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Nil clean rings

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ABSTRACT

Many variations of the notions of clean and strongly clean have been studied by a variety of authors. We develop a general theory, based on idempotents and direct sum decompositions, that unifies several of these existing concepts. As a specific case, we also investigate a new class of clean rings.

1. Introduction

Following [Nic77], we define an element r of a ring R to be clean if there is an idempotent $e \in R$ such that $r - e$ is a unit of R. A clean ring is defined to be one in which every element is clean. Clean rings were initially developed in [Nic77] as a natural class of rings which have the exchange property. In [Nic99], an element r of a ring R is defined to be strongly clean if there is an idempotent $e \in R$, which commutes with r, such that $r - e$ is invertible in R. Analogously, a strongly clean ring is one in which every element is strongly clean. In [Nic99], it is shown that a strongly clean element of a ring satisfies a generalized version of the classical Fitting's Lemma. This implies (see [Bor05, Section 10]) that every strongly π-regular element of a ring is strongly clean. See [CKL+06] for an excellent overview of the relationship between the clean property and other classical ring theory notions.

In the past ten years, there have been many investigations concerning variants of the clean and strongly clean properties. Additionally, several authors have studied versions of such properties in the case of non-unital rings. In this paper, we begin by developing a general result which uses the idea of direct sum decompositions to unify some of these variants of the strongly clean property. This leads us to two new variants, the *nil clean rings* and the *strongly nil clean rings*, which we study in depth. We begin by establishing the basic properties of nil clean rings and strongly nil clean rings, and

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then investigate the behavior of these properties under various ring extensions. Finally, we provide a complete classification of the uniquely nil clean rings.

In general, all rings will be associative with identity. We will, however, on occasion consider non-unital rings (rings which do not *necessarily* have an identity). Such instances are rare and will always be specifically announced. We also note that many of the notions in this paper are most naturally defined in the language of module endomorphisms. Recall that any ring R can be represented as the full endomorphism ring of R_R. Therefore, any result that we prove for endomorphisms will have a natural analog in terms of ring elements. This translation will be used freely throughout the paper, without further mention. The notations $J(R)$ and $U(R)$ will be used, respectively, for the Jacobson radical and the unit group of the ring R. Other general ring theory terms can be found in [Lam01].

2. A decomposition theorem

The following important result about strongly clean endomorphisms is proved in [Nic99].

Lemma 2.1. *(See [Nic99].) Let R be a ring, and let M_R be a right R-module. An element $\varphi \in \operatorname{End}(M_R)$ is strongly clean if and only if there exists a direct sum decomposition $M = A \oplus B$ such that A and B are φ-invariant and such that $\varphi|_A \in \operatorname{End}(A)$ is an isomorphism and $(1-\varphi)|_B \in \operatorname{End}(B)$ is an isomorphism.*

We express Lemma 2.1 with the following diagram.

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & \varphi\Big\downarrow\cong & & (1-\varphi)\Big\downarrow\cong \\ M & = & A & \oplus & B \end{array}$$

Thus, a strongly clean endomorphism satisfies a generalized version of Fitting's Lemma. This connects strongly clean elements to the classical strongly π-regular elements. For completeness, we will record some of the essential definitions and facts.

Definition 2.2. Let R be any ring.

(1) An element $a \in R$ is called strongly π-regular if $a^n \in Ra^{n+1} \cap a^{n+1}R$ for some positive integer n.
(2) An element $a \in R$ is called strongly regular if $a \in Ra^2 \cap a^2R$.

A ring is called strongly π-regular (respectively, strongly regular) if each of its elements is strongly π-regular (respectively, strongly regular).

The following established results show how to characterize strongly π-regular and strongly regular endomorphisms in terms of direct sum decompositions.

Lemma 2.3. *Let R be a ring, and let M_R be a right R-module.*

(1) *([Bor05, Theorem 10.6]) An element $\varphi \in \operatorname{End}(M_R)$ is strongly π-regular if and only if there exists a direct sum decomposition $M = A \oplus B$ such that A and B are φ-invariant and such that $\varphi|_A \in \operatorname{End}(A)$ is an isomorphism and $\varphi|_B \in \operatorname{End}(B)$ is nilpotent.*
(2) *([Nic99, Theorem 3]) An element $\varphi \in \operatorname{End}(M_R)$ is strongly regular if and only if there exists a direct sum decomposition $M = A \oplus B$ such that A and B are φ-invariant and such that $\varphi|_A \in \operatorname{End}(A)$ is an isomorphism and $\varphi|_B \in \operatorname{End}(B)$ is zero.*

We once again include a diagram so that it may be compared with the diagram following Lemma 2.1.

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & \varphi \big\downarrow \cong & & \varphi \big\downarrow \text{nilpotent or 0, respectively} \\ M & = & A & \oplus & B \end{array}$$

Lemma 2.1 and Lemma 2.3 (and the corresponding diagrams) now suffice to give a straightforward and satisfying proof of the following fact.

Corollary 2.4. *Let R be a ring, let M_R be a right R-module and let $\varphi \in \operatorname{End}(M_R)$ be an endomorphism of M_R. Consider the following conditions.*

(1) *φ is strongly regular.*
(2) *φ is strongly π-regular.*
(3) *φ is strongly clean.*

Then we have the sequence of implications

$$(1) \Rightarrow (2) \Rightarrow (3).$$

In general, none of these is reversible.

Lemma 2.1 shows that there is a strong connection between a direct sum decomposition related to a strongly clean element, and the decomposition of that element as the sum of a unit and an idempotent. In fact, we can use Lemma 2.3 to give alternate definitions of strongly regular and strongly π-regular elements.

