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Isomorphisms of quantizations via quantization of resolutions ☆

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Abstract

In this paper, we prove the existence of isomorphisms between certain non-commutative algebras that are interesting from the representation theoretic perspective and arise as quantizations of certain Poisson algebras. We show that quantizations of Kleinian singularities obtained by three different constructions are isomorphic to each other. The constructions are via symplectic reflection algebras, quantum Hamiltonian reduction, and W-algebras. Next, we prove that parabolic W-algebras in type A are isomorphic to quantum Hamiltonian reductions associated to quivers of type A. Finally, we show that the symplectic reflection algebras for wreath-products of the symmetric group and a Kleinian group are isomorphic to certain quantum Hamiltonian reductions. Our results involving W-algebras are new, while for those dealing with symplectic reflection algebras we just find new proofs. A key ingredient in our proofs is the study of quantizations of symplectic resolutions of appropriate Poisson varieties.

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Keywords: Quantization; Symplectic resolution; Symplectic reflection algebra; Quiver variety; Quantum hamiltonian reduction; W-algebra

Contents

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1. Introduction

In this paper, we prove the existence of isomorphisms between quantizations of certain graded Poisson algebras. We start by giving all necessary general definitions.

A base field will be the field $\mathbb{C}$ of complex numbers. Let A be a commutative associative unital algebra over $\mathbb{C}$ equipped with a Poisson bracket $\{\cdot,\cdot\}$. We suppose that A is graded, $A=\bigoplus_{i\in\mathbb{Z}_{\geqslant 0}} A_i$, and that the Poisson bracket has degree $-d$ with $d>0$, $\{A_i,A_j\}\subset A_{i+j-d}$ (in most examples, $d=2$).

Now let $\mathcal{A}$ be an associative unital algebra equipped with an increasing exhaustive filtration $\mathrm{F}_i\mathcal{A}$, $i\geqslant 0$. Suppose that $[\mathrm{F}_i\mathcal{A},\mathrm{F}_j\mathcal{A}]\subset \mathrm{F}_{i+j-d}\mathcal{A}$. Then the associated graded algebra $\operatorname{gr}\mathcal{A}$ is commutative. Moreover, it comes equipped with a natural Poisson bracket of degree $-d$ induced by taking the top degree term in the Lie bracket of elements of $\mathcal{A}$. We say that $\mathcal{A}$ is a *quantization* of A if the graded Poisson algebras $\operatorname{gr}\mathcal{A}$ and A are isomorphic.

Perhaps, the easiest example is as follows. Let V be a vector space equipped with a symplectic (= skew-symmetric and non-degenerate) form ω. Then SV becomes a Poisson algebra with bracket uniquely determined from $\{u, v\} = \omega(u, v), u, v \in V$. This algebra is graded in the standard way, the Poisson bracket has degree -2. It admits a (unique, in fact) quantization called the *Weyl algebra* $\mathbb{A}(V)$ of V. This is the quotient of the tensor algebra $T(V)$ by the relations $u \otimes v - v \otimes u - \omega(u, v) = 0$. The algebra $\mathbb{A}(V)$ is filtered by the order of a monomial.

In general, a graded Poisson algebra A admits many non-isomorphic quantizations and it is difficult to describe them. Many constructions (such as a Hamiltonian reduction, see Section 3.1 for the definition) produce quantizations depending on a parameter (for a quantum Hamiltonian reduction a parameter will be a character of a Lie algebra used to reduce). Basically, this paper concentrates on the following problem: suppose we have two families of quantizations of A parameterized by two (isomorphic) parameter spaces. Given a parameter for the first family determine a parameter for the second one producing an isomorphic quantization.

Of course, this problem is too vague and too general to approach. We will deal with some cases that are interesting from the representation theoretic perspective. Also in all of the cases we are going to consider that algebra A will have some nice algebro-geometric properties. Namely, we will assume that A is finitely generated and integral, and that the Poisson variety $X_0 := \mathrm{Spec}(A)$ admits a symplectic resolution of singularities; see Section 4.1 for a definition. The existence of a symplectic resolution is crucial for us, because we will approach quantizations of A via the quantization of the structure sheaf or more general vector bundles on the resolution.

For different classes of varieties X_0 we have various classes of quantizations. Let us describe them briefly.

The first class of varieties comes from Lie theory. A *Slodowy slice* $\mathrm{S}(=\mathrm{S}(\mathbb{O}))$ in a semisimple Lie algebra $\mathfrak{g}$ is a transversal slice to a nilpotent orbit $\mathbb{O}$. The algebra $\mathbb{C}[\mathrm{S}]$ comes equipped with a natural grading and a natural Poisson bracket of degree -2. Choose a parabolic subgroup $P \subset G$, where G is a connected algebraic group corresponding to $\mathfrak{g}$. Then $X_0 := \mathrm{S} \cap G\mathfrak{p}^{\perp}$ is a Poisson subvariety of S (a parabolic Slodowy slice). The algebra $\mathbb{C}[\mathrm{S}]$ admits a natural quantization—a finite W-algebra $\mathcal{W}$. We will recall two definitions of $\mathcal{W}$ in Section 5.1. So one can hope to construct quantizations of X_0 as quotients of $\mathcal{W}$. This can be done under some restrictions on $\mathfrak{g}, \mathbb{O}$ and P, the corresponding quotients (*parabolic W-algebras*) are defined in Section 5.2. In this paper, we are concerned with two cases:

(1) $\mathfrak{g}$ is of type A, D or E, and $\mathbb{O}$ is a subprincipal nilpotent orbit.
(2) $\mathfrak{g}$ is of type A, $\mathbb{O}$ is any nilpotent orbit, and P is a certain parabolic subgroup (of course, we want $\mathrm{S} \cap G\mathfrak{p}^{\perp}$ to be non-empty).

Let us proceed to the second class of varieties. Now suppose that X_0 is obtained by Hamiltonian reduction of a symplectic vector space by a reductive group, the definition is recalled in Section 3.1. One can hope to get a quantization of $\mathbb{C}[X_0]$ by replacing a classical Hamiltonian reduction with a quantum one, Section 3.3, and this hope is fulfilled under some technical assumptions on the action. One special case of reductive group actions on symplectic vector spaces comes from quivers. The corresponding classical reductions are Nakajima's quiver varieties; see, for example, [38]. These varieties together with the corresponding quantizations are recalled in Section 4.2. Each quiver variety is constructed from a quiver together with two dimension vectors. In this paper, we are concerned with two classes of quivers and of dimension vectors:

(1) Finite Dynkin quivers of type A and dimension vectors subject to certain inequalities.
(2) Affine Dynkin quivers and dimension vectors as in [12].

Finally, the third class of varieties we consider is as follows. Let V be a symplectic vector space, and $\mathcal{G}$ be a finite subgroup of $\mathrm{Sp}(V)$. Consider the symplectic quotient singularity $X_0 = V^*/\mathcal{G}$. Here one can produce quantizations of $\mathbb{C}[X_0]$ by studying deformations of the "orbifold" algebra $SV\#\mathcal{G}$, that is the semidirect product of the symmetric algebra SV and the group algebra $\mathbb{C}\mathcal{G}$. These deformations are known as symplectic reflection algebras (SRAs) and appeared first in this context in [8] in the special case when $V = \mathbb{C}^2$ and in [13] in the general case. For a deformation of $\mathbb{C}[X_0]$ one takes a so called *spherical subalgebra* $e\mathbf{H}e$, where e is the trivial idempotent in $\mathcal{G}$. For details see Section 6.1. We are interested in the case when $V = \mathbb{C}^{2n}$ and $\mathcal{G}$ is the wreath-product $\Gamma_n := S_n \ltimes \Gamma^n$, where Γ is a Kleinian group, i.e., a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.

In some cases one Poisson variety can be obtained by using two different constructions. For example, the Kleinian singularity $\mathbb{C}^2/\Gamma$ can be obtained both as a quiver variety and as the intersection of the Slodowy slice with the null-cone in an appropriate Lie algebra $\mathfrak{g}$. In this case all three families of quantizations are the same; see Theorems 5.3.1 and 6.2.2 for the precise statements including the explicit correspondence between the parameters.

Parabolic Slodowy varieties for $\mathfrak{g} = \mathfrak{sl}_N$ can be realized as quiver varieties for the Dynkin quiver of type A, this was first discovered by Nakajima, [38]. The corresponding quantizations are also the same; see Theorem 5.3.3.

Finally, the quotient singularity $\mathbb{C}^{2n}/\Gamma_n$ for $n > 1$ can also be constructed as a quiver variety (but no Slodowy type construction is known, in general). Again, the two families of quantizations are the same; see Theorem 6.2.1. We remark that the results relating the SRA quantizations with quantum Hamiltonian reductions were known before; see Section 6.2 for references. However, our proofs here are new.

1.1. *Content of the paper*

Our arguments are based on the study of partially non-commutative deformations of the symplectic resolutions of the Poisson varieties of interest. Such deformations were studied in [22] (purely commutative deformations) and in [2] (quantizations = non-commutative deformations). The results of these papers (with appropriate modifications we need) are explained in Section 2. One of the very pleasant features of resolutions is that their deformations are very easy to control, they are basically parameterized by the second cohomology via a certain *period map*. It is this feature that was a reason for us to consider deformations of the resolutions. In Section 2.1, we explain results of [22] including the existence of a universal commutative formal deformation. In Section 2.2, we explain some general definitions and constructions including various compatibilities with group actions, in particular, notions of *graded* and *even* quantizations. In Section 2.3, we recall the main construction of [2], the *non-commutative period map*, that classifies quantizations of symplectic varieties under certain vanishing assumptions on a variety. We also study the compatibility of this map with the group actions introduced in Section 2.2. The compatibility results seem to be pretty standard but we could not find a reference.

All Poisson varieties we consider and most of their quantizations are obtained by Hamiltonian reduction. In Section 3, we recall general definitions and results regarding both classical and quantum Hamiltonian reductions. This section again basically does not contain new results. We recall the definition of a classical Hamiltonian reduction both for Poisson algebras and for Poisson varieties in Section 3.1. In Section 3.2, we establish an algebro-geometric version of the Duistermaat–Heckman theorem, [10]. For us this theorem is a recipe to compute the period map for Hamiltonian reductions. The reason why we need this result is as follows. If a symplectic variety is obtained by Hamiltonian reduction, then one can produce its (formal) deformations also using the Hamiltonian reduction. The Duistermaat–Heckman theorem can be viewed as a tool to identify two commutative deformations related to different constructions of a given variety by Hamiltonian reduction. In the next two Sections 3.3 and 3.4, we recall generalities on quantum Hamiltonian reduction.

Our goal in Section 4 is to recall several examples of symplectic resolutions of singularities as well as isomorphisms between them. There are no new results in this section either. In Section 4.1, we recall the general definition and some general vanishing properties for them. In Section 4.2, we provide necessary information on Nakajima quiver varieties, including the definitions of both affine and non-affine quiver varieties. Also we recall sufficient conditions to obtain symplectic resolutions using the quiver variety construction, as well as sufficient conditions for a quantum Hamiltonian reduction to provide a quantization of an affine quiver variety. Then we recall the Slodowy slices and their ramifications, Section 4.3. After this, in Section 4.4, we recall different facts about the quotients $\mathbb{C}^{2n}/\Gamma_n$ and their resolutions including the construction via quiver varieties and the existence of some special vector bundles ("weakly Procesi bundles"). In Section 4.5, we consider the $n = 1$ case in more detail recalling the constructions of the Kleinian singularities and of their minimal (= symplectic) resolutions via the Slodowy varieties. One of the important results here is the comparison between two families of line bundles on the resolution: one coming from the Slodowy construction and the other from the quiver varieties construction. This comparison is one of the ingredients in constructing a correspondence between the quantization parameters in Theorem 5.3.1. Finally, in Section 4.6 we recall the isomorphisms between parabolic Slodowy varieties in type A and quiver varieties of type A due to Maffei, [34], and establish some easy properties of these isomorphisms.

A common feature of our main results is that one of the families of quantizations we consider always comes from quiver varieties, while the other is obtained by a different (W-algebra or symplectic reflection algebra) construction. In Section 5, we establish results involving W-algebras. In Section 5.1, we recall the definitions of W-algebras due to the author, [27] and Premet, [41]. Then in Section 5.2 we introduce certain ramifications of W-algebras, the *parabolic* W-algebras, and relate them to quantizations of appropriate parabolic Slodowy varieties. Next, in Section 5.3, we state two main results relating parabolic W-algebras to quantum Hamiltonian reductions of quiver varieties: Theorem 5.3.1 (Kleinian case) and Theorem 5.3.3 (type A case).

The strategy of the proofs of these theorem is the same. First, we have two constructions of commutative formal deformations of the resolutions, both obtained via Hamiltonian reduction. We use the Duistermaat–Heckman theorem and the results on the relation between appropriate line bundles to establish an isomorphism between the two commutative deformations. Then the algebras, for which we need to establish an isomorphism, are, roughly speaking, the algebras of global sections of quantizations of the two commutative deformations. The noncommutative deformations are again obtained by two different quantum Hamiltonian reductions. Isomorphisms of interest follow from the claim that the corresponding quantizations are isomorphic. Thanks to results of Section 2.3, it is enough to show that both quantizations are *canonical* in the sense of Bezrukavnikov and Kaledin or, equivalently, even. So we need to understand when a quantization obtained by a quantum Hamiltonian reduction is even. This is done in Section 5.4. There we obtain a general result answering this question, Theorem 5.4.1, which seems to be of independent interest. After this completing the proofs of Theorems 5.3.1 and 5.3.3 is pretty easy; see Section 5.5.

The SRA side is studied in Section 6. The definition of the symplectic reflection algebras is recalled in Section 6.1. The main result, Theorem 6.2.1, comparing the family of quantizations of $\mathbb{C}^{2n}/\Gamma_n$ coming from an SRA with that obtained by quantum Hamiltonian reduction is stated in Section 6.2. This theorem is proved in the rest of Section 6. The scheme of this proof will be explained at the end of Section 6.2.

1.2. Conventions and notation

Let us list some conventions we use in the paper.

Quivers. Recall that by a quiver one means an oriented graph possibly with multiple arrows or loops. More precisely, a quiver Q consists of two sets, a set Q_0 of vertices and a set Q_1 of arrows, and two maps t (tail) and h (head) from Q_1 to Q_0.

Associated to a quiver is the vector space $\mathbb{C}^{Q_0}$ with the "scalar product" $\alpha \cdot \beta = \sum_{i \in Q_0} \alpha_i \beta_i$. Let $\epsilon_i, i \in Q_0$, denote the tautological basis in $\mathbb{C}^{Q_0}$.

Quotients. In this paper, we deal with several kinds of quotients. Geometric quotients for an action of an algebraic group G on a scheme X (e.g., quotients for finite group actions or quotients by free group actions) are denoted by X/G. The categorical quotient of an affine variety X by an action of a reductive group G is denoted by $X /\!\!/ G$. The GIT quotient of X corresponding to a stability condition θ will be denoted by $X /\!\!/^{\theta} G$. The set of θ-semistable points in X will be denoted by $X^{\theta, ss}$.

Finally, "$/\!\!/\!\!/$" means a (classical or quantum) Hamiltonian reduction (either of an algebra or of a variety).

Sheaves. We usually denote sheaves as $\mathcal{A}_X$, where X is a (formal) scheme, where the sheaf lives. For a sheaf $\mathcal{A}_X$ by $\mathcal{A}(X)$ we denote its global sections.

$\widehat{\otimes}$	Completed tensor product of complete topological vector spaces/modules.
(X)	The two-sided ideal in an associative algebra generated by a subset X.
$\mathbb{A}_h(V)$	The homogenized Weyl algebra of a symplectic vector space V.
$\mathbb{A}_{2n,h}$	$:=\mathbb{A}_h(\mathbb{C}^{2n})$.
$\mathrm{Aut}(\mathcal{A})$	The automorphism group of an algebra (or a sheaf of algebras) $\mathcal{A}$.
$\mathcal{D}_{h,X}$	The sheaf of "homogenized" differential operators on a smooth scheme X.
$\mathrm{Der}(A)$	The Lie algebra of derivations of an algebra A.
G_x	The stabilizer of x in G.
$\mathrm{gr}\mathcal{A}$	The associated graded vector space of a filtered vector space $\mathcal{A}$.
$H^i_{\mathrm{DR}}(X)$	ith De Rham cohomology of a smooth (formal) scheme X.
$\mathbb{C}[X]$	The algebra of regular functions on a (formal) scheme X.
$\mathcal{O}_X$	The structure sheaf of a (formal) scheme X.
$R\Gamma$	The group algebra of a group Γ with coefficients in a ring R.
SV	The symmetric algebra of a vector space V.
$\mathcal{T}_{X/S}$	The relative tangent bundle of a (formal) scheme X/S.
T^*X	The cotangent bundle of a smooth scheme X.
$U_h(\mathfrak{g})$	Homogenized universal enveloping algebra of a Lie algebra $\mathfrak{g}$, i.e., the quotient of $T(\mathfrak{g})[h]$ by the relations $\xi \otimes \eta - \eta \otimes \xi - h[\xi, \eta] = 0$.
$U^{\perp}$	The annihilator of a subspace $U \subset V$ in V^*.
X^G	G-invariants in a G-set X.
$X^{\wedge_Y}$	The completion of a scheme X along a subscheme Y.
ι_ξ	The contraction with a vector field ξ.
$\Omega^i_{X/S}$	The bundle of relative i-forms on a (formal) scheme X/S.

2. Deformations of symplectic schemes

2.1. Commutative deformations

In this subsection, we will discuss the existence of a universal (commutative) deformation of a symplectic variety X_0 over $\mathbb{C}$. Basically, all results here are taken from [22]. The deformations are controlled by a certain *period map* which has a very explicit description. Throughout the section we suppose that $H^1(X_0, \mathcal{O}_{X_0}) = H^2(X_0, \mathcal{O}_{X_0}) = \{0\}$, although most of the results we provide hold in greater generality. Let Ω_0 denote the symplectic form on X_0.

Let S be the n-dimensional formal polydisc, i.e., the completion $(\mathbb{C}^n)^{\wedge_0}$ of the affine space $\mathbb{C}^n$ at 0. Let X be a formal scheme over S with zero fiber X_0. More precisely, we suppose that X is a formal scheme equipped with a morphism $\rho : X \to S$ such that $\rho^{-1}(0) = X_0$, and X coincides with its completion along X_0. In this case we will say that X is a *formal deformation* of X_0 over S. We also assume that the deformation X is symplectic, i.e., X/S is equipped with a symplectic form $\Omega \in \Omega^2(X/S)$, whose restriction to X_0 coincides with Ω_0. Here "symplectic", as usual, stands for "closed and non-degenerate", where the latter means that the natural map between the relative tangent and cotangent bundles induced by Ω is an isomorphism.

The universal deformation of X_0 will be a formal scheme X over the formal scheme Per that, by definition, is the formal neighborhood of the cohomology class $[\Omega_0]$ in $H^2_{\mathrm{DR}}(X_0)$.

The $\mathbb{C}[S]$-module $H^2_{\mathrm{DR}}(X/S)$ is naturally isomorphic to $H^2_{\mathrm{DR}}(X_0)\widehat{\otimes}\mathbb{C}[S]$ (via the Gauss–Manin connection to be recalled at the end of the subsection). The cohomology class $[\Omega]$ is then an element of $H^2_{\mathrm{DR}}(X_0)\widehat{\otimes}\mathbb{C}[S]$ and as such defines a linear map $H^2_{\mathrm{DR}}(X_0)^* \to \mathbb{C}[S]$. This linear map produces a morphism $S \to \mathrm{Per}$ of formal schemes. So to a formal symplectic deformation $(X/S, \Omega)$ of (X_0, Ω_0) one assigns a morphism $p_{X/S} : S \to \mathrm{Per}$.

Proposition 2.1.1. *Recall that X_0 is a symplectic variety with $H^1(X_0, \mathcal{O}_{X_0}) = H^2(X_0, \mathcal{O}_{X_0}) = \{0\}$. The map $X/S \mapsto p_{X/S}$ is a bijection between*

- *the set of isomorphism classes of symplectic formal deformations X/S of X_0*
- *and the set of formal scheme morphisms $S \to \mathrm{Per}$.*

The inverse map assigns to $p : S \to \mathrm{Per}$ the pull-back $p^*(X_{\mathrm{univ}}/\mathrm{Per})$ of some universal *deformation $X_{\mathrm{univ}}/\mathrm{Per}$.*

This statement is a variant of Theorem 3.6 from [22] (that theorem deals with deformations of X_0 over local Artinian schemes so our statement is the "inverse limit" of theirs).

Below we will need a "graded" version of Proposition 2.1.1. Suppose that $\mathbb{C}^\times$ acts on X_0 such that $t \cdot \Omega_0 = t^2\Omega_0$. In particular, Ω_0 is exact. Equip S $(=(\mathbb{C}^n)^{\wedge_0})$ with the $\mathbb{C}^\times$-action induced from the action $t \cdot v = tv, t \in \mathbb{C}^\times, v \in \mathbb{C}^n$. We say that a symplectic formal deformation X/S is *graded* if X is equipped with a $\mathbb{C}^\times$-action such that the morphism $X \to S$ is equivariant and the symplectic form Ω on X/S satisfies $t \cdot \Omega = t^2\Omega$. Then $p_{X/S}$ is restricted from a linear map $\mathbb{C}^n \to H^2_{\mathrm{DR}}(X_0)$ also denoted by $p_{X/S}$.

Performing an easy modification of the argument in [22] one gets the following result.

Proposition 2.1.2. *We preserve the assumptions of Proposition 2.1.1 and of the previous paragraph. The universal deformation $X_{\mathrm{univ}}/\mathrm{Per}$ can be made graded in such a way that the map $X/S \to p_{X/S}$ is a bijection between*

- *the set of isomorphism classes of graded symplectic formal deformations X/S of X_0.*
- *and the set of linear maps $\mathbb{C}^n \to H^2_{\mathrm{DR}}(X_0)$.*

To a $\mathbb{C}^\times$*-equivariant morphism* $p : S \to \mathrm{Per}$ *the inverse map assigns the pull-back* $p^*(X_{\mathrm{univ}}/\mathrm{Per})$ *of* $X_{\mathrm{univ}}/\mathrm{Per}$.

Now let us recall the Gauss–Manin connection on $H^2_{\mathrm{DR}}(X/S)$, where S still denotes $(\mathbb{C}^n)^\wedge_0$. Let $[\alpha]$ be an element of $H^2_{\mathrm{DR}}(X/S)$ and $\xi \in \mathbb{C}^n$. We need to define the covariant derivative $\xi \cdot [\alpha]$.

Let $X = \bigcup_i X^i$ be an open covering by affine formal subschemes. Pick sections $\alpha^i \in \Omega^2(X^i), \alpha^{ij} \in \Omega^1(X^{ij}), \alpha^{ijk} \in \mathbb{C}[X^{ijk}]$, such that the images of these forms in $\bigwedge^\bullet \mathcal{T}^*(X/S)$ form a Čech–De Rham cocycle representing $[\alpha]$. Here, as usual, $X^{ij} := X^i \cap X^j, X^{ijk} := X^i \cap X^j \cap X^k$.

Fix liftings ξ^i of $\xi \in \mathbb{C}^n$ to X^i. Then set

$$\begin{aligned}
&\beta^i := \iota_{\xi^i} d\alpha^i,\\
&\beta^{ij} := \iota_{\xi^i}(d\alpha^{ij} - \alpha^i + \alpha^j),\\
&\beta^{ijk} := \iota_{\xi^i}(d\alpha^{ijk} - \alpha^{ij} - \alpha^{jk} - \alpha^{ki}).
\end{aligned} \tag{2.1}$$

Here $\iota_\bullet$ stands for the contraction with a vector field. Further, let $\underline{\beta}^i, \underline{\beta}^{ij}, \underline{\beta}^{ijk}$ denote the image of $\beta^i, \beta^{ij}, \beta^{ijk}$ in $\bigwedge^\bullet \mathcal{T}^*(X/S)$. From the claim that the image of $(\alpha^i, \alpha^{ij}, \alpha^{ijk})$ in $\bigwedge^\bullet \mathcal{T}^*(X/S)$ is closed it is easy to deduce that $\underline{\beta}^{ij}, \underline{\beta}^{ijk}$ are skew-symmetric. So $(\underline{\beta}^i, \underline{\beta}^{ij}, \underline{\beta}^{ijk})$ is a cochain in the Čech–De Rham complex of X/S. Similarly, we see that $\underline{\beta}^i, \underline{\beta}^{ij}, \underline{\beta}^{ijk}$ do not depend on the choice of the local liftings ξ^i.

Now let us check that $(\underline{\beta}^i, \underline{\beta}^{ij}, \underline{\beta}^{ijk})$ is a cocycle. This, by definition, means that $d\beta^i, d\beta^{ij} - \beta^i + \beta^j, d\beta^{ijk} - (\beta^{ij} + \beta^{jk} + \beta^{ki})$ vanish on the vector fields that are tangent to the fibers of $X \to S$, while $\beta^{ijk} - \beta^{ijl} + \beta^{ikl} - \beta^{jkl} = 0$. The last equality follows easily from the observation that $\beta^{ijk} (= \underline{\beta}^{ijk})$ does not depend on the choice of the liftings. Let us check that $d\beta^i$ vanishes on the tangent vector fields. The proofs of the other two claims are similar but more involved computationally.

Each X^i can be identified with $X^0_i \times S$, where X^i_0 is an open affine subvariety of X_0. Fix such an identification. We may assume that ξ^i is tangent to S in $X^i = X^i_0 \times S$. Further, we can write α^i as $\sum_p f_p \otimes \alpha^i_p + \overline{\alpha}^i$, where f_p's form a topological basis in $\mathbb{C}[S], \alpha^i_p$ are some 2-forms on X^i_0, and $\overline{\alpha}^i$ is a 2-form vanishing on all vector fields tangent to X^i_0. Then $\beta^i = \sum_p \partial_{\xi^i} f_p \otimes \alpha^i_p + \iota_{\xi^i} d\overline{\alpha}^i = \sum_p \partial_{\xi^i} f_p \otimes \alpha^i_p - d\iota_{\xi^i}\overline{\alpha}_i - \mathcal{L}_{\xi^i}\overline{\alpha}_i$, where $\mathcal{L}_{\xi^i}$ denotes the Lie derivative. We remark that $\mathcal{L}_{\xi^i}\overline{\alpha}_i$ vanishes on the vector fields tangent to X^i_0. Therefore the image of $d\beta^i$ in $\Omega^3(X^i/S)$ equals $\sum_p \partial_\xi f_p \otimes d\alpha^i_p$. The image of $d\alpha^i$ in $\Omega^3(X^i/S)$ equals $\sum_p f_p \otimes d\alpha^i_p$. Therefore $d\alpha^i_p = 0$ and so $d\beta^i = 0$ in $\Omega^3(X^i/S)$, and we are done.

Now we need to check that the class of $(\underline{\beta}^i, \underline{\beta}^{ij}, \underline{\beta}^{ijk})$ does not depend on the choice of $(\alpha^i, \alpha^{ij}, \alpha^{ijk})$. This will follow if we check that $(\underline{\beta}^i, \underline{\beta}^{ij}, \underline{\beta}^{ijk})$ is exact provided $\alpha^{ijk} = 0$ and α^i, α^{ij} vanish on the tangent vectors. This is checked analogously to the previous paragraph.

So for $\xi \cdot [\alpha]$ we take the class of $(\underline{\beta}^i, \underline{\beta}^{ij}, \underline{\beta}^{ijk})$. Similarly to the above, one can check that $\xi \cdot \eta \cdot [\alpha] - \eta \cdot \xi \cdot [\alpha] = [\xi, \eta] \cdot [\alpha]$. So $[\alpha] \mapsto \xi \cdot [\alpha]$ defines a flat connection on the $\mathbb{C}[S]$-module $H^2_{\mathrm{DR}}(X/S)$. This is the Gauss–Manin connection. It is a standard fact that a flat connection defines an identification $H^2_{\mathrm{DR}}(X/S) \cong H^2_{\mathrm{DR}}(X_0) \widehat{\otimes} \mathbb{C}[S]$.

2.2. Generalities on deformation quantization

Let S be a scheme of finite type over $\mathbb{C}$ or a completion of such a scheme.

Let X be a smooth scheme of finite type over S, $\rho : X \to S$ be a projection. We assume that X is symplectic with symplectic form $\Omega \in \Omega^2(X/S)$. The form Ω induces a $\rho^{-1}(\mathcal{O}_S)$-linear Poisson bracket on $\mathcal{O}_X$.

Let h be a formal variable. Let $\mathcal{D}$ be a sheaf of associative $\rho^{-1}(\mathcal{O}_S)[[h]]$-algebras flat over $\mathbb{C}[[h]]$, complete in the h-adic topology, and equipped with an isomorphism $\theta : \mathcal{D}/h\mathcal{D} \xrightarrow{\sim} \mathcal{O}_X$ (of sheaves of $\rho^{-1}(\mathcal{O}_S)$-algebras). There is a Poisson structure on $\mathcal{D}/h\mathcal{D}$: for local sections a, b of $\mathcal{O}_X$ we pick their local liftings $\tilde{a}, \tilde{b}$ to sections of $\mathcal{D}$ and define the bracket $\{a, b\}$ as the class of $\frac{1}{h}[\tilde{a}, \tilde{b}]$. We say that $\mathcal{D}$ (or, more precisely, the pair $(\mathcal{D}, \theta)$) is a *quantization* of $\mathcal{O}_X$ (or of X) if the isomorphism $\theta : \mathcal{D}/h\mathcal{D} \to \mathcal{O}_X$ intertwines the Poisson brackets.

We say that two quantizations $(\mathcal{D}_1, \theta_1), (\mathcal{D}_2, \theta_2)$ of X are isomorphic if there is a $\rho^{-1}(\mathcal{O}_S)[[h]]$-linear isomorphism $\varphi : \mathcal{D}_1 \to \mathcal{D}_2$ of sheaves of algebras such that the induced isomorphism $\mathcal{D}_1/h\mathcal{D}_1 \to \mathcal{D}_2/h\mathcal{D}_2$ intertwines θ_1, θ_2. The set of isomorphism classes of quantizations of X will be denoted by $\mathcal{Q}(X/S)$.

Below we will need to consider two special types of quantizations: *graded* and *even* quantizations.

The former is defined when we have a $\mathbb{C}^\times$-action on X with some special properties. Namely, suppose that X and S are acted on by $\mathbb{C}^\times$ in such a way that $\rho : X \to S$ is $\mathbb{C}^\times$-equivariant and Ω has degree 2 with respect to the $\mathbb{C}^\times$-action: the form $t \cdot \Omega$ obtained from Ω by the push-forward with respect to the automorphism induced by $t \in \mathbb{C}^\times$ equals $t^2\Omega$. We say that a quantization $\mathcal{D}$ of X is *graded* if $\mathcal{D}$ is equipped with a $\mathbb{C}^\times$-action by algebra automorphisms such that $t \cdot h = t^2 h$ and the isomorphism $\mathcal{D}/h\mathcal{D} \cong \mathcal{O}_X$ is $\mathbb{C}^\times$-equivariant.

Moreover, there is a natural action of $\mathbb{C}^\times$ on $\mathcal{Q}(X/S)$. Namely, pick a quantization $\mathcal{D}$ of X/S. For $t \in \mathbb{C}^\times$ define a sheaf ${}^t\mathcal{D}$ on X as $t_*(\mathcal{D})$, where t_* stands for the sheaf-theoretic push-forward with respect to the automorphism of X induced by t. Clearly, ${}^t\mathcal{D}$ is a sheaf of $\mathbb{C}$-algebras on X. Turn it into a sheaf of $\rho^{-1}(\mathcal{O}_S)[[\hbar]]$-algebras by setting $sd := (t \cdot s)d$, $hd := (t^2 h)d$ (in the right hand side the products are taken in $\mathcal{D}$) for local sections s of $\mathcal{O}_S$, d of ${}^t\mathcal{D}$. It is straightforward to check that ${}^t\mathcal{D}$ is again a quantization of X. This defines a $\mathbb{C}^\times$-action on $\mathcal{Q}(X/S)$.

Clearly, an isomorphism class of a graded quantization is a $\mathbb{C}^\times$-stable point in $\mathcal{Q}(X/S)$. Conversely, let ${}^t\mathcal{D} \cong \mathcal{D}$ for all $t \in \mathbb{C}^\times$. Pick an isomorphism $\varphi_t : \mathcal{D} \to {}^t\mathcal{D}$. Define a map $a : \mathbb{C}^\times \times \mathbb{C}^\times \to \operatorname{Aut}(\mathcal{D})$ by $\varphi_{t_1 t_2} = t_{1*}(\varphi_{t_2})\varphi_{t_1} a(t_1, t_2)$. Since the group $\operatorname{Aut}(\mathcal{D})$ is non-commutative, the map a is, in general, not a 2-cocycle. However, every element of $\operatorname{Aut}(\mathcal{D})$ has the form $\exp(hd)$, where d is a $\rho^{-1}(S)[[h]]$-linear derivation of $\mathcal{D}$. Indeed, for $\varphi \in \operatorname{Aut}(\mathcal{D})$ we can put $d = \frac{1}{h}\ln(\varphi)$.

So, as a group, $\operatorname{Aut}(\mathcal{D})$ is just $h\operatorname{Der}(\mathcal{D})$, where the multiplication is given by the Campbell–Hausdorff series. In particular, this multiplication is commutative modulo h. Therefore modulo h the function a is a 2-cocycle. By the Hilbert theorem 90, this cocycle is a coboundary. After modifying φ_t in an appropriate way we get $\varphi_{t_1 t_2} = t_{1*}(\varphi_{t_2})\varphi_{t_1}$ modulo h. Now repeat the same procedure modulo h^2. So we see that we can choose isomorphisms $\varphi_t : \mathcal{D} \to {}^t\mathcal{D}$ in such a way that $\varphi_{t_1 t_2} = t_{1*}(\varphi_{t_2})\varphi_{t_1}$ for all $t_1, t_2 \in \mathbb{C}^\times$. The existence of such isomorphisms means that $\mathcal{D}$ is graded.

A similar argument implies that two graded quantizations are isomorphic if and only if they are $\mathbb{C}^\times$-equivariantly isomorphic.

Let us present a very standard example of a quantization. Let S be a single point, $\underline{X}$ a smooth variety and $X := T^*\underline{X}$ be the cotangent bundle of $\underline{X}$ equipped with a standard symplectic form. Consider the sheaf $\mathcal{D}_{h,\underline{X}}$ of "homogeneous" differential operators. This is a sheaf of $\mathbb{C}[h]$-

algebras on $\underline{X}$ generated by $\mathcal{O}_{\underline{X}}$ and the tangent sheaf $\mathcal{T}_X$ modulo the following relations

$$\begin{aligned} &f * g = fg,\\ &f * v = fv,\\ &v * f = fv + hv \cdot f,\\ &u * v - v * u = h[u, v]. \end{aligned} \tag{2.2}$$

Here f, g are local sections of $\mathcal{O}_X$, while u, v are local sections of $\mathcal{T}_X$. In the left hand side $*$ means a product in $\mathcal{D}_{h,\underline{X}}$. We remark that $\mathcal{D}_{h,\underline{X}}/(h-1)$ is the usual sheaf of linear differential operators on $\underline{X}$. Also $\mathcal{D}_{h,\underline{X}}/(h)$ is naturally identified with $p_*(\mathcal{O}_X)$, where $p : X \twoheadrightarrow \underline{X}$ is a natural projection.

We remark that $\mathcal{D}_{h,\underline{X}}$ is not a quantization of X in the above sense because this is a sheaf on $\underline{X}$ and not on X and it is not complete in the h-adic topology. To fix this we can replace $\mathcal{D}_{h,\underline{X}}$ with its h-adic completion and localize it to X (compare with [2, Remark 1.6]). Abusing the notation, we still denote this sheaf on X by $\mathcal{D}_{h,\underline{X}}$. Consider a $\mathbb{C}^\times$-action on $\mathcal{D}_{h,\underline{X}}$ that is uniquely determined by $t \cdot f = f, t \cdot v = t^2 v, t \cdot h = t^2 h$ for any $t \in \mathbb{C}^\times$. To recover the initial sheaf on $\underline{X}$ one takes $\mathbb{C}^\times$-finite sections in $p_*(\mathcal{D}_{h,\underline{X}})$.

More generally, we can consider the homogeneous analogs of twisted differential operators. By definition, a sheaf of homogeneous twisted differential operators is a pair of

- a sheaf $\mathcal{D}$ of $\mathbb{C}[h]$-algebras on $\underline{X}$ equipped with a $\mathbb{C}^\times$-action by algebra automorphisms and
- a $\mathbb{C}^\times$-equivariant $\mathbb{C}[h]$-algebra embedding $\mathcal{O}_{\underline{X}}[h] \to \mathcal{D}$ (where $\mathbb{C}^\times$ acts trivially on $\mathcal{O}_{\underline{X}}$ and $t \cdot h = th$)

subject to the following condition. There are

- an open covering $\underline{X} := \bigcup_i \underline{X}^i$ and
- $\mathbb{C}^\times$-equivariant isomorphisms $\iota^i : \mathcal{D}|_{\underline{X}^i} \to \mathcal{D}_{h,\underline{X}^i}$ intertwining the embeddings of $\mathcal{O}_{X_0^i}[h]$

that produce a global isomorphism $\mathcal{D}/(h) \xrightarrow{\sim} p_*(\mathcal{O}_X)$. As with usual twisted differential operators, the sheaves of homogeneous differential operators are classified up to an isomorphism by $H^1(X, \Omega^1_{\mathrm{cl}})$, where Ω^1_{cl} denotes the sheaf of closed 1-forms on $\underline{X}$ (an isomorphism is supposed to intertwine the $\mathcal{O}_{\underline{X}}[h]$-embeddings).

