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# Ehrhart polynomials of convex polytopes with small volumes

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ABSTRACT

Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension $d$ and $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ be its $\delta$-vector. By using the known inequalities on $\delta$-vectors, we classify the possible $\delta$-vectors of convex polytopes of dimension $d$ with $\sum_{i=0}^{d} \delta_i \leq 3$.



## 0. Introduction

One of the most attractive problems on enumerative combinatorics of convex polytopes is to find a combinatorial characterizations of the Ehrhart polynomials of integral convex polytopes. First of all, we recall what the Ehrhart polynomial of a convex polytope is.

Let $\mathcal{P} \subset \mathbb{R}^N$ be an *integral* convex polytope; i.e., a convex polytope any of whose vertices has integer coordinates, of dimension $d$, and let $\partial\mathcal{P}$ denote the boundary of $\mathcal{P}$. Given a positive integer $n$ we define the numerical functions $i(\mathcal{P}, n)$ and $i^*(\mathcal{P}, n)$ by setting

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N|, \qquad i^*(\mathcal{P}, n) = |n(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|.$$

Here $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$ and $|X|$ is the cardinality of a finite set $X$.

The systematic study of $i(\mathcal{P}, n)$ originated in the work of Ehrhart [1], who established the following fundamental properties:

(0.1) $i(\mathcal{P}, n)$ is a polynomial in $n$ of degree $d$ (and thus in particular $i(\mathcal{P}, n)$ can be defined for *every* integer $n$);
(0.2) $i(\mathcal{P}, 0) = 1$;
(0.3) *(loi de réciprocité)* $i^*(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n)$ for every integer $n > 0$.

We say that $i(\mathcal{P}, n)$ is the *Ehrhart polynomial* of $\mathcal{P}$. An introduction to the theory of Ehrhart polynomials is discussed in [6, pp. 235–241] and [2, Part II].

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We define the sequence $\delta_0, \delta_1, \delta_2, \ldots$ of integers by the formula

$$(1-\lambda)^{d+1}\left[1+\sum_{n=1}^{\infty} i(\mathcal{P}, n)\lambda^n\right] = \sum_{i=0}^{\infty} \delta_i \lambda^i. \tag{1}$$

Then the basic facts (0.1) and (0.2) on $i(\mathcal{P}, n)$ together with a fundamental result on the generating function [6, Corollary 4.3.1] guarantee that $\delta_i = 0$ for every $i > d$. We say that the sequence

$$\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$$

which appears in Eq. (1) is the *$\delta$-vector* of $\mathcal{P}$. Thus $\delta_0 = 1$ and $\delta_1 = |\mathcal{P} \cap \mathbb{Z}^N| - (d+1)$.

It follows from *loi de réciprocité* (0.3) that

$$\sum_{n=1}^{\infty} i^*(\mathcal{P}, n)\lambda^n = \frac{\sum_{i=0}^{d} \delta_{d-i}\lambda^{i+1}}{(1-\lambda)^{d+1}}. \tag{2}$$

In particular $\delta_d = |(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|$. Hence $\delta_1 \geq \delta_d$. Moreover, each $\delta_i$ is nonnegative [7]. In addition, if $(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N$ is nonempty, i.e., $\delta_d \neq 0$, then one has $\delta_1 \leq \delta_i$ for every $1 \leq i < d$ [3]. When $d = N$, the leading coefficient $(\sum_{i=0}^{d} \delta_i)/d!$ of $i(\mathcal{P}, n)$ is equal to the usual volume $\mathrm{vol}(\mathcal{P})$ of $\mathcal{P}$ [6, Proposition 4.6.30].

It follows from Eq. (2) that

$$\max\{i : \delta_i \neq 0\} + \min\{i : i(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N \neq \emptyset\} = d + 1.$$

Let $s = \max\{i : \delta_i \neq 0\}$. Stanley [8] shows the inequalities

$$\delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_s + \delta_{s-1} + \cdots + \delta_{s-i}, \quad 0 \leq i \leq [s/2] \tag{3}$$

by using the theory of Cohen–Macaulay rings. On the other hand, the inequalities

$$\delta_{d-1} + \delta_{d-2} + \cdots + \delta_{d-i} \leq \delta_2 + \delta_3 + \cdots + \delta_i + \delta_{i+1}, \quad 1 \leq i \leq [(d-1)/2] \tag{4}$$

appear in [3, Remark (1.4)]. These inequalities (3) and (4) are discussed in detail by Stapledon [5].

