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The isometry group of Outer Space

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Abstract

We prove analogues of Royden's Theorem for the Lipschitz metrics of Outer Space, namely that $Isom(CV_n) = Out(F_n)$.

Keywords: Outer space; Free group; Stretching factors; Isometries; Lipschitz distances

Contents

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1. Introduction

For $n \geq 2$ let F_n be the free group of rank n, and $Out(F_n)$ be the group of outer automorphisms of F_n. The Culler–Vogtmann *Outer Space*, CV_n, is the analogue of Teichmuller space for $Out(F_n)$ and is a space of metric graphs with fundamental group of rank n.

As for Teichmuller space, one can define the Lipschitz metric of CV_n with a resulting metric which is not symmetric. This non-symmetric metric is geodesic and seems natural in terms of capturing the dynamics of free group automorphisms; for instance the axes of iwip automorphisms [1]. However the non-symmetric version also lacks some properties one might want; it fails to be complete, for instance, while the symmetrised version turns CV_n into a proper metric space (see [12,1,2], and also [13] for a different approach).

The group $Out(F_n)$ naturally acts on CV_n and the action is by isometries. It is also easy to see that this action is faithful for $n \geq 3$ but not faithful for $n = 2$. The reason for this is that $Out(F_2) \simeq GL(2, \mathbb{Z})$ has a central element of order 2, namely $-I_2$, which is in the kernel of the action. If one picks a basis, x_1, x_2 for F_2 the automorphism which sends each x_i to ${x_i}^{-1}$ is a pre-image in $Aut(F_2)$ of $-I_2$.

In this paper, we prove an analogue of Royden's Theorem for both metrics, and any rank, so that $Isom(CV_n) = Out(F_n)$ (see below for exact statements).

There are many such kind of results in the literature; for instance we have the following.

- The Fundamental Theorem of projective geometry (if a field F has no non-trivial automorphisms, the group of incidence-preserving bijections of the projective space of dimension n over F is precisely $PGL(n, F)$).
- Tits Theorem: under suitable hypotheses, the full group of simplicial automorphisms of the spherical building associated to an algebraic group is equal to the algebraic group [24].
- Ivanov's Theorem: the group of simplicial automorphisms of the curve-complex of a surface S of genus at least two is the mapping class group of S [16].
- Royden's Theorem: the isometry group of the Teichmuller space of S is the mapping class group of S [22].
- Bridson and Vogtmann's Theorem: for $n \geq 3$ the group of simplicial automorphisms of the spine of CV_n is $Out(F_n)$ [6].
- Aramayona and Souto's Theorem: for $n \geq 3$, the group of simplicial automorphisms of the free splitting graph is $Out(F_n)$; see [3].

Our main results are the following.

Theorem 1.1. *With respect to the symmetric Lipschitz distance,*

$$Isom(CV_n) = Out(F_n) \quad \text{for } n \geq 3.$$

For $n = 2$,

$$Isom(CV_2) = PGL(2, \mathbb{Z}).$$

We note that by replacing the symmetric distance by its non-symmetrised version one gets the same result.

Theorem 1.2. *For both non-symmetric Lipschitz distances d_R and d_L, $Isom(CV_n)$ is $Out(F_n)$ for $n \geq 3$ and $PGL(2, \mathbb{Z})$ for $n = 2$.*

This kind of result has immediate corollaries of fixed-point type (see for example [5,6]).

Corollary 1.3. *Let G be a semisimple Lie group of real rank at least two that has finite centre. Let Γ be an irreducible lattice in G. Then every isometric action of Γ on CV_n has a global fixed point.*

As we note above, there already exists a result of this kind for the *spine* of CV_n, [6], which states that the simplicial automorphism group of the spine of CV_n is equal to $Out(F_n)$ for $n \geq 3$. At a first glance, Theorem 1.1 could appear to be a direct consequence of [6] after some easy remarks (using, for instance, Lemma 4.1) and in fact that was exactly the thought of the authors when this work was started.

However, the main difficulty in the paper is precisely moving from a statement that an isometry preserves the simplicial structure of CV_n to the statement that it is the identity. For instance, once one knows that an isometry leaves some simplex invariant, it is not clear, a priori, that the centre of the simplex is fixed (in fact it is not true in general if one simply looks at isometries of a simplex rather than the restriction of a global isometry). And even when one has that a given isometry leaves *every* simplex invariant, it is not clear how to deduce that the isometry is in fact the identity—obviously, this is in sharp contrast to the piecewise Euclidean metric (for which a Royden theorem seems to be a direct consequence of the Bridson–Vogtmann result).

There are four key facts in Theorem 1.1. First, the study of local isometries. The main point is that in general, the isometry group of a fixed simplex of CV_n is in fact much bigger than its stabiliser in $Out(F_n)$.

The second fact is that CV_n is highly non-homogeneous. This allows one to find particular points in simplices of CV_n that are invariant under isometries, so that one can characterise those isometries that are restrictions of global ones.

Third, there is the fact that a point in CV_n is determined by the distances from it to points in a sub-complex containing only simplices whose underlying graph is a rose. The main consequence of this fact is that one can deduce that an isometry that does not permute simplices is in fact the identity.

Finally, there is a permutation issue, similar to the one faced in [6], that we solve metrically using a particular version of "Busemann functions".

We also remark that Theorem 1.1 holds for any rank and includes the study of simplices with disconnecting edges. The complete schema of the proof of Theorem 1.1 is described in Section 3.

2. Preliminaries

In this section we fix terminology, give basic definitions, and recall some known facts (and prove some easy ones) that we shall need for the rest of the paper. Experienced readers may skip directly to next section and refer to present one just for notation.

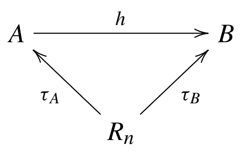
2.1. Outer Space

First of all, we recall what Culler–Vogtmann space or "Outer Space" is. We refer to the pioneer work [10] and beautiful surveys [25,26] for more details.

For any $n \geq 2$ let F_n be the free group of rank n which we identify with the fundamental group of $R_n = S^1 \wedge \cdots \wedge S^1$ (the wedge taken n times).

Consider finite graphs X whose vertices have valence at least three, this means that each vertex has at least three germs of incident edges. We require that X has rank n, that is to say, $\pi_1(X) \simeq F_n$ and that X comes equipped with a metric. Giving a metric on X is equivalent to giving positive lengths for the edges of X.

We also require X to be a *marked graph*, which is to say that it comes with a fixed *marking*. A marking on X is a continuous map $\tau : R_n \to X$ which induces an isomorphism $\tau_* : F_n \simeq \pi_1(R_n) \to \pi_1(X)$. Two marked metric graphs (A, τ_A) and (B, τ_B) are considered equivalent if there exists a homothety, $h : A \to B$, such that the following diagram commutes up to free homotopy,



Culler–Vogtmann Space of F_n or *Outer Space of rank n* is the set CV_n of equivalence classes of marked metric graphs of rank n.

It is common to consider the standard representative of a given class by taking volume one graphs (here volume means total edge length).

However, we usually do not normalise metric graphs, and when we will do it we will use different normalisations depending on the calculations we are making.

We note that since the equivalence allows homothety, given a point $[X]$ in CV_n, we only have the metric on X up to scaling constants. If one instead only considers the equivalence up to isometry, then one obtains *unprojectivised* CV_n and the metric on the graph corresponding to a point there is determined by the point.

Remark 2.1. In the following, if there is no ambiguity, we will not distinguish between a metric graph X and its class $[X]$. If we need to choose a particular representative of $[X]$ we will explicitly declare that.

2.2. The topology of CV_n

Outer Space is endowed with the topology induced by edge-lengths of graphs. Given any marked graph A, we can look at the universal cover T_A which is an $\mathbb{R}$-tree on which $\pi_1(R_n)$ acts by isometries, via the marking τ_A. Conversely, given any minimal free action of F_n by isometries on a simplicial $\mathbb{R}$-tree, we can look at the quotient object, which will be a graph, A, and produce a homotopy equivalence $\tau_A : R_n \to A$ via the action. Equivalence of graphs in CV_n corresponds to actions which are equivalent up to equivariant homothety.

Thus, points in CV_n can be thought of as equivalence classes of minimal free isometric actions on simplicial $\mathbb{R}$-trees. Given an element w of F_n and a point A of the unprojectivised CV_n, with universal cover T_A whose metric we denote by d_A, we may consider,

$$L_A(w) := \inf_{p \in T_A} d_A(p, wp).$$

It is well known that this infimum is always obtained and that, for a free action, it is non-zero for the non-identity elements of the group. In this context, $L_A(w)$ is called the translation length of the element w in the corresponding tree and clearly depends only on the conjugacy class of w in F_n. If we look at graph A, then $L_A(w)$ is the length of the geodesic representative of w in A, that is to say, the length of shortest closed loop representing free homotopy class of $\tau_{A*}(w)$ as an element of $\pi_1(A)$. Thus for any point, A, in CV_n we can associate the sequence $(L_A(w))_{w\in F_n}$ and it is clear that equivalent marked metric graphs will produce two sequences, one of which is a multiple of the other by a positive real number (the homothety constant). Moreover, it is also the case that inequivalent points in CV_n will produce sequences which are not multiples of each other [9]. Thus, we have an embedding of CV_n into $\mathbb{R}^{F_n}/\sim$, where $\sim$ is the equivalence relation of homothety. The space CV_n is given the subspace topology induced by this embedding.

Finally it is clear that we can realise any automorphism, ϕ, of F_n as a homotopy equivalence, also called ϕ, of R_n. Thus the automorphism group of F_n acts on CV_n by changing the marking. That is, given a point (A, τ_A) of CV_n the image of this point under ϕ is $(A, \tau_A\phi)$.

$$R_n \xrightarrow{\ \phi\ } R_n \xrightarrow{\ \tau_A\ } A, \qquad \tau_A\phi : R_n \to A.$$

Since two automorphisms which differ by an inner automorphism always send equivalent points in CV_n to equivalent points, we actually have an action of $Out(F_n)$ on CV_n, and this space is called *Outer Space* for this reason.

2.3. Simplicial subdivision of CV_n

Given a rank-n, marked, metric graph X whose edges are labelled $e_1, \dots, e_k$, we can consider all marked metric graphs homeomorphic to X and with same marking. Such subset of CV_n can be embedded in $\mathbb{R}^k$ by

$$X \mapsto (L_X(e_1), \dots, L_X(e_k)).$$

If we consider standard normalisation with volume one, we obtain standard open $(k-1)$-simplex of $\mathbb{R}^k$, i.e. the set $\{(x_1, \dots, x_k) \in \mathbb{R}^k : x_i > 0, \sum x_i = 1\}$.

This gives us a natural subdivision of CV_n into open simplices.

Definition 2.2. Let Δ be an open simplex of CV_n. The *(marked) graph underlying* of Δ is the (marked) topological type of graphs corresponding to points of Δ.

Simplices of CV_n will have some ideal faces and some true faces. More precisely, in an abstract way, if Δ is a simplex with underlying graph X, a face δ of Δ is obtained by setting to zero the lengths of some of the edges of X. This topologically corresponds to collapsing such edges. If the resulting graph has still rank n, then δ exists as a simplex of CV_n, and in this sense it is a true face. On the other hand, if the rank decreases, then δ is not in CV_n (and in fact belongs to the boundary at infinity of CV_n) and in this case we say that δ is an ideal face of Δ.

In what follows we always deal with true faces.

Definition 2.3. Let Δ be a simplex of CV_n with underlying marked graph X. A *face* of Δ is a simplex of CV_n whose underlying marked graph is obtained from X by collapsing some edges. The codimension of the face of Δ is the number of collapsed edges.

It is readily checked by an Euler characteristic count that simplices of maximal dimension of CV_n correspond to trivalent graphs, and that such graphs have $3n-3$ edges and $2n-2$ vertices. Therefore their dimension is $3n-4$ as CV_n is the projectivised Outer Space. Looking at the topology of graphs we see the following lemma in general.

Lemma 2.4. *k-dimensional simplices of (projectivised) CV_n correspond to graphs with $k+1$ edges and $k-n+2$ vertices.*

Next we consider the i-skeleton of CV_n.

Definition 2.5. For $i \leq 3n-4$, the *i-skeleton* CV_n^i of CV_n is the set of simplices of CV_n of dimension at most i.

An easy but important fact is that i-simplices correspond to smooth points of the i-skeleton.

Definition 2.6. A point $x \in CV_n^i$ is *smooth* if it has a neighbourhood in CV_n^i homeomorphic to $\mathbb{R}^i$.

Lemma 2.7. *Open i-simplices of CV_n are exactly the connected components of the set of smooth points of CV_n^i. That is to say*

$$\{x \in CV_n^i : x \text{ is smooth}\} = \bigsqcup_{\Delta \text{ open } i\text{-simplex}} \Delta.$$

Proof. It is enough to show that any $i-1$ simplex is the face of at least three different i-simplices. Let X be a point of an $(i-1)$-simplex. Then X is obtained by collapsing to zero an edge e of a point $\bar{X}$ of an i-simplex. Let v_- and v_+ be the endpoints of e. Clearly $v_- \neq v_+$ because otherwise the collapse would decrease the rank. By definition, both v_- and v_+ have valence at least three, and they are identified in X to the same vertex v which therefore has valence at least four.