For example, let $\mathcal{L}$ be a line bundle. Let $\underline{X}'$ denote the complement to $\underline{X}$ in the total space of $\mathcal{L}$. This is a principal $\mathbb{C}^\times$-bundle on $\underline{X}$, let $\tau : \underline{X}' \twoheadrightarrow \underline{X}$ denote the canonical projection. The $\mathbb{C}^\times$-action on $\underline{X}'$ gives rise to the Euler vector field **eu** on $\underline{X}'$ so that the eigensheaf of **eu** in $\tau_*(\mathcal{L})$ with eigenvalue 1 is nothing else but $\mathcal{L}$. For $\alpha \in \mathbb{C}$ define the sheaf $\mathcal{D}^{\alpha\mathcal{L}}_{h,\underline{X}}$ as the quantum Hamiltonian reduction

$$\tau_*(\mathcal{D}_{h,\underline{X}'})^{\mathbb{C}^\times} / \tau_*(\mathcal{D}_{h,\underline{X}'})^{\mathbb{C}^\times}(\mathbf{eu} - \alpha h).$$

For example, when $\mathcal{L}$ is the canonical bundle $\Omega^{\mathrm{top}}_{\underline{X}}$ of $\underline{X}$, the sheaf $\mathcal{D}^{\alpha\Omega^{\mathrm{top}}_{\underline{X}}}_{h,\underline{X}}$ is generated by $\mathcal{O}_{\underline{X}}, \mathcal{T}_{\underline{X}}$ subject to the 1st and 4th relations (2.2) and to

$$\begin{aligned} &f * v = fv - \alpha hv \cdot f,\\ &v * f = fv + (1-\alpha)hv \cdot f. \end{aligned} \tag{2.3}$$

Below we write $D^\alpha_{h,\underline{X}}$ instead of $D^{\alpha\Omega^{\mathrm{top}}_{\underline{X}}}_{h,\underline{X}}$ and view it as a sheaf defined by generators and relations as above.

Proceed to the definition of even quantizations. We say that a quantization $\mathcal{D}$ is even, if there is an involutory antiautomorphism $\sigma : \mathcal{D} \to \mathcal{D}$ (to be referred to as a "parity antiautomorphism") that is trivial modulo h and maps h to $-h$. If $\mathcal{D}$ was $\mathbb{C}^\times$-equivariant we additionally require that σ is $\mathbb{C}^\times$-equivariant.

We can define an action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{Q}(X/S)$. Namely, let σ denote the non-trivial element in $\mathbb{Z}/2\mathbb{Z}$. For $\mathcal{D} \in \mathcal{Q}(X/S)$ set ${}^\sigma\mathcal{D} = \mathcal{D}^{\mathrm{op}}$ and equip $\mathcal{D}^{\mathrm{op}}$ with a $\rho^{-1}(\mathcal{O}_S)[[h]]$-algebra structure by preserving the same $\rho^{-1}(\mathcal{O}_S)$-algebra structure but making h act by $-h$. Similarly to the above, a quantization is even if and only if it is a fixed point for the $\mathbb{Z}/2\mathbb{Z}$-action on $\mathcal{Q}(X/S)$.

Let us present an (again, pretty standard) example of an even quantization. Let $\underline{X}$, X be such as above. Consider the sheaf $\mathcal{D}_h^{1/2}(\underline{X})$. Define its anti-automorphism σ by $\sigma(f) = f, \sigma(v) = v, \sigma(h) = -h$ for local sections f of $\mathcal{O}_{\underline{X}}$ and v of $\mathcal{T}_{\underline{X}}$. It is clear that $\mathcal{D}_h^{1/2}(\underline{X})$ is even (with parity antiautomorphism σ).

We will also need some more straightforward compatibility of quantizations with group actions. Namely, let G be an algebraic group acting on X such that $\rho : X \to S$ is G-invariant. We say that a quantization $\mathcal{D}$ is G-equivariant if $\mathcal{D}$ is equipped with an action of G by algebra automorphisms such that h is G-invariant and the isomorphism $\mathcal{D}/h\mathcal{D} \cong \mathcal{O}_X$ is G-equivariant.

2.3. Non-commutative period map

Let X/S, Ω be such as in the previous subsection. In this subsection, we will explain the approach of [2] to the problem of classifying quantizations of X/S. Following [2] we will produce a certain natural map $\mathrm{Per} : \mathcal{Q}(X/S) \to h^{-1}H^2_{\mathrm{DR}}(X/S)[[h]] \subset H^2_{\mathrm{DR}}(X/S)[h^{-1}, h]]$ (the non-commutative period map). Under some cohomology vanishing conditions on X this map is a bijection. As Bezrukavnikov and Kaledin point out in their paper, their approach is not essentially new, but the language they use turns out to be very convenient for our purposes. The only (somewhat) new feature in our exposition is the presence of a $\mathbb{C}^\times \times \mathbb{Z}/2\mathbb{Z}$-action.

Let us recall the main result of [2] providing a classification of quantizations. We say that X is *admissible* if the natural homomorphism $H^i_{\mathrm{DR}}(X/S) \to H^i(X, \mathcal{O}_X)$ is surjective for $i = 1, 2$. Let K^2 denote the kernel of $H^2_{\mathrm{DR}}(X/S) \to H^2(X, \mathcal{O}_X)$.

Proposition 2.3.1. *For any $\mathcal{D} \in \mathcal{Q}(X/S)$ we have* $\mathrm{Per}(\mathcal{D}) \in h^{-1}[\Omega] + H^2_{\mathrm{DR}}(X/S)[[h]]$. *Further, fix a splitting $P : H^2_{\mathrm{DR}}(X/S) \twoheadrightarrow K^2$ of the inclusion $K^2 \hookrightarrow H^2_{\mathrm{DR}}(X/S)$. Then $\mathcal{D} \mapsto P(\mathrm{Per}(\mathcal{D}) - [\Omega])$ defines a bijection $\mathcal{Q}(X/S) \xrightarrow{\sim} K^2[[h]]$.*

Now let us discuss the compatibility of Per with group actions considered in the previous subsection. Let $\mathbb{C}^\times$ act on X and S as above. We remark that Ω is exact provided S is a point. The following proposition establishes a relation between Per and the $\mathbb{C}^\times \times \mathbb{Z}/2\mathbb{Z}$-action on $\mathcal{Q}(X/S)$.

Proposition 2.3.2. (1) *If $\mathcal{D} \in \mathcal{Q}(X/S)$ is graded, then* $\mathrm{Per}(\mathcal{D}) \in H^2_{\mathrm{DR}}(X/S) \subset h^{-1}H^2_{\mathrm{DR}}(X/S)[[h]]$.
(2) *If $\mathcal{D}$ is, in addition, even, then* $\mathrm{Per}(\mathcal{D}) = 0$.

The proof of Proposition 2.3.2 follows fairly easily from the constructions of [2] and will be given after we recall all necessary definitions and constructions below in this subsection.

Following [2], we call a quantization $\mathcal{D}$ of X/S *canonical* if $\mathrm{Per}(\mathcal{D}) = [\Omega]$.

Corollary 2.3.3. *Suppose $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Let X/S be equipped with a $\mathbb{C}^\times$-action as above. Then* Per *identifies the set of isomorphism classes of graded quantizations of*

X/S with $H^2_{\mathrm{DR}}(X/S) \subset h^{-1}H^2_{\mathrm{DR}}(X/S)[[h]]$. Further, an even graded quantization of X/S is canonical.

An essential notion in the approach of [2] is that of a *Harish-Chandra torsor*. For reader's convenience, we will sketch basic definitions and results here. For a more detailed exposition, the reader is referred to [2].

Let G be a (pro)algebraic group and $\mathfrak{h}$ a (pro)finite dimensional Lie algebra equipped with a G-action and with a G-equivariant embedding $\mathfrak{g} \hookrightarrow \mathfrak{h}$ that are compatible in the sense that the differential of the G-action coincides with the adjoint action of $\mathfrak{g}$ on $\mathfrak{h}$. A pair $(G, \mathfrak{h})$ is called a Harish-Chandra pair. It is straightforward to define the notions of a module over a Harish-Chandra pair and of a homomorphism of Harish-Chandra pairs.

Examples we need in this paper (and which were considered in [2]) include the following.

(1) Consider the Poisson bracket on $A := \mathbb{C}[[x_1, \ldots, x_n, y_1, \ldots, y_n]]$ given by $\{x_i, x_j\} = \{y_i, y_j\} = 0, \{x_i, y_j\} = \delta_{ij}$. Consider the subgroup $\operatorname{Symp} A$ of $\operatorname{Aut} A$ consisting of all Poisson automorphisms and the subalgebra $\mathrm{H} \subset \operatorname{Der} A$ consisting of all Hamiltonian (= annihilating the Poisson bracket) derivations. Then $(\operatorname{Symp} A, \mathrm{H})$ is a Harish-Chandra pair.

(2) Let $D := \mathbb{A}^{\wedge 0}_{2n,h}$ be the completed Weyl algebra $\mathbb{C}[[x_1, \ldots, x_n, y_1, \ldots, y_n, h]]$ with the multiplication given by the Moyal–Weyl formula $f * g = m(\exp(\frac{Ph}{2}) f \otimes g)$, where m stands for the standard commutative multiplication map $D \otimes D \to D$ and $P : D \otimes D \to D \otimes D$ is given by

$$f \otimes g \mapsto \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \otimes \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \otimes \frac{\partial g}{\partial x_i}.$$

A Harish-Chandra pair under consideration is $(\operatorname{Aut} D, \operatorname{Der} D)$, where both automorphisms and derivations are supposed to be $\mathbb{C}[[h]]$-linear.

The next notion we need is that of a *Harish-Chandra torsor*. Let G be a pro-algebraic group. By a torsor over X we mean a G-scheme M over X such that the structure morphism $\pi : M \to X$ is G-invariant and the action map $G \times M \to M$ together with a projection $G \times M \to M$ gives rise to an isomorphism $G \times M \xrightarrow{\sim} M \times_X M$. We have the short exact sequence (the Atiyah extension) of $\mathcal{O}_X$-modules

$$0 \to \mathfrak{g}_M \to \mathcal{E}_M \to \mathcal{T}_{X/S} \to 0,$$

where $\mathfrak{g}_M := \pi_*(\mathcal{O}_M \widehat{\otimes} \mathfrak{g})$ and $\mathcal{E}_M := \pi_*(\mathcal{T}_{M/S})$. By an $\mathfrak{h}$-valued connection on M one means a G-equivariant bundle map $\theta_M : \mathcal{E}_M \to \mathfrak{h}_M := \pi_*(\mathcal{O}_M \widehat{\otimes} \mathfrak{h})$ such that the composition $\mathfrak{g}_M \to \mathcal{E}_M \to \mathfrak{h}_M$ coincides with the embedding $\mathfrak{g}_M \to \mathfrak{h}_M$ induced by the embedding $\mathfrak{g} \hookrightarrow \mathfrak{h}$. An $\mathfrak{h}$-valued connection θ_M is said to be *flat* if the $\mathfrak{h}$-valued 1-form $\Omega := \pi^*(\theta_M)$ on M satisfies the Maurer–Cartan equation $2d\Omega + \Omega \wedge \Omega = 0$. The pair (M, θ_M) of a G-torsor M and an $\mathfrak{h}$-valued flat connection θ_M is said to be a Harish-Chandra $\langle G, \mathfrak{h}\rangle$-torsor. The Harish-Chandra $\langle G, \mathfrak{h}\rangle$-torsors on X naturally form a groupoid (= category with invertible morphisms), let $H^1(X, \langle G, \mathfrak{h}\rangle)$ denote the set of isomorphism classes of Harish-Chandra torsors.

We will need some functoriality properties for Harish-Chandra torsors. Now let $\langle G_1, \mathfrak{h}_1\rangle, \langle G_2, \mathfrak{h}_2\rangle$ be two Harish-Chandra torsors and let $\varphi : \langle G_1, \mathfrak{h}_1\rangle \to \langle G_2, \mathfrak{h}_2\rangle$ be a homomorphism. Pick a Harish-Chandra $\langle G_1, \mathfrak{h}_1\rangle$-torsor M_1 and define its push-forward $\varphi_*(M)$ as follows. As a torsor $\varphi_*(M_1)$ is the quotient of $G_2 \times M_1$ by the diagonal action of G_1 ($g_1 \in G_1$ acts on G_2 via $g_1 \cdot g_2 = g_2 \varphi(g_1)^{-1}$). The bundle $\mathcal{E}_{\varphi_*(M_1)}$ is the quotient of $\mathfrak{g}_{2M_1} \oplus \mathcal{E}_{M_1}$ modulo the image of $\mathfrak{g}_{1M_1}$ under $(-\varphi_{M_1}, \iota_{M_1})$, where $\varphi_{M_1} : \mathfrak{g}_{1M_1} \to \mathfrak{g}_{2M_2}$ is the map induced by φ and

$\iota_{M_1} : \mathfrak{g}_{1M_1} \to \mathcal{E}_{M_2}$ is the inclusion above. Define a map $\mathfrak{g}_{2M_1} \oplus \mathcal{E}_{M_1} \to \mathfrak{h}_{2M_1}$ as the direct sum of the inclusion $g_{2M_2} \hookrightarrow \mathfrak{h}_{2M_2}$ and of $\varphi_{M_1} \circ \theta_{M_1}$. This map vanishes on the image of $\mathfrak{g}_{1M_1}$ and hence defines a map $\mathcal{E}_{M_2} \to \mathfrak{h}_{2M_2}$. It is straightforward to check that this map is a flat connection.

We will need two appearances of this construction. First of all, let a group Γ act by automorphisms of a Harish-Chandra pair $\langle G, \mathfrak{h}\rangle$ and also by automorphisms on X. Then the previous construction gives rise to a Γ-action on $H^1(X, \langle G, \mathfrak{h}\rangle)$.

Second, let $\varphi : \langle G_1, \mathfrak{h}_1\rangle \to \langle G, \mathfrak{h}\rangle$ be an epimorphism and M be a Harish-Chandra $\langle G, \mathfrak{h}\rangle$-torsor. By a *lifting* of M to $\langle G_1, \mathfrak{h}_1\rangle$ we mean a Harish-Chandra $\langle G_1, \mathfrak{h}_1\rangle$-torsor M_1 equipped with an isomorphism $\pi_*(M_1) \xrightarrow{\sim} M$. Let $H^1_M(X, \langle G_1, \mathfrak{h}_1\rangle)$ denote the set of isomorphism classes of liftings of M to $\langle G_1, \mathfrak{h}_1\rangle$.

The constructions considered in the previous two paragraphs combine together as follows. Suppose that Γ acts by automorphisms on both $\langle G, \mathfrak{h}\rangle$ and $\langle G_1, \mathfrak{h}_1\rangle$ such that the epimorphism φ is Γ-equivariant. Further, suppose that M is Γ-stable. Then we get a natural action of Γ on $H^1_M(X, \langle G_1, \mathfrak{h}_1\rangle)$.

Let us proceed to some examples of Harish-Chandra torsors and their liftings we need. Set $n := \frac{1}{2} \dim X/S$.

Recall the Harish-Chandra pair $\langle \mathrm{Symp}\, A, \mathrm{H}\rangle$. Then X has a canonical Harish-Chandra torsor $\mathcal{M}_{\mathrm{symp}}$ of symplectic coordinate systems. Following [2, Lemma 3.2], we define it as the (pro-finite) scheme of all morphisms $\varphi : (\mathbb{C}^{2n})^{\wedge_0} \to X$ that are compatible with the symplectic structure, i.e., $\varphi^*(\Omega)$ coincides with the symplectic form $\sum_{i=1}^n dx_i \wedge dy_i$.

Assume now that $\mathbb{C}^\times$ acts on X satisfying the conventions of the previous section. The action of $\mathbb{C}^\times$ on A given by $t \cdot x_i = tx_i, t \cdot y_i = ty_i$ gives rise to a $\mathbb{C}^\times$-action on $\langle \mathrm{Symp}\, A, \mathrm{H}\rangle$ and hence on $H^1(X, \langle \mathrm{Symp}\, A, \mathrm{H}\rangle)$. Also we remark that under the $\mathbb{C}^\times$-actions both on X and $(\mathbb{C}^{2n})^{\wedge_0}$ an element $t \in \mathbb{C}^\times$ multiplies the symplectic forms by t^2. Thus $\mathcal{M}_{\mathrm{symp}} \in H^1(X, \langle \mathrm{Symp}\, A, \mathrm{H}\rangle)$ is $\mathbb{C}^\times$-stable.

Now recall the Harish-Chandra pair $\langle \mathrm{Aut}\, D, \mathrm{Der}\, D\rangle$. The group $\mathbb{C}^\times$ acts on D by automorphisms: $t \cdot x_i = tx_i, t \cdot y_i = ty_i, t \cdot h = t^2 h$. The group $\mathbb{Z}/2\mathbb{Z}$ acts on D by antiautomorphisms: $\sigma \cdot x_i = x_i, \sigma \cdot y_i = y_i, \sigma \cdot h = -h$ (here θ stands for the non-unit element of $\mathbb{Z}/2\mathbb{Z}$). This gives rise to an action of $\mathbb{C}^\times \times \mathbb{Z}/2\mathbb{Z}$ on $\langle \mathrm{Aut} D, \mathrm{Der}\, D\rangle$ and hence to the action of $\mathbb{C}^\times \times \mathbb{Z}/2\mathbb{Z}$ on $H^1_{\mathcal{M}_{\mathrm{symp}}}(X, \langle \mathrm{Aut}\, D, \mathrm{Der}\, D\rangle)$.

The reason why we are interested in the set $H^1_{\mathcal{M}_{\mathrm{symp}}}(X, \langle \mathrm{Aut}\, D, \mathrm{Der}\, D\rangle)$ is that it is in a natural bijection with $\mathcal{Q}(X/S)$; see [2, Lemma 3.3]. To define the bijection we need the *localization* construction.

Given a module V over a Harish-Chandra pair $\langle G, \mathfrak{h}\rangle$ and a Harish-Chandra torsor M over X one can define a flat bundle $\mathrm{Loc}(M, V)$ (the localization of V with respect to M) as follows. As a bundle, $\mathrm{Loc}(M, V)$ is the associated bundle of the principal bundle M with fiber V. By the construction of $\mathrm{Loc}(M, V)$, both $\mathcal{E}_M, \mathfrak{h}_M$ act on $\mathrm{Loc}(M, V)$. A flat connection ∇ on $\mathrm{Loc}(M, V)$ is defined by (see [2, (2.2)])

$$\nabla_\xi(a) = \widetilde{\xi} a - \theta_M(\widetilde{\xi})a,$$

where a is a local section of $\mathrm{Loc}(M, V)$, ξ is a local vector field on X, and $\widetilde{\xi}$ is a local section of $\mathcal{E}_M$ lifting ξ. We remark that the right hand side does not depend on the choice of $\widetilde{\xi}$.

Lemma 2.3.4. *The map* $M \mapsto \mathrm{Loc}(M, D)^\nabla$*, where* $\mathrm{Loc}(M, D)^\nabla$ *stands for the sheaf of* ∇*-flat sections, defines a* $\mathbb{C}^\times \times \mathbb{Z}/2\mathbb{Z}$*-equivariant bijection between* $H^1_{\mathcal{M}_{\mathrm{symp}}}(X, \langle \mathrm{Aut}\, D, \mathrm{Der}\, D\rangle)$ *and* $\mathcal{Q}(X/S)$.

The claim that the map is a bijection is [2, Lemma 3.3]. The equivariance part is verified directly from the definitions of the $\mathbb{C}^\times \times \mathbb{Z}/2\mathbb{Z}$-actions.

To define the non-commutative period map we need some more discussion on the localization functor and extensions of Harish-Chandra torsors.

The localization functor $V \mapsto \operatorname{Loc}(M, V)$ is exact and hence it gives rise to a natural map $\operatorname{Loc}(M, \bullet) : H^i(\langle G, \mathfrak{h}\rangle, V) \to H^i_{DR}(X/S, \operatorname{Loc}(M, V))$. Here on the left hand side $H^i(\langle G, \mathfrak{h}\rangle, V)$ means $\operatorname{Ext}^i(\mathbb{C}, V)$, where Ext^i is taken in the category of $\langle G, \mathfrak{h}\rangle$-modules, while in the right hand side H^i_{DR} means De Rham hyper-cohomology with respect to the flat connection on $\operatorname{Loc}(M, V)$.

Let $\langle G, \mathfrak{h}\rangle$ be a Harish-Chandra pair and V be a $\langle G, \mathfrak{h}\rangle$-module. Let $\langle G_1, \mathfrak{h}_1\rangle$ be an extension of $\langle G, \mathfrak{h}\rangle$ by the Harish-Chandra pair (V, V). Further, let M be a Harish-Chandra torsor over $\langle G, \mathfrak{h}\rangle$. We need an obstruction for extending M to a Harish-Chandra torsor over $\langle G_1, \mathfrak{h}_1\rangle$.

Following [2] we will state the corresponding result under some restriction on M. Namely, we say that M is *transitive* if the connection map $\theta_M : \mathcal{E}_M \to \mathfrak{h}_M$ is an isomorphism. In particular, $\mathcal{M}_{\text{symp}}$ is transitive (compare with the discussion in the beginning of Section 3 in [2]).

Proposition 2.3.5 (*[2], Proposition 2.7*)**.** *In the above notation, suppose M is transitive.*

(1) *There exists a canonical cohomology class $c \in H^2(\langle G, \mathfrak{h}\rangle, V)$ with the following property: $H^1_M(\langle G_1, \mathfrak{h}_1\rangle, V) \neq \varnothing$ if and only if $\operatorname{Loc}(M, c) \in H^2_{DR}(X/S, \operatorname{Loc}(M, V))$ vanishes.*
(2) *If $\operatorname{Loc}(M, c) = 0$, then $H^1_M(X, \langle G_1, \mathfrak{h}_1\rangle)$ has a natural structure of an affine space with the underlying vector space $H^1_{DR}(X/S, \operatorname{Loc}(M, V))$.*

We will need compatibility of constructions of the previous proposition with group actions. Namely, let Γ be a group that acts:

- on X and on S by scheme automorphisms such that the morphism $\pi : X \to S$ is Γ-equivariant.
- on $\langle G_1, \mathfrak{h}_1\rangle$ and $\langle G, \mathfrak{h}\rangle$ by Harish-Chandra pair automorphisms such that the projection $\langle G_1, \mathfrak{h}_1\rangle \twoheadrightarrow \langle G, \mathfrak{h}\rangle$ is Γ-equivariant.

In particular, Γ acts on V by automorphisms of a $\langle G, \mathfrak{h}\rangle$-module. Hence Γ acts on $H^2(\langle G, \mathfrak{h}\rangle, V)$. Being canonical (see the proof of Proposition 2.7 in [2]), the class c is Γ-equivariant. The actions of an element $\gamma \in \Gamma$ on $H^1(X, \langle G, \mathfrak{h}\rangle)$ and on V define a push-forward isomorphism $\gamma_* : \operatorname{Loc}(M, V) \to \operatorname{Loc}({}^\gamma M, V)$ of sheaves that is compatible with the automorphism γ_* of $\mathcal{O}_X$ and intertwines the flat connections. This follows directly from the definition of $\operatorname{Loc}(\bullet, \bullet)$ given above. So we have the induced linear map $\gamma_* : H^i(X/S, \operatorname{Loc}(M, V)) \to H^i(X/S, \operatorname{Loc}({}^\gamma M, V))$. Again, from the construction of $\operatorname{Loc}(\bullet, \bullet)$ and the observation that c is Γ-invariant one deduces that $\gamma_*(\operatorname{Loc}(M, c)) = \operatorname{Loc}({}^\gamma M, c)$.

Now suppose that M is Γ-stable and $\operatorname{Loc}(M, c) = 0$. Then Γ acts on both the affine space $H^1_M(X, \langle G_1, \mathfrak{h}_1\rangle)$ and the underlying vector space $H^1_{DR}(X/S, \operatorname{Loc}(M, V))$ and these actions are compatible.

Proceed to the definition of a non-commutative period map (see [2, Section 4]). For this we will need one more Harish-Chandra pair related to the Weyl algebra D. Consider the subspace $h^{-1}D \subset D[h^{-1}]$. This subspace is closed with respect to the Lie bracket on $D[h^{-1}]$. There is a Lie algebra homomorphism $\eta : h^{-1}D \to \operatorname{Der} D$ given by $a \mapsto [a, \cdot]$. It is a standard fact that this map is surjective, its kernel coincides with $h^{-1}\mathbb{C}[h]$.

There is also an extension G of $\operatorname{Aut} D$ by the abelian group $h^{-1}\mathbb{C}[h]$ constructed as follows. The group $\operatorname{Aut} D$ is the semidirect product of Sp_{2d} (acting linearly on the span of

$x_1, \ldots, x_n, y_1, \ldots, y_n$) and of the normal subgroup $\mathrm{Aut}_+ D$ that acts by the identity modulo the ideal generated by $h, x_i x_j, y_i y_j, x_i y_j$. The Lie algebra $\mathrm{Lie}(\mathrm{Aut}\, D)$ of $\mathrm{Aut}\, D$ coincides with the algebra $\mathrm{Der}_0 D$ of derivations preserving the maximal ideal of D. The preimage $(h^{-1}D)_0$ of $\mathrm{Der}_0 D$ in $h^{-1}D$ is spanned by $h^{-1}\mathbb{C}[h]$ and all monomials $h^i x_1^{i_1} \ldots x_m^{i_m} y_1^{j_1} \ldots y_m^{j_m}$ where either $i \geqslant 0$ or $\sum_{k=1}^m i_k + j_k > 1$. The Lie algebra $\mathfrak{sp}_{2d}$ embeds naturally to $(h^{-1}D)_0$ and $(h^{-1}D)_0 = \mathfrak{sp}_{2d} \ltimes \eta^{-1}(D_+)$. The algebra $\eta^{-1}(D_+)$ is pro-nilpotent and so integrates to a pro-unipotent group G_+. Set $\mathrm{G} := \mathrm{Sp}_{2n} \leftthreetimes \mathrm{G}_+$. We have a natural projection $\mathrm{G} \twoheadrightarrow \mathrm{Aut}\, D$, whose kernel is the abelian group $h^{-1}\mathbb{C}[[h]]$. The group G acts on $h^{-1}D$ via the epimorphism $\mathrm{G} \twoheadrightarrow \mathrm{Aut}\, D$. So $(\mathrm{G}, h^{-1}D)$ becomes a Harish-Chandra pair and there is a natural epimorphism $\langle \mathrm{G}, h^{-1}D\rangle \twoheadrightarrow \langle \mathrm{Aut}\, D, \mathrm{Der}\, D\rangle$ with kernel $(h^{-1}\mathbb{C}[[h]], h^{-1}\mathbb{C}[[h]])$.

Definition 2.3.6. Let $c \in H^2(\langle \mathrm{Aut}\, D, \mathrm{Der}\, D\rangle, h^{-1}\mathbb{C}[[h]])$ denote the canonical class of the extension $0 \to \langle h^{-1}\mathbb{C}[[h]], h^{-1}\mathbb{C}[[h]]\rangle \to \langle \mathrm{G}, h^{-1}D\rangle \to \langle \mathrm{Aut}\, D, \mathrm{Der}\, D\rangle \to 0$; see Proposition 2.3.5. Let $M \in H^1_{\mathcal{M}_{\mathrm{symp}}}(X, \langle \mathrm{Aut}\, D, \mathrm{Der}\, D\rangle)$. To M assign an element $\mathrm{Per}(M) \in H^2_{\mathrm{DR}}(X/S, \mathrm{Loc}(M, h^{-1}\mathbb{C}[[h]])) = h^{-1}H^2_{\mathrm{DR}}(X/S)[[h]]$ by $\mathrm{Per}(M) := \mathrm{Loc}(M, c)$. The map

$$\mathrm{Per} : \mathcal{Q}(X/S) = H^1_{\mathcal{M}_{\mathrm{symp}}}(X, \langle \mathrm{Aut}\, D, \mathrm{Der}\, D\rangle) \to h^{-1}H^2_{\mathrm{DR}}(X/S)[[h]]$$

is called the *non-commutative period map*.

Recall that by Lemma 2.3.4, the set $H^1_{\mathcal{M}_{\mathrm{sympl}}}(X, \langle \mathrm{Aut}\, D, \mathrm{Der}\, D\rangle)$ is in bijection with $\mathcal{Q}(X/S)$. For $\mathcal{D} \in \mathcal{Q}(X/S)$ we write $\mathrm{Per}(\mathcal{D})$ for the image of the corresponding torsor under Per.

Proof of Proposition 2.3.2. Let us introduce a $\mathbb{C}^\times \times \mathbb{Z}/2\mathbb{Z}$-action on the Harish-Chandra pair $(\mathrm{G}, h^{-1}D)$. Take a $\mathbb{C}^\times$-action on $h^{-1}D$ restricted from $D[h^{-1}]$. An action of $\mathbb{Z}/2\mathbb{Z}$ we need is defined as follows. A non-trivial element σ acts on $h^{-1}D$ as a unique continuous anti-automorphism mapping x_i to x_i, y_i to y_i and h to $-h$. Equip G with a unique $\mathbb{C}^\times \times \mathbb{Z}/2\mathbb{Z}$-action compatible with the action on $h^{-1}D$. It is checked directly that the epimorphism $\langle \mathrm{G}, h^{-1}D\rangle \twoheadrightarrow \langle \mathrm{Aut}\, D, \mathrm{Der}\, D\rangle$ is $\mathbb{C}^\times \times \mathbb{Z}/2\mathbb{Z}$-equivariant. The induced action on $h^{-1}\mathbb{C}[[h]]$ is given by: $t \cdot h^i = t^i h^i, \sigma \cdot h^i = (-1)^{i+1} h^i$. Both claims of the proposition follow from the definition of Per and the above discussion of the compatibility of $\mathrm{Loc}(\bullet, \bullet)$ with group actions. □

Remark 2.3.7. All results quoted above transfer to the formal scheme setting (e.g., to formal deformations of symplectic varieties) directly without any noticeable modifications.

3. Hamiltonian reduction

3.1. Classical reduction

An important construction of symplectic varieties is that of a Hamiltonian reduction. In this subsection, we will recall it. All results gathered in this subsection are very standard. The proofs are the same as in the C^∞-setting; see, for example, [20].

First of all, let A be a Poisson algebra, $\mathfrak{g}$ be a Lie algebra equipped with a Lie algebra homomorphism $\varphi : \mathfrak{g} \to A$. Extend φ to a Poisson algebra homomorphism $S\mathfrak{g} \to A$. Pick $\chi \in \mathfrak{z}$. The ideal $A\varphi(\mathfrak{g}_\chi)$, where $\mathfrak{g}_\chi := \{\xi - \langle \chi, \xi\rangle, \xi \in \mathfrak{g}\} \subset S\mathfrak{g}$, is stable with respect to the action of $\mathfrak{g}$ on A given by $\xi \mapsto [\varphi(\xi), \cdot]$. Define the algebra $A/\!\!/\!\!/_\chi \mathfrak{g} := (A/A\varphi(\mathfrak{g}_\chi))^{\mathfrak{g}}$. By definition, this is a classical Hamiltonian reduction of A (with respect to $\varphi : \mathfrak{g} \to A$).

There is one important special case of this construction. Suppose that a connected algebraic group G with Lie algebra $\mathfrak{g}$ acts on A in such a way that the derivation $[\varphi(\xi), \cdot]$ is the image of $\xi \in \mathfrak{g}$ under the differential of the action. Then G acts on $A/A\varphi(\mathfrak{g}_\chi)$ and the G-invariants are the same as the $\mathfrak{g}$-invariants. Let $\mathfrak{z}$ be $\mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{X}(G)$, where $\mathfrak{X}(G) = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^\times)$ is the group of characters of G. In this case we denote the reduction by $A/\!\!/\!\!/_\chi G$. Also for a subspace $U \subset (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$ we can define the reduction $A/\!\!/\!\!/_U \mathfrak{g} := (A/A\varphi(U^\perp))^{\mathfrak{g}}$. Here $U^\perp$ stands for the annihilator of U in $\mathfrak{g}$. When $U = \mathfrak{z}$ we write $A/\!\!/\!\!/\mathfrak{g}$ instead of $A/\!\!/\!\!/_U \mathfrak{g}$. We remark that φ induces an algebra homomorphism $\mathbb{C}[U] \to A/\!\!/\!\!/_U \mathfrak{g}$, whose image lies in the Poisson center. Clearly, $A/\!\!/\!\!/_U \mathfrak{g} := \mathbb{C}[U] \otimes_{\mathbb{C}[(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*]} A/\!\!/\!\!/\mathfrak{g}$.

Proceed to the definition of reduction for Poisson varieties. Let $\mathcal{X}$ be a Poisson algebraic variety. Let G be a connected algebraic group acting on $\mathcal{X}$ by Poisson automorphisms. Suppose that the action admits a moment map $\mu : \mathcal{X} \to \mathfrak{g}^*$, a G-equivariant map such that the following condition holds:

- $\{\mu^*(\xi), \cdot\} = \xi_{\mathcal{X}}$, where $\xi_{\mathcal{X}}$ denotes the derivation of $\mathcal{O}_{\mathcal{X}}$ (the velocity vector field) corresponding to ξ.

We suppose that the G-action is free, that the quotient $\mathcal{X}/G$ exists and that the quotient morphism $\pi_G : \mathcal{X} \to \mathcal{X}/G$ is affine.

Consider the inverse image $\mu^{-1}(\chi) \subset \mathcal{X}$. This is a smooth complete intersection. Set $\mathcal{X}/\!\!/\!\!/_\chi G := \mu^{-1}(\chi)/G \subset \mathcal{X}/G$. Thanks to the discussion above, $\mathcal{X}/\!\!/\!\!/_\chi G$ has a natural structure of a Poisson variety. Similarly, we can define the Poisson varieties $\mathcal{X}/\!\!/\!\!/_U G$, $\mathcal{X}/\!\!/\!\!/G$.

Now suppose that $\mathcal{X}$ is a smooth variety and that the Poisson bivector is non-degenerate, so that $\mathcal{X}$ is symplectic. Let ω denote the symplectic form on $\mathcal{X}$. There is a unique 2-form Ω on $\mathcal{X}/\!\!/\!\!/G$ such that $\mu^*(\Omega)$ coincides with the restriction of ω to $\mu^{-1}(\mathfrak{z})$. The natural morphism $\pi : \mathcal{X}/\!\!/\!\!/G \to \mathfrak{z}$ induced from μ is smooth of relative dimension $\dim \mathcal{X} - 2\dim G$. The form Ω belongs to $\Omega^2(\mathcal{X}/\!\!/\!\!/G, \mathfrak{z})$ and is a symplectic form on the $\mathfrak{z}$-scheme $\mathcal{X}/\!\!/\!\!/G$. The reduction $\mathcal{X}/\!\!/\!\!/_\eta G$ is a symplectic variety. We remark that $\mathcal{X}/\!\!/\!\!/_U G$ equals $U \times_{\mathfrak{z}} \mathcal{X}/\!\!/\!\!/G$ and so is a symplectic scheme over U.

Suppose that $\mathbb{C}^\times$ acts on $\mathcal{X}$ such that $t \cdot \omega = t^2\omega$ and $\mu(t \cdot x) = t^{-2}\mu(x)$ for all $x \in \mathcal{X}$. Then the $\mathbb{C}^\times$-action descends to $\mathcal{X}/\!\!/\!\!/G$. We have $t \cdot \Omega = t^2\Omega$. Equip $\mathfrak{z}$ with a $\mathbb{C}^\times$-action by $t \cdot \alpha = t^{-2}\alpha$. Then the morphism $\pi : \mathcal{X}/\!\!/\!\!/G \to \mathfrak{z}$ becomes $\mathbb{C}^\times$-equivariant. In particular, we have natural $\mathbb{C}^\times$-actions on $\mathcal{X}/\!\!/\!\!/_0 G$, $\mathcal{X}/\!\!/\!\!/_U G$.

Finally, let V be a G-module. Consider the G-equivariant vector bundle $\mathcal{X} \times V \to \mathcal{X}$, where the G-action on $\mathcal{X} \times V$ is supposed to be diagonal. Since the action of G on $\mathcal{X}$ is free, this G-equivariant bundle descends to the bundle on $\mathcal{X}/G$ to be denoted by $\mathcal{V}$. Restricting the latter to $\mathcal{X}/\!\!/\!\!/G$ or to $\mathcal{X}/\!\!/\!\!/_0 G$ we get bundles on these varieties.