Proposition 2.5. *Let R be a ring. An element $a \in R$ is strongly π-regular if and only if there is an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$, $ae = ea$ and eae is nilpotent. The element a is strongly regular if and only if there is an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$, $ae = ea$ and eae is zero.*

In what follows, we will refer to this elementwise decomposition of a strongly π-regular element as a *strongly π-regular decomposition*. It is interesting to note that a strongly π-regular element has only one such decomposition. For completeness, we include a quick proof of this statement.

Proposition 2.6. *Suppose that R is a ring, and that $a \in R$ is a strongly π-regular element, with strongly π-regular decomposition $a = e + u$. If $a = f + v$ is another strongly π-regular decomposition of a, then $e = f$ and $u = v$.*

Proof. It clearly suffices to show that $e = f$. Since ea and fa are both nilpotent, we may choose a positive integer n such that $(ea)^n = 0 = (fa)^n$. Let $e' = 1 - e$ and $f' = 1 - f$. Then $e'a = e'u$ and $f'a = f'v$, which implies, upon taking nth powers, that $e'a^n = e'u^n$ and $f'a^n = f'v^n$. Since $ea^n = (ea)^n = 0 = (fa)^n = fa^n$, we see that

$$e'u^n = e'a^n = e'a^n + ea^n = a^n = f'a^n + fa^n = f'a^n = f'v^n.$$

Multiplying on the left by e and on the right by v^{-n} shows that $e'f = 0$. In a similar fashion (making use of the fact that e commutes with u and that f commutes with v), we see that fe', $f'e$ and $e'f$ are also zero. Therefore $e = f$. □

Remark. We can exploit the connection to Fitting's Lemma to sketch another proof of Proposition 2.6. If φ is a strongly π-regular endomorphism of a module M then, referring to Lemma 2.3, $A = \bigcap_n \operatorname{im}(\varphi^n)$ and $B = \bigcup_n \ker(\varphi^n)$. The idempotent which witnesses the strong π-regularity of φ is the projection to B, along A.

Motivated by this discussion, we formulate the following general definition and lemma.

Definition 2.7. Let $\mathcal{P}$ be a property that a ring element can satisfy. We will call $\mathcal{P}$ an *ABAB-compatible property* if it satisfies the following three conditions.

(1) If $a \in R$ has property $\mathcal{P}$, then $-a$ has property $\mathcal{P}$.
(2) If $a \in R$ has property $\mathcal{P}$ and e is any idempotent in R such that $ea = ae$, then $eae \in eRe$ has property $\mathcal{P}$.
(3) If a is in R and $e \in R$ is an idempotent such that $ea = ae$, and if the elements $eae \in eRe$ and $(1-e)a(1-e) \in (1-e)R(1-e)$ both have property $\mathcal{P}$ as elements of the respective corner rings, then a has property $\mathcal{P}$ in R.

Our choice of the term "ABAB-compatible" is motivated by the next lemma.

Lemma 2.8. *Let R be a ring, and let M_R be a right R-module. Let $\mathcal{P}$ be an ABAB-compatible property. An endomorphism $\varphi \in \operatorname{End}(M_R)$ is then the sum of an idempotent e and an element a with property $\mathcal{P}$ such that $ae = ea$ if and only if there exists a direct sum decomposition $M = A \oplus B$ such that $\varphi|_A$ is an element of $\operatorname{End}(A)$ with property $\mathcal{P}$ and $(1-\varphi)|_B$ is an element of $\operatorname{End}(B)$ with property $\mathcal{P}$.*

Proof. The proof is a generalization of the proof of Lemma 2.1. For the forward direction, suppose that $\varphi = e + a$ is the sum of an idempotent e and an element a with property $\mathcal{P}$, and suppose further that $ae = ea$. Define the decomposition $M = A \oplus B$ by setting $A = (1-e)(M)$ and $B = e(M)$. Then A and B are each clearly φ-invariant since $\varphi e = e\varphi$. Further, if $x \in A$ and $y \in B$, then $\varphi(x) = (a+e)(x) = a(x)$, and $(1-\varphi)(y) = (1-e-a)(y) = -a(y)$. Thus $\varphi|_A = a|_A \in eRe$, and $\varphi|_B = -a|_B \in (1-e)R(1-e)$. Since a has property $\mathcal{P}$ which is preserved under taking additive inverses and passing to corners, we see that $\varphi|_A$ and $(1-\varphi)|_B$ both have property $\mathcal{P}$.

Conversely, suppose that such a decomposition $M = A \oplus B$ exists. We need to then find an idempotent e such that $\varphi e = e\varphi$ and such that $\varphi - e$ has property $\mathcal{P}$. Define e to be the projection onto B with kernel A. Then, as before, defining $a|_A = \varphi|_A = (\varphi - e)|_A$ and $a|_B = -(1-\varphi)|_B = -(1-\varphi-(1-e))|_B$, we see that both $a|_A$ and $a|_B$ have property $\mathcal{P}$. Therefore a has property $\mathcal{P}$. In order to show that $\varphi e = e\varphi$, simply notice that this is the case on A and on B. If $x \in A$, then $\varphi e(x) = 0 = e\varphi(x)$, and if $y \in B$, then $\varphi e(y) = \varphi(y) = e\varphi(y)$. $\square$

Note that "being a unit" is an ABAB-compatible property. This fact allows us to recover the characterization of strongly clean endomorphisms in Lemma 2.1. By analogy, we will refer to an element as $\mathcal{P}$-clean if it is the sum of an idempotent and an element with property $\mathcal{P}$. We will call the element strongly $\mathcal{P}$-clean if, in addition, the idempotent and the element with property $\mathcal{P}$ commute.

Remark. We note that the term "P-clean" is used in [Che06] to refer to a different class of variants of cleanness in which the idempotent, rather than the unit, is replaced with a ring element with a different property. Since we will not refer to such variants, there should be no risk of confusion.