Somewhat surprisingly, when $\sum_{i=0}^{d} \delta_i \leq 3$, the above inequalities (3) together with (4) give a characterization of the possible $\delta$-vectors. In fact,

**Theorem 0.1.** *Let $d \geq 3$. Given a finite sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ of nonnegative integers, where $\delta_0 = 1$ and $\delta_1 \geq \delta_d$, which satisfies $\sum_{i=0}^{d} \delta_i \leq 3$, there exists an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension $d$ whose $\delta$-vector coincides with $(\delta_0, \delta_1, \ldots, \delta_d)$ if and only if $(\delta_0, \delta_1, \ldots, \delta_d)$ satisfies all inequalities (3) and (4).*

The "Only if" part of Theorem 0.1 is obvious. In addition, no discussion will be required for the case of $\sum_{i=0}^{d} \delta_i = 1$. The "If" part of Theorem 0.1 will be given in Section 2 for the case of $\sum_{i=0}^{d} \delta_i = 2$ and in Section 3 for the case of $\sum_{i=0}^{d} \delta_i = 3$.

On the other hand, Example 1.2 shows that Theorem 0.1 is no longer true for the case of $\sum_{i=0}^{d} \delta_i = 4$. Finally, when $d \leq 2$, the possible $\delta$-vectors are known [4].

## 1. Review on the computation of the $\delta$-vector of a simplex

We recall from [2, Part II] the well-known combinatorial technique for computing the $\delta$-vector of a simplex.

- Given an integral $d$-simplex $\mathcal{F} \subset \mathbb{R}^N$ with the vertices $v_0, v_1, \ldots, v_d$, we set $\widetilde{\mathcal{F}} = \{(\alpha, 1) \in \mathbb{R}^{N+1} : \alpha \in \mathcal{F}\}$, which is an integral $d$-simplex in $\mathbb{R}^{N+1}$ with the vertices $(v_0, 1), (v_1, 1), \ldots, (v_d, 1)$ and $\partial\widetilde{\mathcal{F}} = \left\{(\alpha, 1) \in \mathbb{R}^{N+1} : \alpha \in \partial\mathcal{F}\right\}$ is its boundary. Clearly $i(\mathcal{F}, n) = i(\widetilde{\mathcal{F}}, n)$ and $i^*(\mathcal{F}, n) = i^*(\widetilde{\mathcal{F}}, n)$ for all $n$.
- The subset $\mathcal{C} = \mathcal{C}(\widetilde{\mathcal{F}}) \subset \mathbb{R}^{N+1}$ defined by $\mathcal{C} = \{r\beta : \beta \in \widetilde{\mathcal{F}}, 0 \leq r \in \mathbb{Q}\}$ is called *the simplicial cone associated with $\mathcal{F} \subset \mathbb{R}^N$ with apex* $(0, \ldots, 0)$. Its boundary is $\partial\mathcal{C} = \left\{r\beta : \beta \in \partial\widetilde{\mathcal{F}}, 0 \leq r \in \mathbb{Q}\right\}$. One has $i(\mathcal{F}, n) = \left|\left\{(\alpha, n) \in \mathcal{C} : \alpha \in \mathbb{Z}^N\right\}\right|$ and $i^*(\mathcal{F}, n) = \left|\left\{(\alpha, n) \in \mathcal{C} \setminus \partial\mathcal{C} : \alpha \in \mathbb{Z}^N\right\}\right|$.