For any subdivision of the set of germs of edges at v in two subsets of at least two germs, we can form a different i-simplex, having X in one of its faces, by separating such subsets and inserting a new edge between them. Clearly, different subdivisions give different i-simplices, and we have at least three such subdivisions because the valence of v is at least four. □

2.4. Roses and multi-thetas

Our result will be based on a detailed study of isometries of two particular classes of marked graphs. Namely roses and multi-theta graphs.

Definition 2.8. A *rose simplex* is a simplex Δ of CV_n whose underlying graph is a rose, i.e. a bouquet of n copies of S^1. Edges of such a graph are also called *petals*. The *centre* of Δ is the symmetric graph, that is to say one whose petals all have the same length.

One should note that in the definition above, the centre is only defined by specifying that the edges have the same length without saying what that length is. We shall usually take a representative whose petals all have length 1 but the reader should be aware that as long as all the petals have the same length, the point in CV_n will be the same.

By Lemma 2.4, rose simplices are those simplices of lowest dimension of CV_n.

Definition 2.9. A *multi-theta simplex* is a simplex Δ of CV_n whose underlying graph has only two vertices and $n+1$ edges joining them (such graph is called a multi-theta). The centre

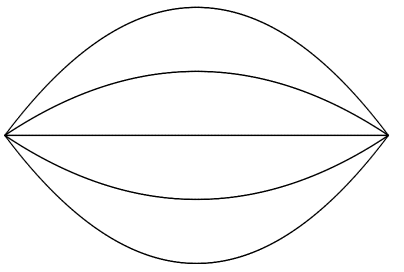


Fig. 1. A multi-theta graph in CV_4.

of Δ is the symmetric graph, that is to say, the one whose edges have all same length (see Fig. 1).

Definition 2.10. A rose-face of a simplex Δ of CV_n is a rose simplex which is a face of Δ.

Formally speaking, simplices are open, so the rose-face of a simplex is not subset of it. Nonetheless, it is readily checked (and we will prove that in Lemma 4.1) that any isometry of a simplex extends to its faces and rose-faces, though it may permute them. As we are interested in studying isometries, by abuse of notation, we will consider the rose-faces of a simplex as subsets of it.

Remark 2.11. Let Δ be a simplex of CV_n with underlying graph X. Then any rose-face of Δ is obtained by collapsing a maximal tree T of X, and different trees give rise to different faces. Therefore rose-faces of Δ are in correspondence with maximal trees of X (for instance, in the case of multi-theta simplices, rose-faces are in correspondence with edges).

2.5. *Distances and stretching factors*

We recall here the definitions of – both the symmetric and non-symmetric – Lipschitz distances on CV_n. These are defined via stretching factors of maps between points of Outer Space. Stretching factors, Outer Space and related topics are widely studied by many authors, and the literature on the matter is huge (see for instance [4,14,17,23,10,12,15,18,11,19,20]).

Definition 2.12. For any two points X and Y in CV_n, normalised to have volume one, we define the right stretching factor as

$$\Lambda_R(X,Y) = \sup_\gamma \frac{L_X(\gamma)}{L_Y(\gamma)}$$

where the supremum is taken over all loops in Y (or, equivalently over all conjugacy classes in F_n). Similarly, the left stretching factor is

$$\Lambda_L(X,Y) = \Lambda_R(Y,X) = \sup_\gamma \frac{L_Y(\gamma)}{L_X(\gamma)}.$$

Definition 2.13. For any two points X and Y in CV_n, normalised to have volume one, the right and left distances are defined by

$$d_R(X,Y) = \log(\Lambda_R(X,Y)) \qquad d_L(X,Y) = \log(\Lambda_L(X,Y)).$$

Definition 2.14. For any two points $X, Y \in CV_n$, not necessarily normalised, the symmetric bi-Lipschitz metric, is defined by

$$d(X, Y) = d_R(X, Y) + d_L(X, Y) = \log \sup_\gamma \frac{L_X(\gamma)}{L_Y(\gamma)} \sup_\gamma \frac{L_Y(\gamma)}{L_X(\gamma)}.$$

We refer the reader to [12] for a detailed discussion on such metrics. We recall some basic facts. First, the suprema in definitions are actually maxima. Also, we recall that $Out(F_n)$ acts faithfully by isometries on CV_n (for $n \geq 3$, in rank two the kernel of the action is $\mathbb{Z}_2$) endowed with any of the above metrics. Finally we note that the symmetric metric is scale invariant, while the non-symmetric ones require normalisation.

The main tool for studying such distances is the so-called Sausages lemma, which allows us to quickly compute stretching factors, and which we will use extensively throughout the paper (see [12] for the proof).

Definition 2.15 (*Almost Simple Closed Curves*)**.** Let X be a point of CV_n. A *simple closed curve* (s.c.c.) is an embedding of S^1 to X. A *figure-eight* curve is an embedding to X of the bouquet $S^1 \wedge S^1$ of two circles. Roughly speaking a *barbell* curve is an embedding to X of the space: O–O. More precisely, let $Q = \{(x, y) \in \mathbb{R}^2 : \sup(|x|, |y|) = 1\}$, then a barbell curve is an immersion $c : Q \to X$ such that $c(x, y) = c(x', y')$ if and only if $x = x'$ and $|y| = |y'| = 1$.

An *almost simple closed curve* (a.s.c.c.) is a curve which is either an s.c.c., or a figure-eight or a barbell curve.

Lemma 2.16 (*Sausages Lemma*)**.** *For any two marked metric graphs X and Y*

$$\sup_\gamma \frac{L_Y(\gamma)}{L_X(\gamma)}$$

is realised by an a.s.c.c. of X. Moreover, If both X and Y are roses, then the supremum is realised by petals.

We notice that the Sausages lemma not only allows to actually compute distances, but is also important from a theoretical view-point. Indeed, the fact that lengths of a.s.c.c. determine distances, and therefore points of Outer Space, is a key-point in the proof of Theorem 1.1 (see in particular Theorems 6.7 and 6.2).

Another simple but somehow surprising result that we will need in the sequel is the following (whose proof can be found in [12]).

Lemma 2.17. *Suppose σ is a d-geodesic between two points X and Y of CV_n. Let Z be a point in σ. A loop γ_0 is maximally (resp. minimally) stretched from X to Y– that is to say, it realises $\sup_\gamma L_Y(\gamma)/L_X(\gamma)$– if and only if the same is true from X to Z and from Z to Y.*

3. Schema of proof of Theorem 1.1

We briefly describe here the strategy for proving our main result. We recall that we aim to show that any isometry Φ of CV_n is induced by some element of $Out(F_n)$. First we show that k-simplices are mapped to k-simplices, and also that rose and multi-theta simplices are mapped to rose and multi-theta simplices (respectively). By composing with an element of $Out(F_n)$ we can arrange that Φ preserves a multi-theta simplex Δ and all of its rose faces. We show that in this case Δ is fixed point-wise and hence all of its rose faces. Next we show that all multi-theta graphs adjacent to Δ are fixed point-wise. Since the sub-complex of multi-theta graphs is path connected, we get that it is point-wise fixed by Φ. We use Theorem 6.9 that asserts that in this case Φ is the identity.

For the ease of the reader, we give a detailed schema of what is proved in the sequel, and where.

(1) Isometries preserve the simplicial structure (Section 4).
(2) Computation of isometry group of rose simplices (it will be $\mathbb{R}^n \rtimes$ a finite group) (Section 5).
(3) Simplices that are possibly permuted by Φ share their rose-faces, and simplices that share rose-faces "have the same set of simple closed curves and the same set of almost simple closed curves" (Section 6).
(4) For a point X in a simplex Δ, the asymptotic behaviour of distances from X to points in rose-faces of Δ determine lengths of simple closed curves of X. This being true not only for points of Δ but also for points in any other simplex having the same rose-faces as Δ (Section 6).
(5) For a point X in a simplex Δ (or in other simplices sharing rose-faces with Δ) the lengths of simple closed curves and the asymptotic behaviour of distances from X to points in rose-faces of Δ, determine lengths of almost simple closed curves of X (whence asymptotic distances determine lengths of a.s.c.c.) (Section 6).
(6) Isometries of multi-theta simplices (Section 7). We show that any isometry of a multi-theta simplex fixes its centre. How:
    (a) Study of those pairs of points joined by a unique geodesic, showing that for any point X in the interior of Δ, there is a standard set of "rigid" geodesics emanating from X.
    (b) Show that for any point other than the centre, there is at least one more "rigid" geodesic, while for the centre, the standard set is all we have. This characterises the centre of Δ from a metric point of view.
    (c) Finally, for the centre of any rose-face of Δ there is a unique "rigid" geodesic joining it to the centre.
    (d) In particular, any isometry fixes the centre, and if it does not permute rose faces, it fixes also such "rigid" geodesics.
(7) Any isometry of a multi-theta simplex Δ that does not permute its rose-faces, it point-wise fixes them and hence point-wise fixes Δ by 5 above. (Section 7.)
(8) Any isometry that fixes a multi-theta simplex, also fixes all rose simplices of CV_n (not only its faces). (Section 7.)

4. Topological constraints for homeomorphisms

In this section we prove the first step of our strategy, that isometries of CV_n respect its simplicial and incidence structure. That result does not require any metric structure, just the fact that isometries are homeomorphisms.

Lemma 4.1. *Any homeomorphism of CV_n maps k-dimensional simplices to k-dimensional simplices.*

Proof. The proof goes by induction on the codimension. Open top dimensional simplices coincide with smooth points (Lemma 2.7).

Clearly, to be a smooth point is invariant under homeomorphisms. Again, by Lemma 2.7, open top-dimensional simplices are exactly connected components of set of smooth points. Therefore homeomorphisms map open top-dimensional simplices to open top-dimensional simplices.

Suppose the claim true for dimensions greater than i. By induction, any homeomorphism Φ of CV_n induces a homeomorphism of i-skeleton CV_n^i. Open codimension-$(n-i)$ simplices are now connected components of smooth part of CV_n^i, and therefore Φ maps i-simplices to i-simplices. □

Lemma 4.2. *Any homeomorphism of CV_n maps rose-simplices to rose-simplices, and multi-theta simplices to multi-theta simplices.*

Proof. This is just a dimensional argument. Clearly, homeomorphisms preserve dimension. By Lemma 2.4, $n-1$-dimensional simplices are exactly rose-simplices, and the first claim follows. If we look at n dimensional simplices, we see that multi-theta simplices are characterised by having exactly $n+1$ rose-faces. So the homeomorphic image of a multi-theta simplex still is a multi-theta simplex. □

5. Isometries of roses

In this section, we compute the isometry groups of rose simplices. In rank two, it is immediate to see that a rose simplex is isometric to $\mathbb{R}$, so its isometries are known. For the general case we prove the following theorem.

Theorem 5.1. *The isometry group of a rose-simplex R of CV_{n+1} is $\mathbb{R}^n \rtimes \mathfrak{F}$, where $\mathfrak{F}$ is finite and stabilises the centre, and $\mathbb{R}^n$ acts transitively. Moreover, for $n \geq 2$, the group $\mathfrak{F}$ is $S_{n+1} \times \mathbb{Z}_2$, where S_{n+1} is the symmetric group on $n+1$ letters and is induced by permutations of petals. For $n=1$ (i.e. in the rank-two case) $\mathfrak{F} = \mathbb{Z}_2 = S_2$.*

Proof. Any point of R is determined by the lengths of its petals, that we label $e_0, \ldots, e_n$. We identify the unprojectivised R with $\mathbb{R}^{n+1}$ as follows. To any $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ is associated the graph X such that

$$L_X(e_i) = e^{x_i}.$$

Note that origin of $\mathbb{R}^{n+1}$ corresponds to centre of R. Moreover, scaling-equivalence on CV_{n+1} descends to the relation

$$x \sim y \quad \text{if and only if} \quad x - y = \lambda(1, \ldots, 1).$$

The pull back of the (pseudo) metric d to $\mathbb{R}^{n+1}$ is then

$$d\big((x_0, \ldots, x_n), (y_0, \ldots, y_n)\big) = \sup_i (x_i - y_i) + \sup_i (y_i - x_i).$$

This immediately implies that translations of $\mathbb{R}^{n+1}$ are isometries, and that translations along vector $(1, \ldots, 1)$ are in fact the only ones inducing the identity of the projectivised R. So we have that

$$\mathbb{R}^n = \mathbb{R}^{n+1} / \langle (1, \ldots, 1) \rangle$$

acts freely and transitively on R.

Thus, it remains to determine the stabiliser of the origin.

Clearly, permutations of coordinates are isometries that fix origin. Finally, we have the reflection

$$\sigma : (x_0, \dots, x_n) \mapsto (-x_0, \dots, -x_n).$$

In the rank-two case, that is to say when $n = 1$, we are studying isometries of $\mathbb{R}$ that fix origin. Therefore in rank-two, the stabiliser of the origin consists of the reflection about the origin and the identity: note that this reflection (the map σ, above, in other words) is induced by the map which interchanges the two petals of our rank 2 rose.

