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Advances in Mathematics 231 (2012) 1786–1817

ADVANCES IN Mathematics

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# Regionally proximal relation of order $d$ is an equivalence one for minimal systems and a combinatorial consequence

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Received 6 November 2010; accepted 16 July 2012
Available online 11 August 2012

Communicated by the Managing Editors of AIM

**Abstract**

Inverse limits of nilsystems in topologically dynamical systems were characterized by Host, Kra and Maass recently. Namely, for each $d \in \mathbb{N}$ a certain generalization of the regionally proximal relation was introduced, and for a distal minimal system it was shown that such a relation is an equivalence one, which determines the maximal $d$-step nilfactor. One of the main results in this article is to show that the above results hold for a general minimal system.

A combinatorial consequence is also deduced, which is the topological correspondence of the result obtained by Host and Kra for positive upper Banach density subsets using ergodic methods.


*MSC:* primary 37B05; 37A99

*Keywords:* Nilsystems; Regionally proximal relation; Minimal systems

## 1. Introduction

### 1.1. Background

In a recent paper [23] by Host et al. the authors characterized inverse limits of nilsystems in topologically dynamical systems, via a structure theorem for topological dynamical systems that

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0001-8708/$ - see front matter 
doi:10.1016/j.aim.2012.07.012

is an analog of the structure theorem for measure preserving systems. The way they achieved this is first to define a certain generalization of the regionally proximal relation for any $d \in \mathbb{N}$ and any topologically dynamical system $(X, T)$, and then to show that this relation is an equivalence one for any minimal distal system. Finally they used the ergodic method to prove that the quotient of $X$ under this relation is an inverse limit of $d$-step nilsystems which is the maximal nilfactor of $(X, T)$. We note that the case $d = 1$ is classic, and the case $d = 2$ was obtained by Host and Maass in [24].

The question if this relation is an equivalence one for general minimal systems remains open in [23]. We aim to study this question in the current paper. To state our main results let us recall the notion of the regionally proximal relation of order $d$ introduced in [23] ($d = 2$ in [24]). Let $(X, T)$ be a topologically dynamical system and let $d \geq 1$ be an integer. A pair $(x, y) \in X \times X$ is said to be *regionally proximal of order* $d$ if for any $\delta > 0$, there exist $x', y' \in X$ and a vector $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that $\rho(x, x') < \delta$, $\rho(y, y') < \delta$, and

$$\rho(T^{\mathbf{n}\cdot\epsilon}x', T^{\mathbf{n}\cdot\epsilon}y') < \delta \quad \text{for any } \epsilon \in \{0,1\}^d, \epsilon \neq (0, \ldots, 0),$$

where $\mathbf{n}\cdot\epsilon = \sum_{i=1}^{d} \epsilon_i n_i$. The set of regionally proximal pairs of order $d$ is denoted by $\mathbf{RP}^{[d]}(X)$, which is called *the regionally proximal relation of order* $d$.

It is easy to see that $\mathbf{RP}^{[d]}(X)$ is a closed and invariant relation for all $d \in \mathbb{N}$. When $d = 1$, $\mathbf{RP}^{[d]}(X)$ is nothing but the classical regionally proximal relation which determines the maximal equicontinuous factor for any minimal system.

### 1.2. Main results

In this article, we completely answered the question remained in [23]. Namely, we show that for each minimal system $(X, T)$, $\mathbf{RP}^{[d]}(X)$ is a closed invariant equivalence relation which is obtained by a deep understanding of the minimal sets in the dynamical parallelepipeds $\mathbf{Q}^{[d]}(X)$ for the actions of face transformations. We also show that for a factor map $\pi : (X, T) \longrightarrow (Y, S)$ between minimal systems, $\pi \times \pi(\mathbf{RP}^{[d]}(X)) = \mathbf{RP}^{[d]}(Y)$ which is interesting itself and also allows us to prove that $X/\mathbf{RP}^{[d]}(X)$ is the maximal $d$-step nilfactor of $(X, T)$ by using some result in [23].

Note that a subset $S$ of $\mathbb{Z}$ is *dynamically syndetic* if there is a minimal system $(X, T)$, $x \in X$ and an open neighborhood $U$ of $x$ such that $S = \{n \in \mathbb{Z} : T^n x \in U\}$. Equivalently, $S \subset \mathbb{Z}$ is dynamically syndetic if and only if $S$ contains $\{0\}$ and $1_S$ is a minimal point of $(\{0,1\}^{\mathbb{Z}}, \sigma)$, where $\sigma$ is the shift map. A subset $S$ of $\mathbb{Z}^d$ is *syndetic* if there exists a finite subset $F \subset \mathbb{Z}^d$ such that $S + F = \mathbb{Z}^d$. In [20, Theorem 1.5] Host and Kra proved the following result. Let $A \subset \mathbb{Z}$ with $\overline{d}(A) > \delta > 0$ and let $d \in \mathbb{N}$, then

$$\left\{ \mathbf{n} = (n_1, n_2, \ldots, n_k) \in \mathbb{Z}^d : \overline{d}\left( \bigcap_{\epsilon \in \{0,1\}^d} (A + \epsilon \cdot \mathbf{n}) \right) \geq \delta^{2^d} \right\}$$

is syndetic, where $\overline{d}(B)$ denotes the upper density of $B \subset \mathbb{Z}$.

A combinatorial consequence of our results is that if $S$ is a dynamically syndetic subset of $\mathbb{Z}$, then for each $d \geq 1$,

$$\{\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d : \epsilon \cdot \mathbf{n} = n_1\epsilon_1 + \cdots + n_d\epsilon_d \in S, \epsilon_i \in \{0,1\}, 1 \leq i \leq d\}$$

is syndetic. In some sense this is the topological correspondence of the above result obtained by Host and Kra for positive upper Banach density subsets using ergodic methods.

In [23] the authors showed that $\mathbf{RP}^{[d]}$ is an equivalence relation for minimal distal systems without using the enveloping semigroup theory explicitly. In our situation we are forced to use the theory. The main idea of the proof is the following. First using the structure theory of a minimal system we show that the diagonal points acting by the face transformations are minimal, and then we prove some equivalence conditions for a pair being regionally proximal of order $d$. A key lemma here is to switch from a cubic point to a face point. Combining the minimality with the conditions we show that $\mathbf{RP}^{[d]}$ is an equivalence relation for minimal systems. Finally we show that $\mathbf{RP}^{[d]}$ can be lifted up from a factor to an extension between two minimal systems, which implies that the factor induced by $\mathbf{RP}^{[d]}$ is the maximal $d$-step nilfactor by using some result in [23].

We remark that the main results of the paper can be extended to abelian group actions without difficulty.

### *1.3. Historic remarks*

The study of the regionally proximal relation has a long history in topological dynamics. One of the first problems in the study of topological dynamics was to characterize the equicontinuous structure relation $S_{eq}(X)$ of a system $(X, T)$; i.e. to find the smallest closed invariant equivalence relation $R(X)$ on $(X, T)$ such that $(X/R(X), T)$ is equicontinuous. A natural candidate for $R(X)$ is the so-called regionally proximal relation $\mathbf{RP}(X)$ [7]. By the definition, $\mathbf{RP}(X)$ is closed, invariant, and reflexive, but not necessarily transitive. The problem was then to find conditions under which $\mathbf{RP}(X)$ is an equivalence relation. It turns out to be a difficult problem. Starting with Veech [29], various authors, including MacMahon [27], Ellis–Keynes [9], came up with various sufficient conditions for $\mathbf{RP}(X)$ to be an equivalence relation. For somewhat different approach, see [2]. Note that in our case, $T : X \to X$ being homeomorphism and $(X, T)$ being minimal, $\mathbf{RP}(X)$ is always an equivalence relation.

In the 1970s Furstenberg gave a beautiful proof of Szemerédi's theorem via ergodic theory [11]. It remains a question if the multiple ergodic averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^n x)\dots f_d(T^{dn}x),$$

converges in $L^2(X, \mu)$ for $f_1, \dots, f_d \in L^\infty(X, \mu)$. This question was finally answered by Host and Kra in [20]. For the later development along the line see [32,28,3,19].

In the paper by Host–Kra [20] the authors defined for each $d \in \mathbb{N}$ and each measure-preserving transformation on the probability space $(X, \mathcal{B}, \mu)$ a factor $\mathcal{Z}_d$ which is characteristic and is an inverse limit of $d$-step nilsystems. Those factors have many important applications. Since topological dynamics and ergodic theory are twins, it is natural to ask how to obtain similar factors in topological dynamics. In the pioneer paper [23] ([24] for $d = 2$) the authors succeeded doing the job for minimal distal systems. So the main results in [23] and in the current paper can be seen as the topological correspondence of the $\mathcal{Z}_d$ factors in ergodic theory. We note that the counterpart of the characteristic factors in topological dynamics was also studied by Glasner [15,16]. For applications of the main results of the paper see [4,25].

### *1.4. Organization of the paper*

In Section 2, we introduce some basic notions used in the paper. Since we will use tools from abstract topological dynamics, we collect basic facts about them in Appendix A. In Section 3, main results of the paper are discussed. The three sections followed are devoted to give proofs of main results. Notice that lots of results obtained there have their independent interest.

## 2. Preliminaries

In this section we introduce notions about dynamical parallelepipeds and nilsystems etc. For more details see [20–23].

### 2.1. Topological dynamical systems

A *transformation* of a compact metric space X is a homeomorphism of X to itself. A *topological dynamical system*, referred to more succinctly as just a *system*, is a pair $(X, T)$, where $X$ is a compact metric space and $T : X \to X$ is a transformation. We use $\rho(\cdot, \cdot)$ to denote the metric in $X$. We also make use of a more general definition of a topological system. That is, instead of just a single transformation $T$, we will consider a countable abelian group of transformations. We collect basic facts about topological dynamics under general group actions in Appendix A.

A system $(X, T)$ is *transitive* if there exists some point $x \in X$ whose orbit $\mathcal{O}(x, T) = \{T^n x : n \in \mathbb{Z}\}$ is dense in $X$ and we call such a point a *transitive point*. The system is *minimal* if the orbit of any point is dense in $X$. This property is equivalent to say that X and the empty set are the only closed invariant sets in $X$.

### 2.2. Cubes and faces

Let $X$ be a set, let $d \geq 1$ be an integer, and write $[d] = \{1, 2, \dots, d\}$. We view $\{0, 1\}^d$ in one of two ways, either as a sequence $\epsilon = (\epsilon_1, \dots, \epsilon_d)$ of 0's and 1's; or as a subset of $[d]$. A subset $\epsilon$ corresponds to the sequence $(\epsilon_1, \dots, \epsilon_d) \in \{0, 1\}^d$ such that $i \in \epsilon$ if and only if $\epsilon_i = 1$ for $i \in [d]$. For example, $\mathbf{0} = (0, 0, \dots, 0) \in \{0, 1\}^d$ is the same to $\emptyset \subset [d]$.

If $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ and $\epsilon \in \{0, 1\}^d$, we define

$$\mathbf{n} \cdot \epsilon = \sum_{i=1}^{d} n_i \epsilon_i .$$

If we consider $\epsilon$ as $\epsilon \subset [d]$, then $\mathbf{n} \cdot \epsilon = \sum_{i \in \epsilon} n_i$.

We denote $X^{2^d}$ by $X^{[d]}$. A point $\mathbf{x} \in X^{[d]}$ can be written in one of two equivalent ways, depending on the context:

$$\mathbf{x} = (x_\epsilon : \epsilon \in \{0, 1\}^d) = (x_\epsilon : \epsilon \subset [d]).$$

Hence $x_\emptyset = x_\mathbf{0}$ is the first coordinate of $\mathbf{x}$. As examples, points in $X^{[2]}$ are like

$$(x_{00}, x_{10}, x_{01}, x_{11}) = (x_\emptyset, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}),$$

and points in $X^{[3]}$ are like

$$\begin{aligned}&(x_{000}, x_{100}, x_{010}, x_{110}, x_{001}, x_{101}, x_{011}, x_{111})\\ &\quad = (x_\emptyset, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}, x_{\{3\}}, x_{\{1,3\}}, x_{\{2,3\}}, x_{\{1,2,3\}}).\end{aligned}$$

For $x \in X$, we write $x^{[d]} = (x, x, \dots, x) \in X^{[d]}$. The diagonal of $X^{[d]}$ is $\Delta^{[d]} = \{x^{[d]} : x \in X\}$. Usually, when $d = 1$, one denotes the diagonal by $\Delta_X$ or $\Delta$ instead of $\Delta^{[1]}$.

A point $\mathbf{x} \in X^{[d]}$ can be decomposed as $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$ with $\mathbf{x}', \mathbf{x}'' \in X^{[d-1]}$, where $\mathbf{x}' = (x_{\epsilon 0} : \epsilon \in \{0,1\}^{d-1})$ and $\mathbf{x}'' = (x_{\epsilon 1} : \epsilon \in \{0,1\}^{d-1})$. We can also isolate the first coordinate, writing $X_*^{[d]} = X^{2^d-1}$ and then writing a point $\mathbf{x} \in X^{[d]}$ as $\mathbf{x} = (x_\emptyset, \mathbf{x}_*)$, where $\mathbf{x}_* = (x_\epsilon : \epsilon \neq \emptyset) \in X_*^{[d]}$.

Identifying $\{0,1\}^d$ with the set of vertices of the Euclidean unit cube, a Euclidean isometry of the unit cube permutes the vertices of the cube and thus the coordinates of a point $x \in X^{[d]}$. These permutations are the *Euclidean permutations* of $X^{[d]}$. For details see [20].

### 2.3. Dynamical parallelepipeds

**Definition 2.1.** Let $(X, T)$ be a topological dynamical system and let $d \geq 1$ be an integer. We define $\mathbf{Q}^{[d]}(X)$ to be the closure in $X^{[d]}$ of elements of the form

$$(T^{\mathbf{n}\cdot\epsilon}x = T^{n_1\epsilon_1+\cdots+n_d\epsilon_d}x : \epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \{0,1\}^d),$$

where $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $x \in X$. When there is no ambiguity, we write $\mathbf{Q}^{[d]}$ instead of $\mathbf{Q}^{[d]}(X)$. An element of $\mathbf{Q}^{[d]}(X)$ is called a (dynamical) parallelepiped of dimension $d$.

It is important to note that $\mathbf{Q}^{[d]}$ is invariant under the Euclidean permutations of $X^{[d]}$.

As examples, $\mathbf{Q}^{[2]}$ is the closure in $X^{[2]} = X^4$ of the set

$$\{(x, T^m x, T^n x, T^{n+m}x) : x \in X, m, n \in \mathbb{Z}\}$$

and $\mathbf{Q}^{[3]}$ is the closure in $X^{[3]} = X^8$ of the set

$$\{(x, T^m x, T^n x, T^{m+n}x, T^p x, T^{m+p}x, T^{n+p}x, T^{m+n+p}x) : x \in X, m, n, p \in \mathbb{Z}\}.$$

**Definition 2.2.** Let $\phi : X \to Y$ and $d \in \mathbb{N}$. Define $\phi^{[d]} : X^{[d]} \to Y^{[d]}$ by $(\phi^{[d]}\mathbf{x})_\epsilon = \phi x_\epsilon$ for every $\mathbf{x} \in X^{[d]}$ and every $\epsilon \subset [d]$.

Let $(X, T)$ be a system and $d \geq 1$ be an integer. The *diagonal transformation* of $X^{[d]}$ is the map $T^{[d]}$.