3.2. The Duistermaat–Heckman theorem

If a symplectic variety is obtained by Hamiltonian reduction, then there is an easy way to produce its formal deformation. The period map (see Section 2.1) for this deformation can be computed with the help of (an algebraic variant of) the Duistermaat–Heckman theorem.

Let $\mathcal{X}, \omega, G, \mathfrak{z}$ be as in Section 3.1. Suppose that $\mathbb{C}^\times$ acts on $\mathcal{X}$ as explained at the end of that subsection.

Let us produce a graded symplectic formal deformation of $X_0 := \mathcal{X}/\!\!/\!\!/_0 G$. Namely, take the symplectic scheme $X := \mathcal{X}/\!\!/\!\!/G$ over $\mathfrak{z}$ and consider the completion of X along X_0 to be denoted by $\mathcal{X}\widehat{/\!\!/\!\!/}G$. This is a graded formal deformation of interest. According to Section 2.1, this deformation gives rise to a linear map $p : \mathfrak{z} \to H^2_{\mathrm{DR}}(X_0)$. Our goal is to describe this map.

Replacing G with $G/(G,G)$ and $\mathcal{X}$ with $\mathcal{X}/\!\!/\!\!/(G,G)$, we may assume that G is a torus. We have the principal G-bundle $\mu^{-1}(0)\to\mathcal{X}/\!\!/\!\!/_0G$. Let $c\in H^2_{DR}(X_0)\otimes\mathfrak{g}=\operatorname{Hom}(\mathfrak{g}^*,H^2_{DR}(X_0))$ denote its Chern class (we will recall the definition suitable for our purposes below).

The following statement is an algebro-geometric version of the Duistermaat–Heckman theorem.

Proposition 3.2.1. *The map $\mathfrak{g}^*\to H^2_{DR}(X_0)$ induced by the deformation $\mathcal{X}\widehat{/\!\!/\!\!/}G$ of X_0 coincides with c.*

Let us recall one of the possible definitions of c. In the C^∞-setting the Chern class of a line bundle can be defined as the cohomology class of the curvature form of a connection on this bundle. A similar definition can be given in the algebraic situation.

Namely, let Y be a smooth algebraic variety and $\widetilde{Y}\to Y$ be a principal G-bundle, where G is still a torus. Then $\widetilde{Y}\to Y$ is locally trivial in the Zariski topology. In particular, one can find an open affine covering $Y=\bigcup_i Y^i$ such that the restriction $\widetilde{Y}^i\to Y$ of $\widetilde{Y}\to Y$ admits a connection. Let $\alpha^i\in\Omega^1(Y^i)\otimes\mathfrak{g}$ be the connection form. Then both $d\alpha^i$ and $\alpha^i-\alpha^j$ descend to Y and $(d\alpha^i,\alpha^i-\alpha^j)$ form a Čech–De Rham 2-cocycle on Y. It is straightforward to check that the cohomology class of this cocycle does not depend on the choices we made. By definition, this cohomology class is the Chern class of the bundle $\widetilde{Y}\to Y$.

Proof of Proposition 3.2.1. Our proof is a slight modification of the original proof by Duistermaat and Heckman.

Let $\underline{\Omega}$ stand for the symplectic form on $X\widehat{/\!\!/\!\!/}G/\mathfrak{g}^*$. We need to check that $\xi\cdot[\underline{\Omega}]=\langle c,\xi\rangle$ for any $\xi\in\mathfrak{g}^*$.

Let $\widehat{\mathcal{X}}$ denote the formal neighborhood of $\mu^{-1}(0)$ in $\mathcal{X}$. Consider a covering $\widehat{\mathcal{X}}=\bigcup_i\widehat{\mathcal{X}}^i$ by open affine G-stable formal subschemes. Fix an identification $\widehat{\mathcal{X}}^i\cong Y^i\times(\mathfrak{g}^*)^\wedge_0$, where Y^i is an open affine subvariety in $\mu^{-1}(0)$. Moreover, shrinking Y^i's if necessary, we may assume that the restriction of the principal bundle $\mu^{-1}(0)\to X_0$ to Y^i/G is trivial. So $Y^i:=Y^i_0\times G\times(\mathfrak{g}^*)^\wedge_0$. The formal scheme $G\times(\mathfrak{g}^*)^\wedge_0$ can be thought as the completion $(T^*G)^\wedge_G$ of the cotangent bundle T^*G along the base G.

Following the definition of the Gauss–Manin connection recalled in Section 2.1, to compute $\xi\cdot[\underline{\Omega}]$ we need to lift $\underline{\Omega}$ to a Čech–De Rham cochain on $\mathcal{X}\widehat{/\!\!/\!\!/}G$. Let π denote the quotient morphism $\widehat{\mathcal{X}}\to\mathcal{X}\widehat{/\!\!/\!\!/}G$ and p_i be the projection $\widehat{\mathcal{X}}^i\to(T^*G)^\wedge_G$ induced by the decomposition $\widehat{\mathcal{X}}^i=Y^i_0\times(T^*G)^\wedge_G$ introduced in the previous paragraph. Further, let γ_i stand for the canonical 1-form on T^*G (so that $d\gamma_i$ is the natural symplectic form on T^*G).

Consider the 2-form $\omega-p_i^*(d\gamma_i)$ on $\widehat{\mathcal{X}}^i$. It is easy to see that this form is G-invariant and vanishes on the vector fields tangent to the fibers of π. So there is a unique 2-form ω_i on $Y^i=Y^i_0\times(\mathfrak{g}^*)^\wedge_0$ such that $\pi^*\omega_i=\omega-p_i^*(d\gamma_i)$. Also it is easy to see that the restriction of ω_i to the vector fields tangent to the fibers of the projection $Y^i\to(\mathfrak{g}^*)^{\wedge_0}$ coincides with $\underline{\Omega}$. So the cochain $(\omega_i,0,0)$ represents $\underline{\Omega}$.

Clearly, $d\omega_i=0$, so the Čech–De Rham differential of $(\omega_i,0,0)_i$ is nothing else but $(0,\omega_i-\omega_j,0)_{ij}$. Now $\pi^*(\omega_i-\omega_j)=dp_i^*(\gamma_i)-dp_j^*(\gamma_j)$. The form $p_i^*(\gamma_i)$ is G-equivariant. Moreover, for $\xi\in\mathfrak{g}^*,\eta\in\mathfrak{g}$ we have $dp_i^*(\gamma_i)(\xi,\eta_X)=d\gamma_i(\xi,\eta_G)=\langle\xi,\eta\rangle$. Analogously to the original proof in [10], we see that the map $\mathfrak{g}^*\to\Omega^1(Y^i)$ given by $\xi\mapsto\iota_\xi dp_i^*(\gamma_i)$ is a connection form on $Y^i\times(\mathfrak{g}^*)^\wedge_0$. Now the claim of the proposition follows from the definition of the Chern class recalled above. □

3.3. Quantum reduction: algebra level

Let $\mathcal{A}$ be an associative algebra, $\mathfrak{g}$ be a Lie algebra equipped with a Lie algebra homomorphism $\Phi : \mathfrak{g} \to \mathcal{A}$. Similarly to Section 3.1, we can define the spaces $\mathcal{A}/\!\!/\!\!/_\chi \mathfrak{g}, \mathcal{A}/\!\!/\!\!/_U \mathfrak{g}, \mathcal{A}/\!\!/\!\!/ \mathfrak{g}$ and $\mathcal{A}/\!\!/\!\!/_\chi G, \mathcal{A}/\!\!/\!\!/_U G, \mathcal{A}/\!\!/\!\!/ G$ (for a connected group G with Lie algebra $\mathfrak{g}$). We remark that all these spaces have natural algebra structures. For example, $\mathcal{A}/\!\!/\!\!/_U \mathfrak{g}, \mathcal{A}/\!\!/\!\!/_U G$ are algebras over $\mathbb{C}[U]$. The algebras above are called *quantum Hamiltonian reductions* of $\mathcal{A}$.

We are mostly interested in the following special case. Suppose that the algebra $\mathcal{A}$ comes equipped with an exhaustive increasing filtration $\mathrm{F}_i\mathcal{A}, i \in \mathbb{Z}$. Suppose that $[\mathrm{F}_i\mathcal{A}, \mathrm{F}_j\mathcal{A}] \subset \mathrm{F}_{i+j-2}\mathcal{A}$ for all i, j and that $\operatorname{im}\Phi \subset \mathrm{F}_2\mathcal{A}$. Then $A := \operatorname{gr}\mathcal{A}$ is a graded commutative algebra and has a natural Poisson bracket of degree -2. Then the quantum reductions $\mathcal{A}/\!\!/\!\!/_\chi \mathfrak{g}$ etc. inherit a filtration from $\mathcal{A}$. On the other hand, set $\varphi := \operatorname{gr}\Phi : \mathfrak{g} \to \mathcal{A}$. The algebras $A/\!\!/\!\!/_0 \mathfrak{g}, A/\!\!/\!\!/_U \mathfrak{g}$ are graded with the Poisson brackets of degree -2.

We say that *quantization commutes with reduction* for χ if the following two conditions hold:

(1) $\operatorname{gr}\mathcal{A}\Phi(\mathfrak{g}_\chi) = A\varphi(\mathfrak{g})$.
(2) Any $\mathfrak{g}$-invariant in $A/A\varphi(\mathfrak{g})$ lifts to a $\mathfrak{g}$-invariant in $\mathcal{A}/\mathcal{A}\Phi(\mathfrak{g}_\chi)$.

One can give a similar definition for U. Clearly, if quantization commutes with reduction, then $\operatorname{gr}\mathcal{A}/\!\!/\!\!/_\chi \mathfrak{g} = A/\!\!/\!\!/_0 \mathfrak{g}, \operatorname{gr}\mathcal{A}/\!\!/\!\!/_U \mathfrak{g} = A/\!\!/\!\!/_U \mathfrak{g}$.

Here are some conditions that guarantee that quantization commutes with reduction.

Lemma 3.3.1. *Let G be an algebraic group that acts on $\mathcal{A}$ rationally by filtration preserving automorphisms. Suppose that $A = \operatorname{gr}\mathcal{A}$ is finitely generated. Let $U \subset \mathfrak{z}$ be a subspace. Suppose that the elements $\varphi(\xi_1), \dots, \varphi(\xi_k)$ form a regular sequence in A, where $\xi_1, \dots, \xi_k$ is a basis in $U^\perp \subset \mathfrak{g}$. Then* (1) *holds. Finally, suppose that one of the following conditions holds:*

(A) *G is reductive.*
(B) *The G-action on $\mu^{-1}(U)$ (where $\mu : \operatorname{Spec}(A) \to \mathfrak{g}^*$ is the moment map) is free, and $A/\!\!/\!\!/_U G$ is finitely generated.*

Then (2) *holds.*

Proof. (1) is pretty standard; see, for instance, proof of Lemma 3.6.1 in [27]. (A) easily implies (2). If (B) holds, then (2) also is pretty standard, compare with [28, proof of Proposition 3.4.1]. □

We will need a ramification of the above construction. Namely, let $\mathcal{A}_h$ be a flat $\mathbb{C}[h]$-algebra such that $A := \mathcal{A}_h/(h)$ is commutative. Suppose that $\mathbb{C}^\times$ acts on $\mathcal{A}_h$ by automorphisms such that $t \cdot h = t^2 h$ for all $t \in \mathbb{C}^\times$. Let $\mathcal{A}_h$ be equipped with a $\mathbb{C}[h]$-linear map $\Phi_h : \mathfrak{g} \to \mathcal{A}_h$ with $t \cdot \Phi_h(\xi) = t^2 \Phi_h(\xi)$ for all $\xi \in \mathfrak{g}$. On $\mathcal{A}_h$ we have a Lie bracket $[\cdot,\cdot]_h$ given by $[a, b]_h = \frac{1}{h}[a, b]$. We suppose, in addition, that Φ_h is a Lie algebra homomorphism. For $\chi \in (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$ form the shift $\mathfrak{g}_{\chi h} := \{\xi - \langle\chi, \xi\rangle h\} \subset \mathfrak{g} \oplus \mathbb{C}h$ and set $\mathcal{A}_h/\!\!/\!\!/_{\chi h} \mathfrak{g} := (\mathcal{A}_h/\mathcal{A}_h\Phi_h(\mathfrak{g}_{\chi h}))^{\mathfrak{g}}$. The reduction $\mathcal{A}_h/\!\!/\!\!/_U \mathfrak{g}$ is defined in the same way as above.

The quantization commutes with reduction condition is stated analogously to the filtered situation (taking the associated graded should be replaced with taking the quotient by h).

The relation between the $\mathbb{C}[h]$-algebra setting and the filtered algebra setting is as follows. Suppose the $\mathbb{C}^\times$-action comes from a grading on $\mathcal{A}_h$. Then $\mathcal{A} := \mathcal{A}_h/(h-1)$ is a filtered algebra with $\operatorname{gr}\mathcal{A} = \mathcal{A}_h/(h)$. The homomorphism $\Phi_h : \mathfrak{g} \to \mathcal{A}_h$ induces a homomorphism $\Phi : \mathfrak{g} \to \mathcal{A}$. It is straightforward to see that $\left[\mathcal{A}_h/\!\!/\!\!/_{\chi h} \mathfrak{g}\right]/(h-1) = \mathcal{A}/\!\!/\!\!/_\chi \mathfrak{g}$ and $\mathcal{A}_h/\!\!/\!\!/_U \mathfrak{g}/(h-1) = \mathcal{A}/\!\!/\!\!/_U \mathfrak{g}$.

We remark that there is a way to replace a filtered algebra with a graded algebra over the polynomial ring: the Rees algebra construction. However, in this construction the independent variable is of degree 1, not 2 that we need. Of course, $\mathcal{A}_h$ is closely related to the Rees algebra of $\mathcal{A}$ but we will not need this relation in the sequel.

3.4. Quantum reduction: sheaf level

Let $\mathcal{X}$, G be as in Section 3.1. We suppose that $\mathbb{C}^\times$ acts on $\mathcal{X}$ as at the end of that subsection. Let $\mathcal{D}$ be a quantization of $\mathcal{X}$. For convenience we suppose that $\mathcal{D}$ is graded.

Suppose further that $\mathcal{D}$ is G-equivariant; see Section 2.2. The G-action defines a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Der}_{\mathbb{C}[[h]]}(\mathcal{D})$, $\xi \mapsto \xi_{\mathcal{D}}$. Finally, we suppose that the G-action on $\mathcal{D}$ admits a quantum comoment map, i.e., a G-equivariant linear map $\Phi : \mathfrak{g} \to \Gamma(X, \mathcal{D})$ such that

- $\frac{1}{h}[\Phi(\xi), \cdot] = \xi_{\mathcal{D}}$,
- the image of $\Phi(\xi)$ in $\mathcal{O}_X$ coincides with $\mu^*(\xi)$,
- $\Phi(\xi)$ has degree 2 with respect to the $\mathbb{C}^\times$-action.

We can sheafify the constructions of the previous paragraph. From Lemma 3.3.1 we see that the sheaf $\mathcal{D}/\!\!/\!\!/G$ on the $\mathfrak{z}$-scheme $X/\!\!/\!\!/G$ is a graded quantization. Similarly, we can define the graded quantizations $\mathcal{D}/\!\!/\!\!/_{\chi h}G$, $\mathcal{D}/\!\!/\!\!/_U G$.

Example 3.4.1. Let G be an algebraic group, H be its algebraic subgroup. Then $\mathcal{D}_h(G/H)$ is naturally identified with $\mathcal{D}_h(G)/\!\!/\!\!/_0 H$. This is well known for the usual differential operators. The proof in the homogenized setting is similar.

Finally, we will need the completion $\mathcal{D}\widehat{/\!\!/\!\!/}G$ of $\mathcal{D}/\!\!/\!\!/G$ along $X/\!\!/\!\!/_0 G$. By definition, this completion is the inverse limit $\varprojlim [\mathcal{D}/\!\!/\!\!/G]/\mathcal{J}^k$, where $\mathcal{J}$ is the kernel of the composition $\mathcal{D}/\!\!/\!\!/G \twoheadrightarrow \mathcal{O}_{X/\!\!/\!\!/G} \twoheadrightarrow \mathcal{O}_{X/\!\!/\!\!/_0 G}$. One can view $\mathcal{D}\widehat{/\!\!/\!\!/}G$ as a quantization of the scheme $X\widehat{/\!\!/\!\!/}G/\mathfrak{z}^{\wedge_0}$.

4. Symplectic resolutions

4.1. Generalities

We start by recalling the definition of a symplectic resolution.

Let X_0 be a Poisson (possibly singular) normal affine variety. Further, suppose that X_0 is a scheme over a smooth affine variety S and the subalgebra $\mathbb{C}[S] \subset \mathbb{C}[X_0]$ is central. Suppose that the restriction of the Poisson bivector to X_0^{reg} is a non-degenerate section of $\bigwedge^2 \mathcal{T}_{X_0^{\mathrm{reg}}/S}$, let Ω_0 be the corresponding 2-form in $\Omega^2(X_0^{\mathrm{reg}}/S)$. A smooth S-scheme X equipped with a morphism $\pi : X \to X_0$ over S and with a fiberwise symplectic form $\Omega \in \Omega^2(X/S)$ is said to be a *symplectic resolution* of X_0 if

(1) π is projective and is birational fiberwise.
(2) $\pi^*(\Omega_0) = \Omega$.

Of course, when S is a point, we get the usual definition of a symplectic resolution.

We will need some general properties of symplectic resolutions. The following lemma is very standard.

Lemma 4.1.1. *Let X, X_0, S be as above. Then $H^i(X, \mathcal{O}_X) = \{0\}$ for $i > 0$.*

Proof. Let $\rho : X \to S$ denote the structure morphism and set $X_s := \rho^{-1}(s)$, $s \in S$. Each X_s is a symplectic resolution of a normal affine variety. So thanks to the Grauert–Riemenschneider theorem, $H^i(X_s, \mathcal{O}_{X_s}) = \{0\}$. It follows that the fiber of $R^i_*\rho(\mathcal{O}_X)$ at s coincides with $H^i(X_s, \mathcal{O}_{X_s})$ and hence is zero. Since S is affine, we are done. □

The main application of this lemma for us is as follows. Let $\mathcal{D}$ be a quantization of X/S. Then using Lemma 4.1.1 it is easy to show that $H^i(X, \mathcal{D}) = 0$ for $i > 0$, while $\Gamma(X, \mathcal{D})/(h) = \mathbb{C}[X]$.

4.2. Quiver varieties

The basic notation related to quivers was introduced in Section 1.2.

To a quiver Q one assigns its double quiver $\mathrm{D}Q = (\mathrm{D}Q_0, \mathrm{D}Q_1)$ with $\mathrm{D}Q_0 := Q_0$ and $\mathrm{D}Q_1 := Q_1 \sqcup Q_1^{\mathrm{op}}$, where Q_1^{op} is identified with Q_1, an element of Q_1^{op} corresponding to $a \in Q_1$ is usually denoted by a^*. The maps $t, h : Q_1 \to Q_0$ in the double quiver are as before, while $t(a^*) = h(a), h(a^*) = t(a), a \in Q_1^{\mathrm{op}}$. In a sentence, $\mathrm{D}Q$ is obtained from Q by adding a reverse arrow for any arrow in Q.

Fix non-negative integers $v_i, d_i, i \in Q_0$. Set $\mathbf{v} := (v_i)_{i\in Q_0}, \mathbf{d} := (d_i)_{i\in Q_0}$. Following Nakajima consider the spaces

$$R(Q, \mathbf{v}, \mathbf{d}) := \bigoplus_{a\in Q_1} \mathrm{Hom}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{i\in Q_0} \mathrm{Hom}(D_i, V_i),$$

$$\begin{aligned} R(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) &:= \bigoplus_{a\in Q_1} (\mathrm{Hom}(V_{t(a)}, V_{h(a)}) \oplus \mathrm{Hom}(V_{h(a)}, V_{t(a)})) \\ &\quad \oplus \bigoplus_{i\in Q_0} (\mathrm{Hom}(D_i, V_i) \oplus \mathrm{Hom}(V_i, D_i)) \\ &= R(Q, \mathbf{v}, \mathbf{d}) \oplus R(Q, \mathbf{v}, \mathbf{d})^*, \end{aligned}$$

where $V_i, D_i, i \in Q_0$, are vector spaces of dimensions v_i, d_i, respectively. Throughout the paper we skip $\mathbf{d}$ from the notation if $\mathbf{d} = 0$ and write $R(Q, \mathbf{v})$ instead of $R(Q, \mathbf{v}, 0)$, etc.

Being identified with $R(Q, \mathbf{v}, \mathbf{d}) \oplus R(Q, \mathbf{v}, \mathbf{d})^*$, the space $R(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ is symplectic, let ω stand for the symplectic form. The group $\mathrm{GL}(\mathbf{v}) := \prod_{i\in Q_0} \mathrm{GL}(v_i)$ acts naturally on $R(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$. The map $\mu : R(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \to \mathfrak{gl}(\mathbf{v})^* \cong \mathfrak{gl}(\mathbf{v})$ sending $(A_a, B_a, \Gamma_i, \Delta_i)$, $A_a \in \mathrm{Hom}(V_{t(a)}, V_{h(a)})$, $B_a \in \mathrm{Hom}(V_{h(a)}, V_{t(a)})$, $\Gamma_i \in \mathrm{Hom}(D_i, V_i)$, $\Delta_i \in \mathrm{Hom}(V_i, D_i)$ to

$$\left(\sum_{a, i=h(a)} A_a B_{a^*} - \sum_{a, t(a)=i} B_{a^*} A_a + \Gamma_i \Delta_i\right)_{i\in Q_0}$$

is a moment map for the $\mathrm{GL}(\mathbf{v})$-action. Set $\mathfrak{z} := (\mathfrak{gl}(\mathbf{v})/[\mathfrak{gl}(\mathbf{v}), \mathfrak{gl}(\mathbf{v})])^*$. We identify $\mathfrak{z}$ with $\mathbb{C}^{Q_0}$ by setting $\langle \chi, (\xi_i)_{i\in Q_0}\rangle \mapsto \sum_{i\in Q_0} \chi_i \mathrm{tr}(\xi_i)$.

Below for $U \subset \mathfrak{z}$ we set $\Lambda_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) := \mu^{-1}(U)$. We write $\Lambda(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ for $\Lambda_{\mathfrak{z}}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ and $\Lambda_\chi(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ for the fiber of $\chi \in \mathfrak{z}$ of the natural map $\Lambda(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \to \mathfrak{z}$.

So one can form the quotient

$$\mathcal{M}(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) := R(\mathrm{D}Q, \mathbf{v}, \mathbf{d})/\!\!/\!\!/\mathrm{GL}(\mathbf{v}) = \Lambda(\mathrm{D}Q, \mathbf{v}, \mathbf{d})/\!\!/\mathrm{GL}(\mathbf{v}).$$

This is a Poisson algebraic variety to be referred to as a (universal) affine quiver variety. Also we can form the reductions $\mathcal{M}_\chi(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$, $\mathcal{M}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$.

These reductions were studied by Crawley-Boevey; see, in particular, [6,7]. In fact, he worked only with the case when $\mathbf{d} = 0$. However, the general case can be reduced to this one by using the following construction. Consider the quiver $Q^{\mathbf{d}} = (Q_0^{\mathbf{d}}, Q_1^{\mathbf{d}})$, where $Q_0^{\mathbf{d}} := Q_0 \sqcup \{s\}$, where s is a new vertex, and $Q_1^{\mathbf{d}}$ is the disjoint union of Q_1 and the set $Q_1' := \{a^{ij}, i \in Q_0, j = 1, \ldots, d_i\}$

with $t(a^{ij}) = i, h(a^{ij}) = s$. Then $R(Q, \mathbf{v}, \mathbf{d}) = R(Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}})$, where $\mathbf{v}^{\mathbf{d}} := (\mathbf{v}, \epsilon_s)$, ϵ_s being the coordinate vector at the vertex s. The group $\mathrm{GL}(\mathbf{v})$ is included naturally into $\mathrm{GL}(\mathbf{v}^{\mathbf{d}})$. Moreover, $\mathrm{GL}(\mathbf{v}^{\mathbf{d}}) = \mathbb{C}^\times \times \mathrm{GL}(\mathbf{v})$, where $\mathbb{C}^\times = \{x \cdot 1, x \in \mathbb{C}^\times\}$ with 1 standing for the unit in $\mathrm{GL}(\mathbf{v}^{\mathbf{d}})$. The subgroup $\mathbb{C}^\times$ acts trivially on $R(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}})$ and so the action of $\mathrm{GL}(\mathbf{v}^{\mathbf{d}})$ reduces to that of $\mathrm{GL}(\mathbf{v})$. It follows that $\Lambda_U(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}}) = \Lambda_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$, $\mathcal{M}_U(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}}) = \mathcal{M}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ etc.

Following Crawley-Boevey, let us describe the algebro-geometric properties of the varieties $\mathcal{M}_\chi(\mathrm{D}Q, \mathbf{v}), \mathcal{M}_U(\mathrm{D}Q, \mathbf{v})$. Define a quadratic function $p : \mathbb{C}^{Q_0} \to \mathbb{C}, \alpha = \sum_{i \in Q_0} \alpha_i \epsilon_i \mapsto 1 - \sum_{i \in Q_0} \alpha_i^2 + \sum_{a \in Q_1} \alpha_{t(a)}\alpha_{h(a)}$.

Proposition 4.2.1. (1) *Suppose that the following condition holds: for any decomposition of* $\mathbf{v}$ *into the sum* $\mathbf{v} = \mathbf{v}^1 + \cdots + \mathbf{v}^k$, $k > 1$, *of positive roots* $\mathbf{v}^1, \ldots, \mathbf{v}^k$ *of* Q *the inequality* $p(\mathbf{v}) \geqslant \sum_{i=1}^k p(\mathbf{v}^i)$ *holds. Then the moment map* μ *is flat, the schemes* $\Lambda_\chi(\mathrm{D}Q, \mathbf{v}), \Lambda_U(\mathrm{D}Q, \mathbf{v})$ *are non-empty complete intersections provided* $\chi \cdot \mathbf{v} = 0$. *Moreover, if* $p(\mathbf{v}) > \sum_{i=1}^k p(\mathbf{v}^i)$ *for any decomposition as above, then the schemes* $\Lambda_\chi(\mathrm{D}Q, \mathbf{v}), \Lambda_U(\mathrm{D}Q, \mathbf{v})$ *are reduced and irreducible, and each scheme* $\Lambda_\chi(\mathrm{D}Q, \mathbf{v})$ *(with* $\chi \cdot \mathbf{v} = 0$*) contains a closed* $\mathrm{GL}(\mathbf{v})$*-orbit with the stabilizer* $\mathbb{C}^\times = \{x \cdot 1, x \in \mathbb{C}^\times\}$.

(2) *The varieties* $\mathcal{M}_\chi(\mathrm{D}Q, \mathbf{v}), \mathcal{M}_U(\mathrm{D}Q, \mathbf{v})$ *are normal.*

Proof. The statements for $\Lambda_\chi, \mathcal{M}_\chi$ were proved by Crawley-Boevey: (1) in [6, Theorems 1.1, 1.2], and (2) in [7]. The claim in (1) that Λ_U is reduced follows from the reducedness of the Λ_χ's. The irreducibility for Λ_U follows from the irreducibility and the reducedness for Λ_0, thanks to the contracting action of $\mathbb{C}^\times$ on Λ_U. A similar argument proves the normality. □

Now we turn to non-affine quiver varieties. Namely, consider a character θ of $\mathrm{GL}(\mathbf{v})$. Recall that a point $x \in R(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ is said to be θ-*semistable* if there is a homogeneous polynomial $f \in \mathbb{C}[R(\mathrm{D}Q, \mathbf{v}, \mathbf{d})]$ with $f(x) \neq 0$ that is $\mathrm{GL}(\mathbf{v})$-semiinvariant of weight that is a positive multiple of θ. Geometric Invariant Theory implies that there is a categorical quotient $\mathcal{M}^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ for the $\mathrm{GL}(\mathbf{v})$-action on $\Lambda(\mathrm{D}Q, \mathbf{v}, \mathbf{d})^{\theta,ss}$ such that the quotient morphism $\Lambda(\mathrm{D}Q, \mathbf{v}, \mathbf{d})^{\theta,ss} \to \mathcal{M}^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ is affine. The latter implies that $\mathcal{M}^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ has a natural Poisson structure. This structure is symplectic (over $\mathfrak{z}$) whenever the action of $\mathrm{GL}(\mathbf{v})$ on $\Lambda(\mathrm{D}Q, \mathbf{v}, \mathbf{d})^{\theta,ss}$ is free. Introduce the Poisson schemes $\mathcal{M}^\theta_\chi(\mathrm{D}Q, \mathbf{v}, \mathbf{d}), \mathcal{M}^\theta_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ in a similar way.

It is a standard fact from Geometric Invariant Theory that there are natural projective morphisms

$$\mathcal{M}^\theta_\chi(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \to \mathcal{M}_\chi(\mathrm{D}Q, \mathbf{v}, \mathbf{d}), \mathcal{M}^\theta_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \to \mathcal{M}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d}). \tag{4.1}$$

Nakajima in [38] found some conditions on θ guaranteeing that the $\mathrm{GL}(\mathbf{v})$-actions on the varieties $\Lambda_\chi(\mathrm{D}Q, \mathbf{v}, \mathbf{d})^{\theta,ss}$ are free and the projective morphisms (4.1) are birational (and so are symplectic resolutions). His results are summarized below.

To state the proposition we will need to recall some definitions. Let R_+ denote the system of positive roots of the quiver Q. Recall that $\mathfrak{z}$ is identified with $\mathbb{C}^{Q_0}$. We say that $\theta \in \mathbb{C}^{Q_0}$ is *generic* if $\theta \cdot \alpha \neq 0$ for any $\alpha \in R_+$.

The following statement follows from [38, Theorems 2.8, 3.2, 4.1].

Proposition 4.2.2. *Suppose* θ *is generic. Then the* $\mathrm{GL}(\mathbf{v})$*-action on* $\Lambda(\mathrm{D}Q, \mathbf{v}, \mathbf{d})^{\theta,ss}$ *is free, and the morphisms* (4.1) *are birational onto their images provided* $\Lambda_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})^{\theta,ss} \neq \varnothing$.

For example, suppose that

$$\theta((X_i)_{i\in Q_0}) = \prod_{i\in Q_0} \det(X_i)^{-1}. \tag{4.2}$$

The character θ is generic. An element $(A_a, B_{a^*}, \Gamma_i, \Delta_i)$ is semistable if there are no nonzero subspaces $V_i' \subset \ker \Delta_i$ such that the collection $(V_i')_{i\in Q_0}$ is stable under all A_a, B_{a^*}. It is clear that the $\mathrm{GL}(\mathbf{v})$-action on $\Lambda(\mathrm{D}Q, \mathbf{v}, \mathbf{d})^{\theta,ss}$ is free.

In fact, Nakajima's results also allow to determine the number of irreducible components of $\Lambda_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})^{\theta,ss}$. Namely, assume that the quiver Q is either of finite type or affine. Let $\mathfrak{g}(Q)$ be the corresponding (finite dimensional semisimple or affine) Kac–Moody algebra. Then we can view $\mathbf{d}$ as an element of the weight lattice of $\mathfrak{g}(Q)$, the corresponding weight is $\mathrm{d} := \sum_{i\in Q_0} d_i\omega_i$, where ω_i, $i \in Q_0$, are fundamental weights. Also to $\mathbf{v}$ we assign an element $\mathrm{v} := \sum_{i\in Q_0} v_i\epsilon_i$ in the root lattice.

The next proposition follows from Theorem 10.16 in [38].

Proposition 4.2.3. *Let θ be generic. Consider the irreducible highest weight module $L(\mathrm{d})$ of $\mathfrak{g}(Q)$. The number of irreducible components in $\Lambda_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})^{\theta,ss}$ coincides with the dimension of the weight space of weight $\mathrm{d} - \mathrm{v}$ in $L(\mathrm{d})$.*

Finally, let us discuss the quantum analogs of quiver varieties.

Let $\mathbb{A}_h (=\mathbb{A}_h(V^*))$ denote the homogeneous Weyl algebra of V^*, i.e., the quotient of the tensor algebra $T(V^*)[h]$ by the relations $u \otimes v - v \otimes u - h\omega(u, v)$, $u, v \in V^*$, where ω denotes the symplectic form on V^*. This algebra inherits a grading from $T(V^*)[h]$ with $\mathbb{A}_h/h\mathbb{A}_h = \mathbb{C}[V]$.

Set $V := R(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$. Clearly, the group $G := \mathrm{GL}(\mathbf{v})$ acts on $\mathbb{A}_h$ by graded algebra automorphisms. Further, the comoment map $\mu^* : \mathfrak{g} \to \mathbb{C}[V]$ naturally lifts to a quantum comoment map $\Phi : \mathfrak{g} \to \mathbb{A}_h$. Namely, we have a unique (up to a scalar factor) $\mathrm{Sp}(V)$-equivariant homomorphism $\mathfrak{sp}(V) \to \mathbb{A}_h$. The map Φ is obtained by restricting this homomorphism to $\mathfrak{g} \subset \mathfrak{sp}(V)$. So we can define the quantum Hamiltonian reductions $\mathbb{A}_{\chi h}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h := \mathbb{A}_h /\!\!/\!\!/_{\chi h} G$, $\mathbb{A}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h := \mathbb{A}_h /\!\!/\!\!/_U G$ as in Section 3.3.

We can also consider the sheaf version of this construction. Namely, consider the deformation quantization $\mathbb{A}_{V^*,h}$ of V obtained by localizing (the h-adic completion of) the algebra $\mathbb{A}_h$. This quantization is G-equivariant and graded. Now let us compare the algebra $\mathbb{A}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ with the algebra of global sections of $\mathbb{A}_{V^*,h} /\!\!/\!\!/^{\theta}_U G := \mathbb{A}_{V^*,h}|_{V^{\theta,ss}} /\!\!/\!\!/_U G$.

Lemma 4.2.4. *Suppose that*

(i) *quantization commutes with reduction for $\mathbb{A}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$,*
(ii) *the action of G on $\Lambda_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})^{\theta,ss}$ is free,*
(iii) *and the morphism $\mathcal{M}^{\theta}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \to \mathcal{M}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ is a symplectic resolution of singularities.*

Then the natural morphism

$$\mathbb{A}_h^G \to \Gamma(\mathcal{M}^{\theta}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d}), \mathbb{A}_{V^*,h} /\!\!/\!\!/^{\theta}_U G) \tag{4.3}$$

gives rise to an isomorphism between $\mathbb{A}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ and the subalgebra of $\mathbb{C}^\times$-finite elements in $\Gamma(\mathcal{M}^{\theta}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d}), \mathbb{A}_{V^*,h} /\!\!/\!\!/^{\theta}_U G)$.

Proof. Let us construct a homomorphism $\mathbb{A}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h^{\wedge_h} \to \Gamma(\mathcal{M}^{\theta}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d}), \mathbb{A}_{V^*,h} /\!\!/\!\!/^{\theta}_U G)$, where the superscript ${}^{\wedge_h}$ means the h-adic completion. Since G is reductive, and its action on $\mathbb{A}_h(V^*)^{\wedge_h}$ is pro-finite, we see that the natural homomorphism $(\mathbb{A}_h(V^*)^{\wedge_h})^G \to \mathbb{A}_h(V^*)^{\wedge_h} /\!\!/\!\!/_U G = \mathbb{A}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h^{\wedge_h}$ is surjective and its kernel coincides with $(\mathbb{A}_h(V)^{\wedge_h})^G \cap \mathbb{A}_h(V)^{\wedge_h} \Phi_h(U^\perp) = [\mathbb{A}_h(V)^{\wedge_h} \Phi_h(U^\perp)]^G$. However, it is easy to see that this kernel is contained in the kernel of (4.3). So we have an obviously $\mathbb{C}^\times$-equivariant and $\mathbb{C}[U][h]$-linear homomorphism $\mathbb{A}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h^{\wedge_h} \to \Gamma(V /\!\!/\!\!/_U^\theta G, \mathbb{A}_{V^*,h} /\!\!/\!\!/_U^\theta G)$. We are going to show that this homomorphism is an isomorphism.

We have the following commutative diagram, where the vertical arrows are quotients by h.

$$\begin{array}{ccc}
\mathbb{A}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h^{\wedge_h} & \longrightarrow & \Gamma(V /\!\!/\!\!/_U^\theta G, \mathbb{A}_{V^*,h} /\!\!/\!\!/_U^\theta G) \\
\downarrow & & \downarrow \\
\mathbb{C}[\mathcal{M}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})] & \longrightarrow & \mathbb{C}[\mathcal{M}^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})]
\end{array}$$

The bottom horizontal arrow is an isomorphism since the morphism $\mathcal{M}_U^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \to \mathcal{M}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ is a resolution because of (iii). The left vertical arrow is surjective because of (i). Thanks to (ii), $\mathbb{A}_{V^*,h} /\!\!/\!\!/_U^\theta G$ is a quantization of $V /\!\!/\!\!/_U^\theta G$. The right vertical arrow is the quotient by h because of (iii) and the remarks at the end of Section 4.1. So the top horizontal arrow is surjective modulo h and hence is genuinely surjective. Since the sheaf $\mathbb{A}_{V^*,h} /\!\!/\!\!/_U^\theta G$ is $\mathbb{C}[[h]]$-flat, we see that $\Gamma(\mathcal{M}^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d}), \mathbb{A}_{V^*,h} /\!\!/\!\!/_U^\theta G)$ is a flat $\mathbb{C}[[h]]$-algebra. Using this and the observation that the bottom arrow is an isomorphism we see that the top horizontal arrow is injective.