The main focus of this paper will be to investigate the property "being a nilpotent", which is easily seen to be an ABAB-compatible property. We shall call the resulting rings nil clean rings and strongly nil clean rings. Before beginning a study of these rings, however, we will first look at several properties which have been featured in other variants of cleanness, with a view toward determining which are ABAB-compatible.

For example, in [NZ04], a ring is defined to be *semiboolean* if every element is the sum of an idempotent and an element in the Jacobson radical. It is straightforward to check that "being contained in the Jacobson radical" is an ABAB-compatible property, so Lemma 2.8 would apply to these rings.

On the other hand, there are certain variants of cleanness in the literature whose substitute for "being a unit" is not an ABAB-compatible property. In these cases, Lemma 2.8 cannot be used to associate such properties with direct sum decompositions. In [McG03], an element is called *almost clean* if it is the sum of an idempotent and a non-zerodivisor. We claim that "being a non-zerodivisor" is not an ABAB-compatible property. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{pmatrix}$. Let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $a = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Then e is an idempotent which commutes with a, and both eae and $(1-e)a(1-e)$ are non-zerodivisors in their respective corner rings. However, a is, itself, a zerodivisor, as it is annihilated by $\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$.

In [XT05], an element is called 2-*clean* if it is the sum of an idempotent and two units. Note that the property "being the sum of two units" is also not an ABAB-compatible property. Let $S = \mathbb{M}_2(\mathbb{Z}_2)$, and let $e = a = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$. Clearly, e is idempotent and commutes with a. Note that $a = \left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right) + \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ is the sum of two units. However, $ea = e$ is not the sum of two units in eSe.

Before beginning our investigation of the nil clean rings, we first make some preliminary remarks about the non-unital case. Recall that the term "non-unital ring" will refer to a ring that does not necessarily have an identity. For such a ring, there is a standard generalization of the notion of invertibility. Let A be a non-unital ring. We define the operation $*$ as follows

$$p * q = p + q + pq.$$

Then A is a monoid (with identity 0) under the operation $*$. We denote the set of invertible elements of this monoid by $Q(A)$. Note that, if A has an identity, then $a \in Q(A)$ (with inverse b) if and only if $1 + a \in U(A)$ (with inverse $1 + b$). We remark as well that, for a non-unital ring A, $Q(A)$ contains $J(A)$ as well as all nilpotent elements of A.

Following [NZ05], we will define an element x of a non-unital ring A to be *clean general* if there exists an idempotent $e \in A$ and an element $q \in Q(A)$ such that $x = e + q$. Following [CYZ06], an element x will be further called *strongly clean general* in this setting if such an e and q can be chosen so that $qe = eq$.

Remark. Our terminology differs slightly from that used in [NZ05] and [CYZ06], in that we append the word "general" to the adjective "clean". This is meant to help reconcile these two notions in a unital ring. Although it is true that a unital ring is clean (respectively, strongly clean) if and only if it is clean general (respectively, strongly clean general), this is not true elementwise. For example, the element $2 \in \mathbb{Z}$ is (strongly) clean but not (strongly) clean general; on the other hand, $-2 \in \mathbb{Z}$ is (strongly) clean general but not (strongly) clean. More generally, if R is a ring, then $x \in R$ is e-clean (respectively, strongly e-clean) if and only if $-x$ is $1-e$-clean (respectively, strongly $1-e$-clean). See [NZ05] and [CYZ06] for the full details.

The next two results are not entirely new (see, for example, [NZ05, Theorem 8]), but we believe that our proofs are the first which do not rely on a unitalization argument.

Proposition 2.9. *Let A be a non-unital ring and let I be a right ideal of A. Suppose that a is an element of I such that $a = e + q$ for some $e^2 = e \in A$ and $q \in Q(A)$. If either I is a two-sided ideal or $eq = qe$, then $e, q \in I$.*

Proof. Since $q \in Q(A)$, there is an element $p \in Q(A)$ such that $p + q + qp = 0$. Since $a = e + q$, $ea + eap = e + eq + ep + eqp = e$. If I is an ideal, then we see that $e \in I$ and therefore also that $q \in I$. If, on the other hand, $eq = qe$, then $ea = ae$, and therefore $e = ea + eap = a(e + ep) \in I$. Again, we see that $e, q \in I$. □

Corollary 2.10. *If A is a non-unital clean general ring and I is an ideal of A, then I is clean general. If A is a non-unital strongly clean general ring and I is a left or a right ideal of A, then I is strongly clean general.*

Proof. Let $a \in I$. Suppose that $a = e + q$ for some idempotent $e \in A$ and some $q \in Q(A)$ with inverse p. In either case, Proposition 2.9 guarantees that $e, q \in I$. Since $p + q + pq = 0 = p + q + qp$, we see that $p \in I$ as well. This suffices to prove both results. □

3. Basic results on nil clean rings

As noted previously, the property "being nilpotent" is an ABAB-compatible property. We therefore make the following definition.

Definition 3.1. Let R be a ring. An element $r \in R$ is called nil clean if there is an idempotent $e \in R$ and a nilpotent $b \in R$ such that $r = e + b$. The element r is further called strongly nil clean if such an idempotent and nilpotent can be chosen such that $be = eb$. A ring is called nil clean (respectively, strongly nil clean) if every one of its elements is nil clean (respectively, strongly nil clean).

Before investigating further, we give some basic examples of nil clean and strongly nil clean elements and rings. It is clear that every nilpotent, idempotent and unipotent is strongly nil clean (recall that an element is called *unipotent* if it can be written as $1 + b$ for some nilpotent b). Therefore, every Boolean ring is strongly nil clean. Moreover, a reduced ring is nil clean if and only if it is Boolean.

Based on Lemma 2.8, the following corollary is immediate.