**Definition 2.3.** Face transformations are defined inductively as follows: Let $T^{[0]} = T$, $T_1^{[1]} = \mathrm{id} \times T$. If $\{T_j^{[d-1]}\}_{j=1}^{d-1}$ is defined already, then set

$$T_j^{[d]} = T_j^{[d-1]} \times T_j^{[d-1]}, \quad j \in \{1, 2, \ldots, d-1\},$$
$$T_d^{[d]} = \mathrm{id}^{[d-1]} \times T^{[d-1]}.$$

It is easy to see that for $j \in [d]$, the face transformation $T_j^{[d]} : X^{[d]} \to X^{[d]}$ can be defined by, for every $\mathbf{x} \in X^{[d]}$ and $\epsilon \subset [d]$,

$$T_j^{[d]}\mathbf{x} = \begin{cases} (T_j^{[d]}\mathbf{x})_\epsilon = Tx_\epsilon, & j \in \epsilon; \\ (T_j^{[d]}\mathbf{x})_\epsilon = x_\epsilon, & j \notin \epsilon. \end{cases}$$

The *face group* of dimension $d$ is the group $\mathcal{F}^{[d]}(X)$ of transformations of $X^{[d]}$ spanned by the face transformations. The *parallelepiped group* of dimension $d$ is the group $\mathcal{G}^{[d]}(X)$ spanned by the diagonal transformation and the face transformations. We often write $\mathcal{F}^{[d]}$ and $\mathcal{G}^{[d]}$ instead of $\mathcal{F}^{[d]}(X)$ and $\mathcal{G}^{[d]}(X)$, respectively. For $\mathcal{G}^{[d]}$ and $\mathcal{F}^{[d]}$, we use similar notations to that used for $X^{[d]}$: namely, an element of either of these groups is written as $S = (S_\epsilon : \epsilon \in \{0,1\}^d)$. In particular, $\mathcal{F}^{[d]} = \{S \in \mathcal{G}^{[d]} : S_\emptyset = \mathrm{id}\}$.

For convenience, we denote the orbit closure of $\mathbf{x} \in X^{[d]}$ under $\mathcal{F}^{[d]}$ by $\overline{\mathcal{F}^{[d]}}(\mathbf{x})$, instead of $\overline{\mathcal{O}(\mathbf{x}, \mathcal{F}^{[d]})}$.

It is easy to verify that $\mathbf{Q}^{[d]}$ is the closure in $X^{[d]}$ of

$$\{Sx^{[d]} : S \in \mathcal{F}^{[d]}, x \in X\}.$$

If $x$ is a transitive point of $X$, then $\mathbf{Q}^{[d]}$ is the closed orbit of $x^{[d]}$ under the group $\mathcal{G}^{[d]}$.

### 2.4. Nilmanifolds and nilsystems

Let $G$ be a group. For $g, h \in G$, we write $[g,h] = ghg^{-1}h^{-1}$ for the commutator of $g$ and $h$ and we write $[A,B]$ for the subgroup spanned by $\{[a,b] : a \in A, b \in B\}$. The commutator subgroups $G_j$, $j \geq 1$, are defined inductively by setting $G_1 = G$ and $G_{j+1} = [G_j, G]$. Let $k \geq 1$ be an integer. We say that $G$ is *$k$-step nilpotent* if $G_{k+1}$ is the trivial subgroup.

Let $G$ be a $k$-step nilpotent Lie group and $\Gamma$ a discrete cocompact subgroup of $G$. The compact manifold $X = G/\Gamma$ is called a *$k$-step nilmanifold*. The group $G$ acts on $X$ by left translations and we write this action as $(g,x) \mapsto gx$. The Haar measure $\mu$ of $X$ is the unique probability measure on $X$ invariant under this action. Let $\tau \in G$ and $T$ be the transformation $x \mapsto \tau x$ of $X$. Then $(X, T, \mu)$ is called a *basic $k$-step nilsystem*. When the measure is not needed for results, we omit it and write that $(X,T)$ is a basic $k$-step nilsystem.

We also make use of inverse limits of nilsystems and so we recall the definition of an inverse limit of systems (restricting ourselves to the case of sequential inverse limits). If $(X_i, T_i)_{i\in\mathbb{N}}$ are systems with $diam(X_i) \leq M < \infty$ and $\phi_i : X_{i+1} \to X_i$ are factor maps, the *inverse limit* of the systems is defined to be the compact subset of $\prod_{i\in\mathbb{N}} X_i$ given by $\{(x_i)_{i\in\mathbb{N}} : \phi_i(x_{i+1}) = x_i, i \in \mathbb{N}\}$, which is denoted by $\varprojlim\{X_i\}_{i\in\mathbb{N}}$. It is a compact metric space endowed with the distance $\rho(x,y) = \sum_{i\in\mathbb{N}} 1/2^i d_i(x_i, y_i)$. We note that the maps $\{T_i\}$ induce a transformation $T$ on the inverse limit.

In [23] authors characterized inverse limits of nilsystems in topological dynamics, via a structure theorem for topological dynamical systems that is an analog of the structure theorem for measure preserving systems [20].

**Theorem 2.4** (*Host–Kra–Maass [23, Theorem 1.2]*)**.** *Assume that $(X,T)$ is a transitive topological dynamical system and let $d \geq 2$ be an integer. The following properties are equivalent:*

(1) *If $x, y \in \mathbf{Q}^{[d]}(X)$ have $2^d - 1$ coordinates in common, then $x = y$.*
(2) *If $x, y \in X$ are such that $(x, y, \ldots, y) \in \mathbf{Q}^{[d]}(X)$, then $x = y$.*
(3) *$X$ is an inverse limit of basic $(d-1)$-step minimal nilsystems.*

*A transitive system satisfying either of the equivalent properties above is called a $(d-1)$-step nilsystem or a system of order $(d-1)$.*

### 2.5. $\mathbf{RP}^{[d]}$

For a dynamical system, the regionally proximal relation of order $d$, denoted by $\mathbf{RP}^{[d]}$ has been stated in the introduction. It is easy to see that $\mathbf{RP}^{[d]}$ is a closed and invariant relation for all $d \in \mathbb{N}$. Note that

$$\cdots \subseteq \mathbf{RP}^{[d+1]} \subseteq \mathbf{RP}^{[d]} \subseteq \cdots \subseteq \mathbf{RP}^{[2]} \subseteq \mathbf{RP}^{[1]} = \mathbf{RP}(X).$$

By the definition it is easy to verify the following equivalent condition for $\mathbf{RP}^{[d]}$, see [23] for a proof.

**Lemma 2.5.** *Let $(X, T)$ be a minimal system and let $d \geq 1$ be an integer. Let $x, y \in X$. Then $(x, y) \in \mathbf{RP}^{[d]}$ if and only if there is some $\mathbf{a}_* \in X_*^{[d]}$ such that $(x, \mathbf{a}_*, y, \mathbf{a}_*) \in \mathbf{Q}^{[d+1]}$.*

**Remark 2.6.** When $d = 1$, $\mathbf{RP}^{[1]}$ is the classical regionally proximal relation. If $(X, T)$ is minimal, it is easy to verify directly the following useful fact:

$$(x, y) \in \mathbf{RP} = \mathbf{RP}^{[1]} \Leftrightarrow (x, x, y, x) \in \mathbf{Q}^{[2]} \Leftrightarrow (x, y, y, y) \in \mathbf{Q}^{[2]}.$$

## 3. Main results

In this section we will state the main results of the paper. We remark that the main results of the paper can be extended to abelian group actions.

### 3.1. $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$

To show $\mathbf{RP}^{[d]}$ is an equivalence relation we are forced to investigate the $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$ and the equivalent conditions for $\mathbf{RP}^{[d]}$. Those are done in Theorems 3.1 and 3.4 respectively.

First recall that $(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$ is a minimal system, which is discussed in [23]. In our situation we need to understand $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$. Let $(X, T)$ be a system and $x \in X$. Recall that $\overline{\mathcal{F}^{[d]}}(\mathbf{x}) = \overline{\mathcal{O}(\mathbf{x}, \mathcal{F}^{[d]})}$ for $\mathbf{x} \in X^{[d]}$. Set

$$\mathbf{Q}^{[d]}[x] = \{\mathbf{z} \in \mathbf{Q}^{[d]}(X) : z_\emptyset = x\}.$$

We can show the following theorem.

**Theorem 3.1.** *Let $(X, T)$ be a minimal system and $d \in \mathbb{N}$. Then*

(1) *$(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$.*
(2) *$(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is the unique $\mathcal{F}^{[d]}$-minimal subset in $\mathbf{Q}^{[d]}[x]$ for all $x \in X$.*

The above theorem has the following combinatorial consequence.

**Corollary 3.2.** *Let $(X, T)$ be a minimal system, $x \in X$ and $U$ be an open neighborhood of $x$. Put $S = \{n \in \mathbb{Z} : T^n x \in U\}$. Then for each $d \geq 1$,*

$$\{(n_1, \dots, n_d) \in \mathbb{Z}^d : n_1\epsilon_1 + \cdots + n_d\epsilon_d \in S, \epsilon_i \in \{0, 1\}, 1 \leq i \leq d\}$$

*is syndetic.*

**Remark 3.3.** (1) To understand $S$ better we state the following proposition whose proof is similar to [26, Proposition 2.3]: The family of dynamically syndetic subsets is the family generated by the sets $S$ whose indicator functions $1_S$ are the minimal points of $(\{0, 1\}^{\mathbb{Z}}, \sigma)$ and $0 \in S$, where $\sigma$ is the shift. Notice that a collection $\mathcal{F}$ of subsets of $\mathbb{Z}$ is a *family* if it is upwards, i.e. $A \in \mathcal{F}$ and $A \subset B$ imply that $B \in \mathcal{F}$.

(2) We note that if $S$ is a syndetic subset of $\mathbb{Z}$ then $S - S \supset S_1 - S_1$ for some dynamically syndetic subset $S_1$.

### 3.2. $\mathbf{RP}^{[d]}$ *is an equivalence relation*

With the help of Theorem 3.1, we can prove that $\mathbf{RP}^{[d]}$ is an equivalence relation. First we have the following equivalent conditions for $\mathbf{RP}^{[d]}$.

**Theorem 3.4.** *Let $(X,T)$ be a minimal system and $d \in \mathbb{N}$. Then the following conditions are equivalent:*

(1) $(x,y) \in \mathbf{RP}^{[d]}$;
(2) $(x,y,y,\ldots,y) = (x, y_*^{[d+1]}) \in \mathbf{Q}^{[d+1]}$;
(3) $(x,y,y,\ldots,y) = (x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$.

**Proof.** (3) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (1) follows from Lemma 2.5. Hence it suffices to show (1) $\Rightarrow$ (3).

Let $(x,y) \in \mathbf{RP}^{[d]}$. Then by Lemma 2.5 there is some $\mathbf{a}_* \in X_*^{[d]}$ such that $(x,\mathbf{a}_*,y,\mathbf{a}_*) \in \mathbf{Q}^{[d+1]}$. Observe that $(y,\mathbf{a}_*) \in \mathbf{Q}^{[d]}$. By Theorem 3.1-(2), there is a sequence $\{F_k\} \subset \mathcal{F}^{[d]}$ such that $F_k(y,\mathbf{a}_*) \to y^{[d]}$, $k \to \infty$. Hence

$$F_k \times F_k(x,\mathbf{a}_*,y,\mathbf{a}_*) \to (x, y_*^{[d]}, y, y_*^{[d]}) = (x, y_*^{[d+1]}), \quad k \to \infty.$$

Since $F_k \times F_k \in \mathcal{F}^{[d+1]}$ and $(x,\mathbf{a}_*,y,\mathbf{a}_*) \in \mathbf{Q}^{[d+1]}$, we have that $(x, y_*^{[d+1]}) \in \mathbf{Q}^{[d+1]}$.

By Theorem 3.1-(1), $y^{[d+1]}$ is $\mathcal{F}^{[d+1]}$-minimal. It follows that $(x, y_*^{[d+1]})$ is also $\mathcal{F}^{[d+1]}$-minimal. Now $(x, y_*^{[d+1]}) \in \mathbf{Q}^{[d+1]}[x]$ is $\mathcal{F}^{[d+1]}$-minimal and by Theorem 3.1-(2), $\overline{\mathcal{F}^{[d+1]}(x^{[d+1]})}$ is the unique $\mathcal{F}^{[d+1]}$-minimal subset in $\mathbf{Q}^{[d+1]}[x]$. Hence we have that $(x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$, and the proof is completed. $\square$

By Theorem 3.4, we have the following theorem immediately.

**Theorem 3.5.** *Let $(X,T)$ be a minimal system and $d \in \mathbb{N}$. Then $\mathbf{RP}^{[d]}(X)$ is an equivalence relation.*

**Proof.** It suffices to show the transitivity, i.e. if $(x,y),(y,z) \in \mathbf{RP}^{[d]}(X)$, then $(x,z) \in \mathbf{RP}^{[d]}(X)$. Since $(x,y),(y,z) \in \mathbf{RP}^{[d]}(X)$, by Theorem 3.4 we have

$$(y,x,x,\ldots,x),\ (y,z,z,\ldots,z) \in \overline{\mathcal{F}^{[d+1]}}(y^{[d+1]}).$$

By Theorem 3.1 $(\overline{\mathcal{F}^{[d+1]}}(y^{[d+1]}), \mathcal{F}^{[d+1]})$ is minimal, it follows that $(y,z,z,\ldots,z) \in \overline{\mathcal{F}^{[d+1]}(y,x,x,\ldots,x)}$. It follows that $(x,z,z,\ldots,z) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$. By Theorem 3.4 again, $(x,z) \in \mathbf{RP}^{[d]}(X)$. $\square$

**Remark 3.6.** By Theorem 3.4 we know that in the definition of regionally proximal relation of $d$, $x'$ can be replaced by $x$. More precisely, $(x,y) \in \mathbf{RP}^{[d]}$ if and only if for any $\delta > 0$ there exist $y' \in X$ and a vector $\mathbf{n} = (n_1,\ldots,n_d) \in \mathbb{Z}^d$ such that for any nonempty $\epsilon \subset [d]$

$$\rho(y,y') < \delta \quad \text{and} \quad \rho(T^{\mathbf{n}\cdot\epsilon}x, T^{\mathbf{n}\cdot\epsilon}y') < \delta.$$

This conclusion is first given in [23] for a minimal distal system.

### 3.3. $\mathbf{RP}^{[d]}$ *and nilfactors*

A subset $S \subset \mathbb{Z}$ is *thick* if it contains arbitrarily long runs of positive integers, i.e. there is a subsequence $\{n_i\}$ of $\mathbb{Z}$ such that $S \supset \bigcup_{i=1}^{\infty}\{n_i, n_i+1, \ldots, n_i+i\}$.

Let $\{b_i\}_{i\in I}$ be a finite or infinite sequence in $\mathbb{Z}$. One defines

$$FS(\{b_i\}_{i\in I}) = \left\{\sum_{i\in\alpha} b_i : \alpha \text{ is a finite non-empty subset of } I\right\}.$$

Note when $I = [d]$,

$$FS(\{b_i\}_{i=1}^{d}) = \left\{\sum_{i\in I} b_i\epsilon_i : \epsilon = (\epsilon_i) \in \{0,1\}^d \setminus \{\emptyset\}\right\}.$$

$F$ is an *IP set* if it contains some $FS(\{p_i\}_{i=1}^{\infty})$, where $p_i \in \mathbb{Z}$.

**Lemma 3.7.** *Let $(X,T)$ be a system. Then for every $d \in \mathbb{N}$, the proximal relation*

$$\mathbf{P}(X) \subseteq \mathbf{RP}^{[d]}(X).$$

**Proof.** Let $(x,y) \in \mathbf{P}(X)$ and $\delta > 0$. Set

$$N_\delta(x,y) = \{n \in \mathbb{Z} : \rho(T^n x, T^n y) < \delta\}.$$

It is easy to check $N_\delta(x,y)$ is thick and hence an IP set. From this it follows that $\mathbf{P}(X) \subseteq \mathbf{RP}^{[d]}(X)$. More precisely, set $FS(\{p_i\}_{i=1}^{\infty}) \subseteq N_\delta(x,y)$, then for any $d \in \mathbb{N}$,

$$\rho(T^{p_1\epsilon_1+\cdots+p_d\epsilon_d}x, T^{p_1\epsilon_1+\cdots+p_d\epsilon_d}y) < \delta, \quad \epsilon = (\epsilon_1,\ldots,\epsilon_d) \in \{0,1\}^d, \epsilon \neq (0,\ldots,0).$$

That is, $(x,y) \in \mathbf{RP}^{[d]}$ for all $d \in \mathbb{N}$. $\square$

The following corollary was observed in [24] for $d = 2$.