- Each rational point $\alpha \in \mathcal{C}$ has a unique expression of the form $\alpha = \sum_{i=0}^{d} r_i(v_i, 1)$ with each $0 \le r_i \in \mathbb{Q}$. Moreover, each rational point $\alpha \in \mathcal{C} \backslash \partial\mathcal{C}$ has a unique expression of the form $\alpha = \sum_{i=0}^{d} r_i(v_i, 1)$ with each $0 < r_i \in \mathbb{Q}$.
- Let $S$ (resp. $S^*$) be the set of all points $\alpha \in \mathcal{C} \cap \mathbb{Z}^{N+1}$ (resp. $\alpha \in (\mathcal{C} \setminus \partial\mathcal{C}) \cap \mathbb{Z}^{N+1}$) of the form $\alpha = \sum_{i=0}^{d} r_i(v_i, 1)$, where each $r_i \in \mathbb{Q}$ with $0 \le r_i < 1$ (resp. with $0 < r_i \le 1$).
- The degree of an integer point $(\alpha, n) \in \mathcal{C}$ is $\deg(\alpha, n) = n$.

**Lemma 1.1.** (a) *Let $\delta_i$ be the number of integer points $\alpha \in S$ with $\deg \alpha = i$. Then*

$$1 + \sum_{n=1}^{\infty} i(\mathcal{F}, n)\lambda^n = \frac{\delta_0 + \cdots + \delta_d \lambda^d}{(1-\lambda)^{d+1}}.$$

(b) *Let $\delta_i^*$ be the number of integer points $\alpha \in S^*$ with $\deg \alpha = i$. Then*

$$\sum_{n=1}^{\infty} i^*(\mathcal{F}, n)\lambda^n = \frac{\delta_1^* \lambda + \cdots + \delta_{d+1}^* \lambda^{d+1}}{(1-\lambda)^{d+1}}.$$

(c) *One has $\delta_i^* = \delta_{(d+1)-i}$ for each $1 \le i \le d+1$.*

**Example 1.2.** Theorem 0.1 is no longer true for the case of $\sum_{i=0}^{d} \delta_i = 4$. In fact, the sequence $(1, 0, 1, 0, 1, 1, 0, 0)$ cannot be the $\delta$-vector of an integral convex polytope of dimension 7. Suppose, on the contrary, that there exists an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$ with $(\delta_0, \delta_1, \ldots, \delta_7) = (1, 0, 1, 0, 1, 1, 0, 0)$ its $\delta$-vector. Since $\delta_1 = 0$, we know that $\mathcal{P}$ is a simplex. Let $v_0, v_1, \ldots, v_7$ be the vertices of $\mathcal{P}$. By using Lemma 1.1, one has $S = \{(0, \ldots, 0), (\alpha, 2), (\beta, 4), (\gamma, 5)\}$ and $S^* = \{(\alpha', 3), (\beta', 4), (\gamma', 6), (\sum_{i=0}^{7} v_i, 8)\}$. Write $\alpha' = \sum_{i=0}^{7} r_i v_i$ with each $0 < r_i \le 1$. Since $(\alpha', 3) \notin S$, there is $0 \le j \le 7$ with $r_j = 1$. If there are $0 \le k < \ell \le 7$ with $r_k = r_\ell = 1$, say, $r_0 = r_1 = 1$, then $0 < r_q < 1$ for each $2 \le q \le 7$ and $\sum_{i=2}^{7} r_i = 1$. Hence $(\alpha' - v_0 - v_1, 1) \in S$, a contradiction. Thus there is a unique $0 \le j \le 7$ with $r_j = 1$, say, $r_0 = 1$. Then $\alpha = \sum_{i=1}^{7} r_i v_i$ and $\gamma = \sum_{i=1}^{7} (1 - r_i) v_i$. Let $\mathcal{F}$ denote the facet of $\mathcal{P}$ whose vertices are $v_1, v_2, \ldots, v_7$ with $\delta(\mathcal{F}) = (\delta'_0, \delta'_1, \ldots, \delta'_6) \in \mathbb{Z}^7$. Then $\delta'_2 = \delta'_5 = 1$. Since $\delta'_i \le \delta_i$ for each $0 \le i \le 6$, it follows that $\delta(\mathcal{F}) = (1, 0, 1, 0, 0, 1, 0)$. This contradicts the inequalities (3).

