]$ is isomorphic to a completion of an algebra of the form $\Bbbk\langle x_1, \ldots, x_g, y_1, \ldots, y_g\rangle/([x_1, y_1] + \cdots + [x_g, y_g])$.

This example may be generalized to the situation of orbifold surfaces of nonpositive Euler characteristic (the latter also yields NCCI algebras).

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Appendix. On the morphism from periodic cyclic homology to equivariant cohomology, by Boris Tsygan

A.1. Motivation

Since Hochschild chains are a noncommutative analogue of forms, it is natural to ask what algebraic structure that is carried by forms can be defined for chains of an associative algebra. The structure we will discuss involves not only forms but also vector fields (their noncommutative analogue is the Lie algebra of derivations) and functions (their analogue is the algebra itself). It is well known that, for a vector field X, two operators on forms are defined: the contraction ι_X and the Lie derivative L_X. We will write all the formulas for a graded commutative ring, since this will give us an intuition about the signs. For a graded derivation X of such an algebra $\mathcal{A}$, let ι_X be the derivation of $\Omega^*_{\mathcal{A}/k}$ of degree $\deg(X)-1$ sending a to zero and da to $X(a)$ for $a\in\mathcal{A}$. Let L_X be the derivation of degree $\deg(X)$ sending a to $X(a)$ and da to $(-1)^{\deg(X)}dX(a)$. Put $|X|=\deg(X)+1$. Together with the de Rham differential d, the above operations are subject to Cartan relations

$$\begin{aligned} &[L_X, L_Y]=L_{[X,Y]}; \qquad [L_X,\iota_Y]=\iota_{[X,Y]};\\ &[\iota_X,\iota_Y]=0; \qquad [d,L_X]=0; \qquad [d,\iota_X]=(-1)^{|X|-1}L_X. \end{aligned} \tag{A.1.1}$$

All commutators are taken in the graded sense. As a corollary, we get the classical formula for the de Rham differential: for an n-form ω, one has

$$\begin{aligned} \iota_{X_1}\dots\iota_{X_{n+1}}d\omega = \Biggl(&\sum_{j=1}^{n+1}(-1)^{K_j}L_{X_j}\iota_{X_1}\dots\widehat{\iota_{X_j}}\dots\iota_{X_{n+1}}\\ &+\sum_{i<j}(-1)^{M_{ij}}\iota_{[X_i,X_j]}\dots\widehat{\iota_{X_j}}\dots\widehat{\iota_{X_j}}\dots\Biggr)\omega, \end{aligned} \tag{A.1.2}$$

where

$$K_j=\sum_a|X_a|+|X_j|\left(\sum_{b<j}|X_b|+1\right) \tag{A.1.3}$$

and

$$M_{ij}=\sum_a|X_a|+|X_j|\sum_{b<j,b\neq i}|X_b|+|X_i|\left(\sum_{c<i}|X_c|+1\right)+1. \tag{A.1.4}$$

This follows from (A.1.1).

Now, for an element f of $\mathcal{A}$, put $|f|=\deg(f)$, $\iota_f\omega=f\omega$, and $L_f\omega=(-1)^{|f|-1}df\wedge\omega$. In addition to the above equations, the following are true:

$$\begin{aligned} &[L_X,L_f]=L_{X(f)}; \qquad [L_X,\iota_f]=\iota_{X(f)}; \qquad [\iota_f,\iota_g]=0;\\ &[d,\iota_f]=(-1)^{|f|-1}L_f. \end{aligned} \tag{A.1.5}$$

We can combine the two sets of equations (A.1.1) and (A.1.5), into one set that looks same as (A.1.1), but X and Y are now elements of the cross product of the graded Lie algebra $\mathrm{Der}(\mathcal{A})$ by the Abelian graded Lie algebra $\mathcal{A}[1]$.

The algebraic structure described above, and a much stronger additional structure, generalizes to the noncommutative setting. These generalizations become progressively more and more difficult and inexplicit. However, the very first level of noncommutative calculus, namely formula (A.1.2) (and its slight generalization to the case when X_i are vector fields or functions) turns out to be as easy as one can expect.