**Corollary 3.8.** *Let $(X,T)$ be a minimal system and $d \in \mathbb{N}$. Then $(X,T)$ is a weakly mixing system if and only if $\mathbf{RP}^{[d]} = X \times X$.*

**Proof.** Since a minimal system $(X,T)$ is weakly mixing if and only if $\overline{\mathbf{P}(X)} = \mathbf{RP}(X) = X \times X$ (see [1]), so the result follows from Lemma 3.7. $\square$

We remark that more general properties for weakly mixing systems will be shown in Theorem 3.13 in the sequel.

**Proposition 3.9.** *Let $(X,T)$ be a minimal system and $d \in \mathbb{N}$. Then $\mathbf{RP}^{[d]} = \Delta$ if and only if $X$ is a system of order $d$.*

**Proof.** It follows from Theorems 3.4 and 2.4 directly. $\square$

### *3.4. Maximal nilfactors*

Note that the lifting property of $\mathbf{RP}^{[d]}$ between two minimal systems is obtained in the paper. This result is new even for minimal distal systems.

**Theorem 3.10.** *Let $\pi : (X,T) \to (Y,T)$ be a factor map and $d \in \mathbb{N}$. Then*

(1) $\pi \times \pi(\mathbf{RP}^{[d]}(X)) \subseteq \mathbf{RP}^{[d]}(Y)$*;*
(2) *if $(X,T)$ is minimal, then $\pi \times \pi(\mathbf{RP}^{[d]}(X)) = \mathbf{RP}^{[d]}(Y)$.*

**Proof.** (1) It follows from the definition.

(2) It will be proved in Section 6. □

**Theorem 3.11.** *Let $\pi : (X, T) \to (Y, T)$ be a factor map of minimal systems and $d \in \mathbb{N}$. Then the following conditions are equivalent:*

(1) *$(Y, T)$ is a $d$-step nilsystem;*
(2) $\mathbf{RP}^{[d]}(X) \subset R_\pi$.

*Especially the quotient of $X$ under $\mathbf{RP}^{[d]}(X)$ is the maximal $d$-step nilfactor of $X$, i.e. any $d$-step nilfactor of $X$ is the factor of $X/\mathbf{RP}^{[d]}(X)$.*

**Proof.** Assume that $(Y, T)$ is a $d$-step nilsystem. Then we have $\mathbf{RP}^{[d]}(Y) = \Delta_Y$ by Proposition 3.9. Hence by Theorem 3.10-(1),

$$\mathbf{RP}^{[d]}(X) \subset (\pi \times \pi)^{-1}(\Delta_Y) = R_\pi.$$

Conversely, assume that $\mathbf{RP}^{[d]}(X) \subset R_\pi$. If $(Y, T)$ is not a $d$-step nilsystem, then by Proposition 3.9, $\mathbf{RP}^{[d]}(Y) \neq \Delta_Y$. Let $(y_1, y_2) \in \mathbf{RP}^{[d]} \setminus \Delta_Y$. Now by Theorem 3.10, there are $x_1, x_2 \in X$ such that $(x_1, x_2) \in \mathbf{RP}^{[d]}(X)$ with $(\pi \times \pi)(x_1, x_2) = (y_1, y_2)$. Since $\pi(x_1) = y_1 \neq y_2 = \pi(x_2)$, $(x_1, x_2) \notin R_\pi$. This means that $\mathbf{RP}^{[d]}(X) \not\subset R_\pi$, a contradiction! The proof is completed. □

**Remark 3.12.** In [23, Proposition 4.5] it is showed that this proposition holds for minimal distal systems.

### *3.5. Weakly mixing systems*

In this subsection we completely determine $\mathbf{Q}^{[d]}$ and $\overline{\mathcal{F}^{[d]}}(x^{[d]})$ for minimal weakly mixing systems which helps us to understand the proof of Lemma 4.4.

**Theorem 3.13.** *Let $(X, T)$ be a minimal weakly mixing system and $d \geq 1$. Then*

(1) *$(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$ is minimal and $\mathbf{Q}^{[d]} = X^{[d]}$;*
(2) *For all $x \in X$, $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal and*

$$\overline{\mathcal{F}^{[d]}}(x^{[d]}) = \{x\} \times X_*^{[d]} = \{x\} \times X^{2^d-1}.$$

**Proof.** The fact that $(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$ is minimal and $\mathbf{Q}^{[d]} = X^{[d]}$ is followed from (2) easily. Hence it suffices to show (2).

We will show for any point of $\mathbf{x} \in X^{[d]}$ with $x_\emptyset = x$, we have

$$\overline{\mathcal{F}^{[d]}}(\mathbf{x}) = \{x\} \times X_*^{[d]},$$

which obviously implies (2). First note that it is trivial for $d = 1$. Now we assume that (1), and hence (2) holds for $d - 1$, $d \geq 2$.

Let $\mathbf{x} = (\mathbf{x}', \mathbf{x}'') \in \mathbf{Q}^{[d]}$. Since $(X, T)$ is weakly mixing, $(X^{[d-1]}, T^{[d-1]})$ is transitive (see [10]). Let $\mathbf{a} \in X^{[d-1]}$ be a transitive point. By the induction for $d-1$, $\mathbf{Q}^{[d-1]} = X^{[d-1]}$ is $\mathcal{G}^{[d-1]}$-minimal. Hence $\mathbf{a} \in \overline{\mathcal{O}(\mathbf{x}'', \mathcal{G}^{[d-1]})}$ and there is some sequence $F_k \in \mathcal{F}^{[d]}$ and $\mathbf{w} \in X^{[d-1]}$ such that

$$F_k\mathbf{x} = F_k(\mathbf{x}', \mathbf{x}'') \to (\mathbf{w}, \mathbf{a}), \quad k \to \infty.$$

Especially $(\mathbf{w}, \mathbf{a}) \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. Note that

$$(T_d^{[d]})^n(\mathbf{w}, \mathbf{a}) = (\mathbf{w}, (T^{[d-1]})^n \mathbf{a}) \in \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

We have

$$\{\mathbf{w}\} \times \mathcal{O}(\mathbf{a}, T^{[d-1]}) \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

Since $\mathbf{a}$ is a transitive point of $(X^{[d-1]}, T^{[d-1]})$, we have

$$\{\mathbf{w}\} \times X^{[d-1]} = \{\mathbf{w}\} \times \overline{\mathcal{O}(\mathbf{a}, T^{[d-1]})} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}). \tag{3.1}$$

By the induction assumption for $d-1$, $\mathbf{w}$ is minimal for $\mathcal{F}^{[d-1]}$ action and

$$\overline{\mathcal{F}^{[d-1]}}(\mathbf{w}) = \overline{\mathcal{O}(\mathbf{w}, \mathcal{F}^{[d-1]})} = \{x\} \times X_*^{[d-1]}. \tag{3.2}$$

By acting the elements of $\mathcal{F}^{[d]}$ on (3.1), we have

$$\mathcal{O}(\mathbf{w}, \mathcal{F}^{[d-1]}) \times X^{[d-1]} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}). \tag{3.3}$$

By (3.2) and (3.3), we have

$$\{x\} \times X_*^{[d-1]} \times X^{[d-1]} = \{x\} \times X_*^{[d]} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

This completes the proof. □

**Remark 3.14.** Using the so-called natural extension, it can be shown that the main results of the paper hold for continuous surjective maps.

## 4. $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$

In this section we discuss $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$ and prove Theorem 3.1-(1). First we will discuss proximal extensions, distal extensions and weakly mixing extension one by one. They exhibit different properties and satisfy our requests by different reasons. After that, the proof of Theorem 3.1-(1) will be given. The proof of Theorem 3.1-(2) will be given in next section. For notions which are not mentioned before see Appendix A.

### 4.1. *Idea of the proof of Theorem 3.1-(1)*

Before going on let us say something about the idea in the proof of Theorem 3.1-(1). By the structure Theorem A.4, for a minimal system $(X, T)$, we have the following diagram.

$$\begin{array}{ccc} X_\infty & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle \phi} & & \\ Y_\infty & & \end{array}$$

In this diagram $Y_\infty$ is a strictly PI system, $\phi$ is weakly mixing and RIC, and $\pi$ is proximal.

So if we want to show that $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$, it is sufficient to show it holds for $X_\infty$. By the definition of $X_\infty$ and $Y_\infty$, it is sufficient to consider the following cases: (1) proximal extensions; (2) distal or equicontinuous extensions; (3) RIC weakly mixing extensions and (4) the inverse limit. Since the inverse limit is easy to handle, we need only to focus on the three kinds of extensions.

*4.2. Properties about proximal, distal and weakly mixing extensions*

In this subsection we collect some properties about proximal, distal and weakly mixing extensions, which will be used frequently in the sequel. As in Appendix A, $(X, \mathcal{T})$ is a system under the action of a topological group $\mathcal{T}$, and $E(X, \mathcal{T})$ is its enveloping semigroup.

The following two lemmas are folk results.

**Lemma 4.1.** *Let $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ be a proximal extension of minimal systems. Let $x \in X$, $y = \pi(x)$ and let $x_1, x_2, \ldots, x_n \in \pi^{-1}(y)$. Then there is some $p \in E(X, \mathcal{T})$ such that*

$$px_1 = px_2 = \cdots = px_n = x.$$

*Especially, when $x = x_1$, we have that $(x_1, x_2, \ldots, x_n)$ is proximal to $(x, x, \ldots, x)$ in $(X^n, \mathcal{T})$.*

**Proof.** This is a direct consequence of Proposition A.3. □

**Lemma 4.2.** *Let $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ be a distal extension of systems. Then for any $x \in X$, if $\pi(x)$ is minimal in $(Y, \mathcal{T})$, then $x$ is minimal in $(X, \mathcal{T})$. Especially, if $(Y, \mathcal{T})$ is semi-simple (i.e. every point is minimal), then so is $(X, \mathcal{T})$.*

**Proof.** Let $x \in X$ and $y = \pi(x)$. Since $y$ is a minimal point, by Proposition A.2 there is some minimal idempotent $u \in E(X, \mathcal{T})$ such that $uy = y$. Then $\pi(ux) = u\pi(x) = uy = y$. Hence $ux, x \in \pi^{-1}(y)$. Since $(ux, x) \in \mathbf{P}(X, \mathcal{T})$ (Proposition A.3) and $\pi$ is distal, we have $ux = x$. That is, $x$ is a minimal point of $X$ by Proposition A.2. □

Now we discuss weakly mixing extensions. We need Theorem 4.3, which is a generalization of [1, Chapter 14, Theorem 28]. Note that in [18, Theorem 2.7 and Corollary 2.9] Glasner showed that $R_\pi^n$ is transitive. So Theorem 4.3 is a slight strengthening of the results in [18]. Since its proof needs some techniques in the enveloping semigroup theory, we leave it to Appendix A.

**Theorem 4.3.** *Let $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ be a RIC weakly mixing extension of minimal systems, then for all $n \geq 1$ and $y \in Y$, there exists a transitive point $(x_1, x_2, \ldots, x_n)$ of $R_\pi^n$ with $x_1, x_2, \ldots, x_n \in \pi^{-1}(y)$.*

Note that each RIC extension is open, and $\pi : X \to Y$ is open if and only if $Y \to 2^X$, $y \mapsto \pi^{-1}(y)$ is continuous, see for instance [31]. Using Theorem 4.3 we have the following lemma, which will be used in the sequel.

**Lemma 4.4.** *Let $\pi : (X, T) \to (Y, T)$ be a RIC weakly mixing extension of minimal systems. Then for each $y \in Y$ and $d \geq 1$, we have*

(1) $\big(\pi^{-1}(y)\big)^{[d]} = \big(\pi^{-1}(y)\big)^{2^d} \subset \mathbf{Q}^{[d]}(X)$,
(2) *for all $\mathbf{x} \in X^{[d]}$ with $x_\emptyset = x$ and $\pi^{[d]}(\mathbf{x}) = y^{[d]}$*

$$\{x\} \times \Big(\pi^{-1}(y)\Big)_*^{[d]} = \{x\} \times \Big(\pi^{-1}(y)\Big)^{2^d-1} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

**Proof.** The idea of the proof is similar to Theorem 3.13. When $d = 1$, for any $(x, x') \in X^{[1]} = X \times X$, $\overline{\mathcal{F}^{[1]}}(x, x') = \overline{\mathcal{O}}\big((x, x'), \mathrm{id} \times T\big) = \{x\} \times X$ and $\mathbf{Q}^{[1]}(X) = X \times X$. Hence the results hold obviously. Now we show the case $d = 2$. Let $\mathbf{x} = (x_1, x_2, x_3, x_4) \in X^{[2]}$ with $\pi^{[2]}(x_1, x_2, x_3, x_4) = y^{[2]}$. By Theorem 4.3, there is a transitive point $(a, b)$ of $(R_\pi, T \times T)$

with $\pi(a) = \pi(b) = y$. Since $(X, T)$ is minimal, there is some sequence $\{n_i\} \subset \mathbb{Z}$ such that $T^{n_i}x_3 \to a, i \to \infty$. Without loss of generality, assume that $T^{n_i}x_4 \to x'_4, i \to \infty$ for some $x'_4 \in X$. Since $\pi(a) = y$, $\pi(x'_4) = y$ too. So

$$(\mathrm{id} \times \mathrm{id} \times T \times T)^{n_i}(x_1, x_2, x_3, x_4) \to (x_1, x_2, a, x'_4), \quad i \to \infty. \tag{4.1}$$

Since $(X, T)$ is minimal, there is some sequence $\{m_i\} \subset \mathbb{Z}$ such that $T^{m_i}x'_4 \to b, i \to \infty$. Without loss of generality, assume that $T^{m_i}x_2 \to x'_2, i \to \infty$ for some $x'_2 \in X$. Since $\pi(b) = y$, $\pi(x'_2) = y$ too. So

$$(\mathrm{id} \times T \times \mathrm{id} \times T)^{m_i}(x_1, x_2, a, x'_4) \to (x_1, x'_2, a, b), \quad i \to \infty. \tag{4.2}$$

Hence by (4.1) and (4.2),

$$(x_1, x'_2, a, b) \in \overline{\mathcal{F}^{[2]}}(\mathbf{x}). \tag{4.3}$$

Thus for all $n \in \mathbb{Z}$,

$$(x_1, x'_2, T^n a, T^n b) = (\mathrm{id} \times \mathrm{id} \times T \times T)^n(x_1, x'_2, a, b) \in \overline{\mathcal{F}^{[2]}}(\mathbf{x}).$$

Since $(a, b)$ is a transitive point of $(R_\pi, T \times T)$, it follows that

$$\{x_1\} \times \{x'_2\} \times \pi^{-1}(y) \times \pi^{-1}(y) \subset \{x_1\} \times \{x'_2\} \times R_\pi \subset \overline{\mathcal{F}^{[2]}}(\mathbf{x}). \tag{4.4}$$

Now we show that

$$\{x_1\} \times \pi^{-1}(y) \times \pi^{-1}(y) \times \pi^{-1}(y) = \{x_1\} \times (\pi^{-1}(y))^3 \subset \overline{\mathcal{F}^{[2]}}(\mathbf{x}). \tag{4.5}$$

For any $z \in \pi^{-1}(y)$, there is a sequence $k_i \subset \mathbb{Z}$ such that $T^{k_i}x'_2 \to z, i \to \infty$. Thus $T^{k_i}y = T^{k_i}\pi(x'_2) = \pi(T^{k_i}x'_2) \to \pi(z) = y, i \to \infty$. Since $\pi$ is open, we have $T^{k_i}\pi^{-1}(y) = \pi^{-1}(T^{k_i}y) \to \pi^{-1}(y), i \to \infty$ in the Hausdorff metric. Thus

$$\{x_1\} \times \{z\} \times \pi^{-1}(y)^2 \subset \overline{\bigcup_{i=1}^{\infty}(\mathrm{id} \times T \times \mathrm{id} \times T)^{k_i}(\{x_1\} \times \{x'_2\} \times \pi^{-1}(y)^2)} \subset \overline{\mathcal{F}^{[2]}}(\mathbf{x}).$$

Since $z$ is arbitrary, we have (4.5). Similarly, we have $\left(\pi^{-1}(y)\right)^4 \subset \mathbf{Q}^{[2]}(X)$ and we are done for $d = 2$.