Corollary 3.2. *Let R be a ring, and let M_R be a right R-module. An element $\varphi \in \mathrm{End}(M_R)$ is strongly nil clean if and only if there exists a direct sum decomposition $M = A \oplus B$ such that A and B are φ-invariant and such that $\varphi|_A \in \mathrm{End}(A)$ is nilpotent and $(1 - \varphi)|_B \in \mathrm{End}(B)$ is nilpotent.*

For reference, here again is the relevant diagram.

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & \varphi \downarrow \text{nilpotent} & & \varphi \downarrow \text{1-nilpotent} \\ M & = & A & \oplus & B \end{array}$$

As an aside, note that the definitions of nil clean and strongly nil clean elements extend quite naturally to the case of non-unital rings. For example, the following result follows easily.

Proposition 3.3. *Let A be a non-unital ring and let $x \in A$. If x is nil clean (respectively, strongly nil clean), then x is clean general (respectively, strongly clean general).*

Proof. Note that $Q(A)$ contains every nilpotent element of A. □

From now on, we will restrict ourselves to the unital case. However, many of our results will hold just the same for non-unital rings.

The analog of Proposition 3.3 holds in the clean case.

Proposition 3.4. *Every nil clean ring is clean.*

Proof. Suppose that R is a nil clean ring, and let r be an element of R. Then $r - 1 = e + b$ where $e^2 = e$ and b is nilpotent. This implies that $r = e + (1 + b)$ is clean since $1 + b$ is a unit. □

Alternatively, we see that Proposition 3.4 is true by appealing to the fact that, for rings, clean is equivalent to clean general. The situation with elements is less clear. Although we do know that neither "clean" nor "clean general" implies the other for elements, we have not yet been able to find a nil clean element which is not clean.

Question 1. Is every nil clean element also clean?

In the strongly nil clean case, we can say more. The following is an immediate consequence of Lemma 2.3 and Corollary 3.2.

Proposition 3.5. *Any strongly nil clean endomorphism is strongly π-regular. Any strongly nil clean element is strongly π-regular.*

Remark. There is another, more direct, elementwise proof of Proposition 3.5. If e is an idempotent and b is a nilpotent, which commutes with e, such that $a = e + b$, then $a = (1 - e) + (2e - 1 + b)$ is a strongly π-regular decomposition for a.

Corollary 3.6. *Any strongly nil clean element is strongly clean.*

Corollary 3.7. *Every strongly nil clean ring is strongly π-regular and thus strongly clean.*

Remark. Note that the fact that every nil clean ring (respectively, strongly nil clean ring) is clean (respectively, strongly clean) could just as easily have proceeded by writing the element $-r = e' + b'$ as the sum of an idempotent and a nilpotent, and then noticing that $r = (1 - e') + (-1 - b')$ is a clean (respectively, strongly clean) decomposition. This merely reflects the fact that an element r is clean (respectively, strongly clean) if and only if $1 - r$ is clean (respectively, strongly clean). We further remark, based on Proposition 3.4, that a ring is nil clean if and only if every element can be written as the sum of an idempotent and a unipotent.

By Proposition 2.6, we also have the following useful fact.

Corollary 3.8. *If an element of a ring is strongly nil clean, then it has precisely one strongly nil clean decomposition.*

Proof. Since, by Proposition 3.5, any strongly nil clean decomposition is automatically a strongly π-regular decomposition, the result holds by Proposition 2.6. $\square$

We express this fact by saying that any strongly nil clean element (respectively, ring) is *uniquely strongly nil clean*.

We can then strengthen the relationship between "strongly nil clean" and "strongly π-regular".

Proposition 3.9. *Let R be a ring, and let $a \in R$. Suppose that a is strongly π-regular with strongly π-regular decomposition $a = e + u$. Then a is strongly nil clean if and only if $2e - 1 + u$ is nilpotent.*

Proof. Suppose that a is strongly nil clean. Write $a = f + b$ for some idempotent f and some nilpotent b which commutes with f. Then $a = (1 - f) + (2f - 1 + b)$ is a strongly π-regular decomposition. By Proposition 2.6, $e = 1 - f$ and $u = 2f - 1 + b$. But then $2e - 1 + u = 2(1 - f) - 1 + (2f - 1 + b) = b$ is nilpotent.

If, on the other hand, $2e - 1 + u$ is nilpotent, then we see immediately that $a = e + u = (1 - e) + (2e - 1 + u)$ is strongly nil clean. $\square$

We record a few easy corollaries, which will be of use to us later on.

Corollary 3.10. *Let R be a ring. A unit $u \in R$ is strongly nil clean if and only if it is unipotent.*

Proof. Let $a \in U(R)$, and suppose that a is strongly nil clean. Since $a = 0 + a$ is the strongly π-regular decomposition of a, Proposition 3.9 implies that $a - 1$ is nilpotent. Thus, a is unipotent. The reverse implication is clear. $\square$

Corollary 3.11. *Let R be a ring. Then R is strongly nil clean if and only if R is strongly π-regular and every unit of R is unipotent.*

Proof. The forward direction holds by Corollary 3.7 and Corollary 3.10. For the reverse direction, suppose that R is strongly π-regular and that every unit of R is unipotent. Let $a \in R$. By hypothesis, we have a strongly π-regular decomposition $a = e + u$. We claim that the element $2e + u$ is a unit. This follows from a consideration of the Peirce decomposition with respect to the idempotent e (with which $2e+u$ commutes). Indeed, $(1-e)(2e+u)=(1-e)u$ is a unit in $(1-e)R(1-e)$, and $e(2e+u)=e(e+a)=e+ea$ is unipotent in eRe. Since $2e+u$ is a unit, $2e+u-1$ must be nilpotent. By Proposition 3.9, a is strongly nil clean. □

Proposition 3.12. *Let R be a ring, let M_R be a right R-module and let $\varphi \in \operatorname{End}(M_R)$ be an endomorphism. Consider the following conditions.*

(1) *φ is strongly regular.*
(2) *φ is strongly nil clean.*
(3) *φ is strongly π-regular.*
(4) *φ is strongly clean.*

Then we have the sequences of implications

$$(1) \Rightarrow (3) \Rightarrow (4),$$

$$(2) \Rightarrow (3) \Rightarrow (4),$$

and none of these is, in general, reversible. Additionally, representing any ring R as $\operatorname{End}(R_R)$, we get the same implications on the level of rings.

