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Classification of the centers and their isochronicity for a class of polynomial differential systems of arbitrary degree

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Abstract

In this paper we classify the centers localized at the origin of coordinates, and their isochronicity for the polynomial differential systems in $\mathbb{R}^2$ of degree d that in complex notation $z = x + iy$ can be written as

$$\dot{z} = (\lambda + i)z + Az^{(d-n+1)/2}\bar{z}^{(d+n-1)/2} + Bz^{(d+n+1)/2}\bar{z}^{(d-n-1)/2} + Cz^{(d+1)/2}\bar{z}^{(d-1)/2} + Dz^{(d-(2+j)n+1)/2}\bar{z}^{(d+(2+j)n-1)/2},$$

where j is either 0 or 1. If $j = 0$ then $d \geqslant 5$ is an odd integer and n is an even integer satisfying $2 \leqslant n \leqslant (d+1)/2$. If $j = 1$ then $d \geqslant 3$ is an integer and n is an integer with converse parity with d and satisfying $0 < n \leqslant [(d+1)/3]$ where $[\cdot]$ denotes the integer part function. Furthermore $\lambda \in \mathbb{R}$ and $A, B, C, D \in \mathbb{C}$. Note that if $d = 3$ and $j = 0$, we are obtaining the generalization of the polynomial differential systems with cubic homogeneous nonlinearities studied in K.E. Malkin (1964) [17], N.I. Vulpe and K.S. Sibirskii (1988) [25], J. Llibre and C. Valls (2009) [15], and if $d = 2$, $j = 1$ and $C = 0$, we are also obtaining as a particular case the quadratic polynomial differential systems studied in N.N. Bautin (1952) [2], H. Zoladek (1994) [26]. So the class of polynomial differential systems here studied is very general having arbitrary degree and containing the two more relevant subclasses in the history of the center problem for polynomial differential equations.

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1. Introduction and statement of the main results

One of the main problems in the qualitative theory of real planar polynomial differential systems is the center–focus problem; i.e. to distinguish when a singular point is either a focus or a center. The notion of *center* goes back to Poincaré in [20]. He defined it for a vector field on the real plane; i.e. a singular point surrounded by a neighborhood filled with closed orbits with the unique exception of the singular point.

The classification of the centers of the polynomial differential systems started with the quadratic ones with the works of Dulac [7], Kapteyn [11,12], Bautin [2] and others. Schlomiuk, Guckenheimer and Rand in [24] described a brief history of the problem of the center in general, and it includes a list of 30 papers covering the topic and the history of the center for the quadratic case (see pages 3, 4 and 13). Here we are mainly interested in finding new families of centers of polynomial differential systems of arbitrary degree and in studying their cyclicity and isochronicity. There are other interesting problems related with the centers that in this paper we do not consider as for instance, their phase portraits in the Poincaré disc, or the kind of first integrals that the centers can have, or the bifurcation diagram of the different phase portraits of centers in the parameter space, etc. In the case of quadratic centers these last problems were studied by several authors, see for instance Schlomiuk [22,23] and the references therein.

In this paper we consider the polynomial differential systems in the real (x, y)-plane which has a singular point at the origin with eigenvalues $\lambda \pm i$ and that can be written as

$$\begin{aligned}\dot z &= (\lambda + i)z + A z^{(d-n+1)/2}\bar z^{(d+n-1)/2} + B z^{(d+n+1)/2}\bar z^{(d-n-1)/2}\\ &\quad + C z^{(d+1)/2}\bar z^{(d-1)/2} + D z^{(d-(2+j)n+1)/2}\bar z^{(d+(2+j)n-1)/2},\end{aligned} \tag{1}$$

where j is either 0 or 1. If $j = 0$ then $d \geqslant 5$ is an odd integer and $n \geqslant 2$ is an even integer satisfying $n \leqslant (d+1)/2$. If $j = 1$ then $d \geqslant 3$ is an integer and $n > 0$ is an integer with converse parity with d and satisfying $n \leqslant [(d+1)/3]$. Furthermore $\lambda \in \mathbb{R}$, and $A, B, C, D \in \mathbb{C}$. When $j = 0$ we are considering the polynomial differential systems

$$\begin{aligned}\dot z &= (\lambda + i)z + A z^{(d-n+1)/2}\bar z^{(d+n-1)/2} + B z^{(d+n+1)/2}\bar z^{(d-n-1)/2}\\ &\quad + C z^{(d+1)/2}\bar z^{(d-1)/2} + D z^{(d-2n+1)/2}\bar z^{(d+2n-1)/2},\end{aligned}$$

and when $j = 1$ we are considering the polynomial differential systems

$$\begin{aligned}\dot z &= (\lambda + i)z + A z^{(d-n+1)/2}\bar z^{(d+n-1)/2} + B z^{(d+n+1)/2}\bar z^{(d-n-1)/2}\\ &\quad + C z^{(d+1)/2}\bar z^{(d-1)/2} + D z^{(d-3n+1)/2}\bar z^{(d+3n-1)/2}.\end{aligned}$$

The vector field associated to system (1) is formed by the linear part $(\lambda + i)z$ and by a homogeneous polynomial of degree d formed by four monomials in complex notations. Since the eigenvalues at the singular point located at the origin of system (1) are $\lambda \pm i$, the origin is either a weak focus or a center if $\lambda = 0$, see [1,8].

For calculating the Poincaré–Liapunov constants several algorithms have been developed to compute them automatically up to a certain order (see for instance [3,10,16,19,21] and the references therein). The main reason for working with this class of polynomial differential systems is that for such class we can compute the Poincaré–Liapunov constants and afterwards the conditions for center (i.e. to determine the common zeros of the Poincaré–Liapunov constants), in general for polynomial differential systems of arbitrary degree such computations are very difficult or impossible. We also note that if $d = 3$ and $j = 0$, we are obtaining the generalization of the polynomial differential systems with cubic homogeneous nonlinearities studied in [17,25,15], and if $d = 2$, $j = 1$ and $C = 0$, we are obtaining as a particular case the quadratic polynomial differential systems studied in [2,26]. On the other hand there are partial results on the centers of polynomial vector fields formed by a linear center with a homogeneous nonlinearity of degree 4 and 5 (see [4,5]), but these vector fields are not contained in the ones studied here.

The resolution of these problems imply the effective computation of the Poincaré–Liapunov constants. There are different methods for computing the Poincaré–Liapunov constants, see for instance [9,19], but here we prefer the one based on Abel equations because using it, the computation of the isochronous centers becomes easier.

We write

$$A = a_1 + ia_2, \qquad B = b_1 + ib_2, \qquad C = c_1 + ic_2, \qquad D = d_1 + id_2.$$

Indeed writing (1) in polar coordinates, i.e., doing the change of variables $r^2 = z\bar{z}$ and $\theta = \arctan(\operatorname{Im} z/\operatorname{Re} z)$, systems (1) become

$$\frac{dr}{d\theta} = \frac{\lambda r + F(\theta)r^d}{1 + G(\theta)r^{d-1}}, \tag{2}$$

where

$$\begin{aligned} F(\theta) &= (a_1 + b_1)\cos n\theta + (a_2 - b_2)\sin n\theta + c_1 + d_1\cos(2+j)n\theta \\ &\quad + d_2\sin(2+j)n\theta, \\ G(\theta) &= (a_2 + b_2)\cos n\theta - (a_1 - b_1)\sin n\theta + c_2 + d_2\cos(2+j)n\theta \\ &\quad - d_1\sin(2+j)n\theta. \end{aligned} \tag{3}$$

Note that Eq. (2) is well defined in a sufficiently small neighborhood of the origin. Therefore if system (1) has a center, then Eq. (2) defined in the plane (r, θ) when $\dot{\theta} > 0$ also has a center at the origin.