Now assume we have (1) and (2) for $d-1$ already, and show the case for $d$. Let $\mathbf{x} \in X^{[d]}$ with $x_\emptyset = x$ and $\pi^{[d]}(\mathbf{x}) = y^{[d]}$.

Let $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$. Since $\pi$ is weakly mixing, $(R_\pi^{2^{d-1}}, T^{[d-1]})$ is transitive. By Theorem 4.3 there is $\mathbf{a} \in R_\pi^{2^{d-1}}$ which is a transitive point of $(R_\pi^{2^{d-1}}, T^{[d-1]})$ and $\pi^{[d-1]}(\mathbf{a}) = y^{[d-1]}$. Without loss of generality, we may assume that $a_\emptyset = x''_\emptyset$ (i.e. the first coordinate of $\mathbf{a}$ is equal to that of $\mathbf{x}''$), otherwise we may use the face transformation $\mathrm{id}^{[d-1]} \times T^{[d-1]}$ to find some point in $\overline{\mathcal{F}^{[d]}}(\mathbf{x})$ satisfying this property.

By the induction assumption for $d-1$,

$$\mathbf{a} \in \{x''_\emptyset\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}-1} \subset \overline{\mathcal{F}^{[d-1]}}(\mathbf{x}'').$$

Hence there is some sequence $F_k \in \mathcal{F}^{[d-1]}$ and $\mathbf{w} \in X^{[d-1]}$ such that

$$F_k \times F_k(\mathbf{x}) = F_k \times F_k(\mathbf{x}', \mathbf{x}'') \to (\mathbf{w}, \mathbf{a}), \quad k \to \infty.$$

Especially $(\mathbf{w},\mathbf{a}) \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. Since $\pi^{[d]}(\mathbf{x}) = y^{[d]}$ and $\pi^{[d-1]}(\mathbf{a}) = y^{[d-1]}$, it is easy to verify that $\pi^{[d-1]}(\mathbf{w}) = y^{[d-1]}$ and $w_\emptyset = x$. Note that

$$(T_d^{[d]})^n(\mathbf{w},\mathbf{a}) = (\mathbf{w}, (T^{[d-1]})^n\mathbf{a}) \in \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

We have

$$\{\mathbf{w}\} \times \mathcal{O}(\mathbf{a}, T^{[d-1]}) \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

And so

$$\{\mathbf{w}\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}} \subset \{\mathbf{w}\} \times R_\pi^{2^{d-1}} = \{\mathbf{w}\} \times \overline{\mathcal{O}(\mathbf{a}, T^{[d-1]})} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}). \tag{4.6}$$

By the induction assumption for $d-1$, for $\mathbf{w}$ we have

$$\{x\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}-1} \subset \overline{\mathcal{F}^{[d-1]}}(\mathbf{w}). \tag{4.7}$$

Hence for all $\mathbf{z} \in \{x\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}-1}$, there is some sequence $\{H_k\} \subset \mathcal{F}^{[d-1]}$ such that $H_k\mathbf{w} \to \mathbf{z}$, $k \to \infty$. Since $\pi$ is open, similar to the proof of (4.5), we have that $H_k\left(\pi^{-1}(y)\right)^{2^{d-1}} \to \left(\pi^{-1}(y)\right)^{2^{d-1}}$, $k \to \infty$. Hence

$$H_k \times H_k \left(\{\mathbf{w}\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}}\right) \to \{\mathbf{z}\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}}, \quad k \to \infty.$$

Since $H_k \times H_k \in \mathcal{F}^{[d]}$ and $\mathbf{z} \in \{x\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}-1}$ is arbitrary, it follows from (4.6) that

$$\{x\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}-1} \times \left(\pi^{-1}(y)\right)^{2^{d-1}} = \{x\} \times \left(\pi^{-1}(y)\right)^{2^d-1} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

Now by this fact it is easy to get $\left(\pi^{-1}(y)\right)^{[d]} = \left(\pi^{-1}(y)\right)^{2^d} \subset \mathbf{Q}^{[d]}(X)$. So (1) and (2) hold for the case $d$. This completes the proof. $\square$

In fact with a small modification of the above proof one can show that $R_\pi^{2^d} \subset \mathbf{Q}^{[d]}(X)$. We do not know if $\{x\} \times R_\pi^{2^d-1} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x})$.

### 4.3. *Proof of Theorem 3.1-(1)*

A subset $S \subseteq \mathbb{Z}$ is a *central set* if there exists a system $(X,T)$, a point $x \in X$ and a minimal point $y \in X$ proximal to $x$, and a neighborhood $U_y$ of $y$ such that $N(x,U_y) \subset S$, where $N(x,U_y) = \{n \in \mathbb{Z} : T^n x \in U_y\}$. It is known that any central set is an IP-set [12, Proposition 8.10].

**Proposition 4.5.** *Let $\pi : (X,T) \to (Y,T)$ be a proximal extension of minimal systems and $d \in \mathbb{N}$. If $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $y \in Y$, then $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$.*

**Proof.** It is sufficient to show that for any $\mathbf{x} \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$, we have $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. Let $y = \pi(x)$. Then by the assumption $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal. Note that $\pi^{[d]} : (\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]}) \to (\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is a factor map. Especially there is some $\mathbf{y} \in \overline{\mathcal{F}^{[d]}}(y^{[d]})$ such that $\pi^{[d]}(\mathbf{x}) = \mathbf{y}$.

Since $\mathbf{y} \in \overline{\mathcal{F}^{[d]}}(y^{[d]})$ and $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal, there is some sequence $F_k \in \mathcal{F}^{[d]}$ such that

$$F_k\mathbf{y} \to y^{[d]}, \quad k \to \infty.$$

Without loss of generality, we may assume that

$$F_k\mathbf{x} \to \mathbf{z}, \quad k \to \infty. \tag{4.8}$$

Then $\pi^{[d]}(\mathbf{z}) = \lim_k \pi^{[d]}(F_k\mathbf{x}) = \lim_k F_k\mathbf{y} = y^{[d]}$. That is,

$$z_\epsilon \in \pi^{-1}(y), \quad \forall \epsilon \in \{0,1\}^d.$$

Since $\pi$ is proximal, by Lemma 4.1 there is some $p \in E(X,T)$ such that

$$pz_\epsilon = px = x, \quad \forall \epsilon \in \{0,1\}^d.$$

That is, $p\mathbf{z} = x^{[d]} = px^{[d]}$, i.e. $\mathbf{z}$ is proximal to $x^{[d]}$ under the action of $T^{[d]}$. Since $x^{[d]}$ is $T^{[d]}$-minimal, for any neighborhood $\mathbf{U}$ of $x^{[d]}$,

$$N_{T^{[d]}}(\mathbf{z}, \mathbf{U}) = \{n \in \mathbb{Z} : (T^{[d]})^n\mathbf{z} \in \mathbf{U}\}$$

is a central set and hence contains some IP set $FS(\{p_i\}_{i=1}^\infty)$. Particularly,

$$FS(\{p_i\}_{i=1}^d) \subseteq N_{T^{[d]}}(\mathbf{z}, \mathbf{U}).$$

This means for all $\epsilon \in \{0,1\}^d \setminus \{\mathbf{0}\}$,

$$(T^{[d]})^{\mathbf{p}\cdot\epsilon}\mathbf{z} \in \mathbf{U},$$

where $\mathbf{p} = (p_1, p_2, \dots, p_d) \in \mathbb{Z}^d$. Especially,

$$(T^{\mathbf{p}\cdot\epsilon} z_\epsilon)_{\epsilon\in\{0,1\}^d} \in \mathbf{U}.$$

In other words, we have

$$(T_1^{[d]})^{p_1}(T_2^{[d]})^{p_2}\dots(T_d^{[d]})^{p_d}\mathbf{z} \in \mathbf{U}.$$

Since $\mathbf{U}$ is arbitrary, we have that $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{z})$. Combining with (4.8), we have

$$x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

Thus $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal. This completes the proof. $\square$

**Proposition 4.6.** *Let $\pi : (X,T) \to (Y,T)$ be a distal extension of minimal systems and $d \in \mathbb{N}$. If $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $y \in Y$, then $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$.*

**Proof.** It follows from Lemma 4.2, since it is easy to check that $\pi^{[d]} : (\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]}) \to (\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is a distal extension. $\square$

**Proposition 4.7.** *Let $\pi : (X,T) \to (Y,T)$ be a RIC weakly mixing extension of minimal systems and $d \in \mathbb{N}$. If $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $y \in Y$, then $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$.*

**Proof.** It is sufficient to show that for any $\mathbf{x} \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$, we have $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. Let $y = \pi(x)$. Then by the assumption $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal. Note that $\pi^{[d]} : (\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]}) \to (\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is a factor map. Let $\mathbf{y} \in \overline{\mathcal{F}^{[d]}}(y^{[d]})$ such that $\pi^{[d]}(\mathbf{x}) = \mathbf{y}$.

Since $\mathbf{y} \in \overline{\mathcal{F}^{[d]}}(y^{[d]})$ and $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal, there is some sequence $F_k \in \mathcal{F}^{[d]}$ such that

$$F_k \mathbf{y} \to y^{[d]}, \quad k \to \infty.$$

Without loss of generality, we may assume that

$$F_k \mathbf{x} \to \mathbf{z}, \quad k \to \infty. \tag{4.9}$$

Then $\pi^{[d]}(\mathbf{z}) = \lim_k \pi^{[d]}(F_k \mathbf{x}) = \lim_k F_k \mathbf{y} = y^{[d]}$. By Lemma 4.4

$$x^{[d]} \in \{x\} \times \left(\pi^{-1}(y)\right)^{2^d-1} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{z}).$$

Together with (4.9), we have $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. This completes the proof. $\square$

**Proof of Theorem 3.1-(1).** By the structure Theorem A.4, we have the following diagram, where $Y_\infty$ is a strictly PI-system, $\phi$ is RIC weakly mixing extension and $\pi$ is proximal.

$$\begin{array}{ccc} X_\infty & \xrightarrow{\pi} & X \\ \big\downarrow{\scriptstyle\phi} & & \\ Y_\infty & & \end{array}$$

Since the inverse limit of minimal systems is minimal, it follows from Propositions 4.5 and 4.6 that the result holds for $Y_\infty$. By Proposition 4.7 it also holds for $X_\infty$. Since the factor of a minimal system is always minimal, it is easy to see that we have the theorem for $X$. $\square$

### 4.4. *Minimality of* $(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$

We will need the following theorem mentioned in [23], where no proof is included. We give a proof (due to Glasner–Ellis) here for completeness. Note one can also prove this result using the method in the previous subsection.

**Proposition 4.8.** *Let $(X, T)$ be a minimal system and let $d \geq 1$ be an integer. Let $A$ be a $T^{[d]}$-minimal subset of $X^{[d]}$ and set $N = \overline{\mathcal{O}(A, \mathcal{F}^{[d]})} = \mathrm{cl}\big(\bigcup\{SA : S \in \mathcal{F}^{[d]}\}\big)$. Then $(N, \mathcal{G}^{[d]})$ is a minimal system, and $\mathcal{F}^{[d]}$-minimal points are dense in $N$.*

**Proof.** The proof is similar to the one in [17]. Let $E = E(N, \mathcal{G}^{[d]})$ be the enveloping semigroup of $(N, \mathcal{G}^{[d]})$. Let $\pi_\epsilon : N \to X$ be the projection of $N$ on the $\epsilon$-th component, $\epsilon \in \{0,1\}^d$. We consider the action of the group $\mathcal{G}^{[d]}$ on the $\epsilon$-th component via the representation $T^{[d]} \mapsto T$ and

$$T_j^{[d]} \mapsto \begin{cases} T, & j \in \epsilon; \\ \mathrm{id}, & j \notin \epsilon. \end{cases}$$

With respect to this action of $\mathcal{G}^{[d]}$ on $X$ the map $\pi_\epsilon$ is a factor map $\pi_\epsilon : (N, \mathcal{G}^{[d]}) \to (X, \mathcal{G}^{[d]})$. Let $\pi_\epsilon^* : E(N, \mathcal{G}^{[d]}) \to E(X, \mathcal{G}^{[d]})$ be the corresponding homomorphism of enveloping semigroups. Notice that for this action of $\mathcal{G}^{[d]}$ on $X$ clearly $E(X, \mathcal{G}^{[d]}) = E(X, T)$ as subsets of $X^X$.

Let now $u \in E(N, T^{[d]})$ be any minimal idempotent in the enveloping semigroup of $(N, T^{[d]})$. Choose $v$ a minimal idempotent in the closed left ideal $E(N, \mathcal{G}^{[d]})u$. Then $vu = v$. Set for each $\epsilon \in \{0,1\}^d$, $u_\epsilon = \pi_\epsilon^* u$ and $v_\epsilon = \pi_\epsilon^* v$. We want to show that also $uv = u$. Note that as an element of $E(N, \mathcal{G}^{[d]})$ is determined by its projections, thus it suffices to show that for each $\epsilon \in \{0,1\}^d$, $u_\epsilon v_\epsilon = u_\epsilon$.

Since for each $\epsilon \in \{0,1\}^d$ the map $\pi_\epsilon^*$ is a semigroup homomorphism, we have $v_\epsilon u_\epsilon = v_\epsilon$ as $vu = v$. In particular we deduce that $v_\epsilon$ is an element of the minimal left ideal of $E(X,T)$ which contains $u_\epsilon$. In turn this implies

$$u_\epsilon v_\epsilon = u_\epsilon v_\epsilon u_\epsilon = u_\epsilon;$$

and it follows that indeed $uv = u$. Thus $u$ is an element of the minimal left ideal of $E(N, \mathcal{G}^{[d]})$ which contains $v$, and therefore $u$ is a minimal idempotent of $E(N, \mathcal{G}^{[d]})$.

Now let $x$ be an arbitrary point in $A$ and let $u \in E(N, T^{[d]})$ be a minimal idempotent with $ux = x$. By the above argument, $u$ is also a minimal idempotent of $E(N, \mathcal{G}^{[d]})$, whence $N = \overline{\mathcal{O}(A, \mathcal{F}^{[d]})} = \overline{\mathcal{O}(x, \mathcal{G}^{[d]})}$ is $\mathcal{G}^{[d]}$-minimal.

Finally, we show $\mathcal{F}^{[d]}$-minimal points are dense in $N$. Let $B \subseteq N$ be an $\mathcal{F}^{[d]}$-minimal subset. Then $\mathcal{O}(B, T^{[d]}) = \bigcup\{(T^{[d]})^n B : n \in \mathbb{Z}\}$ is a $\mathcal{G}^{[d]}$-invariant subset of $N$. Since $(N, \mathcal{G}^{[d]})$ is minimal, $\mathcal{O}(B, T^{[d]})$ is dense in $N$. Note that every point in $\mathcal{O}(B, T^{[d]})$ is $\mathcal{F}^{[d]}$-minimal, hence the proof is completed. □

Setting $A = \Delta^{[d]}$ we have the following.

**Corollary 4.9.** *Let $(X,T)$ be a minimal system and let $d \geq 1$ be an integer. Then $(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$ is a minimal system, and $\mathcal{F}^{[d]}$-minimal points are dense in $\mathbf{Q}^{[d]}$.*

## 5. Proof of Theorem 3.1-(2)

In this section we prove Theorem 3.1-(2). That is, we show that $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is the unique $\mathcal{F}^{[d]}$-minimal subset in $\mathbf{Q}^{[d]}[x]$ for all $x \in X$.

### *5.1. A useful lemma*

The following lemma is a key step to show the uniqueness of minimal sets in $\mathbf{Q}^{[d]}[x]$ for $x \in X$. Unlike the case when $(X,T)$ is minimal distal, we need to use the enveloping semigroup theory.