Remark. Note that any field with more than two elements is strongly regular but not strongly nil clean, and note also that $\mathbb{Z}_4$ is strongly nil clean but not strongly regular. Since every strongly regular ring is reduced, it is easy to see that a ring is both strongly regular and strongly nil clean if and only if it is Boolean.

We now record some basic properties of nil clean rings.

Proposition 3.13. *Any quotient of a nil clean ring (respectively, a strongly nil clean ring) is nil clean (respectively, strongly nil clean). Any finite direct product of nil clean rings (respectively, of strongly nil clean rings) is nil clean (respectively, strongly nil clean).*

Remark. However, one can construct an infinite direct product of strongly nil clean rings that is not, itself, even nil clean. Consider $R=\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \cdots$. The element $r=(0,2,2,2,\ldots) \in R$ is not nil clean. On the other hand, if $\{R_\alpha\}$ is a family of rings for which there is a fixed number N such that no nilpotent in any R_α has index of nilpotence larger than N, then the product $\prod_\alpha R_\alpha$ is nil clean (respectively, strongly nil clean) if and only if each R_α is nil clean (respectively, strongly nil clean). In particular, if R is a nil clean (respectively, strongly nil clean) ring with bounded index of nilpotence, then an arbitrary direct product of copies of R is nil clean (respectively, strongly nil clean).

Proposition 3.14. *Let R be a nil clean ring. Then the element 2 is a (central) nilpotent and, as such, is always contained in $J(R)$.*

Proof. Note that, if one can write $2=e+b$ as a sum of an idempotent and a nilpotent, then $1-e=b-1$ is both an idempotent and a unit. Therefore, $b-1=1$ and $2=b$ is nilpotent. □

A non-zero Jacobson-semisimple nil clean ring thus always has characteristic 2.

Proposition 3.15. *Let R be a ring, and let I be any nil ideal of R. Then R is nil clean if and only if R/I is nil clean.*

Proof. The forward direction has been shown; it remains only to show the converse. Suppose that $r \in R$, and suppose that $\bar{r}=\bar{e}+\bar{b}$ for some idempotent $\bar{e}$ in R/I and some nilpotent $\bar{b} \in R/I$. Since idempotents lift modulo any nil ideal, we lift $\bar{e}$ to an idempotent $e \in R$. Then $r - e$ is nilpotent, modulo I. Since I is nil, this means that $r - e$ is nilpotent, which shows that r is nil clean. □

Proposition 3.16. *If R is a nil clean ring, then $J(R)$ is nil.*

Proof. By Proposition 2.9, $J(R)$ is nil clean, as a non-unital ring. Since $J(R)$ contains no non-zero idempotents, it must be the case that $J(R)$ is nil. □

As a corollary, we have a basic structure theorem.

Corollary 3.17. *A ring R is nil clean if and only if $J(R)$ is nil and $R/J(R)$ is nil clean.*

If R is abelian, we can say more.

Proposition 3.18. *If R is a nil clean abelian ring, then every nilpotent element of R is contained in $J(R)$.*

Proof. It is a well-known result of Nicholson that any clean ring is potent (for an easy proof, see [NZ04, Lemma 17]). This means that if R is a clean ring (in particular if R is nil clean) and $T \nsubseteq J(R)$ is any right ideal of R, then T contains a non-zero idempotent. Given this result, suppose that $a \in R$ is a non-zero nilpotent but that $a \notin J(R)$. Let $T = aR$; this is a right ideal which is not contained in $J(R)$. This means that there is a non-zero idempotent $e = ar$ in T. However, R is abelian and therefore Dedekind finite. This further implies that eRe is Dedekind finite. Thus eae is actually a unit (with inverse ere) in eRe. But this is absurd, as eae is also nilpotent. This gives a contradiction and the proof. □

The following corollary will be useful in characterizing the uniquely nil clean rings, which will be investigated in Section 5.

Corollary 3.19. *If R is a nil clean abelian ring with $J(R) = 0$, then R is reduced and therefore Boolean.*

Proof. By the above, every nilpotent of R must be in $J(R)$, which is zero. This gives that R is reduced, and the conclusion that reduced nil clean rings are Boolean has already been established. □

Corollary 3.20. *Let R be a commutative ring. Then R is nil clean if and only if $R/J(R)$ is Boolean and $J(R)$ is nil.*

Proof. Suppose that R is nil clean. By Proposition 3.16, $J(R)$ is nil. By Proposition 3.18 and Corollary 3.19, $R/J(R)$ is Boolean. If, on the other hand, $R/J(R)$ is Boolean and $J(R)$ is nil, then R is nil clean by Proposition 3.15. □

One can prove a version of Proposition 3.15 for strongly nil clean rings. The proof, however, relies on the fact that strong π-regularity lifts modulo a nilpotent ideal.