The transformation $(r, \theta) \mapsto (\rho, \theta)$ defined by

$$\rho = \frac{r^{d-1}}{1 + G(\theta)r^{d-1}}$$

is a diffeomorphism from the region $\dot{\theta} > 0$ into its image. If we write Eq. (2) in the variable ρ, we obtain the following Abel differential equation

$$\begin{aligned}\frac{d\rho}{d\theta} &= (d-1)G(\theta)\big[\lambda G(\theta) - F(\theta)\big]\rho^3 \\ &\quad + \big[(d-1)\big(F(\theta) - 2\lambda G(\theta)\big) - G'(\theta)\big]\rho^2 + (d-1)\lambda\rho \\ &= A(\theta)\rho^3 + B(\theta)\rho^2 + C\rho. \end{aligned} \tag{4}$$

The solution $\rho(\theta,\gamma)$ of (4) satisfying that $\rho(0,\gamma)=\gamma$ can be expanded in a convergent power series of $\gamma \geqslant 0$ sufficiently small. Thus

$$\rho(\theta,\gamma) = \rho_1(\theta)\gamma + \rho_2(\theta)\gamma^2 + \rho_3(\theta)\gamma^3 + \cdots \tag{5}$$

with $\rho_1(\theta)=1$ and $\rho_k(0)=0$ for $k \geqslant 2$. Let $P:[0,\gamma_0] \to \mathbb{R}$ be the Poincaré map defined by $P(\gamma)=\rho(2\pi,\gamma)$ and for a convenient $\gamma_0>0$. Then, the values of $\rho_k(2\pi)$ for $k \geqslant 2$ control the behavior of the Poincaré map in a neighborhood of $\rho=0$. Then clearly systems (1) have a center at the origin if and only if $\rho_1(2\pi)=1$ and $\rho_k(2\pi)=0$ for every $k \geqslant 2$. Assuming that $\rho_2(2\pi)=\cdots=\rho_{m-1}(2\pi)=0$ we say that $\rho_m(2\pi)$ is the m-th *Poincaré–Liapunov* or *Poincaré–Liapunov–Abel* constant of system (1).

The problem of computing the Poincaré–Liapunov constants for determining a center goes back to the very beginning of the qualitative theory of differential equations, see for instance [20] and [14]. In the case of polynomial differential systems each of the Poincaré–Liapunov constants is a polynomial in the coefficients of the system. The set of coefficients for which all the Poincaré–Liapunov constants vanish is called the *center space* of the family of polynomial differential systems. By the Hilbert Basis Theorem, the center space is an algebraic set. Furthermore, the center space, i.e., the space of systems (1) with a center at the origin is invariant with respect to the action group C^* of changes of variables $z \to \xi z$:

$$\begin{aligned} &A \to \xi^{(d-n-1)/2}\bar{\xi}^{(d+n-1)/2}A, \qquad &&B \to \xi^{(d+n-1)/2}\bar{\xi}^{(d-n-1)/2}B, \\ &C \to \xi^{(d-1)/2}\bar{\xi}^{(d-1)/2}\bar{\xi}^{4}C, \qquad &&D \to \xi^{(d-(2+j)n-1)/2}\bar{\xi}^{(d+(2+j)n-1)/2}D. \end{aligned} \tag{6}$$

A natural question arises: *how to characterize the center space of a given family of polynomial differential systems*?

To distinguish between the centers and the foci is a very difficult problem. In general it is not easy to compute Poincaré–Liapunov constants. These constants are polynomials in the coefficients of the systems. They generate an ideal in the ring of polynomials having as variables the coefficients of the systems over the real field. So they determine an algebraic set in the affine space of the real coefficients of the system. This is the algebraic set of the systems with center. There are two things which are difficult in the problem of the center.

(i) Finding this algebraic set, i.e. explicitly finding where to stop in the calculations of the Poincaré–Liapunov constants.
(ii) Splitting this algebraic set into its irreducible components. In other words finding the algebraic varieties which build this algebraic set.

Of the two parts the second one is more difficult.

We recall that systems (1) are *reversible* with respect to a straight line if they are invariant under the change of variables $\bar{w}=e^{-i\gamma}z$, $\tau=-t$ for some γ real. For systems (1) we have the following result.

Proposition 1. *Systems* (1) *are reversible if and only if* $A = -\bar{A}e^{-ni\gamma}$, $B = -\bar{B}e^{ni\gamma}$, $C = -\bar{C}$ *and* $D = -\bar{D}e^{-(2+j)ni\gamma}$, *for some* $\gamma \in \mathbb{R}$. *Furthermore, in this situation the origin of system* (1) *is a center.*

Proof of Proposition 1. The proof follows directly from its statement. For more details see [3]. □

Once we have proved the existence of the so-called center space of systems (1) we also want to determine which of the centers are isochronous. In that case let $z = 0$ be a center (that is, we assume that we are under the hypothesis which guarantees that $z = 0$ is a center) and let V be a neighborhood of $z = 0$ such that $V \setminus \{0\}$ is covered with periodic orbits surrounding $z = 0$. We can define a function, *the period function* of $z = 0$ by associating to every point z of V the minimal period of the periodic orbits passing through z. The center $z = 0$ of system (1) is *isochronous* if the period of all integral curves in $V \setminus \{0\}$ is constant. The study of the isochronous centers started with Huygens when he studied the cycloidal pendulum. The existence of isochronous centers for several classes of polynomial differential systems has been studied in [6].

If we take the equation of $\theta' = d\theta/dt$ we obtain

$$T = \int_0^{2\pi} \frac{d\theta}{\theta'} = \int_0^{2\pi} \frac{1}{1 + G(\theta)r(\theta)^{d-1}}\, d\theta$$
$$= \int_0^{2\pi} \big(1 - G(\theta)\rho(\theta)\big)\, d\theta = 2\pi - \int_0^{2\pi} G(\theta)\rho(\theta)\, d\theta,$$

where $\rho(\theta) = \sum_{j\geqslant 1} \rho_j(\theta)\gamma^j$ is given in (5), and $\rho_j(\theta)$ are the terms giving rise to the Poincaré–Liapunov–Abel constants. Then systems (1) have an isochronous center at the origin if it is a center and satisfies

$$\int_0^{2\pi} G(\theta)\varrho(\theta)\, d\theta = \sum_{j\geqslant 1}\left(\int_0^{2\pi} G(\theta)\rho_j(\theta)\, d\theta\right)\gamma^j = 0,$$

that is if

$$T = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}} = 2\pi - \sum_{j\geqslant 1} T_j\gamma^j = 2\pi, \tag{7}$$

with

$$T_j = \int_0^{2\pi} G(\theta)\rho_j(\theta)\, d\theta = 0, \quad \text{for } j \geqslant 1. \tag{8}$$

The constants T_j are called the *period Abel constants*.

The main result in this paper is the following

Theorem 2. *The following statements hold.*

(a) *A system* (1) *has a center at the origin if and only if one of the following conditions hold*:
  (a.1) $\lambda = c_1 = A + (3 - j)\bar{B} = 0$,
  (a.2) $\lambda = c_1 = \mathrm{Im}(AB) = \mathrm{Re}(A^2\bar{D}) = \mathrm{Re}(\bar{B}^2\bar{D}) = 0$ *when* $j = 0$,
  (a.3) $\lambda = c_1 = \mathrm{Im}(AB) = \mathrm{Im}(A^3\bar{D}) = \mathrm{Im}(\bar{B}^3\bar{D}) = 0$ *when* $j = 1$.

(b) *A system* (1) *has an isochronous center at the origin if and only if one of the following conditions hold*:
  (b.1) $\lambda = C = D = 0$ *and* $A = \bar{B}$,
  (b.2) $\lambda = C = D = 0$ *and* $(n + d - 1)A = (n - d + 1)\bar{B}$,
  (b.3) $\lambda = A = C = 0$, $D = -\bar{B}^2/B$, $d = 3n + 1$ *and* $j = 1$.

The proof of Theorem 2(a) is given in Sections 2 and 3 and the proof of Theorem 2(b) is provided in Sections 4 and 5.

2. Proof of Theorem 2(a): Necessary conditions for a center

We prove in this section that conditions (a.1), (a.2) and (a.3) are necessary conditions to have a center at the origin. For this we compute the Poincaré–Liapunov constants up to some order and then show that the zeroes of those Poincaré–Liapunov constants are precisely conditions (a.1), (a.2) and (a.3).

Proposition 3. *The Poincaré–Liapunov constants of systems* (1), *when* $j = 0$ *with* $d \geqslant 5$ *odd*, $n \geqslant 2$ *even and* $n \leqslant (d + 1)/2$, *are*

$$
\begin{aligned}
V_1 &= e^{2\pi(d-1)\lambda},\\
V_2 &= c_1,\\
V_3 &= -\mathrm{Im}(AB),\\
V_4 &= \mathrm{Re}\big((A + 3\bar{B})\bar{D}\big[(d + 2n - 1)A + (d - 2n - 1)\bar{B}\big]\big),\\
V_5 &= \mathrm{Im}\big((A + 3\bar{B})\bar{D}C(A + \bar{B})\big).
\end{aligned}
$$

We remark that $V_k \equiv \rho_k(2\pi)$ (mod. $\{V_1, V_2, \dots, V_{k-1}\}$), *for* $k = 2, \dots, 5$ *and also modulo a positive constant.*

Proof. Solving $\rho_1'(\theta) = (d - 1)\lambda\rho_1(\theta)$ and evaluating at $\theta = 2\pi$ we obtain $\rho_1(2\pi) = e^{2\pi(d-1)\lambda}$. Then $V_1 = e^{2\pi(d-1)\lambda}$. In what follows we take $\lambda = 0$.