**Lemma 5.1.** *Let $(X,T)$ be a minimal system and let $d \geq 1$ be an integer. If $(x^{[d-1]}, \mathbf{w}) \in \mathbf{Q}^{[d]}(X)$ for some $\mathbf{w} \in X^{[d-1]}$ and it is $\mathcal{F}^{[d]}$-minimal, then*

$$(x^{[d-1]}, \mathbf{w}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]}).$$

**Proof.** Since $(x^{[d-1]}, \mathbf{w}) \in \mathbf{Q}^{[d]}(X)$ and $(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$ is a minimal system by Corollary 4.9, $(x^{[d-1]}, \mathbf{w})$ is in the $\mathcal{G}^{[d]}$-orbit closure of $x^{[d]}$, i.e. there are sequences $\{n_k\}_k, \{n_k^1\}_k, \ldots, \{n_k^d\}_k \subseteq \mathbb{Z}$ such that

$$(T_d^{[d]})^{n_k}(T_1^{[d]})^{n_k^1}\ldots(T_{d-1}^{[d]})^{n_k^{d-1}}(T^{[d]})^{n_k^d}(x^{[d-1]}, x^{[d-1]}) \to (x^{[d-1]}, \mathbf{w}), \quad k \to \infty.$$

Let

$$\mathbf{a_k} = (T_1^{[d-1]})^{n_k^1}\ldots(T_{d-1}^{[d-1]})^{n_k^{d-1}}(T^{[d-1]})^{n_k^d}(x^{[d-1]}),$$

then the above limit can be rewritten as

$$(T_d^{[d]})^{n_k}(\mathbf{a_k}, \mathbf{a_k}) = (\mathrm{id}^{[d-1]} \times T^{[d-1]})^{n_k}(\mathbf{a_k}, \mathbf{a_k}) \to (x^{[d-1]}, \mathbf{w}), \quad k \to \infty. \tag{5.1}$$

Let

$$\pi_1 : (X^{[d]}, \mathcal{F}^{[d]}) \to (X^{[d-1]}, \mathcal{F}^{[d]}), \qquad (\mathbf{x}', \mathbf{x}'') \mapsto \mathbf{x}',$$
$$\pi_2 : (X^{[d]}, \mathcal{F}^{[d]}) \to (X^{[d-1]}, \mathcal{F}^{[d]}), \qquad (\mathbf{x}', \mathbf{x}'') \mapsto \mathbf{x}'',$$

be projections to the first $2^{d-1}$ coordinates and last $2^{d-1}$ coordinates respectively. For $\pi_1$ we consider the action of the group $\mathcal{F}^{[d]}$ on $X^{[d-1]}$ via the representation $T_i^{[d]} \mapsto T_i^{[d-1]}$ for $1 \le i \le d-1$ and $T_d^{[d]} \mapsto \mathrm{id}^{[d-1]}$. For $\pi_2$ we consider the action of the group $\mathcal{F}^{[d]}$ on $X^{[d-1]}$ via the representation $T_i^{[d]} \mapsto T_i^{[d-1]}$ for $1 \le i \le d-1$ and $T_d^{[d]} \mapsto T^{[d-1]}$.

Denote the corresponding semigroup homomorphisms of enveloping semigroups by

$$\pi_1^* : E(X^{[d]}, \mathcal{F}^{[d]}) \to E(X^{[d-1]}, \mathcal{F}^{[d]}), \qquad \pi_2^* : E(X^{[d]}, \mathcal{F}^{[d]}) \to E(X^{[d-1]}, \mathcal{F}^{[d]}).$$

Notice that for this action of $\mathcal{F}^{[d]}$ on $X^{[d-1]}$ clearly

$$\pi_1^*(E(X^{[d]}, \mathcal{F}^{[d]})) = E(X^{[d-1]}, \mathcal{F}^{[d-1]}) \quad \text{and} \quad \pi_2^*(E(X^{[d]}, \mathcal{F}^{[d]})) = E(X^{[d-1]}, \mathcal{G}^{[d-1]})$$

as subsets of $(X^{[d-1]})^{X^{[d-1]}}$. Thus for any $p \in E(X^{[d]}, \mathcal{F}^{[d]})$ and $\mathbf{x} \in X^{[d]}$, we have

$$p\mathbf{x} = p(\mathbf{x}', \mathbf{x}'') = (\pi_1^*(p)\mathbf{x}', \pi_2^*(p)\mathbf{x}'').$$

Now fix a minimal left ideal $\mathbf{L}$ of $E(X^{[d]}, \mathcal{F}^{[d]})$. By (5.1), $\mathbf{a_k} \to x^{[d-1]}, k \to \infty$. Since $(\mathbf{Q}^{[d-1]}(X), \mathcal{G}^{[d-1]})$ is minimal, there exists $p_k \in \mathbf{L}$ such that $\mathbf{a_k} = \pi_2^*(p_k)x^{[d-1]}$. Without loss of generality, we assume that $p_k \to p \in \mathbf{L}$. Then

$$\pi_2^*(p_k)x^{[d-1]} = \mathbf{a_k} \to x^{[d-1]} \quad \text{and} \quad \pi_2^*(p_k)x^{[d-1]} \to \pi_2^*(p)x^{[d-1]}.$$

Hence

$$\pi_2^*(p)x^{[d-1]} = x^{[d-1]}. \tag{5.2}$$

Since $\mathbf{L}$ is a minimal left ideal and $p \in \mathbf{L}$, by Proposition A.1 there exists a minimal idempotent $v \in J(\mathbf{L})$ such that $vp = p$. Then we have

$$\pi_2^*(v)x^{[d-1]} = \pi_2^*(v)\pi_2^*(p)x^{[d-1]} = \pi_2^*(vp)x^{[d-1]} = \pi_2^*(p)x^{[d-1]} = x^{[d-1]}.$$

Let

$$F = \mathfrak{G}(\overline{\mathcal{F}^{[d-1]}}(x^{[d-1]}), x^{[d-1]}) = \{\alpha \in v\mathbf{L} : \pi_2^*(\alpha)x^{[d-1]} = x^{[d-1]}\}$$

be the Ellis group. Then $F$ is a subgroup of the group $v\mathbf{L}$. By (5.2), we have that $p \in F$.

Since $F$ is a group and $p \in F$, we have

$$pFx^{[d]} = Fx^{[d]} \subset \pi_2^{-1}(x^{[d-1]}). \tag{5.3}$$

Since $vx^{[d]} \in Fx^{[d]}$, there is some $\mathbf{x_0} \in Fx^{[d]}$ such that $vx^{[d]} = p\mathbf{x_0}$. Set $\mathbf{x_k} = p_k\mathbf{x_0}$. Then

$$\mathbf{x_k} = p_k\mathbf{x_0} \to p\mathbf{x_0} = vx^{[d]} = (\pi_1^*(v)x^{[d-1]}, x^{[d-1]}), \quad k \to \infty,$$

and

$$\pi_2(\mathbf{x_k}) = \pi_2(p_k\mathbf{x_0}) = \pi_2^*(p_k)x^{[d-1]} = \mathbf{a_k} \to x^{[d-1]}, \quad k \to \infty.$$

Let $\mathbf{x_k} = (\mathbf{b_k}, \mathbf{a_k}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$. Then $\lim_k \mathbf{b_k} = \pi_1^*(v)x^{[d-1]}$.

By (5.1), we have $(T^{[d-1]})^{n_k}\mathbf{a_k} \to \mathbf{w}, k \to \infty$. Hence

$$(\mathrm{id}^{[d-1]} \times T^{[d-1]})^{n_k}(\mathbf{b_k}, \mathbf{a_k}) = (\mathbf{b_k}, (T^{[d-1]})^{n_k}\mathbf{a_k}) \to (\pi_1^*(v)x^{[d-1]}, \mathbf{w}), \quad k \to \infty. \tag{5.4}$$

Since $\mathrm{id}^{[d-1]} \times T^{[d-1]} = T_d^{[d]} \in \mathcal{F}^{[d]}$ and $(\mathbf{b_k}, \mathbf{a_k}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$, we have

$$(\pi_1^*(v)x^{[d-1]}, \mathbf{w}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]}). \tag{5.5}$$

Since $(x^{[d-1]}, \mathbf{w})$ is $\mathcal{F}^{[d]}$ minimal by assumption, by Proposition A.2 there is some minimal idempotent $u \in J(\mathbf{L})$ such that

$$u(x^{[d-1]}, \mathbf{w}) = (\pi_1^*(u)x^{[d-1]}, \pi_2^*(u)\mathbf{w}) = (x^{[d-1]}, \mathbf{w}).$$

Since $u, v \in \mathbf{L}$ are minimal idempotents in the same minimal left ideal $\mathbf{L}$, we have $uv = u$ by Proposition A.1. Thus

$$\begin{aligned} u(\pi_1^*(v)x^{[d-1]}, \mathbf{w}) &= (\pi_1^*(u)\pi_1^*(v)x^{[d-1]}, \pi_2^*(u)\mathbf{w}) \\ &= (\pi_1^*(uv)x^{[d-1]}, \mathbf{w}) = (\pi_1^*(u)x^{[d-1]}, \mathbf{w}) = (x^{[d-1]}, \mathbf{w}). \end{aligned}$$

By (5.5), we have

$$(x^{[d-1]}, \mathbf{w}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]}).$$

The proof is completed. □

*5.2. Proof of Theorem 3.1-(2)*

Let $(X, T)$ be a system and $x \in X$. Recall

$$\mathbf{Q}^{[d]}[x] = \{\mathbf{z} \in \mathbf{Q}^{[d]}(X) : z_\emptyset = x\}.$$

With the help of Lemma 5.1 we have the following.

**Proposition 5.2.** *Let $(X, T)$ be a minimal system and let $d \geq 1$ be an integer. If $\mathbf{x} \in \mathbf{Q}^{[d]}[x]$, then*

$$x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

*Especially, $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is the unique $\mathcal{F}^{[d]}$-minimal subset in $\mathbf{Q}^{[d]}[x]$.*

**Proof.** It is sufficient to show the following claim:

**S(d):** *If $\mathbf{x} \in \mathbf{Q}^{[d]}[x]$, then there exists a sequence $F_k \in \mathcal{F}^{[d]}$ such that $F_k(\mathbf{x}) \to x^{[d]}$.*

The case $\mathbf{S(1)}$ is trivial. To make the idea clear, we show the case $d = 2$. Let $(x, a, b, c) \in \mathbf{Q}^{[2]}(X)$. We may assume that $(x, a, b, c)$ is $\mathcal{F}^{[2]}$-minimal, or we replace it by some $\mathcal{F}^{[2]}$-minimal point in its $\mathcal{F}^{[2]}$ orbit closure. Since $(X, T)$ is minimal, there is a sequence $\{n_k\} \subset \mathbb{Z}$ such that $T^{n_k}a \to x$. Without loss of generality we assume $T^{n_k}c \to c'$. Then we have

$$(T_1^{[2]})^{n_k}(x, a, b, c) = (\mathrm{id} \times T \times \mathrm{id} \times T)^{n_k}(x, a, b, c) \to (x, x, b, c'), \quad k \to \infty.$$

Since $(x, a, b, c)$ is $\mathcal{F}^{[2]}$-minimal, $(x, x, b, c')$ is also $\mathcal{F}^{[2]}$-minimal. By Lemma 5.1, $(x, x, b, c') \in \overline{\mathcal{F}^{[2]}}(x^{[2]})$. Together with $\mathrm{id} \times T \times \mathrm{id} \times T = T_1^{[2]} \in \mathcal{F}^{[2]}$ and the minimality of the system $(\overline{\mathcal{F}^{[2]}}(x^{[2]}), \mathcal{F}^{[2]})$ (Theorem 3.1-(1)), it is easy to see there exists a sequence $F_k \in \mathcal{F}^{[2]}$ such that $F_k(x, a, b, c) \to x^{[2]}$. Hence we have $\mathbf{S(2)}$.

Now we assume $\mathbf{S(d)}$ holds for $d \geq 1$. Let $\mathbf{x} \in \mathbf{Q}^{[d+1]}[x]$. We may assume that $\mathbf{x}$ is $\mathcal{F}^{[d+1]}$-minimal, or we replace it by some $\mathcal{F}^{[d+1]}$-minimal point in its $\mathcal{F}^{[d+1]}$-orbit closure. Let

$\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$, where $\mathbf{x}', \mathbf{x}'' \in X^{[d]}$. Then $\mathbf{x}' \in \mathbf{Q}^{[d]}[x]$. By $\mathbf{S(d)}$, there is a sequence $F_k \in \mathcal{F}^{[d]}$ such that $F_k\mathbf{x}' \to x^{[d]}$. Without loss of generality, we assume that $F_k\mathbf{x}'' \to \mathbf{w}$, $k \to \infty$. Then

$$(F_k \times F_k)\mathbf{x} = (F_k \times F_k)(\mathbf{x}', \mathbf{x}'') \to (x^{[d]}, \mathbf{w}) \in \mathbf{Q}^{[d+1]}(X), \quad k \to \infty.$$

Since $F_k \times F_k \in \mathcal{F}^{[d+1]}$ and $\mathbf{x}$ is $\mathcal{F}^{[d+1]}$-minimal, $(x^{[d]}, \mathbf{w})$ is also $\mathcal{F}^{[d+1]}$-minimal. By Lemma 5.1, $(x^{[d]}, \mathbf{w}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$. Since $(\overline{\mathcal{F}^{[d+1]}}(x^{[d+1]}), \mathcal{F}^{[d+1]})$ is minimal by Theorem 3.1-(1), we have $x^{[d+1]}$ is in the $\mathcal{F}^{[d+1]}$-orbit closure of $\mathbf{x}$. Hence we have $\boldsymbol{S(d+1)}$, and the proof of claim is completed.

Since $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{Q}^{[d]}[x]$ and $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal, it is easy to see that $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ intersects all $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}[x]$ and hence it is the unique $\mathcal{F}^{[d]}$-minimal set in $\mathbf{Q}^{[d]}[x]$. The proof is completed. □

## 6. Lifting $\mathbf{RP}^{[d]}$ from factors to extensions

In this section, first we give some equivalent conditions for $\mathbf{RP}^{[d]}$, and give the proof of Theorem 3.10-(2), i.e. lifting $\mathbf{RP}^{[d]}$ from factors to extensions.

### *6.1. Equivalent conditions for* $\mathbf{RP}^{[d]}$

In this subsection we collect some equivalent conditions for $\mathbf{RP}^{[d]}$.

**Proposition 6.1.** *Let $(X, T)$ be a minimal system and $d \in \mathbb{N}$. Then the following conditions are equivalent:*

(1) $(x, y) \in \mathbf{RP}^{[d]}$;
(2) $(x, y, y, \ldots, y) = (x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$;
(3) $(x, x_*^{[d]}, y, x_*^{[d]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$.

**Proof.** By Theorem 3.4, we have (1) $\Leftrightarrow$ (2). By Lemma 2.5 we have (3) $\Rightarrow$ (1). Now show (2) $\Rightarrow$ (3).

If (2) holds, then $(x, y, y, \ldots, y) = (x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$ and $(x, y) \in \mathbf{RP}^{[d]}$. Since $(x, y) \in \mathbf{RP}^{[d]} \subset \mathbf{RP}^{[d-1]}$, $(x, y_*^{[d]}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$. By Theorem 3.1, $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal. So there is some sequence $F_k \in \mathcal{F}^{[d]}$ such that $F_k(x, y_*^{[d]}) \to x^{[d]}$, $k \to \infty$. Then

$$F_k \times F_k(x, y_*^{[d]}, y, y_*^{[d]}) \to (x, x_*^{[d]}, y, x_*^{[d]}), \quad k \to \infty.$$

Thus we have (3), and the proof is completed. □

**Lemma 6.2.** *Let $(X, T)$ be a minimal system. Then $(x, y) \in \mathbf{RP}^{[d]}(X)$ if and only if $(x, x, \ldots, x, y) \in \mathbf{Q}^{[d+1]}$.*

**Proof.** If $(x, y) \in \mathbf{RP}^{[d]}$, then by Proposition 6.1, we have $(x, x_*^{[d]}, y, x_*^{[d]}) = (x^{[d]}, y, x_*^{[d]}) \in \mathbf{Q}^{[d+1]}$. Since $\mathbf{Q}^{[d+1]}$ is invariant under the Euclidean permutation of $X^{[d+1]}$, we have $(x, x, \ldots, x, y) \in \mathbf{Q}^{[d+1]}$.