For $n > 1$, our claim is that the stabiliser of origin is

$$\mathfrak{F} = S_{n+1} \times \langle\sigma\rangle.$$

For that, we need some work. First of all, note that the (pseudo) metric d on $\mathbb{R}^{n+1}$ is induced by the (pseudo) norm

$$\|x\| = d(0, x).$$

In order to make $\|\cdot\|$ a norm and d a metric, for any point $x \in \mathbb{R}^{n+1}$ we choose the $\sim$-representative of $x + \mathbb{R}(1, \dots, 1)$ that has 0 as the first coordinate. We can do that because $(x_0, \dots, x_n) \sim (x_0, \dots, x_n) - x_0(1, \dots, 1)$. This gives an isometry between R and $\mathbb{R}^n$ with the following metric (still denoted by d)

$$d\big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) := d\big((0, x_1, \dots, x_n), (0, y_1 \dots, y_n)\big).$$

We give now a more explicit description of that metric.

Lemma 5.2. *For any set $I \subseteq \{1, \dots, n\}$ let R^I be the sector of $\mathbb{R}^n$ such that either $x_i \geq 0$ for all $i \in I$ and $x_i \leq 0$ for all $i \notin I$, or vice versa. Then, for $x \in R^I$*

$$\|x\| = \|x\|_{\infty,I} + \|x\|_{\infty,I^c}$$

where $\|x\|_{\infty,I} = \sup_{i\in I} |x_i|$ and I^c is the complement of I in $\{1, \dots, n\}$.

Proof. This is a straightforward calculation. Indeed, by definition

$$\|x\| = \sup\{0, \sup_{i=1,\dots,n} x_i\} + \sup\{0, \sup_{i=1,\dots,n} -x_i\}$$

and, when $x \in R^I$, that equals $\|x\|_{\infty,I} + \|x\|_{\infty,I^c}$. □

Our next step is an idea that we will return to throughout the paper, and it is that the "unique" geodesics are rather rare and allow one to determine the possible isometries.

Remark 5.3. Note that l^1-norms naturally present phenomena of non-uniqueness of geodesics. Namely, consider two geodesic spaces (X_1, d_1) and (X_2, d_2), and their cartesian product equipped with the sum metric $d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$. Then any geodesic $\gamma : [0, 1] \to X_1 \times X_2$ is of the form $\gamma = (\gamma_1, \gamma_2)$, and, up to reparametrisation,

$$t \mapsto \begin{cases} (\gamma_1(t), \gamma_2(0)) & t \in [0, 1] \\ (\gamma_1(1), \gamma_2(t-1)) & t \in [1, 2] \end{cases} \qquad t \mapsto \begin{cases} (\gamma_1(0), \gamma_2(t)) & t \in [0, 1] \\ (\gamma_1(t-1), \gamma_2(t)) & t \in [1, 2] \end{cases}$$

are two different geodesics whenever neither γ_1 nor γ_2 is the constant map. This situation is exactly the one arising in each sector R^I as above, where, by Lemma 5.2, we have the sum of two l^∞-norms.

Proposition 5.4. *Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, equipped with the metric d above. Then there exists a unique geodesic joining the origin to x if and only if there exists a real number λ such that for all i, $x_i = \lambda$ or $x_i = 0$. This geodesic is given (up to reparametrisation) by the path γ_x whose ith coordinate at time t is tx_i.*

Equivalently, a point $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ represents a point in $\mathbb{R}^n$ joined to the origin by a unique geodesic if and only if there exist λ, μ such that each x_i is equal to either λ or μ.

Proof. The last statement follows trivially from the first, on taking the representative with $x_0 = 0$, obtained by subtracting one of λ or μ from each coordinate.

Next, let O denote the origin of $\mathbb{R}^n$. For any x, y let $\overline{xy}$ denote the path whose ith coordinate at time t is $x_i + t(y_i - x_i)$, $t \in [0, 1]$. By Remark 5.3, if there is a set of indices I such that $\|x\|_{\infty,I}\|x\|_{\infty,I^c} \neq 0$ then x is joined to O by at least two different geodesics. Thus, up to rearranging coordinates and possibly applying the isometry σ above, we can suppose $0 \le x_1 \le \cdots \le x_n$. Clearly, $d(\gamma_x(s), \gamma_x(t)) = x_n|t-s|$, so that $\gamma_x = \overline{Ox}$ is a geodesic. Suppose there is i such that $0 < x_i < x_n$. Then, consider the point $x_\varepsilon = (x_1/2, \ldots, x_i/2 + \varepsilon, \ldots, x_n/2)$. For small enough ε the path γ_ε resulting in the union of $\overline{Ox_\varepsilon}$ and $\overline{x_\varepsilon x}$ is a geodesic from x to O as $d(\gamma_\varepsilon(s), \gamma_\varepsilon(t)) = x_n|t-s|/2$. Also, γ_ε is not a reparametrisation of γ_x because they differ in their middle points.

Conversely, suppose that there is i so that $x_j = 0$ for $j < i$ and $x_j = x_n$ for $j \ge i$. Let γ be a geodesic between O and x. If there is a time t such that the jth coordinate of $\gamma(t)$ is different from 0 for some $j < i$, then a direct calculation shows that $d(0, \gamma(t)) + d(\gamma(t), x)$ is strictly bigger than x_n (while $d(O, x) = x_n$.) Thus, the jth coordinates of $\gamma(t)$ all vanish for $j < i$. The very same argument shows that for $j \ge t$ the jth coordinate of $\gamma(t)$ equals the nth one, this showing that γ_x is the unique geodesic from 0 to x. □

Proposition 5.4 is a translation of the fact that two roses in the same simplex are joined by a unique geodesic if and only if there are only two possible stretching factors for petals.

We note that Proposition 5.4 gives us a collection of geodesics which are permuted by any isometry fixing the origin. Using this fact, we now proceed to calculate the stabiliser of the origin. Since we already have that these geodesics must be permuted by any isometry fixing the origin, we shall proceed by studying points on these geodesics at fixed distance 1 from the origin. These are also permuted and will give us the information we need about the stabiliser.

For any $I \subseteq \{1, \ldots, n\}$ we define points p_I^+ and p_I^- in $\mathbb{R}^n$ by

$$p_I^{\pm} = (x_1, \ldots, x_n): \quad x_i = \begin{cases} \pm 1 & i \in I \\ 0 & i \in I^c \end{cases}$$

such points are equivalents to points P_I of $\mathbb{R}^{n+1}$

$$P_I = (x_0, \ldots, x_n): \quad x_i = \begin{cases} 1 & i \in I \\ 0 & i \in I^c \end{cases}$$

where p_I^+ is equivalent to $P_{0\cup I}$, and p_I^- is equivalent to P_{I^c} (the complement here is made in $\{0, \ldots, n\}$.)

Lemma 5.5. *For any distinct $I, J \subseteq \{0, \ldots, n\}$ we have*

$$d(P_I, P_J) = \begin{cases} 1 & \textit{if } I \subseteq J \textit{ or } J \subseteq I \\ 2 & \textit{otherwise.} \end{cases}$$

Moreover, if $d(P_I, P_J) = 2$, then the points of $\mathbb{R}^n$ corresponding to P_I and P_J are joined by a unique geodesic if and only and $I = J^c$.

Proof. The first part is a simple calculation. For the second part, we use the fact that translations are isometries. Translate the point P_I to the origin and look at the image of P_J, which we denote $x = (x_0, \dots, x_n)$. Then there will be a unique geodesic between P_I and P_J if and only if there is a unique geodesic between x and the origin. However, it is clear what each x_i will be. Namely,

$$x_i = \begin{cases} 0 & \text{if } i \in I \cap J \\ 0 & \text{if } i \in I^c \cap J^c \\ 1 & \text{if } i \in I^c \cap J \\ -1 & \text{if } i \in I \cap J^c. \end{cases}$$

As $d(P_I, P_J) = 2$, we cannot have either $I \subseteq J$ or $J \subseteq I$; hence both 1 and -1 must be taken by some of the x_i. So by Proposition 5.4, P_I and P_J will be joined by a unique geodesic if and only if no x_i is equal to zero, which is the same as saying $I \cap J = \emptyset = I^c \cap J^c$. Equivalently, $I = J^c$. □

As stated, by Proposition 5.4 any isometry that fixes the origin must permute the P_I's. For such an isometry F and $I \subset \{0, \dots, n\}$, we denote by $F(I)$ the set corresponding to point $F(P_I)$.

From Lemma 5.5, we get

$$\big(I \subseteq J \text{ or } J \subseteq I\big) \Leftrightarrow \big(F(I) \subseteq F(J) \text{ or } F(J) \subseteq F(I)\big) \tag{1}$$

and

$$F(I^c) = F(I)^c. \tag{2}$$

Remark 5.6. The isometry σ corresponds to $I \mapsto I^c$.

Lemma 5.7. *For any isometry F, the cardinality $|F(I)|$ is either $|I|$ or $n + 1 - |I|$.*

Proof. By (1) sets I and $F(I)$ must have the same number of subsets and supersets. For I such number is $2^{|I|} + 2^{n+1-|I|} - 1$, whence

$$2^{|I|} + 2^{n+1-|I|} = 2^{|F(I)|} + 2^{n+1-|F(I)|}.$$

Set $x = \min\{|I|, n + 1 - |I|\}$ and $y = \min\{|F(I)|, n + 1 - |F(I)|\}$. We have

$$2^x(1 + 2^k) = 2^y(1 + 2^h)$$

for some non-negative integers k, h. Since x, y are both non-negative integers, the above gives us a factorisation of an *integer* and, since $1 + 2^k$ and $1 + 2^h$ are both odd, the exponent of 2 in the factorisation must be the same; therefore $x = y$ and the claim follows. □

Remark 5.8. Up to possibly composing with σ we may suppose, as we do, that there is i_0 such that $|F(\{i_0\})| = 1$.

Lemma 5.9. *If there is i_0 such that $|F(\{i_0\})| = 1$, then for all i we have that $|F(\{i\})| = 1$.*

Proof. Note that by (2), $|F(\{i_0\}^c)| = |F(\{i_0\})^c| = n$. Now consider some $i \neq i_0$, whence $\{i\} \subseteq \{i_0\}^c$. If $\{i\} = \{i_0\}^c$ then $n = 1$ and the lemma is proved. So we can suppose $\{i\} \neq \{i_0\}^c$, so $F(\{i\}) \neq F(\{i_0\}^c) = F(\{i_0\})^c$ (latter equality is by (2)). Thus, by (1) and Lemma 5.7 we have that $F(\{i\})$ is strictly contained in $F(\{i_0\})^c$. We therefore have $|F(\{i\})| \leq n - 1$, which implies that $|F(\{i\})| = 1$ because of Lemma 5.7. □

Remark 5.10. When $|F(\{i\})| = 1$ for all i, we can define an element f of S_{n+1} by

$$F(\{i\}) = \{f(i)\}.$$

We show now that the permutation F is actually induced by f.

Lemma 5.11. *Suppose $|F(\{i\})| = 1$ for all i. For all $I \subseteq \{0, \dots, n\}$ we have*

$$F(I) = \{f(i) : i \in I\}.$$

Proof. For any $i \in I$ we have that $\{f(i)\}$ is either contained in or contains $F(I)$, so we must have $f(i) \in F(I)$. The same holds for I^c. □

An immediate consequence of all these facts is the following fact.

Proposition 5.12. *Up to possibly composing with σ and an element of S_{n+1}, any isometry of R that fixes origin also fixes all points $p_I^{\pm}$.*

Proof. Let F be an isometry of R fixing the origin. We shall also use F to denote the induced permutation of $\{0, \dots, n\}$, so that $F(P_I) = P_{F(I)}$.

By Remark 5.8 and Lemma 5.9, we may suppose that for all i we have $|F(\{i\})| = 1$. Hence by Lemma 5.11, F is induced by some permutation, f. We can think of this permutation as an isometry of R which permutes the petals of the rose. By composing F with the inverse of this isometry, we get that $F(I) = I$ for all subsets I of $\{0, \dots, n\}$. Thus F fixes all the points P_I and thus all the $p_I^{\pm}$. □

Next lemma is a simple case of a general asymptotic argument (see Section 6 and compare in particular with Proposition 6.4).

Lemma 5.13. *For any $i = 1, \dots, n$ and $t \in \mathbb{R}$, let $x_i(t)$ be the point of rose R, identified with $\mathbb{R}^n$, whose coordinates are zero except for ith which is t:*

$$x_i(t) = (0, \dots, t, \dots, 0), \quad t \text{ at the } i\text{th place}$$

and let $x_0(t) = (t, \dots, t)$. Then, points of R are determined by distances from points $x_i(t)$'s.