Theorem 3.21. *Let R be a ring, and I be a nilpotent ideal of R. Let $\overline{R} = R/I$. If a is an element of R such that $\bar{a}$ is strongly nil clean in $\overline{R}$, then a is strongly nil clean in R.*

Proof. Since $\bar{a}$ is strongly nil clean, we may write $\bar{a} = \bar{e} + \bar{b}$ for some idempotent $\bar{e}$ and some nilpotent $\bar{b}$ which commute. By Proposition 3.5, $a = \overline{1-e} + \overline{2e-1+b}$ is thus a strongly π-regular decomposition of $\bar{a}$. Following the first main result of [DDGK12], there is an idempotent f, lifting $1-e$, and a unit u such that $a = f + u$ is a strongly π-regular decomposition. By Proposition 3.9, we need only show that $2f - 1 + u$ is nilpotent. Since $\bar{f} = \overline{1-e}$ and $\bar{u} = \overline{2e-1+b}$, we can calculate that $\overline{2f-1+u} = \overline{2(1-e) - 1 + 2e - 1 + b} = \bar{b}$. Since b is nilpotent, modulo I, $2f - 1 + u$ is nilpotent. □

Corollary 3.22. *Suppose that R is a ring with a nilpotent ideal I. Then R is strongly nil clean if and only if R/I is strongly nil clean.*

Proposition 3.15 and Corollary 3.22 can be used to produce more examples of nil clean and strongly nil clean rings.

Corollary 3.23. *Let R be a ring.*

(1) *Let ${}_RM_R$ be an (R,R)-bimodule. The trivial extension $R \propto M$ is nil clean (respectively, strongly nil clean) if and only if R is nil clean (respectively, strongly nil clean).*
(2) *Let σ be a ring endomorphism of R, and let $R[x;\sigma]$ denote the ring of skew (left) polynomials over R. Then $R[x;\sigma]/(x^n)$ is nil clean (respectively, strongly nil clean) if and only if R is nil clean (respectively, strongly nil clean).*

Proposition 3.24. *Let R be a ring with only trivial idempotents. Then R is nil clean if and only if R is a local ring with $J(R)$ nil and $R/J(R) \cong \mathbb{F}_2$.*

Proof. We start with the reverse direction. Suppose that R is a local ring such that $J(R)$ is nil and such that $R = J(R) \cup (1 - J(R))$. If $r \in J(R)$, then r is nilpotent, and $r = 0 + r$ is the desired decomposition. On the other hand, if $r \in 1 - J(R)$, then $r - 1$ is nilpotent and so r is nil clean via $r = 1 + (r - 1)$.

Conversely, suppose that R is nil clean and that R has only trivial idempotents. If $r \in R$, either r is nilpotent or $r - 1$ is nilpotent. This implies that either $1 - r$ or r is a unit, showing that R is local. By Proposition 3.16, $J(R)$ is nil. Finally, $R/J(R)$ is a nil clean division ring, and must therefore be isomorphic to $\mathbb{F}_2$. □

Remark. Using the characterization of uniquely clean local rings proved in [NZ04, Theorem 15], we have shown that a local ring is strongly nil clean if and only if it is strongly π-regular and uniquely clean. See Section 5 for further details.

Proposition 3.25. *Let R be a ring, and let $f \in R$ be any idempotent. An element $a \in fRf$ is strongly nil clean in R if and only if it is strongly nil clean in fRf.*

Proof. Suppose that a is strongly nil clean in R. Write $a = e + b$ for some idempotent e and some nilpotent b, which commutes with e. Noting that Rf is a left ideal of R and that fRf is a right ideal of Rf, we can use Proposition 2.9 to show that $e, b \in fRf$. Thus a is strongly nil clean in fRf. The reverse implication is clear. □

This leads immediately to the following.

Corollary 3.26. *If R is a strongly nil clean ring and $f \in R$ is any idempotent, then the corner ring fRf is strongly nil clean.*

Remark. Although strong nil cleanness passes to corners, we will see in the next section that it is not a Morita invariant.

It has been recently shown in [Šte12] that a corner of a clean ring need not be clean. One might therefore suspect that a corner of a nil clean ring need not be nil clean, but this question remains open at this time.

Question 2. If R is nil clean and e is an idempotent (or a full idempotent), is the corner ring eRe nil clean?

4. Matrix extensions of nil clean rings

Matrix constructions will provide us with new sources of examples of nil clean and strongly nil clean rings. In this section, we will develop results which allow us to study triangular and full matrices over nil clean and strongly nil clean rings.

We begin with triangular matrix rings.

Theorem 4.1. *Let R be a ring, and let n be a positive integer. Then R is nil clean if and only if $\mathbb{T}_n(R)$ is nil clean. Further, R is strongly nil clean if and only if $\mathbb{T}_n(R)$ is strongly nil clean.*

Proof. Let $S = \mathbb{T}_n(R)$, and let I be the ideal of S which consists of all matrices with zeroes along the main diagonal. Note that I is nilpotent and that S/I is isomorphic to the direct product of n copies of R. We can then apply Proposition 3.15 in the nil clean case, and Corollary 3.22 in the strongly nil clean case, noting that a finite product of copies of R is nil clean (respectively, strongly nil clean) if and only if R is nil clean (respectively, strongly nil clean). □

We now proceed to the case of full matrix rings. In [BDD08], techniques are developed which allow one to study strongly clean (and strongly π-regular) matrices over commutative local rings through certain factorizations of the characteristic polynomials. We easily adapt these results to the strongly nil clean case.

In what follows, we will view the ring $\mathbb{M}_n(R)$ as the endomorphism ring of a rank n free module over R, and exploit Corollary 3.2.

We begin with the following result from [BDD08].

Proposition 4.2. *(See [BDD08, Proposition 41].) Let R be a non-zero commutative ring, and let $\varphi \in \operatorname{End}(R^n)$. Let $f(t) \in R[t]$ be the characteristic polynomial of φ. Then φ is nilpotent if and only if $f(t) \equiv t^n \pmod{\operatorname{Nil}(R)}$.*

Borrowing notation from [BDD08], we make the following definition.