Substituting (5) into (4) we get that the functions $\rho_k(\theta)$ must satisfy

$$
\begin{aligned}
\rho_2' &= B\rho_1^2,\\
\rho_3' &= A\rho_1^3 + 2B\rho_1\rho_2,\\
\rho_4' &= 3A\rho_1^2\rho_2 + B\big(\rho_2^2 + 2\rho_1\rho_3\big),\\
\rho_5' &= 3A\big(\rho_1\rho_2^2 + \rho_1^2\rho_3\big) + 2B(\rho_2\rho_3 + \rho_1\rho_4),
\end{aligned}
\tag{9}
$$

where we have omitted that all the functions depend on θ. Note that all these differential equations can be solved recursively doing an integral between 0 and θ, and recalling that $\rho_k(0) = 0$ for $k \geqslant 2$. We have done all the computations of this paper with the help of the algebraic manipulator mathematica. These computations are not difficult but are long and tedious.

Solving the equation $\rho_2' = B\rho_1^2$ we get that $\rho_2(2\pi) = 2\pi(d-1)c_1$. Then we take $c_1 = 0$.

Now we compute the solution $\rho_3(\theta)$ of $\rho_3' = A\rho_1^3 + 2B\rho_1\rho_2$, and we get that $\rho_3(2\pi) = 2\pi(1-d)\operatorname{Im}(AB)$. Then $V_3 = -\operatorname{Im}(AB)$.

Solving the differential equation for $\rho_4(\theta)$ we get $\rho_4(\theta)$ and in particular we obtain from the expression of $\rho_4(2\pi)$ the value of V_4 given in the statement of Proposition 3 modulo $\rho_2(2\pi) = \rho_3(2\pi) = 0$ and a positive constant factor. More precisely we can check that

$$\frac{4\pi}{(d-1)\pi}\rho_4(2\pi) = V_4 + 4V_3(n+d-1)d_2.$$

We compute the solution $\rho_5(\theta)$ from the differential equation for $\rho_5(\theta)$, we get $\rho_5(2\pi)$, and in particular we obtain the value of V_5 given in the statement of Proposition 3 modulo $\rho_2(2\pi) = \rho_3(2\pi) = \rho_4(2\pi) = 0$ and modulo a positive constant. This completes the proof of the proposition. □

Now we want to check which are the solutions of $V_1 = 1$, $V_2 = V_3 = V_4 = V_5 = 0$ in terms of λ, a_1, a_2, b_1, b_2, c_1, c_2, d_1 and d_2.

Proposition 4. *For $j = 0$, $d \geqslant 5$ odd, $n \geqslant 2$ even with $n \leqslant (d+1)/2$, $V_1 = 1$, $V_2 = V_3 = V_4 = V_5 = 0$ if and only if either* (a.1) *with $j = 0$, or* (a.2)*, or*

(c.1) *$j = 0$, $\lambda = C = 0$, $(d+2n-1)A + (d-2n-1)\bar{B} = 0$ and $\operatorname{Re}(\bar{B}^2\bar{D}) \neq 0$*

holds.

Proof. From the fact that $V_1 = 1$ we get that $\lambda = 0$. Since $V_2 = 0$ we have $c_1 = 0$. To make $V_3 = 0$ we will consider two different cases: $B = 0$ and $B \neq 0$.

Case 1: $B = 0$. In that case

$$V_4 = (d+2n-1)\operatorname{Re}\bigl(A^2\bar{D}\bigr).$$

So $\operatorname{Re}(A^2\bar{D}) = 0$ and we are under the assumptions (a.2).

Case 2: $B \neq 0$. Then $A = \mu\bar{B}$ with $\mu \in \mathbb{R}$. In this case

$$V_4 = (3+\mu)\bigl((d+2n-1)\mu + (d-2n-1)\bigr)\operatorname{Re}\bigl(\bar{B}^2\bar{D}\bigr).$$

In view of the factors of V_4 we consider three cases.

Subcase 2.1: $\mu = -3$. In this case we are under the assumptions (a.1).

Subcase 2.2: $\operatorname{Re}(\bar{B}^2 D) = 0$. In this case we are under the assumptions (a.2).

Subcase 2.3: $\mu = -(d-2n-1)/(d+2n-1)$ and $\operatorname{Re}(\bar{B}^2\bar{D}) \neq 0$. In this case since $c_1 = 0$ we have

$$V_5 = \frac{8n(4n+d-1)}{(2n+d-1)^2}\operatorname{Im}\bigl(\bar{B}^2 C\bar{D}\bigr) = \frac{8n(4n+d-1)}{(2n+d-1)^2} c_2 \operatorname{Re}\bigl(\bar{B}^2\bar{D}\bigr),$$

and to have $V_5 = 0$ we must impose $c_2 = 0$ that is $C = 0$. Hence, we are under assumptions (c.1). This concludes the proof of the proposition. □

Now we prove that condition (c.1) is not necessary in order that systems (1) have a center at the origin.

Proposition 5. *Condition* (c.1) *is not necessary for systems* (1) *to have a center at the origin.*

Proof. Systems (1) with the assumptions of condition (c.1) become

$$\dot{z} = iz - \frac{d-2n-1}{d+2n-1}\bar{B}z^{\frac{d-n+1}{2}}\bar{z}^{\frac{d+n-1}{2}} + Bz^{\frac{d+n+1}{2}}\bar{z}^{\frac{d-n-1}{2}} + Dz^{\frac{d-2n+1}{2}}\bar{z}^{\frac{d+2n-1}{2}}, \tag{10}$$

with $\operatorname{Re}(\bar{B}^2\bar{D}) \neq 0$. Now if we do the change $z \to w = \xi z$ with

$$\xi = \frac{\bar{B}^{\frac{d-n-1}{2n(d-1)}}}{B^{\frac{d+n-1}{2n(d-1)}}} \tag{11}$$

then using (6) we have that systems (10) can be written as

$$\dot{w} = iw - \frac{d-2n-1}{d+2n-1}w^{\frac{d-n+1}{2}}\bar{w}^{\frac{d+n-1}{2}} + w^{\frac{d+n+1}{2}}\bar{w}^{\frac{d-n-1}{2}} + \tilde{D}w^{\frac{d-2n+1}{2}}\bar{w}^{\frac{d+2n-1}{2}}, \tag{12}$$

with

$$\tilde{D} = \frac{DB^{1/2}}{\bar{B}^{3/2}} = \tilde{d}_1 + i\tilde{d}_2 \quad \text{and} \quad \tilde{d}_1 \neq 0. \tag{13}$$

For systems (12) (in view of Proposition 3) we have that $V_1 = V_2 = V_3 = V_4 = V_5 = 0$. Now using the expressions of $\rho_1, \dots, \rho_5$ computed in the proof of Proposition 3 and using that

$$\rho_6' = A\bigl(\rho_2^3 + 6\rho_1\rho_2\rho_3 + 3\rho_1^2\rho_4\bigr) + B\bigl(\rho_3^2 + 2\rho_2\rho_4 + 2\rho_1\rho_5\bigr), \tag{14}$$

with $V_6 \equiv \rho_6(2\pi)$ (mod. $\{V_1, V_2, \dots, V_5\}$), and modulo a positive constant we get

$$V_6 = \tilde{d}_1\bigl(R_d^1 + R_d^2|\tilde{D}|^2\bigr),$$

with

$$R_d^1 = 32(d-1)^2n^2,$$
$$R_d^2 = (d-1)^4 + 3n(d-1)^3 - 8(d-1)^2n^2 + 36(d-1)n^3 - 32n^4.$$

Therefore, in order that $V_6 = 0$ we also need to impose that

$$\tilde{D} = \sqrt{\frac{R_d^1}{-R_d^2}} e^{i\psi}, \quad \psi \in [0, 2\pi) \setminus \{\pi/2, 3\pi/2\}.$$

Note that $R_d^1 > 0$ for any values because $d \geqslant 5$ and $n \geqslant 2$. Therefore, in order that $\tilde{D}$ be well defined we need to restrict the values of (d, n) such that $R_d^2 < 0$. For the rest of values of (d, n) such that $R_d^2 > 0$ we have that $V_6 \neq 0$ and thus condition (c.1) is not a necessary condition in order that systems (10) (and consequently of systems (1) under the assumptions (c.1)) have a center at the origin. We note that n is an integer in the interval $[2, (d+1)/2]$. Choosing d sufficiently large and for instance $n = 2$, it is clear that $R_d^2 > 0$ because it is a polynomial of degree 4 in the variable $d-1$ starting with $(d-1)^4$. This concludes the proof of the proposition in these cases. For the other cases condition (c.1) becomes

(c.1)′ $j = 0$, $\lambda = C = 0$, $\tilde{D} = \sqrt{\frac{R_d^1}{-R_d^2}} e^{i\psi}$, $\psi \in [0, 2\pi) \setminus \{\pi/2, 3\pi/2\}$ and (d, n) are such that $R_d^2 < 0$.