To complete the proof we notice that, since the grading on $\mathbb{A}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ is positive, the algebra $\mathbb{A}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ coincides with the subalgebra of $\mathbb{C}^\times$-finite elements in $\mathbb{A}_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h^{\wedge_h}$. $\square$

We note that instead of $\mathcal{M}_U^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ in the previous lemma we could consider the formal neighborhood $V \widehat{/\!\!/\!\!/}_U^\theta G$ of $\mathcal{M}_0^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ in $\mathcal{M}^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$. The claim is still that $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ is the algebra of $\mathbb{C}^\times$-finite elements in the algebra of global sections of $\mathbb{A}_{h,V^*} \widehat{/\!\!/\!\!/}^\theta G$.

4.3. *Slodowy varieties*

In this subsection, we will define *Slodowy varieties* (tracing back to [42]) that are certain smooth symplectic varieties that are related to *Slodowy slices* in reductive Lie algebras.

First of all, let us recall Slodowy slices. Let G be a reductive algebraic group, $\mathfrak{g}$ its Lie algebra, $e \in \mathfrak{g}$ a nilpotent element, and $\mathbb{O} := Ge$. The Slodowy slice $\mathrm{S}(=\mathrm{S}(e) = \mathrm{S}(\mathbb{O}))$ associated to $(\mathfrak{g}, e)$ is a transverse slice to the adjoint orbit $\mathbb{O}$ of $e \in \mathfrak{g}$. It is constructed as follows. Pick a semisimple element $h \in \mathfrak{g}$ and a nilpotent element $f \in \mathfrak{g}$ forming an $\mathfrak{sl}_2$-triple with e. Then set $\mathrm{S} := e + \ker \mathrm{ad}(f)$. In the sequel we will identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using a symmetric non-degenerate invariant form $(\cdot,\cdot)$ and will consider $\mathbb{O}$, S as subvarieties in $\mathfrak{g}^*$. Also set $\chi := (e, \cdot)$.

The algebra $\mathbb{C}[\mathrm{S}]$ has some nice grading (often called the *Kazhdan grading*). Namely, let $\gamma : \mathbb{C}^\times \to G$ be the one-parameter group associated to h (so that $\gamma(t)\cdot\xi = t^i\xi$ for $\xi \in \mathfrak{g}$ with $[h, \xi] = i\xi$). Consider the $\mathbb{C}^\times$-action on $\mathfrak{g}^*$ given by $t \cdot \alpha = t^{-2}\gamma(t)\alpha$, $t \in \mathbb{C}^\times$, $\alpha \in \mathfrak{g}^*$. It is easy to see that $\lim_{t\to\infty} t \cdot s = \chi$ for all $s \in \mathrm{S}$. In other words, the grading on $\mathbb{C}[\mathrm{S}]$ is positive.

Also thanks to the Kazhdan action, we see that S intersects an orbit $\mathbb{O}' \subset \mathfrak{g}^*$ if and only if $\mathbb{O} \subset \overline{\mathbb{C}^\times \mathbb{O}'}$.

In [29,27], we considered a certain symplectic G-variety X, called the *equivariant Slodowy slice*, whose quotient is naturally identified with S. As a variety, $\mathrm{X} := G \times \mathrm{S}$. The group G acts on X by the left translations: $g.(g_1, s) = (gg_1, s)$. A symplectic form on X is defined as follows.

Identify T^*G with $G \times \mathfrak{g}^*$ via the trivialization by means of the left-invariant 1-forms. Then $\mathrm{X} = G \times \mathrm{S}$ is included into $G \times \mathfrak{g}^* = T^*G$. It turns out that the restriction of the canonical symplectic form from T^*G to X is non-degenerate. Denote this restriction by ω. Define the action of $\mathbb{C}^\times$ on T^*G by $t \cdot (g, \alpha) = (g\gamma(t)^{-1}, t^{-2}\gamma(t)\alpha)$. The subvariety $\mathrm{X} \subset T^*G$ is $\mathbb{C}^\times$-stable.

Pick a parabolic subgroup $P \subset G$. By the *Slodowy variety* $\mathrm{S}(e, P)$ corresponding to e and P we mean the reduction $\mathrm{X}/\!\!/\!\!/_0 P$. We will also need the formal deformation $\widetilde{\mathrm{S}}(e, P) := \mathrm{X}\widehat{/\!\!/\!\!/}P$.

It is clear that $\mathrm{S}(e, P)$ can be naturally embedded into $T^*G/\!\!/\!\!/_0 P = T^*(G/P)$ (here we consider the quotient with respect to the action of P from the left). From here it is easy to see $\mathrm{S}(e, B)$ is the resolution of singularities of the intersection $\mathrm{S} \cap \mathcal{N}$ of S with the nil-cone $\mathcal{N} \subset \mathfrak{g}^*$, compare with [16]. For a general parabolic subgroup $P \subset G$, there are projective morphisms $\mathrm{S}(e, P) \to \mathrm{S} \cap \mathcal{N}, \mathrm{X}/\!\!/\!\!/P \to \mathrm{S}$ restricted from $T^*(G/P) \to \mathcal{N}, G *_{P_0} \mathfrak{p}^\perp \to \mathfrak{g}^*$, where P_0 is the solvable radical of P.

Below we will be mostly interested in two cases.

Case 1. $\mathfrak{g}$ is simple of types A, D, E, and e is a subprincipal nilpotent element in $\mathfrak{g}$. The latter means that the orbit $\mathbb{O}$ is of codimension 2 in $\mathcal{N}$.

Case 2. $G = \mathrm{SL}_n$. In this case the moment map $T^*(G/P) \to \mathfrak{g}^*$ is generically injective and its image coincides with $\overline{G\mathfrak{p}^\perp}$. The latter subvariety is the closure of an appropriate nilpotent orbit in $\mathfrak{g}^*$ (the Richardson orbit of $\mathfrak{p}$). So we see that $T^*(G/P)$ is a symplectic resolution of $\overline{G\mathfrak{p}^\perp}$, while $\mathrm{S}(e, P)$ is a symplectic resolution of $\mathrm{S} \cap \overline{G\mathfrak{p}^\perp}$.

In fact, there is an alternative construction of $\mathrm{S}, \mathrm{X}, \mathrm{S}(e, P)$ in terms of a Hamiltonian reduction.

Consider the grading $\mathfrak{g} := \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ given by the eigenvalues of $\mathrm{ad}(h)$. Recall an element $\chi \in \mathfrak{g}^*$. The restriction of the skew-symmetric form $(\xi, \eta) \mapsto \langle \chi, [\xi, \eta] \rangle$ to $\mathfrak{g}(-1)$ is non-degenerate. Following [24,37,41,14], pick a Lagrangian subspace $l \subset \mathfrak{g}(-1)$ and set $\mathfrak{m} := l \oplus \bigoplus_{i \leqslant -2} \mathfrak{g}(i)$.

As Gan and Ginzburg checked in [14], S is naturally identified with the reduction $\mathfrak{g}^*/\!\!/\!\!/_\chi M$, where M is the unipotent subgroup of G corresponding to $\mathfrak{m}$. More precisely, let $\rho : \mathfrak{g}^* \twoheadrightarrow \mathfrak{m}^*$ be the natural projection. Then the natural map $M \times S \to \mathfrak{g}^*, (m, s) \mapsto ms$, is an isomorphism of $M \times S$ and $\rho^{-1}(\chi|_\mathfrak{m})$. From here it also follows that X is naturally (in particular, G- and $\mathbb{C}^\times$-equivariantly and symplectomorphically) identified with $T^*G/\!\!/\!\!/_\chi M$. Hence $\mathrm{S}(e, P) = (T^*G/\!\!/\!\!/_0 P)/\!\!/\!\!/_\chi M = T^*(G/P)/\!\!/\!\!/_\chi M$.

4.4. *Resolutions of quotient singularities*

Set $L := \mathbb{C}^2$ and consider a non-zero $\mathrm{SL}_2(\mathbb{C})$-invariant form ω_0 on L. Equip L^n with the form $\omega = \omega_0^{\oplus n}$ and set $\Gamma_n := \Gamma^n \rtimes S_n$. The group Γ_n acts naturally on L^n (the symmetric group just permutes the copies of L) and this action preserves ω.

Set $X_0 := L^n/\Gamma_n$. There is a $\mathbb{C}^\times$-action on X_0 induced by the action on $\mathbb{C}^{2n}$ given by $(t, v) \mapsto t^{-1}v$.

Consider the set $N_0, \ldots, N_r$ of irreducible Γ-modules, where N_0 is the trivial module. Consider the *McKay quiver*, whose vertices are $0, \ldots, r$ and the number of arrows from i to j equals the dimension of $\mathrm{Hom}_\Gamma(\mathbb{C}^2 \otimes N_i, N_j)$. This quiver is known to be the double of an affine Dynkin quiver Q. Moreover, 0 can be taken for the extending vertex of Q.

Let δ denote the indecomposable positive imaginary root. Set $\mathbf{v} := n\delta, \mathbf{d} = \epsilon_0$. Take a generic character θ of $\mathrm{GL}(\mathbf{v})$, see the discussion before Proposition 4.2.2. Then there is a $\mathbb{C}^\times$-equivariant isomorphism $X_0 \cong \mathcal{M}_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ of schemes (one uses [15] to show that the right scheme is reduced and then argues as in the proof of Theorem 11.16 from [13]).

Let us show that this isomorphism can be made Poisson maybe after rescaling. This stems from the following proposition proved in [13, Lemma 2.23].

Proposition 4.4.1. *There is a unique (up to rescaling) Poisson bracket of degree* -2 *on* $\mathbb{C}[X_0]$. *Furthermore, there are no (not necessarily Poisson) brackets of degree i with* $i < -2$ *on* $\mathbb{C}[X_0]$.

We fix a $\mathbb{C}^\times$-equivariant Poisson isomorphism $X_0 \cong \mathcal{M}_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$.

According to [15, Theorem 1.4.1], the variety $\Lambda_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ (and hence any $\Lambda_\chi(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$, $\Lambda_U(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$) is a non-empty reduced complete intersection and the action of $\mathrm{GL}(\mathbf{v})$ on each component is generically free. By Proposition 4.2.2, we have a $\mathbb{C}^\times$-equivariant symplectic resolution $X := M_0^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \to X_0$ because the dimensions coincide.

Consider the graded symplectic deformation $\widehat{X} := \mathcal{M}^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ of X.

In the sequel we will need a certain vector bundle on X to be referred to as a *weakly Procesi* bundle whose existence was proved by Bezrukavnikov and Kaledin in [3]. Namely, there is a $\mathbb{C}^\times$-equivariant vector bundle $\mathcal{P}$ on X with the following properties:

(P1) There is a graded $\mathbb{C}[X] = \mathbb{C}[L^n]^{\Gamma_n}$-algebra isomorphism $\mathrm{End}_{\mathcal{O}_X}(\mathcal{P}) \cong \mathbb{C}[L^n]\#\Gamma_n$.
(P2) $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{P}, \mathcal{P}) = 0$ for $i > 0$.

In particular, (P1) implies that Γ_n acts on $\mathcal{P}$ fiberwise and each fiber is isomorphic to $\mathbb{C}\Gamma_n$ as a Γ_n-module.

Thanks to (P2) we can uniquely extend $\mathcal{P}$ to a $\mathbb{C}^\times$-equivariant vector bundle $\widehat{\mathcal{P}}$ on $\widehat{X}$. This vector bundle automatically satisfies the following three conditions.

($\widehat{\mathrm{P}}$0) $\mathrm{End}_{\mathcal{O}_{\widehat{X}}}(\widehat{\mathcal{P}})$ is flat over $\mathbb{C}[\mathfrak{z}]$.
($\widehat{\mathrm{P}}$1) $\mathrm{End}_{\mathcal{O}_{\widehat{X}}}(\widehat{\mathcal{P}})/(\mathfrak{z}) = \mathbb{C}[L^n]\#\Gamma_n$.
($\widehat{\mathrm{P}}$2) $\mathrm{Ext}^i_{\mathcal{O}_{\widehat{X}}}(\widehat{\mathcal{P}}, \widehat{\mathcal{P}}) = 0$.

The group Γ_n still acts on $\widehat{\mathcal{P}}$ fiberwise and each fiber is isomorphic to $\mathbb{C}\Gamma_n$ as a Γ_n-module.

4.5. Kleinian case

An important special case of the quotient singularity considered in the previous section is that of the Kleinian singularities, i.e., $n = 1$ and $\Gamma_n = \Gamma$. The reader is referred to [39] for generalities on the Kleinian singularities and their resolutions.

Set $X_0 = \mathbb{C}^2/\Gamma$ and let $\pi : X \to X_0$ be the minimal resolution of X_0. Then X is a symplectic variety with symplectic form, say, Ω.

Let $D_1, \ldots, D_r$ be the irreducible components of the exceptional fiber $\pi^{-1}(0)$. It is well-known (see, for example, [17]) that $D_1, \ldots, D_r$ can be identified with simple roots $\alpha_1, \ldots, \alpha_r$ of a simple root system of type A, D, E. Moreover, the intersection pairing between D_i, D_j equals $-a_{ij}$, where $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ is the corresponding entry of the Cartan matrix.

As we have seen in the previous subsection, one can construct X_0 and X as quiver varieties. The quiver Q is the affine quiver of the Dynkin diagram of $\alpha_1, \ldots, \alpha_r$.

Alternatively, X can be realized as a Slodowy variety. Namely, let $\mathfrak{g}$ be the simple Lie algebra with system $\alpha_1, \ldots, \alpha_r$ of simple roots, G the corresponding simply connected group. Let $e \in \mathfrak{g}$ be a subprincipal nilpotent element and construct the Slodowy slice S and the equivariant Slodowy slice X from e. Let B denote the Borel subgroup of G corresponding to the choice of $\alpha_1, \ldots, \alpha_r$.

According to Brieskorn, the intersection $S \cap \mathcal{N}$ is isomorphic to $\mathbb{C}^2/\Gamma$. The isomorphism can be made $\mathbb{C}^\times$-equivariant. This follows, for instance, from the construction explained in [42]. By Proposition 4.4.1, we may assume, in addition, that an isomorphism is Poisson.

Consider the resolution $\pi : \mathrm{S}(e, B) \to \mathrm{S} \cap \mathcal{N} = X_0$. Its exceptional fiber again consists of r divisors, say D'_i, $i = 1, \ldots, r$, that are in one-to-one correspondence with the set of simple roots of $\mathfrak{g}$. More precisely, consider the line bundle $\mathcal{L}_i$ on G/B corresponding to the fundamental weight ω_i (given by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$). Lift the bundles $\mathcal{L}_i$ to $T^*(G/B)$ and then restrict them $\mathrm{S}(e, B) \subset T^*(G/B)$. It is known that the restriction is the line bundle corresponding to a unique component D'_i of the exceptional divisor. Moreover, the intersection pairing between D'_i, D'_j is again $-a_{ij}$.

Now both $\mathrm{S}(e, B)$, $\mathcal{M}_0^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ are symplectic (= minimal) resolutions of $X_0 = \mathrm{S} \cap \mathcal{N} = \mathcal{M}_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$. There is only one minimal resolution of X_0. So there is an isomorphism $\varphi : \mathrm{S}(e, B) \to \mathcal{M}_0^\theta(\mathrm{D}Q, \mathbf{v})$ of schemes over X_0. This isomorphism is automatically a $\mathbb{C}^\times$-equivariant symplectomorphism. We may assume, in addition, that $\varphi(D'_i) = D_i$. If not, we can enumerate D_i's differently.

We will need to understand the behavior of some natural line bundles on the varieties $\mathrm{S}(e, B)$, $\mathcal{M}_0^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ under the isomorphism φ. On the Slodowy side we have line bundles $\mathcal{L}_i$ corresponding to the fundamental weights. On the quiver variety side we also have $r + 1$ line bundles constructed as follows. Let $\mathcal{L}'_i$, $i = 0, \ldots, r$, be the line bundle on $\mathcal{M}^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ induced by the 1-dimensional $\mathrm{GL}(\mathbf{v})$-module $\bigwedge^{\delta_i} N_i$, compare with the last paragraph of Section 3.1.

Proposition 4.5.1. *Let $C = (a_{ij})_{i,j=1}^r$ be the Cartan matrix of Q. Then $\mathcal{L}'_i = \prod_{j=1}^r \varphi_*(\mathcal{L}_i)^{a_{ij}}$ for $i = 1, \ldots, r$.*

The proposition simply means that if we identify the free group spanned by $\mathcal{L}_i$ (in fact, this group coincides with $\mathrm{Pic}(X)$) with the weight lattice of Q by sending $\mathcal{L}_i$ to ω_i, then the bundles $\mathcal{L}'_i$ get identified with the simple roots.

Proof. Consider the bundle $\mathcal{N}_i$ on $X \cong R(\mathrm{D}Q, \mathbf{v}, \mathbf{d})^{\chi,ss} /\!\!/\!\!/_0 \mathrm{GL}(\mathbf{v})$ associated to N_i. Gonzales-Springberg and Verdier in [17] computed the first Chern classes $c_1(\mathcal{N}_i)$ of $\mathcal{N}_i$ (of course, $c_1(\mathcal{N}_i) = c_1(\mathcal{L}'_i)$). The required result is the direct corollary of their computation (and the well-known fact that $H^2_{\mathrm{DR}}(X) = \mathbb{C} \otimes_{\mathbb{Z}} \mathrm{Pic}(X)$). □

Finally, we will need an explicit construction of the bundle $\widehat{\mathcal{P}}$ on $\widehat{X}$. Namely, consider the $\mathrm{GL}(\mathbf{v})$-module $P := \bigoplus_{i=0}^l N_i^{\oplus \delta_i}$. Let $\widehat{\mathcal{P}}$ denote the corresponding bundle on $\widetilde{X}$.

Consider the restriction $\mathcal{P}$ of $\widehat{\mathcal{P}}$ to X. It is known (see, for example, Section 1.5 in [23]) that $\mathcal{P}$ satisfies (P1), (P2). Being a $\mathbb{C}^\times$-equivariant extension of $\mathcal{P}$, the bundle $\widehat{\mathcal{P}}$ satisfies $(\widehat{\mathrm{P}}0)$, $(\widehat{\mathrm{P}}1)$, $(\widehat{\mathrm{P}}2)$.

4.6. Quiver varieties in type A vs. Slodowy varieties

In [34], Maffei established isomorphisms between quiver varieties and Slodowy varieties in type A. In this subsection, we are going to recall his construction and deduce some of its easy corollaries. We remark that results closely related to Maffei's were also obtained in [36].

First, let us fix some notation. Let N be a positive integer and $\mathfrak{g} = \mathfrak{sl}_N$. Fix $n > 0$ and $r_1, \ldots, r_n \in \mathbb{Z}_{\geqslant 0}$ with $\sum_{i=1}^n r_i = N$. The numbers $r_1, r_2, \ldots, r_n$ define a parabolic subgroup P in $G := \mathrm{SL}_N$, namely, for P we take the stabilizer of a partial flag $\mathcal{F} = (0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^N)$ with $\dim F_j = \sum_{i=1}^j r_i$.

Also pick $\mathbf{d} = (d_1, \dots, d_{n-1})$ with $\sum_{i=1}^{n-1} i d_i = N$ and let $e \in \mathfrak{g}$ be a nilpotent element whose Jordan type is $(1^{d_1}, 2^{d_2}, \dots, (n-1)^{d_{n-1}})$. From these data we can construct the Slodowy slice $\mathrm{S} \subset \mathfrak{g}$, and the Slodowy variety $\mathrm{S}(e, P)$.

Define $\mathbf{v} := (v_1, \dots, v_{n-1})$ by

$$v_i := \sum_{j=i+1}^{n} r_i - \sum_{j=i+1}^{n-1} (j-i) d_i. \tag{4.4}$$

Below we assume that all v_i's are positive. Finally, let Q be the quiver of type A_{n-1} and a character θ of $\mathrm{GL}(\mathbf{v})$ be as in (4.2).

Maffei proved in [34, Theorem 8], that the algebraic varieties $\mathcal{M}_0^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ and $\mathrm{S}(e, P)$ are isomorphic. His construction is pretty technical but we will need to recall it to establish some additional properties of his isomorphism, for example, that it is a symplectomorphism.

Let us proceed to recalling the construction of a required isomorphism. First of all, there is a special case when an isomorphism is easy: when $e = 0$ or, equivalently, $d_1 = N, d_2 = \dots = d_{n-1} = 0$; see [38, Theorem 7.2], or [34, Lemma 15]. In this case, $\mathrm{S}(e, P)$ is nothing else but the cotangent bundle $T^*(G/P)$. A point in $T^*(G/P)$ can be thought as a pair $(x, \mathcal{F})$, where $\mathcal{F}$ is a partial flag as above and $x \in \mathfrak{g}$ is such that $x(F_i) \subset F_{i-1}$. An isomorphism $\widetilde{\varphi} : \mathcal{M}_0^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \to T^*(G/P)$ is given by

$$\begin{aligned}&\mathrm{GL}(\mathbf{v}).[(A_i), (B_i), \Gamma_1, \Delta_1]\\ &\quad \mapsto (\Delta_1 \Gamma_1, 0 \subset \ker \Gamma_1 \subset \ker A_1 \Gamma_1 \subset \dots \subset \ker A_{n-2} \dots A_1 \Gamma_1 \subset \mathbb{C}^N).\end{aligned}$$

Now proceed to the general case. Following [34], we will first introduce some "transversal subvariety" in $T^*(G/P)$.

For this we need some notation. Set

$$\begin{aligned}&\widetilde{d}_1 := N, \qquad \widetilde{d}_i := 0, \quad i > 0,\\ &\widetilde{v}_i := v_i + \sum_{j=i+1}^{n-1} d_j, \quad i = 1, \dots, n-1,\\ &\widetilde{\mathbf{d}} := (\widetilde{d}_1, \dots, \widetilde{d}_{n-1}), \qquad \widetilde{\mathbf{v}} := (\widetilde{v}_1, \dots, \widetilde{v}_{n-1}).\end{aligned}$$

Further, set

$$\widetilde{D}_1 := \sum_{1 \leqslant k \leqslant j < n} D_i^{(k)}, \qquad D_i' := \bigoplus_{1 \leqslant k \leqslant j-i \leqslant n-i-1} D_j^{(k)}, \qquad \widetilde{V}_i := V_i \oplus D_i', \tag{4.5}$$

where $D_i^{(k)}$ means a copy of D_i. For the notational convenience we write $\widetilde{V}_0 = D_0' := \widetilde{D}_1$.

Let $(\widetilde{A}_i, \widetilde{B}_i, \widetilde{\Gamma}_1, \widetilde{\Delta}_1)_{i=1,\dots,n-2}$ be an element of $R(\mathrm{D}Q, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})$. Put $\widetilde{A}_0 := \widetilde{\Gamma}_1, \widetilde{B}_0 := \widetilde{\Delta}_1$. As in [34] we will write the elements $\widetilde{A}_i, \widetilde{B}_i$ in the block form as follows:

$$\begin{aligned}&\pi_{D_j^{(h)}} \widetilde{A}_i|_{D_{j'}^{(h')}} = \mathbb{T}_{i,j,h}^{j',h'}, \qquad \pi_{D_j^{(h)}} \widetilde{B}_i|_{D_{j'}^{(h')}} = \mathbb{S}_{i,j,h}^{j',h'},\\ &\pi_{D_j^{(h)}} \widetilde{A}|_{V_i} = \mathbb{T}_{i,j,h}^{V}, \qquad \pi_{D_j^{(h)}} \widetilde{B}|_{V_i} = \mathbb{S}_{i,j,h}^{V},\\ &\pi_{V_{i+1}} \widetilde{A}|_{D_{j'}^{(h')}} = \mathbb{T}_{i,V}^{j',h'}, \qquad \pi_{V_i} \widetilde{B}_i|_{D_{j'}^{(h')}} = \mathbb{S}_{i,V}^{j',h'},\\ &\pi_{V_{i+1}} \widetilde{A}_i|_{V_i} = \mathbb{A}_i, \qquad \pi_{V_i} \widetilde{B}_i|_{V_{i+1}} = \mathbb{B}_i,\end{aligned} \tag{4.6}$$

where $\pi_\bullet$ stand for the projections to the summands in (4.5).

Embed $\mathrm{GL}(\mathbf{v})$ into $\mathrm{GL}(\widetilde{\mathbf{v}})$ by making $\mathrm{GL}(\mathbf{v})$ act trivially on D_i' and as before on V_i. Let $\widetilde{\theta}$ denote the character of $\mathrm{GL}(\widetilde{\mathbf{v}})$ defined analogously to θ. Clearly, θ coincides with the restriction of $\widetilde{\theta}$ to $\mathrm{GL}(\mathbf{v})$.

Further, choose $\mathfrak{sl}_2$-triples $(e_i, [e_i, f_i], f_i)$ in $\mathfrak{gl}(D_i')$, $i = 0, \ldots, n-1$ as follows:

$$\begin{aligned} &e_i|_{D_j^{(1)}} := 0, && e_i|_{D_j^{(h)}} := \mathrm{id}_{D_j} : D_j^{(h)} \to D_j^{(h-1)}, \\ &f_i|_{D_j^{j-i}} = 0, && f_i|_{D_j^{(h)}} := h(j-i-h)\mathrm{id}_{D_j} : D_j^{(h)} \to D_j^{(h+1)}. \end{aligned} \tag{4.7}$$

In particular, the nilpotent element $e_0 \in \mathfrak{g}$ corresponds to the partition $\mathbf{d}$. Under the isomorphism $T^*(G/P) \cong M(\mathrm{D}Q, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})$ described above the variety $\mathrm{S}(e, P)$ is identified with

$$\{(\widetilde{A}_i, \widetilde{B}_i)_{i=0}^{n-2} \in \Lambda(\mathrm{D}Q, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})^{ss,\widetilde{\theta}} : [\widetilde{B}_0\widetilde{A}_0 - e_0, f_0] = 0\}/\mathrm{GL}(\widetilde{\mathbf{v}}).$$

Now we are ready to define *transversal* elements. For this we need to assign degrees (denoted by grad) to the blocks $\mathbb{T}_\bullet^\bullet, \mathbb{S}_\bullet^\bullet$ as follows.

$$\begin{aligned} &\mathrm{grad}(\mathbb{T}_{i,j,h}^{j',h'}) := \min(h-h'+1, h-h'+1+j'-j), \\ &\mathrm{grad}(\mathbb{S}_{i,j,h}^{j',h'}) := \min(h-h', h-h'+j'-j), \end{aligned} \tag{4.8}$$

An element $((\widetilde{A}_i), (\widetilde{B}_i))_{i=0}^{n-2} \in \Lambda_0(\mathrm{D}Q, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})$ is said to be *transversal* if it satisfies the following relations for $i = 0, 1, \ldots, n-2$:

$$\begin{aligned} &\mathbb{T}_{i,j,h}^{j',h'} = 0 && \text{if } \mathrm{grad}(\mathbb{T}_{i,j,h}^{j',\,h'}) < 0, \\ & && \text{or if } \mathrm{grad}(\mathbb{T}_{i,j,h}^{j',h'}) = 0 \text{ and } (j',\ h') \neq (j, h+1), \\ &\mathbb{T}_{i,j,h}^{j',h'} = \mathrm{id}_{D_j} && \text{if } \mathrm{grad}(\mathbb{T}_{i,j,h}^{j',h'}) = 0 \text{ and } (j', h') = (j, h+1), \\ &\mathbb{T}_{i,j,h}^{V} = 0, && \\ &\mathbb{T}_{i,V}^{j',h'} = 0 && \text{if } h' \neq 1, \\ &\mathbb{S}_{i,j,h}^{j',h'} = 0 && \text{if } \mathrm{grad}(\mathbb{S}_{i,j,h}^{j',h'}) < 0, \\ & && \text{or if } \mathrm{grad}(\mathbb{S}_{i,j,h}^{j',h'}) = 0 \text{ and } (j', h') \neq (j, h), \\ &\mathbb{S}_{i,j,h}^{j',h'} = \mathrm{id}_{D_j} && \text{if } \mathrm{grad}(\mathbb{T}_{i,j,h}^{j',h'}) = 0 \text{ and } (j', h') = (j, h), \\ &\mathbb{S}_{i,j,h}^{V} = 0 && \text{if } h \neq j-i, \\ &\mathbb{S}_{i,V}^{j',h'} = 0. && \end{aligned} \tag{4.9}$$

$$[\pi_{D_i'}\widetilde{B}_i\widetilde{A}_i|_{D_i'} - e_i, f_i] = 0. \tag{4.10}$$

The subvariety of $\Lambda(\widetilde{D}, \widetilde{V})$ consisting of the transversal elements will be denoted by $\mathfrak{T}$.

To establish an isomorphism $\mathcal{M}_0^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \xrightarrow{\sim} \mathrm{S}(e, P)$ Maffei constructs a morphism $\Lambda_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \to \mathfrak{T}$.

The main technical step in Maffei's construction is the following lemma that is a union of Lemmas 17–19 from [34].

Lemma 4.6.1. *Let* $x = ((A_i), (B_i), (\Gamma_j), (\Delta_j)) \in \Lambda_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$. *Then the following claims hold:*

(1) *there is a unique element* $\widetilde{x} = ((\widetilde{A}_i), (\widetilde{B}_i)) \in \mathfrak{T}$ *satisfying the following equalities:*

$$\begin{aligned} &\mathbb{A}_i = A_i, \\ &\mathbb{B}_i = B_i, \\ &\mathbb{T}^{i+1,1}_{i,V} = \Gamma_{i+1}, \\ &\mathbb{S}^{V}_{i,i+1,1} = \Delta_{i+1}. \end{aligned} \tag{4.11}$$

for all $i = 0, \dots, n-2$ *(where we set* $A_0 := \Gamma_1$, $B_0 := \Delta_1$*).*

(2) *The map* $\Phi : \Lambda_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \to \mathfrak{T}, x \mapsto \widetilde{x}$*, is a* $\mathrm{GL}(\mathbf{v})$*-equivariant isomorphism.*

(3) $\Phi(\Lambda_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})^{\theta,ss}) = \mathfrak{T} \cap R(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})^{\widetilde{\theta},ss}$.

So we can define the morphism $\varphi : \mathcal{M}^{\theta}_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \to \mathcal{M}^{\widetilde{\theta}}_0(\mathrm{D}Q, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}}) = T^*(G/P)$ sending an orbit $\mathrm{GL}(\mathbf{v})x$ to $\mathrm{GL}(\widetilde{\mathbf{v}})\Phi(x)$.

The following proposition is the main result of [34].

Proposition 4.6.2. *The morphism* φ *is an isomorphism of* $\mathcal{M}^{\theta}_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ *onto* $\mathrm{S}(e, P)$.

Now we will use the Maffei construction to establish some properties of the morphism φ: namely, that this morphism is $\mathbb{C}^\times$-equivariant, is a symplectomorphism and is compatible, in an appropriate sense, with natural line bundles. This is done in the next three lemmas.

Lemma 4.6.3. *The morphism* φ *is* $\mathbb{C}^\times$*-equivariant.*

Proof. Let us define a certain $\mathbb{C}^\times$-action on $R(\mathrm{D}Q, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})$. For this consider the element $\mathbf{h} = ([e_0, f_0], [e_1, f_1], \dots, [e_{n-2}, f_{n-2}])$. The element $[e_i, f_i]$ acts by $j - i + 1 - 2h$ on $D^{(h)}_j \subset D'_i$. Let $\gamma : \mathbb{C}^\times \to \mathrm{GL}(\widetilde{\mathbf{v}})$ denote the one-parameter subgroup corresponding to $\mathbf{h}$. Consider the action of $\mathbb{C}^\times$ on $R(\mathrm{D}Q, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})$ given by $t \cdot \widetilde{x} = t^{-1}\gamma(t)x$. The following claims are checked directly:

(i) The $\mathbb{C}^\times$-action preserves the affine subspace given by (4.9) and also the subvariety of solutions of (4.10). So $\mathfrak{T}$ is $\mathbb{C}^\times$-stable.

(ii) The induced $\mathbb{C}^\times$-action on $T^*(G/P) = \mathcal{M}^{\widetilde{\theta}}_0(\mathrm{D}Q, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})$ is the Kazhdan action.

(iii) The blocks $\mathbb{A}_i, \mathbb{B}_i, \mathbb{T}^{i+1,1}_{i,V}, \mathbb{S}^{V}_{i,i+1,1}$ are multiplied by t^{-1}.

(i), (iii) and assertion (1) of Lemma 4.6.1 imply that the morphism $\Phi : \Lambda_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \to \mathfrak{T}$ is $\mathbb{C}^\times$-invariant. Now (ii) and the construction of φ complete the proof of the present lemma. □

Lemma 4.6.4. *The isomorphism* φ *is a symplectomorphism.*

Proof. Let $\omega, \widetilde{\omega}$ denote the symplectic forms on the spaces $R(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ and $R(\mathrm{D}Q, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})$. Explicitly, for $v^\alpha = ((A^\alpha_i), (B^\alpha_i), (\Gamma^\alpha_j), (\Delta^\alpha_j)) \in R(\mathrm{D}Q, \mathbf{v}, \mathbf{d}), \alpha = 1, 2$ we have $\omega(v^1, v^2) = \beta(v^1, v^2) - \beta(v^2, v^1)$, where $\beta(v^1, v^2) = \sum_{i=1}^{n-2} \mathrm{tr}(B^1_i A^2_i) + \sum_{i=1}^{n-1} \mathrm{tr}(\Delta^1_i \Gamma^2_i)$. Analogously, $\widetilde{\omega}(\widetilde{v}^1, \widetilde{v}^2) = \widetilde{\beta}(\widetilde{v}^1, \widetilde{v}^2) - \widetilde{\beta}(\widetilde{v}^2, \widetilde{v}^1)$, where $\widetilde{\beta}$ is defined similarly to β.

Let us show that

$$\Phi^*(\widetilde{\beta}|_{\mathfrak{T}}) = \beta|_{\Lambda_0(\mathrm{D}Q,\mathbf{v},\mathbf{d})}. \tag{4.12}$$

First of all, pick $x \in \Lambda_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ and $v \in T_x\Lambda_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$. Write the element $d_x\Phi(v)$ in the block form $(\mathbb{T}^{\bullet}_{\bullet}, \mathbb{S}^{\bullet}_{\bullet}, \mathbb{A}_\bullet, \mathbb{B}_\bullet)$ as above. Then $\mathbb{T}^{j',h'}_{i,j,h} = 0$ if $h < h'$ and $\mathbb{S}^{j',h'}_{i,j,h} = 0$ if $h \leqslant h'$.

Moreover, $\mathbb{T}^V_{i,j,h} = 0$, $S^{j',h'}_{i,V} = 0$ for all i, j, h, j', h', and $\mathbb{T}^{j',h'}_{i,V} = 0$ if $h' \neq 1$, $\mathbb{S}^V_{i,j,h} = 0$ if $h \neq j - i$. It follows that for $v^1, v^2 \in T_x \Lambda_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ we have

$$\widetilde{\beta}(d_x \Phi(v^1), d_x \Phi(v^2)) = \sum_{i=1}^{n-2} \mathrm{tr}(\mathbb{B}^1_i \mathbb{A}^2_i) + \sum_{i=0}^{n-2} \mathrm{tr}((\mathbb{S}^V_{i,i+1,1})^1 (\mathbb{T}^{i+1,1}_{i,V})^2).$$

From assertion (1) of Lemma 4.6.1 we deduce that $\widetilde{\beta}(d_x \Phi(v^1), d_x \Phi(v^2)) = \beta(v^1, v^2)$. This is equivalent to (4.12).

It follows that

$$\Phi^*(\widetilde{\omega}|_{\mathfrak{T}}) = \omega|_{\Lambda_0(\mathrm{D}Q,\mathbf{v},\mathbf{d})}. \tag{4.13}$$

In particular, we see that for $x \in \mathfrak{T} \cap R(\mathrm{D}Q, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})^{\widetilde{\theta},ss}$ the kernel of the restriction of $\widetilde{\omega}$ to $T_x\mathfrak{T}$ coincides with $T_x\mathrm{GL}(\mathbf{v})x$. So the pull-back of the symplectic form from $\mathrm{S}(e, P) = (\mathfrak{T} \cap R(\mathrm{D}Q, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})^{\widetilde{\theta},ss})/\mathrm{GL}(\mathbf{v})$ to $\mathfrak{T} \cap R(\mathrm{D}Q, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})^{\widetilde{\theta},ss}$ coincides with the restriction of $\widetilde{\omega}$. Using the definition of the symplectic form on a reduction, we see that φ is a symplectomorphism. □

Below we will need to understand the behavior of some natural line bundles under the isomorphism φ. Let L_i, $i = 1, \ldots, n-1$, be the 1-dimensional $\mathrm{GL}(\mathbf{v})$-module, where $\mathrm{GL}(\mathbf{v})$ acts by $(X_1, \ldots, X_{n-1}) \mapsto \det(X_i)$. Let $\mathcal{L}_i$ denote the corresponding line bundle on $\mathcal{M}^\theta_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$.

