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A partial order on classical and quantum states

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A PARTIAL ORDER ON CLASSICAL AND QUANTUM STATES

A Thesis

Submitted to the Graduate Faculty of the
Louisiana State University and
Agriculture and Mechanical College
in partial fulfillment of the
requirement for the degree of
Master of Science in System Science

in

The Department of Computer Science

By
Arka Bandyopadhyay
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ABSTRACT

In this work we extend the work done by Bob Coecke and Keye Martin in their paper “Partial Order on Classical States and Quantum States (2003)”. We review basic notions involving elementary domain theory, the set of probability measures on a finite set $\{a_1, a_2, \dots, a_n\}$, which we identify with the standard $(n-1)$ -simplex Δ^n and Shannon Entropy. We consider partial orders on Δ^n , which have the *Entropy Reversal Property* (ERP) : elements lower in the order have higher (Shannon) entropy or equivalently less information . The ERP property is important because of its applications in quantum information theory. We define a new partial order on Δ^n , called Stochastic Order , using the well-known concept of *majorization order* and show that it has the ERP property and is also a continuous domain. In contrast, the *bayesian order* on Δ^n defined by Coecke and Martin has the ERP property but is *not* continuous.

CHAPTER 1

INTRODUCTION

The notion of a “domain” was introduced more than forty years ago by Dana Scott as an appropriate mathematical universe for the semantics of programming languages. In simple terms, a domain is a poset (partially ordered set) with the intrinsic notions of completeness and approximation. For example, the powerset of the set of natural numbers ordered by inclusion or the binary strings (possibly infinite) under prefix order. These are classic examples of partially ordered sets. In the former, the only approximants of an infinite set (e.g. $\{2, 4, 6, \dots\}$) are its finite subsets. In the second poset, any finite or infinite string $x < y$ is approximant to y .

We explore various properties and interrelationships of partial orders that one may define on the set of classical information states $\Delta^n = \{\vec{x} \in [0, 1]^n : \sum_{i=1}^n x_i = 1\}$, $n \geq 2$, including the the Bayesian order, the majorization order, and the stochastic order.

CHAPTER 2

TERMINOLOGY

2.1 Posets and Preorders

A set P with a *binary relation* \sqsubseteq is called a *partially ordered set* or *poset* if the following holds $\forall x, y, z \in P$.

- $x \sqsubseteq x$ (Reflexivity)
- $x \sqsubseteq y \wedge y \sqsubseteq z \implies x \sqsubseteq z$ (Transitivity)
- $x \sqsubseteq y \wedge y \sqsubseteq x \implies x = y$ (Anti-symmetry)

P is called a preorder if we only have reflexivity and transitivity.

2.2 Notation from Order Theory ^[1]

We will use the following notations for a poset (P, \sqsubseteq) .

- The *upper set* of an element $x \in P$ is $\uparrow\{x\} = \{y \in P : y \sqsupseteq x\} \subseteq P$. When no confusion is likely, we abbreviate $\uparrow\{x\}$ as $\uparrow x$. The upper set of a subset A of P is the set of all elements above some element of A , i.e., $\uparrow A = \bigcup_{x \in A} \uparrow\{x\}$. The dual notion is lower set $\downarrow A$ which is the set of all elements which are below some element of A . Clearly, $y \in \downarrow\{x\} \implies x \in \uparrow\{y\}$

- An element $x \in P$ is called an *upper bound* for a subset $A \subseteq P$, if x is above every element of A . We often write $A \sqsubseteq x$ in this situation. We denote by $\text{ub}(A)$ the set of all upper bounds of A . Dually, $\text{lb}(A)$ denotes the set of all *lower bounds* of A .
- An element $x \in A$ is called a *maximal* element of A if there is no other element of A above it: $\uparrow x \cap A = \{x\}$. *Minimal* elements are defined similarly. For a subset $A \subseteq P$ the minimal elements of $\text{ub}(A)$ are called *minimal upper bounds* of A . The set of all minimal upper bounds of A is denoted by $\text{mub}(A)$.
- If all elements of P are below a single element $x \in P$, then x is said to be the *largest element*. The dually defined least element of a poset is also called *bottom* and is commonly denoted by \perp . In the presence of a least element we speak of a *pointed poset*.
- If for a subset $A \subseteq P$, the set $\text{ub}(A)$ of A has a least element x , then x is called the *supremum* or *join* of A . We write $x = \sqcup A$ in this case. In the other direction we speak of *infimum* or *meet* and write $x = \sqcap A$.
- A partially ordered set P is a \sqcup -*semilattice* (\sqcap -*semilattice*) if the supremum (infimum) for each pair of elements exists. If P is both a \sqcup -semilattice and a \sqcap -semilattice, then P is called a *lattice*. A *lattice* is complete if suprema and infima exist for all subsets.

2.3 Directed Set

Let P be a poset. A subset A of P is *directed* if it is nonempty and each pair of elements of A has an upper bound in A . If a directed set A has a supremum then this is denoted by $\sqcup\uparrow A$

2.4 Directed Complete Partial Orders

2.4.1 Definition

A poset D in which every directed subset has a supremum is called a *directed-complete partial order*, or *dcpo* for short. Equivalently, a poset D is a dcpo if and only if each *chain* in D has a supremum.^[1]

2.5 Approximation

The notion of approximation is central to our study of partial orders on Δ^n . In Computer Science, the notion of approximation applies to machine learning, where one tries to learn the probability distribution on a finite set from a sequence of individual observations and the resulting frequency distribution. Each frequency distribution is a point in Δ^n . The notion of approximation is inherent to domain theory but can have largescale applications to a lot of fields which need to be explored. In general this idea can be applied to any recursive process where you cannot actually reach the limit of computation.

2.5.1 Definition

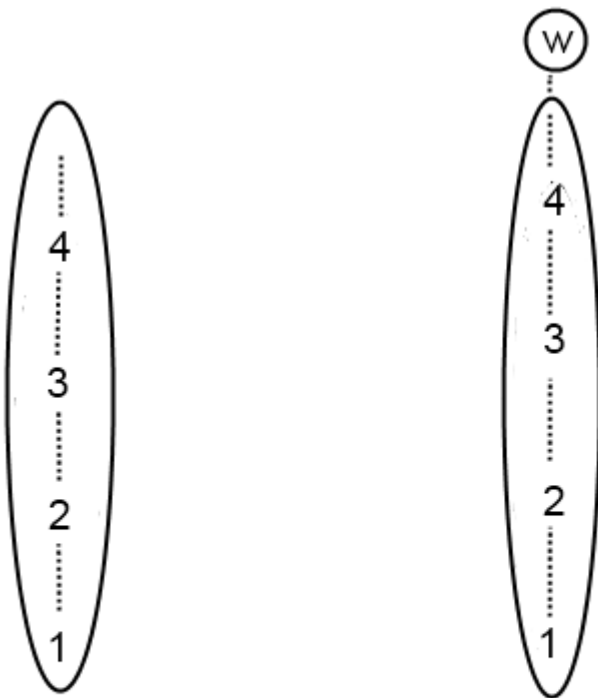
Let $x \sqsubseteq y$ be elements of a dcpo D . (The definition below actually applies to any poset.) We say that x *approximates* y if \forall directed $A \subseteq D$, $y \sqsubseteq \sqcup \uparrow A \Rightarrow x \sqsubseteq a$ for some $a \in A$. In other words, x *approximates* y if $x \sqsubseteq y$ and for every chain $x_1 \sqsubseteq x_2 \sqsubseteq \dots \sqsubseteq \sqcup \uparrow x_i \sqsupseteq y \Rightarrow x \sqsubseteq x_i$ for some i . We use the notation $x \ll y$ to indicate x approximates y . The relation ' \ll ' is traditionally called the “way below relation”.

2.5.2 Compact Elements

We say that x is compact (or finite or isolated) when x approximates itself. In other words, there exists no chain $x_1 \sqsubseteq x_2 \sqsubseteq \dots \sqsubseteq \sqcup \uparrow x_i = x$

2.5.3 Examples

1. Consider a finite set of elements $\{a_1, a_2, \dots, a_n\}$, with the only “ \leq ” relation $a_i \leq a_i$ for all i . Here, every element is an approximant to itself and is also equal to its lower set. So, $\text{approximants}(a_i) = \downarrow a_i = \{a_i\}$ for all $1 \leq i \leq n$.
2. Consider the poset of natural numbers $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ with $i \sqsubseteq i+1$. It is not complete because the chain $1 \sqsubseteq 2 \sqsubseteq 3 \sqsubseteq \dots$ has no supremum but satisfies $\text{approximants}(x) = \downarrow x$.
3. If we add an element w to \mathbb{N} with $i \sqsubseteq w$ for $i \in \mathbb{N}$, $\text{approximants}(x) = \downarrow x \forall x \neq w$ and $\text{approximants}(w) = \{1, 2, 3, 4, \dots\} \subset \downarrow w$.



not complete

complete but
w not compact

Figure 1 : Directed Complete Partial Order

2.6 Entropy

We study partial order on probability distributions (on a finite set) where the ordering is related to the randomness (entropy) of the distribution.