Proof. Let $y = (y_1, \dots, y_n)$ be a point of R. Clearly, for large enough t, we have

$$d(y, x_i(t)) = t - y_i + \max\{0, \sup_{j \neq i} y_j\}.$$

Therefore, by knowing such distances, we know for each i

$$-y_i + \max\{0, \sup_{j \neq i} y_j\}.$$

Note that the y_i's are all negative numbers if and only if such quantities are all positive, and in that case they give exactly $-y_i$. On the other hand, if some non-positive quantity appears, then indices i for which y_i is maximum are characterised by the fact that ith quantity is not positive. Thus, varying $i > 0$ we know all differences $y_i - y_j$ for any i, j.

Finally, consider distances from $x_0(t)$ as $t \to -\infty$

$$d(y, x_0(t)) = \max\{0, \sup_i (t - y_i)\} + \max\{0, \sup_i (y_i - t)\} = \max_i y_i - t.$$

This gives knowledge of $\max_i y_i$, and since we know those indices for which y_i is maximum, and all differences $y_i - y_j$, we get all the y_i's. □

Note that such a result can be re-paraphrased by saying that Busemann functions of ideal vertices determines points.

We are now able to finish the proof of Theorem 5.1. Let ϕ be an isometry of R. Up to composing with a translation of $\mathbb{R}^n$, we can suppose that ϕ fixes the origin. By Proposition 5.12 after possibly composing with elements of $S_{n+1} \times \langle\sigma\rangle$, we can suppose that ϕ fixes all the points $p_I^{\pm}$. Therefore, by Proposition 5.4, ϕ must fix all the points $x_i(t)$ and $x_0(t)$. Lemma 5.13 now implies that ϕ is the identity. □

We conclude this part anticipating results of subsequent sections. We have seen what the isometry group of a rose simplex is, and we have seen in particular that there are isometries which are not induced by elements of $Out(F_{n+1})$. This seems, a priori, to count as evidence against our final result. However, no such isometry arises as the restriction of a global isometry of CV_{n+1}. Indeed, we will show that translations of $\mathbb{R}^n$ and reflection σ are not restrictions of global isometries. On the other hand, any isometry in S_{n+1} is induced by a permutation of generators and hence by an element of $Out(F_{n+1})$ (see Sections 6 and 7, in particular Remark 7.8 and Lemma 7.9).

Note that this is not enough to show that $Isom(CV_{n+1})$ is $Out(F_{n+1})$. Indeed, there are two issues to be dealt with. First, we have to show that up to composing with an element of $Out(F_{n+1})$ we can reduce to the case where all simplices are left invariant. Second, it could happen that the restrictions of an isometry to different simplices are given by different elements of $Out(F_{n+1})$. We will show that this does not, in fact, happen.

6. Asymptotic distances from roses and global isometries

In this section we generalise calculations made in Section 5 about the asymptotic behaviour of distances. The underlying philosophy is that Busemann functions of ideal vertices are enough to distinguish points of Outer Space.

What we have in mind is to prove the following fact, that if an isometry fixes all rose-simplices of CV_n then it must be the identity. This, together with results of the next section, opens the way towards Theorem 1.1. The first point in proving that result is that a priori, an isometry that fixes all rose-faces, could permute other simplices. For that, we have to understand simplices that are possibly not invariant under the action of such an isometry. Lemma 6.1 below will tell us that any two putatively permuted simplices must have the same rose-faces.

Then, our aim will be to show that a point X is determined by asymptotic distances from points in the rose-faces of the simplex containing X. More precisely, we show how such distances determine the lengths, in X, of every almost simple closed curve.

We emphasise that the results we are proving here (out of necessity, due to Lemma 6.1) depend only on the set of rose-faces, and not on the simplex containing X.

That is to say, suppose X and Y are points of simplices Δ_1 and Δ_2 who share their rose-faces. If for any p in every rose-face we have $d(X, p) = d(Y, p)$, then we show that for any two a.s.c.c.

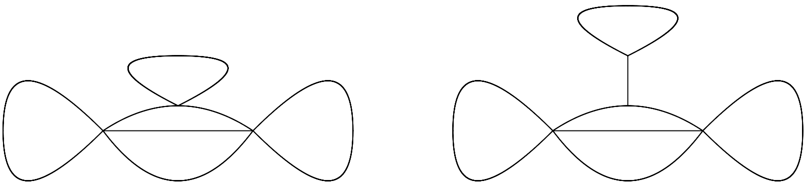


Fig. 2. Two graphs whose corresponding simplices share their four rose faces.

γ_1 and γ_2, $L_X(\gamma_1)/L_X(\gamma_2) = L_Y(\gamma_1)/L_Y(\gamma_2)$ (so lengths of a.s.c.c. are equal up to scaling). Of course, we need also to show that whenever Δ_1 and Δ_2 share rose-faces, then a loop γ is a.s.c.c. in Δ_1 if and only if the same happens in Δ_2. Since the distance from X to Y is computed using only a.s.c.c. (because of Lemma 2.16) we deduce that this implies $X = Y$.

We start by studying simplices possibly permuted by isometries that fix roses.

Lemma 6.1. *Let Φ be an isometry of CV_n that fixes all rose-simplices pointwise. If Δ is any simplex of CV_n, then Δ and $\Phi(\Delta)$ have the same rose-faces.*

Proof. Note that the closure of an open simplex Δ consists of Δ along with all of its faces. Therefore, R is a rose-face of Δ if and only if $d(x, \Delta) = 0$, for every $x \in R$.

Hence,

$$0 = d(x, \Delta) = d(\Phi(x), \Phi(\Delta)) = d(x, \Phi(\Delta)),$$

and R must also be a rose-face of $\Phi(\Delta)$. Using Φ^{-1} we get the converse. □

In order to provide simple examples of simplices having the same rose faces, let us describe a procedure for constructing them.

Start with a finite tree with no vertices of valence two. Now attach loops to the leaves so that in the resulting graph, all vertices have valence at least 3. This graph has an associated rose face, which corresponds to collapsing the original tree. Given two such graphs whose fundamental groups have the same rank, the corresponding rose faces are the same topologically, and we can mark the graphs so that they share a common rose face. The point here is that the rose face constructed is in fact unique, since the maximal tree is unique, and so the open simplices defined by these graphs share their only rose face, but are not in general the same simplex.

One can generalise this construction somewhat. Consider a simplex Δ with underlying graph G and suppose that v is a vertex in G of valence at least 4 whose removal disconnects G. Then we can construct a new graph G' by 'blowing up' the vertex v to an edge and dividing up the edges which start at v into two sets of size at least two. More formally, suppose that the edges starting at v are $e_1, \dots, e_k, f_1, \dots, f_r$ (we are dividing the edges into two sets of at least two edges, in an arbitrary fashion), then we make the new graph G' whose vertex set is $V(G) - \{v\} \cup \{v^+, v^-\}$. The edge set of G' is the same as that of G plus a single new edge e_-^+, which starts at v^- and ends at v^+. The incidence relations are as before except when they involve v, which is exactly for the edges $e_1, \dots, e_k, f_1, \dots, f_r$, where we (arbitrarily) decree that the edges e_i start at v^- and the edges f_j start at v^+. In this new graph, the edge e_-^+ is a part of any maximal tree, as it is separating. And we have an obvious map from G' to G which just collapses the edge e_-^+. Therefore, we can mark the simplex defined by G' so that it has Δ as a face. It is then clear that these two simplices have the same rose faces (see Fig. 2).

Now, we show that two simplices that share rose-faces have the same a.s.c.c.

Theorem 6.2. *Let Δ_1 and Δ_2 be two simplices of CV_n that share their rose-faces. Then they have the same set of almost simple closed curves. More precisely, if γ is a conjugacy-class in F_n, then its geodesic representative in Δ_1 is simple if and only if it is simple in Δ_2, and it is a figure-eight or bar-bell curve in Δ_1 if and only if the same is true in Δ_2 (possibly bar-bells become figure-eight curves and vice versa).*

Proof. Let G_1 and G_2 be marked graphs corresponding to simplices Δ_1 and Δ_2. Any rose-face of Δ_i is obtained by collapsing a maximal tree in G_i.

We first prove that a loop is simple in G_1 if and only if it is simple in G_2. Let γ be a simple loop in G_1, and let e_1 be an edge of γ. As e_1 is part of a simple loop, it does not disconnect G_1. Extend $\gamma \setminus e_1$ to a maximal tree T_1 in G_1. Let R be the rose obtained by collapsing T_1. The class of γ in R is represented by a petal p (the image of e_1). As Δ_1 and Δ_2 share rose-faces, R is obtained by collapsing a maximal tree T_2 in G_2. So the class of γ in G_2 is represented by an edge e_2 corresponding to the petal p plus a path in T_2. As T_2 is a tree, such path is unique and its union with e_2 is simple. Thus, γ is represented by a simple loop also in G_2.

Now, we deal with figure-eight and barbell curves. Let γ be such a curve in G_1. Let α and β be the two simple loops of γ.

Lemma 6.3. *Let R be any rose-face of G_1, then $\alpha \cup \beta$ is represented in R by a union of petals, each petal appearing at most once. In particular the representatives of α and β in R have no common petal.*

Proof. Let T be the maximal tree of G_1 collapsed in order to obtain R. Since T is a tree, and it is maximal, it cannot contain the whole α, nor the whole β. As α and β have no common edge, their images in R share no petal. Moreover, since α and β are simple, no petal can occur twice. □

Note that Lemma 6.3 would fail if $\alpha \cup \beta$ were a theta curve. We can now conclude the proof of Theorem 6.2. Let e_α be an edge of α and e_β be an edge of β. As e_α and e_β are parts of simple loops with no common edges, we have that $e_\alpha \cup e_\beta$ does not disconnect G_1. Extend $\gamma \setminus (e_\alpha \cup e_\beta)$ to a maximal tree T_1, and let R be the rose obtained by collapsing T_1. Let T_2 be the tree of G_2 whose collapsing gives R. The loop α is represented in G_2 by an edge corresponding to e_α, which we still denote by e_α, and a path σ_α in T_2 joining the end-points of e_α. The same (with the same notation) for β. The paths σ_α and σ_β have connected intersection because T_2 is a tree. It follows that the representative of γ in G_2 is either a figure-eight or a barbell, or a theta-curve. We show now that the case of theta-curve cannot arise.

Indeed, suppose representative of γ in G_2 is a theta-curve. This is equivalent to saying that $\sigma_\alpha \cap \sigma_\beta$ contains at least one edge e_0. Clearly, e_0 does not disconnect G_2. We can therefore find a maximal tree T_0 not containing e_0. Collapsing T_0 we get a rose R with a petal p_0 corresponding to e_0. In R, loops representing α and β share petals p_0. By Lemma 6.3 R cannot be obtained from G_1, in contradiction with the hypothesis that Δ_1 and Δ_2 have same rose-faces. □

Our next goal is to show that asymptotic distances from rose-faces determine points. First, we show how to determine lengths of simple closed curves using distances from rose-faces. After, we will deal with a.s.c.c.

Proposition 6.4 (*Distances from Roses Determine Simple Loops*)**.** *Let G_1 and G_2 be the underlying graphs of two simplices Δ_1 and Δ_2 having the same rose-faces. Let $X_1, X_2 \in \Delta_1 \cup \Delta_2$ such that $d(X_1, Y) = d(X_2, Y)$ for any point Y of any rose face of Δ_1 (or Δ_2).*

Now fix a conjugacy class ξ which is a simple loop in G_1 (and hence G_2) and suppose that X_1, X_2 are the representatives for which $L_{X_1}(\xi) = L_{X_2}(\xi) = 1$.

Then, for any *conjugacy class γ in F_n which is represented by a simple loop in G_1,*

$$L_{X_1}(\gamma) = L_{X_2}(\gamma).$$

Recall that γ is simple in G_1 if and only if it is simple in G_2 because of Theorem 6.2. Proposition 6.4 will follow from the next lemma.

Lemma 6.5. *Let R be a rose simplex in CV_n and let e, e_0 be petals in the underlying graph of R. Set Y_t to be the ray in R, consisting of roses in R all of whose edges except e, e_0 have length 1, and such that at time t*

$$L_{Y_t}(e) = t \qquad L_{Y_t}(e_0) = \frac{1}{t}.$$

Now consider an $X \in \Delta$, where Δ is a simplex of CV_n whose underlying graph is G and such that R is a rose-face of Δ.

Let γ_0 be the simple closed curve in X corresponding to e_0. Also let γ_e be an a.s.c.c. in X which minimises $\frac{L_X(\gamma_e)}{n_e(\gamma_e)}$, where $n_e(\gamma_e)$ is the number of times γ_e crosses e (when projected to R). Then,

$$L_X(\gamma_0) = C(e, X) \lim_{t\to\infty} \frac{e^{d(X,Y_t)}}{t^2} \tag{3}$$

where $C(e, X) = \frac{L_X(\gamma_e)}{n_e(\gamma_e)}$ as above.

NOTE: The ray Y_t depends only on the edges e, e_0, and the rose simplex R.

Proof. Let T be the maximal tree in G corresponding to the projection of Δ to R. We can find lifts of the edges e, e_0 in G. We continue to call these edges e and e_0.

Now consider the ray Y_t. We let $t \to \infty$ and study the asymptotic behaviour of $d(X, Y_t)$.

