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Ellipsoidal billiards in pseudo-Euclidean spaces and relativistic quadrics[☆]

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Abstract

We study the geometry of confocal quadrics in pseudo-Euclidean spaces of arbitrary dimension d and any signature, and related billiard dynamics. The goal is to give a complete description of periodic billiard trajectories within ellipsoids. The novelty of our approach is the introduction of a new discrete combinatorial geometric structure associated with a confocal pencil of quadrics, a colouring in d colours. This is used to decompose quadrics of $d+1$ geometric types of a pencil into new relativistic quadrics of d relativistic types. A study of what we term discriminant sets of tropical lines Σ^+ and Σ^- and their singularities provides insight into the related geometry and combinatorics. This yields an analytic criterion describing all periodic billiard trajectories, including light-like trajectories as a case of special interest.

Keywords: Confocal quadrics; Poncelet theorem; Periodic billiard trajectories; Minkowski space; Light-like billiard trajectories; Tropic curves

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Contents

1. Introduction

Pseudo-Euclidean spaces and pseudo-Riemannian manifolds occupy a very important position in science as a geometric background for general relativity. A modern account of the mathematical aspects of the theory of relativity was recently published [8]. From a mathematical point of view, in comparison with Euclidean and Riemannian cases, apart from a natural similarity that includes some rather technical adjustments, there are some cases in which a pseudo-Euclidean setting creates new situations and challenging problems. The aim of this paper is to report on such cases in a study of the geometry of confocal quadrics and related billiard dynamics in pseudo-Euclidean spaces.

Recall that in Euclidean d-dimensional space, a general family of confocal quadrics contains exactly d geometric types of non-degenerate quadrics. Moreover, each point is the intersection of d quadrics of different types. Together with some other properties (E1–E5 in Section 4), these facts are crucial for the introduction of Jacobi coordinates and for applications in the theory of separable systems, including billiards. The case of d-dimensional pseudo-Euclidean space highlights a striking difference, since a confocal family of quadrics has $d+1$ geometric types of quadrics. In addition, quadrics of the same type have a nonempty intersection. These seem to be impregnable obstacles to the extension of methods for applying Jacobi type coordinates from the Euclidean case [9] to pseudo-Euclidean spaces.

To overcome this crucial problem, we have created a new feature of confocal pencils of geometric quadrics in pseudo-Euclidean d-dimensional spaces. The novelty of our approach is the introduction of a new discrete, combinatorial-geometric structure associated with a confocal pencil, a colouring in d colours, that transforms a geometric quadric from a pencil into a union of several *relativistic quadrics*. These relativistic quadrics satisfy properties PE1–PE5, analogues of

E1–E5, and lead to a new notion of decorated Jacobi coordinates. A decorated Jacobi coordinate is a pair comprising a number and a type-colour. These coordinates are used to develop methods for use in a further study of billiards within confocal quadrics in pseudo-Euclidean spaces of arbitrary dimension.

The study of colouring and relativistic types of quadrics, which is the main focus of the present paper, is closely related to the study of what we call *discriminant sets* Σ^+ and Σ^- attached to a confocal pencil of quadrics in pseudo-Euclidean space, as sets of the tropical lines of quadrics. They are developable, with light-like generatrices. Their swallowtail-type singularities [1] are placed at the vertices of curvilinear tetrahedra $\mathcal{T}^+$ and $\mathcal{T}^-$.

Khesin and Tabachnikov introduced billiards within ellipsoids in pseudo-Euclidean space [16] and their study motivated the present investigation. Sections 2 and 3 discuss, clarify and slightly improve some of the properties described by Khesin and Tabachnikov [16].

Section 2 describes pseudo-Euclidean spaces and their confocal families of geometric quadrics. In this section, Theorem 2.3 gives a complete description of structures of types of quadrics from a confocal pencil in pseudo-Euclidean space, which touch a given line. This theorem plays an essential role in proving properties PE3–PE5 in Section 5. Section 3 discusses the geometric properties of elliptical billiards in the two-dimensional case. An elementary but complete description of periodic light-like trajectories is derived in Theorem 3.3 and Proposition 3.6. In Section 4 we suggest a new setting for types of confocal quadrics, an essential novelty of pseudo-Euclidean geometry. For the three-dimensional case, we give a detailed description of discriminant surfaces $\Sigma^{\pm}$, the unions of tropical lines of geometric quadrics from a pencil (Propositions 4.3, 4.7, 4.9 and 4.11). We describe their singularity subsets, curvilinear tetrahedra $\mathcal{T}^{\pm}$, in Proposition 4.5. We introduce decorated Jacobi coordinates in Section 4.5 for three-dimensional Minkowski space, and provide a detailed description of colouring in three colours, with a complete description of all three relativistic types of quadrics. Section 4.6 provides a generalized definition of decorated Jacobi coordinates to arbitrary dimension, and Proposition 4.21 proves properties PE1 and PE2. In Section 5 we apply relativistic quadrics and decorated Jacobi coordinates to provide an analytic description of periodic billiard trajectories. Theorem 5.1 describes an effective criterion for determining the type of a billiard trajectory. Proposition 5.2 proves properties PE3–PE5. Finally, Theorem 5.3 provides an analytic description of all periodic billiard trajectories in pseudo-Euclidean spaces. As a corollary, Theorem 5.4 proves a full Poncelet-type theorem for pseudo-Euclidean spaces.

2. Pseudo-Euclidean spaces and confocal families of quadrics

This section first describes the basic notions connected with pseudo-Euclidean spaces. Then some basic facts on confocal families of quadrics in such spaces are reviewed and improved. Our main result in this section is a complete analysis of quadrics from a confocal family that are touching a given line, as formulated in Theorem 2.3.

2.1. Pseudo-Euclidean spaces

Definition of pseudo-Euclidean space. *A pseudo-Euclidean space* $\mathbf{E}^{k,l}$ is a d-dimensional space $\mathbf{R}^d$ with *pseudo-Euclidean scalar product*:

$$\langle x, y\rangle_{k,l} = x_1y_1 + \cdots + x_ky_k - x_{k+1}y_{k+1} - \cdots - x_dy_d, \tag{2.1}$$

where $k, l \in \{1, \dots, d-1\}$, $k + l = d$. The pair (k, l) is called the *signature* of the space. We denote $E_{k,l} = \mathrm{diag}(1, 1, \dots, 1, -1, \dots, -1)$, with k 1s and l $-$1s. Then the pseudo-Euclidean scalar product is:

$$\langle x, y\rangle_{k,l} = E_{k,l}x \circ y,$$

where $\circ$ is the standard Euclidean product.

The pseudo-Euclidean distance between points x and y is:

$$\mathrm{dist}_{k,l}(x, y) = \sqrt{\langle x - y, x - y\rangle_{k,l}}.$$

Since the scalar product can be negative, note that the pseudo-Euclidean distance can take imaginary values.

Let ℓ be a line in pseudo-Euclidean space and let v be its vector. ℓ is called:

- *space-like* if $\langle v, v\rangle_{k,l} > 0$;
- *time-like* if $\langle v, v\rangle_{k,l} < 0$; and
- *light-like* if $\langle v, v\rangle_{k,l} = 0$.

Two vectors x and y are *orthogonal* in pseudo-Euclidean space if $\langle x, y\rangle_{k,l} = 0$. Note that a light-like line is orthogonal to itself.

For a given vector $v \neq 0$, consider a hyper-plane $v \circ x = 0$. Vector $E_{k,l}v$ is orthogonal to the hyper-plane; moreover, all other orthogonal vectors are collinear with $E_{k,l}v$. If v is light-like, then so is $E_{k,l}v$, and $E_{k,l}v$ belongs to the hyper-plane.

Billiard reflection in pseudo-Euclidean space. Let v be a vector and α a hyper-plane in pseudo-Euclidean space. We decompose vector v into the sum $v = a + n_\alpha$ of a vector n_α orthogonal to α and a belonging to α. Then vector $v' = a - n_\alpha$ is a *the billiard reflection* of v on α. It is easy to see that v is also a billiard reflection of v' with respect to α.

Moreover, note that lines containing vectors v, v', a and n_α are harmonically conjugated [16].

Note that $v = v'$ if v is contained in α and $v' = -v$ if v is orthogonal to α. If n_α is light-like, which means that it belongs to α, then the reflection is not defined.

Line ℓ' is a billiard reflection of ℓ off a smooth surface $\mathcal{S}$ if their intersection point $\ell \cap \ell'$ belongs to $\mathcal{S}$ and the vectors of ℓ and ℓ' are reflections of each other with respect to the tangent plane of $\mathcal{S}$ at this point.

Remark 2.1. It is directly evident from the definition of reflection that the type of line is preserved by a billiard reflection. Thus, the lines containing segments of a given billiard trajectory within $\mathcal{S}$ are all of the same type: they are all either space-like, time-like, or light-like.

If $\mathcal{S}$ is an ellipsoid, then it is possible to extend the reflection mapping to points where the tangent planes contain the orthogonal vectors. At such points, a vector reflects onto the opposite one, i.e. $v' = -v$ and $\ell' = \ell$. According to the explanation of this phenomenon by Khesin and Tabachnikov [16], it is natural to consider each such reflection as two reflections.

2.2. *Families of confocal quadrics*

For a given set of positive constants $a_1, a_2, \dots, a_d$, an ellipsoid is given by:

$$\mathcal{E} : \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \dots + \frac{x_d^2}{a_d} = 1. \tag{2.2}$$

Note that the equation for any ellipsoid in pseudo-Euclidean space can be changed to the canonical form (2.2) using transformations that preserve the scalar product (2.1).

The family of quadrics confocal with $\mathcal{E}$ is:

$$\mathcal{Q}_\lambda : \frac{x_1^2}{a_1-\lambda}+\cdots+\frac{x_k^2}{a_k-\lambda}+\frac{x_{k+1}^2}{a_{k+1}+\lambda}+\cdots+\frac{x_d^2}{a_d+\lambda}=1, \quad \lambda\in\mathbf{R}. \tag{2.3}$$

Unless stated otherwise, we consider the non-degenerate case, when set $\{a_1,\dots,a_k,-a_{k+1},\dots,-a_d\}$ consists of d different values:

$$a_1>a_2>\cdots>a_k>0>-a_{k+1}>\cdots>-a_d.$$

For $\lambda\in\{a_1,\dots,a_k,-a_{k+1},\dots,-a_d\}$, the quadric $\mathcal{Q}_\lambda$ is degenerate and coincides with the corresponding coordinate hyper-plane.

It is natural to join one more degenerate quadric to the family (2.3): the one corresponding to the value $\lambda=\infty$, which is the hyper-plane at the infinity.

For each point x in the space, there are exactly d values of λ, such that (2.3) is satisfied. However, not all the values are necessarily real: either all d of them are real or there are $d-2$ real and 2 conjugate complex values. Thus, through every point in the space, there are either d or $d-2$ quadrics from the family (2.3) [16].

