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Geometric plurisubharmonicity and convexity: An introduction

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Abstract

This is an essay on potential theory for geometric plurisubharmonic functions. It begins with a given closed subset $\mathbb{G}$ of the Grassmann bundle $G(p, TX)$ of tangent p-planes to a riemannian manifold X. This determines a nonlinear partial differential equation which is convex but never uniformly elliptic ($p < \dim X$). A surprising number of results in complex analysis carry over to this more general setting. The notions of: a $\mathbb{G}$-submanifold, an upper semi-continuous $\mathbb{G}$-plurisubharmonic function, a $\mathbb{G}$-convex domain, a $\mathbb{G}$-harmonic function, and a $\mathbb{G}$-free submanifold, are defined. Results include a restriction theorem as well as the existence and uniqueness of solutions to the Dirichlet Problem for $\mathbb{G}$-harmonic functions on $\mathbb{G}$-convex domains.
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1. Introduction

In a recent series of papers [5–13] the authors have studied certain aspects of degenerate nonlinear elliptic partial differential equations and "subequations". The results include the development of a generalized potential theory, a restriction theorem, and solutions to the Dirichlet Problem. An important special case – and, in fact, the motivating case – of all these results is the "geometric" one, in which the equation is determined by a distinguished family $\mathbb{G}$ of tangent p-planes on a manifold (as we explain below). There are many interesting geometric cases coming, for instance, from the theory of calibrations, from almost complex and quaternionic geometry, and from p-convexity in riemannian and hermitian geometry. However, these examples will not be emphasized here since they occur in profusion in the earlier papers.

One aim of this paper is to collect together the various results in the geometric case. Because of their importance as motivation and their usefulness in non-geometric cases, we thought it would be helpful to present them in a coordinated fashion. This exposition also includes several new theorems.

Given an n-dimensional riemannian manifold X, let $G(p, TX)$ denote the Grassmann bundle whose fiber at a point x is the set of p-dimensional subspaces of the tangent space T_xX. The starting point is to distinguish a subset $\mathbb{G} \subset G(p, TX)$ determining the particular "geometry". Then, for example, one defines the $\mathbb{G}$-*submanifolds* to simply be those p-dimensional submanifolds M of X with $T_xM \in \mathbb{G}$ for all $x \in M$. There is also the analytical notion of a $\mathbb{G}$-*plurisubharmonic function*, defined for smooth functions u by using the riemannian hessian $\mathrm{Hess}_x u$. For each $W \in G(p, T_xX)$, one can restrict this quadratic form on T_xX to W and take its trace. We then define $u \in \mathrm{PSH}^\infty_{\mathbb{G}}(X)$, the set of *smooth* $\mathbb{G}$-*plurisubharmonic functions on* X, by requiring that:

$$\mathrm{tr}_W \mathrm{Hess}_x u \ \geq\ 0 \quad \forall W \in \mathbb{G}_x, \forall x \in X. \tag{1.1}$$

The set $\mathcal{P}(\mathbb{G}_x) \subset \mathrm{Sym}^2(T_x X)$ of $\mathbb{G}$-positive quadratic forms (i.e., those satisfying (1.1)) is a closed convex cone with vertex at the origin but it is never uniformly elliptic (unless $p = \dim X$). In Appendix A we prove (Theorem A.7) that the set $\mathcal{P}(\mathbb{G})$ with fiber $\mathcal{P}(\mathbb{G}_x)$ is a subequation as defined in [10] if and only if the natural projection from $\mathbb{G}$ to X is a local surjection.

The smooth theory, i.e., the study of $\mathrm{PSH}^{\infty}_{\mathbb{G}}(X)$, is for the most part a straightforward extension of standard results in complex analysis — where $\mathbb{G}$ is simply the set of complex lines in $\mathbf{C}^n$, considered as a subset $\mathbb{G} \equiv G_{\mathbf{C}}(1, \mathbf{C}^n) \subset G(2, \mathbf{R}^{2n})$, and the functions $u \in \mathrm{PSH}^{\infty}_{\mathbb{G}}(X)$ are the standard classical smooth plurisubharmonics on a domain $X \subset \mathbf{C}^n$. In Section 4 the existence of various kinds of exhaustion functions for X are characterized in terms of $\mathbb{G}$-convex hulls and the $\mathbb{G}$-core. The $\mathbb{G}$-core is empty if and only if X admits a smooth strictly $\mathbb{G}$-plurisubharmonic function (Definition 4.1 and Theorem 4.2). We recall the notion of a $\mathbb{G}$-*free submanifold* which generalizes the notion of a totally real submanifold in complex analysis. The maximal possible dimension of such submanifolds provides an upper bound on the homotopy type of strictly $\mathbb{G}$-convex manifolds (Theorem 4.16). In Section 5 the $\mathbb{G}$-convexity of the boundary of a domain is defined and related to the second fundamental form of the boundary, and also to properties of local defining functions for the boundary.