We claim that for sufficiently large t, the loop γ_0 is maximally shrunk from X to Y_t. Indeed, if $\sigma = e_1 \dots e_k$ is a loop, then

$$\frac{L_{Y_t}(\sigma)}{L_X(\sigma)} = \frac{\sum_{i:e_i=e_0} \frac{1}{t} + \sum_{i:e_i=e} t + \sum_{i:e_i\notin(T\cup e\cup e_0)} 1}{\sum_i L_X(e_i)}. \tag{4}$$

Since T is a tree, it cannot contain loops. Thus, if in σ there is some $e_0 \neq e_i \notin T$ the above stretching factor is bounded below uniformly on t. On the other side, if $\sigma = \gamma_0$, the stretching factor goes to zero as $t \to \infty$. Finally, if in σ there is no edge $e_i \notin T \cup e_0$, then σ is a multiple of γ_0 because T is a tree, and so γ_0 is the only way to obtain a simple loop from e_0 by adding edges of T.

Now, we look for maximally stretched loops. As above, we compute $L_{Y_t}(\sigma)/L_x(\sigma)$ for a generic a.s.c.c. σ using (4). If σ does not contain e, then there is an upper bound to the stretching factor and, as $t \to \infty$, it is readily checked that if (4) is maximised, then for big enough t the ratio of $L_X(\sigma)$ over the number of occurrences of e in σ is minimised; hence

$$\frac{L_X(\sigma)}{n_e(\sigma)} = \frac{L_X(\gamma_e)}{n_e(\gamma_e)} := C(e, X).$$

It follows, that for sufficiently large t, if σ is maximally stretched we have

$$d(X, Y_t) = \log \frac{L_{Y_t}(\sigma)}{L_X(\sigma)} \frac{L_X(\gamma_0)}{L_{Y_t}(\gamma_0)} = \log \frac{L_X(\gamma_0)}{L_X(\sigma)} t \left(n_e(\sigma)t + b + n_{e_0}(\sigma)\frac{1}{t} \right) \tag{5}$$

where $n_e(\sigma)$, $n_{e_0}(\sigma)$ are either 1 or 2 as σ is a.s.c.c., b is the number of edges of σ not belonging to $T \cup e_0 \cup e$. Whence,

$$\frac{L_X(\gamma_0)}{L_X(\sigma)} = \lim_{t\to\infty} \frac{e^{d(X,Y_t)}}{n_e(\sigma)t^2}$$

so

$$\begin{aligned} L_X(\gamma_0) &= \frac{L_X(\sigma)}{n_e(\sigma)} \lim_{t\to\infty} \frac{e^{d(X,Y_t)}}{t^2} \\ &= \frac{L_X(\gamma_e)}{n_e(\gamma_e)} \lim_{t\to\infty} \frac{e^{d(X,Y_t)}}{t^2} \\ &= C(e, X) \lim_{t\to\infty} \frac{e^{d(X,Y_t)}}{t^2} \end{aligned}$$

and the lemma is proved. □

Proof of Proposition 6.4. Let X be a point of either Δ_1 or Δ_2. Let γ and η be two simple closed curves in G_1. Choose an edge e_0 in γ (but not in η) and an edge f_0 in η (but not in γ). We can then find a maximal tree T in G which extends $\gamma \cup \eta - (e_0 \cup f_0)$. Let R be the corresponding rose face of Δ. Note that in any rose within this simplex γ and η each project to a single petal, which we will call e_0 and f_0 (these petals are also the projections of those edges).

Now assume that $n \geq 3$ so that we can find yet another petal, e, distinct from e_0, f_0.

By Lemma 6.5, there is a ray Y_t such that,

$$L_X(\gamma) = C(e, X) \lim_{t\to\infty} \frac{e^{d(X,Y_t)}}{t^2}.$$

Similarly, there is a ray Z_t such that,

$$L_X(\eta) = C(e, X) \lim_{t\to\infty} \frac{e^{d(X,Z_t)}}{t^2}.$$

Hence,

$$\frac{L_X(\gamma)}{L_X(\eta)} = \lim_{t\to\infty} \frac{e^{d(X,Y_t)}}{e^{d(X,Z_t)}}. \tag{6}$$

Moreover, by Lemma 6.5, this last equation must hold for *any* X which has R as a rose face (where we simply interpret γ, η as conjugacy classes of F_n) and thus certainly for any $X \in \Delta_1 \cup \Delta_2$. Thus, for the X_1, X_2 in the statement of the Proposition,

$$\frac{L_{X_1}(\gamma)}{L_{X_1}(\eta)} = \frac{L_{X_2}(\gamma)}{L_{X_2}(\eta)}$$

for any two loops γ, η which are simple in G_1 (and hence G_2). Putting $\eta = \xi$ proves Proposition 6.4 when $n \geq 3$.

Now consider the case $n = 2$. Note that here, distinct simplices have different collections of rose faces (so we need not worry about Δ_2). If the underlying graph of X is a rose, Proposition 6.4 is trivially true.

If the underlying graph of X is a barbell, then there is exactly one rose face and exactly two simple loops, γ, η. The limits from (6) will give $\frac{L_X(\gamma)}{L_X(\eta)}$ and $\frac{L_X(\eta)}{L_X(\gamma)}$ and the lemma is again true in this case.

Finally, if the underlying graph of X is a theta curve, then X has exactly 3 edges, x, y, z, 3 rose faces and 3 simple closed curves, $x\overline{y}, x\overline{z}, y\overline{z}$. There are then 6 possible rays as in Lemma 6.5. However, each limit,

$$\lim_{t\to\infty} \frac{e^{d(X,Y_t)}}{t^2},$$

is equal to one of $\frac{L_X(x\overline{y})}{C(z,X)}, \frac{L_X(x\overline{z})}{C(y,X)}, \frac{L_X(y\overline{z})}{C(x,X)}$. Also note that $C(x, X)$ is simply the length of the shortest simple loop in X which crosses x, since an a.s.c.c. in X is actually a simple loop. Hence, $C(x, X)$ is equal to either $L_X(x\overline{y})$ or $L_X(x\overline{z})$.

Thus, if $x\overline{y}$ is the shortest simple loop in X, then $\frac{L_X(x\overline{y})}{C(z,X)}$ will be the smallest of the three limits, $C(x, X) = C(y, X) = L_X(x\overline{y})$ and conversely. From these observations, the Proposition follows easily. Take X_1, X_2 with the same distances to rose faces. Then the limits above, for X_1, X_2 respectively, produce the same ordered results (however, the C terms need to be evaluated in different X_i's).

Nevertheless, if without loss of generality, $x\overline{y}$ is the shortest loop in X_1, then the limit $L_{X_1}(x\overline{y})/C(z, X_1)$ will be least, and thus so will $L_{X_2}(x\overline{y})/C(z, X_2)$ and hence $x\overline{y}$ must also be the shortest loop in X_2. The Proposition now readily follows. □

Remark 6.6. We note that the constant C depends on X and on e, but not on γ_0 or e_0. Such a dependence is thus cancelled when we consider the ratio $L_X(\gamma)/L_X(\eta)$, which therefore actually depends only on asymptotic distances from X to rose-faces.

We now have sufficient tools for proving that asymptotic distances from X to rose-faces determine the lengths of all a.s.s.c., whence determine X.

Theorem 6.7. *Let Δ_1 and Δ_2 be simplices of CV_n with the same set of rose-faces. Let G_1 and G_2 be the underlying graphs of Δ_1 and Δ_2 respectively. Let ξ be a simple loop in G_1 (whence its representative in G_2 is a simple loop as well). For any class $[X]$ of metric graphs in $\Delta_1 \cup \Delta_2$ consider the representative X so that $L_X(\xi) = 1$. Now consider two such representatives, $X_1, X_2 \in \Delta_1 \cup \Delta_2$ such that $d(X_1, Y) = d(X_2, Y)$ for any Y in any rose face of Δ_1. Then, for any a.s.c.c. γ in G_1 (and hence G_2),*

$$L_{X_1}(\gamma) = L_{X_2}(\gamma).$$

Proof. The proof is in the same spirit as Proposition 6.4, but the situation now is a little more complicated.

Let X be a point in either Δ_1 or Δ_2. It will be sufficient to show that we can calculate the length of any a.s.c.c. in X by only using distances to rose faces.

By Proposition 6.4, we know that lengths in X of simple loops are determined via asymptotic distances to particular sequences of points, not depending on X. Thus, we can suppose that we already know the lengths of all simple loops in X, because we have normalised so that $L_X(\xi) = 1$. Thus what remains is to deal with figure-eight and barbell curves. Clearly, the length of a figure-eight curve is determined via Proposition 6.4. On the other hand, Theorem 6.2 tells that a figure-eight curve in Δ_1 may become a barbell in Δ_2. For this reason we treat figure-eight and barbell curves at the same time, considering a figure-eight as a barbell whose central segment is reduced to a point.

In order to do this, we proceed as in Proposition 6.4; for any given barbell curve, we build an appropriate sequence of points Y_t in some rose-face, such that the asymptotic distances from Y_t determine the length of the barbell.

Remark 6.8. At this point, the reader should be aware of the subtle difference in the argument from that in Proposition 6.4. Indeed, the points Y_t we constructed in Lemma 6.5 do not depend on X, but just on e_0 (hence on γ_0) and e. Here, the ray Y_t we shall define will actually depend on X, or at least seem to, and thus present a logical obstacle to our argument.

More precisely, the ray Y_t here will depend on lengths of simple loops in X. Intuitively speaking, the ray Y_t escapes to infinity in a rose face and the "slope" of this ray is determined by the lengths of simple loops in X. However, this is sound because of Proposition 6.4. So for any barbell curve, the ray we chose for computing its length is the same for both X_1 and X_2, thus barbells have same lengths in X_1 and X_2, and Theorem 6.7 will be proved.

The rank-two case is easy and left to the reader (just use the following argument without the need to introduce the edge e and the loop γ_e). Suppose $n \geq 3$.

Let γ be a barbell curve, possibly degenerate to a figure-eight curve, say in G_1. Let γ_1 and γ_2 be the two simple loops of γ, and let $e_1 \in \gamma_1$ and $e_2 \in \gamma_2$ be two edges. Clearly, $G_1 \setminus (e_1 \cup e_2)$ is connected. Extend $\gamma \setminus (e_1 \cup e_2)$ to a maximal tree T, and consider the rose R_T obtained by collapsing T. Since $n > 2$ there is an edge $e \notin (T \cup \gamma)$. Also, there is a simple loop not containing e (for instance, γ_1).

In R_T we still denote by e, e_1, e_2 the petals corresponding to e, e_1, e_2 respectively.

Now, look at simplex Δ_2. Since R_T is a rose-face also of G_2, it is obtained by collapsing a maximal tree T' in G_2. Therefore, petals e, e_1, e_2 correspond to edges of $G_2 \setminus T'$, and the representative of γ in G_2 is disjoint from e.

Now let Y_t be the point of R_T whose petals have length 1 except e, e_1, e_2 for which we set

$$L_{Y_t}(e) = t \qquad L_{Y_t}(e_1) = \frac{L_X(\gamma_1)}{t} \qquad L_{Y_t}(e_2) = \frac{L_X(\gamma_2)}{t}.$$

We now let $t \to \infty$. If $\sigma = l_1 \dots l_k$ is a loop, then (replace T with T' if $X \in \Delta_2$)

$$\frac{L_{Y_t}(\sigma)}{L_X(\sigma)} = \frac{\sum_{i:l_i=e_1} \frac{L_X(\gamma_1)}{t} + \sum_{i:l_i=e_2} \frac{L_X(\gamma_2)}{t} + \sum_{i:l_i=e} t + \sum_{i:l_i \notin (T \cup e_1 \cup e_2)} 1}{\sum_i L_X(e_i)}. \tag{7}$$

Note that by Lemma 2.16, the loop σ minimising the equation above is realised by an a.s.c.c. in Y_t (note that this statement is independent of t) and inspection of the Eq. (7) shows that the only possible candidates are e_1, e_2 and e_1e_2. It is then easy to see that e_1e_2 is a loop which realises the minimum, and this is exactly the realisation of γ in Y_t. (Note that when the barbell is actually a figure-eight, all three loops give the same answer, but our statement remains true.)

As in Proposition 6.4, one also checks that, for large enough t, any maximally stretched loop σ from X to Y_t (*i.e.* one that maximises (7)) must minimise the ratio of $L_X(\sigma)$ over the number of occurrences of e in σ, among all loops. Such a ratio is exactly the constant $C(e, X)$ introduced in Proposition 6.4.

Distances may then be computed, and we obtain an expression of the form,

$$d(X, Y_t) = \log \frac{L_{Y_t}(\sigma)}{L_X(\sigma)} \frac{L_X(\gamma)}{L_{Y_t}(\gamma)} = \log \frac{L_X(\gamma)}{L_X(\sigma)} \frac{t\left(at + b + c\frac{1}{t}\right)}{L_X(\gamma_1) + L_X(\gamma_2)}$$

where a is the number of occurrences of e in σ. Thus

$$\frac{L_X(\gamma)}{L_X(\gamma_1)+L_X(\gamma_2)} = \frac{L_X(\sigma)}{a}\lim_{t\to\infty}\frac{e^{d(X,Y_t)}}{t^2} = C(e,X)\lim_{t\to\infty}\frac{e^{d(X,Y_t)}}{t^2}.$$

Since $L_X(\gamma_1)$ and $L_X(\gamma_2)$ are known, we just need to determine $C(e,X)$, which is given by Lemma 6.5 in terms of asymptotic distances. Namely, if (Z_t) is the sequence of points given by Lemma 6.5 for computing the length of γ_1 we get $L_X(\gamma_1) = C(e,X)\lim_{t\to\infty} e^{d(X,Z_t)}/t^2$.