The Shannon entropy was first introduced by Claude E. Shannon in 1948 in his landmark paper “A Mathematical Theory of Communication.” Shannon entropy is a measure of the uncertainty of (individual observations of) a probability distribution as well as a measure of information content. Uffink(1990) provides an axiomatic

characterization of measures of uncertainty, deriving a general class of measures $U_r(\vec{p})$, of which the Shannon entropy is one (see also Maassen and Uffink, 1989). The key property possessed by these measures is Schur Concavity.

2.6.1 Entropy of a Discrete Random Variable

Let A be a discrete random variable on a finite set $\{a_1, a_2, \dots, a_n\}$ with probabilities $p(a_i) = \Pr(A = a_i)$. The Shannon entropy $H_n(p_1, p_2, \dots, p_n) = H(A)$ of the random variable A is defined as $H(A) = - \sum_i p(a_i) \log p(a_i)$. The convention $0 \log 0 = 0$ is adopted in the definition. The logarithm is usually taken to the base 2, in which case the entropy is measured in “bits,” or to the base e , in which case $H(X)$ is measured in “nats.”

2.6.2 Relative Entropy

Kullback–Leibler Divergence^[3] or Information Divergence or Information Gain or Relative Entropy $H(A||B)$ is a non-symmetric measure of the difference between two probability distributions P_A and P_B on the same set of events $a_i = b_i = e_i, 1 \leq i \leq n$. It measures the expected number of extra bits required to code samples from P_A when using a code based on P_B , rather than using a code based on P_A . $H(A||B) = \sum_{i=1}^n P_A(e_i) [\log P_A(e_i) - \log P_B(e_i)] = \sum_{i=1}^n P_A(e_i) \log \frac{P_A(e_i)}{P_B(e_i)} = -H(A) - \sum_{i=1}^n P_A(e_i) \cdot \log P_B(e_i) = H(P_A, P_B) - H(A)$ where $H(P_A, P_B) = - \sum_{i=1}^n P_A(e_i) \cdot \log P_B(e_i)$ is called the Cross Entropy.

2.6.2.1 Properties

1. $H(A|A) = 0$ and $H(A|B) \geq 0$

Proof: We need to prove only the second part. Since $-\log x \geq \frac{1}{\ln 2}(1-x)$ for $x > 0$,
$$H(A|B) = \sum_{i=1}^n P_A(e_i) \log \frac{P_A(e_i)}{P_B(e_i)} \geq \sum_{i=1}^n P_A(e_i) \frac{1 - \frac{P_A(e_i)}{P_B(e_i)}}{\ln 2} = \sum_{i=1}^n \frac{[P_A(e_i) - P_B(e_i)]}{\ln 2} = 0$$

Corollary. $\log(n)$ is the maximal entropy and it happens only for the uniform distribution.

Proof: Consider the uniform probabilities $P_B(e_i) = \frac{1}{n}$. Then, $H(B) = \log(n)$ and from above we get $H(A|B) = -H(A) + \log n \geq 0 \Rightarrow H(A) \leq \log(n)$. So any probability distribution that is not uniform is not going to have maximal entropy.

2.7 Characterization of Entropy Function and Uniqueness as a Consequence

The Shannon entropy satisfies the following properties^[5].

1. For any n , $H_n(p_1, p_2, \dots, p_n)$ is a continuous and symmetric function on variables p_1, p_2, \dots, p_n , i.e, $H_n(p_{\sigma(1)}, p_{\sigma(2)}, \dots, p_{\sigma(n)}) = H_n(p_1, p_2, \dots, p_n)$ for any permutation σ of indices.
2. Event of probability zero does not contribute to the entropy, i.e. $\forall n$,

$$H_{n+1}(p_1, p_2, \dots, p_n, 0) = H_n(p_1, p_2, \dots, p_n).$$

3. Entropy is maximized when the probability distribution is uniform, i.e., $\forall n$, $H_n(p_1, p_2, \dots, p_n) \leq H_n(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$. This follows from Jensen inequality, $H(A) = E[\log(1/p(A))] \leq \log(E[1/p(A)]) = \log(n)$.
4. If $p_{ij} \geq 0$, $1 \leq i \leq m$, $1 \leq j \leq n$ where $\sum_{j=1}^n p_{ij} = 1$ and $q_i = \sum_{j=1}^n p_{ij}$, then $H_{mn}(p_{11}, \dots, p_{mn}) = H_m(q_1, q_2, \dots, q_m) + \sum_{i=1}^m q_i H_n(\frac{p_{i1}}{q_i}, \frac{p_{i2}}{q_i}, \dots, \frac{p_{in}}{q_i})$. If we partition the mn outcomes of the random experiment into m groups, each group contains n elements, we can do the experiment in two steps: first determine the group to which the actual outcome belongs to and second find the outcome in this group. The probability that you will observe group i is q_i . The conditional probability distribution of given group i is $(\frac{p_{i1}}{q_i}, \frac{p_{i2}}{q_i}, \dots, \frac{p_{in}}{q_i})$. The entropy $H_n(\frac{p_{i1}}{q_i}, \frac{p_{i2}}{q_i}, \dots, \frac{p_{in}}{q_i})$ is the entropy of the probability distribution conditioned on group i . Property 4 says that the total information is the sum of the information you gain in the first step, $H_m(q_1, q_2, \dots, q_m)$, and a weighted sum of the entropies conditioned on each group. A. I. Khinchin in 1957 showed that the only function satisfying the above assumptions is of the form: $H(A) = -k \sum_i p_i \log p_i$; where k is a positive constant, essentially a choice of unit of measure.

2.8 Barycentric Subdivision ^[7]

Creating a barycentric subdivision is a recursive process. In dimension one, start with an interval; the mid-point of an interval cuts the interval into two intervals, giving a barycentric subdivision in one dimension. In dimension two, we start with a triangle,

subdivide each side at its mid-point to obtain its barycentric subdivision and then draw lines from the centre of the triangle to the mid-points and to the corners. This cuts the triangle into 6 smaller triangles, giving the barycentric subdivision of the triangle. Note that each of the smaller triangles is itself a simplex in dimension two. Given a pyramid in dimension three, subdivide each triangular face as above, then draw walls from the center of the pyramid to the lines that subdivide each triangular face. In general, an n -simplex is cut in this way into $(n+1)!$ smaller simplexes of the same dimension giving the barycentric subdivision of the n - simplex.

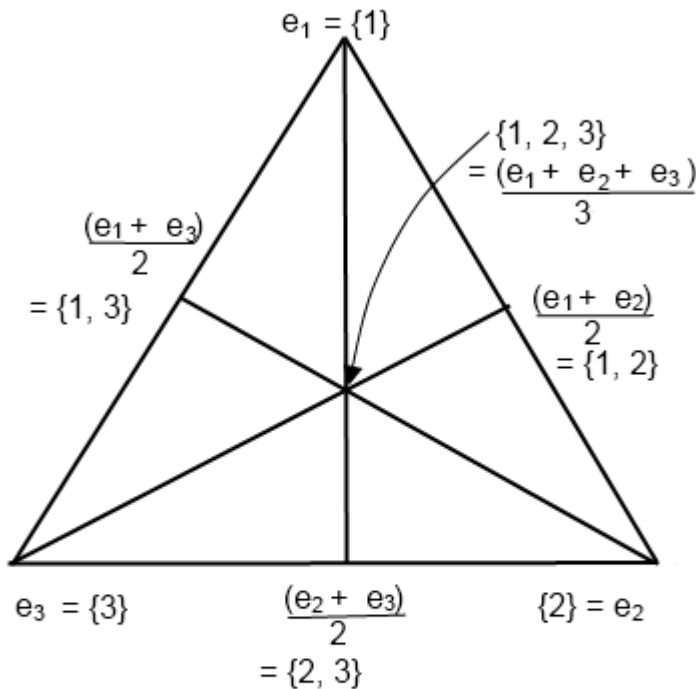


Figure 2 : Barycentric Subdivision

CHAPTER 3

BAYESIAN ORDER^[9]

Consider a point (state) $\vec{x} = (x_1, x_2, \dots, x_{n+1}) \in \Delta^{n+1}$ as the probabilities $P_X(a_i) = x_i$ for an experiment with the possible outcomes $\{a_1, a_2, \dots, a_{n+1}\}$. Given that the outcome $\{a_i\}$ has not occurred in a particular experiment (hence $x_i < 1$), we can update the probabilities of other a_j as : $P_X(\{a_j\} \mid \neg\{a_i\}) = \frac{Prob(a_j \cap (\neg\{a_i\}))}{Prob(\neg\{a_i\})} = \frac{Prob(a_j)}{Prob(\neg\{a_i\})} = \frac{x_j}{1-x_i}$, following Bayes' Rule. This leads us to the definition $p_i : \Delta^{n+1} \setminus \{e_i\} \rightarrow \Delta^n$, where $p_i(\vec{x}) = \frac{1}{1-x_i}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) \in \Delta^n$. These mappings p_i are called Bayesian Projections and they lead one to the following partial order on Δ^{n+1} .