The line $x+tv$ ($t\in\mathbf{R}$) is tangential to quadric $\mathcal{Q}_\lambda$ if the quadratic equation

$$A_\lambda(x+tv)\circ(x+tv)=1 \tag{2.4}$$

has a double root, where

$$A_\lambda=\operatorname{diag}\left(\frac{1}{a_1-\lambda},\dots,\frac{1}{a_k-\lambda},\frac{1}{a_{k+1}+\lambda},\dots,\frac{1}{a_d+\lambda}\right).$$

Calculating the discriminant of (2.4), we obtain:

$$(A_\lambda x\circ v)^2-(A_\lambda v\circ v)(A_\lambda x\circ x-1)=0, \tag{2.5}$$

which is equivalent to

$$\sum_{i=1}^{d}\frac{\varepsilon_i F_i(x,v)}{a_i-\varepsilon_i\lambda}=0, \tag{2.6}$$

where

$$F_i(x,v)=\varepsilon_i v_i^2+\sum_{j\neq i}\frac{(x_iv_j-x_jv_i)^2}{\varepsilon_j a_i-\varepsilon_i a_j} \tag{2.7}$$

with ε values given by

$$\varepsilon_i=\begin{cases}1, & 1\le i\le k;\\ -1, & k+1\le i\le d.\end{cases}$$

Eq. (2.6) can be transformed to

$$\frac{\mathcal{P}(\lambda)}{\prod_{i=1}^{d}(a_i-\varepsilon_i\lambda)}=0, \tag{2.8}$$

where the coefficient of λ^{d-1} in $\mathcal{P}(\lambda)$ is equal to $\langle v, v\rangle_{k,l}$. Thus, polynomial $\mathcal{P}(\lambda)$ is of degree $d-1$ for space-like and time-like lines, and of smaller degree for light-like lines. However, in the latter case, it is natural to consider the polynomial $\mathcal{P}(\lambda)$ also as of degree $d-1$, taking the corresponding roots to be equal to infinity. Thus, light-like lines are characterized by being tangential to the quadric $\mathcal{Q}_\infty$.

With this setting in mind, we note that Khesin and Tabachnikov proved that the polynomial $\mathcal{P}(\lambda)$ has at least $d-3$ roots in $\mathbf{R}\cup\{\infty\}$ [16].

Thus, we have the following.

Proposition 2.2. *Any line in the space is tangential to either $d-1$ or $d-3$ quadrics of the family (2.3). If this number is equal to $d-3$, then there are two conjugate complex values of λ such that the line is also tangential to these two quadrics in $\mathbf{C}^d$.*

This was stated and proved by Khesin and Tabachnikov [16], who claimed that light-like line have only $d-2$ or $d-4$ caustic quadrics because they did not consider $\mathcal{Q}_\infty$ as a member of the confocal family.

A line with a non-empty intersection with an ellipsoid from (2.3) will be tangential to $d-1$ quadrics from the confocal family [16]. However, we prove this in another way to provide greater insight into the distribution of the caustic parameters along the real axis. The next theorem provides a detailed description of the distribution of parameters of quadrics containing a given point placed inside an ellipsoid from (2.3).

Theorem 2.3. *In pseudo-Euclidean space $\mathbf{E}^{k,l}$, consider a line intersecting ellipsoid $\mathcal{E}$ (2.2). Then this line touches $d-1$ quadrics from (2.3). If we denote their parameters by $\alpha_1,\dots,\alpha_{d-1}$ and take*

$$\{b_1,\dots,b_p,c_1,\dots,c_q\}=\{\varepsilon_1a_1,\dots,\varepsilon_da_d,\alpha_1,\dots,\alpha_{d-1}\},$$
$$c_q\le\cdots\le c_2\le c_1<0<b_1\le b_2\le\cdots\le b_p,\qquad p+q=2d-1,$$

we also have the following:

- *If the line is space-like, then $p=2k-1$, $q=2l$, $a_1=b_p$, $\alpha_i\in\{b_{2i-1},b_{2i}\}$ for $1\le i\le k-1$, and $\alpha_{j+k-1}\in\{c_{2j-1},c_{2j}\}$ for $1\le j\le l$.*
- *If the line is time-like, then $p=2k$, $q=2l-1$, $c_q=-a_d$, $\alpha_i\in\{b_{2i-1},b_{2i}\}$ for $1\le i\le k$, and $\alpha_{j+k}\in\{c_{2j-1},c_{2j}\}$ for $1\le j\le l-1$.*
- *If the line is light-like, then $p=2k$, $q=2l-1$, $b_p=\infty=\alpha_k$, $b_{p-1}=a_1$, $\alpha_i\in\{b_{2i-1},b_{2i}\}$ for $1\le i\le k-1$, and $\alpha_{j+k}\in\{c_{2j-1},c_{2j}\}$ for $1\le j\le l-1$.*

Moreover, for each point on ℓ inside $\mathcal{E}$, there are exactly d distinct quadrics from (2.3) that contain it. More precisely, there is exactly one parameter of these quadrics in each of the intervals

$$(c_{2l-1},c_{2l-2}),\dots,(c_3,c_2),(c_1,0),(0,b_1),(b_2,b_3),\dots,(b_{2k-2},b_{2k-1}).$$

Proof. Let ℓ be a line that intersects $\mathcal{E}$, let $x=(x_1,\dots,x_d)$ be a point on ℓ placed inside $\mathcal{E}$, and let $v=(v_1,\dots,v_d)$ bet a vector of the line. Then the parameters of quadrics touching ℓ are solutions of (2.5), i.e. they are roots of polynomial $\mathcal{P}(\lambda)$.

If ℓ is space-like or time-like, that is $\langle v, v\rangle_{k,l}\ne 0$, we then have that polynomial $\mathcal{P}$ is of degree $d-1$ and we want to prove that all its $d-1$ roots are real. By contrast, if ℓ is light-like, $\mathcal{P}$ is of degree $d-2$ and we want to prove that all $d-2$ roots are real, and to add ∞ as the $(d-1)$-th root.

Note that for the real solutions of the equation $A_\lambda v \circ v = 0$, the right-hand side of (2.5) is positive. Thus, we first examine the roots of $A_\lambda v \circ v$.

We have

$$A_\lambda v \circ v = \sum_{i=1}^{d} \frac{v_i^2}{a_i - \varepsilon_i \lambda} = \frac{\mathcal{R}(\lambda)}{\prod_{i=1}^{d}(a_i - \varepsilon_i \lambda)}$$

with

$$\mathcal{R}(\lambda) = \sum_{i=1}^{d} v_i^2 \prod_{j \neq i}(a_j - \varepsilon_j \lambda).$$

We calculate

$$\begin{aligned}
&\operatorname{sign} \mathcal{R}(\varepsilon_i a_i) = \varepsilon_i (-1)^{k+i},\\
&\operatorname{sign} \mathcal{R}(-\infty) = (-1)^l \operatorname{sign} \langle v, v\rangle_{k,l},\\
&\operatorname{sign} \mathcal{R}(+\infty) = (-1)^{k-1} \operatorname{sign} \langle v, v\rangle_{k,l}.
\end{aligned}$$

Thus, it is evident that polynomial $\mathcal{R}$, which is of degree $d-1$ for space-like or time-like ℓ and of degree $d-2$ for light-like ℓ, changes sign at least $d-1$ times along the real axis if $\langle v, v\rangle_{k,l} \neq 0$, and at least $d-2$ times otherwise, so all its roots are real. Moreover, there is one root in each of the $d-2$ intervals:

$$(\varepsilon_{i+1} a_{i+1}, \varepsilon_i a_i), \quad i \in \{1, \dots, k-1, k+1, \dots, d-1\},$$

and one more in $(-\infty, -a_d)$ or $(a_1, +\infty)$ if ℓ is space-like or time-like, respectively.

We denote the roots of $\mathcal{R}$ by $\zeta_0, \zeta_1, \dots, \zeta_{d-2}$, and order them in the following way:

$$\begin{aligned}
&\zeta_i \in (a_{i+1}, a_i), \quad \text{for } 1 \le i \le k-1,\\
&\zeta_j \in (-a_{j+2}, -a_{j+1}), \quad \text{for } k \le j \le d-2,\\
&\zeta_0 \in (-\infty, -a_d) \quad \text{for space-like } \ell,\\
&\zeta_0 \in (a_1, +\infty) \quad \text{for time-like } \ell,\\
&\zeta_0 = \infty \quad \text{for light-like } \ell.
\end{aligned}$$

The right-hand side of (2.5) is positive for $\zeta_0, \dots, \zeta_{d-2}$. It will be also be positive for $\lambda = 0$ because $(A_0 x \circ v)^2 \ge 0$, $A_0 v \circ v > 0$ and, since x is inside $\mathcal{E} = \mathcal{Q}_0$, $A_0 x \circ x < 1$. Note that (2.5) and the equivalent expression (2.8) change sign at points $\varepsilon_i a_i$ and for roots of $\mathcal{P}$ only. Thus, these expressions have positive values at the endpoints of each of the $d-2$ intervals $(\zeta_{d-2}, \zeta_{d-3}), \dots, (\zeta_{k+1}, \zeta_k), (\zeta_k, 0), (0, \zeta_{k-1}), (\zeta_{k-1}, \zeta_{k-2}), \dots, (\zeta_2, \zeta_1)$ and in one of (ζ_0, ζ_{d-2}) or (ζ_1, ζ_0), depending if ℓ is space-like or time-like, respectively. Each of these $d-1$ intervals contains one point from $\{\varepsilon_i a_i\}$, and thus each of them needs to contain at least one more point at which (2.8) changes its sign, that is, a root of $\mathcal{P}$. We conclude that all roots of $\mathcal{P}$ are real, and that they are distributed exactly as stated in this proposition.

Now consider quadrics from the confocal family containing point x. Their parameters are solutions of the equation $A_\lambda x \circ x = 1$. Observe that $A_\lambda x \circ x - 1$ is strictly monotonous and changes sign inside each of the following intervals:

$$(-a_d, -a_{d-1}), \dots, (-a_{k+2}, -a_{k+1}), (a_k, a_{k-1}), \dots, (a_2, a_1),$$

and thus it has one root in each of them. In addition, for such solutions, the right-hand side of (2.5) is positive and thus there is one solution in each of the following:

$$(c_{2l-1}, c_{2l-2}), \dots, (c_3, c_2), (b_2, b_3), \dots, (b_{2k-2}, b_{2k-1}),$$

which makes $d-2$ solutions. Two more solutions are placed in (c_1, b_1), because $A_0 x \circ x - 1 < 0$ and $A_\lambda x \circ x - 1 > 0$ close to the endpoints of this interval. This concludes the proof. □

The analogue of Theorem 2.3 for Euclidean space was proved by Audin [2].

Corollary 2.4. *For each point placed inside an ellipsoid in pseudo-Euclidean space, there are exactly two other ellipsoids from the confocal family containing this point.*

3. Elliptical billiard in the Minkowski plane

In this section we study the properties of confocal families of conics in the Minkowski plane. We derive focal properties of such families and the corresponding elliptical billiards. We then address light-like trajectories of such billiards and derive a periodicity criterion in a simple form in Theorem 3.3. Proposition 3.6 proves that the flow of light-like elliptical billiard trajectories is equivalent to a certain rectangular billiard flow.