Now (12) becomes

$$\dot{w} = iw - \frac{d-2n-1}{d+2n-1} w^{\frac{d-n+1}{2}} \bar{w}^{\frac{d+n-1}{2}} + w^{\frac{d+n+1}{2}} \bar{w}^{\frac{d-n-1}{2}} + \sqrt{\frac{R_d^1}{-R_d^2}} e^{i\psi} w^{\frac{d-2n+1}{2}} \bar{w}^{\frac{d+2n-1}{2}},$$

with $\psi \in [0, 2\pi) \setminus \{\pi/2, 3\pi/2\}$ and (d, n) are such that $R_d^2 < 0$. For these systems and using that

$$\begin{aligned}
\rho_7' &= 3A\big(\rho_2^2\rho_3 + \rho_1\rho_3^2 + 2\rho_1\rho_2\rho_4 + \rho_1^2\rho_5\big) + 2B(\rho_3\rho_4 + \rho_2\rho_5 + \rho_1\rho_6),\\
\rho_8' &= 3A\big(\rho_2\rho_3^2 + \rho_2^2\rho_4 + 2\rho_1\rho_3\rho_4 + 2\rho_1\rho_2\rho_5 + \rho_1^2\rho_6\big)\\
&\quad + B\big(\rho_4^2 + 2\rho_3\rho_5 + 2\rho_2\rho_6 + 2\rho_1\rho_7\big),
\end{aligned} \tag{15}$$

and proceeding as in the proof of Proposition 3 we get that $V_7 = 0$ and $V_8 = \cos\psi$. However by hypothesis we have that $\cos\psi \neq 0$ and thus $V_8 \neq 0$. This implies that neither systems (10) have a center at the origin, nor systems (1) under condition (c.1). This concludes the proof of the proposition. □

Proposition 6. *The Poincaré–Liapunov constants of systems* (1) *with* $j = 1$, $d \geqslant 3$ *integer and* $n > 0$ *an integer with converse parity with* d *satisfying* $n \leqslant [(d+1)/3]$, *are*

$$\begin{aligned}
V_1 &= e^{2\pi(d-1)\lambda},\\
V_2 &= c_1,\\
V_3 &= -\operatorname{Im}(AB),\\
V_4 &= 0,\\
V_5 &= -\operatorname{Im}\big((A+2\bar{B})\bar{D}\big[(d+n-1)A + (d-n-1)\bar{B}\big]\\
&\qquad \times \big[(d+3n-1)A + (d-3n-1)\bar{B}\big]\big),\\
V_6 &= \operatorname{Re}\big((A+2\bar{B})\bar{D}C\big(A^2u + \bar{B}^2v\big)\big),
\end{aligned}$$

with

$$u = -\frac{1}{(d-n-1)^2}\big(2 - 8d + 12d^2 - 8d^3 + 2d^4 + n - 3dn + 3d^2n - d^3n - 2n^2 + 2dn^2 + 2d^2n^2 - 2d^3n^2 - 3n^3 + 3dn^3 - 2n^4 + 6dn^4\big)$$

and

$$v = 2d^2 - 4d + 2 + 5n - 5dn + 4n^2.$$

We remark that $V_k \equiv \rho_k(2\pi)$ (mod. $\{V_1, V_2, \dots, V_{k-1}\}$), *for* $k = 2, \dots, 6$ *and also modulo a positive constant.*

Proof. Proceeding in the same way as in the proof of Proposition 3 using (9) we get V_1, V_2, V_3, V_4 and V_5 given in the statement of Proposition 6. More precisely, we can check that

$$\frac{6n^2}{(d-1)\pi}\rho_5(2\pi) = V_5 - V_3\big(8(1-d)^2a_2d_2 - 5(1-d)^2b_2d_2 - 16(1-d)na_2d_2 - 4(1-d)nb_2d_2 + 9n^2b_2d_2 + 4(1-d)^2a_1d_1 - 8(1-d)na_1d_1 + 10(1-d)^2b_1d_1 + 8(1-d)nb_1d_1 - 18n^2b_1d_1\big).$$

We compute the solution $\rho_6(\theta)$ from the differential equation for $\rho_6(\theta)$ in (14), we get $\rho_6(\theta)$, and in particular we obtain V_6 where V_6 is $\rho_6(2\pi)$ modulo $\rho_2(2\pi) = \rho_3(2\pi) = \rho_4(2\pi) = \rho_5(2\pi) = 0$ and modulo a positive constant. This completes the proof of the proposition. □

Now we want to check which are the solutions of $V_1 = 1$, $V_2 = V_3 = V_5 = V_6 = 0$ in terms of λ, a_1, a_2, b_1, b_2, c_1, c_2, d_1 and d_2.

Proposition 7. *For* $j = 1$, $d \geqslant 3$ *an integer and* $n > 0$ *an integer with converse parity with* d *satisfying* $n \leqslant [(d+1)/3]$, $V_1 = 1$, $V_2 = V_3 = V_5 = V_6 = 0$ *if and only if either* (a.1) *with* $j = 1$, *or* (a.3), *or*

(c.2) $j = 1$, $\lambda = C = 0$, $(d+n-1)A + (d-n-1)\bar{B} = 0$ *and* $\mathrm{Im}(\bar{B}^3\bar{D}) \neq 0$, *or*
(c.3) $j = 1$, $\lambda = C = 0$, $(d+3n-1)A + (d-3n-1)\bar{B} = 0$ *and* $\mathrm{Im}(\bar{B}^3\bar{D}) \neq 0$,

holds.

Proof. We proceed as in the proof of Proposition 4 and we have that $\lambda = 0$, $c_1 = 0$ and either B is 0, or $B \neq 0$.

Case 1: $B = 0$. In that case

$$V_5 = -(d+n-1)(d+3n-1)\,\mathrm{Im}\big(A^3\bar{D}\big).$$

Therefore $\mathrm{Im}(A^3\bar{D}) = 0$ and we are under the assumptions of condition (a.3).

Case 2: $B \neq 0$. Then $A = \mu\bar{B}$ with $\mu \in \mathbb{R}$. In this case

$$V_5 = (\mu+2)\big((d+n-1)\mu + (d-n-1)\big)\big((d+3n-1)\mu + (d-3n-1)\big)\operatorname{Im}\big(\bar{B}^3\bar{D}\big).$$

In view of the factors of V_5 we consider four cases.

Subcase 2.1: $\mu = -2$. Then we are under the assumptions (a.1).

Subcase 2.2: $\operatorname{Im}(\bar{B}^3\bar{D}) = 0$. In this case we are under the assumptions (a.3).

Subcase 2.3: $\mu = -(d-n-1)/(d+n-1)$ and $\operatorname{Im}(\bar{B}^3\bar{D}) \neq 0$. In this case since $c_1 = 0$ we have

$$V_6 = \frac{(d-1)^3(d+3n-1)}{(d+n-1)^3} c_2 \operatorname{Im}\big(\bar{B}^3\bar{D}\big)$$

and to have $V_6 = 0$ we must impose $c_2 = 0$ that is $C = 0$. Hence, we are under assumptions (c.2).

Subcase 2.4: $\mu = -(d-3n-1)/(d+3n-1)$ and $\operatorname{Im}(\bar{B}^3\bar{D}) \neq 0$. In this case since $c_1 = 0$ we have

$$V_6 = -\frac{(d-1)^3(d+9n-1)}{(d+3n-1)^3} c_2 \operatorname{Im}\big(\bar{B}^3\bar{D}\big)$$

and to have $V_6 = 0$ we must impose $c_2 = 0$ that is $C = 0$. Hence, we are under assumptions (c.3). This concludes the proof of the proposition. $\square$

Proposition 8. *Conditions* (c.2) *and* (c.3) *are not necessary in order that systems* (1) *have a center at the origin.*