Definition 4.3. Let R be a commutative ring, and let $r \in R$. Define

$$\mathbb{P}_r = \left\{ f \in R[t] \;\middle|\; f \text{ monic, and } f - (t-r)^{\deg(f)} \in \operatorname{Nil}(R)[t] \right\}.$$

As in [BDD08], it is immediate that $\varphi - rI$ is nilpotent if and only if the characteristic polynomial of φ is in $\mathbb{P}_r$.

We can now prove our main result on matrix rings (which improves significantly on [Che11, Theorem 2.5]).

Theorem 4.4. *Let R be a commutative local ring, and let $h(t) \in R[t]$ be monic of degree n. The following are equivalent.*

(1) *Every $\varphi \in \mathbb{M}_n(R)$ with characteristic polynomial $h(t)$ is strongly nil clean.*
(2) *There exists $\varphi \in \mathbb{M}_n(R)$ with characteristic polynomial $h(t)$ such that φ is strongly nil clean.*
(3) *$h(t)$ has a factorization $h(t) = h_0(t)h_1(t)$ such that $h_0 \in \mathbb{P}_0$ and $h_1 \in \mathbb{P}_1$.*

Proof. (1) ⇒ (2): The companion matrix of $h(t)$ is such an example.

(2) ⇒ (3): By Corollary 3.2, we can write $R^n = A \oplus B$ as a direct sum of φ-invariant R-submodules such that $\varphi|_A$ and $(1-\varphi)|_B$ are nilpotent. Since a local ring is projective-free, each of A and B is a finite-rank free R-module. We may therefore factor $h(t)$ as the product of the characteristic polynomials of $\varphi|_A$ and $(1-\varphi)|_B$. Thus, $h(t) = h_0(t)h_1(t)$, such that $h_0 \in \mathbb{P}_0$ and $h_1 \in \mathbb{P}_1$.

(3) ⇒ (1): Suppose that $\varphi \in \mathbb{M}_n(R)$, and suppose that the characteristic polynomial $h(t)$ of φ factors as $h(t) = h_0(t)h_1(t)$ such that $h_0 \in \mathbb{P}_0$ and $h_1 \in \mathbb{P}_1$. Following [BDD08, Proposition 44], we let

$A = \ker(h_0(\varphi))$ and $B = \ker(h_1(\varphi))$. Then both A and B are φ-invariant, and $R^n = A \oplus B$. It is easy to see that φ is nilpotent on A and that $1 - \varphi$ is nilpotent on B. By Corollary 3.2, φ is strongly nil clean. □

The next example shows that "nil clean" and "strongly nil clean" are not the same, as well as demonstrating that "strongly nil clean" is not a Morita invariant.

Example 4.5. Let $\mathbb{F}_2$ be the field with two elements. By inspection, the ring $\mathbb{M}_2(\mathbb{F}_2)$ is nil clean. By Proposition 3.15, $\mathbb{M}_2(\mathbb{Z}/2^n\mathbb{Z})$ is a nil clean ring for every positive integer n.

On the other hand, if R is any non-zero ring, then Corollary 3.10 implies that $\mathbb{M}_n(R)$ cannot be strongly nil clean for any $n > 1$ since the matrix ring contains units, such as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, which are not unipotent. Note that this observation improves substantially upon [Che11, Corollary 2.3].

It is proved in [HN01, Corollary 1] that any matrix ring over a clean ring is clean. The above examples motivate us to ask whether the same is true in the nil clean case.

Question 3. Let R be a nil clean ring, and let n be a positive integer. Is $\mathbb{M}_n(R)$ nil clean?

We can also use matrix examples to establish the relationship between "strongly clean" and "nil clean".

Example 4.6. Consider the matrix $A = \left(\begin{smallmatrix} 1 & -2 \\ 1 & 0 \end{smallmatrix}\right) \in \mathbb{M}_2(\mathbb{Z}_{(2)})$. By [BDD08, Theorem 12], A is not strongly clean. However,

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix},$$

which shows that A is nil clean.

We have shown previously that a strongly clean ring need not be nil clean. In light of Example 4.5, we ask whether the opposite is also true.

Question 4. Does there exist a nil clean ring that is not strongly clean (or even strongly π-regular)?

5. Uniquely nil clean rings

Finally, we include a classification theorem for uniquely nil clean rings. This will require several preliminary results (mostly following [NZ04], at times with the nil clean hypothesis yielding slightly easier proofs) and culminate in the (somewhat surprising) fact that all uniquely nil clean rings are actually uniquely clean.

Definition 5.1. An element a in a ring R is called uniquely nil clean if there is a unique idempotent e such that $a - e$ is nilpotent. We will say that a ring is uniquely nil clean if each of its elements is uniquely nil clean.

This next result needs no proof.

Proposition 5.2. *Let $R_1, \dots, R_n$ be rings. Then $R_1 \times \cdots \times R_n$ is uniquely nil clean if and only if each R_i is uniquely nil clean.*

Proposition 5.3. *Let R be a ring. Every element of R which is both central and nil clean is uniquely nil clean.*

Proof. If $a \in R$ is central, then any nil clean decomposition is a strongly nil clean decomposition. By Corollary 3.8, a has exactly one strongly nil clean decomposition. Thus a is uniquely nil clean. □

Corollary 5.4. *Every central idempotent and every central nilpotent is uniquely nil clean.*

Lemma 5.5. *Let R be uniquely nil clean. Then all idempotents of R are central.*

Proof. Let $e \in R$ be an idempotent and let r be any element of R. Notice that the element $e+er(1-e)$ can be written as $e+(er(1-e))$ or as $(e+er(1-e))+0$, each time as the sum of an idempotent and a nilpotent. Since R is uniquely nil clean, $e = e+er(1-e)$, implying that $er(1-e)=0$. It can likewise be shown that $(1-e)re=0$. Thus e is central. □