The notion of $\mathbb{G}$-plurisubharmonicity for a general upper semi-continuous function u is defined in Section 6 by requiring that each "viscosity" test function φ for u at each point $x \in X$ satisfies (1.1) (cf. [1,2]). A key *positivity condition* (Remark 6.3) is satisfied, which ensures that smooth $\mathbb{G}$-plurisubharmonic functions are also $\mathbb{G}$-plurisubharmonic in the second sense (cf. Lemma 6.2). A surprising number of the basic properties (see Theorem 6.5) of plurisubharmonic functions in complex analysis carry over to the general geometric case, provided that $\mathbb{G}$ is a closed set which locally surjects onto X, i.e., provided that $\mathcal{P}(\mathbb{G})$ is a subequation as defined in Appendix A.

Under the stronger (but still quite weak) assumption that $\mathbb{G}$ admits a smooth neighborhood retraction which preserves the fibers of the projection $\pi : G(p, TX) \to X$, *restriction holds* in the sense that for any upper semi-continuous $u \in \mathrm{PSH}_{\mathbb{G}}(X)$ and any minimal $\mathbb{G}$-submanifold $M \subset X$, *the restriction* $u\big|_M$ is subharmonic for the riemannian Laplacian Δ_M on M (Theorem 6.7). That is, $u\big|_M$ is subharmonic in any of the many (equivalent) classical senses. For instance, $u\big|_M$ is "sub-the-Δ_M-harmonics". Finally, if each $W \in \mathbb{G}$ is the tangent space to some minimal $\mathbb{G}$-submanifold M, then the converse to restriction also holds. This justifies the terminology "plurisubharmonic".

Next we discuss the solution to the Dirichlet problem on domains $\Omega \subset\subset X$ with smooth strictly $\mathbb{G}$-convex boundary and no core.

A smooth function u is $\mathbb{G}$-*harmonic* if in addition to the inequality (1.1) holding, at each point x there exists a $W \in \mathbb{G}_x$ such that equality holds, i.e., $\mathrm{tr}_W \mathrm{Hess}_x u = 0$. In terms of the set $\mathcal{P}(\mathbb{G}_x)$ defined by (1.1), this is the requirement that $\mathrm{Hess}_x u \in \partial\mathcal{P}(\mathbb{G}_x)$ at each point x.

The notion of the Dirichlet dual $\widetilde{\mathcal{P}(\mathbb{G})}$ of $\mathcal{P}(\mathbb{G})$, defined in (7.1), enables one to extend this notion of $\mathbb{G}$-harmonicity to general continuous functions since $\partial\mathcal{P}(\mathbb{G}) = \mathcal{P}(\mathbb{G}) \cap (-\widetilde{\mathcal{P}(\mathbb{G})})$ and $\widetilde{\mathcal{P}(\mathbb{G})}$ satisfies the positivity condition required of a subequation (see Section 7). First, we give a proof of the maximum principle for any upper semi-continuous function u which is $\widetilde{\mathcal{P}(\mathbb{G})}$-subharmonic (much weaker than $\mathbb{G}$ = plurisubharmonic) under our hypothesis that the $\mathbb{G}$-core is empty (see Theorem 7.2). This easily established result is a precursor to comparison. This notion of $\widetilde{\mathcal{P}(\mathbb{G})}$-subharmonic is referred to as *dually* $\mathbb{G}$-plurisubharmonic in this paper.

As long as $\mathbb{G}$ is in a weak sense modeled on a euclidean case $\mathbb{G}_0 \subset G(p, \mathbf{R}^n)$, *both existence and uniqueness hold for the Dirichlet Problem for $\mathbb{G}$-harmonic functions on Ω* (see Definition 7.5 and Theorem 7.6). An outline of our proof from [10] is provided in Section 7.

Since each closed convex set in a vector space V (in our case $\mathrm{Sym}^2(T_xX)$) is the intersection of its supporting closed half-spaces, linear subequations can be made to play a special role in understanding our $\mathbb{G}$-subequations. This is seen in Sections 8 and 9. The results in these two sections will be extended to general subequations in [14].

In Section 8 we consider the case where each $\mathbb{G}_x$ *involves all the variables* in the tangent space T_xX. This means there does not exist a proper linear subspace $W \subset T_xX$ with $\mathbb{G}_x \subset \mathrm{Sym}^2(W)$, and it is equivalent (see Lemma 8.1) to the condition that there exists $A \in \mathrm{Span}\,\mathbb{G}$ with $A > 0$. Under the mild condition of regularity (Definition 6.8), this enables one to write the subequation $\mathcal{P}(\mathbb{G})$ locally as the intersection of a family of uniformly elliptic subequations (Theorem 8.3), a fact that has many consequences. One is the Strong Maximum Principle for $\mathbb{G}$-plurisubharmonic functions (see Theorem 8.5).