Proof. Systems (1) with conditions either (c.2) or (c.3) become

$$\dot{z} = iz - T_d \bar{B} z^{\frac{d-n+1}{2}} \bar{z}^{\frac{d+n-1}{2}} + B z^{\frac{d+n+1}{2}} \bar{z}^{\frac{d-n-1}{2}} + D z^{\frac{d-3n+1}{2}} \bar{z}^{\frac{d+3n-1}{2}}, \tag{16}$$

where

$$T_d = \begin{cases} (d-n-1)/(d+n-1) & \text{if (c.2) holds,} \\ (d-3n-1)/(d+3n-1) & \text{if (c.3) holds,} \end{cases}$$

and $\operatorname{Im}(\bar{B}^3\bar{D}) \neq 0$. Now if we make the change $z \to w = \xi z$ with

$$\xi = \frac{\bar{B}^{\frac{d-n-1}{2n(d-1)}}}{B^{\frac{d+n-1}{2n(d-1)}}} \tag{17}$$

then using (6) we have that systems (16) can be written as

$$\dot{w} = iw - T_d w^{\frac{d-n+1}{2}} \bar{w}^{\frac{d+n-1}{2}} + w^{\frac{d+n+1}{2}} \bar{w}^{\frac{d-n-1}{2}} + \tilde{D} w^{\frac{d-3n+1}{2}} \bar{w}^{\frac{d+3n-1}{2}}, \tag{18}$$

with

$$\tilde{D} = \frac{DB}{\bar{B}^2} = \tilde{d}_1 + i\tilde{d}_2 \quad \text{and} \quad \tilde{d}_2 \neq 0. \tag{19}$$

For systems (18) (in view of Proposition 6) we have that $V_1 = V_2 = V_3 = V_5 = V_6 = 0$. Now using $\rho_1, \ldots, \rho_6$ computed in the proof of Proposition 3 and using $\rho_7'(\theta)$ in (15) and proceeding as in the proof of Proposition 6, we get that

$$V_7 = \tilde{d}_2\big(R_d^1 + R_d^2|\tilde{D}|^2\big),$$

with

$$R_d^1 = \begin{cases} 0 & \text{if (c.2) holds,} \\ 240(d-1)^2n^2 & \text{if (c.3) holds,} \end{cases}$$

and

$$R_d^2 = 1, \quad \text{if condition (c.2) holds,}$$

while if condition (c.3) holds, then

$$R_d^2 = (1-d)^4 + 4(1-d)^3 n - 102(1-d)^2n^2 + 396(1-d)n^3 - 459n^4.$$

Therefore, since $\tilde{d}_2 \neq 0$ we get that if condition (c.2) holds, then $V_7 \neq 0$. Hence, condition (c.2) is not a necessary condition in order that systems (16) (and consequently (1) under condition (c.2)) have a center at the origin. This completes the proof of the proposition for condition (c.2). On the other hand, if condition (c.3) holds, in order that $V_7 = 0$ we also need to impose that

$$\tilde{D} = \sqrt{\frac{R_d^1}{-R_d^2}}e^{i\psi}, \quad \psi \in (0, 2\pi) \setminus \{\pi\}.$$

Note that $R_d^1 > 0$ for any values of $d \geqslant 3$ and $n > 0$. Therefore in order that $\tilde{D}$ be well defined we need to restrict the values of (d, n) such that $R_d^2 < 0$. For the rest of values of (d, n) such that $R_d^2 > 0$ we have that $V_7 \neq 0$ and thus condition (c.3) is not a necessary condition in order that systems (16) (and consequently of system (1) under condition (c.3)) have a center at the origin. This concludes the proof of the proposition in these cases. For the other cases condition (c.3) becomes

(c.3)′ $j = 1$, $\lambda = C = (d+3n-1)A + (d-3n-1)\bar{B} = 0$, $\tilde{D} = \sqrt{\frac{R_d^1}{-R_d^2}}e^{i\psi}$, $\psi \in (0, 2\pi) \setminus \{\pi\}$ and (d, n) are such that $R_d^2 < 0$.

Now (18) becomes

$$\dot{w} = iw - T_d w^{\frac{d-n+1}{2}} \bar{w}^{\frac{d+n-1}{2}} + w^{\frac{d+n+1}{2}} \bar{w}^{\frac{d-n-1}{2}} + \sqrt{\frac{R_d^1}{-R_d^2}}e^{i\psi} w^{\frac{d-3n+1}{2}} \bar{w}^{\frac{d+3n-1}{2}},$$

with $\psi \in (0, 2\pi) \setminus \{\pi\}$ and (d, n) are such that $R_d^2 < 0$. For these systems and using $\rho_8'(\theta)$ in (15) and

$$\rho_9' = A\big(\rho_3^3 + 6\rho_2\rho_3\rho_4 + 3\rho_1\rho_4^2 + 3\rho_2^2\rho_5 + 6\rho_1\rho_3\rho_5 + 6\rho_1\rho_2\rho_6 + 3\rho_1^2\rho_7\big)$$
$$+ 2B(\rho_4\rho_5 + \rho_3\rho_6 + \rho_2\rho_7 + \rho_1\rho_8),$$

we get that $V_8 = 0$ and $V_9 = \sin\psi$. However by hypothesis we have that $\sin\psi \neq 0$ and thus $V_9 \neq 0$. This implies that neither systems (16) have a center at the origin, nor systems (1) under condition (c.3). This concludes the proof of the proposition. □

3. Proof of Theorem 2(a): Sufficient condition for a center

We prove in this section that for $j = 0$, $d \geqslant 5$ odd, $n \geqslant 2$ even with $n \leqslant (d+1)/2$, conditions (a.1) and (a.2) are also sufficient for having a center, and for $j = 1$, $d \geqslant 3$ an integer with converse parity with d satisfying $n \leqslant [(d+1)/3]$, conditions (a.1) and (a.3) are also sufficient to have a center at the origin.

Proposition 9. *Under conditions* (a.1) *systems* (1) *have a center at the origin.*

Proof. We write systems (1) with condition (a.1). Then

$$\dot z = iz - (3-j)\bar B z^{\frac{d-n+1}{2}}\bar z^{\frac{d+n-1}{2}} + Bz^{\frac{d+n+1}{2}}\bar z^{\frac{d-n-1}{2}} + ic_2 z^{\frac{d+1}{2}}\bar z^{\frac{d-1}{2}}$$
$$+ Dz^{\frac{d-(2+j)n+1}{2}}\bar z^{\frac{d+(2+j)n-1}{2}}. \tag{20}$$

Then if we multiply it $1/(z\bar z)^{\frac{d-(2+j)n+1}{2}}$ it becomes

$$\dot z = \frac{iz}{(z\bar z)^{(d-(2+j)n+1)/2}} - (3-j)\bar B z^{\frac{n(j+1)}{2}}\bar z^{\frac{n(3+j)-2}{2}} + Bz^{\frac{n(j+3)}{2}}\bar z^{\frac{n(1+j)-2}{2}}$$
$$+ ic_2 z^{\frac{(2+j)n}{2}}\bar z^{\frac{n(2+j)-2}{2}} + D\bar z^{n(2+j)-1} = i\frac{\partial H}{\partial \bar z},$$

where for $d \neq (2+j)n+1$ we have

$$\begin{aligned}
H &= \frac{-2}{d-(2+j)n-1}(z\bar z)^{-(d-(2+j)n-1)/2} \\
&\quad + \frac{2i}{n}\left(\frac{3-j}{3+j}\bar B z^{\frac{n(j+1)}{2}}\bar z^{\frac{n(3+j)}{2}} - \frac{B}{1+j}z^{\frac{n(3+j)}{2}}\bar z^{\frac{n(j+1)}{2}}\right) \\
&\quad + \frac{2c_2}{n(2+j)}z^{\frac{(2+j)n}{2}}\bar z^{\frac{(2+j)n}{2}} - \frac{i}{n(2+j)}\big(D\bar z^{n(2+j)} - \bar D z^{n(2+j)}\big) \\
&= \frac{-2}{d-(2+j)n-1}(z\bar z)^{-(d-(2+j)n-1)/2} \\
&\quad + \frac{2i}{n(1+j)}\big(\bar B z^{\frac{n(j+1)}{2}}\bar z^{\frac{n(3+j)}{2}} - Bz^{\frac{n(3+j)}{2}}\bar z^{\frac{n(j+1)}{2}}\big) \\
&\quad + \frac{2c_2}{n(2+j)}z^{\frac{(2+j)n}{2}}\bar z^{\frac{(2+j)n}{2}} - \frac{i}{n(2+j)}\big(D\bar z^{n(2+j)} - \bar D z^{n(2+j)}\big),
\end{aligned}$$

and for $d = (2+j)n+1$ we have

$$H = \log(z\bar{z}) + \frac{2i}{n(j+1)}\big(\bar{B}z^{\frac{n(j+1)}{2}}\bar{z}^{\frac{n(3+j)}{2}} - Bz^{\frac{n(3+j)}{2}}\bar{z}^{\frac{n(j+1)}{2}}\big)$$
$$+ \frac{2c_2}{n(2+j)}z^{\frac{(2+j)n}{2}}\bar{z}^{\frac{(2+j)n}{2}} - \frac{i}{n(2+j)}\big(D\bar{z}^{n(2+j)} - \bar{D}z^{n(2+j)}\big).$$