3.1 Definition of Bayesian Order

For $\vec{x}, \vec{y} \in \Delta^2$, $x \sqsubseteq y \equiv (y_1 \leq x_1 \leq 1/2) \text{ or } (1/2 \leq x_1 \leq y_1) \dots$ (1)

The ordering (1) is derived from the graph of Shannon entropy H as follows:

For $\vec{x}, \vec{y} \in \Delta^{n+1}$, $n \geq 2$, $x \sqsubseteq y \equiv (\forall i)(x, y \in \text{dom}(p_i) \Rightarrow p_i(x) \sqsubseteq p_i(y)) \dots$ (2)

The relation \sqsubseteq on Δ^{n+1} is called the Bayesian order.

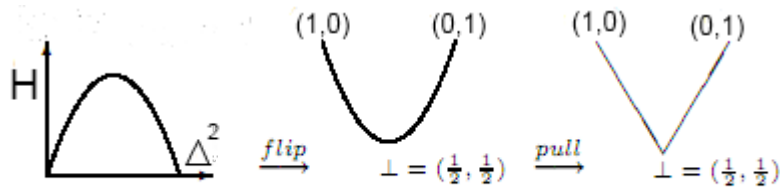


Figure 3 : Ordering of Distributions in 1 - simplex

For the next few sections, we shall denote \vec{x} just by 'x'.

To motivate (2), if $x \sqsubseteq y$, then observer x has more equidistribution than observer y. If something transpires which enables each observer to rule out exactly e_i as a possible state of the system, then the first now knows $p_i(x)$ while the second knows $p_i(y)$.

Example One can actually see from Fig. 5 that $p_1(x)$ and $p_1(x')$ are on the opposite sides of $(1/2, 1/2)$ the bottom element of Δ^2 . So, they are not comparable. In fact we will find out later that, for two different points in Δ^3 to be comparable, they have to belong to the same barycentric subdivision of the simplex Δ^3 (Δ^n , in general).

3.1.1 Proposition

There is a unique partial order on Δ^2 which has $\perp \doteq (1/2, 1/2)$ and satisfies the mixing law $x \sqsubseteq y$ and $p \in [0, 1] \Rightarrow x \sqsubseteq (1 - p)x + py \sqsubseteq y$. It is the Bayesian order on classical two states. The least element in a poset is denoted \perp , when it exists.

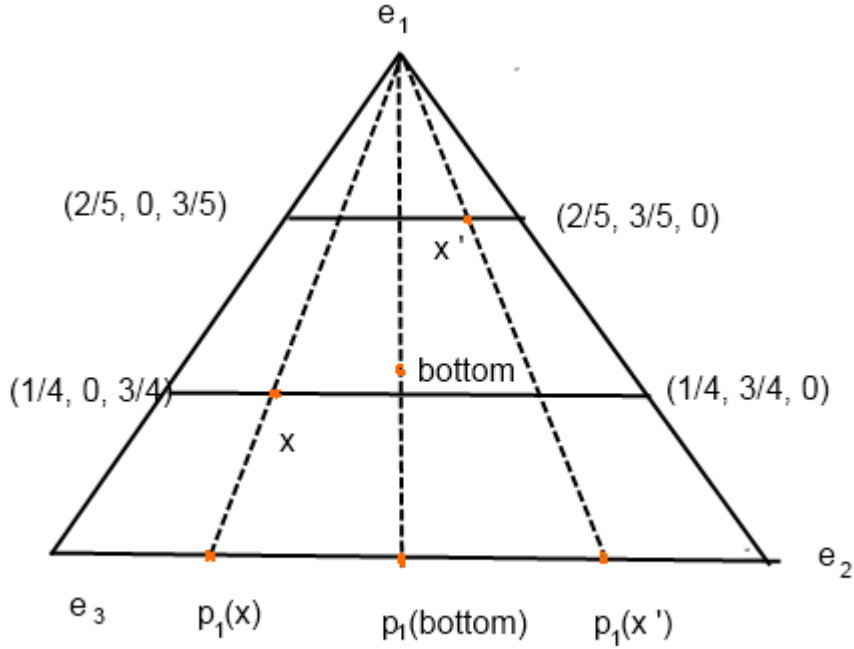


Figure 4 : Projection of Incomparable Points

3.1.2 Observation

(Δ^n, \sqsubseteq) is a dcpo with maximal elements $\max(\Delta^n) = \{e_i : 1 \leq i \leq n\}$ and least element $\perp = (1/n, \dots, 1/n)$. The next theorem shows that the Bayesian order \sqsubseteq can also be described in a more direct manner, called the symmetric characterization. Let $S(n)$ denote the group of permutations on $\{1, \dots, n\}$ and $\Lambda^n := \{x \in \Delta^n : (\forall i < n) x_i \geq x_{i+1}\}$ denote the collection of monotone classical states.

3.1.3 Theorem

For $x, y \in \Delta^n$, we have $x \sqsubseteq y$ iff there is a permutation $\sigma \in S(n)$ such that

$x \hat{A} \cdot \sigma, y \hat{A} \cdot \sigma \in \Lambda^n$ and $(x \hat{A} \cdot \sigma)_i (y \hat{A} \cdot \sigma)_{i+1} \leq (x \hat{A} \cdot \sigma)_{i+1} (y \hat{A} \cdot \sigma)_i \forall i$ with $1 \leq i < n$.

It is important to note here that there is nothing special about $\sigma \in S(n)$. This inequality needs to be satisfied for any pair of points in Λ^n to be in Bayesian Order and it is immaterial which permutation takes the original pair of points $x, y \in \Delta^n$ to Λ^n . So, $\sigma \in S(n)$ is just a representative of any common permutation among possible permutations $\{\sigma_{\vec{x}} : \overrightarrow{x \cdot \sigma} \in \Lambda^n\}$ and $\{\sigma_{\vec{y}} : \overrightarrow{y \cdot \sigma} \in \Lambda^n\}$.

Thus, the Bayesian order is order isomorphic to $n!$ many copies of Λ^n identified along their common boundaries. This fact, together with the pictures of $\uparrow x$ and $\downarrow x$ at representative states x will give a good feel for the geometric nature of the Bayesian order.

3.2 Lemma

Consider Δ^n with the Bayesian order. Each $\vec{y} \in (\Lambda^n)^o$, has an approximation, where $(\Lambda^n)^o = := \{\vec{x} \in \Delta^n : (\forall i < n) x_i > x_{i+1}\}$ denote the collection of *strictly* monotone classical states. .

Proof :

Let $\vec{w}^{(k)} \rightarrow \vec{w} \vec{y} \leq \vec{w} \in (\Lambda^n)^o$ and $\vec{x} = (1-t)\vec{y} + t\perp, 0 < t < 1$. Now from Symmetric Characterization of Bayesian Order,

$$y_i w_{i+1} \leq y_{i+1} w_i \forall i \text{ with } 1 \leq i < n.$$

$$\text{Now, } \forall i \text{ with } 1 \leq i < n : x_i w_{i+1} = [(1-t) y_i + t (\frac{1}{n})] w_{i+1}$$