A.2. The map χ

Let $\mathcal{A}$ be a graded associative algebra. Let $\mathscr{A}[1] \oplus \mathrm{Der}(\mathscr{A})$ be the differential graded Lie algebra defined as follows: $\mathscr{A}[1]$ is Abelian, $\mathrm{Der}(\mathscr{A})$ is a subalgebra whose adjoint action on $\mathscr{A}[1]$ is the natural one, the differential δ sends $a \in \mathscr{A}[1]$ to $(-1)^{|a|}\mathrm{ad}(a)$ and is zero on $\mathrm{Der}(\mathscr{A})$. Let $\mathcal{L}$ be a DG Lie subalgebra of $\mathscr{A}[1] \oplus \mathrm{Der}(\mathscr{A})$. Let K be a graded space on which $\mathcal{L}$ acts so that elements of $\mathscr{A}[1]$ act by zero. Let $\tau : \mathscr{A} \to K$ be an $\mathcal{L}$-equivariant trace. We extend it by zero to the entire Hochschild complex $C_{-\bullet}(\mathscr{A})$.

Define for $a \in \mathscr{A}[1]$

$$\iota_a(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^{|a||a_0|} a_0 a \otimes a_1 \otimes \cdots \otimes a_n \tag{A.2.1}$$

and for $D \in \mathrm{Der}(\mathscr{A})$

$$\iota_D(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^{|D||a_0|} a_0 D(a_1) \otimes a_2 \otimes \cdots \otimes a_n. \tag{A.2.2}$$

Define, for $X_1, \ldots, X_n \in \mathcal{L}$, and for a Hochschild chain c,

$$\chi(X_1, \ldots, X_n)(c) = \sum_{\sigma \in S_n} \pm \tau(\iota_{X_{\sigma(1)}} \ldots \iota_{X_{\sigma(n)}} c); \tag{A.2.3}$$

the sign is computed as follows: a permutation of X_i and X_j introduces a sign $(-1)^{|X_i||X_j|}$. (Note that $|X|$ is the degree of the operator ι_X.) It turns out that χ defines a cocycle of the complex

$$C^\bullet(\mathcal{L}, \mathrm{Hom}(C_{-\bullet}(A), K)[[u]])$$

with the differential $b + uB + \delta + u\partial_{\mathrm{Lie}}$; the action of L on $\mathrm{Hom}(C_{-\bullet}, K)$ is induced by the action on K. In other words,

Proposition A.2.4.

$$\begin{aligned}\chi(X_1, \ldots, X_n)((b + uB)(c)) = \frac{1}{n!}\Big(&\sum (-1)^{L_i} \chi(X_1, \ldots, \delta X_i, \ldots, X_n) \\ &+ u \sum (-1)^{M_{ij}} \chi([X_i, X_j], \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_n) \\ &+ u \sum (-1)^{K_j} X_i \chi(X_1 \ldots, \widehat{X_i}, \ldots, X_n)\Big)(c)\end{aligned}$$

(cf. [25,26]). Here $L_i = \sum_{a \geq i} |X_a|$; K_j is as in (A.1.3) and M_{ij} as in (A.1.4).

Proof. We observe directly that $[b, \iota_X] = \iota_{\delta X}$. This implies immediately that the formula in the statement of the proposition is true modulo u, since τ is zero on the image of b. We denote $\chi(X_1, \ldots, X_n)$ by $\chi(X_1 \ldots X_n)$. Let $c = a_0 \otimes \cdots \otimes a_n$. Let D be an odd derivation and x an even element of $\mathcal{A}$. We are going to prove Proposition A.2.4 first in the case when all X_i are equal to D

or x, so $|X_i| = 0$ and the signs in the statement of the proposition are $K_i = L_j = +1$; $M_{ij} = -1$. By definition,

$$\chi(D^n x^N) = \frac{n!N!}{(N+n)!} \sum_{N_0+N_1+\cdots+N_n=N} \tau(a_0 x^{N_0} D(a_1) x^{N_1} \dots D(a_n) x^{N_n})$$

therefore

$$\chi(D^{n+1} x^N)(Bc) = \frac{(n+1)!N!}{(N+n+1)!} \sum_{\sum_0^{n+1} N_k=N} \sum_{i=0}^{n} \pm\tau(x^{N_0} D(a_{i+1}) x^{N_1} \times D(a_{i+2}) x^{N_2} \dots D(a_i) x^{N_{n+1}});$$