Conversely, assume that $(x, x, \ldots, x, y) \in \mathbf{Q}^{[d+1]}$. Since $\mathbf{Q}^{[d+1]}$ is invariant under the Euclidean permutation of $X^{[d+1]}$, we have $(x, x_*^{[d]}, y, x_*^{[d]}) \in \mathbf{Q}^{[d+1]}$. This means that $(x, y) \in \mathbf{RP}^{[d]}$ by Lemma 2.5. □

### 6.2. *Lifting* $\mathbf{RP}^{[d]}$ *from factors to extensions*

In this section we will show Theorem 3.10-(2). First we need a lemma.

**Lemma 6.3.** *Let* $\pi : (X, T) \to (Y, T)$ *be an extension of minimal systems. If* $(y_1, y_2) \in \mathbf{P}(Y, T)$ *and* $x_1 \in \pi^{-1}(y_1)$ *then there exists* $x_2 \in \pi^{-1}(y_2)$ *such that* $(x_1, x_2) \in \mathbf{P}(X, T)$.

**Proof.** Since $(y_1, y_2) \in \mathbf{P}(Y, T)$, by Proposition A.3 there is a minimal idempotent $u \in E(X, T)$ such that $uy_1 = uy_2 = y_2$. Let $x_2 = ux_1$, then $\pi(x_2) = uy_1 = y_2$. By Proposition A.3 $(x_1, x_2) \in \mathbf{P}(X, T)$ and $\pi \times \pi(x_1, x_2) = (y_1, y_2)$. $\square$

**Theorem 6.4.** *Let* $\pi : (X, T) \to (Y, T)$ *be an extension of minimal systems. If* $(y_1, y_2) \in \mathbf{RP}^{[d]}(Y)$*, then there is* $(x_1, x_2) \in \mathbf{RP}^{[d]}(X)$ *such that*

$$\pi \times \pi(x_1, x_2) = (y_1, y_2).$$

**Proof.** First we claim that it is sufficient to show the result when $(y_1, y_2)$ is a minimal point of $(Y \times Y, T \times T)$. As a matter of fact, by Proposition A.3 there is a minimal point $(y_1', y_2') \in \overline{\mathcal{O}((y_1, y_2), T \times T)}$ such that $(y_1', y_2')$ is proximal to $(y_1, y_2)$. Now $(y_1', y_2')$ is minimal and $(y_1', y_2') \in \mathbf{RP}^{[d]}(Y)$. If we have the claim already, then there is $(x_1', x_2') \in \mathbf{RP}^{[d]}(X)$ with $\pi \times \pi(x_1', x_2') = (y_1', y_2')$. Since $(y_1, y_1'), (y_2, y_2') \in \mathbf{P}(Y, T)$, then by Lemma 6.3 there are $x_1, x_2 \in X$ with $\pi \times \pi(x_1, x_2) = (y_1, y_2)$ such that $(x_1', x_1), (x_2', x_2) \in \mathbf{P}(X, T)$. This implies that $(x_1, x_2) \in \mathbf{RP}^{[d]}(X)$ by Theorem 3.5. Hence we have the result for the general case.

So we may assume that $(y_1, y_2)$ is a minimal point of $(Y \times Y, T \times T)$. Since the case $d = 2$ illustrates the idea of the proof better, we start from $d = 2$ (see Fig. 2). For the case $d = 1$, see Fig. 1.

Let $(y_1, y_2) \in \mathbf{RP}^{[2]}(Y)$, then by Proposition 6.1,

$$(y_1^{[2]}, y_2, (y_1^{[2]})_*) = (y_1, y_1, y_1, y_1, y_2, y_1, y_1, y_1) \in \overline{\mathcal{F}^{[3]}}(y_1^{[3]}).$$

So there is some sequence $F_k \in \mathcal{F}^{[3]}$ such that

$$F_k y_1^{[3]} \to (y_1^{[2]}, y_2, (y_1^{[2]})_*), \quad k \to \infty.$$

Take a point $x_1 \in \pi^{-1}(y_1)$. Without loss of generality, we may assume that

$$F_k x_1^{[3]} \to \mathbf{x} = (x_{000}, x_{100}, x_{010}, x_{110}, x_{001}, x_{101}, x_{011}, x_{111}), \quad k \to \infty.$$

Then $x_\emptyset = x_{000} = x_1$ and $\pi^{[3]}(\mathbf{x}) = (y_1^{[2]}, y_2, (y_1^{[2]})_*)$. Obviously, $\mathbf{x} \in \overline{\mathcal{F}^{[3]}}(x_1^{[3]})$.

By Proposition 5.2, there is some sequence $F_k^1 \in \mathcal{F}^{[2]}$ such that

$$F_k^1(x_{000}, x_{100}, x_{010}, x_{110}) \to x_1^{[2]} = (x_1, x_1, x_1, x_1), \quad k \to \infty.$$

We may assume that

$$F_k^1(x_{001}, x_{101}, x_{011}, x_{111}) \to (x_{001}, x_{101}', x_{011}', x_{111}'), \quad k \to \infty.$$

Note that $\pi^{[2]}(x_{001}, x_{101}, x_{011}, x_{111}) = \pi^{[2]}(x_{001}, x_{101}', x_{011}', x_{111}') = (y_2, (y_1^{[2]})_*)$. Since $F_k^1 \times F_k^1$ is an element of the group generated by $T_1^{[3]}$ and $T_2^{[3]}$ which is in $\mathcal{F}^{[3]}$, one has that

$$(x_1, x_1, x_1, x_1, x_{001}, x_{101}', x_{011}', x_{111}') \in \overline{\mathcal{F}^{[3]}}(x_1^{[3]}).$$

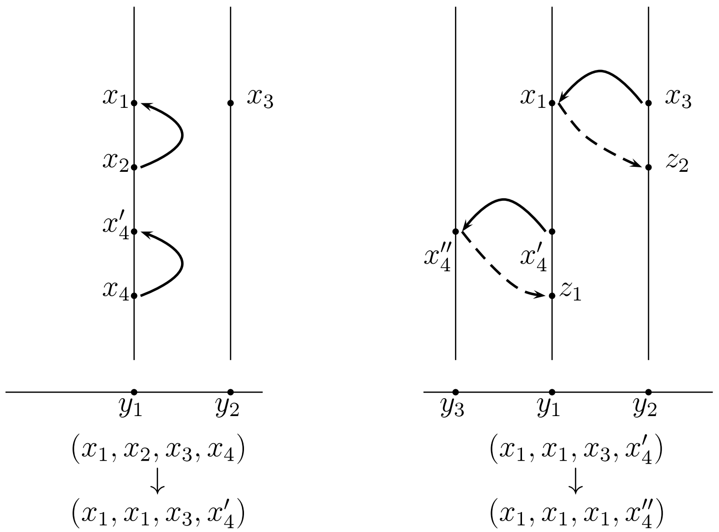


Fig. 1. The case $d = 1$.

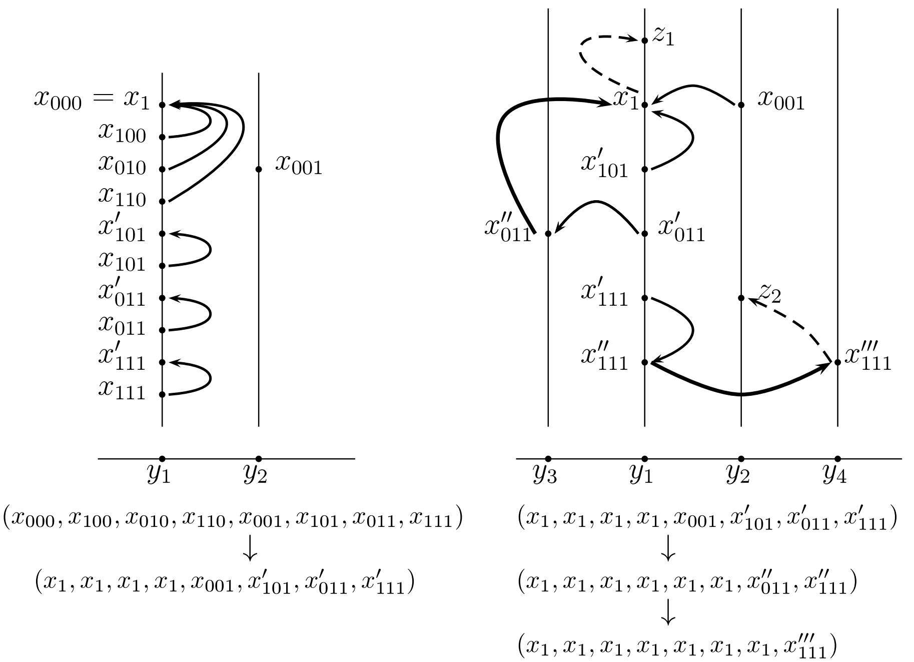


Fig. 2. The case $d = 2$.

By Proposition 5.2, there is some sequence $F_k^2 \in \mathcal{F}^{[2]}$ such that

$$F_k^2(x_1, x_1, x_{001}, x_{101}') \to x_1^{[2]} = (x_1, x_1, x_1, x_1), \quad k \to \infty.$$

We may assume that

$$F_k^2(x_1, x_1, x_{011}', x_{111}') \to (x_1, x_1, x_{011}'', x_{111}''), \quad k \to \infty.$$

Let $\pi^{[2]}(x_1, x_1, x''_{011}, x''_{111}) = (y_1, y_1, y_3, y_1)$ for some $y_3 \in Y$. It is easy to see that $(y_3, y_1) \in \overline{\mathcal{O}((y_1, y_2), T \times T)}$. Let $F_k^2 = (g_k^1, g_k^2, g_k^3, g_k^4)$. Since $(g_k^1, g_k^2, g_k^1, g_k^2, g_k^3, g_k^4, g_k^3, g_k^4)$ is an element of the group generated by $T_1^{[3]}$ and $T_3^{[3]}$ which is in $\mathcal{F}^{[3]}$, one has that

$$(x_1, x_1, x_1, x_1, x_1, x_1, x''_{011}, x''_{111}) \in \overline{\mathcal{F}^{[3]}}(x_1^{[3]}).$$

Again by Proposition 5.2, there is some sequence $F_k^3 \in \mathcal{F}^{[2]}$ such that

$$F_k^3(x_1, x_1, x_1, x''_{011}) \to x_1^{[2]} = (x_1, x_1, x_1, x_1), \quad k \to \infty.$$

We may assume that

$$F_k^3(x_1, x_1, x_1, x''_{111}) \to (x_1, x_1, x_1, x'''_{111}), \quad k \to \infty.$$

Let $\pi^{[2]}(x_1, x_1, x_1, x'''_{111}) = (y_1, y_1, y_1, y_4)$ for some $y_4 \in Y$. The it is easy to see that $(y_1, y_4) \in \overline{\mathcal{O}((y_3, y_1), T \times T)}$, and hence $(y_1, y_4) \in \overline{\mathcal{O}((y_1, y_2), T \times T)}$. Let $F_k^3 = (f_k^1, f_k^2, f_k^3, f_k^4)$. Then $(f_k^1, f_k^1, f_k^2, f_k^2, f_k^3, f_k^3, f_k^4, f_k^4)$ is an element of the group generated by $T_2^{[3]}$ and $T_3^{[3]}$ which is in $\mathcal{F}^{[3]}$, and one has that

$$(x_1, x_1, x_1, x_1, x_1, x_1, x_1, x'''_{111}) \in \overline{\mathcal{F}^{[3]}}(x_1^{[3]}).$$

By Lemma 6.2, $(x_1, x'''_{111}) \in \mathbf{RP}^{[d]}(X)$.

Since $(y_1, y_2)$ is $T \times T$-minimal, there is some $\{n_k\} \subset \mathbb{Z}$ such that $(T \times T)^{n_k}(y_1, y_4) \to (y_1, y_2)$, $k \to \infty$. Without loss of generality, we assume that

$$(T \times T)^{n_k}(x_1, x'''_{111}) \to (z_1, z_2), \quad k \to \infty.$$

Since $\mathbf{RP}^{[d]}(X)$ is closed and invariant, we have

$$(z_1, z_2) \in \overline{\mathcal{O}((x_1, x'''_{111}), T \times T)} \subset \mathbf{RP}^{[d]}(X).$$

And

$$\pi \times \pi(z_1, z_2) = \lim_k (T \times T)^{n_k}(\pi(x_1), \pi(x'''_{111})) = \lim_k (T \times T)^{n_k}(y_1, y_4) = (y_1, y_2).$$

This ends the proof of the case $d = 2$.

The idea of the proof in the general case is the following. For a point $\mathbf{x} \in \overline{\mathcal{F}^{[d+1]}}(x_1)$ we apply face transformations $F_1^k$ such that the first $2^d$-coordinates of $\mathbf{x}_1 = \lim F_1^k \mathbf{x}$ will be $x_1^{[d]}$. Then apply face transformations $F_2^k$ such that the first $2^d + 2^{d-1}$-coordinates of $\mathbf{x}_2 = \lim F_2^k \mathbf{x}_1$ will be $(x_1^{[d]}, x_1^{[d-1]})$. Repeating this process we get a point $((x_1^{[d+1]})_*, x_2) \in \overline{\mathcal{F}^{[d+1]}}(x_1)$ which implies that $(x_1, x_2) \in \mathbf{RP}^{[d]}(X)$. Then we use the same idea used in the proof when $d = 1, 2$ to trace back to find $(z_1, z_2)$. Here are the details.

Let $(y_1, y_2) \in \mathbf{RP}^{[d]}(Y)$, then by Proposition 6.1, $(y_1^{[d]}, y_2, (y_1^{[d]})_*) \in \overline{\mathcal{F}^{[d+1]}}(y_1^{[d+1]})$. So there is some sequence $F_k \in \mathcal{F}^{[d+1]}$ such that

$$F_k y_1^{[d+1]} \to (y_1^{[d]}, y_2, (y_1^{[d]})_*), \quad k \to \infty.$$

Take a point $x_1 \in \pi^{-1}(y_1)$. Without loss of generality, we may assume that

$$F_k x_1^{[d+1]} \to \mathbf{x}, \quad k \to \infty. \tag{6.1}$$

Then $x_\emptyset = x_1$ and $\pi^{[d+1]}(\mathbf{x}) = (y_1^{[d]}, y_2, (y_1^{[d]})_*)$.

Let $\mathbf{x_I} = (x_\epsilon : \epsilon(d+1) = 0) \in X^{[d]}$ and $\mathbf{x_{II}} = (x_\epsilon : \epsilon(d+1) = 1) \in X^{[d]}$. Then $\mathbf{x} = (\mathbf{x_I}, \mathbf{x_{II}})$. Note that

$$\pi^{[d]}(\mathbf{x_I}) = \pi^{[d]}(x_1^{[d]}) = y_1^{[d]}, \quad \text{and} \quad \pi^{[d]}(\mathbf{x_{II}}) = (y_2, (y_1^{[d]})_*).$$

By Proposition 5.2, there is some sequence $F_k^1 \in \mathcal{F}^{[d]}$ such that

$$F_k^1(\mathbf{x_I}) \to x_1^{[d]}, \quad k \to \infty.$$

We may assume that

$$F_k^1(\mathbf{x_{II}}) \to \mathbf{x_{II}}', \quad k \to \infty.$$

Note that $\pi^{[d]}(\mathbf{x_{II}}) = \pi^{[d]}(\mathbf{x}'_{\mathbf{II}}) = (y_2, (y_1^{[d]})_*)$.

Let $F_k^1 = (S_{\epsilon'}^k : \epsilon' \in \{0,1\}^d)$ and $H_k^1 = (S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}) \in \mathcal{F}^{[d+1]}$ such that

$$(S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}, \epsilon(d+1) = 0) = (S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}, \epsilon(d+1) = 1) = F_k^1.$$

Then

$$H_k^1(\mathbf{x}) = F_k^1 \times F_k^1(\mathbf{x_I}, \mathbf{x_{II}}) \to (x_1^{[d]}, \mathbf{x_{II}}') \triangleq \mathbf{x^1} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]}), \quad k \to \infty.$$

Let $\mathbf{y^1} = \pi^{[d+1]}(\mathbf{x^1})$. It is easy to see that $x_\epsilon^1 = x_1$ if $\epsilon(d+1) = 0$. For $\mathbf{y^1}$, $y_{\{d+1\}}^1 = y_{00\dots01}^1 = y_2$ and $y_\epsilon^1 = y_1$ for all $\epsilon \neq \{d+1\}$.