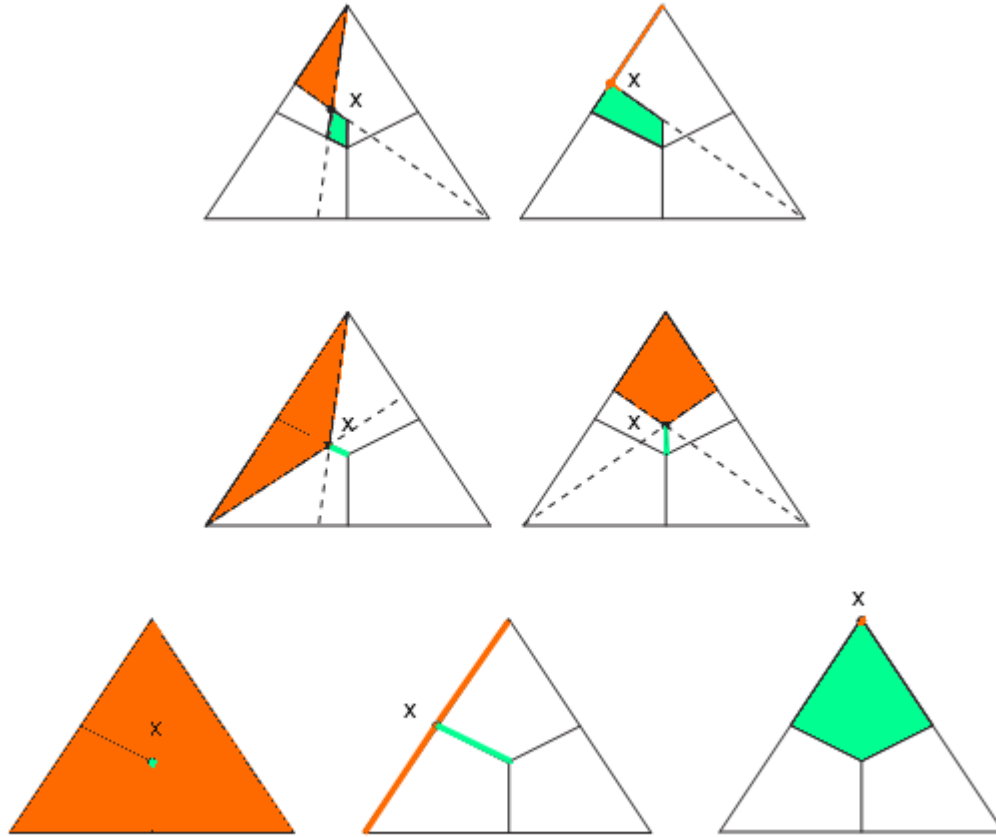




Figure 6 :  $\uparrow x$  $\downarrow x$ for $x \in \Delta^3$.

$$= (1-t) y_i w_{i+1} + t \left(\frac{1}{n}\right) w_{i+1} \leq (1-t) y_{i+1} w_i + t \left(\frac{1}{n}\right) w_{i+1}$$

[Using the given condition $y_i w_{i+1} \leq y_{i+1} w_i$]

$$< (1-t) y_{i+1} w_i + t \left(\frac{1}{n}\right) w_i \text{ [Using the fact that } \vec{w} \in (\Lambda^n)^o \text{ whence } w_{i+1} < w_i \text{]}$$

$$= x_{i+1} w_i . \text{ Thus, } x_i w_{i+1} < x_{i+1} w_i .$$

$$\Rightarrow \forall \varepsilon > 0, \exists N \text{ such that } |x_i w_{i+1}^{(k)} - x_i w_{i+1}| < \frac{\varepsilon}{2} \ \& \ |x_{i+1} w_i^{(k)} - x_{i+1} w_i| < \frac{\varepsilon}{2}, \ k \geq N$$

$$\Rightarrow \exists N \text{ such that } x_i w_{i+1}^{(k)} \leq x_{i+1} w_i^{(k)}, \ k \geq N$$

$$\Rightarrow \exists N \text{ such that } \vec{x} \subseteq \vec{w}^{(k)}, \ k \geq N.$$

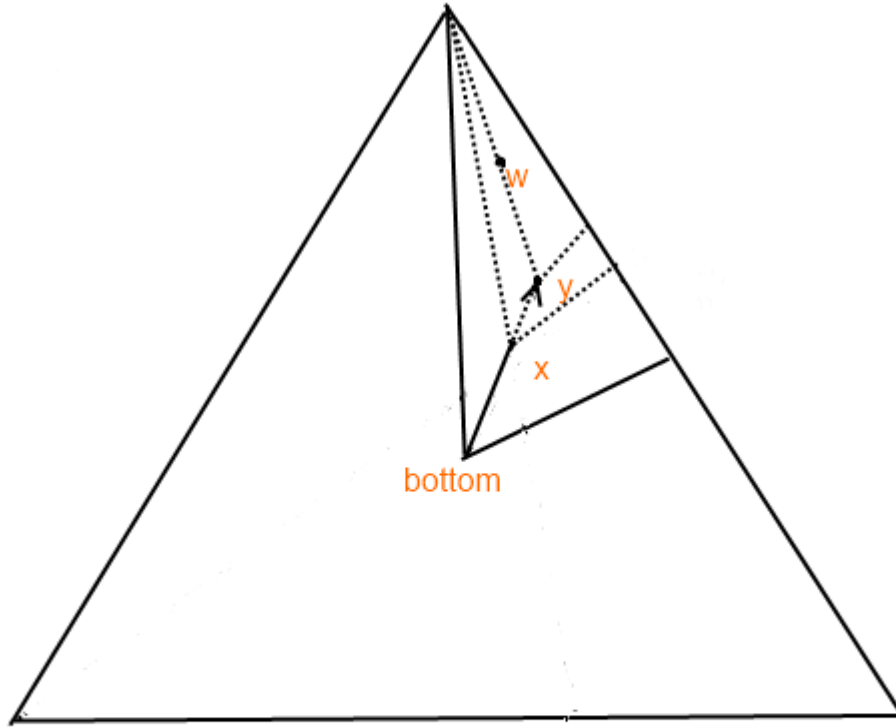


Figure 6 : The Parts of the Barycentric Subdivision in the Direction of the arrow bounded by the Dotted Lines denote the Upper Sets of x , y . w is an Element of the Upper Set of y , so it is also in the Upper Set of x .

3.2.1 Observations

The lemma works for $\vec{y} \in (\Lambda^3)^o$, $\vec{x} = (1-t)\vec{y} + t\perp$ and $\vec{y} \leq \vec{w} \in (\Lambda^3) \setminus \{\text{two closed boundary lines } \perp \text{ to } e_1 \text{ and } \perp \text{ to } (e_1 + e_2)/2\}$ in the barycentric subdivision. The bottom element \perp is special to the proof, in the sense, that if we replace \perp by $\vec{z} = (z_1, z_2, \dots, z_n) \neq \perp$ the proof fails because we no longer have $z_i = 1/n$ for all i .

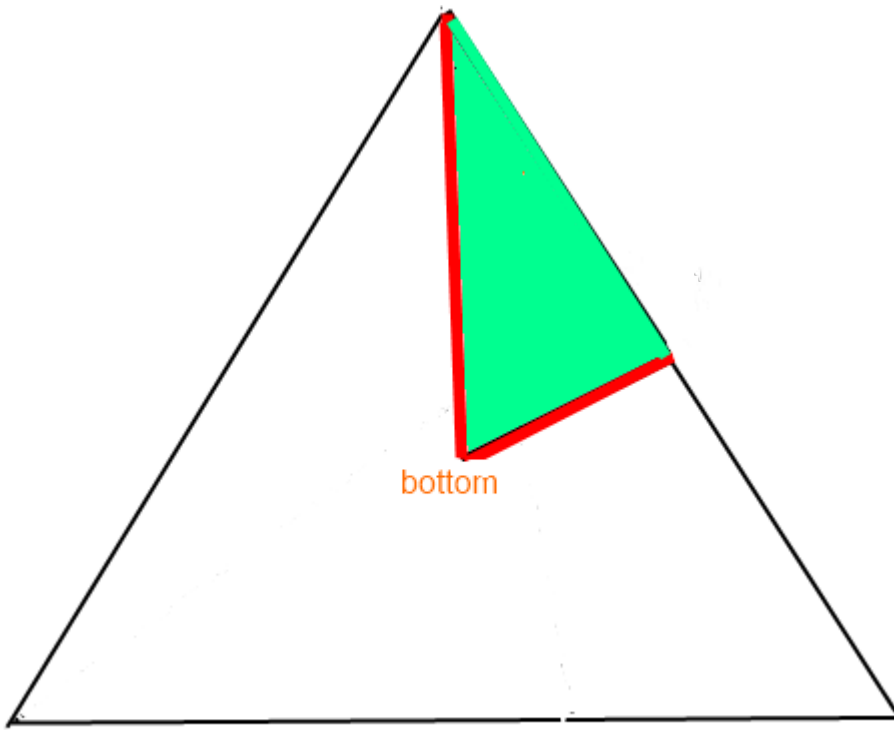


Figure 7 : The Feasible Region for w is marked green whereas the forbidden region red.

CHAPTER 4

MAJORIZATION [6]

We begin by introducing the theory of majorization, a mathematical relation that has recently been shown to have applications to quantum information theory. Majorization constraints have been shown to govern transformations of quantum entanglement, to restrict the spectra of separable quantum states, and to characterize how quantum states change as a result of mixing or measurement. It has even been suggested that all efficient quantum algorithms must respect a majorization principle.

4.1 Definition and Motivation

From the historic perspective, the concept of *Majorization* evolved in Economics out of the need to quantify or have a measure for *Distribution in Wealth* in a fixed population over a period of time with the assumption that wealth of the entire population also remains fixed. If the total wealth is not fixed, we can normalize the wealth of each person in a population by the total wealth of that population, so that the sum of the ratios of wealth is one in both populations. Now suppose the richest person in the first population is richer than the richest person in the second pop-

ulation, the combined wealth of the two richest persons in the first population is more than the combined wealth of the two richest persons in the second population (higher entropy) and so on. Then, it is but natural to argue that the second population has more *equi-distribution* of wealth than the first one and hence better.