Note that the first integral $\exp(H)$ is real and well defined at the origin. Therefore the origin is a center. $\square$

Proposition 10. *Under conditions* (a.2) *or* (a.3) *systems* (1) *have a center at the origin.*

Proof. We will see that if conditions (a.2) or (a.3) are satisfied then systems (1) are reversible systems and thus the proof of this case will follow from Proposition 1. We consider that condition either (a.2) or (a.3) in Theorem 2 holds. Rewriting these conditions as

$$C = -\bar{C}, \qquad \frac{\bar{A}}{A} = \frac{B}{\bar{B}}, \qquad \left(-\frac{\bar{A}}{A}\right)^{2+j} = -\frac{\bar{D}}{D}, \qquad \left(-\frac{\bar{B}}{B}\right)^{2+j} = -\frac{D}{\bar{D}}. \tag{21}$$

Now let θ_1, θ_2 and θ_3 such that $e^{i\theta_1} = -\bar{A}/A$, $e^{i\theta_2} = -\bar{B}/B$ and $e^{i\theta_3} = -D/\bar{D}$. Then by (21) we obtain

$$\theta_1 = -\theta_2 \ (\text{mod. } 2\pi) \quad \text{and} \quad \theta_2 = \frac{1}{2+j}\theta_3 \ (\text{mod. } 2\pi). \tag{22}$$

Now, take $\gamma = \theta_1/n$. Using (22) we have

$$e^{-in\gamma} = e^{-i\theta_1} = -\frac{A}{\bar{A}}, \qquad e^{in\gamma} = e^{i\theta_1} = e^{-i\theta_2} = -\frac{B}{\bar{B}},$$

and

$$e^{-(2+j)in\gamma} = e^{-(2+j)i\theta_1} = e^{(2+j)i\theta_2} = e^{i\theta_3} = -\frac{D}{\bar{D}},$$

which clearly implies, using Proposition 1, that systems (1) under conditions (a.2) or (a.3) are reversible and thus have a center at the origin. $\square$

4. Proof of Theorem 2(b): Sufficient condition for an isochronous center

From the introduction for proving the sufficient conditions for an isochronous center, it is enough to show that

$$\int_0^{2\pi} \frac{d\theta}{\dot{\theta}} = 2\pi, \tag{23}$$

where $\dot{\theta}$ can be obtained writing systems (1) in polar coordinates under conditions (b.k.) for $k = 1, 2, 3$.

Systems (1) with the hypotheses (b.k.) for $k = 1, 2$ have always $A \neq 0$, otherwise it would be linear systems. Furthermore we can do the change of variables

$$\omega = \xi z \quad \text{where } \xi = \frac{A^{(d-n-1)/(2n(d-1))}}{\bar{A}^{(d+n-1)/(2n(d-1))}}. \tag{24}$$

Then systems (1) under the hypotheses (b.1) become

$$\dot{w} = iw + w^{\frac{d-n+1}{2}} \bar{w}^{\frac{d+n-1}{2}} + w^{\frac{d+n+1}{2}} \bar{w}^{\frac{d-n-1}{2}}, \tag{25}$$

while systems (1) under the hypotheses (b.2) can be written as

$$\dot{w} = iw + w^{\frac{d-n+1}{2}} \bar{w}^{\frac{d+n-1}{2}} + \frac{n+d-1}{n-d+1} w^{\frac{d+n+1}{2}} \bar{w}^{\frac{d-n-1}{2}}. \tag{26}$$

Systems (1) under the hypotheses (b.3) and after the changes of variables given by (17) and (19) can be written as

$$\dot{z} = iz + z^{2n+1}\bar{z}^n - z\bar{z}^{3n}. \tag{27}$$

We rewrite systems (25) in polar coordinates and we obtain

$$\dot{r} = 2r^d \cos(n\theta) \quad \text{and} \quad \dot{\theta} = 1.$$

Then clearly (23) holds and thus systems (1) under condition (b.1) have an isochronous center at the origin.

We write systems (26) in polar coordinates and we get

$$\dot{r} = \frac{2n}{n-d+1} r^d \cos(n\theta) \quad \text{and} \quad \dot{\theta} = 1 + \frac{2(d-1)}{n-d+1} r^{d-1} \sin(n\theta). \tag{28}$$

Therefore

$$\frac{dr}{d\theta} = \frac{2nr^d \cos(n\theta)}{n-d+1+2(d-1)r^{d-1}\sin(n\theta)} \quad \text{with } r(0) = r_0.$$

Integrating this differential equation and since $r(\theta) \geqslant 0$ for any θ we get that

$$r(\theta) = \left(\frac{-2(d-1)\sin(n\theta) + \sqrt{(n-d+1)^2 r_0^{2-2d} + 4(d-1)^2 \sin^2(n\theta)}}{n-d+1} \right)^{1/(1-d)}. \tag{29}$$

Note that

$$\sqrt{(n-d+1)^2 r_0^{2-2d} + 4(d-1)^2 \sin^2(n\theta)} \geqslant \left| 2(d-1)\sin(n\theta) \right|$$

and thus $r(\theta)$ given in (29) is positive. Therefore introducing (29) into $\dot{\theta}$ given by (28) we have that

$$\int_0^{2\pi} \frac{d\theta}{\dot\theta} = \int_0^{2\pi} \left(1 - \frac{2(d-1)\sin(n\theta)}{\sqrt{4(d-1)^2\sin^2(n\theta) + (n-d+1)^2 r_0^{2-2d}}}\right) d\theta = 2\pi, \tag{30}$$

because the function $2(d-1)\sin(n\theta)/\sqrt{4(d-1)^2\sin^2(n\theta) + (n-d+1)^2 r_0^{2-2d}}$ is odd in θ. Therefore systems (1) under condition (b.2) have an isochronous center at the origin.

We rewrite systems (27) in polar coordinates and we obtain

$$\dot r = r^{1+3n}\big(\cos(n\theta) - \cos(3n\theta)\big), \qquad \dot\theta = 1 + r^{3n}\big(\sin(n\theta) + \sin(3n\theta)\big). \tag{31}$$

We introduce the change of variables

$$w = r^n, \qquad \varphi = n\theta, \qquad \tau = nt.$$

Then systems (31) become

$$\begin{aligned} w' &= w^4\big(\cos\varphi - \cos(3\varphi)\big) = 4w^4\cos\varphi\sin^2\varphi, \\ \varphi' &= 1 + w^3\big(\sin\varphi + \sin(3\varphi)\big) = 1 + 4w^3\sin\varphi\cos^2\varphi, \end{aligned} \tag{32}$$

where $'$ denotes the derivative with respect to τ.

We note that systems (32) have an invariant (i.e. the first integral depending on the new time τ) of the form

$$I = I(w, \varphi, \tau) = 3(\varphi - \tau) + 4w^3\cos^3\varphi. \tag{33}$$

Indeed we have

$$3\varphi' - 3 + 12w^2w'\cos^3\varphi - 12w^3\cos^2\varphi\sin\varphi\,\varphi' = 0.$$

Now the invariant in the original coordinates becomes

$$I = I(r, \theta, t) = 3n(\theta - t) + 4r^{3n}\cos^3(n\theta).$$

Then the time in function of the variables (r, θ, I) is

$$t = \theta + \frac{4}{3n}r^{3n}\cos^3(n\theta) - \frac{I}{3n}. \tag{34}$$

Then taking into account that systems

$$\frac{dr}{d\theta} = \frac{r^{1+3n}(\cos(n\theta) - \cos(3n\theta))}{1 + r^{3n}(\sin(n\theta) + \sin(3n\theta))}$$

have a center at the origin, we have that $r(0) = r(2\pi)$. Therefore it follows from (34) that the period T is given by

$$T = t(2\pi) - t(0) = 2\pi + \frac{4}{3n} r(2\pi)^{3n} - \frac{I}{3n} - \left[\frac{4}{3n} r(0)^{3n} - \frac{I}{3n}\right] = 2\pi.$$

Therefore systems (1) under condition (b.3) have an isochronous center at the origin.

5. Proof of Theorem 2(b): Necessary conditions for an isochronous center

In this section we compute the necessary conditions to have an isochronous center at the origin of systems (1). We note that since $\rho_1(\theta) = 1$ then from (8) and (3) we have $T_1 = 2\pi c_2$. Therefore in order to have $T_1 = 0$ we must impose $c_2 = 0$. Moreover since either (a.1), or (a.2), or (a.3) holds, we get that $c_1 = 0$. From now on we take $C = 0$. We consider two cases: $j = 0$ and $j = 1$.

Case 1: $j = 0$. We also distinguish two different subcases.