3.1. Confocal conics in Minkowski plane

We first review the basic properties of families of confocal conics in the Minkowski plane. Let

$$\mathcal{E} : \frac{x^2}{a} + \frac{y^2}{b} = 1 \tag{3.1}$$

denote an ellipse in the plane, where a and b are fixed positive numbers.

The associated family of confocal conics is:

$$\mathcal{C}_\lambda : \frac{x^2}{a-\lambda} + \frac{y^2}{b+\lambda} = 1, \quad \lambda \in \mathbf{R}. \tag{3.2}$$

The family is shown in Fig. 1. We can distinguish the following three subfamilies in (3.2):

- For $\lambda \in (-b, a)$, conic $\mathcal{C}_\lambda$ is an ellipse.
- For $\lambda < -b$, conic $\mathcal{C}_\lambda$ is a hyperbola with the x-axis as the major axis.
- For $\lambda > a$, conic $\mathcal{C}_\lambda$ is also a hyperbola, but its major axis is the y-axis.

In addition, there are three degenerated quadrics $\mathcal{C}_a$, $\mathcal{C}_b$ and $\mathcal{C}_\infty$ corresponding to the y-axis, the x-axis and the line at infinity, respectively. Note the three pairs of foci $F_1(\sqrt{a+b}, 0)$, $F_2(-\sqrt{a+b}, 0)$; $G_1(0, \sqrt{a+b})$, $G_2(0, -\sqrt{a+b})$; and $H_1(1 : -1 : 0)$, $H_2(1 : 1 : 0)$ are on the line at the infinity.

We can distinguish four lines:

$$\begin{aligned} x + y &= \sqrt{a+b}, & x + y &= -\sqrt{a+b}, \\ x - y &= \sqrt{a+b}, & x - y &= -\sqrt{a+b}. \end{aligned}$$

These lines are common tangents to all conics in the confocal family.

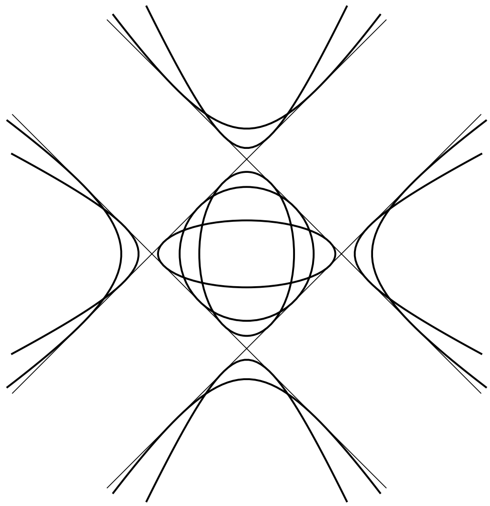

Fig. 1. Family of confocal conics in the Minkowski plane.

It is elementary and straightforward to prove the following.

Proposition 3.1. *For each point on ellipse $\mathcal{C}_\lambda$, $\lambda \in (-b, a)$, either the sum or the difference of its Minkowski distances from the foci F_1 and F_2 is equal to $2\sqrt{a-\lambda}$; either the sum or the difference of the distances from the other pair of foci G_1, G_2 is equal to $2i\sqrt{b+\lambda}$.*

Either the sum or the difference of the Minkowski distances of each point of the hyperbola $\mathcal{C}_\lambda$, $\lambda \in (-\infty, -b)$, from the foci F_1 and F_2 is equal to $2\sqrt{a-\lambda}$; for the other pair of foci G_1, G_2, it is equal to $2\sqrt{-b-\lambda}$.

Either the sum or the difference of the Minkowski distances of each point of the hyperbola $\mathcal{C}_\lambda$, $\lambda \in (a, +\infty)$, from the foci F_1 and F_2 is equal to $2i\sqrt{\lambda-a}$; for the other pair of foci G_1, G_2, it is equal to $2i\sqrt{b+\lambda}$.

A billiard within an ellipse also has the well-known focal property.

Proposition 3.2. *Consider a billiard trajectory within ellipse $\mathcal{E}$ given by (3.1) in the Minkowski plane such that the line containing the initial segment of the trajectory passes through a focus of confocal family (3.2), say F_1, G_1, or H_1. If the tangent to $\mathcal{E}$ at the reflection point of this segment is not light-like, then the line containing the next segment will pass through F_2, G_2, or H_2, respectively.*

In other words, the segments of one billiard trajectory will alternately contain foci of one of the pairs (F_1, F_2), (G_1, G_2) or (H_1, H_2). The only exceptions are successive segments obtained by the reflection on the light-like tangent. Such segments coincide.

3.2. Light-like trajectories of the elliptical billiard

Now we study the light-like trajectories of an elliptical billiard in the Minkowski plane. An example of such a billiard trajectory is shown in Fig. 2.

Successive segments of such trajectories are orthogonal to each other. This implies that a trajectory can close only after an even number of reflections.

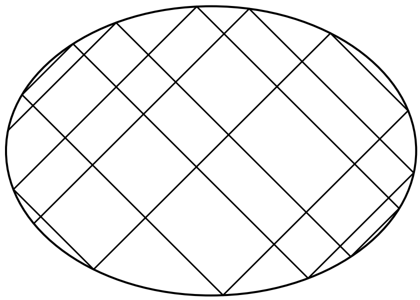

Fig. 2. Light-like billiard trajectory.

Periodic light-like trajectories. The analytic condition for n-periodicity of a light-like billiard trajectory within the ellipse $\mathcal{E}$ given by (3.1) can be written by applying the more general Cayley's condition for closedness of a polygonal line inscribed in one conic and circumscribed about another one [4,5,17,14]:

$$\det\begin{pmatrix} B_3 & B_4 & \dots & B_{m+1} \\ B_4 & B_5 & \dots & B_{m+2} \\ \dots & \dots & \dots & \dots \\ B_{m+1} & B_{m+2} & \dots & B_{2m-1} \end{pmatrix} = 0, \quad \text{with } n = 2m, \tag{3.3}$$

where

$$\sqrt{(a-\lambda)(b+\lambda)} = B_0 + B_1\lambda + B_2\lambda^2 + \cdots$$

is the Taylor expansion around $\lambda = 0$.

Now we derive an analytic condition for periodic light-like trajectories in another way that leads to a more compact form of (3.3).

Theorem 3.3. *A light-like billiard trajectory within ellipse $\mathcal{E}$ is periodic with period n, where n is an even integer, if and only if*

$$\operatorname{arc\,tan}\sqrt{\frac{a}{b}} \in \left\{ \frac{k\pi}{n} \;\middle|\; 1 \le k < \frac{n}{2}, \left(k, \frac{n}{2}\right) = 1 \right\}. \tag{3.4}$$

Proof. Applying the following affine transformation

$$(x, y) \mapsto (x\sqrt{b}, y\sqrt{a}),$$

the ellipse is transformed into a circle. The light-like lines are transformed into lines parallel to two directions, with the angle between them equal to $2\operatorname{arc\,tan}\sqrt{a/b}$. Since the dynamics on the boundary is rotation by this angle, the proof is complete. $\square$

As an immediate consequence, we obtain the following.

Corollary 3.4. *For a given even integer n, the number of different ratios of the axes of ellipses with n-periodic light-like billiard trajectories is equal to:*

$$\begin{cases} \varphi(n)/2 & \textit{if } n \textit{ is not divisible by } 4, \\ \varphi(n)/4 & \textit{if } n \textit{ is divisible by } 4. \end{cases}$$

φ is the Euler's totient function, i.e. the number of positive integers not exceeding n that are relatively prime to n.

Remark 3.5. There are four points on $\mathcal{E}$ at which the tangents are light-like. Those points cut four arcs on $\mathcal{E}$. An n-periodic trajectory within $\mathcal{E}$ hits each one of a pair of opposite arcs exactly k times, and hits the arcs from the other pair $\frac{n}{2} - k$ times.

Light-like trajectories in ellipses and rectangular billiards

Proposition 3.6. *The flow of light-like billiard trajectories within ellipse $\mathcal{E}$ is trajectorially equivalent to the flow of those billiard trajectories within a rectangle whose angle with the sides is $\frac{\pi}{4}$. The ratio of the sides of the rectangle is equal to*

$$\frac{\pi}{2\operatorname{arc\,tan}\sqrt{\frac{a}{b}}} - 1.$$

Proof. For ellipses with periodic light-like trajectories, the proof follows from Theorem 3.3 and Remark 3.5.

In other cases, the number $\frac{\pi}{2\operatorname{arc\,tan}\sqrt{\frac{a}{b}}} - 1$ is not rational. Thus, the statement holds for these cases because of the density of rational numbers. $\square$

Remark 3.7. The flow of light-light billiard trajectories within a given oval in the Minkowski plane will be trajectorially equivalent to the flow of certain trajectories within a rectangle whenever invariant measure m on the oval exists such that $m(AB) = m(CD)$ and $m(BC) = m(AD)$, where A, B, C and D are points on the oval at which the tangents are light-like.

4. Relativistic quadrics

We now introduce *relativistic quadrics* as a tool for further investigation of billiard dynamics. The geometric quadrics of a confocal pencil and their types in pseudo-Euclidean spaces do not satisfy the usual properties of confocal quadrics in Euclidean spaces, including those necessary for applications in billiard dynamics. For example, we have already mentioned that in d-dimensional Euclidean space there are d geometric types of quadrics, while in d-dimensional pseudo-Euclidean space there are $d + 1$ geometric types of quadrics. Thus, we first select the important properties of confocal families in Euclidean space as E1–E5 in Section 4.1. Section 4.2 considers the two-dimensional case, the Minkowski plane. We study appropriate relativistic conics, as first described by Birkhoff and Morris [3]. Section 4.3 investigates the geometric types of quadrics in a confocal family in three-dimensional Minkowski space. Section 4.4 analyses tropic curves on quadrics in the three-dimensional case and introduces an important notion of discriminant sets $\Sigma^{\pm}$ corresponding to a confocal family. The main facts for discriminant sets are proved in Propositions 4.3, 4.7, 4.9 and 4.11. Then we study curved tetrahedra $\mathcal{T}^{\pm}$, which represent the singularity sets of $\Sigma^{\pm}$, and collect related results in Proposition 4.5. Section 4.5 introduces decorated Jacobi coordinates for three-dimensional Minkowski space and provides a detailed description of colouring in three colours. Each colour corresponds to a relativistic type, and we describe decomposition of a geometric quadric of each of the four geometric types into relativistic quadrics. This complex combinatorial geometric problem is solved using our analysis of discriminant surfaces. We give a complete description of all three relativistic types of quadrics. Section 4.6 provides a generalized definition of decorated Jacobi coordinates in arbitrary dimensions. Proposition 4.21 proves properties PE1 and PE2, the pseudo-Euclidean analogues of E1 and E2.