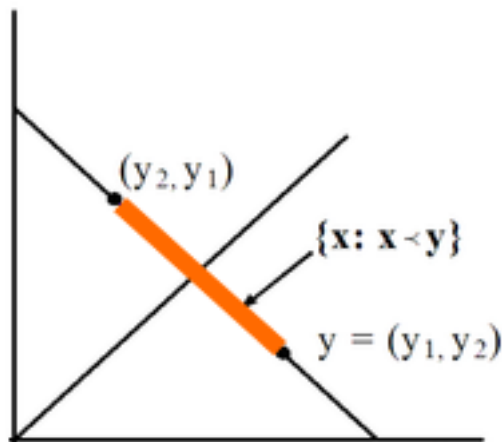


Figure 9 : Majorization Example

Fig. 10 shows that given a \vec{y} in Δ^2 , the set $\vec{x} : \vec{x} \prec \vec{y}$ is the convex hull of the set of all vectors obtained by permutation of the co-ordinates of \vec{y} . Intuitively, if \vec{x} and \vec{y} are probability vectors such that $\vec{x} \prec \vec{y}$ (\prec means less than or equal to in majorization order), then \vec{x} describes a more equidistribution than does \vec{y} . For example, in \mathbb{R}^2 , we have that $(0.5, 0.5) \prec (0.8, 0.2)$ and $(0.5, 0.5) \prec (0.2, 0.8)$. In fact, $(0.5, 0.5)$ is majorized by every vector in \mathbb{R}^2 whose components sum to unity. In particular \prec is a preorder ($(0.8, 0.2) \prec (0.2, 0.8) \prec (0.8, 0.2)$). But in a later section, we will define such an order that will help eradicate this issue and actually give us a partial order instead of just a preorder.

4.1.1 Definition

Let $\vec{x} = (x_1, \dots, x_d)$ and $\vec{y} = (y_1, \dots, y_d) \in \mathbb{R}^d$. We will be most interested in the case where x and y are d -dimensional probability vectors; in other words, their components are nonnegative and sum to unity. However, for most results in the theory of majorization, this restriction is not needed. Let x^\downarrow denote the d -dimensional vector obtained by arranging the components of x in non-increasing order: $\vec{x}^\downarrow = (x_1^\downarrow, \dots, x_d^\downarrow)$, where $x_1^\downarrow \geq x_2^\downarrow \geq \dots \geq x_d^\downarrow$. Then we say that x is majorized by y , written $\vec{x} \prec \vec{y}$, if the following relation holds: $\sum_{i=1}^j x_i^\downarrow \leq \sum_{i=1}^j y_i^\downarrow$ ($1 \leq j \leq d$) (Note that for classical probability states we also have $\sum_{i=1}^d x_i^\downarrow = \sum_{i=1}^d y_i^\downarrow = 1$).

4.1.2 T-transformations

A linear map $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a T-transformation if $\exists 0 \leq t \leq 1$, indices j, k and $T(\vec{y}) = (y_1, y_2, \dots, y_{j-1}, ty_j + (1-t)y_k, y_{j+1}, \dots, (1-t)y_j + ty_k, y_{k+1}, \dots, y_d)$.

The T-transform for $t = 1/2$, $d = 3$, $j = 2$ and $k = 3$ takes, for example, $(0.4, 0.4, 0.2)$ to $(0.4, 0.3, 0.3)$. It is not hard to see that if $\vec{y} \in \Lambda^3$ and $0 < t < 1$, then $T(\vec{y}) \prec \vec{y}$. Note that for $t = 1$, T is an identity transformation and for $t = 0$, T is a permutation with y_j and y_k interchanged.

4.1.3 Theorem [6]:

Let $\vec{x}, \vec{y} \in \mathbb{R}^d$, then, $\vec{x} \prec \vec{y} \iff \vec{x}$ can be obtained from \vec{y} by a finite number of T-transforms.

In fact it can also be shown that such finite composition of T-transforms is what

is called a doubly stochastic matrix, the row sums and column sums for each row and column being one. It is pertinent to note that such matrices have already been characterized and so the theory of majorization gets a giant leap in that regard while applying to partial orders.

4.2 Upper Sets of a Generic Point in Λ^3 under Majorization Order

Let $\vec{x} = (x_1, x_2, x_3)$ be a generic point in the barycentric subdivision Λ^3 under Majorization Order.

- Fix x_1 and try see a line of points (y_1, y_2, y_3) where:

1. $y_1 = x_1$;
2. $y_1 + y_2 \geq x_1 + x_2$;
3. $y_1 + y_2 + y_3 = x_1 + x_2 + x_3 = 1$;

4.2.1 Conclusion Keeping First Co-ordinate Fixed

Clearly this line is parallel to the boundary where $x_1 = 0$

- Fix x_3 and try see a line of points (y_1, y_2, y_3)

where:

1. $y_3 = x_3$;
2. $y_1 \geq x_1$;
3. $y_1 + y_2 \geq x_1 + x_2$;
4. $y_1 + y_2 + y_3 = x_1 + x_2 + x_3 = 1$;

4.2.2 Conclusion Keeping Third Co-ordinate Fixed

Clearly this line is parallel to the boundary where $x_3 = 0$. But this line ends as it is intercepted by a line equidistant from two extreme points and passing through one of them. The point where it ends has a co-ordinate $(1 - 2x_3, x_3, x_3)$. And from here on, really the order goes up to $(1, 0, 0)$, both the 2^{nd} and 3^{rd} co-ordinate diminishing equally and 1^{st} co-ordinate rising to highest value one.

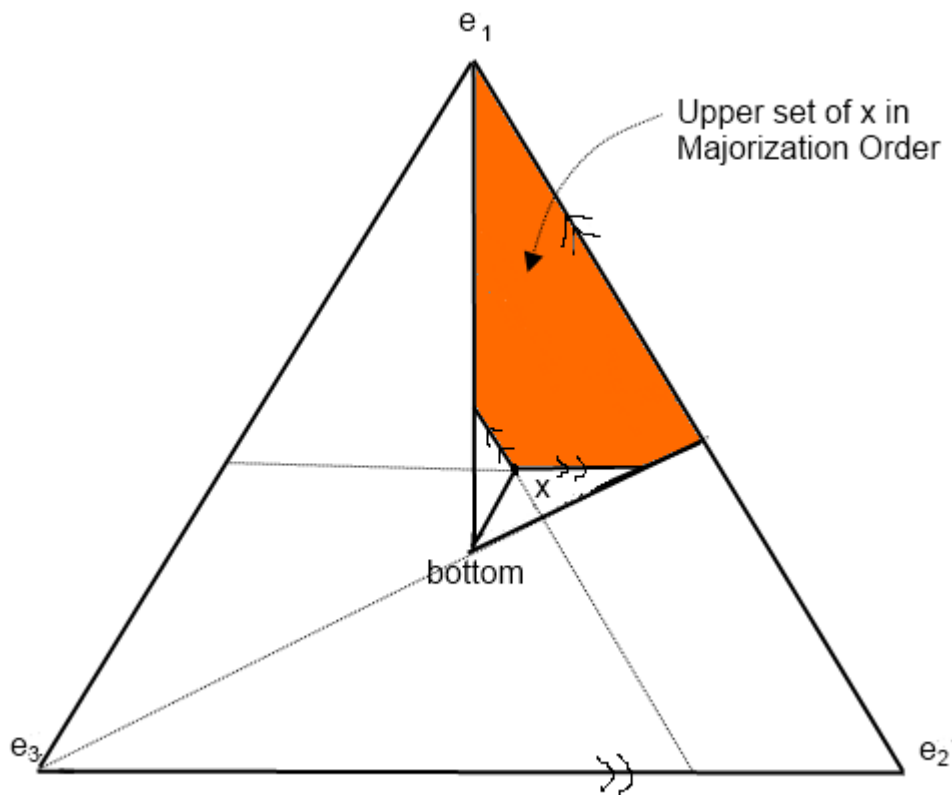


Figure 10 : Upper Sets of Distributions in Majorization Order

4.3 Theorem

Consider Δ^n with the Majorization order. Then $\vec{y} \in (\Lambda^n)^o$ has an approximation.