Now let us define certain line bundles on $\mathrm{S}(e, P)$. Let $\mathcal{F} = (0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{C}^N)$ be the flag stabilized by P. Consider the P-modules $\widetilde{L}_i := \bigwedge^{\mathrm{top}} F_i$. Let $\widetilde{\mathcal{L}}_i$ denote the corresponding bundles on $T^*(G/P)$, $\mathrm{S}(e, P)$.

Lemma 4.6.5. $\varphi^*(\widetilde{\mathcal{L}}_i) \cong \mathcal{L}_i$.

Proof. From the construction of the isomorphism $\mathcal{M}^{\widetilde{\theta}}_0(\mathrm{D}Q, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}}) \cong T^*(G/P)$ produced above, we see that one can interpret the line bundles $\widetilde{\mathcal{L}}_i$ in a different way. Namely, $\widetilde{\mathcal{L}}_i$ coincides with the line bundle on $R(\mathrm{D}Q, \widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})/\!\!/\!\!/^{\widetilde{\chi}}_0 \mathrm{GL}(\widetilde{\mathbf{v}})$ induced by the 1-dimensional $\mathrm{GL}(\widetilde{\mathbf{v}})$-module $\widetilde{L}_i$ defined analogously to L_i.

Now the isomorphism of the lemma follows from the fact that the restriction of the character $(\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}) \mapsto \det(\widetilde{X}_i)$ of $\mathrm{GL}(\widetilde{\mathbf{v}})$ to $\mathrm{GL}(\mathbf{v})$ coincides with the character $(X_1, \ldots, X_{n-1}) \mapsto \det(X_i)$. □

5. *W*-algebras

5.1. Definitions

Let G be a reductive algebraic group, $\mathfrak{g}$ be the Lie algebra of G. Pick a nilpotent element $e \in \mathfrak{g}$ and choose $f, [e, f]$ forming an $\mathfrak{sl}_2$-triple with e. Recall the Slodowy slice S and the equivariant Slodowy slice $\mathrm{X} = G \times \mathrm{S}$.

A (finite) W-algebra is a quantization of the graded Poisson algebra $\mathbb{C}[S]$. In full generality, it was first defined by Premet in [41]. In this subsection, we will recall the definitions of a W-algebra following [27,14]. For details the reader is referred to the review [30]. We remark, however, that here we will need homogenized versions of W-algebras, i.e., our algebras will be graded algebras over $\mathbb{C}[h]$.

The variety X is affine and hence admissible in the sense of Section 2.3. So we can consider the canonical quantization $\widetilde{\mathcal{W}}_h$ of X. Consider the algebra $\Gamma(X, \widetilde{\mathcal{W}}_h)^G$. This algebra is complete in the h-adic topology, and $\Gamma(X, \widetilde{\mathcal{W}}_h)^G / h\Gamma(X, \widetilde{\mathcal{W}}_h)^G = \mathbb{C}[\mathrm{X}]$. Let $\mathcal{W}_h$ denote the subalgebra of $\mathbb{C}^\times$-finite vectors in $\Gamma(X, \widetilde{\mathcal{W}}_h)^G$. Since the $\mathbb{C}^\times$-action on S is contracting, we see that $\mathcal{W}_h/h\mathcal{W}_h = \mathbb{C}[S]$.

We remark that the quantization $\widetilde{\mathcal{W}}_h$ of X admits a quantum comoment map $\mathfrak{g} \to \Gamma(\mathrm{X}, \widetilde{\mathcal{W}}_h)$; see [31]. This gives rise to a $G \times \mathbb{C}^\times$-equivariant algebra homomorphism $U_h(\mathfrak{g}) \to \Gamma(\mathrm{X}, \widetilde{\mathcal{W}}_h)$. Restricting the latter to the G-invariants we get a monomorphism $U_h(\mathfrak{g})^G \hookrightarrow \mathcal{W}_h$. Since the G-action on X is free, it is easy to see that $U_h(\mathfrak{g})^G$ coincides with the center of $\mathcal{W}_h$. An alternative proof is given in [31, Section 2.2].

In the sequel we will need an extension of $\mathcal{W}_h$. Namely, pick a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let W denote the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ and $\Delta \subset \mathfrak{h}^*$ be the root system. Pick a Borel subgroup $B \subset G$. This choice defines a system Π of simple roots in Δ. Let, as usual, ρ stand for half the sum of all positive roots. Then one defines the $\cdot$-action of W on $\mathfrak{h}^*$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$. Consider the induced action of W on $S\mathfrak{h} = \mathbb{C}[\mathfrak{h}^*]$. Recall the Harish-Chandra isomorphism $U(\mathfrak{g})^G \cong S\mathfrak{h}^W$. We will use its homogenized version: we identify $U_h(\mathfrak{g})^G$ with $\mathbb{C}[\mathfrak{h}^*, h]^W$, where the action of W on the latter algebra is given by $w \cdot f(\lambda) = f(w^{-1}(\lambda + \rho h) - \rho h)$.

Below we will need to consider the algebra $\mathcal{W}_{h,\mathfrak{h}} := \mathcal{W}_h \otimes_{U_h(\mathfrak{g})^G} \mathbb{C}[\mathfrak{h}^*, h]$. For $\lambda \in \mathfrak{h}^*$ let $\mathcal{W}_\lambda$ denote the quotient of $\mathcal{W}_\mathfrak{h}$ by the ideal in $\mathbb{C}[\mathfrak{h}^*]$ corresponding to λ and $h = 1$. It is easy to see that the natural homomorphism $\mathcal{W}_h \to \mathcal{W}_\lambda$ is an epimorphism.

Now let us explain the approach to W-algebras of Premet, [41], in the version of Gan and Ginzburg, [14]. Recall the grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$, the subalgebra $\mathfrak{m} \subset \mathfrak{g}$, the element $\chi \in \mathfrak{g}^*$ from Section 4.3.

Consider the quantum Hamiltonian reduction $U_h(\mathfrak{g})/\!\!/\!\!/_\chi M$ equipped with the so called *Kazhdan grading*. The latter is defined as follows: for $\xi \in \mathfrak{g}(i)$ the Kazhdan degree of ξ is, by definition, $i + 2$ (and the degree of h is 2, as usual). The algebra $U_h(\mathfrak{g})/\!\!/\!\!/_\chi M$ inherits the grading from $U_h(\mathfrak{g})$.

In [27], the author checked that the filtered algebras $\mathcal{W}_h/(h-1)\mathcal{W}_h$ and $U(\mathfrak{g})/\!\!/\!\!/_\chi M$ are isomorphic. This does not automatically imply that the graded algebras $\mathcal{W}_h$ and $U_h(\mathfrak{g})/\!\!/\!\!/_\chi M$ are isomorphic. However, the existence of the latter isomorphism can be easily deduced from [27, Remark 3.1.4].

5.2. Parabolic W-algebras and quantizations of Slodowy varieties

Let us define the parabolic analogs of $\mathcal{W}_{h,\mathfrak{h}}$. Namely, let P be a parabolic subgroup of G and let $\mathfrak{a}$ stand for the quotient of $\mathfrak{p}$ by its solvable radical. Consider a $\mathbb{C}[\mathfrak{a}^*, h]$-algebra $\mathcal{A}_h$ that is, by definition, the algebra of $\mathbb{C}^\times$-finite elements in $\Gamma(G/P, \mathcal{D}_h(G/P_0)^{P/P_0})$, where the action of P/P_0 on $\mathcal{D}_h(G/P_0)$ is induced from the action on G/P_0 by right translations. This algebra comes equipped with a homomorphism $U_h(\mathfrak{g}) \to \mathcal{A}_h$ induced by the quantum comoment map. So we can consider the quantum Hamiltonian reduction $\mathcal{W}^P_{h,\mathfrak{a}} := (\mathcal{A}_h/\mathcal{A}_h\mathfrak{m}_\chi)^M$. This is also a graded algebra over $\mathbb{C}[\mathfrak{a}^*, h]$.

We will modify the $\mathbb{C}[\mathfrak{a}^*, h]$-algebra structure on $\mathcal{W}^P_{\mathfrak{a},h}$ as follows. Assume that $B \subset P$ and let L stand for the Levi subalgebra of P containing the maximal torus corresponding to $\mathfrak{h}$. We may and will identify $\mathfrak{a}$ with the center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l} = \mathrm{Lie}(L)$. Consider the map $\iota : \mathfrak{a} \to \mathcal{A}_h$ defined by $\iota(\xi) = \Phi(\xi) - h\langle\rho, \xi\rangle$, where Φ is the initial map $\mathfrak{a} \to \mathcal{A}_h$. The new $\mathbb{C}[\mathfrak{a}^*, h]$-algebra structure on $\mathcal{W}^P_{h,\mathfrak{a}}$ we need is induced by ι. The reason why we need this shift will become clear later.

We remark that the identification of $\mathfrak{a}$ with the center of $\mathfrak{l}$ gives rise to the direct sum decomposition $\mathfrak{h} = \mathfrak{a} \oplus (\mathfrak{h} \cap [\mathfrak{l}, \mathfrak{l}])$ and hence to the projection $\mathfrak{h} \twoheadrightarrow \mathfrak{a}$. So we can set $\mathcal{W}_{\mathfrak{a},h} := \mathbb{C}[\mathfrak{a}^*, h] \otimes_{\mathbb{C}[\mathfrak{h}^*,h]} \mathcal{W}_{h,\mathfrak{h}}$.

Lemma 5.2.1. (1) $\mathcal{W}^P_{h,\mathfrak{a}}/h\mathcal{W}^P_{h,\mathfrak{a}} = \mathbb{C}[\widetilde{\mathrm{S}}(e,P)]$.
(2) *$\mathcal{W}^P_{h,\mathfrak{a}}$ coincides with the subalgebra of $\mathbb{C}^\times$-finite elements in $\Gamma(\widetilde{\mathrm{S}}(e,P), \widetilde{\mathcal{W}}_h\widehat{/\!\!/\!\!/}P)$.*
(3) *There is a natural graded $\mathbb{C}[h]$-algebra homomorphism $\mathcal{W}_{\mathfrak{a},h} \to \mathcal{W}^P_{h,\mathfrak{a}}$.*
(4) *This homomorphism is bijective when $P = B$.*
(5) *This homomorphism is surjective provided $\mathfrak{g}$ is of type A.*

Proof. The M-action on $\mu^{-1}(\chi|_\mathfrak{m})$ is free, [16, Corollary 1.3.8] (here μ is the moment map $\mathrm{Spec}(\mathcal{A}_h/(h)) \to \mathfrak{m}^*$). So the algebra $\mathcal{W}^P_{h,\mathfrak{a}}$ satisfies quantization commutes with reduction condition by Lemma 3.3.1. But the algebra of global functions on $\mathrm{X}/\!\!/\!\!/P = (T^*G/\!\!/\!\!/P)/\!\!/\!\!/_\chi M$, by definition, is just $(\mathcal{A}_h/(h))/\!\!/\!\!/_\chi M$. This implies (1). The proof of (2) is now analogous to that of Lemma 4.2.4.

Let us prove (3). We need to establish a homomorphism $\mathbb{C}[\mathfrak{a}^*, h]\otimes_{U_h(\mathfrak{g})^G} U_h(\mathfrak{g}) \twoheadrightarrow \mathcal{A}_h$, then we will apply the reduction by M.

We have the quantum comoment map homomorphism $U_h(\mathfrak{g}) \to \mathcal{A}_h$ together with a homomorphism $\mathbb{C}[\mathfrak{a}^*, h] \to \mathcal{A}_h$ specified above. Let us show that these two homomorphisms agree on $U_h(\mathfrak{g})^G$ (the latter maps to $\mathbb{C}[\mathfrak{a}^*, h]$ via the composition $U_h(\mathfrak{g})^G \cong \mathbb{C}[\mathfrak{h}^*, h]^W \hookrightarrow \mathbb{C}[\mathfrak{h}^*, h] \twoheadrightarrow \mathbb{C}[\mathfrak{a}^*, h]$). This is a pretty standard fact but we will provide its proof for reader's convenience. First of all, since both homomorphisms $U_h(\mathfrak{g})^G \to \mathcal{A}_h$ are graded it is enough to prove that they coincide modulo $h-1$. But the algebra $\mathcal{A} := \mathrm{D}(G/P_0)^{P/P_0}$ acts on $\mathbb{C}[G/P_0]$ and the subalgebra $S\mathfrak{a} \subset \mathcal{A}$ acts faithfully. Now the claim that the natural action of $U(\mathfrak{g})^G$ on $\mathbb{C}[G/P_0]$ factors through the homomorphism $U(\mathfrak{g})^G \to S\mathfrak{a}$ is just part of the construction of the Harish-Chandra isomorphism $U(\mathfrak{g})^G \cong S\mathfrak{h}^W$. This completes the proof of (3).

So we have constructed a homomorphism $\mathbb{C}[\mathfrak{a}^*, h]\otimes_{U_h(\mathfrak{g})^G} U_h(\mathfrak{g}) \to \mathcal{A}_h$. For $P = B$ the natural homomorphism $\mathbb{C}[\mathfrak{h}^*, h]\otimes_{U_h(\mathfrak{g})^G} U_h(\mathfrak{g}) \to \mathcal{A}_h$ is a bijection. Indeed, since the right hand side is $\mathbb{C}[h]$-flat, it is enough to show that this homomorphism is an isomorphism modulo h. This follows from the fact that $\mathcal{A}_h/(h) = \mathbb{C}[T^*G/\!\!/\!\!/B] = \mathbb{C}[\mathfrak{h}^*\times_{\mathfrak{g}^*/\!\!/G}\mathfrak{g}^*] = \mathbb{C}[\mathfrak{h}^*, h]\otimes_{U_h(\mathfrak{g})^G} U_h(\mathfrak{g})/(h)$.

Proceed to assertion (5). Again, it is again to prove that the homomorphism is surjective modulo h. The homomorphism becomes $\mathbb{C}[\mathfrak{a}^*\times_{\mathfrak{g}^*/\!\!/G}\mathrm{S}] \to \mathbb{C}[\mathrm{X}/\!\!/\!\!/P]$. Again, the homomorphism is the identity on $\mathbb{C}[\mathfrak{a}^*]$ and both algebras are flat (= graded free) over $\mathbb{C}[\mathfrak{a}^*]$. So it is enough to prove that the homomorphism is surjective modulo $(\mathfrak{a})$. But modulo $(\mathfrak{a})$ the left algebra is just $\mathbb{C}[\mathrm{S}\cap\mathcal{N}]$, where $\mathcal{N}$ denotes the nilpotent cone in $\mathfrak{g}^*$. The right algebra is $\mathbb{C}[\mathrm{S}(e,P)]$. The image of $\mathrm{S}(e,P)$ in $S\cap\mathcal{N}$ coincides with $S\cap G\mathfrak{p}^\perp$.

Let us show that the latter is a normal Poisson variety. First of all, $G\mathfrak{p}^\perp$ is the closure of a nilpotent orbit and hence is normal, thanks to the results of Kraft and Procesi, [25]. The intersection $\mathrm{S}\cap G\mathfrak{p}^\perp$ is transversal at χ so $\mathrm{S}\cap G\mathfrak{p}^\perp$ is normal at χ. To prove the normality at the other points we notice that the Kazhdan action contracts $\mathrm{S}\cap G\mathfrak{p}^\perp$ to χ. Also the morphism $\mathrm{S}(e,P)\to \mathrm{S}\cap G\mathfrak{p}^\perp$ is birational because the natural morphism $T^*(G/P)\to G\mathfrak{p}^\perp$ is birational and G-equivariant. So we see that the morphism $\mathrm{S}(e,P)\to\mathrm{S}\cap\mathcal{N}$ gives rise to an isomorphism $\mathbb{C}[\mathrm{S}(e,P)] \cong \mathbb{C}[\mathrm{S}\cap G\mathfrak{p}^\perp]$. So the morphism $\mathbb{C}[\mathrm{S}\cap\mathcal{N}] \to \mathbb{C}[\mathrm{S}(e,P)]$ is surjective, as required. □

Remark 5.2.2. In general, the homomorphism $\mathcal{W}_{h,\mathfrak{a}} \to \mathcal{W}^P_{\mathfrak{a},h}$ is not surjective but is surjective modulo $(h-1)$. Let us sketch a proof. First, we need to show that the homomorphism $S\mathfrak{a}\otimes_{U(\mathfrak{g})^G} U(\mathfrak{g}) \to \mathrm{D}(G)/\!\!/\!\!/P$ is surjective. This homomorphism is the identity on $S\mathfrak{a}$ so it is enough to show that the induced homomorphism of the fibers at $\lambda\in\mathfrak{a}^*$ is surjective for any $\lambda\in\mathfrak{a}^*$. But this follows from results of Borho and Brylinski, [4, Theorem 3.8 and Remark 3.9].

5.3. Main theorems

First of all let us state a result on the isomorphism of deformations of Kleinian singularities.

Let $\Gamma, Q, \mathbf{v}, \mathbf{d}, \mathfrak{g}, \mathbb{O}, \delta$ be as in Section 4.5. Pick a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, a system $\alpha_1, \ldots, \alpha_r$ of simple roots in $\mathfrak{h}$. Recall that the space $\mathfrak{z} = \mathfrak{gl}(\mathbf{v})^{*\mathrm{GL}(\mathbf{v})}$ is identified with $\mathbb{C}^{Q_0}$ by $\sum_{i\in Q_0} \chi_i\epsilon_i \mapsto [\xi \mapsto \sum_{i\in Q_0}\chi_i \mathrm{tr}(\xi_i)]$. Here $\epsilon_i, i \in Q_0$, is the tautological basis in $\mathbb{C}^{Q_0}$. Consider the subspace $\mathfrak{z}_0 \subset \mathfrak{z}$ of all vectors orthogonal to δ. Identify $\mathfrak{h}^*$ with $\mathfrak{z}_0$ by $\lambda \mapsto \sum_{i=0}^r \lambda_i\epsilon_i$, where $\lambda_1, \ldots, \lambda_r$ are defined from $\lambda = \sum_{i=1}^r \lambda_i\alpha_i$, and $\lambda_0 = -\sum_{i=1}^r \delta_i\epsilon_i$.

Recall that we have the $\mathbb{C}[\mathfrak{h}^*][h]$-algebra $\mathcal{W}_{h,\mathfrak{h}}$ (with the modified structure map $\mathbb{C}[\mathfrak{h}^*][h] \to \mathcal{W}_{\mathfrak{h},h}$; see Section 5.2) and also a $\mathbb{C}[\mathfrak{z}_0][h]$-algebra $\mathbb{A}_{\mathfrak{z}_0}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$.

Theorem 5.3.1. *There is a $\mathbb{C}[\mathfrak{h}^*][h]$-linear isomorphism $\mathbb{A}_{\mathfrak{z}_0}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h \xrightarrow{\sim} \mathcal{W}_{h,\mathfrak{h}}$ of graded associative algebras.*

Remark 5.3.2. Usually one considers the $\mathbb{C}[\mathfrak{z}_0][h]$-algebra $\mathbb{A}_{\mathfrak{z}_0}(\mathrm{D}Q, \mathbf{v})_h$ instead of $\mathbb{A}_{\mathfrak{z}_0}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$. However, it is pretty straightforward to see that these two algebras are naturally isomorphic.

Now let $G = \mathrm{SL}_N, n, r_1, \ldots, r_n, P, \mathbf{v}, \mathbf{d}, Q, \mathbb{O}$ have the same meaning as in Section 4.6. Let $\mathfrak{a}$ be constructed from P as in Section 5.2. Let us relate the algebras $\mathcal{W}^P_{\mathfrak{a},h}$ and $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$. First, we need to identify $\mathfrak{a}$ with $\mathfrak{z}^*$. We can view $\mathfrak{a}$ as the space $\{\mathrm{diag}(x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n)\}$ of matrices, where x_i appears r_i times with $\sum_{i=1}^n r_i x_i = 0$. Map $\mathrm{diag}(x_1, \ldots, x_n) \in \mathfrak{a}$ to $\sum_{i=1}^{n-1}(\sum_{j=1}^i r_j x_j)\epsilon_i$.

We assume that $d_i \geqslant 2v_i - v_{i+1} - v_{i-1}$ for all i (we set $v_0 = v_n = 0$). This is equivalent to $r_i \geqslant r_{i+1}$ for all i. Also recall that in Section 4.2 we assigned elements d to $\mathbf{d}$ and v to $\mathbf{v}$. Our condition is equivalent to saying that d − v is dominant.

The reason for this assumption is that it guarantees that quantization commutes with reduction holds for $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$. If we remove this assumption then one can still show that we have an epimorphism $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h \twoheadrightarrow \mathcal{W}^P_{h,\mathfrak{a}}$ that is an isomorphism if and only if the scheme $\mathcal{M}_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ is reduced.

Theorem 5.3.3. *There is a $\mathbb{C}[\mathfrak{a}^*][h]$-linear isomorphism $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h \xrightarrow{\sim} \mathcal{W}^P_{h,\mathfrak{a}}$ of graded associative algebras.*

Remark 5.3.4. The previous theorem provides the third realization of the W-algebra in type A. The first one was that of Premet, while the second one was the Yangian type presentation of $\mathcal{W}$ by generators and relations due to Brundan–Kleshchev, [5]. At the moment, an analogous presentation of the general parabolic W-algebras in type A is not known, but should not be difficult to obtain. We also remark that Mirkovic and Vybornov related parabolic Slodowy slice and Slodowy varieties to affine Grassmanians, [36]. One could speculate that their results are a classical analog of the Brundan–Kleshchev presentation.

5.4. Reduction of even quantizations

Let $\mathcal{X}, G$ be as in Section 3.1 and let $\mathcal{D}$ be an even graded G-equivariant quantization of $\mathcal{X}$ with parity antiautomorphism σ. Also we suppose that $\mathbb{C}^\times$ acts on $\mathcal{X}$ as in Section 3.1. Let $\Phi : \mathfrak{g} \to \Gamma(\mathcal{X}, \mathcal{D})$ be the quantum comoment map, we assume that $\Phi(t^2\xi) = t \cdot \Phi(\xi)$. The goal of this subsection is to obtain a criterium for $\mathcal{D}/\!/\!/G$ to be even. Of course, this depends on the choice of Φ: for two quantum comoment maps Φ, Φ' we have $\Phi(\xi) - \Phi'(\xi) = \langle \alpha, \xi \rangle h$ for some $\alpha \in \mathfrak{g}^{*G}$.

Let α be the character of the G-action on $\bigwedge^{\mathrm{top}} \mathfrak{g}$. The main result of this subsection is the following theorem.

Theorem 5.4.1. *Suppose that $\sigma(\Phi(\xi)) - \Phi(\xi) = \langle \alpha, \xi \rangle h$. Then $\mathcal{D}/\!\!/\!\!/G$ is an even quantization.*

For example, if the group G has no characters, e.g. is semisimple or unipotent, then the reduction of an even quantization is always even. Also if G is reductive and the quantum comoment map is *symmetrized*, i.e., $\sigma(\Phi(\xi)) = \Phi(\xi)$, then $\mathcal{D}/\!\!/\!\!/G$ is an even quantization of $X/\!\!/\!\!/G$, and $\mathcal{D}/\!\!/\!\!/_0 G$ is an even quantization of $\mathcal{X}/\!\!/\!\!/_0 G$. Now if we replace a reductive group G with its parabolic subgroup P, then the quantization $\mathcal{X}/\!\!/\!\!/_{\rho h} P$ is even. This motivates the shift we made in Section 5.2 (see also the proofs of Theorems 5.3.1 and 5.3.3 in the next subsection).

The proof of Theorem 5.4.1 is organized as follows. First, we prove two auxiliary lemmas. Then we prove Theorem 5.4.1 in the case when G is reductive, which is the most technical part of the proof. Next, we deal with the case of the 1-dimensional unipotent group. And then we complete the proof of Theorem 5.4.1.

First of all, let us investigate a relationship between Φ and the antiautomorphism σ participating in the definition of an even quantization.

Lemma 5.4.2. $\sigma(\Phi(\xi)) - \Phi(\xi) \in \mathbb{C}h$ *for all* $\xi \in \mathfrak{g}$. *Moreover, if* $\xi \in \mathfrak{g}$ *vanishes on* $\mathfrak{g}^{*G}$, *then* $\sigma(\Phi(\xi)) = \Phi(\xi)$.

Proof. For any local section f or $\mathcal{D}$ we have

$$[\sigma(\Phi(\xi)), \sigma(f)] = -\sigma([\Phi(\xi), f]) = -\sigma(h\xi_{\mathcal{D}} f) = h\xi_{\mathcal{D}}\sigma(f) = [\Phi(\xi), \sigma(f)],$$

where $\xi_{\mathcal{D}}$ stands for the derivation of $\mathcal{D}$ induced by ξ. So $\Phi(\xi) - \sigma(\Phi(\xi))$ lies in the center of $\mathcal{D}$. Since $\mathcal{X}$ is symplectic, the center coincides with $\mathbb{C}[[h]]$. Since both $\Phi(\xi), \sigma(\Phi(\xi))$ have degree 2 with respect to the $\mathbb{C}^{\times}$-action, we see that $\sigma(\Phi(\xi)) - \Phi(\xi) \in \mathbb{C}h$.

The G-action on $\mathbb{C}[[h]]$ is trivial. So $\xi \mapsto \Phi(\xi) - \sigma(\Phi(\xi))$ is both G-invariant and G-equivariant map. Hence for any ξ in the annihilator of $\mathfrak{g}^{*G}$ we have $\sigma(\Phi(\xi)) = \Phi(\xi)$. □

Recall that $\pi : \mathcal{X} \to \mathcal{X}/G$ stands for the quotient morphism. By our assumptions, this morphism is locally trivial in etale topology.

Lemma 5.4.3. (1) *The natural morphism* $\pi_*(\mathcal{D})^G \to \mathcal{D}/\!\!/\!\!/G$ *induces an isomorphism* $\pi_*(\mathcal{D})^G/\pi_*(\mathcal{D}\Phi(\mathfrak{g}_0))^G \xrightarrow{\sim} \mathcal{D}/\!\!/\!\!/G$ *(were* $\mathfrak{g}_0$ *is the annihilator of* $\mathfrak{g}^{*G}$ *in* $\mathfrak{g}$*).*
(2) $\pi_*(\mathcal{D})$ *is a flat (left or right) module over* $\pi_*(\mathcal{D})^G$.

Proof. Let us prove the first assertion. It is enough to prove that the homomorphism $\pi_*(\mathcal{D})^G \to \mathcal{D}/\!\!/\!\!/G$ is surjective. Similarly to the proof of Proposition 3.4.1 in [28], we can show that $\pi_*(\mathcal{D})^G/(h) = \pi_*(\mathcal{O}_X)^G$. So to prove that $\pi_*(\mathcal{D})^G \to \mathcal{D}/\!\!/\!\!/G$ is surjective it is enough to verify that the natural morphism $\pi_*(\mathcal{O}_{\mathcal{X}})^G \to \mathcal{O}_{\mathcal{X}/\!\!/\!\!/G}$ is surjective. The surjectivity property is preserved by an etale base change. So we may assume that $\mathcal{X} \to \mathcal{X}/G$ is a free principal G-bundle, so that $\mathcal{X} = (\mathcal{X}/G) \times G$. It follows that any G-stable ideal in $\mathcal{O}_{\mathcal{X}}$ is the pullback of an ideal in $\mathcal{O}_{\mathcal{X}/G}$. The surjectivity claim follows.

The second assertion follows from the fact that the morphism $\mathcal{X} \to \mathcal{X}/G$ is flat. □

Let us proceed to proving Theorem 5.4.1 in the case when G is reductive.

Proposition 5.4.4. *If G is reductive and $\sigma(\Phi(\xi)) = \Phi(\xi)$, then $\mathcal{D}/\!/\!/G$ is even.*

Proof. The proof is divided into several steps.

Step 1. Set $\mathcal{I} := \mathcal{D}\Phi(\mathfrak{g}_0)$. The antiautomorphism σ descends to $\pi_*(\mathcal{D})^G$. Assume for a moment that the following claim holds

$(*)$ $\pi_*(\mathcal{I})^G$ is σ stable.

Then σ induces an antiautomorphism of $\mathcal{D}/\!/\!/G = \pi_*(\mathcal{D})^G/\pi_*(\mathcal{I})^G$ and this antiautomorphism is a parity antiautomorphism.

We remark that $(*)$ is local. So we may assume that $\mathcal{X}/G$ is affine. It follows that $\mathcal{X}$ is affine. Then $\Gamma(\mathcal{X},\mathcal{I}) = \Gamma(\mathcal{X},\mathcal{D})\mathrm{Span}(\Phi(\xi) : \xi \in \mathfrak{g}_0)$. Abusing the notation, we will write $\mathcal{D},\mathcal{I}$ for $\Gamma(\mathcal{X},\mathcal{D})$, $\Gamma(\mathcal{X},\mathcal{I})$. Set $\mathcal{J} := \mathcal{I}^G$. Then $(*)$ can be rewritten as

$(*)$ $\sigma(\mathcal{J}) = \mathcal{J}$.

In the subsequent steps we will prove $(*)$ in this form.

Step 2. On this step we will reduce the proof to the case when G is connected and semisimple. Set $G' := (G^\circ, G^\circ)$. Assume that $(*)$ holds for G' instead of G. Then we have a parity antiautomorphism σ' of $\mathcal{D}/\!/\!/G'$. Replacing G with G/G' and $\mathcal{D}$ with $\mathcal{D}/\!/\!/G'$ we may assume that G° is a torus.

We remark that any $\Phi(\xi)$ is G°-invariant. Since G° is reductive, $\mathcal{I}^{G^\circ}$ coincides with the left ($=$ right) ideal in $\mathcal{D}^{G^\circ}$ generated by $\Phi(\xi)$. For $f \in \mathcal{D}^{G^\circ}$ and $\xi \in \mathfrak{g}_0$ we have $\sigma(f\Phi(\xi)) = \Phi(\xi)\sigma(f) = \sigma(f)\Phi(\xi)$. So $\mathcal{D}^{G^\circ}\Phi(\mathfrak{g}_0)$ is σ-stable. Now it is easy to see that $\mathcal{I}^G$ is σ-stable.

Step 3. So we assume that G is semisimple and connected. On this step we are going to reduce the proof of $(*)$ for $\mathcal{D}$ to the proof of an analog of $(*)$ for certain completions of $\mathcal{D}$.

Pick a point $y \in \mu^{-1}(0)/G \subset \mathcal{X}/G$. Let $\mathfrak{m}_y$ denote the ideal of $\pi_G^{-1}(y)$ in $\mathcal{O}_\mathcal{X}$. Denote by $\widetilde{\mathfrak{m}}_y$ the inverse image of $\mathfrak{m}_y$ in $\mathcal{D}$. Consider the completion $\mathcal{D}^{\wedge_y} := \varprojlim_{n\to\infty} \mathcal{D}/\widetilde{\mathfrak{m}}_y^n$. This is a complete and separated topological $\mathbb{C}[[h]]$-algebra equipped with a natural G-action. The quotient of $\mathcal{D}^{\wedge_y}$ modulo h is naturally identified with the completion $\mathbb{C}[X]^{\wedge_y} := \mathbb{C}[X^{\wedge_{\pi^{-1}(y)}}]$. Also we can define the completion $(\mathcal{D}^G)^{\wedge_y} := \varprojlim_{n\to\infty} \mathcal{D}^G/\mathfrak{m}_y^n$. We remark that σ induces an antiautomorphism of $\mathcal{D}^{\wedge_y}$ to be denoted by the same letter.

Consider the natural map $\mathcal{D}\widehat{\otimes}_{\mathcal{D}^G}(\mathcal{D}^G)^{\wedge_y} \to \mathcal{D}^{\wedge_y}$. Here in the left hand side $\mathcal{D}$ is equipped with the h-adic topology, while $(\mathcal{D}^G)^{\wedge_y}$ has the topology of a completion. We claim that this map is bijective. Indeed, analogously to Lemma A2 in [33], we see that both $\mathcal{D}^{\wedge_y}$, $(\mathcal{D}^G)^{\wedge_y}$ are flat over $\mathbb{C}[[h]]$ and complete in the h-adic topology. By assertion 2 of Lemma 5.4.3, $\mathcal{D}\widehat{\otimes}_{\mathcal{D}^G}(\mathcal{D}^G)^{\wedge_y}$ is also flat over $\mathbb{C}[[h]]$. So it is enough to check that our map is an isomorphism modulo h, i.e., a natural map $\mathbb{C}[X]\widehat{\otimes}_{\mathbb{C}[X]^G}(\mathbb{C}[X]^G)^{\wedge_y} \to \mathbb{C}[X]^{\wedge_y}$ is an isomorphism. But this is straightforward from the construction.

It follows that $(\mathcal{D}^{\wedge_y})^G = (\mathcal{D}^G)^{\wedge_y}$. Consider the closure $\mathcal{J}^{\wedge_y}$ of $\mathcal{J}$ in $(\mathcal{D}^{\wedge_y})^G$. The isomorphism from the previous paragraph implies that $\mathcal{J}_y^\wedge = [\mathcal{D}^{\wedge_y}\mathrm{Span}_\mathbb{C}(\Phi(\xi))]^G$. Assume for a moment that $\sigma(\mathcal{J}^{\wedge_y}) = \mathcal{J}^{\wedge_y}$ for all y. Let us deduce from this that $\sigma(\mathcal{J}) = \mathcal{J}$.

Consider the functor of $\widetilde{\mathfrak{m}}_y^G$-adic completion on the category of finitely generated left $\mathcal{D}^G$-modules. As in [33, Lemma A2], one can show that this functor is exact. So $\mathcal{J}^{\wedge_y} = \mathcal{J}^{\wedge_y} + \sigma(\mathcal{J}^{\wedge_y})$ coincides with the completion of $\mathcal{J} + \sigma(\mathcal{J})$. Set $\mathcal{N} := (\sigma(\mathcal{J}) + \mathcal{J})/\mathcal{J}$. The completion of this module at y vanishes for all $y \in \mu^{-1}(0)/G$. On the other hand, $\mathcal{N} \subset \mathcal{D}^G/\mathcal{J}$ is supported on $\mu^{-1}(0)/G$. It follows that $\mathcal{N} = \{0\}$.

Step 4. Let us consider a special case of $\mathcal{X}$ and $\mathcal{D}$.

Consider the quantization $\mathcal{D} := \mathcal{D}_h^{1/2}(G)$ of $\mathcal{X} := T^*G$. Equip the algebra $\mathcal{D}$ with a G-action induced from the action on G by left translations. Then $\Phi(\xi) := \xi_G$ defines a quantum comoment map.

The algebra $\mathcal{D}^G$ is generated (as a $\mathbb{C}[[h]]$-algebra) by the left invariant vector fields and is naturally isomorphic to the h-adic completion $U_h(\mathfrak{g})^{\wedge_h}$ of $U_h(\mathfrak{g})$. The quotient $U_h(\mathfrak{g})^{\wedge_h}/\mathcal{J}$ is flat over $\mathbb{C}[h]$ and $(U_h(\mathfrak{g})^{\wedge_h}/\mathcal{J})/h(U_h(\mathfrak{g})^{\wedge_h}/\mathcal{J}) = \mathbb{C}[T^*G /\!\!/\!\!/_0 G] = \mathbb{C}$. So $U_h(\mathfrak{g})^{\wedge_h}/\mathcal{J} = \mathbb{C}[[h]]$. From here it is easy to deduce that $\mathcal{J} = U_h(\mathfrak{g})^{\wedge_h}\mathfrak{g}$. In particular, $\mathcal{J}$ is σ-stable.

We will need a trivial generalization of this result. Namely, the ideal $\mathcal{J}$ remains σ-stable if we replace $\mathcal{D}_h^{1/2}(G)$ with the tensor product $\mathcal{D}_h^{1/2}(G)\widehat{\otimes}_{K[[h]]}\mathbb{A}_{h,2m}^{\wedge_h}$, where $\mathbb{A}_{h,2m}^{\wedge_h}$ is equipped with a trivial G-action.

Step 5. On this step we are dealing with general $\mathcal{X}, \mathcal{D}$. Our goal is to describe the structure of the triple $\mathcal{D}^{\wedge_y}, \Phi : \mathfrak{g} \to \mathcal{D}^{\wedge_y}, \sigma : \mathcal{D}^{\wedge_y} \to \mathcal{D}^{\wedge_y}$. Then we will deduce the equality $\sigma(\mathcal{J}^{\wedge_y}) = \mathcal{J}^{\wedge_y}$ from this description.