Let $\mathbf{x_I^1} = (x_\epsilon : \epsilon \in \{0,1\}^{d+1}, \epsilon(d) = 0) \in X^{[d]}$ and $\mathbf{x_{II}^1} = (x_\epsilon : \epsilon \in \{0,1\}^{d+1}, \epsilon(d) = 1) \in X^{[d]}$. By Proposition 5.2, there is some sequence $F_k^2 \in \mathcal{F}^{[d]}$ such that

$$F_k^2(\mathbf{x_I^1}) \to x_1^{[d]}, \qquad F_k^2(\mathbf{x_{II}^1}) \to \mathbf{x_{II}^1}', \quad k \to \infty$$

and $\pi^{[d]}(\mathbf{x_{II}^1}') = (y_1^{[d-1]}, y_3, (y_1^{[d-1]})_*)$ for some $y_3 \in Y$.

Let $F_k^2 = (S_{\epsilon'}^k : \epsilon' \in \{0,1\}^d)$ and $H_k^2 = (S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}) \in \mathcal{F}^{[d+1]}$ such that

$$(S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}, \epsilon(d) = 0) = (S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}, \epsilon(d) = 1) = F_k^2.$$

Then

$$H_k^2(\mathbf{x^1}) \to \mathbf{x^2} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]}), \quad k \to \infty.$$

Consider $\mathbf{y^2} = \pi^{[d+1]}(\mathbf{x^2})$. Then $H_k^2(\mathbf{y^1}) \to \mathbf{y^2}$, $k \to \infty$. From this one has that $(y_3, y_1) \in \overline{\mathcal{O}((y_1, y_2), T \times T)}$. By the definition of $\mathbf{x^2}, \mathbf{y^2}$, it is easy to see that $x_\epsilon^2 = x_1$ if $\epsilon(d+1) = 0$ or $\epsilon(d) = 0$; $y_{\{d,d+1\}}^2 = y_{00\dots011}^2 = y_3$ and $y_\epsilon^2 = y_1$ for all $\epsilon \neq \{d, d+1\}$.

Now assume that we have $\mathbf{x^j} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]})$ for $1 \le j \le d$ with $\pi^{[d+1]}(\mathbf{x^j}) = \mathbf{y^j}$ such that $x_\epsilon^j = x_1$ if there exists some $k$ with $d-j+2 \le k \le d+1$ such that $\epsilon(k) = 0$; $y_{\{d-j+2,\dots,d,d+1\}}^j = y_{j+1}$ and $y_\epsilon^j = y_1$ for all $\epsilon \neq \{d-j+2, \dots, d, d+1\}$, and $(y_{j+1}, y_1) \in \overline{\mathcal{O}((y_1, y_j), T \times T)}$.

Let $\mathbf{x_I^j} = (x_\epsilon : \epsilon \in \{0,1\}^{d+1}, \epsilon(d-j+1) = 0) \in X^{[d]}$ and $\mathbf{x_{II}^j} = (x_\epsilon : \epsilon \in \{0,1\}^{d-j+1}, \epsilon(d-j+1) = 1) \in X^{[d]}$. By Proposition 5.2, there is some sequence $F_k^{j+1} \in \mathcal{F}^{[d]}$ such that

$$F_k^{j+1}(\mathbf{x_I^j}) \to x_1^{[d]}, \qquad F_k^{j+1}(\mathbf{x_{II}^j}) \to \mathbf{x_{II}^j}', \quad k \to \infty.$$

Let $F_k^{j+1} = (S_{\epsilon'}^k : \epsilon' \in \{0,1\}^d)$ and $H_k^{j+1} = (S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}) \in \mathcal{F}^{[d+1]}$ such that

$$(S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}, \epsilon(d-j+1) = 0) = (S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}, \epsilon(d-j+1) = 1) = F_k^{j+1}.$$

Then

$$H_k^{j+1}(\mathbf{x^j}) \to \mathbf{x^{j+1}} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]}), \quad k \to \infty.$$

It is easy to see that $x_\epsilon^{j+1} = x_1$ if there exists some $k$ with $d - j + 1 \le k \le d + 1$ such that $\epsilon(k) = 0$.

Let $\mathbf{y^{j+1}} = \pi^{[d+1]}(\mathbf{x^{j+1}})$. Then $y_\epsilon^{j+1} = y_1$ for all $\epsilon \neq \{d-j+1, d-j+2, \dots, d+1\}$, and denote $y^j_{\{d-j+1,d-j+2,\dots,d+1\}} = y_{j+2}$. Note that $H_k^2(\mathbf{y^j}) \to \mathbf{y^{j+1}},\ k \to \infty$. From this one has that $(y_{j+2}, y_1) \in \overline{\mathcal{O}((y_1, y_{j+1}), T \times T)}$.

Inductively we get $\mathbf{x^1}, \dots, \mathbf{x^{d+1}}$ and $\mathbf{y^1}, \dots, \mathbf{y}^{d+1}$ such that for all $1 \le j \le d + 1 \mathbf{x^j} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]})$ with $\pi^{[d+1]}(\mathbf{x^j}) = \mathbf{y^j}$. And $x_\epsilon^j = x_1$ if there exists some $k$ with $d - j + 2 \le k \le d+1$ such that $\epsilon(k) = 0$; $y^j_{\{d-j+2,\dots,d,d+1\}} = y_{j+1}$ and $y_\epsilon^j = y_1$ for all $\epsilon \neq \{d-j+2, \dots, d, d+1\}$, and $(y_{j+1}, y_1) \in \overline{\mathcal{O}((y_1, y_j), T \times T)}$.

For $\mathbf{x^{d+1}}$, we have that $x_\epsilon^{d+1} = x_1$ if there exists some $k$ with $1 \le k \le d + 1$ such that $\epsilon(k) = 0$. That means there is some $x_2 \in X$ such that

$$\mathbf{x^{d+1}} = (x_1, x_1, \dots, x_1, x_2) \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]}).$$

By Lemma 6.2, $(x_1, x_2) \in \mathbf{RP}^{[d]}(X)$. Note that $\pi(x_2) = y_{d+2}$.

Since $(y_{j+1}, y_1) \in \overline{\mathcal{O}((y_1, y_j), T \times T)}$ for all $1 \le j \le d + 1$, we have $(y_{d+2}, y_1) \in \overline{\mathcal{O}((y_1, y_2), T \times T)}$ or $(y_1, y_{d+2}) \in \overline{\mathcal{O}((y_1, y_2), T \times T)}$. Without loss of generality, we assume that $(y_1, y_{d+2}) \in \overline{\mathcal{O}((y_1, y_2), T \times T)}$. Since $(y_1, y_2)$ is $T \times T$-minimal, there is some $\{n_k\} \subset \mathbb{Z}$ such that $(T \times T)^{n_k}(y_1, y_{d+2}) \to (y_1, y_2), k \to \infty$. Without loss of generality, we assume that

$$(T \times T)^{n_k}(x_1, x_2) \to (z_1, z_2), \quad k \to \infty.$$

Since $\mathbf{RP}^{[d]}(X)$ is closed and invariant, we have

$$(z_1, z_2) \in \overline{\mathcal{O}((x_1, x_2), T \times T)} \subset \mathbf{RP}^{[d]}(X).$$

And

$$\pi \times \pi(z_1, z_2) = \lim_k (T \times T)^{n_k}(\pi(x_1), \pi(x_2)) = \lim_k (T \times T)^{n_k}(y_1, y_{d+2}) = (y_1, y_2).$$

The proof is completed. $\square$


## Acknowledgments

We thank V. Bergelson, E. Glasner, W. Huang, Hanfeng Li, A. Maass for helpful discussions. Particularly we thank E. Glasner for sending us his note on the topic, and Hanfeng Li for the very careful reading which helped us to correct misprints and simplify some proofs.

We thank the referee for the careful reading and many useful comments.

S. Shao is supported by NNSF of China (11171320, 10871186) and "the Fundamental Research Funds for the Central Universities" (WK0010000015), and X. Ye is supported by NNSF of China (11071231, 10531010).


## Appendix A. Basic facts about abstract topological dynamics

In Appendix A we recall some basic definitions and results in abstract topological systems, which are used in the article. For more details, see [1,5,14,17,30,31].

### A.1. Topological transformation groups

A *topological dynamical systems* is a triple $\mathcal{X} = (X, \mathcal{T}, \Pi)$, where $X$ is a compact $T_2$ space, $\mathcal{T}$ is a $T_2$ topological group and $\Pi : T \times X \to X$ is a continuous map such that $\Pi(e, x) = x$ and $\Pi(s, \Pi(t, x)) = \Pi(st, x)$. We shall fix $\mathcal{T}$ and suppress the action symbol. In lots of papers, $\mathcal{X}$ is also called a *topological transformation group* or a *flow*. Usually we omit $\Pi$ and denote a system by $(X, \mathcal{T})$.

Let $(X, \mathcal{T})$ be a system and $x \in X$, then $\mathcal{O}(x, \mathcal{T})$ denotes the *orbit* of $x$, which is also denoted by $\mathcal{T}x$. A subset $A \subseteq X$ is called *invariant* if $ta \subseteq A$ for all $a \in A$ and $t \in \mathcal{T}$. When $Y \subseteq X$ is a closed and $\mathcal{T}$-invariant subset of the system $(X, \mathcal{T})$ we say that the system $(Y, \mathcal{T})$ is a *subsystem* of $(X, \mathcal{T})$. If $(X, \mathcal{T})$ and $(Y, \mathcal{T})$ are two dynamical systems their *product system* is the system $(X \times Y, \mathcal{T})$, where $t(x, y) = (tx, ty)$.

A system $(X, \mathcal{T})$ is called *minimal* if $X$ contains no proper closed invariant subsets. $(X, \mathcal{T})$ is called *transitive* if every invariant open subset of $X$ is dense. An example of an transitive system is a *point-transitive* system, which is a system with a dense orbit. It is easy to verify that a system is minimal iff every orbit is dense. The system $(X, \mathcal{T})$ is *weakly mixing* if the product system $(X \times X, \mathcal{T})$ is transitive.

A *homomorphism* (or *extension*) of systems $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ is a continuous onto map of the phase spaces such that $\pi(tx) = t\pi(x)$ for all $t \in \mathcal{T}, x \in X$. In this case one says that $(Y, \mathcal{T})$ is a *factor* of $(X, \mathcal{T})$ and also that $(X, \mathcal{T})$ is an *extension* of $(Y, \mathcal{T})$. Define

$$R_\pi = \{(x_1, x_2) : \pi(x_1) = \pi(x_2)\},$$

then $Y = X/R_\pi$. For $n \geq 2$, define

$$R_\pi^n = \{(x_1, x_2, \dots, x_n) \in X^n : \pi(x_1) = \pi(x_2) = \cdots = \pi(x_n)\},$$

and let $R_\pi^1 = X$.

### A.2. Enveloping semigroups

Given a system $(X, \mathcal{T})$ its *enveloping semigroup* or *Ellis semigroup* $E(X, \mathcal{T})$ is defined as the closure of the set $\{t : t \in \mathcal{T}\}$ in $X^X$ (with its compact, usually non-metrizable, pointwise convergence topology). For an enveloping semigroup, $E \to E : p \mapsto pq$ and $p \mapsto tp$ is continuous for all $q \in E$ and $t \in \mathcal{T}$. Note that $(X^X, \mathcal{T})$ is a system and $(E(X, \mathcal{T}), \mathcal{T})$ is its subsystem.

Let $(X, \mathcal{T})$, $(Y, \mathcal{T})$ be systems and $\pi : X \to Y$ be an extension. Then there is a unique continuous semigroup homomorphism $\pi^* : E(X, \mathcal{T}) \to E(Y, \mathcal{T})$ such that $\pi(px) = \pi^*(p)\pi(x)$ for all $x \in X$, $p \in E(X, \mathcal{T})$. When there is no confusion, we usually regard the enveloping semigroup of $X$ as acting on $Y$: $p\pi(x) = \pi(px)$ for $x \in X$ and $p \in E(X, \mathcal{T})$.

### A.3. Idempotents and ideals

For a semigroup the element $u$ with $u^2 = u$ is called an *idempotent*. Ellis–Numakura Theorem says that for any enveloping semigroup $E$ the set $J(E)$ of idempotents of $E$ is not empty [5]. A non-empty subset $I \subset E$ is a *left ideal* (resp. *right ideal*) if it $EI \subseteq I$ (resp. $IE \subseteq I$). A *minimal left ideal* is the left ideal that does not contain any proper left ideal of $E$. Obviously every left ideal is a semigroup and every left ideal contains some minimal left ideal.

An idempotent $u \in J(E)$ is *minimal* if $v \in J(E)$ and $vu = v$ implies $uv = u$. The following results are well-known [6,13]: let $L$ be a left ideal of enveloping semigroup $E$ and $u \in J(E)$. Then there is some idempotent $v$ in $Lu$ such that $uv = v$ and $vu = v$; an idempotent is minimal if and only if it is contained in some minimal left ideal.

Minimal left ideals have very rich algebraic properties. For example, we have the following proposition.

**Proposition A.1.** *Let $I$ be a minimal left ideal, then*

(1) *$I = \bigcup_{u\in J(I)} uI$ is its partition and every $uI$ is a group with identity $u \in J(I)$.*
(2) *For all $u, v \in J(I)$, one has that $uv = u$ and $vu = v$.*

A useful result about minimal point is as follows:

**Proposition A.2.** *Let $I$ be a minimal left ideal. A point $x \in X$ is minimal if and only if $ux = x$ for some $u \in I$.*

*A.4. Universal point transitive system and universal minimal system*

For fixed $\mathcal{T}$, there exists a universal point-transitive system $\mathcal{S}_\mathcal{T} = (S_\mathcal{T}, \mathcal{T})$ such that $\mathcal{T}$ can densely and equivariantly be embedded in $S_\mathcal{T}$. The multiplication on $\mathcal{T}$ can be extended to a multiplication on $S_\mathcal{T}$, then $S_\mathcal{T}$ is a closed semigroup with continuous right translations. The universal minimal system $\mathfrak{M} = (\mathbf{M}, \mathcal{T})$ is isomorphic to any minimal left ideal in $S_\mathcal{T}$ and $\mathbf{M}$ is a closed semigroup with continuous right translations. Hence $J = J(\mathbf{M})$ of idempotents in $\mathbf{M}$ is nonempty. Moreover, $\{v\mathbf{M} : v \in J\}$ is a partition of $\mathbf{M}$ and every $v\mathbf{M}$ is a group with unit element $v$. Sometimes if there are chances of being confused then we will use $\mathbf{M}_\mathcal{T}$ instead of $\mathbf{M}$.

The sets $S_\mathcal{T}$ and $\mathbf{M}$ act on $X$ as semigroups and $S_\mathcal{T} x = \overline{\mathcal{T}x}$, while for a minimal system $(X, \mathcal{T})$ we have $\mathbf{M}x = \overline{\mathcal{T}x} = X$ for every $x \in X$. A necessary and sufficient condition for $x$ to be minimal is that $ux = x$ for some $u \in J$.

*A.5. All kinds of extensions*

Two points $x_1$ and $x_2$ are called *proximal* iff

$$\overline{\mathcal{T}(x_1, x_2)} \cap \Delta_X \neq \emptyset.$$

Let $\mathcal{U}_X$ be the unique uniform structure of $X$, then

$$\mathbf{P} = \mathbf{P}(X, \mathcal{T}) = \bigcap \{\mathcal{T}\alpha : \alpha \in \mathcal{U}_X\}$$

is the collection of proximal pairs in $X$, the *proximal relation*.