Proof : (We use \sqsubseteq in place of \prec in this proof to simplify notation)

Let $\vec{x} = (1-t)\vec{y} + t\mathbf{1}$, $0 < t < 1$ and $\vec{w}^{(k)}$ is an increasing chain with

$\sqcup\uparrow\vec{w}^{(k)} = \vec{w} \sqsupseteq \vec{y}$ where $\vec{w}^{(k)}, \vec{w} \in (\Lambda^n)^o$. For $0 \leq j \leq n$,

$$\sum_{i=1}^j x_i \leq \sum_{i=1}^j [(1-t) y_i + t (\frac{1}{n})] \leq \sum_{i=1}^j [(1-t) w_i + t (\frac{1}{n})] = (1-t) \sum_{i=1}^j w_i + t (\frac{j}{n}).$$

We show $\frac{j}{n} < \sum_{i=1}^j w_i$. Suppose not. Then, $\frac{j}{n} \geq \sum_{i=1}^j w_i \Rightarrow w_j < \frac{1}{n}$

$$\Rightarrow 1 = \sum_{i=1}^n w_i = \sum_{i=1}^j w_i + \sum_{i=j+1}^n w_i < \frac{j}{n} + \frac{n-j}{n} \text{ (since } w_i > w_{i+1}, 0 \leq i < n) = 1$$

$\Rightarrow \Leftarrow$ contradiction.

$$\text{So indeed we have : } \sum_{i=1}^j x_i \leq (1-t) \sum_{i=1}^j w_i + t (\frac{j}{n}) < (1-t) \sum_{i=1}^j w_i + t \sum_{i=1}^j w_i = \sum_{i=1}^j w_i$$

So, $\vec{x} \sqsubseteq \vec{w}^{(k)} \forall k \geq N$ for some $N > 0$. Hence $\vec{x} \ll \vec{y}$. The last lemma clearly implies that Λ^n under Majorization Order is a continuous dcpo.

4.4 Unrestricted (Classical) Majorization Order

In this order, we drop the assumption that there has to always be a common permutation $\sigma \in S(n)$ such that $\forall \vec{x}, \vec{y} \in \Delta^n$, the permuted states $\vec{x} \cdot \vec{\sigma}, \vec{y} \cdot \vec{\sigma} \in \Lambda^n$. In this case, clearly, for Δ^3 , the bottom element is $(0, 0, 1)$ and the top element is $(1, 0, 0)$.

4.4.1 Upper Sets in Δ^3 under Unrestricted Majorization Order.

The upper sets of a generic point in Δ^3 under unrestricted majorization order are lot different in the sense that the bottom element is changed now to e_3 and the elements in different subdivisions can now easily be compared because we lift the restriction

of the existence of a common permutation which has to take a pair of states to the smaller subset of classical monotone states. This is much general and comes in right from the perspective of "Unequal Distribution of Wealth" which I have talked about in the beginning of this chapter.

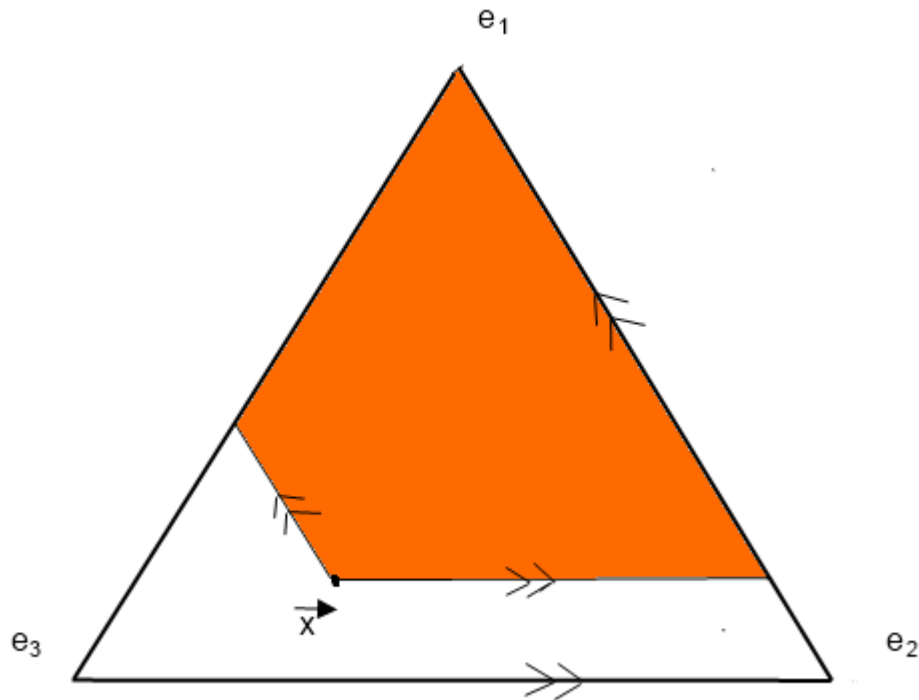


Figure 11 : Upper Set of Unrestricted Majorization Order

4.5 Theorem

$$\vec{x} \prec \vec{y} \Rightarrow \mathbf{H}(\vec{x}) \geq \mathbf{H}(\vec{y})$$

Proof : By Theorem 5.1.2, it is enough to prove the above where $\vec{x} = \mathbf{T}(\vec{y})$ for a T-transform. Now, to prove $\vec{x} = \mathbf{T}(\vec{y}) \Rightarrow \mathbf{H}(\vec{x}) \geq \mathbf{H}(\vec{y})$, we seek resort to the fact that entropy is a concave function. Now, let $\vec{y} = (y_1, y_2, \dots, y_{j-1}, y_j, \dots, y_k, y_{k+1},$

$\dots, y_d)$ and let $\vec{w} = (w_1, w_2, \dots, w_{j-1}, w_j, \dots, w_k, w_{k+1}, \dots, w_d)$ So, $\exists 0 \leq t \leq 1$, and indices j, k such that:

$$\begin{aligned} \vec{x} &= T(\vec{y}) = (y_1, y_2, \dots, y_{j-1}, ty_j + (1-t)y_k, y_{j+1}, \dots, (1-t)y_j + ty_k, y_{k+1}, \dots, y_d) \\ &= t\vec{y} + (1-t)\vec{w}. \end{aligned}$$

Now from concavity of Shannon entropy, the inequality is trivial.

CHAPTER 5

THE RESTRICTED STOCHASTIC ORDER

We again start with the classical probability states $\Delta^n = \{\vec{x} \in [0, 1]^n : \sum_{i=1}^n x_i = 1\}$, $n \geq 2$ where $\vec{x} = (x_1, x_2, x_3, \dots, x_n) = \sum_{i=1}^n x_i \vec{e}_i$. Let $V(\Delta^n) := \{e_1, e_2, \dots, e_n\}$ denote the standard basis of unit vectors in \mathbb{R}^n . Note that $V(\Delta^n)$ is the set of vertices or extreme points of the simplex Δ^n . We will denote them from now on as E_n . We may alternatively identify a standard basis vector e_i with the point measure δ_{e_i} and a classical n-state $(x_1, x_2, x_3, \dots, x_n)$ with the probability measure $\sum_{i=1}^n x_i \delta_{e_i}$ on E_n , and we pass freely between the two characterizations. We denote by $\beta V(\Delta^n)$ or equivalently P^*E_n the set of vertices of the barycentric subdivision of the simplex Δ^n . These points are obtained by taking any non-empty subset $J \subseteq \{1, 2, \dots, n\}$ and setting $e_J = \sum_{j \in J} (\frac{1}{|J|}) e_j \in \Delta^n$. We define a partial order on the set $\beta V(\Delta^n)$ (or equivalently P^*E_n) by $e_I \sqsubseteq e_J$ iff $I \supseteq J$. Note that $e_{\{i\}} = e_i$. We fix n and denote Δ^n , E_n and P^*E_n simply by Δ , E and P^*E respectively. We will now take the example of E_3 and show the partial ordering of its subsets and how it is a probabilistic powerdomain. This helps us define Stochastic Order on entire Δ^3 .

5.1 Hasse Diagram for P^*E_3

Let $\{x, y, z\}$ be a finite set with three elements. Then the subsets of $\{x, y, z\}$ form a partial order. If x, y, z are identified with E_3 , then every subset of $\{x, y, z\}$ can be identified with the corresponding barycentre in the barycentric subdivision of Δ^3 .