Subcase 1.1: $B = 0$. In this case using $\rho_2(\theta)$ computed in the proof of Proposition 3 and also using (8) and (3), T_2 becomes

$$T_2 = -\frac{\pi}{2n}\big(2|A|^2(d+n-1) + |D|^2(d+2n-1)\big).$$

In order that $T_2 = 0$ we must impose $A = D = 0$. Then $A = B = C = D = 0$ which is a linear system. Therefore this case does not provide an isochronous center.

Subcase 1.2: $B \neq 0$. Then since from $V_2 = 0$ we have $\mathrm{Im}(AB) = 0$, we can write $A = \mu \bar{B}$ with $\mu \in \mathbb{R}$. We consider two different subcases.

Subcase 1.2.1: $\mu = -3$. In this case $A = -3\bar{B}$ and we are under the hypothesis (a.1). Then T_2 becomes

$$T_2 = -\frac{\pi(d+2n-1)}{2n}\big(16|B|^2 + |D|^2\big).$$

In order that $T_2 = 0$ we must impose $B = 0$, a contradiction. Therefore this case does not provide an isochronous center.

Subcase 1.2.2: $\mu \in \mathbb{R} \setminus \{-3\}$. In this case $A = \mu \bar{B}$ and we are under the hypothesis (a.2). By the change of variables in (11) and (13) we can rewrite systems (1) as

$$\dot{w} = iw + \mu w^{\frac{d-n+1}{2}} \bar{w}^{\frac{d+n-1}{2}} + w^{\frac{d+n+1}{2}} \bar{w}^{\frac{d-n-1}{2}} + \tilde{D} w^{\frac{d-2n+1}{2}} \bar{w}^{\frac{d+2n-1}{2}}, \tag{35}$$

with $\tilde{D} = DB^{1/2}/\bar{B}^{3/2}$. Since we are under the assumptions (a.2) we have that

$$\tilde{d}_1 = \mathrm{Re}(\bar{\tilde{D}}) = \frac{1}{B^{3/2}\bar{B}^{3/2}} \mathrm{Re}\big(\bar{B}^2 \bar{D}\big) = 0.$$

Computing the period constants of (35) we get

$$\begin{aligned}
T_2 &= -\pi\left((d-1)\big(2\mu^2 + \tilde{d}_2^2 - 2\big) + 2\big(\tilde{d}_2^2 + (\mu-1)^2\big)n\right)/(2n),\\
T_3 &= -\pi \tilde{d}_2\big(2(d-1)^2\big(-2 + \tilde{d}_2^2 + 2\mu + 4\mu^2\big)\\
&\quad + (d-1)\big(8\tilde{d}_2^2 + (\mu-1)(7+23\mu)\big)n + 2\big(4\tilde{d}_2^2 + 7(\mu-1)^2\big)n^2\big)/\big(4n^2\big),
\end{aligned}$$

$$\begin{aligned}T_4 = &-\pi\big(9\big(5\tilde{d}_2^4 + 4(\mu+1)(9\mu-1)\tilde{d}_2^2 + 8(\mu-1)(\mu+1)^3\big)(d-1)^3 \\ &+ 2\big(135\tilde{d}_2^4 + 4\big(\mu(201\mu-7)-66\big)\tilde{d}_2^2 + 156\big(\mu^2-1\big)^2\big)n(d-1)^2 \\ &+ 4\big(135\tilde{d}_2^4 + 2(\mu-1)(299\mu+15)\tilde{d}_2^2 + 78(\mu-1)^3(\mu+1)\big)n^2(d-1) \\ &+ 72\big(5\tilde{d}_2^4 + 14(\mu-1)^2\tilde{d}_2^2 + (\mu-1)^4\big)n^3\big)/\big(96n^3\big).\end{aligned}$$

Now if we compute, using the Gröebner basis of T_2, T_3 and T_4 with respect to the variables μ and $\tilde{d}_2$ we get that the following expression must be zero:

$$(d-1)^6\tilde{d}_2(-3+3d-2n)(-7+7d-2n)(d-n-1)(2d+n-2)^2 = 0. \tag{36}$$

Since $d \geqslant 5$ and $n \leqslant (d+1)/2$ we have that the unique solution of (36) is $\tilde{d}_2 = 0$. Then, using the Gröebner basis obtained with $\tilde{d}_2 = 0$, we also have

$$(\mu-1)\big((d+n-1)\mu + (d-n+1)\big) = 0, \quad \text{that is} \quad \mu = 1, \qquad \mu = \frac{n-d+1}{d+n-1}.$$

Therefore, if $\mu = 1$, $\tilde{d}_2 = 0$ we are under the assumptions (b.1), and if $\mu = (n-d+1)/(d+n-1)$ and $\tilde{d}_2 = 0$ we are under the assumptions (b.2). This completes the proof when $j = 0$.

Case 2: $j = 1$. We also consider two different subcases.

Subcase 2.1: $B = 0$. In this case using $\rho_2(\theta)$ computed in the proof of Proposition 6 and also using (8) and (3), T_2 becomes

$$T_2 = -\frac{\pi}{3n}\big(3|A|^2(d+n-1) + |D|^2(d+3n-1)\big).$$

In order that $T_2 = 0$ we must impose $A = D = 0$. Then $A = B = C = D = 0$ which is a linear system. Therefore this case does not provide an isochronous center.

Subcase 2.2: $B \neq 0$. Then since from $V_2 = 0$ we have $\mathrm{Im}(AB) = 0$, then $A = \mu\bar{B}$ with $\mu \in \mathbb{R}$. We consider two different subcases.

Subcase 2.2.1: $\mu = -2$. In this case $A = -2\bar{B}$ and we are under the hypothesis (a.1). Then T_2 becomes

$$T_2 = -\frac{\pi(d+3n-1)}{3n}\big(9|B|^2 + |D|^2\big).$$

In order that $T_2 = 0$ we must impose $B = 0$, a contradiction. Therefore this case does not provide an isochronous center.

Subcase 2.2.2: $\mu \in \mathbb{R} \setminus \{-2\}$. In this case $A = \mu\bar{B}$ and we are under the hypothesis (a.3). By the change of variables in (17) and (19) we can rewrite systems (1) as

$$\dot{w} = iw + \mu w^{\frac{d-n+1}{2}}\bar{w}^{\frac{d+n-1}{2}} + w^{\frac{d+n+1}{2}}\bar{w}^{\frac{d-n-1}{2}} + \tilde{D}w^{\frac{d-3n+1}{2}}\bar{w}^{\frac{d+3n-1}{2}} \tag{37}$$

with $\tilde{D} = DB/\bar{B}^2$. Since we are under the assumptions (a.3) we have that

$$-\tilde{d}_2 = \operatorname{Im}(\bar{\tilde{D}}) = \frac{1}{B^2\bar{B}^2}\operatorname{Im}\big(\bar{B}^3\bar{D}\big) = 0.$$

Computing the period constants of (37) we get

$$\begin{aligned}
T_2 &= -\pi\big((d-1)\big(\tilde{d}_1^2+3\mu^2-3\big)+3\big(\tilde{d}_1^2+(\mu-1)^2\big)n\big)/(3n),\\
T_3 &= 0,\\
T_4 &= -\pi\big((n-1)(3n-1)(9n-1)\tilde{d}_1^4-6(\mu+2)n(3n-1)\tilde{d}_1^3\\
&\quad+6\big(18(\mu-1)^2n^3+\big((31-47\mu)\mu+16\big)n^2+\big(\mu(28\mu+13)-8\big)n\\
&\quad-2(\mu+1)(2\mu+1)\big)\tilde{d}_1^2+6\big(-6(\mu-1)^3n^3+2(\mu-1)^2(7\mu+1)n^2\\
&\quad-(\mu-1)(\mu+1)(11\mu-4)n+3\mu(\mu+1)^2\big)\tilde{d}_1\\
&\quad+d^3\big(\tilde{d}_1^4+12(\mu+1)(2\mu+1)\tilde{d}_1^2-18\mu(\mu+1)^2\tilde{d}_1+27(\mu-1)(\mu+1)^3\big)\\
&\quad+9(\mu-1)\big(\mu(n-3)-n-3\big)\big(\mu(n-1)-n-1\big)\big(-\mu+3(\mu-1)n-1\big)\\
&\quad+d\big(\big(13n(3n-2)+3\big)\tilde{d}_1^4+6(\mu+2)n(3n-2)\tilde{d}_1^3\\
&\quad+6\big((\mu-1)(47\mu+16)n^2-2\mu(28\mu+13)n+16n+6\mu(2\mu+3)+6\big)\tilde{d}_1^2\\
&\quad+6\big(-9\mu(\mu+1)^2+2(\mu-1)(11\mu-4)n(\mu+1)\\
&\quad-2(\mu-1)^2(7\mu+1)n^2\big)\tilde{d}_1+9\big(13(\mu+1)n^2(\mu-1)^3\\
&\quad+9(\mu+1)^3(\mu-1)-26\big(\mu^2-1\big)^2n\big)\big)+d^2\big((13n-3)\tilde{d}_1^4+6(\mu+2)n\tilde{d}_1^3\\
&\quad+6\big(-8n+\mu\big(13n+4\mu(7n-3)-18\big)-6\big)\tilde{d}_1^2\\
&\quad-6(\mu+1)\big((\mu-1)(11\mu-4)n-9\mu(\mu+1)\big)\tilde{d}_1-81(\mu-1)(\mu+1)^3\\
&\quad+117\big(\mu^2-1\big)^2n\big)\big)/\big(36n^3\big),
\end{aligned}$$

$T_5 = T_7 = 0$ and T_6 and T_8 have expressions two much long and here we do not give them.