If one likes exact formulae, one would have to introduce sequences (Z_t^1) and (Q_t^1), given by Lemma 6.5 for the ratio $L_X(\gamma_1)/L_X(\xi)$; then look at sequences (Z_t^2) and (Q_t^2), given by Proposition 6.4 for the ratio $L_X(\gamma_2)/L_X(\xi)$, and get (remembering the normalisation $L_X(\xi)=1$, and noting that the edge e may occur in ξ so that all the sequences below may be different)

$$L_X(\gamma) = \lim_{t\to\infty}\frac{e^{d(X,Y_t)}}{e^{d(X,Z_t)}}\frac{e^{d(X,Z_t^1)}}{e^{d(X,Q_t^1)}}\left(\frac{e^{d(X,Z_t^1)}}{e^{d(X,Q_t^1)}}+\frac{e^{d(X,Z_t^2)}}{e^{d(X,Q_t^2)}}\right). \quad \square$$

Finally, we are able to deal with global isometries of CV_n, proving that isometries are determined by their restrictions to rose-simplices.

Theorem 6.9. *The only isometry of CV_n that fixes pointwise all rose-simplices is the identity.*

Proof. Let Φ be such an isometry. Let X be a point of a simplex Δ_1 of CV_n, and let $\Delta_2 = \Phi(\Delta_1)$. By Lemma 6.1, Δ_1 and Δ_2 share their rose-faces. By Theorem 6.2 simple loops in Δ_1 are also simple in Δ_2. In particular we can choose a simple loop ξ and consider representatives of metric graphs of Δ_1 and Δ_2 by imposing that the length of ξ is 1.

By Theorem 6.2, Δ_1 and Δ_2 have the same almost simple closed curves. Since Φ fixes points in rose simplices, for any Y in a rose-face of Δ_1, we have $d(X,Y) = d(\Phi(X),\Phi(Y)) = d(\Phi(X),Y)$. Then, Theorem 6.7 says that the lengths of almost simple closed curves are the same in X and $\Phi(X)$. Therefore, the Sausages Lemma 2.16 implies $X = \Phi(X)$ (whence $\Delta_1 = \Delta_2$). $\square$

7. Isometries of multi-theta simplices and their extensions

Recall that our main result is that isometries of Outer Space are all induced by automorphisms of the free group. By Theorem 6.9, it is enough to show that up to composing with automorphisms, we can reduce to the case of isometries that point-wise fix every rose-simplex, and we do that by studying isometries of multi-theta simplices.

Our first main result of this section is that isometries of multi-theta simplices are induced by permutations of edges. Thus we have no translations or inversions as in rose-simplices. In particular this also shows that translations and inversions of rose-simplices cannot arise as restrictions of global isometries of CV_n.

Then, we will prove that situation is in fact even more rigid. Indeed, we show that if two isometries coincide on a multi-theta simplex, then they coincide on all rose-simplices of CV_n (not only on faces of that simplex). This will basically conclude Theorem 1.1.

We start by proving the following theorem.

Theorem 7.1. *Let Δ be a multi-theta simplex, and let Φ be an isometry of Δ. Then Φ fixes the centre of Δ (recall Definition 2.9). Moreover, if Φ leaves invariant all the rose-faces of Δ, then it actually fixes them point-wise, and in that case Φ is the identity map on Δ.*

Before proving Theorem 7.1, we need to establish some preliminary technical lemmas. We follow the strategy sketched in schema of Section 3, focusing on the study of those pairs of points that are joined by a unique geodesic. We recall that Outer Space is not a geodesic space; nevertheless, in any simplex, segments (for the linear structure of the simplex) are geodesic. More precisely, if we are in CV_n, then the points within a multi-theta simplex Δ are specified by $n+1$ positive reals (giving an open n-simplex, since one further needs to projectivise), corresponding to the lengths of the $n+1$ edges. Then, given $x = (x_1, \dots, x_{n+1})$ and $y = (y_1, \dots, y_{n+1})$ we can consider the segment $\overline{xy} := (1-t)x + ty$ in Δ. This turns out to be a geodesic with respect to the symmetric Lipschitz metric. See [12] for details and proofs.

However geodesics, even within a given simplex, are in general not unique. Our strategy is broadly to determine sufficiently many "unique" geodesics.

Definition 7.2. A geodesic segment σ of CV_n is *rigid* if for any two points on it, the restriction of σ is the unique (unparameterised) geodesic joining them.

We fix now a multi-theta simplex Δ, and we denote by $e_0, \dots, e_n$ the (oriented) edges of underlying graph of Δ. Any point x in Δ is thus determined by lengths $L_x(e_i)$ of e_i in x. As usual, we denote by $\bar{e}_i$ the edge e_i with the inverse orientation.

We begin by describing a set of standard rigid geodesics of Δ.

Lemma 7.3 (*Standard Rigid Geodesics*). *Let $x \neq y$ be metric graphs in Δ. For any i let λ_i be the stretching factor of e_i from x to y:*

$$\lambda_i = \frac{L_y(e_i)}{L_x(e_i)}.$$

If the set of such stretching factors contains exactly two elements, none of them with multiplicity 2*, then the segment between x and y is rigid.*

Proof. This is a consequence of Lemmas 2.16 and 2.17.

Indeed, up to rearranging edges, we can suppose $\lambda_0 = \cdots = \lambda_k = \mu$ and $\lambda_{k+1} = \cdots = \lambda_n = \lambda$ for two numbers $\mu < \lambda$. By scaling the graph y by μ, we may reduce to the case where $\lambda_0 = \cdots = \lambda_k = 1 < \lambda_{k+1} = \cdots = \lambda_n = \lambda$. In particular, we have scaled y so that the edges $e_0, \dots, e_k$ have the same length in both x and y.

Let z be a point in a geodesic joining x and y. We claim that, possibly up to scaling, the edges $e_0, \dots, e_k$ are not stretched from x to z, while the edges $e_{k+1}, \dots, e_n$ are stretched all by the same amount between 1 and λ. That is to say, we scale z so that the length of e_0 in z is equal to the length of e_0 in both x and y. Now we claim that if z belongs to a geodesic joining x and y, then it belongs to the segment between x and y, which therefore is rigid.

Let us examine our claim. First, suppose that $k > 1$. Then, the loops $e_i\bar{e}_j$ with $i, j \leq k$ are minimally stretched from x to y. Thus, by Lemma 2.17 the same must be true from x to z. In particular all such loops are stretched by the same amount from x to z. As we have at least three such loops (because $k > 1$) this implies that the edge-stretching factors $L_z(e_0)/L_x(e_0), \dots, L_z(e_k)/L_x(e_k)$ all coincide.

This fact is also trivially true if $k = 0$, while the case $k = 1$ is impossible because the multiplicity of μ was supposed to be different from 2. So, possibly up to scaling, the edges $e_0, \dots, e_k$ are not stretched from x to z (they have the same length in each metric graph).

The same argument, now with maximally stretched loops, shows that edges $e_{k+1}, \dots, e_n$ are all stretched by the same amount (as above, $k \neq n-2$ because the multiplicity of λ is not 2) and by an amount which is between 1 and λ. $\square$

Note that rigid segments of the type just described, always emanate from any point x of Δ. Indeed it suffices to consider a set I of edges and consider a point y whose edge-lengths are equal to those of x for edges in I and, say, double those of x for remaining edges. As above, this will be a rigid geodesic which obviously extends to a rigid geodesic ray. This is why we call such geodesic "standard".

One can think these geodesics as being a standard set in the tangent space at x. Our objective now is to see that points of Δ can have more rigid geodesics emanating from them, and that such a set of "rigid" directions is minimal when x is the centre of Δ. We notice that this is a substantial difference with respect to case of rose-simplices, which are homogeneous as there is transitive action of translations.

Lemma 7.4 (*Rigid Geodesics from the Centre, in the Case of Rank at Least 3*)**.** *Suppose x is the centre of Δ. If $n \geq 3$, then any rigid geodesic through x is of the type described in Lemma 7.3.*

Proof. We scale x so that its edges have length one. We have to show that for any point y, if the segment $\overline{xy}$ is rigid, then the set of edge-stretching factors contains exactly two elements, none of them with multiplicity two.

Suppose first that we have two edge-stretching factors, one of them with multiplicity two. Up to scaling y and rearranging edges, we can suppose that the stretching factors of edges e_i are 1 for $i = 0, \ldots, n-2$ and λ for $i = n-1, n$. We show that in that case the segment from x to y is not rigid.

Without loss of generality we can suppose $\lambda > 1$. Let z be the middle point of such segments, that is to say

$$1 = L_z(e_0) = \cdots = L_z(e_{n-2}) \qquad L_z(e_{n-1}) = L_z(e_n) = \frac{1+\lambda}{2}.$$

Since $\lambda > 1$, the loops $e_i\bar{e}_j$ with $i, j < n-1$ (whose existence is guaranteed because $n \geq 3$) are minimally stretched, and $e_{n-1}\bar{e}_n$ is maximally stretched, both from x to y, from x to z and from z to y.

Moreover, since the inequalities in play are all strict, the same remains true if we slightly perturb the length of e_n (note that maximally and minimally stretched loops have no common edges). That is to say, if z_ε denote the graph whose edge-lengths are equal to those of z except for e_n, for which we set $L_{z_\varepsilon}(e_n) = L_z(e_n) + \varepsilon$, for small enough ε, it is still true that loops $e_i\bar{e}_j$ with $i, j < n-1$ are minimally stretched, and $e_{n-1}\bar{e}_n$ is maximally stretched, both from x to y, from x to z_ε and from z_ε to y. This implies

$$d(x, z_\varepsilon) + d(z_\varepsilon, y) = d(x, y).$$

Thus, as segments are geodesics, the union σ_ε of segments $\overline{xz_\varepsilon}$ and $\overline{z_\varepsilon y}$ is a geodesic between x and y. On the other hand it is clear that z_ε does not belong to segment $\overline{xy}$, so σ_ε is different from $\overline{xy}$ which is therefore not rigid.

It now remains to show that if we have at least three different stretching factors, then we can find a geodesic between x and y, different from a segment. As above, we can scale y, and rearrange edges so that $1 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n$.

Since $L_x(e_i) = 1$ for all i, the minimally stretched loops from x to y are all the $e_i\bar{e}_j$ for which $\lambda_i = \lambda_0 = 1$ and $\lambda_j = \lambda_1$, and maximally stretched ones are those $e_i\bar{e}_j$ for which $\lambda_i = \lambda_{n-1}$ and $\lambda_j = \lambda_n$.

Let z be the middle point of the segment from x to y. Let $\lambda \in \{\lambda_i\}$ be an edge-stretching factor such that $1 \neq \lambda \neq \lambda_n$. Let z_ε be a metric graph whose edge-lengths are equal to those of z, except that for edges stretched by λ, which differ by ε

$$L_{z_\varepsilon}(e_i) = \begin{cases} L_z(e_i) & \lambda_i \neq \lambda \\ L_z(e_i) + \varepsilon & \lambda_i = \lambda \end{cases}$$

and let σ_ε be the union of segments $\overline{xz_\varepsilon}$ and $\overline{z_\varepsilon y}$.

It is clear – because we have at least three stretching factors – that z_ε does not belong to the segment $\overline{xy}$, whence $\sigma_\varepsilon \neq \overline{xy}$. If we show that σ_ε is a geodesic we are done. As above, it is enough to show that

$$d(x, z_\varepsilon) + d(z_\varepsilon, y) = d(x, y).$$

For that, we have to prove that there are loops γ_0 and γ_1 that are respectively minimally and maximally stretched from x to z_ε and from z_ε to y. This easily follows, for small enough ε, by the choice of λ. Indeed, it suffices (since the other cases are easier) to look at the situation when the stretching factors are $1, \lambda, \dots, \lambda, \lambda_n$. Here, min. and max. lops-stretching factors from x to y are $(1+\lambda)/2$ and $(\lambda+\lambda_n)/2$, realised by $e_0\bar{e}_i$ and $e_i\bar{e}_n$ for $i = 1, \dots, n$. Such loops are therefore min and max stretched both from x to z and from z to y, and perturbing λ a little such loops remain min. and max. stretched. □

Now, we show how Lemma 7.4 provides (in rank bigger than two) a metric characterisation of the centre of Δ as the point having the minimum number of rigid geodesics passing through it.