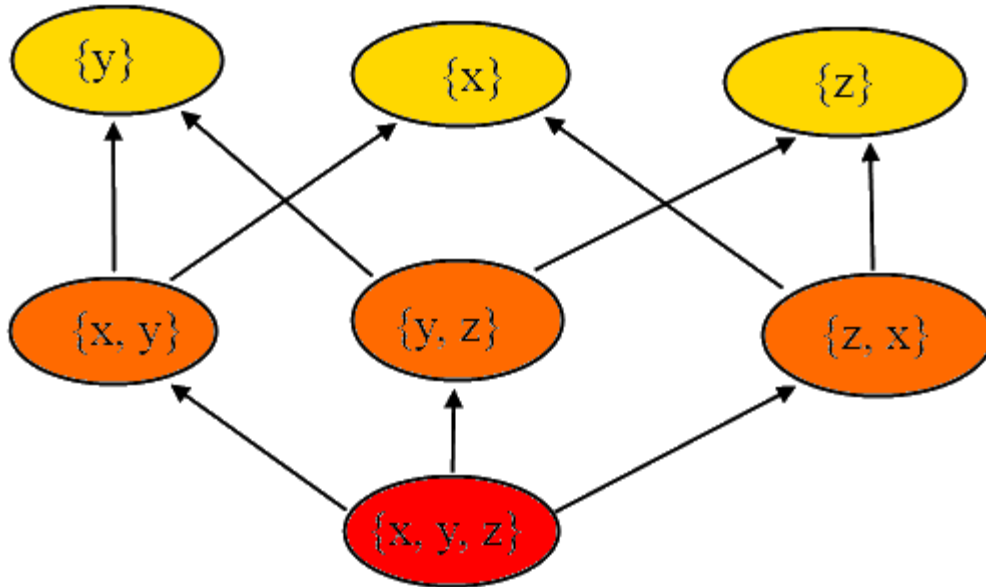


Figure 12 : Hasse Diagram for P^*E_3

5.2 Construction of Probabilistic Power Domain

Consider the set P^*E of all nonempty subsets of E ordered by reverse inclusion: $F_1 \sqsubseteq F_2$ iff $F_1 \supseteq F_2$. We view P^*E as the free meet-semilattice on E (the meet operation being union), where we identify E with the singleton subsets. The probabilistic power domain $P_1(P^*E)$ on P^*E can then be identified with the set of all probability measures

on P^*E equipped with the *stochastic – order*: $\mu \leq \nu$ if and only if $\mu(U) \leq \nu(U)$ for every upper set U in P^*E . We define a map $\beta: P_1(P^*E) \rightarrow \Delta$ by taking the unique convex extension of the restriction of β to P^*E , where β is defined for $F \in P^*E$ by $\beta(F) = \sum_{e \in F} \frac{1}{k} \delta_e$, where $k = |F|$. Since β carries the singleton subset $\{e_i\} \in P^*E$ to δ_{e_i} , the image of β is a convex set containing the set of extreme points of Δ . We thus have:

5.2.1 Lemma

The map β is surjective.

Any member of $P_1(P^*E)$ is a convex sum of point measures of P^*E . The support of μ consists of all points for which the coefficient in the convex sum is nonzero. We are particularly interested in those probability measures on P^*E for which the support is a chain: $\widetilde{P}_1(P^*E) := \{\mu \in P_1(P^*E) : \text{supp is a chain}\}$.

5.2.2 Proposition

The restriction of β to $\widetilde{P}_1(P^*E)$ is a bijection from $\widetilde{P}_1(P^*E)$ to Δ

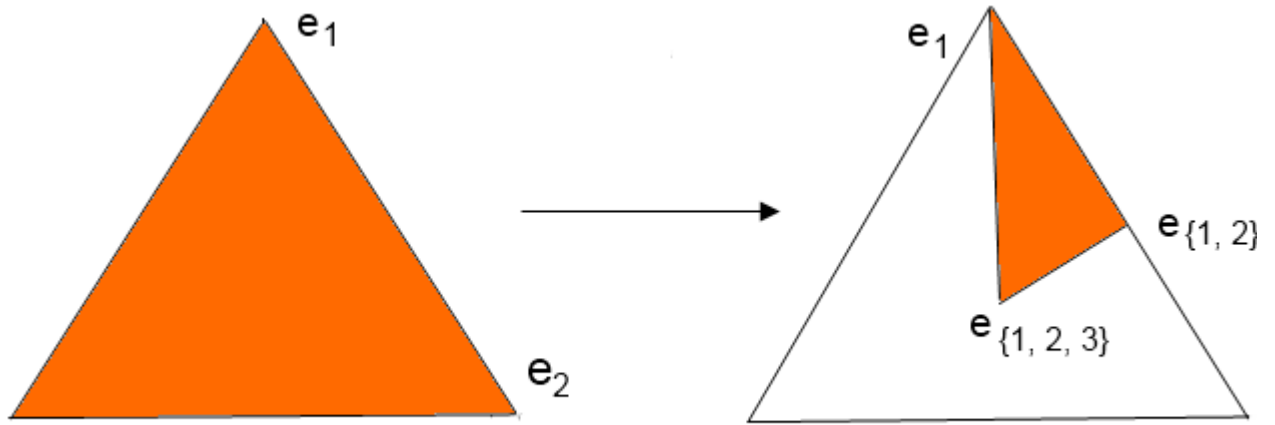
Proof : Every element of Δ^3 (or Δ^n) can be written uniquely as a convex combination of the vertices of the barycentric subsimplex in which it lies. That these vertices form a maximal chain in (P^*E) is a standard fact of simplicial geometry.

The next section shows that we improved our order one step towards obtaining the Entropy Reversing Property (ERP) and also the space of probability distributions has also been reduced to only the monotone classical states.

5.3 Connection between Unrestricted Majorization Order and Stochastic Order

Consider the map from $\Theta : \Delta^3(\text{under unrestricted majorization}) \rightarrow \Lambda^3(\text{under stochastic order})$ defined by :

$$(w_1, w_2, w_3) \mapsto w_1\delta_{e_1} + w_2\delta_{e_{\{1,2\}}} + w_3\delta_{e_{\{1,2,3\}}}$$



e₃ **Figure 13 : Connection between Majorization and Stochastic Order**

One can easily see that this is an order isomorphism. So, we have got hold of an order whereby we can restrict our computations only to the classical monotone states and still get hold of a continuous domain.

CHAPTER 6

VERIFICATION OF CLOSEDNESS OF THREE ORDERS

It is of utmost importance to Computer Science, specifically in Semantics that the computational limit exists in every directed set of a partial order. Indeed we will show here for the three orders described above : namely, Bayesian Order, Majorization Order and Stochastic Order, the supremum exists for every directed set.

6.1 Lemma

The Bayesian Order, Majorization Order and Stochastic Order are all closed, i.e., the set $\{(x, y) : x \leq y\}$ is closed in $\Delta^n \times \Delta^n$.

Let $(\Lambda^n)^o = \{\vec{x} \in \Lambda^n : x_1 > x_2 > \dots > x_n\}$. The next lemma shows that elements on the (open) line segment between \perp and \vec{y} are almost approximating elements of \vec{y} . The lemma is verified with all the three different orders that have been defined in previous chapters and the techniques for proof have also been shown in rigorous details.

6.2 Closedness in Bayesian Order

First we test closedness in Λ^n . Let us take two sequences in $\vec{x}^{(k)}$ and $\vec{y}^{(k)} \in \Lambda^n$ where $\vec{x}^{(k)} \leq \vec{y}^{(k)}$ such that $\vec{x}^{(k)} \rightarrow \vec{x}$ and $\vec{y}^{(k)} \rightarrow \vec{y}$. Let us use the symmetric characterization of the Bayesian Order : $\vec{x}^{(k)} \leq \vec{y}^{(k)} \Leftrightarrow x_i^{(k)} y_{i+1}^{(k)} \leq x_{i+1}^{(k)} y_i^{(k)} \forall i$ with $1 \leq i < n$. Since, $\vec{x}^{(k)} \rightarrow \vec{x}$ and $\vec{y}^{(k)} \rightarrow \vec{y}$, the convergence is co-ordinate wise, i.e, $x_i^{(k)} \rightarrow x_i$ and $y_i^{(k)} \rightarrow y_i$. Now we use the fact that convergence is preserved under product. So, $x_i y_{i+1} \leq x_{i+1} y_i$ in the limit $\forall i$ with $1 \leq i < n \Leftrightarrow \vec{x} \leq \vec{y}$. Now, we extend this to entire Δ^n . Let us take two sequences in $\vec{x}^{(k)}$ and $\vec{y}^{(k)} \in \Delta^n$ where $\vec{x} \cdot \vec{\sigma}^{(k)} \leq \vec{y} \cdot \vec{\sigma}^{(k)}$, where we implicitly assume σ is the common permutation which takes both $\vec{x}^{(k)}$ and $\vec{y}^{(k)}$ in Λ^n . Now from our earlier argument, $\vec{x} \cdot \vec{\sigma} \leq \vec{y} \cdot \vec{\sigma}$ as the convergence is taking place in Λ^n . Now, we just take the inverse permutation to get $\vec{x} \leq \vec{y}$.