## 2. A proof of Theorem 0.1 when $\sum_{i=0}^{d} \delta_i = 2$

The goal of this section is to prove the "If" part of Theorem 0.1 when $\sum_{i=0}^{d} \delta_i = 2$. First of all, we recall the following well-known lemma:

**Lemma 2.1.** *Suppose that $(\delta_0, \delta_1, \ldots, \delta_d)$ is the $\delta$-vector of an integral convex polytope of dimension $d$. Then there exists an integral convex polytope of dimension $d+1$ whose $\delta$-vector is $(\delta_0, \delta_1, \ldots, \delta_d, 0)$.*

**Proof.** Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension $d$ and $\mathcal{Q} \subset \mathbb{R}^{N+1}$ the convex hull of $\{(\alpha, 0) \in \mathbb{R}^{N+1} : \alpha \in \mathcal{P}\}$ together with $(0, \ldots, 0, 1) \in \mathbb{R}^{N+1}$. Then $\mathcal{Q}$ is an integral convex polytope of dimension $d+1$. It follows that

$$i(\mathcal{Q}, n) = \sum_{q=0}^{n} i(\mathcal{P}, q), \quad n = 0, 1, 2, \ldots,$$

where $i(\mathcal{P}, 0) = i(\mathcal{Q}, 0) = 1$. Hence

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} i(\mathcal{Q}, n)\lambda^n &= \left[1 + \sum_{n=1}^{\infty} i(\mathcal{P}, n)\lambda^n\right](1 + \lambda + \lambda^2 + \cdots) \\ &= \left[1 + \sum_{n=1}^{\infty} i(\mathcal{P}, n)\lambda^n\right](1-\lambda)^{-1}. \end{aligned}$$

Thus

$$(1-\lambda)^{d+2}\left[1+\sum_{n=1}^{\infty} i(\mathcal{Q},n)\lambda^n\right] = (1-\lambda)^{d+1}\left[1+\sum_{n=1}^{\infty} i(\mathcal{P},n)\lambda^n\right],$$

as desired. □

Let $d \geq 3$. We study a finite sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ of nonnegative integers with $\delta_0 = 1$ and $\delta_1 \geq \delta_d$ which satisfies all inequalities (4) together with $\sum_{i=0}^{d} \delta_i = 2$. Since $\delta_0 = 1$, $\delta_1 \geq \delta_d$ and $\sum_{i=0}^{d} \delta_i = 2$, one has $\delta_d = 0$. Hence there is an integer $i \in \{1, \ldots, [(d+1)/2]\}$ such that $(\delta_0, \delta_1, \ldots, \delta_d) = (1, 0, \ldots, 0, \underbrace{1}_{i\text{-th}}, 0, \ldots, 0)$, where $\underbrace{1}_{i\text{-th}}$ stands for $\delta_i = 1$. By virtue of Lemma 2.1 our work is to find an integral convex polytope $\mathcal{P}$ of dimension $d$ with $(1, 0, \ldots, 0, \underbrace{1}_{((d+1)/2)\text{-th}}, 0, \ldots, 0) \in \mathbb{Z}^{d+1}$ its $\delta$-vector.

Let $\mathcal{P} \subset \mathbb{R}^d$ be the integral simplex of dimension $d$ whose vertices $v_0, v_1, \ldots, v_d$ are

$$v_i = \begin{cases} (0, \ldots, 0, \underbrace{1}_{i\text{-th}}, \underbrace{1}_{(i+1)\text{-th}}, 0, \ldots, 0), & i = 1, \ldots, d-1, \\ (1, 0, \ldots, 0, 1), & i = d, \\ (0, 0, \ldots, 0), & i = 0. \end{cases}$$

When $d$ is odd, one has $\mathrm{vol}(\mathcal{P}) = 2/d!$ by using an elementary linear algebra. Since

$$\frac{1}{2}\{(v_0, 1) + (v_1, 1) + \cdots + (v_d, 1)\} = (1, 1, \ldots, 1, (d+1)/2) \in \mathbb{Z}^{d+1},$$

Lemma 1.1 says that $\delta_{(d+1)/2} \geq 1$. Thus, since $\mathrm{vol}(\mathcal{P}) = 2/d!$, one has

$$\delta(\mathcal{P}) = (1, 0, \ldots, 0, \underbrace{1}_{((d+1)/2)\text{-th}}, 0, \ldots, 0),$$

as desired.