Lemma 7.5. *For any point x other than the centre of Δ, there is at least one rigid geodesic emanating from x which is not of the type described in Lemma 7.3.*

Proof. We denote by x_i the lengths $L_x(e_i)$. Up to scaling x and rearranging edges, we can suppose that

$$1 = x_0 \geq x_1 \geq \cdots \geq x_n.$$

We want to find stretching factors

$$1 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n$$

at least three of them being different, such that segment between x and point y corresponding to graph whose edges have length $\lambda_i x_i$, is rigid. As three of the λ_i are different, this will prove the lemma.

Let us start by making the simplifying assumption that $x_n \neq x_1$.

Stretching factors, from x to y, of loops $e_i\bar{e}_j$ are $\frac{\lambda_i x_i + \lambda_j x_j}{x_i + x_j}$, and if $1 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n$, an immediate calculation shows that whenever $j \geq i$ we have

$$\frac{1 + \lambda_i x_i}{1 + x_i} \leq \frac{\lambda_i x_i + \lambda_j x_j}{x_i + x_j} \leq \frac{\lambda_j x_j + \lambda_n x_n}{x_j + x_n}.$$

This implies that if we are searching for minimally (respectively maximally) stretched loops, we can restrict to loops of the form $e_0\bar{e}_i$ (respectively $e_i\bar{e}_n$).

The idea is now to force such loops to have the same stretching factors. We impose conditions

$$\lambda_1 = \frac{1 + x_1}{x_1}$$

and, for $i > 0$

$$\frac{1 + \lambda_i x_i}{1 + x_i} = \frac{1 + \lambda_1 x_1}{1 + x_1} = \frac{2 + x_1}{1 + x_1}. \tag{8}$$

We remark that the assumption on λ_1 is for simplifying calculations, we only need $\lambda_1 > 1$.

We can solve these equations getting

$$\lambda_i = \frac{(2 + x_1)(1 + x_i)}{(1 + x_1)x_i} - \frac{1}{x_i} = 1 + \frac{1 + x_i}{(1 + x_1)x_i}$$

thus $\lambda_i \geq \lambda_1$, with equality if and only if $x_i = x_1$, and $\lambda_i \leq \lambda_j$ for $j \geq i$, with equality if and only if $x_i = x_j$. In particular, under our simplifying assumption, we have $\lambda_0 = 1 < \lambda_1 < \lambda_n$, so at least three of the λ_i's are different.

So we get numbers λ_i's with the requested properties. Now, let y be the point of Δ given by

$$L_y(e_i) = \lambda_i x_i$$

and let z be any point in a geodesic between x and y, scaled so that $L_z(e_0) = 1$. We define μ_i by

$$L_z(e_i) = \mu_i x_i.$$

Loops $e_0\bar{e}_i$ are minimally stretched from x to y. Thus, we must have that such loops are minimally stretched from x to z and from z to y. This forces the edge-stretching factors μ_i to satisfy condition (8), which allows us to obtain μ_i as a function of μ_1 exactly as λ_i is obtained from λ_1. This implies that, if z' is the point in the geodesic line between x and y with first edge-stretching factor equal to μ_1, we have that $z = z'$.

So z belongs to the segment $\overline{xy}$ which is hence rigid, and not of the type described in Lemma 7.3.

We are now left with the case in which $x_n = x_1$ and so $x_i = x_j$ for any $i, j \neq 0$. As we are supposing that x is not the centre of Δ, we must have $x_0 \neq x_1$. Up to scaling x and rearranging edges, this case is equivalent to

$$(x_0, \dots, x_n) = (1, \dots, 1, c)$$

with $c > 1$.

We choose y of the form

$$y = (1, \lambda, \dots, \lambda, \mu c).$$

Stretching factors of simple loops are

$$\frac{1 + \lambda}{2}, \qquad \frac{1 + \mu c}{1 + c}, \qquad \lambda, \qquad \frac{\lambda + \mu c}{1 + c}.$$

Now, we impose conditions

$$\mu c = 1, \qquad \frac{1 + \lambda}{2} = \frac{1 + \mu c}{1 + c}$$

which imply that $\lambda \neq \mu$ because $c \neq 1$, and $\lambda < 1$. Whence

$$\frac{1 + \lambda}{2} = \frac{1 + \mu c}{1 + c} > \max\left(\lambda, \frac{\lambda + \mu c}{1 + c}\right).$$

So all the loops $e_0\bar{e}_i$ are maximally stretched from x to y (and in particular, stretched by the same amount). Now we argue as before: the same must be true for any point z on any geodesic from x to y, and this forces z to be of the form (once scaled so that $L_z(e_0) = 1$)

$$z = (1, \bar{\lambda}, \dots, \bar{\lambda}, \bar{\mu}c)$$

with

$$\frac{1+\bar{\lambda}}{2} = \frac{1+\bar{\mu}c}{1+c}.$$

As above, this implies that z belongs to the segment $\overline{xy}$, which is then rigid and it is not of the type described in Lemma 7.3 because $1 \neq \lambda \neq \mu \neq 1$. □

Lemma 7.6 (*Rigid Geodesics in Rank Two*)**.** *Let $x \neq y$ be two marked metric graphs in Δ. Suppose $n = 2$, so that Δ has exactly three different (unoriented) simple loops. Then the segment $\overline{xy}$ is rigid if and only if two of the three simple loops are stretched by the same amount from x to y.*

Proof. The proof uses the same arguments of higher rank case, but takes in account the peculiarities of rank two.

If the three simple loops are stretched by three different factors, then for any point w close enough to the middle point z of $\overline{xy}$, the maximally and minimally stretched loops do not change from x to w, from w to y and from x to y, so that $\overline{xy}$ is not rigid.

On the other hand, if two simple loops are stretched by the same factor, we may rearrange the edges so that e_0 is the edge shared by such loops, and scale graphs so that $L_x(e_0) = L_y(e_0) = 1$. Moreover, as we have only three simple loops, $e_0\bar{e}_1$ and $e_0\bar{e}_2$ are either maximally or minimally stretched from x to y. So the same must be true from x to z and from z to y for any point z in a geodesic between x and y. If $x = (1, a, b)$ and $y = (1, \lambda a, \mu b)$, we have

$$\frac{1+\lambda a}{1+a} = \frac{1+\mu b}{1+b}$$

and the same relation holds for the edge stretching factors of point z which therefore belongs to the segment $\overline{xy}$. □

Lemma 7.7. *For any rose-face of Δ there is a unique rigid geodesic from the centre of Δ to that face.*

Proof. By Lemmas 7.4 and 7.6, a rigid geodesic emanating from the centre is of the type described in Lemma 7.3 (and Lemma 7.6 in the rank-2 case). A rose-face corresponds to collapsing an edge, say e_0. So in a rigid geodesic from the centre to that face we have $\lambda_0 = t$ and $\lambda_i = 1$ for $i > 0$, with $t \in [1, 0]$. Therefore such geodesic is unique. □

Now, we continue with the proof of Theorem 7.1. We begin by examining the first claim in the rank-two case. Since permutations of edges of Δ are isometries that fix its centre and permute its rose-faces, up to composing Φ with such a permutation we can suppose that Φ does not permute rose-faces of Δ. If the restriction of Φ to a rose-face has a translational part, then for any point x in that face we see that the distance of $\Phi^n(x)$ from at least one of the remaining two rose faces of Δ goes to infinity, this being impossible because Φ is an isometry. It follows that Φ fixes the centres of rose-faces of Δ. Explicit calculations (using Lemma 7.6, see the Appendix) show that the centre of Δ is the unique point which is joined to the centres of the three rose-faces by rigid geodesics. Thus Φ fixes the centre of Δ, and the first claim of Theorem 7.1 is proved for $n = 2$.

If $n \geq 3$, Lemmas 7.4 and 7.5 imply that any isometry Φ of Δ must fix its centre; so the first claim of the theorem is proved. Moreover, if Φ does not permute rose-faces, then by Lemma 7.7

it must fix point-wise rigid geodesics emanating from x and going to rose-faces. In particular, Φ fixes centres of rose-faces.

Remark 7.8. Note that we have proved that if Φ does not permute rose-faces of Δ, then its restriction to any rose-face has no translational parts, which is to say that it fixes the centre of rose-faces.

Therefore, by Theorem 5.1, restriction of Φ to rose-faces of Δ is an element of $S_n \times \langle\sigma\rangle$. In the next lemma we show that such an element must be the identity. We first introduce some terminology.

Let R_i denote the rose-face of Δ obtained by collapsing edge e_i, and let C_i denote its centre. Also, for $i \neq j$ we let $\Gamma^i_j(\epsilon)$ denote the point of R_j all of whose petals have length 1 except for e_i which has length ϵ. For $i \neq j$, straightforward calculations show that

$$d(C_i, \Gamma^k_j(2)) = \begin{cases} \log 6 & i = k \quad \text{in any rank} \\ \log 6 & i \neq k \quad \text{in rank bigger that 2} \\ \log 3 & i \neq k \quad \text{in rank 2} \end{cases} \tag{9}$$

and

$$d(C_i, \Gamma^k_j(0.5)) = \begin{cases} \log 3 & i = k \quad \text{in any rank} \\ \log 8 & i \neq k \quad \text{in rank bigger that 2} \\ \log 6 & i \neq k \quad \text{in rank 2.} \end{cases} \tag{10}$$

Note that in the rank-2 case, for $i \neq j \neq k$ we have $\Gamma^k_j(2) = \Gamma^i_j(0.5)$, up to scaling.

Lemma 7.9. *Let Φ be an isometry of a multi-theta simplex Δ which fixes the centres of its rose-faces. Then Φ is the identity on each rose face.*

Proof. By Theorem 5.1, the restriction of Φ to any rose face is an element of $S_n \times \langle\sigma\rangle$. Hence the image of the point $\Gamma^i_j(2)$ is either $\Gamma^k_j(2)$ or $\Gamma^k_j(0.5)$ for some k. However, by (9) and (10), since each C_i is fixed by Φ the distances to $\Gamma^i_j(2)$ are preserved and we must have that $\Gamma^i_j(2)$ is actually fixed by Φ. Since this is true for every $i \neq j$, and since the only element of $S_n \times \langle\sigma\rangle$ which fixes all these is the identity, we get that Φ restricts to the identity on any R_j. □

We can now finish the proof of Theorem 7.1. We proved that any isometry of Δ fixes its centre, and that if it does not permute rose-faces R_i, then it fixes their centres C_i. By Lemma 7.9 this implies that Φ fixes rose-faces of Δ point-wise. Now let $x \in \Delta$. For any y in some R_i, we have $d(x, y) = d(\Phi(x), y)$ (because R_i is fixed.) Therefore, Theorem 6.7 tells us that lengths of a.s.c.c. in x and $\Phi(x)$ coincide. Thus, by Lemma 2.16 we have $d(x, \Phi(x)) = 0$. It follows that Φ is the identity of Δ, and the proof of Theorem 7.1 is concluded. □

Now we come to the other main result of this section, that is that for an isometry of CV_n, what happens on a single multi-theta simplex determines the isometry on the whole CV_n. The first step is to show that if an isometry of a multi-theta simplex is the identity on a rose-face, then it is the identity of the multi-theta simplex. Our claim will follow then by a connectivity argument.

Lemma 7.10. *Let Φ be an isometry of a multi-theta simplex Δ which restricts to the identity on one of the rose-face of Δ. Then Φ restricts to the identity on each rose-face of Δ.*

Proof. Let R_0 be the rose-face fixed by hypothesis. By Lemma 7.9, it is sufficient to show that Φ fixes each centre C_i. By first claim of Theorem 7.1, we know that the centre of Δ is fixed.

By Lemma 7.7, there is a unique rigid geodesic from the centre to each rose face, ending in C_i. Hence, the C_i are permuted by Φ.

However, the stabiliser in $Out(F_n)$ of Δ contains a subgroup isomorphic to S_{n+1}, by simply permuting the edges of the underlying graph of Δ, and this subgroup will induce every permutation of the $n+1$ rose-faces of Δ. Hence, by Lemma 7.9, Φ is equal to the restriction of some element of S_{n+1} (in fact, some such element which fixes the edge corresponding to the fixed rose-face). But the only element of this sort which restricts to the identity in a rose face is the identity. (This also follows from (9) and (10).) □

Theorem 7.11. *Let Φ be an isometry of CV_n that point-wise fixes a multi-theta simplex. Then it point-wise fixes all rose and multi-theta simplices of CV_n.*

Proof. We start by doing a simple calculation. Let Δ be a multi-theta simplex of CV_n, with edges oriented and labelled $e_0, e_1, \dots, e_n$. For an edge e, we denote by $\overline{e}$ the edge e with inverse orientation. Let R_i be the rose face of Δ obtained by collapsing e_i. We will label the edges of R_i, $e_0^i, e_1^i, \dots, e_{i-1}^i, e_{i+1}^i, \dots, e_n^i$.

Now let us explicitly write down the homotopy equivalences between R_0 and R_i in terms of these edges. The map from R_0 to R_i is given by the following:

$$\begin{aligned} e_j^0 &\mapsto e_j^i\overline{e_0^i}, \quad j \neq i \\ e_i^0 &\mapsto \overline{e_0^i}. \end{aligned} \tag{11}$$

Similarly, the map from R_i to R_0 is given by

$$\begin{aligned} e_j^i &\mapsto e_j^0\overline{e_i^0}, \quad j \neq 0 \\ e_0^i &\mapsto \overline{e_i^0}. \end{aligned} \tag{12}$$

This in particular implies that the sub-complex of CV_n consisting of multi-theta and rose-simplices is connected, as we realised Nielsen automorphisms passing from a rose-face to another in a multi-theta simplices.