6.3 Closedness in Majorization Order

First we test closedness in Λ^n . Let us take two sequences in $\vec{x}^{(k)}$ and $\vec{y}^{(k)} \in \Lambda^n$: $\vec{x}^{(k)} \prec \vec{y}^{(k)}$ such that $\vec{x}^{(k)} \rightarrow \vec{x}$ and $\vec{y}^{(k)} \rightarrow \vec{y}$. Then we say that x is majorized by y , written $x \prec y$, if $\sum_{i=1}^j x_i^{(k)} \leq \sum_{i=1}^j y_i^{(k)} (1 \leq j \leq n)$ and $\sum_{i=1}^n x_i^{(k)} = \sum_{i=1}^n y_i^{(k)} = 1$. Again, taking limit on both sides of the inequality, as convergence is co-ordinate wise, we have : $\sum_{i=1}^j x_i \leq \sum_{i=1}^j y_i (1 \leq j \leq n)$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 1$. So, $\vec{x} \prec \vec{y}$ and hence the majorization order is closed in Λ^n . We apply the same argument as in case of Bayesian Order in extending the closed order from Λ^n to Δ^n . We take the common permutation σ which takes both $\vec{x}^{(k)}$ and $\vec{y}^{(k)} \in \Delta^n$ to Λ^n . Let the new sequences be $\vec{x} \cdot \vec{\sigma}^{(k)}$ and $\vec{y} \cdot \vec{\sigma}^{(k)}$.

So, $\sum_{i=1}^l \vec{x} \cdot \vec{\sigma}_i^{(k)} \leq \sum_{i=1}^l \vec{y} \cdot \vec{\sigma}_i^{(k)}$ ($1 \leq l \leq n$) and $\sum_{i=1}^n \vec{x} \cdot \vec{\sigma}_i^{(k)} = \sum_{i=1}^n \vec{y} \cdot \vec{\sigma}_i^{(k)} = 1$ Now from our earlier argument, $\vec{x} \cdot \vec{\sigma} \prec \vec{y} \cdot \vec{\sigma}$ as the convergence is taking place in Λ^n . Just taking the inverse permutation, we get $\vec{x} \prec \vec{y}$.

6.4 Closedness in Stochastic Order

This is a known result. [10]

6.5 Lemma

1. Both Orders are invariant under permutations, i.e, $\vec{x} \leq \vec{y} \iff \vec{x} \cdot \vec{\sigma} \prec \vec{y} \cdot \vec{\sigma}$ where $\sigma \in S(n)$
2. Permutation gives bijections among Barycentric Subdivisions, i.e, given a fixed Barycentric Subdivision, there is a one to one correspondence between every $\sigma \in S(n)$ and the Barycentric Subdivision which it maps to.
3. The maximal elements are the unit vectors $\{e_i\}_{i \in \{1, 2, \dots, n\}}$ and the smallest element is the uniform element $\perp = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$.

CHAPTER 7

QUANTUM MECHANICS AS A THEORY OF PROBABILITY^[11]

7.1 Introduction

- We develop that the Hilbert Space formalism of Quantum Mechanics is a new theory of probability.
 - The theory like its classical counterpart, consists of an algebra of events and probability measure defined on it.
 - The steps of the construction are given herein :
1. Axioms for the algebra of events are introduced following Birkhoff and von Neumann. All axioms except the one that expresses the uncertainty principle, are shared with the classical event space. The only models for the set of axioms are lattices of closed subspaces of inner product spaces over a field K .
 2. Another axiom due to Solèr forces K to be the field of real or complex numbers or the quaternions.

3. Gleason's Theorem fully characterizes the probability measure on the algebra of events, so that Born's Rule is derived.

7.2 The Axioms

For this we have to make the underlying Hilbert Space as a Lattice structure $(L, 0, 1, \leq, \cup, \cap)$. Let the closed subspaces be events named as x, y, z , etc and let $x \cup y$ denote the closure of the union of two subspaces x and y and $x \cap y$ as the intersection of two such events, x^\perp as the orthogonal complement of the closed subset x . Let the certain event be 1 and null event as 0. Then we can define a probability measure on the closed subspaces of the Hilbert Spaces which are now events. Let them follow axioms :

1. $x \leq x$ (Reflexivity)
2. If $x \leq y$ and $y \leq z$, then, $x \leq z$ (Transitivity)
3. If $x \leq y$ and $y \leq x$, then $x = y$ (Anti-Symmetry)
4. $0 \leq x \leq 1$
5. $x \cap y \leq x$ and $x \cap y \leq y$ and $\forall z \leq x \ \& \ z \leq y$, we have $z \leq x \cap y$
6. $x \leq x \cup y$ and $y \leq x \cup y$ and $\forall x \leq z \ \& \ y \leq z$, we have $x \cup y \leq z$
7. $(x^\perp)^\perp = x$
8. $x \cap x^\perp = 0$ and $x \cup x^\perp = 1$

9. $x \leq y \Rightarrow y^\perp \leq x^\perp$

10. If $x \leq y$, then, $y = x \cup (y \cap x^\perp)$ (Orthomodularity)

11. If $x \leq z$, then, $x \cup (y \cap z) = (x \cup y) \cap z$ (Modularity)

12. If $x \not\leq y$, then, \exists an atom p such that $p \leq y$ and $p \not\leq x$. Here, by atom we mean an element $0 \neq p \in L$ such that $x \leq p$ entails $x = 0$ or $x = p$ (Atomism). If for atom p and event x we have $x \cap p = 0$, then , $x \leq y \leq x \cup p \Rightarrow y = x$ or $y = x \cup p$ (Covering Property).

The first 11 axioms are true in classical system of propositional logic or more precisely Lindenbaum-Tarski algebra of such a logic, when we interpret the operations as logical connectives. Atomism and covering property are introduced to ensure that every element of the lattice is a union of atoms. The atoms whose existence is guaranteed by axiom 12 are maximally informative propositions. In classical case they correspond to points in the phase space whereas in the quantum case, they correspond to one-dimensional subspaces of the Hilbert Space.

7.3 Isomorphism with Lattice of Closed Subspaces

- There is a division ring K (field whose product is not necessarily commutative), with an involutorial automorphism $*$: $K \rightarrow K$, i.e, $\forall \alpha, \beta \in K$, we have $\alpha^{**} = \alpha$,

$$(\alpha + \beta)^* = \alpha^* + \beta^*, (\alpha\beta)^* = \beta^*\alpha^*$$

- There is a left vector space V over K
- There is a Hermitian form $\langle \cdot, \cdot \rangle : V \times V \rightarrow K$ satisfying $\forall u, v, w \in V$ and $\alpha, \beta \in K$,

$$1. \langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$$

$$2. \langle u, \alpha v + \beta w \rangle = \langle u, v \rangle \alpha^* + \langle u, w \rangle \beta^*$$

$$3. \langle u, v \rangle = \langle v, u \rangle^*$$

$$4. \langle u, u \rangle = 0 \Leftrightarrow u = 0$$

7.4 Solèr's Theorem

If L is infinite dimensional and satisfies SO (If x and y are orthogonal atoms, then there is a $z \leq x \cup y$ such that $w = \mathcal{H}(z; x, y)$ is orthogonal to z , i.e, $w = z^\perp \cap (x \cup y)$. Intuitively, such a z bisects the angle between x and y , that defines $\sqrt{2}$ in the field K . The extra axiom connects Projective Geometry concept (Harmonic Conjugation) to the orthogonality structure.), then K is \mathbb{R} or \mathbb{C} or the quaternions.

7.5 Born's Rule

Given a probability function on a space of dimension ≥ 3 , \exists an Hermitian non-negative operator W on H , whose trace is unity such that $P(x) = \langle x, Wx \rangle$ for all atoms $x \in L$, where \langle , \rangle is the inner product and x is the unit vector along the ray it spans. In particular, if some $x_0 \in L$ satisfies $P(x_0) = 1$, then $P(x) = |\langle x_0, x \rangle|^2 \forall x \in L$.

7.6 Alternative Approach of defining Partial Order

Instead of defining Spectral Order^[9] on density operators on Quantum states, we can now define a partial order on the closed subspaces of the actual Quantum states. It is pertinent to note at this point that our intuition for defining such a partial order is for the sole reason of measuring how entangled two or more states are. This stems from the fact that we always start with an entangled state in the EMC model which has been proven to be a universal mode for Measurement Calculus of Quantum Computation^[12].

CHAPTER 8

FUTURE DIRECTION OF WORK

We are planning to define Majorization Order and Stochastic Order on Quantum States of Density Operators of States (vectors in Hilbert Space) by creating a Barycentric Subdivision of that by the aforementioned technique of creating Quantum Events and looking at atoms for extreme points (pure states) with a vision to make it a continuous dcpo. This will need to be validated with Birkhoff and von Neumann Logic and can generate a very powerful semantics for Quantum Computation. This is again in connection to the Spectral Order defined on Quantum States by Coecke and Martin in their paper "Partial Order on Classical and Quantum States" (2003) based on von Neumann entropy. Instead of looking into the spectrum of the density operators, we want to look into the simplex of all possible density operators based on construction probabilistic powerdomain on density operators for atoms and then assign weights to the barycenters (subsets of the set of atoms) to get a handle on all density operators.

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