4.1. Confocal quadrics and their types in Euclidean space

A general family of confocal quadrics in d-dimensional Euclidean space is given by

$$\frac{x_1^2}{b_1-\lambda}+\cdots+\frac{x_d^2}{b_d-\lambda}=1, \quad \lambda \in \mathbf{R} \tag{4.1}$$

for $b_1 > b_2 > \cdots > b_d > 0$.

Such a family has the following properties.

- E1 Each point of the space $\mathbf{E}^d$ is the intersection of exactly d quadrics from (4.1); moreover, all these quadrics are of different geometrical types.
- E2 Family (4.1) contains exactly d geometric types of non-degenerate quadrics; each type corresponds to one of the disjoint intervals of the parameter λ: $(-\infty, b_d)$, $(b_d, b_{d-1}), \dots, (b_2, b_1)$.

The parameters $(\lambda_1, \dots, \lambda_d)$ corresponding to the quadrics of (4.1) that contain a given point in $\mathbf{E}^d$ are called *Jacobi coordinates*. We order them as $\lambda_1 > \cdots > \lambda_d$.

Now consider the motion of a billiard ball within an ellipsoid, denoted by $\mathcal{E}$, of family (4.1). Without loss of generality, assume that the parameter λ corresponding to this ellipsoid is equal to 0. Recall that by Chasles' theorem, each line in $\mathbf{E}^d$ touches some $d-1$ quadrics from (4.1). Moreover, for a line and its billiard reflection on a quadric from (4.1), the $d-1$ quadrics are the same. This means that each segment of a given trajectory within $\mathcal{E}$ has the same $d-1$ *caustics*; we denote their parameters by $\beta_1, \dots, \beta_{d-1}$, and introduce the following

$$\{\bar{b}_1, \dots, \bar{b}_{2d}\} = \{b_1, \dots, b_d, 0, \beta_1, \dots, \beta_{d-1}\},$$

such that $\bar{b}_1 \geq \bar{b}_2 \geq \cdots \geq \bar{b}_{2d}$. Thus, we will have $0 = \bar{b}_{2d} < \bar{b}_{2d-1}$, $b_1 = \bar{b}_1$. Moreover, it is always: $\beta_i \in \{\bar{b}_{2i}, \bar{b}_{2i+1}\}$, for each $i \in \{1, \dots, d\}$ [2].

Now we can summarize the main properties of the flow of the Jacobi coordinates along the billiard trajectories.

- E3 Along a fixed billiard trajectory, the Jacobi coordinate λ_i $(1 \leq i \leq d)$ takes values in segment $[\bar{b}_{2i-1}, \bar{b}_{2i}]$.
- E4 Along a trajectory, each λ_i achieves local minima and maxima exactly at touching points with corresponding caustics, intersection points with corresponding coordinate hyper-planes, and, for $i = d$, at reflection points.
- E5 Values of λ_i at these points are $\bar{b}_{2i-1}$ and $\bar{b}_{2i}$; between the critical points, λ_i changes monotonically.

These properties are key in algebrogeometric analysis of billiard flow.

At first glance, it seems that such properties do not hold in the pseudo-Euclidean case. In d-dimensional pseudo-Euclidean space, a general confocal family contains $d+1$ geometric types of quadrics; in addition, quadrics of the same geometric type have a non-empty intersection. Thus, analysis of billiard flow using Jacobi-type coordinates is much more complicated.

We overcome this problem by introducing a new notion of relativistic quadrics. We equip a geometric pencil of quadrics with an additional structure, *a decoration*, that decomposes geometric quadrics of the pencil into coloured subsets that form new types of relativistic quadrics. The new notion of relativistic quadrics, i.e. *coloured geometric quadrics*, is more suitable for pseudo-Euclidean geometry. It is then possible to introduce a new system of coordinates, a nontrivial pseudo-Euclidean analogue of Jacobi elliptic coordinates, that plays a fundamental role in our subsequent study of separable systems.

4.2. *Confocal conics in the Minkowski plane*

Consider first the case of the 2-dimensional pseudo-Euclidean space $\mathbf{E}^{1,1}$, namely the Minkowski plane. We mentioned in Section 3.1 that a family of confocal conics in the Minkowski plane contains conics of three geometric types: ellipses, hyperbolas with the x-axis as the major axis, and hyperbolas with the y-axis as the major axis, as shown on Fig. 1. However, it is more natural to consider *relativistic conics*, as analysed by Birkhoff and Morris [3]. In this section, we give a brief account of that analysis.

Consider points $F_1(\sqrt{a+b}, 0)$ and $F_2(-\sqrt{a+b}, 0)$ in the plane.

For a given constant $c \in \mathbf{R}^+ \cup i\mathbf{R}^+$, a *relativistic ellipse* is the set of points X satisfying

$$\mathrm{dist}_{1,1}(F_1, X) + \mathrm{dist}_{1,1}(F_2, X) = 2c,$$

while a *relativistic hyperbola* is the union of the sets given by the following equations:

$$\mathrm{dist}_{1,1}(F_1, X) - \mathrm{dist}_{1,1}(F_2, X) = 2c,$$
$$\mathrm{dist}_{1,1}(F_2, X) - \mathrm{dist}_{1,1}(F_1, X) = 2c.$$

Relativistic conics can be described as follows.

$0 < c < \sqrt{a+b}$: The corresponding relativistic conics lie on ellipse $\mathcal{C}_{a-c^2}$ from family (3.2). The ellipse $\mathcal{C}_{a-c^2}$ is split into four arcs by touching points with the four common tangents. Thus, the relativistic ellipse is the union of the two arcs intersecting the y-axis, while the relativistic hyperbola is the union of the other two arcs.

$c > \sqrt{a+b}$: The relativistic conics lie on $\mathcal{C}_{a-c^2}$ — a hyperbola with x-axis as the major one. Each branch of the hyperbola is split into three arcs by touching points with common tangents; thus, the relativistic ellipse is the union of the two finite arcs, while the relativistic hyperbola is the union of the four infinite arcs.

c **is imaginary**: The relativistic conics lie on hyperbola $\mathcal{C}_{a-c^2}$, a hyperbola with the y-axis as the major axis. As in the previous case, the branches are split into six arcs by common points for the four tangents. The relativistic ellipse is the union of the four infinite arcs, while the relativistic hyperbola is the union of the two finite arcs.

The conics are shown in Fig. 3.

Remark 4.1. All relativistic ellipses are disjoint from each other, as are all relativistic hyperbolas. Moreover, at the intersection point between a relativistic ellipse that is part of the geometric conic $\mathcal{C}_{\lambda_1}$ belonging to confocal family (3.2) and a relativistic hyperbola belonging to $\mathcal{C}_{\lambda_2}$, $\lambda_1 < \lambda_2$ always holds.

This remark serves as motivation for the introduction of relativistic types of quadrics in higher-dimensional pseudo-Euclidean spaces.

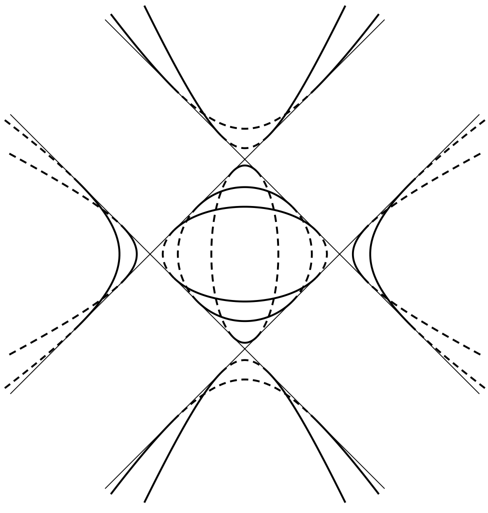

Fig. 3. Relativistic conics in the Minkowski plane. Relativistic ellipses and hyperbolas are represented by full and dashed lines, respectively.

4.3. *Confocal quadrics in three-dimensional Minkowski space and their geometric types*

Consider the three-dimensional Minkowski space $\mathbf{E}^{2,1}$. A general confocal family of quadrics in this space is given by:

$$\mathcal{Q}_\lambda : \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c+\lambda} = 1, \quad \lambda \in \mathbf{R}, \tag{4.2}$$

with $a > b > 0$, $c > 0$.

The family (4.2) contains quadrics of four geometric types:

- 1-sheeted hyperboloids oriented along the z-axis for $\lambda \in (-\infty, -c)$;
- ellipsoids corresponding to $\lambda \in (-c, b)$;
- 1-sheeted hyperboloids oriented along the y-axis for $\lambda \in (b, a)$; and
- 2-sheeted hyperboloids for $\lambda \in (a, +\infty)$; these hyperboloids are oriented along the z-axis.

In addition, there are four degenerate quadrics: $\mathcal{Q}_a$, $\mathcal{Q}_b$, $\mathcal{Q}_{-c}$ and $\mathcal{Q}_\infty$, that is, planes $x = 0$, $y = 0$, $z = 0$, and the plane at infinity, respectively. In the coordinate planes, we single out the following conics:

- hyperbola $\mathcal{C}_a^{yz} : -\frac{y^2}{a-b} + \frac{z^2}{c+a} = 1$ in the plane $x = 0$;
- ellipse $\mathcal{C}_b^{xz} : \frac{x^2}{a-b} + \frac{z^2}{c+b} = 1$ in the plane $y = 0$; and
- ellipse $\mathcal{C}_{-c}^{xy} : \frac{x^2}{a+c} + \frac{y^2}{b+c} = 1$ in the plane $z = 0$.

4.4. *Tropic curves on quadrics in three-dimensional Minkowski space and discriminant sets $\Sigma^\pm$*

For each quadric, note the *tropic curves*, the set of points at which the metrics induced on the tangent plane are degenerate.

Since the tangent plane at point (x_0, y_0, z_0) of $\mathcal{Q}_\lambda$ is given by

$$\frac{xx_0}{a-\lambda} + \frac{yy_0}{b-\lambda} + \frac{zz_0}{c+\lambda} = 1$$

and the induced metric is degenerate if and only if the parallel plane that contains the origin is tangential to the light-like cone $x^2 + y^2 - z^2 = 0$, i.e.

$$\frac{x_0^2}{(a-\lambda)^2} + \frac{y_0^2}{(b-\lambda)^2} - \frac{z_0^2}{(c+\lambda)^2} = 0,$$

we arrive at the statement formulated by Khesin and Tabachnikov [16].

Proposition 4.2. *The tropic curves on $\mathcal{Q}_\lambda$ are the intersection of the quadric with the cone*

$$\frac{x^2}{(a-\lambda)^2} + \frac{y^2}{(b-\lambda)^2} - \frac{z^2}{(c+\lambda)^2} = 0.$$

Now consider the set of the tropic curves on all quadrics of the family (4.2). From Proposition 4.2, we obtain the following.