Set $m := \frac{1}{2}\dim\mathcal{X} - \dim G$. Consider the quantum algebra $\mathcal{D}_h^{1/2}(G)\otimes_{\mathbb{C}[[h]]}\mathbb{A}_{h,2m}^{\wedge_h}$. Let $\mathcal{D}'$ denote its completion with respect to the ideal of the base $G \subset T^*G \hookrightarrow T^*G \times \mathbb{C}^{2m}$. This is an algebra equipped with

- the product G-action, where the action on the Weyl algebra is supposed to be trivial,
- a quantum comoment map $\Phi' : \mathfrak{g} \to \mathcal{D}'$ induced from the quantum comoment map $\mathfrak{g} \to \mathcal{D}_h^{1/2}(G)$,
- a parity antiautomorphism σ' that preserves the tensor product decomposition, and coincides with the antiautomorphisms on the factors that were introduced above.

Lemma 5.4.5. *There is a G-equivariant isomorphism $\iota : \mathcal{D}_y^{\wedge} \to \mathcal{D}'$ of $\mathbb{C}[[h]]$-algebras intertwining the quantum comoment maps and the parity anti-automorphisms.*

Proof of Lemma 5.4.5. Applying (a slight modification of) Theorem 3.3.4 from [28] (without $\mathbb{C}^\times$-actions) we see that there is a G-equivariant isomorphism $\iota_0 : \mathcal{D}_y^{\wedge} \to \mathcal{D}'$ intertwining the quantum comoment maps (since G is supposed to be semisimple the compatibility with quantum comoment maps follows from the G-equivariance). So we only need to prove the following claim:

(∗∗) Let σ_1, σ_2 be two G-equivariant parity anti-automorphisms of $\mathcal{D}'$. Then there is $f \in \mathcal{D}'^G$ such that $\sigma_1 = \exp(\operatorname{ad} f)\sigma_2\exp(-\operatorname{ad} f)$.

First of all, we remark that there is $f' \in \mathcal{D}'^G$ such that $\sigma_1 = \exp(\operatorname{ad}(f')) \circ \sigma_2$. Indeed, $\sigma_1 \circ \sigma_2^{-1}$ is a G-equivariant $\mathbb{C}[[h]]$-linear automorphism of $\mathcal{D}'$ that is the identity modulo h. So $\sigma_1 \circ \sigma_2^{-1} = \exp(hd)$, where d is a $\mathbb{C}[[h]]$-linear derivation of $\mathcal{D}'$. But the completion of $T^*G \times \mathbb{C}^{2m}$ along any G-orbit has the trivial first De Rham cohomology because G is semisimple. This easily implies that any $\mathbb{C}[[h]]$-linear derivation of $\mathcal{D}'$ has the form $\frac{1}{h}\operatorname{ad}(f')$ for some $f' \in \mathcal{D}'$. Since d is G-equivariant, we see that there is a G-invariant element $f' \in \mathcal{D}'$ with $d = \frac{1}{h}\operatorname{ad}(f')$.

So $\sigma_1 = \exp(\operatorname{ad}(f')) \circ \sigma_2$. The equality $\sigma_1^2 = \mathrm{id}$ implies $\exp(\operatorname{ad}(f'))\exp(-\operatorname{ad}(\sigma_2(f'))) = \mathrm{id}$. So $f' - \sigma_2(f') \in \mathbb{C}[[h]]$. Replacing f' with $f' - P$ for an appropriate series $P \in \mathbb{C}[[h]]$, we may assume that $f' = \sigma_2(f')$.

For $f_1, f_2 \in \mathcal{D}'$ let $f_1 \circ f_2$ denote the Campbell–Hausdorff series of f_1, f_2 (the series converges because $[\mathcal{D}', \mathcal{D}'] \subset h\mathcal{D}'$). We have

$$\exp(\operatorname{ad}(f))\sigma_2\exp(-\operatorname{ad}(f)) = \exp(\operatorname{ad}(f))\exp(\operatorname{ad}(\sigma_2(f)))\sigma_2.$$

So it is enough to show that there exists a G-invariant element f with

$$f \circ \sigma_2(f) = f'. \tag{5.1}$$

From the form of the Campbell–Hausdorff series and the inclusion $[\mathcal{D}', \mathcal{D}'] \subset h\mathcal{D}'$ it is easy to deduce that (5.1) has a unique solution that is automatically G-invariant. $\square$

Now to prove that $\mathcal{J}_y^\wedge$ is σ-stable, we use the results of Step 4. $\square$

Now let us proceed to the case when we reduce by the one-dimensional unipotent group.

Proposition 5.4.6. *Let $G_0 = \mathbb{C}$ be the one-dimensional unipotent group, $T := \mathbb{C}^\times$, and $G = T \ltimes G_0$, where $t \in T$ acts on G_0 by $(t, g_0) \mapsto t^a g_0$ for some $a \in \mathbb{Z}$. Suppose that $\sigma(\Phi(\xi_0)) = \Phi(\xi_0)$ for the unit element $\xi_0 \in \mathfrak{g}_0$ (this is automatically true if $a \neq 0$), and $\sigma(\Phi(\eta)) - \Phi(\eta) = a_1 h$ for the unit element $\eta \in \mathfrak{t} = \mathbb{C}$. Then*

(1) *The quantization $\mathcal{D}/\!\!/\!\!/G_0$ is even.*
(2) *Let $\Phi_0 : \mathfrak{t} \to \Gamma(\mathcal{X}/\!\!/\!\!/G_0, \mathcal{D}/\!\!/\!\!/G_0)$ be the quantum comoment map induced by Φ. Then $\sigma(\Phi_0(\eta)) = \Phi_0(\eta) + (a_1 - a)h$.*

Proof. Let us prove assertion 1. Lemma 5.4.3 implies that $\mathcal{D}/\!\!/\!\!/G_0 = \pi_*(\mathcal{D})^{G_0}/\pi_*(\mathcal{D}\Phi(\xi_0))^{G_0}$. But $\Phi(\xi_0)$ is G_0-invariant, and $\pi_*(\mathcal{D})^{G_0}$ is flat over $\mathbb{C}[\Phi(\xi_0)]$. So we have the equality $\pi_*(\mathcal{D}\Phi(\xi_0))^{G_0} = \pi_*(\mathcal{D})^{G_0}\Phi(\xi_0)$ and the right hand side is σ-stable, compare with Step 3 of the proof of Proposition 5.4.4. This implies assertion 1.

Let us prove assertion 2. Since the group G_0 is unipotent, there is an open affine subset $Y^0 \subset \mathcal{X}/G_0$ such that the restriction of the quotient morphism $\pi : \mathcal{X} \to \mathcal{X}/G_0$ is trivial over Y^0. Replacing $\mathcal{X}$ with $\pi^{-1}(Y^0)$ we may assume that $\mathcal{X}/G_0$ is affine and $\mathcal{X} = \mathcal{X}/G_0 \times G_0$. Again we write $\mathcal{D}$ instead of $\Gamma(\mathcal{X}, \mathcal{D})$. Let x denote the coordinate function on $G_0 = \mathbb{C}$ so that $\xi_0 \cdot x = 1$. We remark that the operator $\xi_0 : \mathbb{C}[X] \to \mathbb{C}[X]$ is surjective. It follows that $\xi_0 : \mathcal{D} \to \mathcal{D}$ is surjective. So we can find a lifting $\tilde{x}$ of x to $\mathcal{D}$ such that $\xi_0 \cdot \tilde{x} = 1$.

Consider the element $f = \Phi(\eta) - a\tilde{x}\Phi(\xi_0) \in \mathcal{D}$. Then $[f, \Phi(\xi_0)] = 0$ and so $f \in \mathcal{D}^G$. Moreover, the image of f in $\mathcal{D}/\!\!/\!\!/G_0$ coincides with $\Phi_0(\eta)$. Let us compute $\sigma(f)$. We have

$$\begin{aligned}\sigma(f) &= \sigma(\Phi(\eta)) - a\sigma(\Phi(\xi_0))\sigma(\tilde{x}) = \Phi(\eta) + a_1 h - a\Phi(\xi_0)\sigma(\tilde{x}) \\ &= f + a_1 h - a[\Phi(\xi_0), \sigma(\tilde{x})] + a(\tilde{x} - \sigma(\tilde{x}))\Phi(\xi_0) \\ &= f + a_1 h - ah\xi_0 \cdot \sigma(\tilde{x}) + a(\tilde{x} - \sigma(\tilde{x}))\Phi(\xi_0).\end{aligned}$$

But $\xi_0 \cdot \sigma(\tilde{x}) = \sigma(\xi_0 \cdot x) = 1$ because σ is G_0-equivariant. So we see that $\sigma(f)$ coincides with $f + (a_1 - a)h$ modulo $\mathcal{D}^G\Phi(\xi_0) + h^2\mathcal{D}^G$. It follows that $\sigma(\Phi_0(\eta))$ is congruent $\Phi_0(\eta) + (a_1 - a)h$ modulo $h^2\mathcal{D}^G$. This implies assertion 2. $\square$

Proof of Theorem 5.4.1. Similarly to Step 2 of the proof of Proposition 5.4.4, we may assume that G is connected. The character α does not change if we replace G with its solvable radical. So Proposition 5.4.4 reduces the proof to the case when G is solvable.

Now let G_0 be a one-dimensional normal unipotent subgroup in G. Let α_0 be the character of the action of G on $\mathfrak{g}_0$ and let $\Phi_0 : \mathfrak{g}/\mathfrak{g}_0 \to \Gamma(X/\!\!/\!\!/G_0, \mathcal{D}/\!\!/\!\!/G_0)$ be the induced moment map. Proposition 5.4.6 implies that $\sigma(\Phi_0(\xi)) - \Phi_0(\xi) = \langle \alpha - \alpha_0, \xi \rangle h$ for any $\xi \in \mathfrak{g}/\mathfrak{g}_0$. But $\alpha - \alpha_0$ is nothing else but the character of G/G_0 on $\bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{g}_0)$. So by induction we reduce the proof to the case when G is a torus. Here our claim follows from Proposition 5.4.4. $\square$

5.5. Proofs of the main theorems

Proof of Theorem 5.3.1. Consider the symplectic formal scheme $\widetilde{X} := R(\mathrm{D}Q, \mathbf{v}, \mathbf{d})^{\theta,ss}\widehat{/\!\!/\!\!/}_{\mathfrak{z}_0}\mathrm{GL}(\mathbf{v})$ over the formal neighborhood $\mathfrak{z}_0^{\wedge 0}$ of 0 in $\mathfrak{z}_0$, where θ is as in (4.2). We claim that the symplectic schemes $\widetilde{\mathrm{S}}(e, B)$ and $\widetilde{X}$ are $\mathbb{C}^\times$-equivariantly symplectomorphic in such a way that the symplectomorphism lifts the isomorphism $\mathfrak{h}^* \to \mathfrak{z}_0$ constructed before Theorem 5.3.1.

We have the line bundles $\mathcal{L}'_i, i = 0, \ldots, r$ on $X = \mathcal{M}_0^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$. We claim that $\bigotimes \mathcal{L}_i'^{\otimes\delta_i} \cong \mathcal{O}_X$. This follows, for example, from a well-known fact that if we identify $\mathrm{Pic}(X)$ with the weight lattice of $\mathfrak{g}$, then $\mathcal{L}'_0$ gets identified with $-\delta$, where δ is the maximal root. By Proposition 2.1.2, $\widetilde{X}, \widetilde{\mathrm{S}}(e, B)$ are obtained by pulling back the universal deformation of X by means of certain maps $\mathfrak{z}_0 \to H^2_{\mathrm{DR}}(X), \mathfrak{h}^* \to H^2_{\mathrm{DR}}(X)$. Now Proposition 4.5.1 together with Proposition 3.2.1 implies that the maps $\mathfrak{z}_0 \to H^2_{\mathrm{DR}}(X), \mathfrak{h}^* \to H^2_{\mathrm{DR}}(X)$ are intertwined by the isomorphism $\mathfrak{h}^* \to U$. This shows the claim of the previous paragraph.

Identify $\widetilde{X}$ with $\widetilde{\mathrm{S}}(e, B)$. Consider the canonical quantization $\mathcal{D}$ of this symplectic scheme over $\mathfrak{z}_0^{\wedge 0}$. By Corollary 2.3.3, any even graded quantization of $\widetilde{X}$ is isomorphic to $\mathcal{D}$. So Theorem 5.4.1 specifies the condition on a comoment quantum map for the reduced quantization to be canonical. Thanks to Lemmas 4.2.4 and 5.2.1, the algebras $\mathbb{A}_{\mathfrak{z}_0}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h, \mathcal{W}_{h,\mathfrak{h}}$ are the subalgebras of $\mathbb{C}^\times$-finite global sections of appropriate quantizations of $\widetilde{X} = \widetilde{\mathrm{S}}(e, B)$. Since these quantizations are graded and even by Theorem 5.4.1, they are isomorphic, and so we see that the algebras $\mathbb{A}_{\mathfrak{z}_0}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ and $\mathcal{W}_{h,\mathfrak{h}}$ are isomorphic as $\mathbb{C}[\mathfrak{h}^*, h]$-algebras. □

Proof of Theorem 5.3.3. It is the same as the proof of Theorem 5.3.1, but one has to replace results from Section 4.5 with their counterparts from Section 4.6. Perhaps, the only new part is that we need to check that quantization commutes with reduction for $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$. Thanks to Lemma 3.3.1, we only need to check that Proposition 4.2.1 applies in the present situation.

Consider the quiver $Q^{\mathbf{d}}$ and the dimension vector $\mathbf{v}^{\mathbf{d}}$. Suppose $\mathbf{v}^{\mathbf{d}}$ is decomposed into a sum $\mathbf{v}' + \mathbf{v}^1 + \cdots + \mathbf{v}^k$, with $\mathbf{v}^1, \ldots, \mathbf{v}^k$ being roots for the Dynkin quiver Q. We need to show that $p(\mathbf{v}) > \mathfrak{p}(\mathbf{v}') + \sum_{i=1}^k p(\mathbf{v}^i)$.

We have $p(\mathbf{v}^i) = 0$ for $i = 1, \ldots, k$ so we only need to check that $p(\mathbf{v}) > p(\mathbf{v}')$ for any $\mathbf{v}' \leqslant \mathbf{v}, \mathbf{v}' \neq \mathbf{v}$ and $v'_s = 1$. Let v' be the element of the root lattice associated to $\mathbf{v}'$. We have $p(\mathbf{v}) = (\mathrm{d}, \mathrm{v}) - \frac{1}{2}(\mathrm{v}, \mathrm{v}), p(\mathbf{v}') = (\mathrm{d}, \mathrm{v}') - \frac{1}{2}(\mathrm{v}', \mathrm{v}')$. Here $(\cdot, \cdot)$ is the normalized invariant scalar product, i.e. $(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^k x_i(2y_i - y_{i-1} - y_{i+1})$. Set $\mathbf{u} := \mathbf{v} - \mathbf{v}'$, and u be the corresponding vector in the root lattice. Then we have $p(\mathbf{v}) - p(\mathbf{v}') = (\mathrm{u}, \mathrm{d} - \frac{1}{2}(\mathrm{v} + \mathrm{v}')) = (\mathrm{u}, \mathrm{d} - \mathrm{v} + \frac{1}{2}\mathrm{u}) \geqslant \frac{1}{2}(\mathrm{u}, \mathrm{u})$. The last inequality holds because $\mathrm{d} - \mathrm{v}$ is dominant. □

6. Symplectic reflection algebras

6.1. Definitions

Let $\mathcal{V}$ be a symplectic vector space with symplectic form ω and $\mathcal{G} \subset \mathrm{Sp}(\mathcal{V})$ be a finite group. Let S denote the set of symplectic reflections in $\mathcal{G}$, that is the set of all $g \in \mathcal{G}$ such that $\mathrm{rk}(g - \mathrm{id}) = 2$. Decompose S into the union $\bigsqcup_{i=0}^r S_i$ of $\mathcal{G}$-conjugacy classes. Pick independent variables $c_0, c_1, \ldots, c_r$, one for each conjugacy class in S and also an independent variable h. Let $\mathfrak{c}$ denote the vector space dual to the span of $h, c_0, \ldots, c_r$, with dual basis $\check{h}, \check{c}_0, \ldots, \check{c}_r$. By the (universal) symplectic reflection algebra (SRA) we mean the quotient $\mathbf{H}$ of $\mathbb{C}[\mathfrak{c}] \otimes T\mathcal{V}\#\mathcal{G}$ by the relations

$$[u,v]=h\omega(u,v)+\sum_{i=0}^{r}c_i\sum_{s\in S_i}\omega_s(u,v)s, \tag{6.1}$$

with $\omega_s=\pi^*\omega$, where π stands for the projection $V\twoheadrightarrow \operatorname{im}(s-\mathrm{id})$ along $\ker(s-\mathrm{id})$.

For $\beta\in\mathfrak{c}$ let H_β denote the specialization of $\mathbf{H}$ at β.

Let us list some properties of the algebra $\mathbf{H}$:

(A) $\mathbf{H}/(h,c_0,\ldots,c_r)\mathbf{H}=S\mathcal{V}\#\mathcal{G}$.
(B) $\mathbf{H}$ is graded: $\mathbb{C}\mathcal{G}\subset\mathbf{H}$ has degree 0, $V\subset\mathbf{H}$—degree 1, while $\mathfrak{c}^*$ is of degree 2.
(C) $\mathbf{H}$ is flat over $\mathbb{C}[\mathfrak{c}]$. This is a reformulation of results of Etingof and Ginzburg, [13, Theorem 1.3].
(D) Finally, and, in a sense, most importantly, $\mathbf{H}$ is universal with these three properties under some mild restrictions on $\mathcal{G}$. Namely, assume that $\mathcal{G}$ is symplectically irreducible, that is, there is no proper symplectic $\mathcal{G}$-submodule of $\mathcal{V}$. Then let $\mathfrak{c}'$ be a vector space and $\mathbf{H}'$ be a graded $\mathbb{C}[\mathfrak{c}']$-algebra satisfying the analogs of (A),(B),(C). Then there is a unique linear map $\mathfrak{c}'\to\mathfrak{c}$ such that $\mathbf{H}'\cong\mathbb{C}[\mathfrak{c}']\otimes_{\mathbb{C}[\mathfrak{c}]}\mathbf{H}$.

To prove (D) one argues as follows. The degree -2 component of the Hochshield cohomology group $\mathrm{HH}^2(S\mathcal{V}\#\mathcal{G})$ is identified with $\mathfrak{c}^*$. All graded deformations are unobstructed because the degree -4 component of $\mathrm{HH}^3(S\mathcal{V}\#\mathcal{G})$ vanishes. To compute the Hochshield cohomology of $SV\#\mathcal{G}$ one argues similarly to the proof of Theorem 9.1 in [11].

In fact, below we will be interested mostly in the so called spherical subalgebra of $\mathbf{H}$. Namely, consider the trivial idempotent $e=\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\gamma\in\mathbb{C}\mathcal{G}$ and note that $\mathbb{C}\mathcal{G}\subset\mathbf{H}$. Form the *spherical subalgebra* $e\mathbf{H}e$ with unit e. This is a flat graded deformation of $(S\mathcal{V})^{\mathcal{G}}$.

We are mostly interested in the special case when $\mathcal{G}$ is the wreath-product Γ_n of a Kleinian group $\Gamma\subset\mathrm{SL}_2(\mathbb{C})$ and of the symmetric group S_n, where $n>1$, and $\mathcal{V}=L^{\oplus n}$; see Section 4.4.

Let $\Gamma\setminus\{1\}=\coprod_{i=1}^{l}S_i^0$ be the decomposition into Γ-conjugacy classes. We have $S=S_{\mathrm{sym}}\sqcup\coprod_{i=1}^{l}S_i$, where

$$S_{\mathrm{sym}}:=\{s_{ij}\gamma_{(i)}\gamma_{(j)}^{-1},\ 1\leqslant i<j\leqslant n,\ \gamma\in\Gamma\},$$
$$S_i:=\{\gamma_{(j)},\ 1\leqslant j\leqslant n,\ \gamma\in S_i^0\},\quad i=1,\ldots,l$$

where $\gamma_{(j)}$ means the element $(1,\ldots,1,\gamma,1,\ldots,1)\in\Gamma^n$ with γ on the jth place, and s_{ij} stands for the transposition of the ith and jth elements in S_n.

We remark that Γ_n is symplectically irreducible provided $\Gamma\neq\{1\}$. We can make S_n to act symplectically irreducibly if we replace L^n with the double of the reflection representation of S_n. Below we still write $c_1,\ldots,c_l$ for the independent variables corresponding to S_i, $i=1,\ldots,l$, and we write k for the variable corresponding to S_{sym}. Then (6.1) becomes the same system of relations as (1.2.2), (1.2.3) in [12]. Of course, for $n=1$ we just do not have the class S_{sym}. However, it will be convenient for us to consider the space $\widetilde{\mathfrak{c}}:=\mathfrak{c}$ for $n>1$ and $\widetilde{\mathfrak{c}}:=\mathfrak{c}\oplus\mathbb{C}\check{k}$ for $n=1$ and set $\widetilde{\mathbf{H}}:=\mathbb{C}[\widetilde{\mathfrak{c}}]\otimes_{\mathbb{C}[\mathfrak{c}]}\mathbf{H}$. So $\widetilde{\mathbf{H}}=\mathbf{H}$ for $n>1$ and $\mathbb{C}[k]\otimes\mathbf{H}$ for $n=1$.

6.2. Main result

Our ultimate goal is to reprove results relating $e\mathbf{H}e$ to certain quantum Hamiltonian reductions. The latter is as follows.

Let N_i, Q, δ, $\mathbf{v}$, $\mathbf{d}$ be as in Section 4.4. Set $V := R(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$. Further, set $G := \mathrm{GL}(n\delta)$, $\mathfrak{z} := \mathfrak{g}^{*G}$. Let $\mathbb{A}_h$ be the homogenized Weyl algebra of V. Consider the reduction $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h := \mathbb{A}_h /\!\!/\!\!/ G$. This is a graded algebra over $\mathbb{C}[\mathfrak{z}][h]$.

Let us state our main result. Recall that we have fixed a $\mathbb{C}^\times$-equivariant Poisson isomorphism $\mathbb{C}^{2n}/\Gamma_n \cong \mathcal{M}_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$. Set $\mathbf{c} := h + \sum_{i=1}^r c_i \sum_{\gamma \in S_i^0} \gamma \in \mathfrak{c}^* \otimes Z(\mathbb{C}\Gamma)$. Further, recall the identification $\mathfrak{z} \cong \mathbb{C}^{Q_0}$. It will be convenient for us to change our usual notation and write $\epsilon_0, \ldots, \epsilon_r$ for the tautological basis of $\mathfrak{z}^*$ (and not of $\mathfrak{z}$). Also set $\widehat{\mathfrak{z}}^* := \mathfrak{z}^* \oplus \mathbb{C}h$ and let $\check{\epsilon}_0, \ldots, \check{\epsilon}_r, \check{h}$ be the dual basis in $\widehat{\mathfrak{z}}$.

Theorem 6.2.1. *Suppose $\Gamma \neq \{1\}$. There is a graded algebra isomorphism $e\widetilde{\mathbf{H}}e \to \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$, that maps $\widetilde{\mathfrak{c}}^* \subset e\widetilde{\mathbf{H}}e$ to $\widehat{\mathfrak{z}}^* \subset \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ and induces the fixed isomorphism $\mathbb{C}^{2n}/\Gamma_n \cong \mathcal{M}_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$. The corresponding map $\upsilon : \widetilde{\mathfrak{c}}^* \to \widehat{\mathfrak{z}}^*$ is the inverse of the following map*

$$\begin{aligned} &h \mapsto h, \\ &\epsilon_0 \mapsto \mathrm{tr}_{N_0}\mathbf{c}/|\Gamma| - (k+h)/2. \\ &\epsilon_i \mapsto \mathrm{tr}_{N_i}\mathbf{c}/|\Gamma|, \quad i = 1, \ldots, r. \end{aligned} \tag{6.2}$$

This theorem is similar to the principal result of [12] but there are several differences. First, (6.2) looks different from the analogous formula in [12]. This is because their quantum comoment map differed from ours by a character (our quantum comoment map is, in a sense, symmetrized but theirs is not). Second, our parameters are independent variables, while [12] considers numerical values. Finally, the proof in [12] works only when the quiver Q is bi-partive, which is true for Γ of types D, E and A_l for even l. The cases A_l for all $l > 0$ are covered by the work of Oblomkov, [40] and Gordon, [18]. The case of $n = 1$ follows basically from Holland's paper [21]. Holland's results may be interpreted as follows. The case $\Gamma = \{1\}$ was obtained in [14]. The proof in this case can be obtained by a slight modification of our argument, but we are not going to provide it.

For $n = 1$ Theorem 6.2.1 together with Remark 5.3.2 implies the following result, which essentially was first proved by Holland. Set $\widehat{\mathfrak{z}_0} := \mathfrak{z}_0 \oplus \mathbb{C}h$.

Theorem 6.2.2. *Let $n = 1$. There is a graded algebra isomorphism $e\mathbf{H}e \to \mathbb{A}_{\mathfrak{z}_0}(\mathrm{D}Q, \mathbf{v})_h$ mapping $\mathfrak{c}^*$ to $\widehat{\mathfrak{z}}_0^*$, where the induced map $\upsilon_0 : \mathfrak{c}^* \to \widehat{\mathfrak{z}}_0^*$ is the inverse of the following map*

$$\begin{aligned} &h \mapsto h, \\ &\epsilon_0 \mapsto \mathrm{tr}_{N_0}\mathbf{c}/|\Gamma| - h. \\ &\epsilon_i \mapsto \mathrm{tr}_{N_i}\mathbf{c}/|\Gamma|, \quad i = 1, \ldots, r. \end{aligned} \tag{6.3}$$

Theorem 6.2.1 for $n = 1$ will be proved in Section 6.3. The case $n > 1$ is much more complicated. Its proof will be completed in Section 6.6.

Let us explain the scheme of the proof. To prove the existence of a graded endomorphism $\Upsilon : e\widetilde{\mathbf{H}}e \to \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ mapping $\mathfrak{c}^* = \widetilde{\mathfrak{c}}^*$ to $\widehat{\mathfrak{z}}^*$ is relatively easy. There are two ingredients for this proof: the universality property of SRAs, see (D) in the previous subsection, and the existence of a bundle $\mathcal{P}$ on a resolution of $\mathbb{C}^{2n}/\Gamma_n$. This bundle allows to construct a deformation of $S(\mathbb{C}^{2n})\#\Gamma_n$ whose spherical subalgebra is precisely $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$. In the case $n = 1$ we have enough information about the bundle $\mathcal{P}$ to recover the corresponding map $\mathfrak{c}^* \to \widehat{\mathfrak{z}}^*$ pretty easily. But this is not the case in general, so the most difficult part of the proof is to show that the map $\mathfrak{c}^* \to \widehat{\mathfrak{z}}^*$ is as needed.

Of course, it is enough to show that the restriction $\Upsilon|_{\mathfrak{c}^*}$ differs from υ by a map induced by an automorphism of $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$. We study certain group of automorphisms of the algebras $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ in Section 6.4. First we use Maffei's construction of isomorphisms of quiver varieties, [35], to show that the group we are interested in includes the Weyl group W_{fin} of the Dynkin part of Q. Then we check that for $n = 1$ the group under consideration basically coincides with W_{fin}. Finally, we produce a certain automorphism of $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ for $n > 1$ that (as we will see later) does not belong to W_{fin}.

Then our strategy to prove Theorem 6.2.1 for $n > 1$ is to reduce it to the case $n = 1$, where the result is already known. This is achieved by considering certain completions of $e\mathbf{H}e, \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ and their isomorphism induced by Υ. We will consider two different completions. This will be done in Section 6.5.

In Section 6.6 we complete the proof of Theorem 6.2.1. We will show that the isomorphism of completions introduced in Section 6.5 gives rise to certain endomorphisms of some SRA with $n = 1$. From here, thanks to results of Section 6.4, we will deduce that $\Upsilon|_{\mathfrak{c}^*}$ coincides with υ up to an element of $W_{\mathrm{fin}} \times \mathbb{Z}/2\mathbb{Z}$ acting on $\widehat{\mathfrak{z}}$. Then we will see that any element of this group is actually induced by an automorphism of $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$.

The last subsection of the section has nothing to do with the main theorem. There we use our techniques to establish a result to be used in a subsequent paper [19].

6.3. An isomorphism via a (weakly) Procesi bundle

Here we are going to prove that there is a $\mathbb{C}[h]$-linear graded algebra homomorphism $\Upsilon : e\widetilde{\mathbf{H}}e \to \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ mapping $\widetilde{\mathfrak{c}}^*$ to $\widehat{\mathfrak{z}}^*$. Then we will prove Theorem 6.2.1 for $n = 1$.

Thanks to the universality property of $\mathbf{H}$, the existence of Υ will follow if we produce a graded flat $\mathbb{C}[\widehat{\mathfrak{z}}^*]$-algebra $\widetilde{\mathbf{H}}'$ that deforms $\mathbb{C}[\mathbb{C}^{2n}]\#\Gamma$ and such that $e\widetilde{\mathbf{H}}'e \cong \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$. For the algebra $\widetilde{\mathbf{H}}'$ we basically take the endomorphism algebra of a quantization of a bundle $\widehat{\mathcal{P}}$ on the symplectic variety $\widehat{X}$; see Section 4.4.

So let $\mathcal{D}$ stand for the canonical quantization of $\widehat{X}$. By Proposition 5.4.4, $\mathcal{D} \cong \mathbb{A}_{h,V^*} /\!\!/\!\!/^{\theta} G$. Recall, Lemma 4.2.4, that the subalgebra of $\mathbb{C}^\times$-finite elements in $\mathcal{D}$ is naturally identified with $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$.

Since $\mathrm{Ext}^i_{\mathcal{O}_{\widehat{X}}}(\widehat{\mathcal{P}}, \widehat{\mathcal{P}}) = 0$, one can lift $\widehat{\mathcal{P}}$ to a unique projective $\mathbb{C}^\times \times \Gamma_n$-equivariant right $\mathcal{D}$-module $\widehat{\mathcal{P}}_h$. Automatically, $\mathrm{End}_{\mathcal{D}^{\mathrm{opp}}}(\widehat{\mathcal{P}}_h)/(h) = \mathrm{End}_{\mathcal{O}_{\widehat{X}}}(\widehat{\mathcal{P}})$ and so $\mathrm{End}_{\mathcal{D}^{\mathrm{opp}}}(\widehat{\mathcal{P}}_h)$ is flat over $\mathbb{C}[\widehat{\mathfrak{z}}]$. So we see that $\mathrm{End}_{\mathcal{D}^{\mathrm{opp}}}(\widehat{\mathcal{P}}_h)/(\widehat{\mathfrak{z}}) = \mathrm{End}_{\mathcal{O}_X}(\mathcal{P}) = \mathbb{C}[\mathbb{C}^{2n}]\#\Gamma_n$. Let $\widetilde{\mathbf{H}}'$ be the subalgebra of $\mathbb{C}^\times$-finite elements in $\mathrm{End}_{\mathcal{D}^{\mathrm{opp}}}(\widehat{\mathcal{P}}_h)$. Since $\widehat{\mathcal{P}}^{\Gamma_n} = \mathcal{O}_{\widehat{X}}$, we see that $\widehat{\mathcal{P}}_h^{\Gamma_n} = \mathcal{D}$. So $\mathcal{D} \cong e[\mathcal{E}nd_{\mathcal{D}^{\mathrm{opp}}}(\widehat{\mathcal{P}}_h)]e$, and $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h \cong e\widetilde{\mathbf{H}}'e$.

The graded flat deformation $\widetilde{\mathbf{H}}'$ of $\mathbb{C}[\mathbb{C}^{2n}]\#\Gamma_n$ gives rise to a (unique for $n > 1$) linear map $\widehat{\mathfrak{z}} \to \widetilde{\mathfrak{c}}$ with $\widetilde{\mathbf{H}}' \cong \mathbb{C}[\widehat{\mathfrak{z}}] \otimes_{\mathbb{C}[\widetilde{\mathfrak{c}}]} \mathbf{H}$. Taking the spherical subalgebras we get a homomorphism $\Upsilon : e\widetilde{\mathbf{H}}e \to \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$. Our problem now becomes to determine the restriction $\Upsilon|_{\mathfrak{c}^*}$.

First of all, let us show that $\Upsilon(h) = h$. The algebras $e\widetilde{\mathbf{H}}'e/(h), e\widetilde{\mathbf{H}}e/(h)$ are commutative (for the first one this is evident, and for the second one follows from [13]). On the other hand, for $\beta' \in \widehat{\mathfrak{z}}$ with $\langle \beta', h \rangle \neq 0$ the specialization $\mathbb{A}_{\beta'}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ is not commutative (because the Poisson bracket on $\mathbb{C}[\mathbb{C}^{2n}]^{\Gamma}$ induced by the bracket on the filtered algebra $\mathbb{A}_{\beta'}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ is nonzero). Similarly, for $\beta \in \mathfrak{c}$ with $\langle \beta, h \rangle \neq 0$ the algebra $eH_\beta e$ is non-commutative for the same reason. This implies that $\Upsilon(h)$ is a non-zero multiple of h. Now consider the Poisson brackets on $\mathbb{C}[\mathcal{M}_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})]$ and on $\mathbb{C}[\mathbb{C}^{2n}]^{\Gamma_n}$. Let $\rho : \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h \twoheadrightarrow \mathbb{C}[\mathcal{M}_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})]$ be the canonical projection. The bracket on $\mathbb{C}[\mathcal{M}_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})]$ is given by

$\{a, b\} = \rho(\frac{1}{h}[\iota(a), \iota(b)])$, where $\iota : \mathbb{C}[\mathcal{M}_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})] \to \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ is any section of ρ. The bracket on $\mathbb{C}[\mathbb{C}^{2n}]^{\Gamma}$ can be defined using $e\mathbf{H}e$ in a similar way. Since our identification of $\mathbb{C}[\mathbb{C}^{2n}]^{\Gamma_n}$ and $\mathbb{C}[\mathcal{M}_0(\mathrm{D}Q, \mathbf{v}, \mathbf{d})]$ was Poisson, we see that $\Upsilon(h) = h$.

In fact, the case $n = 1$ is particularly easy because in this case we know much more about $\widehat{\mathcal{P}}$ than in general; see Section 4.5. We use the description of $\widehat{\mathcal{P}}$ given there to prove Theorem 6.2.1 for $n = 1$.

In Section 4.5, we constructed $\widehat{\mathcal{P}}$ explicitly for any generic θ. Now suppose that θ is chosen as explained after Proposition 4.2.2. In fact, one can interpret $\widehat{X}, \widehat{\mathcal{P}}$ in a slightly different way. Namely, consider the variety Y of all Γ-linear algebra homomorphisms $\widetilde{\mathbf{H}}/(h) \to \mathrm{End}(\mathbb{C}\Gamma)$. This variety comes equipped with a $\mathrm{GL}(\mathbf{v}) = \mathrm{GL}(\mathbb{C}\Gamma)$-action and with a canonical $\mathrm{GL}(\mathbf{v})$-equivariant bundle $\mathcal{C}$ of rank $|\Gamma|$. Now $Y^{\theta,ss}$ consists of all homomorphisms having a "cocyclic covector", i.e., an element $\alpha \in (\mathbb{C}\Gamma)^*$ such that $a^*(\alpha)$, $a \in \widetilde{\mathbf{H}}/(h)$ span the whole space $(\mathbb{C}\Gamma)^*$. Consider the variety $\widehat{X}' := Y^{\theta,ss}/\mathrm{GL}(\mathbb{C}\Gamma)$ and the bundle $\widehat{\mathcal{P}}'$ on $\widehat{X}'$ obtained from $\mathcal{C}$ by descend. It is a standard fact on the McKay correspondence that there is a $\mathbb{C}^\times$-equivariant isomorphism $\eta : \widehat{X}' \to \widehat{X}$ such that

- $\widehat{\mathcal{P}}' \cong \eta^*(\widehat{\mathcal{P}})$.
- Ψ lifts the map $\mathfrak{c}^*/\mathbb{C}h \to \mathfrak{z}_0^*/\mathbb{C}h$ induced by (6.3).

By the definition of $\widehat{X}', \widehat{\mathcal{P}}'$, we have a natural homomorphism $\widetilde{\mathbf{H}}/(h) \to \mathrm{End}_{\mathcal{O}_{\widehat{X}'}}(\widehat{\mathcal{P}}')$. This homomorphism is the identity modulo $\mathfrak{z}_0$ and both algebras are flat over $\mathbb{C}[\mathfrak{z}_0]$. It follows that this homomorphism is an isomorphism. So we see that there is an isomorphism $e\mathbf{H}e \cong \mathbb{C}[\mathfrak{c}] \otimes_{\mathbb{C}[\widehat{\mathfrak{z}}_0]} \mathbb{A}_{\mathfrak{z}_0}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ with the isomorphism $\upsilon_0' : \mathfrak{c}^* \to \widehat{\mathfrak{z}}_0^*$ that maps h to h and is congruent to υ_0 modulo $\mathbb{C}h$. We need to show that this isomorphism actually coincides with υ_0.