the sign in the above is $(-1)^{\sum_{k>i}(|a_k|+1)\sum_{k\le i}(|a_k|+1)}$. Using the fact that τ is a trace, re-indexing the N_k, and re-ordering the $D(a_i)$ (which cancels out the previous sign), we see that the above is equal to

$$\begin{aligned} &\frac{(n+1)!N!}{(N+n+1)!} \sum_{\sum_0^n N_k=N} \sum_{i=0}^{n} \sum_{N_i'+N_i''=N_i} \tau(D(a_0) x^{N_0} \dots D(a_n) x^{N_n}) \\ &= (N+n+1)\frac{(n+1)!N!}{(N+n+1)!} \sum_{\sum\limits_0^n N_k=N} \tau(D(a_0) x^{N_0} \dots D(a_n) x^{N_n}); \end{aligned}$$

by the Leibniz identity for D, we rewrite this as

$$\begin{aligned} &\frac{(n+1)!N!}{(N+n)!}\Bigg(\sum_{\sum_0^n N_k=N} D\tau(a_0 x^{N_0} \dots D(a_n) x^{N_n}) \\ &\quad -\frac{1}{2} \sum_{\sum\limits_0^n N_k=N} \sum_{i=1}^{n} \pm\tau(a_0 x^{N_0} \dots [D,D](a_i) x^{N_i} \dots D(a_n) x^{N_n}) \\ &\quad - \sum_{\sum\limits_0^n N_k=N} \sum_{i=0}^{n} \sum_{N_i'+N_i''=N_i} \pm\tau(a_0 x^{N_0} \dots x^{N_i'} D(x) x^{N_i''} \dots D(a_n) x^{N_n})\Bigg); \end{aligned}$$

the signs appearing in the above terms from the Leibniz identity, with an overall minus sign, are precisely the signs from the definition of $\iota_{[D,D]}$ and $\iota_{D(x)}$. We see that the above is equal to

$$(n+1)D\chi(D^n x^N) - \frac{n(n+1)}{2}\chi([D,D]D^{n-1}x^N) - (n+1)N\chi(D^n D(x) x^{N-1}).$$

Therefore

$$\begin{aligned} &\chi(D^{n+1}x^N)(Bc) = (n+1)D\chi(D^n x^N)(c) \\ &\quad - \frac{n(n+1)}{2}\chi([D,D]D^{n-1}x^N)(c) - (n+1)N\chi(D^n D(x) x^{N-1})(c). \end{aligned}$$

This is exactly the equality of the terms of the formula in the statement of the proposition that contain u, in the special cases $X_1 = \cdots = X_{n+1} = D$ and $X_{n+2} = \cdots = X_{N+n+1} = x$ (and with n replaced by $N+n$). To prove the general case, tensor our algebra by $k[t_1, \dots, t_n]$ where

$|t_i| = -|X_i|$, put $X = t_1 X_1 + \cdots t_n X_n$, apply the special case to $\chi(X, \ldots, X)$, and look at the coefficient at $t_1 \ldots t_n$. $\square$

A.3. Construction of the morphism

Now let $\mathscr{A} = \Omega(\mathrm{Rep}(A)) \otimes \mathrm{End}(V)$, $\mathrm{Rep}(A)$ being the scheme of representations of an algebra A in a given finite dimensional space V. Consider the subalgebra $\mathrm{End}(V)$ consisting of constant functions. Let $K = \Omega(\mathrm{Rep}(A))$ and $\tau : \mathscr{A} \to K$ be the matrix trace. Let $\mathfrak{g} = \mathrm{End}(V)$ viewed as a Lie algebra.

Proposition A.3.1. *The map*

$$c, x \to \tau(\exp(\iota_{d+x}))(c),$$

$x \in \mathfrak{g}$ *and* $c \in C_{-\bullet}(\mathscr{A})$, *composed with the morphism induced by* $\mathrm{ev} : A \to \mathscr{A}$, *defines a morphism of complexes*

$$\mathrm{HKR}(x) : C_{-\bullet}(A)((u)), b + uB \to \Omega(\mathrm{Rep}(A))[[\mathfrak{g}]]^G((u)), ud + \iota_x.$$

Here ι_x refers to the contraction of a form by a vector field corresponding to x. (Note that the value of this map at $x = 0$ is the HKR morphism).