Proposition 4.3. *The union of the tropic curves on all quadrics of (4.2) is a union of two ruled surfaces Σ^+ and Σ^-, which can be parametrically represented as:*

$$\Sigma^+ : x = \frac{a-\lambda}{\sqrt{a+c}}\cos t, \qquad y = \frac{b-\lambda}{\sqrt{b+c}}\sin t, \qquad z = (c+\lambda)\sqrt{\frac{\cos^2 t}{a+c} + \frac{\sin^2 t}{b+c}},$$

$$\Sigma^- : x = \frac{a-\lambda}{\sqrt{a+c}}\cos t, \qquad y = \frac{b-\lambda}{\sqrt{b+c}}\sin t, \qquad z = -(c+\lambda)\sqrt{\frac{\cos^2 t}{a+c} + \frac{\sin^2 t}{b+c}},$$

with $\lambda \in \mathbf{R}$, $t \in [0, 2\pi)$.

The intersection of these two surfaces is an ellipse in the xy-plane:

$$\Sigma^+ \cap \Sigma^- : \frac{x^2}{a+c} + \frac{y^2}{b+c} = 1, \qquad z = 0.$$

The two surfaces Σ^+, Σ^- are developable as embedded into Euclidean space. Moreover, their generatrices are all light-like.

Proof. Let $\mathbf{r} = (x, y, z)$ denote an arbitrary point of $\Sigma^+ \cup \Sigma^-$ and let $\mathbf{n}$ be the corresponding unit normal vector, $\mathbf{n} = \mathbf{r}_\lambda \times \mathbf{r}_t / |\mathbf{r}_\lambda \times \mathbf{r}_t|$, where $\times$ denotes the vector product in three-dimensional Euclidean space. Then the Gaussian curvature of the surface is $K = (LN - M^2)/(EG - F^2)$, with $L = \mathbf{r}_{\lambda\lambda} \cdot \mathbf{n} = 0$, $M = \mathbf{r}_{\lambda t} \cdot \mathbf{n} = 0$, $N = \mathbf{r}_{tt} \cdot \mathbf{n}$, $E = \mathbf{r}_\lambda \cdot \mathbf{r}_\lambda$, $F = \mathbf{r}_\lambda \cdot \mathbf{r}_t$ and $G = \mathbf{r}_t \cdot \mathbf{r}_t$. Since

$$EG - F^2 = \frac{(a+b-2\lambda+(b-a)\cos(2t))^2}{2(a+c)(b+c)} \not\equiv 0,$$

the Gaussian curvature K is equal to zero. □

Surfaces Σ^+ and Σ^- in Proposition 4.3 are shown in Fig. 4.

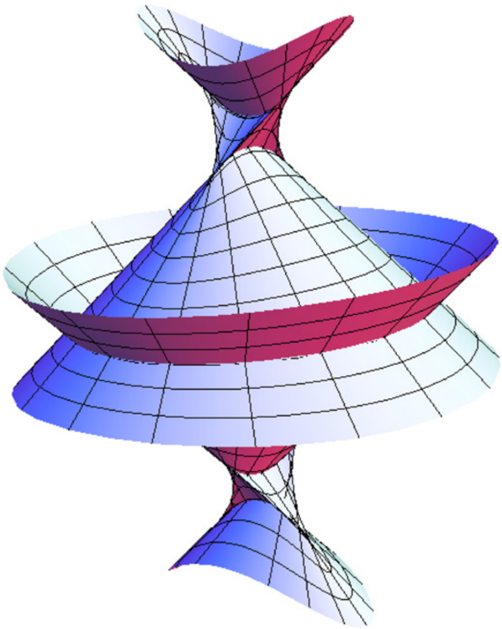

Fig. 4. The union of all tropic curves of a confocal family.

Pei defined a generalization of the Gauss map to surfaces in three-dimensional Minkowski space [19] and introduced the *pseudo vector product* as

$$\mathbf{x} \wedge \mathbf{y} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ e_1 & e_2 & -e_3 \end{vmatrix} = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, -(x_1y_2 - x_2y_1)) = E_{2,1}(\mathbf{x} \times \mathbf{y}).$$

It is easy to check that $\langle \mathbf{x} \wedge \mathbf{y}, \mathbf{x} \rangle_{2,1} = \langle \mathbf{x} \wedge \mathbf{y}, \mathbf{y} \rangle_{2,1} = 0$. Then, for surface $S : U \to \mathbf{E}^{2,1}$, with $U \subset \mathbf{R}^2$, the *Minkowski Gauss map* is defined as

$$\mathcal{G} : U \to \mathbf{RP}^2, \qquad \mathcal{G}(x_1, x_2) = \mathbf{P}\left(\frac{\partial S}{\partial x_1} \wedge \frac{\partial S}{\partial x_2}\right),$$

where $\mathbf{P} : \mathbf{R}^3 \setminus \{(0, 0, 0)\} \to \mathbf{RP}^2$ is the usual projection.

Lemma 4.4. *The Minkowski Gauss map of surfaces $\Sigma^\pm$ is singular at all points.*

Proof. This follows from the fact that $\mathbf{r}_\lambda \wedge \mathbf{r}_t$ is light-like for all λ and t. $\square$

Since the *pseudo-normal vectors* to $\Sigma^\pm$ are all light-like, these surfaces are *light-like developable*, as defined by Chino and Izumiya [7], who classified such surfaces as

- a light-like plane;
- a light-like cone; or
- a tangent surface of a light-like curve.

Since Σ^+ and Σ^- are contained neither in a plane nor in a cone, we expect that they will be tangent surfaces of some light-like curve, as shown in Corollary 4.10.

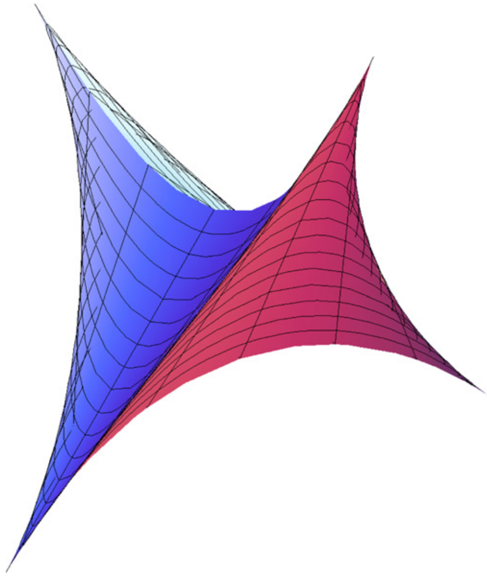

Fig. 5. Curved tetrahedron $\mathcal{T}^+$: the union of all tropic curves on Σ^+ corresponding to $\lambda \in (b, a)$.

For each of the surfaces Σ^+, Σ^-, it is evident that tropic lines corresponding to 1-sheeted hyperboloids oriented along the y-axes form a curved tetrahedron (Fig. 4). We denote the tetrahedra by $\mathcal{T}^+$ and $\mathcal{T}^-$, respectively; they are symmetric with respect to the xy-plane. Fig. 5 shows the tetrahedron $\mathcal{T}^+ \subset \Sigma^+$.

We now summarize the properties of these tetrahedra.

Proposition 4.5. *Consider the subset $\mathcal{T}^+$ of Σ^+ determined by the condition $\lambda \in [b, a]$. This set is a curved tetrahedron with the following properties.*

- *Its vertices are*

$$V_1\left(\frac{a-b}{\sqrt{a+c}}, 0, \frac{b+c}{\sqrt{a+c}}\right), \qquad V_2\left(-\frac{a-b}{\sqrt{a+c}}, 0, \frac{b+c}{\sqrt{a+c}}\right),$$
$$V_3\left(0, \frac{a-b}{\sqrt{b+c}}, \frac{a+c}{\sqrt{b+c}}\right), \qquad V_4\left(0, -\frac{a-b}{\sqrt{b+c}}, \frac{a+c}{\sqrt{b+c}}\right).$$

- *The shorter arcs of conics $\mathcal{C}_b^{xz}$ and $\mathcal{C}_a^{yz}$ determined by V_1, V_2 and V_3, V_4, respectively, are two edges of the tetrahedron.*
- *These two edges represent self-intersection of Σ^+.*
- *The other four edges are determined by the relation*

$$-a - b + 2\lambda + (a-b)\cos 2t = 0. \tag{4.3}$$

- *These four edges are cuspid edges of Σ^+.*
- *Thus, at each vertex of the tetrahedron, a swallowtail singularity of Σ^+ occurs.*

Proof. Eq. (4.3) is obtained from the condition $\mathbf{r}_t \times \mathbf{r}_\lambda = 0$. □

Lemma 4.6. *The tropic curves of the quadric $\mathcal{Q}_{\lambda_0}$ represent exactly the locus of points (x, y, z) for which the equation*

$$\frac{x^2}{a-\lambda}+\frac{y^2}{b-\lambda}+\frac{z^2}{c+\lambda}=1 \tag{4.4}$$

has λ_0 as a multiple root.

Proof. Without loss of generality, take $\lambda_0 = 0$. Eq. (4.4) is equivalent to

$$\lambda^3+q_2\lambda^2+q_1\lambda+q_0=0 \tag{4.5}$$

with

$$q_2=-x^2-y^2+z^2+a+b-c,$$
$$q_1=x^2(b-c)+y^2(a-c)-z^2(a+b)-ab+bc+ac,$$
$$q_0=x^2bc+y^2ac+z^2ab-abc.$$

Polynomial (4.5) has $\lambda_0 = 0$ as a double zero if and only if $q_0 = q_1 = 0$. Obviously, $q_0 = 0$ is equivalent to the condition that (x, y, z) belongs to $\mathcal{Q}_0$. Additionally, we have

$$\begin{aligned} q_1 &= x^2(b-c)+y^2(a-c)-z^2(a+b)-ab+bc+ac \\ &= (ab-bc+ac)\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1\right)-abc\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}\right), \end{aligned}$$

which is needed. □

Proposition 4.7. *A tangent line to the tropic curve of a non-degenerate quadric of the family (4.2) is always space-like, except on a 1-sheeted hyperboloid oriented along the y-axis.*

Tangent lines to a tropic on 1-sheeted hyperboloids oriented along the y-axis are light-like at four points, while at other points of the tropic curve the tangents are space-like.

Moreover, a tangent line to the tropic of a quadric from (4.2) belongs to the quadric if and only if it is light-like.

Proof. The tropic curves on $\mathcal{Q}_\lambda$, similar to the approach in Proposition 4.3, can be represented as

$$x(t)=\frac{a-\lambda}{\sqrt{a+c}}\cos t,\qquad y(t)=\frac{b-\lambda}{\sqrt{b+c}}\sin t,\qquad z(t)=\pm(c+\lambda)\sqrt{\frac{\cos^2 t}{a+c}+\frac{\sin^2 t}{b+c}},$$

with $t \in [0, 2\pi)$.

We calculate

$$\dot{x}^2+\dot{y}^2-\dot{z}^2=\frac{(a+b-2\lambda-(a-b)\cos 2t)^2}{2(a+b+2c-(a-b)\cos 2t)},$$

which is always non-negative and may only be zero if $\lambda \in [b, a]$. For $\lambda \in (b, a)$, there are exactly four values of t in $[0, 2\pi)$ for which the expression is zero.