**Proposition A.3.** *Let $(X, \mathcal{T})$ be a system. Then*

(1) *$x_1, x_2$ are proximal in $(X, \mathcal{T})$ iff $px_1 = px_2$ for some $p \in E(X, \mathcal{T})$.*
(2) *If $x \in X$ and $u$ is an idempotent in $E(X, \mathcal{T})$, then $(x, ux) \in \mathbf{P}$.*
(3) *If $x \in X$, then there is an minimal point $x' \in \overline{\mathcal{O}(x, \mathcal{T})}$ such that $(x, x') \in \mathbf{P}$.*
(4) *If $(X, T)$ is minimal, then $(x, y) \in \mathbf{P}$ if and only if there is some minimal idempotent $u \in E(X, \mathcal{T})$ such that $y = ux$.*

The extension $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ is called *proximal* iff $R_\pi \subseteq \mathbf{P}$ iff $\mathbf{P}_\pi = \bigcap \{\mathcal{T}\alpha \cap R_\pi : \alpha \in \mathcal{U}_X\} = R_\pi$. $\pi$ is *distal* if $\mathbf{P}_\pi = \Delta_X$. An extension $\pi : X \to Y$ of systems is called *equicontinuous* or *almost periodic* if for every $\alpha \in \mathcal{U}_X$ there is $\beta \in \mathcal{U}_X$ such that $\mathcal{T}\alpha \cap R_\pi \subseteq \beta$. In the metric case an equicontinuous extension is also called an *isometric extension*. The extension $\pi$ is a *weakly mixing extension* when $(R_\pi, \mathcal{T})$ as a subsystem of the product system $(X \times X, \mathcal{T})$ is transitive.

*A.6. Vietoris topology and circle operation*

Let $2^X$ be the collection of nonempty closed subsets of $X$ endowed with the Vietoris topology. Note that a base for the Vietoris topology on $2^X$ is formed by the sets

$$\langle U_1, U_2, \dots, U_n\rangle = \{A \in 2^X : A \subseteq \bigcup_{i=1}^n U_i \text{ and } A \cap U_i \neq \emptyset \text{ for every } i\},$$

where $U_i$ is open in $X$. Then $(2^X, \mathcal{T})$ defined by $tA = \{ta : a \in A\}$ is a system again, and $S_\mathcal{T}$ acts on $2^X$ too. To avoid ambiguity we denote the action of $S_\mathcal{T}$ on $2^X$ by the *circle operation* as follows. Let $p \in S_\mathcal{T}$ and $D \in 2^X$, then define $p \circ D = \lim_{2^X} t_i D$ for any net $\{t_i\}_i$ in $\mathcal{T}$ with $t_i \to p$. Moreover

$$p \circ D = \{x \in X : \text{there are } d_i \in D \text{ with } x = \lim_i t_i d_i\}$$

for any net $t_i \to p$ in $S_\mathcal{T}$. We always have $pD \subseteq p \circ D$.

*A.7. Ellis group*

The group of automorphisms of $(\mathbf{M}, \mathcal{T})$, $G = \mathrm{Aut}(\mathbf{M}, \mathcal{T})$ can be identified with any one of the groups $u\mathbf{M}$ $(u \in J)$ as follows: with $\alpha \in uM$ we associate the automorphism $\hat{\alpha} : (\mathbf{M}, \mathcal{T}) \to (\mathbf{M}, \mathcal{T})$ given by right multiplication $\hat{\alpha}(p) = p\alpha$, $p \in \mathbf{M}$. The group $G$ plays a central role in the algebraic theory. It carries a natural $T_1$ compact topology, called by Ellis the *$\tau$-topology*, which is weaker than the relative topology induced on $G = u\mathbf{M}$ as a subset of $\mathbf{M}$.

It is convenient to fix a minimal left ideal $\mathbf{M}$ in $S_\mathcal{T}$ and an idempotent $u \in \mathbf{M}$. As explained above we identify $G$ with $u\mathbf{M}$ and for any subset $A \subseteq G$, *$\tau$-topology* is determined by

$$\mathrm{cl}_\tau A = u(u \circ A) = G \cap (u \circ A).$$

Also in this way we can consider the "action" of $G$ on every system $(X, \mathcal{T})$ via the action of $S_\mathcal{T}$ on X. With every minimal system $(X, T)$ and a point $x_0 \in uX = \{x \in X : ux = x\}$ we associate a $\tau$-closed subgroup

$$\mathfrak{G}(X, x_0) = \{\alpha \in G : \alpha x_0 = x_0\}$$

the *Ellis group* of the pointed system $(X, x_0)$.

For a homomorphism $\pi : X \to Y$ with $\pi(x_0) = y_0$ we have

$$\mathfrak{G}(X, x_0) \subseteq \mathfrak{G}(Y, y_0).$$

It is easy to see that $u\pi^{-1}(y_0) = \mathfrak{G}(Y, y_0)x_0$.

For a $\tau$-closed subgroup $F$ of $G$ the derived group $H(F) = F'$ is given by

$$H(F) = F' = \bigcap \big\{\mathrm{cl}_\tau O : O \quad \text{is a } \tau\text{-open neighborhood of } u \text{ in } F\big\}.$$

$H(F)$ is a $\tau$-closed normal subgroup of $F$ and it is characterized as the smallest $\tau$-closed subgroup $H$ of $F$ such that $F/H$ is a compact Hausdorff topological group.

### A.8. Structure of minimal systems

Let $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ be a homomorphism of minimal systems with $x_0 \in X$ and $y_0 = \pi(x_0) \in Y$. We say that $\pi$ is a *RIC* (relatively incontractible) extension if for every $y = py_0 \in Y$, $p$ an element of $\mathbf{M}$,

$$\pi^{-1}(y) = p \circ u\pi^{-1}(y_0) = p \circ Fx_0,$$

where $F = \mathfrak{G}(Y, y_0)$. One can show that the extension $\pi : X \to Y$ is RIC if and only if it is open and for every $n \geq 1$ the minimal points are dense in the relation $R_\pi^n$. Note that every distal extension is RIC. It then follows that every distal extension is open.

We say that a minimal system $(X, \mathcal{T})$ is a *strictly PI system* if there is an ordinal $\eta$ (which is countable when $X$ is metrizable) and a family of systems $\{(W_\iota, w_\iota)\}_{\iota \leq \eta}$ such that (i) $W_0$ is the trivial system, (ii) for every $\iota < \eta$ there exists a homomorphism $\phi_\iota : W_{\iota+1} \to W_\iota$ which is either proximal or equicontinuous (isometric when $X$ is metrizable), (iii) for a limit ordinal $\nu \leq \eta$ the system $W_\nu$ is the inverse limit of the systems $\{W_\iota\}_{\iota<\nu}$, and (iv) $W_\eta = X$. We say that $(X, \mathcal{T})$ is a *PI-system* if there exists a strictly PI system $\tilde{X}$ and a proximal homomorphism $\theta : \tilde{X} \to X$.

We have the structure theorem for minimal systems, which we will state in its relative form (Ellis–Glasner–Shapiro [8], Veech [30], and Glasner [14]).

**Theorem A.4** (*Structure Theorem for Minimal Systems*)**.** *Given a homomorphism $\pi : X \to Y$ of minimal dynamical system, there exists an ordinal $\eta$ (countable when $X$ is metrizable) and a canonically defined commutative diagram (the canonical PI-Tower)*

$$\begin{tikzcd}
X \arrow[d, "\pi"'] & X_0 \arrow[l, "\theta_0^*"'] \arrow[d, "\pi_0"'] \arrow[dr, "\sigma_1"] & & X_1 \arrow[ll, "\theta_1^*"'] \arrow[d, "\pi_1"'] & \cdots & X_\nu \arrow[d, "\pi_\nu"'] \arrow[dr, "\sigma_{\nu+1}"] & & X_{\nu+1} \arrow[ll, "\theta_{\nu+1}^*"'] \arrow[d, "\pi_{\nu+1}"'] & \cdots & X_\eta = X_\infty \arrow[d, "\pi_\infty"'] \\
Y & Y_0 \arrow[l, "\theta_0"] & Z_1 \arrow[l, "\rho_1"] & Y_1 \arrow[l, "\theta_1"] & \cdots & Y_\nu & Z_{\nu+1} \arrow[l, "\rho_{\nu+1}"] & Y_{\nu+1} \arrow[l, "\theta_{\nu+1}"] & \cdots & Y_\eta = Y_\infty
\end{tikzcd}$$

*where for each $\nu \leq \eta, \pi_\nu$ is RIC, $\rho_\nu$ is isometric, $\theta_\nu, \theta_\nu^*$ are proximal and $\pi_\infty$ is RIC and weakly mixing of all orders. For a limit ordinal $\nu$, $X_\nu, Y_\nu, \pi_\nu$ etc. are the inverse limits (or joins) of $X_\iota, Y_\iota, \pi_\iota$ etc. for $\iota < \nu$. Thus $X_\infty$ is a proximal extension of $X$ and a RIC weakly mixing extension of the strictly PI-system $Y_\infty$. The homomorphism $\pi_\infty$ is an isomorphism (so that $X_\infty = Y_\infty$) if and only if $X$ is a PI-system.*

## Appendix B. Proof of Theorem 4.3

First we need the so-called *Ellis trick* in [14]. Refer to [14, Lemma X.6.1] for the proof. See [18] for more discussions about weakly mixing extensions. Recall that $\mathbf{M}$ is the universal minimal set.

**Lemma B.1** (*Ellis Trick*)**.** *Let $F$ be $\tau$ closed subgroup of $G$ acting on $\mathbf{M}$ by right multiplication, $\mathbf{M} \times F \to \mathbf{M}$, $(p, \alpha) \mapsto p\alpha$.*

(1) *there is a minimal idempotent $\omega \in J(\mathbf{M}) \cap \overline{F}$ such that $\overline{\omega F}$ is $F$-minimal.*
(2) *if $V$ is a open subset of $\overline{wF}$, then* $\mathrm{int}_\tau \mathrm{cl}_\tau (V \cap wF) \neq \emptyset$.

**Lemma B.2.** *Let $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ be a RIC weakly mixing extension of minimal systems and $u \in J(\mathbf{M})$ be a minimal idempotent. Let $x \in uX$, $y = \pi(x)$. Then for all $n \geq 2$, any nonempty open subset $U$ of $\overline{u\pi^{-1}(y)}$ and any transitive point $x' = (x'_1, \ldots, x'_{n-1}) \in R_\pi^{n-1}$ with $\pi(x'_j) = y$, $j = 1, \ldots, n-1$, we have $\overline{\mathcal{T}(\{x'\} \times U)} = R_\pi^n$.*

**Proof.** Note that we have $H(F)A = F$, where $F = \mathfrak{G}(Y, y)$, $A = \mathfrak{G}(X, x)$, since $\pi$ is weakly mixing.

Claim.

$$\{ux'\} \times \pi^{-1}(y) \subset \overline{\mathcal{T}(\{x'\} \times U)}.$$

Proof of the Claim. Set $V = \{p \in \overline{F} : px \in U\}$. Then $V$ is a nonempty open set of $\overline{F}$ and by Ellis trick we have $\widetilde{V} = \mathrm{int}_\tau \mathrm{cl}_\tau (V \cap F) \neq \emptyset$. By the definition of $H(F)$, there exists $\alpha \in F$ such that $\alpha H(F) \subseteq \mathrm{cl}_\tau \widetilde{V}$.

Since $F = AH(F) = H(F)A$, we have

$$\begin{aligned}
\overline{\mathcal{T}(\{x'\} \times U)} &\supseteq u \circ (\{x'\} \times U) \supseteq u \circ (\{x'\} \times Vx) \\
&\supseteq \{ux'\} \times u(u \circ V)x \supseteq \{ux'\} \times u(u \circ (V \cap F))x \\
&= \{ux'\} \times \mathrm{cl}_\tau (V \cap F)x \supseteq \{ux'\} \times \mathrm{cl}_\tau \widetilde{V}x \\
&\supseteq \{ux'\} \times \alpha H(F)x = \{ux'\} \times \alpha H(F)Ax \\
&= \{ux'\} \times \alpha Fx = \{ux'\} \times Fx.
\end{aligned}$$

Since $\pi$ is RIC, we have $u \circ Fx = \pi^{-1}(y)$. Hence

$$\overline{\mathcal{T}(\{x'\} \times U)} \supseteq u \circ (\{ux'\} \times Fx) = \{ux'\} \times \pi^{-1}(y).$$

This ends the proof of the claim. $\square$

Now it is easy to see that $\overline{\mathcal{T}(\{x'\} \times U)} = R_\pi^n$. Let $(x_1, x_2) \in R_\pi^n$, where $x_1 \in R_\pi^{n-1}$. Since $x'$ is a transitive point of $R_\pi^{n-1}$, there exists a $p \in S_\mathcal{T}$ such that $px' = x_1$. Then $x_2 \in \pi^{-1}(py) = p \circ \pi^{-1}(y)$. Thus

$$(x_1, x_2) \in \{px'\} \times p \circ \pi^{-1}(y) \subseteq \overline{\mathcal{T}(\{ux'\} \times \pi^{-1}(y))} \subseteq \overline{\mathcal{T}(\{x'\} \times U)}.$$

Thus we have $R_\pi^n = \overline{\mathcal{T}(\{x'\} \times U)}$. $\square$

**Theorem B.3.** *Let $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ be a RIC weakly mixing extension of minimal metric systems and $y \in Y$. Then for all $n \geq 1$, there exists a transitive point $(x_1, x_2, \ldots, x_n)$ of $R_\pi^n$ with $x_1, x_2, \ldots, x_n \in \pi^{-1}(y)$.*

**Proof.** It is obvious for the case when $n = 1$, since $R_\pi^1 = X$. Now assume it is true for $n - 1$ ($n \geq 2$). Fix a transitive point $x' = (x_1, x_2, \ldots, x_{n-1}) \in R_\pi^{n-1}$ with $x_1, x_2, \ldots, x_{n-1} \in \pi^{-1}(y)$. Assume that $y \in uY$ for some minimal idempotent $u \in J(\mathbf{M})$.

For each $\epsilon > 0$, define

$$V_\epsilon = \{x \in \overline{u\pi^{-1}(y)} : \mathcal{T}(x', x) \text{ is } \epsilon\text{-dense in } R_\pi^n\}.$$

It is easy to verify that $V_\epsilon$ is open. Now we show that $V_\epsilon$ is dense in $\overline{u\pi^{-1}(y)}$. For any $\Lambda \subseteq X^n$, $z \in X^n$, $\delta > 0$, $\Lambda \overset{\delta}{\sim} z$ is defined by $\rho(z, z') < \delta$, $\forall z' \in \Lambda$.

Now let $\{z_1, z_2, \ldots, z_n\}$ be an $\epsilon$-net of $R_\pi^n$, i.e. for each $z \in R_\pi^n$ there is some $z_j$ ($j \in \{1, 2, \ldots, n\}$) such that $\rho(z, z_j) < \epsilon$. Let $U$ be an open subset of $\overline{w\pi^{-1}(y)}$. By Lemma B.2,

$\overline{\mathcal{T}(\{x'\} \times U)} = R_\pi^n$. So there are some open subset $U_1 \supseteq U$ and $t_1 \in \mathcal{T}$ such that $t_1(\{x'\} \times U_1) \overset{\epsilon}{\sim} z_1$. Again, by Lemma B.2, $\overline{\mathcal{T}(\{x'\} \times U_1)} = R_\pi^n$. So there are an open subset $U_2 \supseteq U_1$ and $t_2 \in \mathcal{T}$ such that $t_2(\{x'\} \times U_2) \overset{\epsilon}{\sim} z_2$. Inductively, we have a sequence $U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n$ (relatively open) and $t_1, \ldots, t_n \in \mathcal{T}$ such that $t_j(\{x'\} \times U_n) \overset{\epsilon}{\sim} z_j, \forall j \in \{1, 2, \ldots, n\}$. Hence $U_n \subseteq V_\epsilon$. This means that $V_\epsilon$ is dense in $\overline{u\pi^{-1}(y)}$.

Let $\Gamma = \bigcap_{n=1}^\infty V_{1/n}$. Then $\Gamma$ is a residual set of $\overline{u\pi^{-1}(y)}$, and for all $x \in \Gamma$, we have $\overline{\mathcal{T}(x', x)} = R_\pi^n$. In particular, there exists a transitive point $(x_1, x_2, \ldots, x_n)$ of $R_\pi^n$ with $x_1, x_2, \ldots, x_n \in \pi^{-1}(y)$. The proof is completed. $\square$

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