We are looking for solutions of $\tilde{d}_1$ and μ such that $T_k = 0$ for $k = 2,4,6,8$. Note that these $T_k = T_k(\tilde{d}_1,\mu)$ are polynomials in the variables $\tilde{d}_1$ and μ. So we shall use the properties of the resultants for solving the system $T_k = 0$ for $k = 2,4,6,8$ with respect to the variables $\tilde{d}_1$ and μ. We compute

$$\begin{aligned}
r_1(\mu) &= \operatorname{Resultant}\big(T_2(\tilde{d}_1,\mu), T_4(d_1,\mu), d_1\big),\\
r_2(\mu) &= \operatorname{Resultant}\big(T_2(\tilde{d}_1,\mu), T_6(d_1,\mu), d_1\big),\\
r_3(\mu) &= \operatorname{Resultant}\big(T_2(\tilde{d}_1,\mu), T_8(d_1,\mu), d_1\big).
\end{aligned}$$

Thus $r_l(\mu)$ for $l = 1,2,3$ are polynomials in the variable μ. These three polynomials have in common the factors $(3n+d-1)^2(\mu-1)(d-n-1+\mu(n+d-1))$. Clearly $3n+d-1$ cannot be zero, because $d \geqslant 3$ and $n > 0$; and since

$$T_2(\tilde{d}_1,1) = T_2\left(\tilde{d}_1, \frac{n-d+1}{n+d-1}\right) = -\frac{\tilde{d}_1^2(d+3n-1)\pi}{3n},$$

we must take $\tilde{d}_1 = 0$. Then it is easy to check that $T_k = 0$ for $k = 2, 4, 6, 8$ when $\mu = 1$ and $\mu = (n - d + 1)/(n + d - 1)$, obtaining the cases (b.1) and (b.2) of Theorem 2 respectively.

Now we omit these three common factors from the three polynomials $r_l(\mu)$ for $l = 1, 2, 3$, and thus we get the polynomials $s_l(\mu)$ for $l = 1, 2, 3$. We compute

$$\text{Resultant}\big(s_1(\mu), s_2(\mu), \mu\big) = Kf(d, n),$$
$$\text{Resultant}\big(s_1(\mu), s_3(\mu), \mu\big) = Lg(d, n),$$

where

$$\begin{aligned} K &= (d-1)^{45} n^8 (1-d+n)^3 (-1+d+n)(-3+3d+n)^2 \\ &\quad \times (1-d+3n)^2 (-1+d+3n)^{13}, \\ L &= (d-1)^{65} n^{12} (1-d+n)^3 (-1+d+n)^5 (-3+3d+n)^2 \\ &\quad \times (1-d+3n)^2 (-1+d+3n)^{19} (-1+d+9n)^4, \end{aligned}$$

and $f(d, n)$ and $g(d, n)$ are polynomials in the variables d and n. Since $d \geqslant 3$ and $n > 0$, we have that $K = L = 0$ implies that $n = (d-1)/3$.

Now assume that $n \neq (d-1)/3$. Then doing the

$$\text{Resultant}\big(f(d, n), g(d, n), n\big) = M(d-1)^{2240},$$

where M is a positive integer. Since $d \geqslant 3$, using the properties of the resultant (see for more details [13,18]) it follows that the system $T_k = 0$ for $k = 2, 4, 6, 8$ with respect to the variables $\tilde{d}_1$ and μ has no solution when $n \neq (d-1)/3$.

Suppose that $n = (d-1)/3$. Then $T_k = 0$ for $k = 2, 4, 6, 8$ reduce to

$$\begin{aligned} T_2 &= -2\pi\big(-1 + \tilde{d}_1^2 - \mu + 2\mu^2\big), \\ T_4 &= -\pi\big(16\tilde{d}_1^4 + 6\tilde{d}_1^3(2+\mu) + \tilde{d}_1^2\big(-16 + 50\mu + 173\mu^2\big) \\ &\quad - 2\tilde{d}_1\big(6 + 5m + 5\mu^2 + 38\mu^3\big) + 12\mu\big(-4 - 9\mu + 3\mu^2 + 10\mu^3\big)\big)/2, \\ T_6 &= -\pi\big(5670\tilde{d}_1^6 + 4116\tilde{d}_1^5(2+\mu) + 2\tilde{d}_1^4\big(1309 + 35881\mu + 75022\mu^2\big) \\ &\quad - 7\tilde{d}_1^3\big(776 - 3610\mu - 3046\mu^2 + 15879\mu^3\big) \\ &\quad + 14\tilde{d}_1^2\big(-727 - 5540\mu - 5397\mu^2 + 20980\mu^3 + 29645m^4\big) \\ &\quad - 35\tilde{d}_1\big(64 + 729\mu + 518\mu^2 - 997m^3 + 3630\mu^4 + 5776m^5\big) \\ &\quad + 70\big(35 + 159\mu - 501\mu^2 - 2527\mu^3 - 1734\mu^4 + 2328\mu^5 + 2240\mu^6\big)\big)/140, \\ T_8 &= -\pi\big(16056320\tilde{d}_1^8 + 17147781\tilde{d}_1^7(2+\mu) \\ &\quad + \tilde{d}_1^6\big(52317568 + 440476696\mu + 785186476\mu^2\big) \\ &\quad - 21\tilde{d}_1^5\big(-812342 - 16686977\mu - 14276363m^2 + 38219700\mu^3\big) \\ &\quad + 9\tilde{d}_1^4\big(-8204960 - 36356392\mu + 55565628\mu^2 + 466656722\mu^3 + 540545695\mu^4\big) \end{aligned}$$

$$
\begin{aligned}
&- 126\tilde{d}_1^3\big(444878 + 2448179\mu - 1776946\mu^2 - 9212642m^3 + 19329337\mu^4 \\
&+ 34830922\mu^5\big) + 1764\tilde{d}_1^2\big(6592 - 59116\mu - 728450\mu^2 - 1893296\mu^3 \\
&- 65099\mu^4 + 4700711\mu^5 + 3915370\mu^6\big) - 882\tilde{d}_1\big(-12310 + 8051\mu \\
&+ 412869\mu^2 + 489773\mu^3 - 1245259\mu^4 - 360264\mu^5 + 4430660\mu^6 + 3624800\mu^7\big) \\
&+ 141120\mu\big(320 + 1872\mu + 912\mu^2 - 11656\mu^3 - 21612\mu^4 \\
&- 1731m^5 + 20345\mu^6 + 11550\mu^7\big)\big)/70560.
\end{aligned}
$$

We compute

$$
\begin{aligned}
r_1(\mu) &= \text{Resultant}\big(T_2(\tilde{d}_1, \mu), T_4(d_1, \mu), d_1\big), \\
r_2(\mu) &= \text{Resultant}\big(T_2(\tilde{d}_1, \mu), T_6(d_1, \mu), d_1\big), \\
r_3(\mu) &= \text{Resultant}\big(T_2(\tilde{d}_1, \mu), T_8(d_1, \mu), d_1\big).
\end{aligned}
$$

Then it is easy to check that the unique common factor of $r_l(\mu)$ for $l = 1, 2, 3$ are $(\mu - 1) \times \mu(2\mu + 1)$. Evaluating T_k for $k = 2, 4, 6, 8$ in $\mu = -1/2$, 0 and 1 we obtain the solutions

$$
\mu = -\frac{1}{2}, \quad \tilde{d}_1 = 0; \qquad \mu = 0, \quad \tilde{d}_1 = -1; \qquad \mu = 1, \quad \tilde{d}_1 = 0;
$$

for $n = (d - 1)/3$. Note that the first solution is a particular case of condition (b.2), and that the third solution corresponds to a particular case of the condition (b.1). Finally observe that the second condition corresponds to condition (b.3). This concludes the proof.

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