Now fix $t \in [0, 2\pi)$ and $\lambda \in \mathbf{R}$. The tangent to the tropic of $\mathcal{Q}_\lambda$ at $(x(t), y(t), z(t))$ is completely contained in $\mathcal{Q}_\lambda$ if and only if, for each τ,

$$\frac{(x(t)+\tau\dot{x}(t))^2}{a-\lambda}+\frac{(y(t)+\tau\dot{y}(t))^2}{b-\lambda}+\frac{(z(t)+\tau\dot{z}(t))^2}{c+\lambda}=1,$$

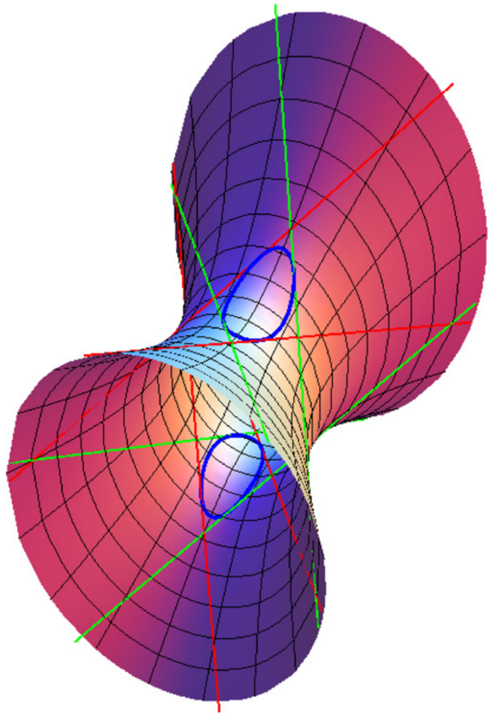

Fig. 6. Tropic curves and light-like tangents on a hyperboloid.

which is equivalent to

$$\frac{a+b-2\lambda-(a-b)\cos 2t}{a+b+2c-(a-b)\cos 2t}=0. \quad \square$$

Remark 4.8. In other words, the only quadrics of family (4.2) that may contain a tangent to their tropic curve are 1-sheeted hyperboloids oriented along the y-axis, and such tangents are always light-like. The tropic curves and their light-like tangents on such a hyperboloid are shown in Fig. 6.

Note that equations obtained in the proof of Proposition 4.7 are equivalent to (4.3) of Proposition 4.5, which leads to the following.

Proposition 4.9. *Each generatrix of Σ^+ and Σ^- is contained in one* 1*-sheeted hyperboloid oriented along the y-axis from* (4.2)*. Moreover, such a generatrix touches one of the tropic curves of the hyperboloid and one of the cusp-like edges of the corresponding curved tetrahedron at the same point.*

Corollary 4.10. *Surfaces Σ^+ and Σ^- are tangent surfaces of the cuspid edges of tetrahedra $\mathcal{T}^+$ and $\mathcal{T}^-$, respectively.*

The next propositions provide further analysis of light-like tangents to tropic curves on a 1-sheeted hyperboloid oriented along the y-axis.

Proposition 4.11. *For fixed $\lambda_0 \in (b, a)$, consider a hyperboloid $\mathcal{Q}_{\lambda_0}$ from (4.2) and an arbitrary point (x, y, z) on $\mathcal{Q}_{\lambda_0}$. Eq. (4.4) has, along with λ_0, two other roots in $\mathbf{C}$, which we denote by λ_1 and λ_2. Then $\lambda_1 = \lambda_2$ if and only if (x, y, z) is placed on a light-like tangent to a tropic curve of $\mathcal{Q}_{\lambda_0}$.*

Proof. The proof follows from the fact that the light-like tangents are contained in $\Sigma^+ \cup \Sigma^-$ (Proposition 4.3, Lemma 4.6 and Proposition 4.9). □

Proposition 4.12. *Two light-like lines on a 1-sheeted hyperboloid oriented along the y-axis from (4.2) are either skew or intersect each other on a degenerate quadric from (4.2).*

Proof. The proof follows from the fact that the hyperboloid is symmetric with respect to the coordinate planes. □

Lemma 4.13. *Consider a non-degenerate quadric $\mathcal{Q}_{\lambda_0}$ that is not a hyperboloid oriented along the y-axis, i.e. $\lambda_0 \notin [b, a] \cup \{-c\}$. Then each point of $\mathcal{Q}_{\lambda_0}$ that is not on one of the tropic curves is contained in two additional distinct quadrics from family (4.2).*

Consider two points A and B of $\mathcal{Q}_{\lambda_0}$ in the same connected component bounded by the tropic curves. Let λ'_A, λ''_A and λ'_B, λ''_B denote the solutions, different to λ_0, of (4.4) corresponding to A and B, respectively. Then, if λ_0 is less than (greater than, between) λ'_A, λ''_A, it is also less than (greater than, between) λ'_B, λ''_B.

Lemma 4.14. *Let $\mathcal{Q}_{\lambda_0}$ be a hyperboloid oriented along the y-axis, $\lambda_0 \in (b, a)$. Let A and B be two points of $\mathcal{Q}_{\lambda_0}$ in the same connected component bounded by the tropic curves and light-like tangents. Then if A is contained in two more quadrics from family (4.2), the same is true for B.*

In this case, let λ'_A, λ''_A and λ'_B, λ''_B denote the real solutions, different to λ_0, of (4.4) corresponding to A and B, respectively. Then if λ_0 is less than (greater than, between) λ'_A, λ''_A, it is also less than (greater than, between) λ'_B, λ''_B.

By contrast, if A is not contained in any other quadric from (4.2), then the same is true for all points of its connected component.

Proof. The proof of Lemmas 4.13 and 4.14 follows from the fact that the solutions of (4.4) continuously change through space and that two of the solutions coincide exactly on tropic curves and their light-like tangents. □

4.5. Generalized Jacobi coordinates and relativistic quadrics in three-dimensional Minkowski space

Definition 4.15. The *generalized Jacobi coordinates* of point (x, y, z) in the 3D Minkowski space $\mathbf{E}^{2,1}$ are the unordered triplet of solutions of (4.4).

Note that any of the following cases may occur:

- Generalized Jacobi coordinates are real and different.
- Only one generalized Jacobi coordinate is real.
- Generalized Jacobi coordinates are real, but two of them coincide; or
- All three generalized Jacobi coordinates are equal.

Lemmas 4.13 and 4.14 help in defining relativistic types of quadrics in 3D Minkowski space. Consider connected components of quadrics from (4.2) bounded by tropic curves and, for

1-sheeted hyperboloids oriented along the y-axis, their light-like tangent lines. Each connected component represents a *relativistic quadric*.

Definition 4.16. A component of quadric $\mathcal{Q}_{\lambda_0}$ is of *relativistic type E* if, at each of its points, λ_0 is less than the other two generalized Jacobi coordinates.

A component of quadric $\mathcal{Q}_{\lambda_0}$ is of *relativistic type* H^1 if, at each of its points, λ_0 is between the other two generalized Jacobi coordinates.

A component of quadric $\mathcal{Q}_{\lambda_0}$ is of *relativistic type* H^2 if, at each of its points, λ_0 is greater than the other two generalized Jacobi coordinates.

A component of quadric $\mathcal{Q}_{\lambda_0}$ is of *relativistic type* 0 if, at each of its points, λ_0 is the only real generalized Jacobi coordinate.

Lemmas 4.13 and 4.14 guarantee that the types of relativistic quadrics are well defined, so that each such quadric can be assigned a unique type E, H^1, H^2 or 0.

Definition 4.17. Suppose (x, y, z) is a point of the 3D Minkowski space $\mathbf{E}^{2,1}$ where (4.4) has real and different solutions. *Decorated Jacobi coordinates* of that point are the ordered triplet of pairs

$$(E, \lambda_1), \qquad (H^1, \lambda_2), \qquad (H^2, \lambda_3)$$

of generalized Jacobi coordinates and the corresponding types of relativistic quadrics.

Now we analyse the arrangement of the relativistic quadrics. We start with their intersections with the coordinate planes.

Intersection with the xy-plane. In the xy-plane, the Minkowski metric is reduced to the Euclidean metric. Family (4.2) intersects this plane in the following family of confocal conics:

$$\mathcal{C}_\lambda^{xy} : \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} = 1. \tag{4.6}$$

We conclude that the xy-plane is divided by ellipse $\mathcal{C}_{-c}^{xy}$ into two relativistic quadrics:

- The region within $\mathcal{C}_{-c}^{xy}$ is a relativistic quadric of E-type.
- The region outside this ellipse is of H^1-type.

Moreover, the types of relativistic quadrics intersecting the xy-plane are as follows:

- The components of ellipsoids are of H^1-type.
- The components of 1-sheeted hyperboloids oriented along the y-axis are of H^2-type.
- The components of 1-sheeted hyperboloids oriented along the z-axis are of E-type.

Intersection with the xz-plane. In the xz-plane, the metric is reduced to the Minkowski metric. The intersection of family (4.2) with this plane is the following family of confocal conics:

$$\mathcal{C}_\lambda^{xz} : \frac{x^2}{a-\lambda} + \frac{z^2}{c+\lambda} = 1. \tag{4.7}$$

The plane is divided by ellipse $\mathcal{C}_b^{xz}$ and the four joint tangents of (4.7) into 13 parts:

- The part within $\mathcal{C}_b^{xz}$ is a relativistic quadric of H^1-type.
- Four parts placed outside of $\mathcal{C}_b^{xz}$ that have a non-empty intersection with the x-axis are of H^2-type.

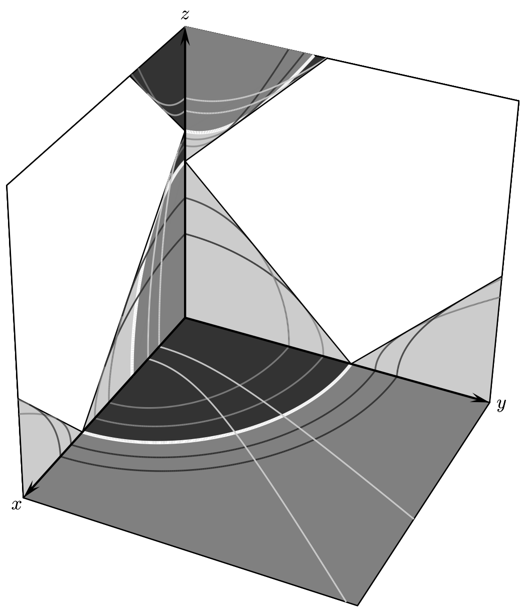

Fig. 7. Intersection of relativistic quadrics with coordinate planes.

- Four parts placed outside of $\mathcal{C}_b^{xz}$ that have a non-empty intersection with the z-axis are of E-type.
- The four remaining parts are of 0-type and no quadric from the family (4.2), except the degenerate $\mathcal{Q}_b$, passes through any of their points.