Corollary 5.6. *A uniquely nil clean ring is Dedekind finite, and every nilpotent of a uniquely nil clean ring is contained in its Jacobson radical.*

Proof. This is just an easy application of Lemma 5.5 (since any abelian ring is Dedekind finite) and Proposition 3.18. □

Corollary 5.7. *Let R be a uniquely nil clean ring, and let $e \in R$ be an idempotent. Then eRe is uniquely clean.*

Proof. Apply Proposition 5.2 and Lemma 5.5. □

In [NZ04], Nicholson and Zhou study uniquely clean rings and characterize them as the rings R for which $R/J(R)$ is Boolean and for which idempotents lift uniquely modulo $J(R)$. We will give a similar characterization for uniquely nil clean rings. It is clear that a nil clean ring that is uniquely clean is uniquely nil clean. In what follows we shall show, among other things, that the converse is also true.

Theorem 5.8. *The following are equivalent for a ring R.*

(1) *R is uniquely nil clean and $J(R)=0$.*
(2) *R is nil clean and reduced.*
(3) *R is Boolean.*

Proof. $(2) \Rightarrow (3)$ and $(3) \Rightarrow (1)$ are both easy verifications. To show $(1) \Rightarrow (2)$, notice first that all idempotents of R are central by Lemma 5.5. This, together with $J(R)=0$ gives that R is reduced by Corollary 3.19. □

This leads us immediately to the following characterization. In any ring R, we say that idempotents lift modulo $J(R)$ if for any $a \in R$ such that $a^2-a \in J(R)$ there exists an idempotent $e \in R$ such that $e-a \in J(R)$. We further say that idempotents lift uniquely modulo $J(R)$ if there is a unique such idempotent $e \in R$ for each $a \in R$ such that $a^2-a \in J(R)$.

Theorem 5.9. *The following are equivalent for a ring R.*

(1) *R is uniquely nil clean.*
(2) *$R/J(R)$ is Boolean, $J(R)$ is nil and idempotents lift uniquely modulo $J(R)$.*
(3) *R is strongly nil clean, $J(R)$ is nil and idempotents lift uniquely modulo $J(R)$.*
(4) *R is strongly π-regular and uniquely clean.*
(5) *R is uniquely clean and $J(R)$ is nil.*

Proof. We will first show the equivalence of (1) and (2). Suppose first that R is uniquely nil clean. Proposition 3.16 implies that $J(R)$ is nil. To show that idempotents lift uniquely modulo $J(R)$, suppose that e and f are idempotents in R such that $e - f \in J(R)$. But then $e - f$ is nilpotent, and so $e + 0 = f + (e - f)$ gives two ways of representing the same element of R as the sum of an idempotent and a nilpotent. Since R is uniquely nil clean, we must therefore have that $e = f$. Finally, to show that $R/J(R)$ is Boolean, notice that by Corollary 5.6, $J(R)$ is precisely the set of nilpotent elements of R. Therefore, $R/J(R)$ is nil clean and reduced, which implies immediately that $R/J(R)$ is Boolean. Thus (1) implies (2).

Suppose, on the other hand, that (2) holds. We will use "bar" notation to denote working modulo $J(R)$. If $a \in R$, then $\bar{a}$ is an idempotent since $R/J(R)$ is Boolean. Since idempotents lift modulo $J(R)$, there exists $e^2 = e \in R$ such that $a - e \in J(R)$ and is therefore nilpotent. This shows that a is nil clean. To show uniqueness, suppose that $a = e + b = f + c$ are two nil clean decompositions. Working modulo $J(R)$, Corollary 5.6 implies that $\bar{e} = \bar{f}$. Since idempotents lift uniquely modulo $J(R)$, we get that $e = f$ and therefore that a is uniquely nil clean.

We can now show that (3) follows from (1). Lemma 5.5 implies that R is abelian and is therefore strongly nil clean. By the equivalence of (1) and (2), we already know that $J(R)$ is nil and that idempotents lift uniquely modulo $J(R)$.

Uniquely clean rings are characterized in [NZ04, Theorem 20] as those rings R for which $R/J(R)$ is Boolean and for which idempotents lift uniquely modulo $J(R)$. This fact, together with the fact that a strongly nil clean ring is strongly π-regular (Corollary 3.7), shows that (3) implies (4).

The implication (4) $\Rightarrow$ (5) is a consequence of the fact that any strongly π-regular ring has nil Jacobson radical, and (5) $\Rightarrow$ (2) is proved by again appealing to the characterization in [NZ04, Theorem 20] of uniquely clean rings. □

Since we have proved that every uniquely nil clean ring is uniquely clean, we can apply results in [NZ04] to prove additional results about uniquely nil clean rings. For example, we have the following pleasing result.

Corollary 5.10. *Every quotient of a uniquely nil clean ring is uniquely nil clean.*

Proof. Since it is proved in [NZ04, Theorem 22] that a quotient of a uniquely clean ring is uniquely clean, and it is known that a quotient of a strongly π-regular ring is strongly π-regular, this result is a consequence of Theorem 5.9. □

We finish with a few examples. If R is any strongly nil clean ring, then $\mathbb{T}_n(R)$ is a strongly nil clean ring (by Theorem 4.1) that is not uniquely nil clean since it contains idempotents that are not central.

For a noncommutative example of a uniquely nil clean ring, let S be the subring of $\mathbb{T}_3(\mathbb{Z}_2)$ consisting of the matrices with constant diagonal. Then S is uniquely clean (see [NZ04, Example 8]) and strongly π-regular. Thus S is uniquely nil clean and noncommutative.

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