Now, it would seem that we are done simply by starting from our initial fixed multi-theta simplex and extending our results, via Lemma 7.10, over the whole of CV_n. The only problem is that we do not know, *a priori*, that Φ does not induce some non-trivial permutation of the multi-theta simplices. Therefore, we need to rule out this possibility.

Remark 7.12. The next Lemma is an "elementary" proof of the fact that permutations of multi-theta simplices do not occur. The calculations it involves are somewhat tedious and the reader may prefer to invoke the result of Bridson and Vogtmann [6] asserting that simplicial actions on the spine of CV_n (see [6] for definitions and details) come from automorphisms (for $n \geq 3$). Then, she could show that isometries naturally induce such actions on the spine, and since the spine encodes the combinatoric of roses and multi-theta incidences, get the desired result. We present here the proof of Lemma 7.13 as follows because it is self-contained and more in the spirit of the techniques of the present work.

Lemma 7.13. *Let Δ be a multi-theta simplex, and R a rose face of it. Suppose that $\Delta_1, \dots, \Delta_k$ are all the other multi-theta simplices in CV_n which are incident to R. Let Φ be an isometry of CV_n which point-wise fixes Δ (and therefore R). Then Φ leaves each Δ_i invariant.*

Proof. Consider our multi-theta simplex Δ which is given by a graph with 2 vertices and $n+1$ edges, ordered and labelled $e_0, \ldots, e_{n+1}$. As usual, for an edge e we denote by $\overline{e}$ the one with inverse orientation. Moreover, we chose orientations so that the e_i's share the same initial vertex (so they also share the terminal vertex). We will let R denote the rose simplex obtained by collapsing the edge e_0.

It is now an easy exercise to see that there are 2^{n-1} multi-theta simplices incident to R. Therefore, the result is trivial in CV_2 and we shall restrict our attention to CV_n for $n \geq 3$.

We shall describe the set of multi-thetas incident to R by listing the homotopy equivalences from Δ. Specifically, choose some $I \subseteq \{1, \ldots, n\}$ and consider the homotopy equivalence on Δ given by

$$\begin{aligned} e_0 &\mapsto e_0 \\ e_i &\mapsto e_i, \quad i \notin I \\ e_i &\mapsto e_0\overline{e_i}e_0, \quad i \in I. \end{aligned}$$

It is then clear that the set of all multi-thetas incident to R will be given by these maps. However, we note that replacing I by its complement gives the same simplex, so we have counted each twice. From now we will make a choice between I and I^c so that $|I| \geq |I^c|$ (or $I = \emptyset$)—if $|I| = |I^c|$ the choice will be arbitrary. Hence if I is not empty it will have at least two elements, and its complement will be non-empty. Let Δ_I denote the multi-theta simplex obtained via the map above. This gives us our 2^{n-1} multi-thetas, with $\Delta = \Delta_\emptyset$.

Now, we will show that the distances from Δ will determine the Δ_I. Note that since we are dealing with multi-theta graphs, by the Sausages Lemma 2.16 the maximally and minimally stretched loops can be taken to be simple closed curves, which are straightforward to enumerate. Below, we present a list of curves. On the left side, we have curves in Δ and on the right side their image in Δ_I so that each simple closed curve in either Δ or Δ_I appears somewhere on the list (up to orientation). Throughout, we have that $i, j \neq 0$.

$$\begin{array}{rcl} \Delta & & \Delta_I \\ e_i\overline{e_0} & \mapsto & e_i\overline{e_0}, \quad i \notin I \\ e_i\overline{e_0} & \mapsto & e_0\overline{e_i}, \quad i \in I \\ e_i\overline{e_j} & \mapsto & e_i\overline{e_j}, \quad i, j \notin I \\ & \mapsto & e_0\overline{e_i}e_j\overline{e_0} \sim \overline{e_i}e_j, \quad i, j \in I \\ & \mapsto & e_0\overline{e_i}e_0\overline{e_j}, \quad i \in I, j \notin I \\ \overline{e_i}e_0\overline{e_j}e_0 & \mapsto & e_i\overline{e_j}, \quad i \in I, j \notin I. \end{array}$$

Now let us assign edge lengths and calculate distances. For each $\Delta_I \neq \Delta$, we will let all edge lengths equal 1, since we know that isometries preserve the centres. Next choose some $J \subseteq \{1, \ldots, n\}$ and let $\Delta(J, 1/3)$ be the graph Δ where each edge has length 1 except for the e_j which has length 1/3 for all $j \in J$. Moreover, let us stipulate that $J \neq \emptyset, \{1, \ldots, n\}$. We want to compute the distance between $\Delta(J, 1/3)$ and Δ_I. It is then an easy exercise to check the stretching factors for each of the simple loops in $\Delta(J, 1/3)$ and Δ_I. Clearly, this depends on the relationship between J and I. We list below, the possible stretching factors from $\Delta(J, 1/3)$ to Δ_I, with the condition which allows it. Some stretching factors can occur in more than one way, in which case we have removed the redundancy (an empty condition means the stretching factor is always realisable).

So distance is computed by taking the log of the ratio of the maximum over the minimum of the allowed factors. Recall that by the choice we made for I, we always have $|I| \geq 2$ and $|I^c| \geq 1$.

Possible stretching factors of loops from $\Delta(J, 1/3)$ to Δ_I	Condition
1	
3/2	
3	$\|I \cap J\| \geq 2$ or $\|I^c \cap J\| \geq 2$
2	$\|I \cap J^c\| \geq 1$ and $\|I^c \cap J^c\| \geq 1$
3	$\|I \cap J\| \geq 1$ and $\|I^c \cap J^c\| \geq 1$
3	$\|I \cap J^c\| \geq 1$ and $\|I^c \cap J\| \geq 1$
6	$\|I \cap J\| \geq 1$ and $\|I^c \cap J\| \geq 1$
1/2	$\|I \cap J^c\| \geq 1$ and $\|I^c \cap J^c\| \geq 1$
3/5	$\|I \cap J\| \geq 1$ and $\|I^c \cap J^c\| \geq 1$
3/5	$\|I \cap J^c\| \geq 1$ and $\|I^c \cap J\| \geq 1$
3/4	$\|I \cap J\| \geq 1$ and $\|I^c \cap J\| \geq 1$.

We now apply these conditions to calculate the distances from $\Delta(J, 1/3)$ to Δ_I when J has exactly 2 elements. Specifically,

- if $|J| = 2$, $|J \cap I| = |J \cap I^c| = 1$, and $|I^c| \geq 2$, then maximal and minimal stretching factors are 6 and 1/2, so the distance is $\log 12$;
- if $|J| = 2$, $|J \cap I| = |J \cap I^c| = 1$, and $|I^c| = 1$, then the max and min stretching factors are 6 and 3/5, whence the distance is $\log 10$;
- if $|J| = 2$ and $J \subseteq I$ or $J \subseteq I^c$, then the maximal stretching factor is always 3, and the distance is $\log 5$ or $\log 6$, depending on the sizes of I and I^c.

Hence, we may determine I and I^c. More precisely, the set $\{1\} \cup \{i \neq 1 : d(\Delta(\{1, i\}, 1/3), \Delta_I) = \log 5\} \cup \{i \neq 1 : d(\Delta(\{1, i\}, 1/3), \Delta_I) = \log 6\}$ is equal to either I or I^c.

Note that this does not let us distinguish which one we picked, but since Δ_I only depends on the pair I, I^c, this is sufficient to distinguish the simplex and proves Lemma 7.13. □

Now Theorem 7.11 follows. □

8. Proof of Theorem 1.1 and main results

We prove here results stated in Section 1.

Proof of Theorem 1.1. Our claim is that the isometry-group of Outer Space of rank-n free group, is just $Out(F_n)$ (for $n \geq 3$). Clearly, $Out(F_n)$ acts faithfully on CV_n for $n \geq 3$ and this action is by isometries (see for instance [12]). Thus, we have an inclusion of $Out(F_n)$ into the group of isometries of CV_n. For $n = 2$, we still have an isometric action, but this is no longer faithful. However, up to a small kernel, a subgroup of order 2, the map from $Out(F_2)$ to $Isom(CV_2)$ is injective. Our goal is to show that this exhausts the isometry group of CV_n (in either case).

Let Φ be an isometry of CV_n. We shall compose Φ with elements of $Out(F_n)$ until we obtain the identity.

By Lemma 4.2, Φ maps multi-theta simplices to multi-theta simplices. Therefore, since the action of $Out(F_n)$ on multi-theta simplices is transitive, we may suppose that Φ leaves invariant a multi-theta simplex Δ. In fact, the stabiliser in $Out(F_n)$ of Δ will induce any permutation of the $n + 1$ rose faces of Δ and so we may also assume that Φ leaves both Δ and every rose-face of Δ invariant.

Theorem 7.1 then implies that Φ is the identity of Δ. Then, by Theorem 7.11 Φ point-wise fixes all rose-simplices. And Theorem 6.9 implies that Φ is the identity. □

Proof of Theorem 1.2. Let us denote by $Isom_R(CV_n)$ the group of isometries of CV_n for the non-symmetric metric d_R, and by $Isom(CV_n)$ the group of isometries of CV_n for the symmetric metric d.

Let Φ be an isometry of CV_n for d_R. Then

$$\forall x, y \quad d_R(x, y) = d_R(\Phi x, \Phi y).$$

Since $d_L(x, y) = d_R(y, x)$, we have that Φ is also an isometry for d_L, whence Φ is an isometry for the symmetric Lipschitz metric d. Thus, $Isom_R(CV_n) \subseteq Isom(CV_n)$.

As for the symmetric case, one has that $Out(F_n) \subseteq Isom_R(CV_n)$ (with a small adjustment for rank 2). By Theorem 1.1 we have that $Isom(CV_n) = Out(F_n)$. Thus

$$Out(F_n) \subseteq Isom_R(CV_n) \subseteq Isom(CV_n) = Out(F_n).$$

The same for d_L. (The argument for rank 2 is the same.) □

Proof of Corollary 1.3. Every homomorphism from Γ to $Out(F_n)$ has finite image by Bridson and Wade [7] (see also [21,5]), and every finite subgroup of $Out(F_n)$ has a fixed point in its action on CV_n by Culler [8]. □

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This work was inspired by the beautiful articles [6,5], and most of the material of the introduction was picked from there.

Appendix. Rigid geodesics in rank two

Here we explicitly calculate rigid geodesics emanating from centres of rose-simplices and pointing into theta-simplices, for the rank-two case. Showing that for any theta-simplex, its centre is the unique point simultaneously joined to centres of all rose-faces by rigid geodesics.

We fix a theta-simplex Δ and we parametrise its points by (projective classes of) triples of positive numbers (x, y, z). Such simplex is a triangle with vertices removed, as can be seen by taking representatives unitary volume.

Let $(1, 0, 1)$ be the centre of a rose face of Δ and let $(1, z, y)$ be a point joined to it by a rigid segment, scaled so that $x = 1$. Stretching factors are

$$1 + z, \qquad z + y, \qquad \frac{1 + y}{2}$$

(the loop with stretching factor $z + (1 + y)/2$ is not relevant).

By Lemma 7.6 we must have only two stretching factors from $(1, 0, 1)$ to $(1, z, y)$. Possible cases are $1 + z = z + y$, $1 + z = \frac{1+y}{2}$, $z + y = \frac{1+y}{2}$. If

$$1 + z = z + y$$

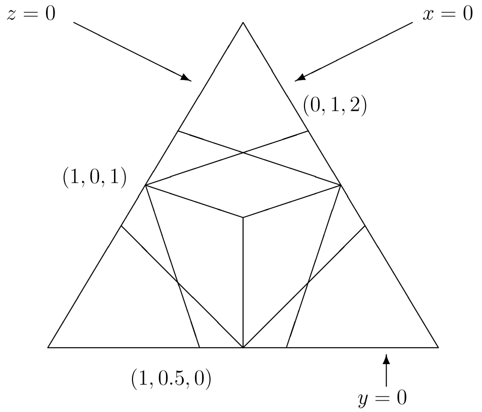
then $y = 1$ and this is the rigid geodesic going to the centre of Δ. If

$$1 + z = \frac{1 + y}{2}$$

then $y = 1 + 2z$, then $(1, y, z) = (1, z, 1 + 2z)$. We want to know where such geodesic hits other rose-faces. Letting $z \to \infty$ and scaling by z we get $(1/z, 1, 2 + 1/z)$ which ends up to the point $(0, 1, 2)$. Finally,

$$z + y = \frac{1 + y}{2}$$

gives $z = (1 - y)/2$, so that $(1, y, z) = (1, (1 - y)/2, y)$. Letting $y \to 0$ we get $(1, 0.5, 0)$. The picture of rigid geodesics through the centres is therefore as follows.



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