Intersection with the yz-plane. As in the previous case, in the yz-plane, the metric is reduced to the Minkowski metric. The intersection of family (4.2) with this plane is the following family of confocal conics:

$$\mathcal{C}_\lambda^{yz} : \frac{y^2}{b-\lambda} + \frac{z^2}{c+\lambda} = 1. \tag{4.8}$$

The plane is divided by hyperbola $\mathcal{C}_a^{yz}$ and joint tangents of (4.8) into 15 parts:

- The two convex parts determined by $\mathcal{C}_a^{yz}$ are relativistic quadrics of H^1-type.
- Five parts placed outside of $\mathcal{C}_a^{yz}$ that have a non-empty intersection with the coordinate axes are of H^2-type.
- Four parts, each placed between $\mathcal{C}_a^{yz}$ and one of the joint tangents of (4.8), are of E-type.
- No quadric from the family (4.2), except the degenerate $\mathcal{Q}_a$, passes through points of the four remaining parts.

The intersection of relativistic quadrics with the coordinate planes is shown in Fig. 7. Type E quadrics are coloured in dark gray, type H^1 medium grey, type H^2 light grey, and type 0 white. Curves $\mathcal{C}_{-c}^{xy}$, $\mathcal{C}_b^{xz}$, $\mathcal{C}_a^{yz}$ are also white in the figure.

From the above analysis, using Lemmas 4.13 and 4.14, we can determine the type of each relativistic quadric with a non-empty intersection with some of the coordinate hyper-planes.

One-sheeted hyperboloids oriented along the z-axis: $\lambda \in (-\infty, -c)$. Such a hyperboloid is divided by its tropic curves into three connected components; two of them are unbounded and mutually symmetric with respect to the xy-plane, while the third is the bounded annulus placed between them. The two symmetric components are of H^1-type, while the third is of E-type.

Ellipsoids: $\lambda \in (-c, b)$. An ellipsoid is divided by its tropic curves into three bounded connected components; two of them are mutually symmetric with respect to the xy-plane, while the third is the annulus placed between them. In this case, the symmetric components represent relativistic quadrics of E-type. The annulus is of H^1-type.

One-sheeted hyperboloids oriented along the y-axis: $\lambda \in (b, a)$. The decomposition of those hyperboloids into relativistic quadrics is more complicated and interesting than for the other types of quadrics from (4.2). By its two tropic curves and their eight light-like tangent lines, such a hyperboloid is divided into 28 connected components:

- Two bounded components placed inside the tropic curves are of H^1-type.
- Four bounded components placed between the tropic curves and light-like tangents such that they have non-empty intersections with the xz-plane are of H^2-type.
- Four bounded components placed between the tropic curves and light-like tangents such that they have non-empty intersections with the yz-plane are of E-type.
- Two bounded components, each limited by four light-like tangents, are of H^2-type.
- Four unbounded components, each limited by two light-like tangents such that they have non-empty intersections with the xy-plane, are of H^2-type.
- Four unbounded components, each limited by two light-like tangents such that they have non-empty intersections with the yz-plane, are of E-type.
- Eight unbounded components, each limited by four light-like tangents, are sets of points not contained in any other quadric from (4.2).

Two-sheeted hyperboloids: $\lambda \in (a, +\infty)$. Such a hyperboloid is divided by its tropic curves into four connected components: two bounded ones are of H^2-type, while the two unbounded are of H^1-type.

4.6. Decorated Jacobi coordinates and relativistic quadrics in d-dimensional pseudo-Euclidean space

Inspired by the results obtained in Sections 4.2 and 4.5, we introduce relativistic quadrics and their types in confocal family (2.3) in the d-dimensional pseudo-Euclidean space $\mathbf{E}^{k,l}$.

Definition 4.18. *Generalized Jacobi coordinates* of point x in the d-dimensional pseudo-Euclidean space $\mathbf{E}^{k,l}$ is the unordered d-tuple of solutions λ of

$$\frac{x_1^2}{a_1-\lambda}+\cdots+\frac{x_k^2}{a_k-\lambda}+\frac{x_{k+1}^2}{a_{k+1}+\lambda}+\cdots+\frac{x_d^2}{a_d+\lambda}=1. \tag{4.9}$$

As already mentioned in Section 2.2, this equation has either d or $d-2$ real solutions. Besides, some of the solutions may be multiple.

The set Σ_d of points x in $\mathbf{R}^d$ where Eq. (4.9) has multiple solutions is an algebraic hypersurface. Σ_d divides each quadric from (2.3) into several connected components. We call these components *relativistic quadrics*.

Since the generalized Jacobi coordinates depend continuously on x, we have the following definition.

Definition 4.19. We say that a relativistic quadric placed on $\mathcal{Q}_{\lambda_0}$ is of *type E* if, at each of its points, λ_0 is less than the other $d-1$ generalized Jacobi coordinates.

We say that a relativistic quadric placed on $\mathcal{Q}_{\lambda_0}$ is of *type* H^i $(1 < i < d-1)$ if, at each of its points, λ_0 is greater than i other generalized Jacobi coordinates and less than $d-i-1$ of them.

We say that a relativistic quadric placed on $\mathcal{Q}_{\lambda_0}$ is of *type* 0^i $(0 < i < d-2)$ if, at each of its points, λ_0 is greater than i other real generalized Jacobi coordinates and less than $d-i-2$ of them.

It would be interesting to analyse the properties of the discriminant manifold Σ_d, as well as the combinatorial structure of the arrangement of relativistic quadrics, as in Section 4.5 for $d=3$. Note that this description would have $[d/2]$ substantially different cases in each dimension, depending on the choice of k and l.

Definition 4.20. Suppose $(x_1,\dots,x_d)$ is a point of the d-dimensional Minkowski space $\mathbf{E}^{k,l}$ where (4.9) has real and different solutions. *Decorated Jacobi coordinates* of that point are the ordered d-tuplet of pairs

$$(E,\lambda_1),(H^1,\lambda_2),\dots,(H^{d-1},\lambda_d)$$

of generalized Jacobi coordinates and the corresponding types of relativistic quadrics.

Since we consider billiard system within ellipsoids in pseudo-Euclidean space, it is of interest to analyse the behaviour of decorated Jacobi coordinates inside an ellipsoid.

Proposition 4.21. *Let $\mathcal{E}$ be the ellipsoid in $\mathbf{E}^{k,l}$ given by (2.2). We have the following:*

PE1 *Each point inside $\mathcal{E}$ is the intersection of exactly d quadrics from (2.3); moreover, all of these quadrics are of different relativistic types.*

PE2 *Types of these quadrics are E, $H^1,\dots,H^{d-1}$; each type corresponds to one of the disjoint intervals of the parameter λ:*

$$(-a_d,-a_{d-1}),(-a_{d-1},-a_{d-2}),\dots,(-a_{k+1},0),(0,a_k),(a_k,a_{k-1}),\dots,(a_2,a_1).$$

Proof. The function given by the left-hand side of (4.9) is continuous and strictly monotonic in each interval $(-a_d,-a_{d-1}),(-a_{d-1},-a_{d-2}),\dots,(-a_{k+2},-a_{k+1}),(a_k,a_{k-1}),\dots,(a_2,a_1)$, with infinite values at their endpoints. Thus, (4.9) has one solution in each of them. Conversely, in $(-a_{k+1},a_k)$ the function tends to $+\infty$ at the endpoints, and has only one extreme value, the minimum. Since the value of the function for $\lambda=0$ is less than 1 for a point inside $\mathcal{E}$, it follows that (4.9) will have two solutions in $(-a_{k+1},a_k)$, one positive and one negative. □

Proposition 4.21 proves the relativistic analogues of properties E1 and E2 from Section 4.1 for the Euclidean case.

5. Billiards within quadrics and their periodic trajectories

In this section we first derive further properties of ellipsoidal billiards in pseudo-Euclidean spaces. In Section 5.1, Theorem 5.1 provides a simple and effective criterion for determining the type of a billiard trajectory, knowing its caustics. Then we derive properties PE3–PE5 in Proposition 5.2. In Section 5.2 we prove a generalization of the Poncelet theorem for ellipsoidal billiards in pseudo-Euclidean spaces and derive the corresponding Cayley-type conditions, giving a complete analytical description of periodic billiard trajectories in an arbitrary dimension. These results are contained in Theorems 5.3 and 5.4.

5.1. Ellipsoidal billiards

Ellipsoidal billiards. Billiard motion within an ellipsoid in pseudo-Euclidean space is uniformly straightforward inside the ellipsoid and obeys the reflection law on the boundary. We consider billiard motion within ellipsoid $\mathcal{E}$, given by (2.2), in $\mathbf{E}^{k,l}$. The family of quadrics confocal with $\mathcal{E}$ is (2.3).

Since functions F_i given by (2.7) are integrals of billiard motion [18,2,16], for each zero λ of (2.6) the corresponding quadric $\mathcal{Q}_\lambda$ is a caustic of billiard motion, i.e. it is tangential to each segment of the billiard trajectory passing through point x with velocity vector v.

Note that, according to Theorem 2.3, for a point placed inside $\mathcal{E}$, there are d real solutions of Eq. (4.9). In other words, there are d quadrics from the family (2.3) containing such a point, although some of them may be multiple. Also, by Proposition 2.2 and Theorem 2.3, a billiard trajectory within an ellipsoid will always have $d-1$ caustics.

According to Remark 2.1, all segments of a billiard trajectory within $\mathcal{E}$ will be of the same type. Now, we can apply the reasoning from Section 2.2 to billiard trajectories.

Theorem 5.1. *In the d-dimensional pseudo-Euclidean space $\mathbf{E}^{k,l}$, consider a billiard trajectory within ellipsoid $\mathcal{E} = \mathcal{Q}_0$, and let quadrics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$ from the family (2.3) be its caustics. Then all billiard trajectories within $\mathcal{E}$ sharing the same caustics are of the same type: space-like, time-like, or light-like, as the initial trajectory. Moreover, the type is determined as follows:*

- *if $\infty \in \{\alpha_1, \dots, \alpha_{d-1}\}$, the trajectories are light-like;*
- *if $(-1)^l \cdot \alpha_1 \cdots \alpha_{d-1} > 0$, the trajectories are space-like;*
- *if $(-1)^l \cdot \alpha_1 \cdots \alpha_{d-1} < 0$, the trajectories are time-like.*

Proof. Since values of functions F_i given by (2.7) are preserved by the billiard reflection and

$$\sum_{i=1}^{d} F_i(x, v) = \langle v, v\rangle_{k,l},$$

the type of billiard trajectory depends on the sign of $\sum_{i=1}^{d} F_i(x, v)$. From the equivalence of relations (2.6) and (2.8), it follows that the sum depends only on the roots of $\mathcal{P}$, i.e. on parameters $\alpha_1, \dots, \alpha_{d-1}$ of the caustics.

Note that the product $\alpha_1 \cdots \alpha_{d-1}$ changes continuously for the variety of lines in $\mathbf{E}^{k,l}$ that intersect $\mathcal{E}$, with infinite singularities at light-like lines. Besides, the subvariety of light-like lines divides the variety of all lines into subsets of space-like and time-like lines. When passing through light-like lines, one of the parameters α_i passes through infinity going from positive to negative real numbers or vice versa; thus, a change in sign for the product occurs simultaneously with a change in the type of line.