## 3. A proof of Theorem 0.1 when $\sum_{i=0}^{d} \delta_i = 3$

The goal of this section is to prove the "If" part of Theorem 0.1 when $\sum_{i=0}^{d} \delta_i = 3$. Let $d \geq 3$. Suppose that a finite sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ of nonnegative integers with $\delta_0 = 1$ and $\delta_1 \geq \delta_d$ satisfies all inequalities (3) and (4) together with $\sum_{i=0}^{d} \delta_i = 3$.

When there is $1 \leq i \leq d$ with $\delta_i = 2$, the same discussion as in Section 1 can be applied. In fact, instead of the vertices of the convex polytope arising in the last paragraph of Section 1, we may consider the convex polytope whose vertices $v_0, v_1, \ldots, v_d$ are

$$v_i = \begin{cases} (0, \ldots, 0, \underbrace{1}_{i\text{-th}}, \underbrace{1}_{(i+1)\text{-th}}, 0, \ldots, 0), & i = 1, \ldots, d-1, \\ (2, 0, \ldots, 0, 1), & i = d, \\ (0, 0, \ldots, 0), & i = 0. \end{cases}$$

Now, in what follows, a sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ with each $\delta_i \in \{0, 1\}$, where $\delta_0 = 1$ and $\delta_1 \geq \delta_d$, which satisfies all inequalities (3) and (4) together with $\sum_{i=0}^{d} \delta_i = 3$ will be considered.

If $\delta_d = 1$, then $\delta_1 = 1$. However, since $d \geq 3$, this contradicts (3). If $\delta_1 = 1$, then $\delta_2 = 1$ by (3). Clearly, $(1, 1, 1, 0, \ldots, 0) \in \mathbb{Z}^{d+1}$ is a possible $\delta$-vector. Thus we will assume that $\delta_1 = \delta_d = 0$. Let $\delta_m = \delta_n = 1$ with $1 < m < n < d$. Let $p = m - 1$, $q = n - m - 1$, and $r = d - n$. By (3) one has $0 \leq q \leq p$. Moreover, by (4) one has $p \leq r$. Consequently,

$$0 \leq q \leq p \leq r, \quad p + q + r = d - 2. \tag{5}$$

Our work is to construct an integral convex polytope $\mathcal{P}$ with dimension $d$ whose $\delta$-vector coincides with $\delta(\mathcal{P}) = (1, \underbrace{0, \ldots, 0}_{p}, 1, \underbrace{0, \ldots, 0}_{q}, 1, \underbrace{0, \ldots, 0}_{r})$ for an arbitrary integer $1 < m < n < d$ satisfying the conditions (5).

**Lemma 3.1.** *Let $d = 3k + 2$. There exists an integral convex polytope $\mathcal{P}$ of dimension $d$ whose $\delta$-vector coincides with*

$$(1, \underbrace{0, \ldots, 0}_{k}, 1, \underbrace{0, \ldots, 0}_{k}, 1, \underbrace{0, \ldots, 0}_{k}) \in \mathbb{Z}^{d+1}.$$

**Proof.** When $k \geq 1$, let $\mathcal{P} \subset \mathbb{R}^d$ be the integral simplex of dimension $d$ with the vertices $v_0, v_1, \ldots, v_d$, where

$$v_i = \begin{cases} (0, \ldots, 0, \underbrace{1}_{i\text{-th}}, \underbrace{1}_{(i+1)\text{-th}}, \underbrace{1}_{(i+2)\text{-th}}, 0, \ldots, 0), & i = 1, \ldots, d-2, \\ (1, 0, \ldots, 0, 1, 1), & i = d-1, \\ (1, 1, 0, \ldots, 0, 1), & i = d, \\ (0, \ldots, 0), & i = 0. \end{cases}$$