To show this let us consider the set $\mathfrak{c}_{\mathrm{sing}}$ of all elements β in $\mathfrak{c}_1 := \{\beta \in \mathfrak{c} | \langle \beta, h \rangle = 1\}$ such that the algebra $eH_\beta e$ has infinite homological dimension. According to [8], this set is a finite union of hyperplanes, whose only common intersection point is the only point β^0 such that the corresponding element $\mathbf{c}^0 := 1 + \sum_{i=1}^r \langle \beta^0, c_i \rangle \sum_{\gamma \in S_i^0} \gamma$ satisfies $\mathrm{tr}_{N_i}(\mathbf{c}^0) = 0$. So to prove $\upsilon_0 = \upsilon_0'$ it is enough to show that $\langle \upsilon'^{-1*}(\beta^0), \epsilon_i \rangle = 0$ for $i = 1, \ldots, r$.

Recall the isomorphism $\mathbb{A}_{\mathfrak{z}_0}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h \cong \mathcal{W}_{h,\mathfrak{h}}$, Theorem 5.3.1. It is known, thanks to the localization theorems from [16] or [9], that the algebra $\mathcal{W}_\lambda$ has finite homological dimension provided $\langle \lambda, \alpha \rangle \neq 0$ for any root α. The number of hyperplanes in $\mathfrak{c}_{\mathrm{sing}}$ coincides with the number of positive roots. So $\upsilon'^{*-1}(\beta^0)$ is the only intersection point of the hyperplanes $\ker \alpha$ and hence vanishes on ϵ_i, $i = 1, \ldots, r$.

We remark that the claim of the previous sentence can be obtained also without using Theorem 5.3.1 and the localization. For this one needs to use the results on automorphisms of $\mathbb{A}_{\widehat{\mathfrak{z}}_0}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ obtained in the next subsection. We will see that a unique fixed point of certain group of automorphisms of $\mathbb{A}_{\widehat{\mathfrak{z}}_0}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ vanishes on ϵ_i, $i = 1, \ldots, r$. On the other hand, this group obviously preserves $\upsilon_0'^{*-1}(\mathfrak{c}_{\mathrm{sing}})$.

6.4. Automorphisms

In this subsection, we will study the graded automorphisms of the algebras $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ that preserve $\widehat{\mathfrak{z}}^*$ and are the identity modulo $\widehat{\mathfrak{z}}^*$. For this we will need to recall a construction appearing in different forms in the work of Nakajima, Lusztig and Maffei. We will follow [35].

Consider, for a moment, an arbitrary quiver $\widetilde{Q} = (\widetilde{Q}_0, \widetilde{Q}_1)$ and a dimension vector $\widetilde{\mathbf{v}}$. Let W be the Weyl group of the quiver $\widetilde{Q}$. The group W acts on $\widetilde{\mathfrak{z}} := \mathbb{C}^{\widetilde{Q}_0}$.

Proposition 6.4.1. *Pick a character θ of $\mathrm{GL}(\widetilde{\mathbf{v}})$ such that $\theta \cdot \widetilde{\mathbf{v}} = 0$. Let i be a loop free vertex of $\widetilde{Q}$, and $s = s_i$, $i \in \widetilde{Q}_0$, be the simple reflection corresponding to i. Suppose that $\theta_i \neq 0$, $s\widetilde{\mathbf{v}} \geqslant 0$. Then there exists a $\mathbb{C}^\times$-equivariant isomorphism $S : \mathcal{M}^\theta(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}) \xrightarrow{\sim} \mathcal{M}^{s\theta}(\mathrm{D}\widetilde{Q}, s\widetilde{\mathbf{v}})$ lifting $s : \mathbb{C}^{\widetilde{Q}_0} \to \mathbb{C}^{\widetilde{Q}_0}$.*

The construction of the isomorphism is due to Maffei, [35]. We will recall his construction in the setting we need.

Without loss of generality we may assume that i is a source in $\widetilde{Q}$ (i.e., $h(a) \neq i$ for all $a \in \widetilde{Q}_1$). Set $\widetilde{\mathbf{v}}' := s\widetilde{\mathbf{v}}$, $\theta' := s\theta$. Also we may assume that $\theta_i > 0$ (because θ_i, θ'_i have different signs).

Pick the spaces V_i of dimension $\widetilde{\mathbf{v}}_i$, $i \in \widetilde{Q}_0$, and set $T := \bigoplus_{a, t(a)=i} V_i$. Also for any i pick a vector space V'_i of dimension $\widetilde{\mathbf{v}}'_i$. We remark that $\dim T = \widetilde{\mathbf{v}}_i + \widetilde{\mathbf{v}}'_i$. Consider the space

$$\begin{aligned}\widetilde{V} := &\bigoplus_{a, t(a)\neq i} \big(\mathrm{Hom}(V_{t(a)}, V_{h(a)}) \oplus \mathrm{Hom}(V_{h(a)}, V_{t(a)})\big)\\ &\oplus \mathrm{Hom}(V_i, T) \oplus \mathrm{Hom}(T, V_i) \oplus \mathrm{Hom}(T, V'_i) \oplus \mathrm{Hom}(V'_i, T).\end{aligned}$$

We write an element of $\widetilde{V}$ as $((A_a), (B_a), A, B, A', B')$ with $A_a \in \mathrm{Hom}(V_{t(a)}, V_{h(a)})$, $B_a \in \mathrm{Hom}(V_{h(a)}, V_{t(a)})$, $A \in \mathrm{Hom}(V_i, T)$, $B \in \mathrm{Hom}(T, V_i)$, $A' \in \mathrm{Hom}(V'_i, T)$, $B' \in \mathrm{Hom}(T, V'_i)$.

The space $\widetilde{V}$ comes with a natural action of a group $\widetilde{G} := G \times \mathrm{GL}(\widetilde{\mathbf{v}}_i) \times \mathrm{GL}(\widetilde{\mathbf{v}}'_i)$, $G := \prod_{j\neq i} \mathrm{GL}(\widetilde{\mathbf{v}}_j)$. Moreover, $\widetilde{V}^{\mathrm{GL}(\widetilde{\mathbf{v}}'_i)} = R(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}})$, $\widetilde{V}^{\mathrm{GL}(\widetilde{\mathbf{v}}_i)} = R(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}')$. Let π, π' denote the natural projections $\widetilde{V} \twoheadrightarrow R(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}), R(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}')$.

Consider the locally closed subvariety $Z \subset \widetilde{V}$ consisting of all vectors $x = ((A_a), (B_a), A, B, A', B')$ such that

(1) The sequence $0 \to V'_i \xrightarrow{A'} T \xrightarrow{B} V_i \to 0$ is exact.
(2) $\pi(x) \in \Lambda_\chi(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}})$, $\pi'(x) \in \Lambda_{s\chi}(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}')$ for some $\chi \in \mathbb{C}^{\widetilde{Q}_0}$.
(3) $A'B' = AB - \chi_i \mathrm{id}_T$.

In particular, we see that π, π' induce $\widetilde{G}$-equivariant projections $Z \to \Lambda(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}), \Lambda(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}')$ denoted by ρ, ρ'. Thanks to [35, Lemma 30], $\rho^{-1}(\Lambda(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}})^{\theta,ss}) = \rho'^{-1}(\Lambda(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}})^{\theta',ss})$ (for applying this lemma we need to assume that $\theta_i > 0$). Denote these equal subvarieties of Z by Z^{ss}. Now thanks to Lemma 33 in [35], the restrictions

$$\begin{aligned}&\rho : Z^{ss} \to \Lambda(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}})^{\theta,ss},\\ &\rho' : Z^{ss} \to \Lambda(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}')^{\theta',ss}\end{aligned} \tag{6.4}$$

are principal $\mathrm{GL}(\widetilde{\mathbf{v}}'_i)$- and $\mathrm{GL}(\widetilde{\mathbf{v}}_i)$-bundles, respectively. This gives isomorphisms $\mathcal{M}^\theta(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}) \cong Z^{ss}/\widetilde{G} \cong \mathcal{M}^{\theta'}(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}')$. For S_i we take the resulting isomorphism $\mathcal{M}^\theta(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}) \xrightarrow{\sim} \mathcal{M}^{\theta'}(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}')$. This isomorphism is $\mathbb{C}^\times$-equivariant by the construction. The claim that it lifts $s_i : \mathbb{C}^{\widetilde{Q}_0} \to \mathbb{C}^{\widetilde{Q}_0}$ follows from the condition (2) in the definition of Z. Most likely, S is always a Poisson isomorphism but we will not need this fact in the whole generality.

Now consider the special case when $s\widetilde{\mathbf{v}} = \widetilde{\mathbf{v}}$. Suppose that the natural morphisms $\mathcal{M}^\theta_0(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}) \to \mathcal{M}_0(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}})$, $M^{s\theta}_0(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}}) \to \mathcal{M}_0(\mathrm{D}\widetilde{Q}, \widetilde{\mathbf{v}})$ are resolutions of singularities.

Lemma 6.4.2. *The following diagram is commutative.*

$$\begin{array}{ccc} \mathcal{M}_0^\theta(\mathrm{D}\widetilde{Q},\widetilde{\mathbf{v}}) & \xrightarrow{S} & \mathcal{M}_0^{s\theta}(\mathrm{D}\widetilde{Q},\widetilde{\mathbf{v}}) \\ & \searrow \quad \swarrow & \\ & \mathcal{M}_0(\mathrm{D}\widetilde{Q},\widetilde{\mathbf{v}}) & \end{array}$$

Proof. The lemma will follow if we check the following claim:

- Suppose that $x = ((A_a), (B_a), A, B, A', B')$ is an element of Z such that $\chi = 0$ (see (1)–(3)) above. Then $f(\pi(x)) = f(\pi'(x))$ for any $f \in \mathbb{C}[R(\mathrm{D}\widetilde{Q},\widetilde{\mathbf{v}})]^{\mathrm{GL}(\widetilde{\mathbf{v}})}$.

Le Bruyn and Procesi found a set of generators for the algebra $\mathbb{C}[R(\mathrm{D}\widetilde{Q},\widetilde{\mathbf{v}})]^{\mathrm{GL}(\widetilde{\mathbf{v}})}$ in [26]. Let $p := (a_0, \ldots, a_k)$ be a cyclic (i.e., $t(a_0) = h(a_k)$) path in $\mathrm{D}\widetilde{Q}$. To this path we can assign a polynomial f_p mapping $x \in R(\mathrm{D}\widetilde{Q},\widetilde{\mathbf{v}})$ to $\mathrm{tr}(x_{a_k} \ldots x_{a_0})$, where x_{a_k} is the component of x corresponding to a_k. The polynomials f_p generate $\mathbb{C}[R(\mathrm{D}\widetilde{Q},\widetilde{\mathbf{v}})]^{\mathrm{GL}(\widetilde{\mathbf{v}})}$. So it remains to check that $f_p(\pi(x)) = f_p(\pi'(x))$ for any path p and any $x \in Z$ with $\chi = 0$, equivalently, with $AB = A'B'$. Since the trace of a product is stable under a cyclic permutation of the factors, we may assume that $h(a_k) = t(a_0) \neq i$. In this case the products for $\pi(x)$ and $\pi'(x)$ are the same, thanks to the equality $AB = A'B'$. $\square$

We will apply the construction above to the special case explained in Section 6.2. Let $Q, n, \widetilde{\mathbf{v}}, \mathbf{d}$ be as in the beginning of that section. We are interested in the quiver $\widetilde{Q} := Q^{\mathbf{d}}$ and the dimension vector $\widetilde{\mathbf{v}} := \mathbf{v}^{\mathbf{d}}$.

Lemma 6.4.3. *Preserve the notation of Proposition 6.4.1. Let $i = 1, 2, \ldots, r$. Then the morphism $S : \mathcal{M}^\theta(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}}) \to \mathcal{M}^{s\theta}(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}})$ is Poisson.*

Proof. Since the morphisms $\mathcal{M}^\theta(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}}), M^{s\theta}(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}}) \to \mathcal{M}(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}})$ are Poisson and birational, it is enough to prove that the morphism $\mathcal{M}(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}}) \to \mathcal{M}(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}})$ induced by S (and also denoted by S) is Poisson. Let $\{\cdot,\cdot\}_\chi$ denote the Poisson bracket on $\mathcal{M}_\chi(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}})$. We need to show that $\{\cdot,\cdot\}_\chi = S^*\{\cdot,\cdot\}_{s\chi}$.

The Poisson algebra $\mathbb{C}[\mathcal{M}_\chi(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}})]$ is filtered and the associated graded algebra is $\mathbb{C}[\mathcal{M}_0(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}})]$. Moreover, $\{\cdot,\cdot\}_\chi$ decreases the degree by 2, and the induced bracket on $\mathbb{C}[\mathcal{M}_0(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}})]$ coincides with $\{\cdot,\cdot\}_0$. The automorphism of $\mathcal{M}_0(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}})$ induced by S is the identity by Lemma 6.4.2. It follows that $\{\cdot,\cdot\}_\chi - S^*\{\cdot,\cdot\}_{s\chi}$ decreases degrees at least by 3. But there are no brackets on $\mathbb{C}[\mathcal{M}_0(\mathrm{D}Q^{\mathbf{d}}, \mathbf{v}^{\mathbf{d}})]$ of degree less than -2; see Proposition 4.4.1. So $\{\cdot,\cdot\}_\chi = S^*\{\cdot,\cdot\}_{s\chi}$. $\square$

Now let us proceed to the quantum situation. Let $\mathcal{D}^\theta$ denote the reduction $\mathbb{A}_h /\!/\!/^\theta \mathrm{GL}(\mathbf{v})$. Consider the sheaves $\mathcal{D}^\theta, S^*(\mathcal{D}^{s\theta})$ on $\mathcal{M}^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$. Thanks to Proposition 5.4.4, both are canonical quantizations of $\mathcal{M}^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})/\mathfrak{z}$ but with respect to different morphisms $M^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \to \mathfrak{z}, \mathcal{D}^\theta$ – for the original map $\mathcal{M}^\theta(\mathrm{D}Q, \mathbf{v}) \to \mathfrak{z}$, while $S^*(\mathcal{D}^{s\theta})$ – for its composition with $s : \mathfrak{z} \to \mathfrak{z}$. It follows that there is a $\mathbb{C}^\times$-equivariant isomorphism $\iota : \mathcal{D}^\theta \xrightarrow{\sim} S^*(\mathcal{D}^{s\theta})$ that induces s on $\mathcal{O}_{\mathfrak{z}}$. Thanks to Lemma 4.2.4 and Proposition 5.4.4, we get a $\mathbb{C}[h]$-linear automorphism $S : \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h \to \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ that acts as s on $\mathfrak{z}^*$. Moreover, S is the identity modulo $\widehat{\mathfrak{z}}^*$.

Let W_{fin} denote the Weyl group of the Dynkin part of Q. Let $\mathfrak{A}$ denote the group of automorphisms of $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ that are $\mathbb{C}^\times$-equivariant, $\mathbb{C}[h]$-linear, preserve $\widehat{\mathfrak{z}}^* \subset \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ and induce the identity modulo $\widehat{\mathfrak{z}}^*$. We have a natural homomorphism $\mathfrak{A} \to \mathrm{GL}(\mathfrak{z})$.

The assignment $s \mapsto S$ extends to a homomorphism $W_{\text{fin}} \to \mathfrak{A}$ whose composition with $\mathfrak{A} \to \mathrm{GL}(\mathfrak{z})$ is the identity. To see this one either applies results of Maffei, [35], or uses the following lemma.

Lemma 6.4.4. *The restriction of the $\mathfrak{A}$-action to $\mathfrak{z}$ defines an embedding $\mathfrak{A} \hookrightarrow \mathrm{GL}(\mathfrak{z})$.*

Proof. Let $a \in \mathfrak{A}$ be an element in the kernel. Then a acts by the identity on both $\widehat{\mathfrak{z}}^*$ and on $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h/(\widehat{\mathfrak{z}}^*)$. So for any choice of $\beta \in \widehat{\mathfrak{z}}^*$ the element a induces a filtration preserving automorphism a_β of $\mathbb{A}_\beta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ (the specialization of $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ at β) such that $\mathrm{gr}\, a_\beta$ is the identity. If all a_β are the identity, then so is a. Indeed, for any different elements $x, y \in \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ there is $\beta \in \widehat{\mathfrak{z}}^*$ such that the images of x, y in $\mathbb{A}_\beta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ are different. So let us prove that a_β is the identity for any β.

Let $\mathrm{F}_i\mathbb{A}_\beta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ denote the natural filtration on $\mathbb{A}_\beta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$. Since the associated graded of a_β is the identity, we see that the restriction of a_β to $\mathrm{F}_i\mathbb{A}_\beta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ is unipotent for any i. So $d = \ln(a_\beta)$ is defined and is a derivation of $\mathbb{A}_\beta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ that reduces the degrees. Let m be the maximal integer such that $d\mathrm{F}_i\mathbb{A}_\beta(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \subset \mathrm{F}_{i-m}\mathbb{A}_\beta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ for all i. Then d induces a derivation d_0 of $\mathrm{gr}\,\mathbb{A}_\beta(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) = (S\mathbb{C}^{2n})^{\Gamma_n}$ of degree $-m$. The derivation d_0 is Poisson by the construction. On the other hand, d_0 can be extended to a (automatically Poisson) derivation of $S(\mathbb{C}^{2n})$. This is because the morphism $\mathbb{C}^{2n} \to \mathbb{C}^{2n}/\Gamma_n$ is étale in codimension 1. But any Poisson derivation of $S(\mathbb{C}^{2n})$ is Hamiltonian. It follows that d_0 is Hamiltonian, i.e., $d_0 = \{f, \cdot\}$ for some $f \in S(\mathbb{C}^{2n})^{\Gamma_n}$. Since d_0 is of degree $-m$, one can choose f of degree $2-m$. But $m \geqslant 1$ so the degree of f is less than 2. Since Γ_n has no fixed points in $\mathbb{C}^{2n}$, we get a contradiction. □

Now consider the case $n = 1$. Consider the subgroup $\widetilde{\mathfrak{A}}$ of $\mathrm{GL}(\widehat{\mathfrak{z}})$ consisting of all maps $\widehat{\mathfrak{z}}^* \to \widehat{\mathfrak{z}}^*$ that send h to h, preserve $\mathfrak{z}_0^*$, and coincide with an element of W_{fin} on $\mathfrak{z}_0^*$. We claim that the image of $\mathfrak{A}$ in $\mathrm{GL}(\widehat{\mathfrak{z}}^*)$ coincides with $\widetilde{\mathfrak{A}}$. This is a consequence of the following more general and more technical statement to be used also later.

Proposition 6.4.5. *Let $n = 1$. Let U_1, U_2 be finite dimensional vector spaces, $\mathcal{A}_i := \mathbb{C}[U_i] \otimes \mathbb{A}_{\mathfrak{z}_0}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$. Let $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ be a $\mathbb{C}[h]$-linear endomorphism that maps $U_1^* \oplus \widehat{\mathfrak{z}}_0^*$ to $U_2^* \oplus \widehat{\mathfrak{z}}_0^*$ and induces the identity map on $S(\mathbb{C}^2)^{\Gamma}$. Then φ maps $\mathfrak{z}_0^*$ to $\mathfrak{z}_0^*$ and induces an element from W_{fin} on $\mathfrak{z}_0^*$.*

Proof. It is enough to consider the case when $U_1 = 0$. Let $(\widehat{\mathfrak{z}_0})_{\text{sing}}$ be the set of all $\beta \in \widehat{\mathfrak{z}}_0$ with $\langle \beta, h \rangle = 1$ and $\langle \beta, \alpha^\vee \rangle = 0$ for some $\alpha \in \Delta$, where $\Delta \subset \mathfrak{z}_0^*$ is the finite part of the root system of Q.

Pick $u \in U_2$. The endomorphism φ induces a filtered algebra homomorphism

$$\mathbb{A}_{\varphi^*(\beta+u)}(\mathrm{D}Q, \mathbf{v}, \mathbf{d}) \to \mathbb{A}_\beta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$$

whose associated graded is the identity. Hence this homomorphism is an isomorphism. It follows that for any $u \in U_2$ and $\beta \in (\widehat{\mathfrak{z}_0})_{\text{sing}}$ we have $\varphi^*(\beta) + \varphi^*(u) = \varphi^*(\beta + u) \in (\widehat{\mathfrak{z}_0})_{\text{sing}}$. Since $(\widehat{\mathfrak{z}_0})_{\text{sing}}$ is not stable under translations by a vector, we see that $\varphi^*(u) = 0$. By the same reason, φ^* is bijective on $\widehat{\mathfrak{z}}_0$.

It remains to show that φ preserves $\mathfrak{z}_0^*$ and induces an element from W_{fin} on this space. To show this it is enough to consider the case when $U_2 = 0$. Here φ is an automorphism of $\mathbb{A}_{\mathfrak{z}_0}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$.

Thanks to Lemma 6.4.4, we can identify φ with its image in $\mathrm{GL}(\widehat{\mathfrak{z}_0^*})$. From the discussion at the end of the previous section we see that φ preserves $(\mathfrak{z}_0^*)_{\text{sing}}$. From here it is easy to deduce that $\varphi \in \mathbb{C}^\times W_{\mathrm{fin}}\mathfrak{A}_0$, where $\mathfrak{A}_0$ stands for the automorphism group of the Dynkin diagram and $\mathbb{C}^\times$ is viewed as the group of scalar matrices. Now, according to [8], the global dimension of $eH_\beta e$ for $\beta \in \mathfrak{c}_1^*$ is bigger than 1 if and only if c is annihilated by some real root of the affine quiver Q. The automorphism φ preserves the corresponding subset of $\widehat{\mathfrak{z}}_0$. From here it is easy to deduce that $\varphi \in W_{\mathrm{fin}}\mathfrak{A}_0$. So it remains to show that $\varphi \in \mathfrak{A}_0$ implies $\varphi = \mathrm{id}$.

Assume the converse, let φ be a nontrivial element of $\mathfrak{A}_0$. Recall the element $\check{h} \in \mathfrak{c}$. The element $\upsilon_0^*(\check{h}) \in \widehat{\mathfrak{z}}_0$ is $\mathfrak{A}_0$-stable. Consider the algebra $H_{\check{h}}$. This algebra is just $\mathbb{A}_2\#\Gamma$, where $\mathbb{A}_2$ stands for the usual (not homogenized) Weyl algebra of $\mathbb{C}^2$. So the spherical subalgebra $e\mathbf{H}_{\check{h}}e$ is just $\mathbb{A}_2^\Gamma$. The restriction of φ to $e\mathbf{H}_{\check{h}}e = \mathbb{A}_2^\Gamma$ is the identity because the associated graded of φ is the identity (compare with the proof of Lemma 6.4.4). On the other hand, according to [13, Theorem 2.16], the completion of $e\mathbf{H}e/(h-1)$ at $\check{h}$ is the universal formal deformation of $\mathbb{A}_2^\Gamma$. In particular, we have an $\mathfrak{A}_0$-equivariant isomorphism $\mathrm{HH}^2(\mathbb{A}_2^\Gamma) \xrightarrow{\sim} \mathfrak{c}_1$ (the r.h.s. is viewed as a vector space with origin $\check{h}$). But the action of φ on $\mathrm{HH}^2(\mathbb{A}_2^\Gamma)$ is trivial because φ restricts to the identity automorphism of $\mathbb{A}_2^\Gamma$. Contradiction. □

Let us proceed to the case of $n > 1$. We still have W_{fin} acting on $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ such that W_{fin} acts on $\widehat{\mathfrak{z}}$ exactly as in $n = 1$ case. We will see below that $\mathfrak{A} = W_{\mathrm{fin}} \times \mathbb{Z}/2\mathbb{Z}$ and describe the action of $\mathbb{Z}/2\mathbb{Z}$. Now let us describe a certain element of $\mathfrak{A}$. We will see later that this element does not lie in W_{fin}. Namely, we have an antiautomorphism τ of $\mathbf{H}$ given by $\tau(v) = v, \tau(g) = g^{-1}$ for $g \in \Gamma_n, \tau(h) = -h, \tau(k) = -k, \tau(c(\gamma)) = -c(\gamma^{-1})$ for $\gamma \in \Gamma$. Then τ fixes e and so descends to $e\mathbf{H}e$. Also we note that τ is the identity on $e\mathbf{H}e/(\mathfrak{c})$. So we get the antiautomorphism $\tau' := \Upsilon \circ \tau \circ \Upsilon^{-1}$ of $\mathbb{A}_h(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$. Set $\nu := \sigma \circ \tau'$, where σ is the parity antiautomorphism induced from $\mathbb{A}_h(V)$. So ν is a $\mathbb{C}[h]$-linear automorphism of $\mathbb{A}_h(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$ that preserves $\widehat{\mathfrak{z}}^*$ and is the identity modulo $\widehat{\mathfrak{z}}^*$. Moreover, the action of $\Upsilon^{-1} \circ \nu \circ \Upsilon$ on $\mathfrak{c}/\mathbb{C}h$ coincides with that of τ.

6.5. Completions

Below $n > 1$.

So we have two tasks: to describe the restriction of Υ to $\mathfrak{c}^*$ and to describe the group $\mathfrak{A}$. The first is our primary goal, but the second is also very important. We will see that we are able to recover $\Upsilon|_{\mathfrak{c}^*}$ only up to an element of $\mathfrak{A}$.

To approach both these questions we will study isomorphisms of certain completions induced by Υ. The completions will have a form $(e\mathbf{H}e)^{\wedge_b}$, $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h^{\wedge_b}$ for certain points $b \in \mathbb{C}^{2n}/\Gamma_n$.

A general definition is as follows. Let A be an algebra such that its center Z is finitely generated, and A is finite over Z. Further, let $\mathcal{A}$ be another algebra equipped with an epimorphism $\mathcal{A} \twoheadrightarrow A$. Pick a point $b \in \mathrm{Spec}(Z)$ and let $\mathfrak{m}_b$ be the ideal in A generated by the maximal ideal of b in Z. Then consider the preimage $\widetilde{\mathfrak{m}}_b$ of $\mathfrak{m}_b$ in $\mathcal{A}$. We put $\mathcal{A}^{\wedge_b} := \varprojlim \mathcal{A}/\widetilde{\mathfrak{m}}_b^n$.

In this subsection, we will study the structure of the completions we need and in the next one we will apply these results to accomplish the two goals mentioned above.

Until a further notice we consider the general SRA $\mathbf{H}$ corresponding to a space V and a group $\mathcal{G}$. Taking $A = SV\#\mathcal{G}, \mathcal{A} := \mathbf{H}$ we get the ideals $\widetilde{\mathfrak{m}}_b \subset \mathbf{H}, e\widetilde{\mathfrak{m}}_b e \subset e\mathbf{H}e$ and the corresponding completions $\mathbf{H}^{\wedge_b}$, $(e\mathbf{H}e)^{\wedge_b}$. The following lemma implies that the completions $(e\mathbf{H}e)^{\wedge_b}$ and $e\mathbf{H}^{\wedge_b}e$ are naturally identified.

Lemma 6.5.1. *The filtrations $e\widetilde{\mathfrak{m}}_b^n e$ and $(e\widetilde{\mathfrak{m}}_b e)^n$ on $e\mathbf{H}e$ are compatible.*

Proof. Let us remark that $(e\widetilde{\mathfrak{m}}_b e)^n \subset e\widetilde{\mathfrak{m}}_b^n e$. On the other hand, as a left $\mathbf{H}$-ideal, $\widetilde{\mathfrak{m}}_b$ is generated by $\mathfrak{c}$ and some elements $x_1, \dots, x_m$ of $\mathbf{H}$ such that each x_i commutes with $\mathcal{G} \subset \mathbf{H}$

and $[x_i, x] \subset \mathfrak{c}\mathbf{H}$ for any $x \in \mathbf{H}$. From this description it is easy to deduce that for any n the ideal $(e\widetilde{\mathfrak{m}}_b e)^n$ contains $e\widetilde{\mathfrak{m}}_b^N e$ for $N \gg 0$. $\square$

The structure of the algebra $\mathbf{H}^{\wedge_b}$ was studied in [32]. Namely, let $\widetilde{b}$ be a point in the preimage of b in V and set $\underline{\mathcal{G}} := \mathcal{G}_{\widetilde{b}}$. Then define the algebra $\underline{\mathbf{H}}$ as the quotient of $TV[\mathfrak{c}]\#\underline{\mathcal{G}}$ by the relations

$$[u, v] = h\omega(u, v) + \sum_{i=1}^{r} c_i \sum_{s\in S_i\cap\underline{\mathcal{G}}} \omega_s(x, y)s. \tag{6.5}$$

Following [1], consider the *centralizer algebra* $Z(\mathcal{G}, \underline{\mathcal{G}}, \underline{\mathbf{H}}^{\wedge_0})$ that comes equipped with an embedding $\mathbb{C}\mathcal{G} \hookrightarrow Z(\mathcal{G}, \underline{\mathcal{G}}, \underline{\mathbf{H}}^{\wedge_0})$. What we need to know about this algebra is that there is a $\mathbb{C}[\mathfrak{c}]\mathcal{G}$-linear isomorphism $\mathbf{H}^{\wedge_b} \xrightarrow{\sim} Z(\mathcal{G}, \underline{\mathcal{G}}, \underline{\mathbf{H}}^{\wedge_0})$, [32], see Theorems 1.2.1, 2.7.3 and that $eZ(\mathcal{G}, \underline{\mathcal{G}}, \underline{\mathbf{H}}^{\wedge_0})e$ is naturally identified with $\underline{e}\underline{\mathbf{H}}^{\wedge_0}\underline{e}$, [1, Lemma 3.1], where $\underline{e}$ is the trivial idempotent in $\mathbb{C}\underline{\mathcal{G}}$. So we get a $\mathbb{C}[\mathfrak{c}]$-linear isomorphism $e\mathbf{H}^{\wedge_b}e \cong \underline{e}\underline{\mathbf{H}}^{\wedge_0}\underline{e}$.

The algebra $\underline{\mathbf{H}}^{\wedge_0}$ can be decomposed into a completed tensor product as follows. There is a unique $\underline{\mathcal{G}}$-stable decomposition $V = V^{\underline{\mathcal{G}}} \oplus V^+$. Consider the Weyl algebra $\mathbb{A}_{V^{\underline{\mathcal{G}}},h}$ and the algebra $\underline{\mathbf{H}}^+$ that is the quotient of $TV^+[\mathfrak{c}]\#\underline{\mathcal{G}}$ by relations (6.5). It is clear that $\underline{\mathbf{H}} = \mathbb{A}_{V^{\underline{\mathcal{G}}},h} \otimes_{\mathbb{C}[h]} \underline{\mathbf{H}}^+$. So $\underline{\mathbf{H}}^{\wedge_0} = \mathbb{A}^{\wedge_0}_{V^{\underline{\mathcal{G}}},h} \widehat{\otimes}_{\mathbb{C}[[h]]} \underline{\mathbf{H}}^{+\wedge_0}$. Summarizing we get a $\mathbb{C}[\mathfrak{c}]$-linear isomorphism

$$(e\mathbf{H}e)^{\wedge_b} \cong \mathbb{A}^{\wedge_0}_{V^{\underline{\mathcal{G}}},h} \widehat{\otimes}_{\mathbb{C}[[h]]} \underline{e}\underline{\mathbf{H}}^{+\wedge_0}\underline{e}. \tag{6.6}$$

Let us proceed to the completions of quantum Hamiltonian reductions. Let V be a symplectic vector space and G be a reductive group acting on V by linear symplectomorphisms. Let $\mu : V \to \mathfrak{g}^*$ denote the moment map. Pick a point $b \in V/\!\!/\!\!/_0 G$. We are interested in the structure of $(\mathbb{A}_h(V)/\!\!/\!\!/G)^{\wedge_b}$. Let $x \in V$ be a point from a unique closed orbit in the fiber of b. Automatically, $\mu(x) = 0$. Set $H := Gx$ and let U denote the symplectic part of the normal space to Gx at x, i.e., $U := V/(T_xGx)^{\angle}$ (since $\mu(x) = 0$, we see that T_xGx is an isotropic subspace in V). Then U is a symplectic H-module. Set $\mathfrak{z} := (\mathfrak{g}^*)^G$, $\mathfrak{z}' := (\mathfrak{h}^*)^H$. We have a natural (restriction) map $\mathfrak{z} \to \mathfrak{z}'$.

Lemma 6.5.2. *We have a $\mathbb{C}[[\mathfrak{z}, h]]$-linear isomorphism*

$$(\mathbb{A}_h(V)/\!\!/\!\!/G)^{\wedge_b} \cong \mathbb{C}[[\mathfrak{z}, h]] \widehat{\otimes}_{\mathbb{C}[[\mathfrak{z}',h]]} (\mathbb{A}_h(U)/\!\!/\!\!/H)^{\wedge_0}.$$

Proof. Set $Y = (T^*G \times U)/\!\!/\!\!/_0 H$, where H acts on T^*G from the right. This is a Hamiltonian G-variety that is naturally isomorphic to the model variety $M_G(H, 0, U)$ from [29]. Let $y \in Y$ be a point corresponding to the orbit of $(1, 0, 0) \in T^*G \times U$. According to the main result of [29], the Hamiltonian formal G-schemes $V^{\wedge_{Gx}}$, $Y^{\wedge_{Gy}}$ are isomorphic. By Proposition 5.4.4, the quantization $(\mathcal{D}_h(G) \otimes_{\mathbb{C}[h]} \mathbb{A}_h(U))/\!\!/\!\!/_0 H$ of Y is even. Also it is graded and therefore canonical. It follows that the topological algebras $\mathbb{A}_h(V)^{\wedge_{Gx}}$, $[(\mathcal{D}_h(G) \otimes_{\mathbb{C}[h]} \mathbb{A}_h(U))/\!\!/\!\!/_0 H]^{\wedge_{Gy}}$ are G-equivariantly isomorphic. Furthermore, similarly to Theorem 3.3.4 in [28], we see that there is a G-equivariant $\mathbb{C}[h]$-linear isomorphism $\mathbb{A}_h(V)^{\wedge_{Gx}} \cong [(\mathcal{D}_h(G) \otimes_{\mathbb{C}[h]} \mathbb{A}_h(U))/\!\!/\!\!/_0 H]^{\wedge_{Gy}}$ intertwining the quantum comoment maps. It follows that the topological $\mathbb{C}[[\mathfrak{z}, h]]$-algebras $\mathbb{A}_h(V)^{\wedge_{Gx}}/\!\!/\!\!/G$, $[(\mathcal{D}_h(G) \otimes_{\mathbb{C}[h]} \mathbb{A}_h(U))/\!\!/\!\!/_0 H]^{\wedge_{Gy}}/\!\!/\!\!/G$ are isomorphic. Similarly to the proof of Proposition 5.4.4, we see that $\mathbb{A}_h(V)^{\wedge_{Gx}}/\!\!/\!\!/G \cong (\mathbb{A}_h(V)/\!\!/\!\!/G)^{\wedge_b}$ and

$$[(\mathcal{D}_h(G) \otimes_{\mathbb{C}[h]} \mathbb{A}_h(U))/\!\!/\!\!/_0 H]^{\wedge_{Gy}}/\!\!/\!\!/G \cong [(\mathcal{D}_h(G) \otimes_{\mathbb{C}[h]} \mathbb{A}_h(U))/\!\!/\!\!/_0 H/\!\!/\!\!/G]^{\wedge_0}.$$

So it remains to identify $(\mathcal{D}_h(G) \otimes_{\mathbb{C}[h]} \mathbb{A}_h(U))/\!\!/\!\!/_0 H/\!\!/\!\!/G$ with $\mathbb{C}[\mathfrak{z}, h] \otimes_{\mathbb{C}[\mathfrak{z}',h]} \mathbb{A}_h(U)/\!\!/\!\!/H$.

First of all, we note that, by the definition of a quantum Hamiltonian reduction, $(\mathcal{D}_h(G)\otimes_{\mathbb{C}[h]}\mathbb{A}_h(U))/\!/\!/_0 H/\!/\!/ G \cong (\mathcal{D}_h(G)\otimes_{\mathbb{C}[h]}\mathbb{A}_h(U))/\!/\!/ G/\!/\!/_0 H$. But it is easy to see that $(\mathcal{D}_h(G)\otimes_{\mathbb{C}[h]}\mathbb{A}_h(U))/\!/\!/ G$ is naturally identified with $\mathcal{D}_h(G)/\!/\!/ G\otimes_{\mathbb{C}[h]}\mathbb{A}_h(U) = (\mathbb{C}[\mathfrak{z}][h]\otimes_{\mathbb{C}[h]}\mathbb{A}_h(U))/\!/\!/_0 H$, where the quantum comoment map $\mathfrak{h}\to\mathbb{C}[\mathfrak{z}][h]\otimes_{\mathbb{C}[h]}\mathbb{A}_h(U)$ has the form $\xi\mapsto -\rho(\xi)\otimes 1+1\otimes\Phi(\xi)$, ρ being the natural projection $\mathfrak{h}\to\mathfrak{z}^*$ and $\Phi:\mathfrak{h}\to\mathbb{A}_h(U)$ being the quantum comoment map for the action of H on U. This implies that $(\mathcal{D}_h(G)\otimes_{\mathbb{C}[h]}\mathbb{A}_h(U))/\!/\!/_0 H/\!/\!/ G$ and $\mathbb{C}[\mathfrak{z},h]\otimes_{\mathbb{C}[\mathfrak{z}',h]}(\mathbb{A}_h(U)/\!/\!/ H)$ are $\mathbb{C}[\mathfrak{z},h]$-linearly isomorphic. □

Now let us specify what choices of $b\in\mathbb{C}^{2n}/\Gamma_n\cong\mathcal{M}_0(\mathrm{D}Q,\mathbf{v},\mathbf{d})$ we need.