Now take $\alpha_j = -a_{k+j}$ for $1 \le j \le l$ and note that all lines placed in the k-dimensional coordinate subspace $\mathbf{E}^k \times \mathbf{0}^l$ will have corresponding degenerate caustics. The pseudo-Euclidean metric is reduced to the Euclidean metric in this subspace, and thus such lines are space-like. Since $\alpha_1, \dots, \alpha_k$ are positive for lines of $\mathbf{E}^k \times \mathbf{0}^l$ that intersect $\mathcal{E}$, the statement is proved. □

Note that, in general, for fixed $d-1$ quadrics from the confocal family we can find joint tangents of different types, which makes Theorem 5.1 somewhat unexpected. However, for fixed caustics, only lines having one type may intersect with a given ellipsoid, and only these lines give rise to billiard trajectories.

Next we investigate the behaviour of decorated Jacobi coordinates along ellipsoidal billiard trajectories.

Proposition 5.2. *Let $\mathcal{T}$ be the trajectory of a billiard within ellipsoid $\mathcal{E}$ in pseudo-Euclidean space $\mathbf{E}^{k,l}$. Let $\alpha_1, \dots, \alpha_{d-1}$ be the parameters of the caustics from the confocal family (2.3) of $\mathcal{T}$ and take $b_1, \dots, b_p$, $c_1, \dots, c_q$ as in Theorem 2.3. Then we have the following:*

PE3 *Along $\mathcal{T}$, each generalized Jacobi coordinate takes values in exactly one of the segments*

$$[c_{2l-1}, c_{2l-2}], \dots, [c_3, c_2], [c_1, 0], [0, b_1], [b_2, b_3], \dots, [b_{2k-2}, b_{2k-1}].$$

PE4 *Along $\mathcal{T}$, each generalized Jacobi coordinate can achieve local minima and maxima only at touching points of the corresponding caustics, points of intersection with the corresponding coordinate hyper-planes, and at reflection points.*

PE5 *Values of the generalized Jacobi coordinates at critical points are $0, b_1, \dots, b_{2k-1}$, $c_1, \dots, c_{2l-1}$; between the critical points, the coordinates change monotonically.*

Proof. Property PE3 follows from Theorem 2.3. Along each line, the generalized Jacobi coordinates change continuously. Moreover, they are monotonic at all points where the line has a transversal intersection with a non-degenerate quadric. Thus, critical points on a line are touching points with the corresponding caustics and points of intersection with the corresponding coordinate hyper-planes.

Note that the reflection points of $\mathcal{T}$ are also points of transversal intersection with all quadrics containing those points, except for $\mathcal{E}$. Thus, at such points, 0 will be a critical value of the corresponding generalized Jacobi coordinate, and all other coordinates are monotonic. This proves PE4 and PE5. $\square$

The properties we obtained are pseudo-Euclidean analogues of properties E3–E5 from Section 4.1, which hold for ellipsoidal billiards in Euclidean spaces.

5.2. Analytic conditions for periodic trajectories

Now we derive analytic conditions of Cayley type for periodic trajectories of an ellipsoidal billiard in pseudo-Euclidean space, and thus obtain a generalization of the Poncelet theorem for pseudo-Euclidean spaces.

Theorem 5.3 (*Generalized Cayley-Type Conditions*)**.** *In the pseudo-Euclidean space $\mathbf{E}^{k,l}$ ($k + l = d$), consider a billiard trajectory $\mathcal{T}$ within ellipsoid $\mathcal{E}$ given by (2.2). Let $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$ from confocal family (2.3) be the caustics of $\mathcal{T}$.*

Then $\mathcal{T}$ is periodic with period n if and only if the following condition is satisfied:

- *for $n = 2m$:*

$$\operatorname{rank}\begin{pmatrix} B_{d+1} & B_{d+2} & \dots & B_{d+m-1} \\ B_{d+2} & B_{d+3} & \dots & B_{d+m} \\ \dots & \dots & \dots & \dots \\ B_{m+1} & B_{m+2} & \dots & B_{2m-1} \end{pmatrix} < m - d + 1 \quad \text{or}$$

$$\operatorname{rank}\begin{pmatrix} B_{d+1} & B_{d+2} & \dots & B_{d+m} \\ \dots & \dots & \dots & \dots \\ B_{m} & B_{m+1} & \dots & B_{2m-1} \\ C_{m} & C_{m+1} & \dots & C_{2m-1} \\ D_{m} & D_{m+1} & \dots & D_{2m-1} \end{pmatrix} < m - d + 2;$$

- *for* $n = 2m + 1$*:*

$$\operatorname{rank}\begin{pmatrix} B_{d+1} & B_{d+2} & \dots & B_{d+m} \\ B_{d+2} & B_{d+3} & \dots & B_{d+m+1} \\ \dots & \dots & \dots & \dots \\ B_{m+1} & B_{m+2} & \dots & B_{2m} \\ C_{m+1} & C_{m+2} & \dots & C_{2m} \end{pmatrix} < m - d + 2 \quad \textit{or}$$

$$\operatorname{rank}\begin{pmatrix} B_{d+1} & B_{d+2} & \dots & B_{d+m} \\ B_{d+2} & B_{d+3} & \dots & B_{d+m+1} \\ \dots & \dots & \dots & \dots \\ B_{m+1} & B_{m+2} & \dots & B_{2m} \\ D_{m+1} & D_{m+2} & \dots & D_{2m} \end{pmatrix} < m - d + 2.$$

Here, (B_i), (C_i), (D_i) *are coefficients in the Taylor expansions around* $\lambda = 0$ *of the functions* $f(\lambda) = \sqrt{(\alpha_1 - \lambda)\cdots(\alpha_{d-1} - \lambda)\cdot(a_1 - \varepsilon_1\lambda)\cdots(a_d - \varepsilon_d\lambda)}$, $\frac{f(\lambda)}{b_1-\lambda}$ *and* $\frac{f(\lambda)}{c_1-\lambda}$, *respectively.*

Proof. Denote

$$\mathcal{P}_1(\lambda) = (\alpha_1 - \lambda)\cdots(\alpha_{d-1} - \lambda)\cdot(a_1 - \varepsilon_1\lambda)\cdots(a_d - \varepsilon_d\lambda).$$

Following Jacobi [15], along a given billiard trajectory we consider the integrals

$$\sum_{s=1}^{d}\int\frac{d\lambda_s}{\sqrt{\mathcal{P}_1(\lambda_s)}}, \sum_{s=1}^{d}\int\frac{\lambda_s d\lambda_s}{\sqrt{\mathcal{P}_1(\lambda_s)}}, \dots, \sum_{s=1}^{d}\int\frac{\lambda_s^{d-2} d\lambda_s}{\sqrt{\mathcal{P}_1(\lambda_s)}}. \tag{5.1}$$

According to PE3 of Proposition 5.2, we can suppose that

$$\lambda_1 \in [0, b_1], \qquad \lambda_i \in [b_{2i-2}, b_{2i-1}] \quad \text{for } 2 \le i \le k;$$
$$\lambda_{k+1} \in [c_1, 0], \qquad \lambda_{k+j} \in [c_{2j-1}, c_{2j-2}] \quad \text{for } 2 \le j \le l.$$

Along a billiard trajectory, according to PE4 and PE5 of Proposition 5.2, each λ_s will pass through the corresponding interval monotonically from one endpoint to another and vice versa, alternately. Note also that values $b_1, \dots, b_{2k-1}$, $c_1, \dots, c_{2l-1}$ correspond to the Weierstrass points of the hyper-elliptic curve

$$\mu^2 = \mathcal{P}_1(\lambda). \tag{5.2}$$

Calculation of integrals (5.1) reveals that the billiard trajectory is closed after n reflections if and only if, for some n_1, n_2 such that $n_1 + n_2 = n$,

$$n\mathcal{A}(P_0) \equiv n_1\mathcal{A}(P_{b_1}) + n_2\mathcal{A}(P_{c_1})$$

on the Jacobian of curve (5.2). Here, $\mathcal{A}$ is the Abel–Jacobi map and P_t is a point on the curve corresponding to $\lambda = t$. Furthermore, in the same manner as in previous studies [10,11], we obtain the conditions as stated in the theorem. □

As an immediate consequence, we obtain the following.

Theorem 5.4 (*Generalized Poncelet Theorem*)**.** *In the pseudo-Euclidean space* $\mathbf{E}^{k,l}$ $(k + l = d)$*, consider a billiard trajectory* $\mathcal{T}$ *within ellipsoid* $\mathcal{E}$*.*

If $\mathcal{T}$ *is periodic and becomes closed after* n *reflections on the ellipsoid, then any other trajectory within* $\mathcal{E}$ *with the same caustics as* $\mathcal{T}$ *is also periodic with period* n*.*

Remark 5.5. A generalization of the full Poncelet theorem of Chang et al. [6] to pseudo-Euclidean spaces was presented by Wang et al. [20]. However, they only discussed space-like and time-like trajectories.

A Poncelet-type theorem for light-like geodesics on the ellipsoid in 3D Minkowski space was proved by Genin et al. [13].

Remark 5.6. Theorems 5.3 and 5.4 also hold in symmetric and degenerate cases, that is, when some of the parameters $\varepsilon_i a_i$, α_j coincide or, in the case of light-like trajectories, when $\infty \in \{\alpha_j \mid 1 \leq j \leq d-1\}$. In such cases, we need to desingularise the corresponding curve, as previously explained in detail [9,12].

When we consider light-like trajectories, the factor containing the infinite parameter is omitted from polynomial $\mathcal{P}_1$.

Example 5.7. We find all 4-period trajectories within ellipse $\mathcal{E}$ given by (3.1) in the Minkowski plane, i.e. all conics $\mathcal{C}_\alpha$ from confocal family (3.2) corresponding to such trajectories.

By Theorem 5.3, the condition is $B_3 = 0$, where

$$\sqrt{(a-\lambda)(b+\lambda)(\alpha-\lambda)} = B_0 + B_1\lambda + B_2\lambda^2 + B_3\lambda^3 + \cdots$$

is the Taylor expansion around $\lambda = 0$. Since

$$B_3 = \frac{(-ab - a\alpha + b\alpha)(-ab + a\alpha + b\alpha)(ab + a\alpha + b\alpha)}{16(ab\alpha)^{5/2}},$$

we obtain the following solutions:

$$\alpha_1 = \frac{ab}{b-a}, \qquad \alpha_2 = \frac{ab}{a+b}, \qquad \alpha_3 = -\frac{ab}{a+b}.$$

Since $\alpha_1 \notin (-b, a)$ and $\alpha_2, \alpha_3 \in (-b, a)$, conic $\mathcal{C}_{\alpha_1}$ is a hyperbola, while $\mathcal{C}_{\alpha_2}$ and $\mathcal{C}_{\alpha_3}$ are ellipses.

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