By using the induction on $k$ it follows that $\mathrm{vol}(\mathcal{P}) = 3/d!$. Since

$$\frac{1}{3}\{(v_0, 1) + (v_1, 1) + \cdots + (v_d, 1)\} = (1, 1, \ldots, 1, k+1) \in \mathbb{Z}^{d+1},$$

Lemma 1.1 now guarantees that $\delta_{k+1} \geq 1$ and $\delta^*_{k+1} \geq 1$. Hence $\delta_{k+1} = 1$ and $\delta_{2k+2} = 1$, as required. $\Box$

**Lemma 3.2.** *Let $d = 3k + 2$, $\ell > 0$ and $d' = d + 2\ell$. There exists an integral simplex $\mathcal{P} \subset \mathbb{R}^{d'}$ of dimension $d'$ whose $\delta$-vector coincides with*

$$(1, \underbrace{0, \ldots, 0}_{k+\ell}, 1, \underbrace{0, \ldots, 0}_{k}, 1, \underbrace{0, \ldots, 0}_{k+\ell}) \in \mathbb{Z}^{d'+1}.$$

**Proof.** *(First Step)* Let $k = 0$. Thus $d = 2$ and $d' = 2\ell + 2$. Let $\mathcal{P} \subset \mathbb{R}^{d'}$ be an integer convex polytope of dimension $d'$ whose vertices $v_0, v_1, \ldots, v_{2\ell+2}$ are

$$v_i = \begin{cases} (2, 1, 0, 0, \ldots, 0), & i = 1, \\ (0, 2, 1, 0, \ldots, 0), & i = 2, \\ (0, \ldots, 0, \underbrace{1}_{i\text{-th}}, \underbrace{1}_{(i+1)\text{-th}}, 0, \ldots, 0) & i = 3, \ldots, 2l+1, \\ (1, 0, \ldots, 0, 1), & i = 2l+2, \\ (0, \ldots, 0), & i = 0. \end{cases}$$

As usual, a routine computation says that $\mathrm{vol}(\mathcal{P}) = 3/d'!$. Let $v$ be the point

$$\frac{1}{3}\{(v_0, 1) + (v_1, 1) + (v_2, 1)\} + \frac{1}{3}\sum_{q=2}^{\ell+1}(v_{2q}, 1) + \frac{2}{3}\sum_{q=2}^{\ell+1}(v_{2q-1}, 1)$$

belonging to $\mathbb{R}^{d'+1}$. Then

$$v = (1, 1, \ldots, 1, \ell + 1) \in \mathbb{Z}^{d'+1}.$$

Thus Lemma 1.1 guarantees that $\delta_{\ell+1} \geq 1$ and $\delta^*_{\ell+1} \geq 1$. Hence $\delta_{\ell+1} = \delta_{\ell+2} = 1$, as required.

*(Second Step)* Let $k \geq 1$. We write $\mathcal{P} \subset \mathbb{R}^{d'}$ for the integral simplex of dimension $d'$ with the vertices $v_0, v_1, \ldots, v_{3k+2\ell+2}$ as follows:

- $v_0 = (0, 0, \ldots, 0)$,

- $v_1 = (1, 1, 1, 0, 0, \ldots, \underbrace{0}_{(3k+2)\text{-th}}, 1, 1, \ldots, 1)$,
- $v_2 = (0, 1, 1, 1, 0, \ldots, \underbrace{0}_{(3k+2)\text{-th}}, 1, 1, \ldots, 1)$,
- $v_i = (0, \ldots, 0, \underbrace{1}_{i\text{-th}}, \underbrace{1}_{(i+1)\text{-th}}, \underbrace{1}_{(i+2)\text{-th}}, 0, 0, \ldots, \underbrace{0}_{(3k+2)\text{-th}}, 1, 0, 1, 0, \ldots, 1, 0)$, for $i = 3, 4, 5, \ldots, 3k$,
- $v_{3k+1} = (1, 0, 0, \ldots, 0, 1, \underbrace{1}_{(3k+2)\text{-th}}, 1, 0, 1, 0, \ldots, 1, 0)$,
- $v_{3k+2} = (1, 1, 0, 0, \ldots, 0, \underbrace{1}_{(3k+2)\text{-th}}, 1, 0, 1, 0, \ldots, 1, 0)$,
- $v_i = (0, 0, \ldots, \underbrace{0}_{(3k+2)\text{-th}}, \ldots, 0, \underbrace{1}_{i\text{-th}}, 0, 1, 0, \ldots, 1, 0)$, for $i = 3k+3, 3k+5, \ldots, 3k+2\ell+1$,
- $v_i = (0, 0, \ldots, \underbrace{0}_{(3k+2)\text{-th}}, \ldots, 0, \underbrace{1}_{i\text{-th}}, 1, 0, 1, 0, \ldots, 1, 0)$, for $i = 3k+4, 3k+6, \ldots, 3k+2\ell+2$.

Let $A$ denote the $(3k+2) \times (3k+2)$ matrix

$$\mathrm{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 1 & \ddots & & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & 0 & & \ddots & 1 & 1 & 1 \\ 1 & & & & & 0 & 1 & 1 \\ 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Then a simple computation on determinants enables us to show that

$$d!\,\mathrm{vol}(\mathcal{P}) = \underbrace{\begin{vmatrix} \mathrm{A} & & * & \\ & 1 & & \\ 0 & & \ddots & \\ & & & 1 \end{vmatrix}}_{(3k+2+2\ell)\times(3k+2+2\ell)} = |\mathrm{A}| = 3.$$

One has

$$\begin{aligned} &\frac{1}{3}\{(v_0, 1) + (v_1, 1) + \cdots + (v_{3k+4}, 1)\} + \frac{2}{3}\{(v_{3k+5}, 1) + (v_{3k+7}, 1) + \cdots + (v_{3k+2\ell+1}, 1)\} \\ &\quad + \frac{1}{3}\{(v_{3k+6}, 1) + (v_{3k+8}, 1) + \cdots + (v_{3k+2\ell+2}, 1)\} \\ &= (1, \ldots, 1, k+1, 1, k+2, 1, \ldots, k+\ell, 1, k+\ell+1) \in \mathbb{Z}^{d'+1}. \end{aligned}$$

Hence $\delta_{k+\ell+1} = \delta_{2k+\ell+2} = 1$, as required. $\square$

In order to complete a proof of the "If" part of Theorem 0.1 when $\sum_{i=0}^{d} \delta_i = 3$, we must show the existence of an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension $d$ whose $\delta$-vector coincides with $(1, 0, \ldots, 0, \underbrace{1}_{m\text{-th}}, 0, \ldots, 0, \underbrace{1}_{n\text{-th}}, 0, \ldots, 0)$, where $1 < m < n < d$ and $n - m - 1 \le m - 1 \le d - n$.

First, Lemma 3.1 says that there exists an integral convex polytope whose $\delta$-vector coincides with

$$(1, 0, \ldots, 0, \underbrace{1}_{(n-m)\text{-th}}, 0, \ldots, 0, \underbrace{1}_{(2n-2m)\text{-th}}, 0, \ldots, 0) \in \mathbb{Z}^{3n-3m}.$$

Second, Lemma 3.2 guarantees that there exists an integral convex polytope whose $\delta$-vector coincides with

$$(1, 0, \ldots, 0, \underbrace{1}_{m\text{-th}}, 0, \ldots, 0, \underbrace{1}_{n\text{-th}}, 0, \ldots, 0) \in \mathbb{Z}^{n+m}.$$

Finally, by using Lemma 1.1, there exists an integral convex polytope $\mathcal{P}$ of dimension $d$ with

$$\delta(\mathcal{P}) = (1, 0, \ldots, 0, \underbrace{1}_{m\text{-th}}, 0, \ldots, 0, \underbrace{1}_{n\text{-th}}, 0, \ldots, 0) \in \mathbb{Z}^{d+1},$$

as desired.

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