The stratification of $\mathbb{C}^{2n}/\Gamma_n$ by the symplectic leaves is the same as the stabilizer stratification. In particular, there are two symplectic leaves of codimension 2: the first one, $\mathcal{L}_\Gamma$ corresponds to the first copy of Γ inside $\Gamma^n\subset\Gamma_n$, while the second one, $\mathcal{L}_{\mathrm{sym}}$,—to the subgroup of order 2 generated by the transposition $s_{12}\in S_n\subset\Gamma_n$. In other words, $\overline{\mathcal{L}}_\Gamma,\overline{\mathcal{L}}_{\mathrm{sym}}$ are the images of $\{0,u_2,\dots,u_n\},\{u_1,u_1,u_3,\dots,u_n\}$ in $\mathbb{C}^{2n}/\Gamma_n$.

We will need points $b^1\in\mathcal{L}_\Gamma, b^2\in\mathcal{L}_{\mathrm{sym}}$. The algebra $\underline{\mathbf{H}}^{+1}$ constructed from b^1 is just $\mathbb{C}[k]\otimes\mathbf{H}(\Gamma)$, where $\mathbf{H}(\Gamma)$ is the SRA constructed from Γ (over $\mathbb{C}[h,c_1,\dots,c_r]$). Similarly, the algebra $\underline{\mathbf{H}}^{+2}$ constructed from b^2 is $\mathbb{C}[c_1,\dots,c_r]\otimes\mathbf{H}(S_2)$, where the algebra $\mathbf{H}(S_2)$ is constructed from S_2 (over $\mathbb{C}[k]$).

Now let us consider the Hamiltonian reduction side. Recall that we reduce the space $V:=R(\mathrm{D}Q^{\mathbf{d}},\mathbf{v}^{\mathbf{d}})$ by the action of $G:=\mathrm{GL}(\mathbf{v})$. Pick an element $x\in R(\mathrm{D}Q,\mathbf{v})$ that is decomposed into the sum $\bigoplus_{i=1}^{n-1}x_i\oplus 0$, where x_i are generic non-isomorphic elements of $R(\mathrm{D}Q,\delta)$ mapping to 0 under the moment map. Since the representation of x is semisimple, its G-orbit is closed. The stabilizer $H:=G_x$ is naturally identified with $\mathrm{GL}(\delta)\times\mathbb{C}^{\times(n-1)}$. The symplectic part U of the slice module $T_xV/\mathfrak{g}x$ is identified with $R(\mathrm{D}Q^{\mathbf{d}},\delta^{\mathbf{d}})\oplus\mathbb{C}^{2n-2}\oplus\mathbb{C}^{2n-2}$, where the stabilizer H acts via the projection $H\twoheadrightarrow\mathrm{GL}(\delta)$ on the first summand, trivially on the second one, and via the projection $H\twoheadrightarrow\mathbb{C}^{\times(n-1)}$ on the third one.

It is easy to see that x maps into $\mathcal{L}_\Gamma$ under the quotient map (in fact, any point of $\mathcal{L}_\Gamma$ is obtained in this way but we will not need this fact). So we may assume that b^1 coincides with the image of x.

Now let us study the structure of the completion $(\mathbb{A}_h(V)/\!/\!/ G)^{\wedge_b}$ in more detail. By Lemma 6.5.2, $(\mathbb{A}_h(V)/\!/\!/ G)^{\wedge_b}\cong\mathbb{C}[[\mathfrak{z},h]]\widehat{\otimes}_{\mathbb{C}[[\mathfrak{z}',h]]}(\mathbb{A}_h(U)/\!/\!/ H)^{\wedge_0}$. Recall the basis $\check{\epsilon}_i, i=0,\dots,r$ in $\mathfrak{z}$. Let $\check{\epsilon}'_i, i=0,\dots,r$ be the analogous basis in $\mathfrak{gl}(\delta)/[\mathfrak{gl}(\delta),\mathfrak{gl}(\delta)]\subset\mathfrak{z}'$. Also let $\lambda_1,\dots,\lambda_{n-1}$ be the elements in $\mathfrak{z}'$ corresponding to the $n-1$ copies of $\mathbb{C}^\times$. The map $\mathfrak{z}\to\mathfrak{z}'$ is given by $\epsilon_i\mapsto\epsilon'_i+\delta_i\sum_{j=1}^{n-1}\lambda_j$ and, in particular, is an embedding.

Lemma 6.5.3. *There is a $\mathbb{C}[\mathfrak{z}]$-linear isomorphism $\mathbb{C}[\mathfrak{z}]\otimes_{\mathbb{C}[\mathfrak{z}_0]}\mathbb{A}_{\mathfrak{z}_0}(\mathrm{D}Q,\delta,\mathbf{d})_h\cong\mathbb{C}[\mathfrak{z},h]\otimes_{\mathbb{C}[\mathfrak{z}',h]}\mathbb{A}_h(U)/\!/\!/ H$ of graded algebras.*

Proof. Let $\mathfrak{z}'_0\subset\mathfrak{z}'$ be defined analogously to $\mathfrak{z}_0\subset\mathfrak{z}$. The spaces $\mathfrak{z}'_0$ and $\mathfrak{z}_0$ are naturally identified. We need to show that $\mathbb{A}_{\mathfrak{z}_0}(\mathrm{D}Q,\delta,\mathbf{d})_h\cong\mathbb{A}_h(U)/\!/\!/_{\mathfrak{z}_0}H$. This follows from the observation that $U/\!/\!/_{\mathfrak{z}_0}Z(H)^\circ$ is naturally identified with $R(\mathrm{D}Q,\mathbf{v})$. □

So we see that $\mathbb{A}(\mathrm{D}Q,\mathbf{v},\mathbf{d})_h^{\wedge_{b^1}}\cong\mathbb{A}_{2n-2,h}^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[h]]}(\mathbb{C}[\mathfrak{z}]\otimes_{\mathbb{C}[\mathfrak{z}_0]}\mathbb{A}_{\mathfrak{z}_0}(\mathrm{D}Q,\delta,\mathbf{d})_h^{\wedge_0})$.

Now let us proceed to the second leaf. Pick an element $x\in R(\mathrm{D}Q,n\delta)$ that is decomposed into the sum $\bigoplus_{i=1}^{n-2}x_i\oplus x_{n-1}^{\oplus 2}$, where $x_i, i=1,\dots,n-1$, are generic non-isomorphic elements of $R(\mathrm{D}Q,\delta)$ mapping to 0 under the moment map. The stabilizer $H:=G_x$ is naturally identified with $\mathrm{GL}(2) \times \mathbb{C}^{\times(n-2)}$. The symplectic part U of the slice module $T_xV/\mathfrak{g}x$ is identified with

$$(\mathrm{End}(\mathbb{C}^2) \oplus \mathrm{End}(\mathbb{C}^2)^* \oplus \mathbb{C}^2 \oplus \mathbb{C}^{2*}) \oplus (\mathbb{C}^{n-2} \oplus \mathbb{C}^{n-2*}) \oplus \mathbb{C}^{2n-2}$$

(where $\mathbb{C}^2$ should be viewed as the multiplicity space for the representation x_{n-1}). The stabilizer H acts via the projection $H \twoheadrightarrow \mathrm{GL}(2)$ on the first summand (denote it by U_1), via the projection $H \twoheadrightarrow \mathbb{C}^{\times(n-2)}$ on the second one, and trivially on the third one.

Again, it is easy to see that x maps into $\mathcal{L}_{\mathrm{sym}}$ under the quotient map and hence we may assume that b^2 coincides with the image of x. Set $\mathfrak{z}_1 := \mathfrak{gl}(2)^{*\mathrm{GL}(2)}$. We have a basis $\lambda, \lambda_1, \ldots, \lambda_{n-2}$ in $\mathfrak{z}'$, where λ corresponds to the determinant of GL(2) and $\lambda_1, \ldots, \lambda_{n-2}$ correspond to the $n-2$ copies of $\mathbb{C}^\times$. A natural map $\mathfrak{z} \to \mathfrak{z}'$ is given by $\epsilon_i \mapsto \delta_i(\lambda + \sum_{i=1}^{n-2}\lambda_i)$. Again, $\mathbb{C}[\mathfrak{z}] \otimes_{\mathbb{C}[\mathfrak{z}']} \mathbb{A}_h(U)/\!\!/\!\!/H = \mathbb{C}[\mathfrak{z}] \otimes_{\mathbb{C}[\mathfrak{z}_1]} \mathbb{A}_h(U_1)/\!\!/\!\!/\mathrm{GL}(2)$, where the map $\mathfrak{z} \to \mathfrak{z}_1$ is given by $\chi \mapsto (\delta \cdot \chi)\lambda$.

6.6. Reduction to the Kleinian case

The following proposition constitutes a reduction procedure.

Proposition 6.6.1. *There are isomorphisms* $\Upsilon_1 : \mathbb{C}[k] \otimes \underline{e}_1\underline{\mathbf{H}}^{+1}\underline{e}_1 \to \mathbb{A}(\mathrm{D}Q, \delta, \mathbf{d})_h$ *and* $\Upsilon_2 : \mathbb{C}[c_1, \ldots, c_r] \otimes \underline{e}_2\mathbf{H}(S_2)\underline{e}_2 \to \mathbb{C}[\mathfrak{z}] \otimes_{\mathbb{C}[\mathfrak{z}_1]} \mathbb{A}_h(U_1)$ *such that* $\Upsilon_1|_{\mathfrak{c}^*}, \Upsilon_2|_{\mathfrak{c}^*} = \Upsilon|_{\mathfrak{c}^*}$.

Proof. Let us prove the existence of Υ_1.

Step 1.

Pick a point b_1 as in the previous subsection. The isomorphism $\Upsilon : e\mathbf{H}e \to \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ induces the isomorphism of completions $\Upsilon : e\mathbf{H}^{\wedge_{b_1}}e \to \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h^{\wedge_{b_1}}$. As we have seen in the previous subsection we have isomorphisms

$$e\mathbf{H}^{\wedge_{b_1}}e \cong \mathbb{A}_{2n-2,h}^{\wedge 0}\widehat{\otimes}_{\mathbb{C}[[h]]}\mathbb{C}[[k]]\widehat{\otimes}\underline{e}_1\underline{\mathbf{H}}^{+1\,\wedge 0}\underline{e}_1, \tag{6.7}$$

$$\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h^{\wedge_{b_1}} \cong \mathbb{A}_{2n-2,h}^{\wedge 0}\widehat{\otimes}_{\mathbb{C}[[h]]}\mathbb{A}(\mathrm{D}Q, \delta, \mathbf{d})_h^{\wedge 0}. \tag{6.8}$$

So we get an isomorphism $\Upsilon^{\wedge_{b_1}}$ of the right hand sides. We are going to show that there is an isomorphism

$$\Upsilon_1^{\wedge} : \mathbb{C}[[k]]\widehat{\otimes}\underline{e}_1(\underline{\mathbf{H}}^{+1})^{\wedge 0}\underline{e}_1 \to \mathbb{A}(\mathrm{D}Q, \delta, \mathbf{d})_h^{\wedge 0} \tag{6.9}$$

that coincides with $\Upsilon^{\wedge_{b_1}}$ on $\mathfrak{c}^*$. This follows from the next step.

Step 2. Let Z be a commutative algebra, and let $\mathcal{A}_h$ be a $Z[[h]]$-algebra such that:

- $\mathcal{A}_h$ is flat over $\mathbb{C}[h]$.
- $\mathcal{A}_h$ is complete in the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is a maximal ideal of $\mathcal{A}_h$ containing h and such that $\mathcal{A}_h/\mathfrak{m} = \mathbb{C}$.
- $\mathcal{A}_h/h\mathcal{A}_h$ is commutative and Noetherian.

Let $\iota_1, \iota_2 : \mathbb{A}_{2n-2,h}^{\wedge 0} \to \mathcal{A}_h$ be $\mathbb{C}[[h]]$-linear homomorphisms such that the images of the maximal ideal of $\mathbb{A}_{2n-2,h}^{\wedge 0}$ lie in $\mathfrak{m}$. There is a $Z[h]$-linear automorphism A of $\mathcal{A}_h$ such that $\iota_1 = A \circ \iota_2$.

Let us prove this claim. Using an easy induction, we reduce the proof to the case when $n = 2$. So we have elements $u_i, v_i \in \mathcal{A}_h$, $i = 1, 2$, with $[u_i, v_i] = h$. First of all, we notice that $\mathcal{A}_h$ is decomposed into the completed tensor product

$$\mathbb{A}_h(U)^{\wedge 0}\widehat{\otimes}_{\mathbb{C}[[h]]}\underline{\mathcal{A}}_h, \tag{6.10}$$

where $U \subset \mathcal{A}_h$ with $u_1, v_1 \in U$ and $\underline{\mathcal{A}}_h$ is some subalgebra of $\mathcal{A}_h$ such that the maximal ideal in $\underline{\mathcal{A}}_h/h\underline{\mathcal{A}}_h$ is Poisson, compare with [28, Section 7.2]. In particular, for $a, b \in \underline{\mathcal{A}}_h$ we have $[a, b] \in h\mathfrak{m}$. From here it is easy to see that $u_2, v_2 \in U + \mathfrak{m}^2$ and modulo $\mathfrak{m}^2$ the elements u_2, v_2 are linearly independent. So applying the automorphism of $\mathcal{A}_h$ induced by a linear symplectomorphism of U, we may assume that $u_1 \equiv u_2, v_1 \equiv v_2 \mod \mathfrak{m}^2$.

Now we claim that there is an element $a \in \mathfrak{m}^3$ such that

$$\exp\left(\frac{1}{h}\mathrm{ad}(a)\right) u_1 = u_2, \exp\left(\frac{1}{h}\mathrm{ad}(a)\right) v_1 = v_2$$

(the series in the left hand sides automatically converges because $a \in \mathfrak{m}^3$). First of all, from the fact that u_1 is a generator in the Weyl algebra and the decomposition (6.10), we can deduce that the map $\frac{1}{h}\mathrm{ad}(u_1) : \mathfrak{m}^{i+1} \to \mathfrak{m}^i$ is surjective. So if $u_2 - u_1 \in \mathfrak{m}^i$, then there is $a_1 \in \mathfrak{m}^{i+1}$ such that $u_2 - u_1 = \frac{1}{h}[a_1, u_1]$. Now $u_2 - \exp(\frac{1}{h}a_1)u_1 \in \mathfrak{m}^{i+1}$. Thanks to the Campbell–Hausdorff theorem, we can use an induction to prove that there is $a' \in \mathfrak{m}^3$ with $\exp(\frac{1}{h}\mathrm{ad}(a'))u_1 = u_2$. So we may assume that there is $u_1 = u_2$.

Now we need to show that there is a'' with $[a'', u_1] = 0$ and $\exp(\frac{1}{h}\mathrm{ad}(a''))v_1 = v_2$. Let us note that $[u_1, v_2 - v_1] = 0$. In other words, $v_2 - v_1$ is expressed as a (non-commutative) power series of u_1, the elements of the skew-orthogonal complement to u_1, v_1 in U, and the elements of $\underline{\mathcal{A}}_h$. Now we can find a'' using the argument of the previous paragraph.

Step 3. So we have proved the existence of an isomorphism in (6.9). Now we are going to show that there is an isomorphism

$$\Upsilon_1 : \mathbb{C}[k] \otimes \underline{e}_1\underline{\mathbf{H}}^{+1}\underline{e}_1 \to \mathbb{A}(\mathrm{D}Q, \delta, \mathbf{d})_h \tag{6.11}$$

that coincides with $\Upsilon_1^\wedge$ on $\mathfrak{c}^*$. On both algebras in (6.9) there are $\mathbb{C}^\times$-actions such that the algebras of $\mathbb{C}^\times$-finite vectors coincide with the algebras in (6.11).

So we only need to prove the following claim:

Let $(\mathbb{C}[k] \otimes \underline{e}_1\underline{\mathbf{H}}^{+1}\underline{e}_1)^{\wedge_0}$ be equipped with a $\mathbb{C}^\times$-action such that

(i) $t \cdot \alpha = t^2\alpha$ for any $\alpha \in \mathfrak{c}, t \in \mathbb{C}^\times$
(ii) and the induced action on $\mathbb{C}[\mathbb{C}^2]^{\Gamma\wedge_0} = (\mathbb{C}[k] \otimes \underline{e}_1\underline{\mathbf{H}}^{+1}\underline{e}_1)^{\wedge_0}/(\mathfrak{c})$ is the standard one.

Then the algebra of $\mathbb{C}^\times$-finite vectors in $(\mathbb{C}[k] \otimes \underline{e}_1\underline{\mathbf{H}}^{+1}\underline{e}_1)^{\wedge_0}$ is isomorphic to $\mathbb{C}[k] \otimes \underline{e}_1\underline{\mathbf{H}}^{+1}\underline{e}_1$. Moreover, the two gradings on the last algebra (the standard one, and one induced by the $\mathbb{C}^\times$-action) differ by a compatible inner grading.

We say that a grading of $\mathbb{C}[k] \otimes \underline{e}_1\underline{\mathbf{H}}^{1+}\underline{e}_1$ is compatible and inner if it is the grading by eigenvalues of a derivation of the form $\frac{1}{h}\mathrm{ad}(a)$, where a is an element of degree 2 with respect to the standard grading on this algebra.

Let E, E_{st} be the derivations induced by the $\mathbb{C}^\times$-action under consideration and by the standard $\mathbb{C}^\times$-action. Then $E - E_{st}$ is a $\mathbb{C}[\mathfrak{c}]$-linear derivation of $\mathbb{C}[k] \otimes \underline{e}_1\underline{\mathbf{H}}^{1+\wedge_0}\underline{e}_1$. Every Poisson derivation of $(\mathbb{C}[\mathbb{C}^2]^\Gamma)^{\wedge_0} = \mathbb{C}[[k]]\widehat{\otimes}\underline{e}_1(\underline{\mathbf{H}}^{1+})^{\wedge_0}\underline{e}_1/(\mathfrak{c})$ is inner. It follows that $E - E_{st} = \frac{1}{h}\mathrm{ad}(a)$ for some $a \in \mathbb{C}[[k]]\widehat{\otimes}\underline{e}_1\underline{\mathbf{H}}^{1+\wedge_0}\underline{e}_1$, compare with Lemma 2.11.2 in [32]. The algebra $\mathbb{C}[k] \otimes \underline{e}_1\underline{\mathbf{H}}^{1+}\underline{e}_1$ has no elements of degree 1, so we can assume that $a = \sum_{i\geqslant 2} a_i$, where a_i has degree i with respect to the standard grading.

We claim that a_2 is central. Indeed, when Γ is of type D or E, the algebra $\mathbb{C}[\mathbb{C}^2]^\Gamma$ has no nonzero elements of degree 2. Now consider the case when $\Gamma = \mathbb{Z}_n$. Choose an eigenbasis $x, y \in \mathbb{C}^{2*}$ for Γ. The algebra $\mathbb{C}[\mathbb{C}^2]^\Gamma$ is generated by x^n, y^n, xy. The subspace of elements of degree 2 in $\mathbb{C}[\mathbb{C}^2]^\Gamma$ is one-dimensional and is spanned by xy. Therefore $a_2 = cxy$ for some

$c \in \mathbb{C}$. Consider the actions of E, E_{st} on the (three-dimensional) cotangent space to 0 in $\mathbb{C}^2/\Gamma$. The eigenvalues of these operators are $n, n, 2$ (because of (ii)) and for both the image of xy has eigenvalue 2. On the other hand, the eigenvalues of $E_{st} - \{cxy, \cdot\}$ equal $n+c, n-c, 2$ and again xy corresponds to the eigenvalue 2. But these eigenvalues coincide with those for E. So we see that $c = 0$ and so a_2 is central.

So $E = E_{st} + \sum_{i \geqslant 3} \frac{1}{h}\mathrm{ad}(a_i)$. Since $E_{st}(\frac{1}{h}a_i) = (i-2)\frac{1}{h}a_i$, it follows that there is $b \in \mathbb{C}[[k]]\widehat{\otimes}\underline{e}_1\underline{\mathbf{H}}^{+1\wedge_0}\underline{e}_1$ with $E = \exp(-\frac{1}{h}\mathrm{ad}b)E_{st}\exp(\frac{1}{h}\mathrm{ad}b)$. This implies the claim of this step and completes the proof of the existence of Υ_1.

The proof that Υ_2 exists is completely analogous. □

The existence of Υ_1 together with Proposition 6.4.5 implies that Υ maps $h, c_1, \dots, c_r$ to $\widehat{\mathfrak{z}}_0^*$ and there is $w \in W_{\mathrm{fin}}$ such that $\Upsilon(c_i) = w\upsilon(c_i)$. Similarly the existence of Υ_2 implies that $\Upsilon(k) = \pm\upsilon(k)$.

Define an action of $W_{\mathrm{fin}} \times \mathbb{Z}/2\mathbb{Z}$ on $\widehat{\mathfrak{z}}$ in the following way: $w \in W_{\mathrm{fin}}$ fixes δ and acts on $\widehat{\mathfrak{z}}_0$ as before, while a non-trivial element $\varsigma \in \mathbb{Z}/2\mathbb{Z}$ maps δ to $-\delta$ and is the identity on $\widehat{\mathfrak{z}}_0$. From the previous paragraph we see that $\Upsilon|_{\mathfrak{c}^*}$ coincides with $g\upsilon$ for some $g \in W_{\mathrm{fin}} \times \mathbb{Z}/2\mathbb{Z}$.

So it remains to check that the group $\mathfrak{A}$ established in Section 6.4 coincides with $W_{\mathrm{fin}} \times \mathbb{Z}/2\mathbb{Z}$.

First of all, let us show that $\mathfrak{A} \subset W_{\mathrm{fin}} \times \mathbb{Z}/2\mathbb{Z}$. Any element $a \in \mathfrak{A}$ acts on $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})^{\wedge_{b_1}}$ preserving $\widehat{\mathfrak{z}}^*$ and coinciding with the identity modulo $\widehat{\mathfrak{z}}^*$. Following the proof of Proposition 6.6.1, we see that there is an automorphism of $\mathbb{A}(\mathrm{D}Q, \delta, \mathbf{d})_h$ that coincides with a on $\widehat{\mathfrak{z}}^*$. Thanks to Proposition 6.4.5, we see that a preserves $\widehat{\mathfrak{z}}_0$ and acts on this space as an element of W_{fin}. Similarly, we can consider, b_2 instead of b_1 and get that a preserves $\mathbb{C}\delta$ and acts on this line by ± 1. This proves the inclusion $\mathfrak{A} \subset W_{\mathrm{fin}} \times \mathbb{Z}/2\mathbb{Z}$.

Let us prove now that $\mathfrak{A} = W_{\mathrm{fin}} \times \mathbb{Z}/2\mathbb{Z}$. Recall the automorphism $\nu \in \mathfrak{A}$. It is enough to check that $\nu \notin W_{\mathrm{fin}}$.

Transport the action of $\mathfrak{A}$ to $e\mathbf{H}e$ by means of Υ, and the action of $W_{\mathrm{fin}} \times \mathbb{Z}/2\mathbb{Z}$ by means of υ. Since $\Upsilon|_{\mathfrak{c}^*} = g \circ \upsilon$ for some $g \in W_{\mathrm{fin}} \times \mathbb{Z}/2\mathbb{Z}$, we see that the image of $\mathfrak{A}$ in $\mathrm{GL}(\mathfrak{c}^*)$ is still contained in $W_{\mathrm{fin}} \times \mathbb{Z}/2\mathbb{Z}$. From the description of $\Upsilon^{-1} \circ \nu \circ \Upsilon$ given in the end of Section 6.4 we see that this automorphism does not lie in W_{fin}. Hence our claim.

This completes the proof of Theorem 6.2.1.

Remark 6.6.2. It seems to be impossible to produce explicit formulas for the isomorphism Υ using our approach. However, there is a uniqueness property for Υ: this is a unique isomorphism satisfying the conditions of Theorem 6.2.1. This follows from Lemma 6.4.4.

6.7. S_l-invariance property for the Euler element

Here we consider the SRA $\mathbf{H}$ for the group Γ_n, where $\Gamma = \mathbb{Z}/l\mathbb{Z}$. Below $\mathfrak{h} = \mathbb{C}^n$ is the reflection representation of Γ_n. Recall that the Euler element $\mathbf{eu} \in \mathbf{H}$ is defined by

$$\mathbf{eu} = \sum_{i=1}^{n} x_i y_i + \dim \mathfrak{h}/2 + \sum_{s \in S} \frac{c_s}{1-\lambda_s} s,$$

where $x_1, \dots, x_n$ is a basis of $\mathfrak{h}^*$, $y_1, \dots, y_n$ is the dual basis of $\mathfrak{h}$, $c_s = c_i$ for $c \in S_i, i = 1, \dots, r$, $c_s = k$ for $s \in S_{\mathrm{sym}}$, and λ_s is the only non-unit eigenvalue of s in $\mathfrak{h}^*$. Our formula looks different from a usual one (see, for instance, [1]) because our parameters c_i's differ from ones used in the standard presentation of the rational Cherednik algebras (by the factor of -2).

We remark that **eu** is independent of the choice of $x_1, \dots, x_n$. The reason to consider **eu** is that $[\mathbf{eu}, x] = x, [\mathbf{eu}, y] = -y$ for all $x \in \mathfrak{h}^*, y \in \mathfrak{h}$, where we consider $\mathfrak{h}, \mathfrak{h}^*$ as Lagrangian subspaces in $\mathcal{V} = \mathbb{C}^{2n}$. Also **eu** is Γ_n-invariant.

Recall that the group W_{fin} acts on $e\widetilde{\mathbf{H}}e \cong \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ as was explained in Section 6.4. In our case $W_{\text{fin}} = S_l$. The main result of this appendix to be used in a subsequent paper is the following proposition.

Proposition 6.7.1. *The element* $\mathbf{eu}^{\text{sph}} := e\mathbf{eu} \in e\widetilde{\mathbf{H}}e$ *is* S_l*-invariant.*

One of the corollaries of this proposition is that the S_l-action commutes with the $\mathbb{C}^\times$-action on $e\widetilde{\mathbf{H}}e$ induced from the action by algebra automorphisms on $\widetilde{\mathbf{H}}$ given by $t \cdot x = tx, t \cdot y = t^{-1}y, t \cdot w = w, t \cdot h = h, t \cdot c_i = c_i$, where $x \in \mathfrak{h}^*, y \in \mathfrak{h}, w \in W$. However, this can also be easily deduced from Lemma 6.4.4: for each $a \in \mathfrak{A}$ the automorphism tat^{-1} also lies in $\mathfrak{A}$ and the restrictions of a and tat^{-1} to $\widehat{\mathfrak{z}}^*$ (or to $\widetilde{\mathfrak{c}}^*$) coincide.

Our proof is basically in two steps. First we prove the invariance for a similar element $\tilde{\mathbf{eu}} \in \mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ and then relate $e\mathbf{eu}$ with $\tilde{\mathbf{eu}}$.

Let us define $\tilde{\mathbf{eu}}$. In our case the quiver Q is the affine Dynkin quiver of type A_{l-1}. Consider the action of $\mathbb{C}^\times$ on V given by $t \cdot ((A_i), (B_i), \Gamma_0, \Delta_0) := ((tA_i), (t^{-1}B_i), \Gamma_0, \Delta_0)$. This action commutes with G and preserves the symplectic form on V. So it gives rise to a quantum comoment map $\mathbb{C} \to \mathbb{A}_h(V)$ sending 1 to the *symmetrized Euler element* $\tilde{\mathbf{eu}} \in \mathbb{A}_h(V)^G$. Abusing the notation, by $\tilde{\mathbf{eu}}$ we also denote the image of $\tilde{\mathbf{eu}}$ in $\mathbb{A}_h(V)/\!\!/\!\!/G$. Recall that S_l acts on $\mathbb{A}_h(V)/\!\!/\!\!/G$ by $\mathbb{C}[h]$-linear automorphisms.

Lemma 6.7.2. *The element* $\tilde{\mathbf{eu}} \in \mathbb{A}_h(V)/\!\!/\!\!/G$ *is* S_l*-invariant.*

Proof. The Hamiltonian $\mathbb{C}^\times$-action on V considered in the previous paragraph induces Hamiltonian actions on all reductions $\mathcal{M}^\theta(\mathrm{D}Q, \mathbf{v}, \mathbf{d})$. Inspecting the Maffei construction recalled in Section 6.4, we see that the isomorphisms constructed there are equivariant with respect to the Hamiltonian $\mathbb{C}^\times$-actions. It follows that S_l acts on $\mathbb{A}_h(V)/\!\!/\!\!/G$ by $\mathbb{C}^\times$-equivariant automorphisms. Now all $\sigma(\tilde{\mathbf{eu}}), \sigma \in S_l$, are elements of degree 2 defining quantum comoment maps for the $\mathbb{C}^\times$-action. So they all differ from one another by elements of $\widehat{\mathfrak{z}}^*$.

It is enough to prove that $\sigma(\tilde{\mathbf{eu}}) - \tilde{\mathbf{eu}} \in \mathbb{C}h$ because h is S_l-invariant. This will follow if we check that the image of $\tilde{\mathbf{eu}}$ (also denoted by $\tilde{\mathbf{eu}}$) in $\mathbb{C}[\Lambda(\mathbf{v}, \mathbf{d})]^{\mathrm{GL}(\mathbf{v})}$ is S_l-stable. We may assume that the affine quiver Q is oriented counterclockwise and the framing is attached to vertex 0. Then $\tilde{\mathbf{eu}} = \sum_{i=0}^{l-1} \operatorname{tr}(A_i B_i)$. The condition that the point $x = ((A_i), (B_i), \Gamma_0, \Delta_0)$ lies in $\Lambda_\chi(\mathbf{v}, \mathbf{d})$ is $B_0 A_0 - A_{l-1} B_{l-1} + \Gamma_0 \Delta_0 = \chi_0$ and $B_i A_i - A_{i-1} B_{i-1} = \chi_i$ for $i \neq 0$. Let us prove that $s(\tilde{\mathbf{eu}}) = \tilde{\mathbf{eu}}$ for the reflection s constructed from the vertex $i \neq 0$. Recall the variety Z introduced in Section 6.4 and the projections $\pi, \pi' : Z \to \Lambda(\mathbf{v}, \mathbf{d})$. We remark that in the definition of Z we required i to be the source. With the orientation we have chosen this is not so and to fix this we need to replace the pair (A_{i-1}, B_{i-1}) with $(-B_{i-1}, A_{i-1})$. Condition (3) in the definition of Z then implies (in the notation of Section 6.4) that $B'_i A'_i = B_i A_i - \chi_i$, $A'_{i-1} B'_{i-1} = A_{i-1} B_{i-1} + \chi_i$. Also we remark that for $j \neq i, i-1$ we have $B'_j A'_j = B_j A_j$. So we have $\sum_i \operatorname{tr}(B'_i A'_i) = \sum_i \operatorname{tr}(B_i A_i)$. But this precisely means that $S^*(\tilde{\mathbf{eu}}) = \tilde{\mathbf{eu}}$. Since this equality holds for all generators of S_l, we see that $\tilde{\mathbf{eu}}$ is S_l-stable. $\square$

Identify $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h$ and $e\widetilde{\mathbf{H}}e$ by means of the isomorphism Υ from Theorem 6.2.1. Again, for any $\sigma \in S_l$ the operators $[\sigma(\mathbf{eu}^{\text{sph}}), \cdot], [\mathbf{eu}^{\text{sph}}, \cdot]$ are the same because S_l commutes with the $\mathbb{C}^\times$-action introduced above in this subsection. Since $\mathbf{eu}^{\text{sph}}$ is homogeneous of degree 2, this implies $\sigma(\mathbf{eu}^{\text{sph}}) - \mathbf{eu}^{\text{sph}} \in \widehat{\mathfrak{z}}^*$. From the explicit form of the correspondence between the parameters (see Theorem 6.2.1) we see that h, k are S_l-invariant. So it is enough to show that $\tilde{\mathbf{eu}}$ lies in the linear span of $\mathbf{eu}^{\mathrm{sph}}$ and 1 modulo the ideal generated by h, k.

The algebra $e\tilde{\mathbf{H}}e/(h,k)$ is just $(\underline{e}\underline{\mathbf{H}}_0^+\underline{e}^{\otimes n})^{S_n}$, where $\underline{\mathbf{H}}_0^+$ is the SRA for Γ at $h=0$, and $\underline{e}$ is the trivial idempotent in this algebra. On the other hand, $\mathbb{A}(\mathrm{D}Q, \mathbf{v}, \mathbf{d})_h/(k,h)$ is the algebra of functions on the reduced scheme $R(\mathrm{D}Q, n\delta)/\!\!/\!\!/G'$. Here G' is the quotient of G by the subgroup $\{x1, x \in \mathbb{C}^\times\}$. The induced isomorphism

$$(\underline{e}\underline{\mathbf{H}}_0^+\underline{e}^{\otimes n})^{S_n} \cong \mathbb{C}[R(\mathrm{D}Q, n\delta)/\!\!/\!\!/G']$$

can be described as follows.

First, consider the case $n=1$. Here $\mathrm{Spec}(\underline{e}\underline{\mathbf{H}}_0^+\underline{e})$ is the moduli space parameterizing semisimple $\mathbf{H}_0^+$-modules that are Γ-equivariantly isomorphic to $\mathbb{C}\Gamma$. To a module we assign the pair of operators $x \in \mathfrak{h}^*, y \in \mathfrak{h}$. This assignment is known to define an isomorphism $\mathrm{Spec}(\underline{e}\underline{\mathbf{H}}_0^+\underline{e}) \to R(\mathrm{D}Q, \delta)/\!\!/\!\!/G'$ coinciding with the isomorphism we need. The element $\mathbf{eu}^{\mathrm{sph}}$ corresponds to $xy + \frac{1}{2} + \sum_{s\in S}\frac{c_s}{1-\lambda_s}s$, where x, y are basis elements in $\mathfrak{h}^*, \mathfrak{h}$ subject to $\langle x, y\rangle = 1$ and viewed as operators on $\mathbb{C}\Gamma$. These operators are subject to the relation $[y,x] = \sum_{s\in S} c_s s$, where S is just $\Gamma \setminus \{1\}$. In other words, x, y are just ℓ-tuples $(x^0, \ldots, x^{\ell-1})$, $(y^0, \ldots, y^{\ell-1})$ (that represent maps between isotypic components of $\mathbb{C}\mathbb{Z}/\ell\mathbb{Z}$) subject to

$$x^i y^i - x^{i-1}y^{i-1} = -\sum_{j=1}^{\ell-1} c_j \eta^{ji}, \tag{6.12}$$

where η is the primitive ℓth root of 1 defining the action of Γ on $\mathfrak{h}^*$. The element $\mathbf{eu}$ is central and so acts by the same scalar on all isotypic components. The scalar equals $\frac{1}{\ell}\sum_{i=0}^{\ell-1}(x^i y^i + \frac{1}{2} + \sum_{j=1}^{\ell-1}\frac{c_j}{1-\eta^j}\eta^{ji})$. Changing the summation order, we see that the last sum is just $\frac{1}{\ell}\sum_{i=0}^{\ell-1} x^i y^i + \frac{1}{2} = \frac{1}{\ell}\tilde{\mathbf{eu}} + \frac{1}{2}$.

Now let us consider the case of arbitrary n. Here the isomorphism $e\mathbf{H}e/(k,h) \cong \mathbb{C}[R(\mathrm{D}Q, n\delta)/\!\!/\!\!/G']$ is given as follows. Recall that

$$e\mathbf{H}e/(h,k) = (\underline{e}\underline{\mathbf{H}}_0^+\underline{e}^{\otimes n})^{S_n} = (\mathbb{C}[R(\mathrm{D}Q, \delta)/\!\!/\!\!/G_0']^{\otimes n})^{S_n},$$

where G_0' is the quotient of $\mathrm{GL}(\delta)$ analogous to G'. So we need to define a morphism from $(R(\mathrm{D}Q, \delta)/\!\!/\!\!/G_0')^n/S_n \to R(\mathrm{D}Q, n\delta)/\!\!/\!\!/G'$. This morphism just sends the n-tuple of representations $(z_1, \ldots, z_n)$ to their direct sum. The morphism of interest is an isomorphism and the corresponding homomorphism of algebras is the required one. Under the isomorphism $(R(\mathrm{D}Q, \delta)/\!\!/\!\!/G_0')^n/S_n \xrightarrow{\sim} R(\mathrm{D}Q, n\delta)/\!\!/\!\!/G'$ the element $\tilde{\mathbf{eu}}$ on the right hand side corresponds to the sum $\sum_{i=1}^n \tilde{\mathbf{eu}}_i$, where $\tilde{\mathbf{eu}}_i$, $i = 1, \ldots, n$, are analogous trace polynomials for the n copies of $\mathbb{C}[R(\mathrm{D}Q, \delta)/\!\!/\!\!/G_0']$. But we also have $\mathbf{eu} = \sum_{i=1}^n \mathbf{eu}_i$. Now using the previous paragraph we prove our claim for arbitrary n.

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