Proof. By Proposition A.2.4, taking into account that $[d, d] = 0$ and $d(x) = 0$,

$$\mathrm{HKR}(x)((b + uB)(c)) = ud\mathrm{HKR}(x)(c) + \sum_{K=0}^{\infty} \frac{1}{K!} \chi(\delta(x)(d + x)^K)(c).$$

If we put $c = \alpha_0 \otimes \cdots \otimes \alpha_n,\ \alpha_i \in \mathcal{A}$, then the second summand is equal to

$$\sum_{N=0}^{\infty} \sum_{\sum_0^n N_k = N} \sum_{i=1}^{N} \mathrm{tr}(\alpha_0 x^{N_0} d\alpha_1 x^{N_1} \ldots [x, \alpha_i] x^{N_i} \ldots d\alpha_n x^{N_n}).$$

Now observe that for $x \in \mathfrak{g}$, if $L_x \mathrm{ev}(a) = [x, \mathrm{ev}(a)]$ where L_x denotes the Lie derivative by the vector field on $\mathrm{Rep}(A)$ corresponding to x. Therefore, for $\alpha_i = \mathrm{ev}(a_i)$, the above formula turns into

$$\sum_{N=0}^{\infty} \sum_{\sum_0^n N_k = N} \sum_{i=1}^{N} \mathrm{tr}(\alpha_0 x^{N_0} d\alpha_1 x^{N_1} \ldots L_x \alpha_i x^{N_i} \ldots d\alpha_n x^{N_n})$$
$$= \iota_x \sum_{N=0}^{\infty} \sum_{\sum_0^n N_k = N} \mathrm{tr}(\alpha_0 x^{N_0} d\alpha_1 x^{N_1} \ldots d\alpha_n x^{N_n}) = \iota_x \mathrm{HKR}(x)(c). \quad \square$$

Remark A.3.2. As for the Hochschild homology, the fact that the map

$$\mathrm{HKR} : C_{-\bullet}(A) \to \Omega(\mathrm{Rep}(A))$$

(the value of $\mathrm{HKR}(x)$ at $x = 0$) sends Hochschild cycles to basic forms follows from the formula (used in the proof above) $\iota_x \mathrm{HKR}(x)(c) = \mathrm{HKR}(x\iota_\Delta(c))$, and the fact that ι_Δ kills the image of b.

A.4. More on noncommutative calculus

Let us finish by saying a few words about noncommutative analogues of formulas (A.1.1) and other algebraic properties of forms on manifolds. Note first that operators ι_X and L_X can be defined not just for vector fields and functions but for multivector fields; they satisfy (A.1.1), the bracket being the Schouten–Nijenhuis bracket. Since all the equations (A.1.1) are in terms of commutators, they can be interpreted as an action of certain differential graded Lie algebras on the complex of forms. This latter formulation has a noncommutative analogue as follows. For any (graded) associative algebra A, let $\mathfrak{g}(A)$ be the graded Lie algebra of its Hochschild cochains, with the Gerstenhaber bracket and the Hochschild differential δ. Then on $(C_{-\bullet}(A)[[u]], b+uB)$ there is a $k[[u]]$-linear, (u)-adically continuous structure of an L_∞ module over $\big(\mathfrak{g}(A)[\epsilon, u], \delta + u\frac{\partial}{\partial\epsilon}\big)$, such that, for a cochain D of degree ≤ 1, $D\epsilon$ acts by $-\iota_D$ as in (A.2.1), (A.2.2). From that, one deduces a construction of a flat superconnection on the periodic cyclic complex of a family of algebras [8, Proposition 2]. The formulas for the L_∞ action are somewhat more complicated than the definition of χ in (A.2.3). It would be interesting to better understand the relation between the above and the construction of the Gauss–Manin connection in the present paper.

Let us finish by mentioning one more feature of the classical calculus. The space of multivector fields is in fact a Gerstenhaber algebra, in particular a graded commutative algebra; one has

$$\iota_X\iota_Y = \iota_{XY}; \qquad L_{XY} = (-1)^{|Y|}L_X\iota_Y + \iota_X L_Y. \tag{A.4.1}$$

An algebraic system uniting (A.1.1) and (A.4.1) is called a calculus; it was shown in [20,28,9] that, for an associative algebra A, the pair of complexes $(C^*(A, A), C_*(A, A))$ is always a homotopy calculus. This structure quite inexplicit and not canonical; it depends on a choice of a Drinfeld associator.

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