There is a distributional notion of $\mathbb{G}$-plurisubharmonicity (but not of $\mathbb{G}$ harmonicity). In Section 9 we prove that $\mathbb{G}$-plurisubharmonic functions and distributionally $\mathbb{G}$-plurisubharmonic functions are equivalent in a sense that will be made very precise in Theorem 9.2. An hypothesis that $\mathbb{G}$ *involves all the variables* is required. Strict $\mathbb{G}$-pluri-subharmonicity can also be defined distributionally and is again equivalent to the viscosity definition (Theorem 9.8). Section 9 concludes with a local-to-global result (of Richberg type [16]) for C^∞ approximation of strictly $\mathbb{G}$-plurisubharmonic functions.

Regarding the various technical assumptions on the closed subset $\mathbb{G} \subset G(p, TX)$, the reader may wish to simply assume that $\mathbb{G}$ admits a smooth fiber-preserving neighborhood retract. In this case it follows easily that $\mathbb{G}$ is regular (Definition 6.8), which was needed for restriction to $\mathbb{G}$-flat submanifolds (Theorem 6.10) and for the description (Theorem 8.3) of $\mathbb{G}$-pluri-subharmonic functions locally as classical subharmonics for a family of "A-Laplacians" associated with $\mathbb{G}$. Finally, $\mathbb{G}$-regularity automatically ensures that projection from $\mathbb{G}$ to X is a local surjection, the property shown in Theorem A.7 to be equivalent to $\mathcal{P}(\mathbb{G})$ being a subequation.

In Appendix B we characterize the subequations which are both linear and geometric under the weak notion of local jet equivalence (Proposition B.4).

Finally we note that the extreme case, where $\mathbb{G} = G(p, TX)$ is chosen to be the full Grassmann bundle, is a basic $\mathbb{G}$-geometry. There are many additional results specific to this case which are discussed in a separate but companion paper [12]. In that paper we use the classical terminology: p-plurisubharmonicity, p-convexity, etc.

2. $\mathbb{G}$-plurisubharmonicity for smooth functions

This concept will be developed in stages. We begin with the basic case.

2.1. Euclidean space

Suppose V is an n-dimensional real inner product space, and fix an integer p, with $1 \le p \le n$. Let $\mathrm{Sym}^2(V)$ denote the space of symmetric endomorphisms of V. Using the inner product, this space is identified with the space of quadratic forms on V. Let $G(p, V)$ denote the set of p-dimensional subspaces of V. For $W \in G(p, V)$, the *W-trace* of A, denoted $\mathrm{tr}_W A$, is the trace of the restriction $A\big|_W$ of A to W.

We identify the Grassmannian $G(p, V)$ with a subset of $\mathrm{Sym}^2(V)$ by identifying a subspace W with orthogonal projection P_W onto the subspace W. The natural inner product on $\mathrm{Sym}^2(V)$ is defined by using the trace, namely $\langle A, B\rangle = \mathrm{tr}(AB)$. Under this identification we have

$$\mathrm{tr}_W A = \langle A, P_W\rangle . \tag{2.1}$$

Let $D_x^2 u$ denote the second derivative of a function u at $x \in V$.

Definition 2.1. Suppose that $\mathbb{G}$ is a closed subset of the Grassmannian $G(p, V)$.

(a) A form $A \in \mathrm{Sym}^2(V)$ is $\mathbb{G}$-*positive* if

$$\mathrm{tr}_W A \geq 0 \quad \forall\, W \in \mathbb{G}. \tag{2.2}$$

(b) A smooth function u defined on an open subset $X \subset V$ is said to be $\mathbb{G}$-*plurisubharmonic* if

$$\mathrm{tr}_W D_x^2 u \geq 0 \quad \forall\, W \in \mathbb{G} \quad \text{and} \quad \forall\, x \in X. \tag{2.3}$$

Let $\mathcal{P}(\mathbb{G})$ denote the set of all $\mathbb{G}$-positive forms $A \in \mathrm{Sym}^2(V)$, and let $\mathrm{PSH}^\infty_{\mathbb{G}}(X)$ denote the set of all smooth $\mathbb{G}$-plurisubharmonic function on X. If $\mathrm{tr}_W A > 0$ for all $W \in \mathbb{G}$, then A is said to be $\mathbb{G}$-*strict*. Similarly, if the inequalities in (2.3) are all strict, then u is said to be *strictly* $\mathbb{G}$-*plurisubharmonic*.

Note that:

$$u \in \mathrm{PSH}^\infty_{\mathbb{G}}(X) \iff D_x^2 u \in \mathcal{P}(\mathbb{G}) \quad \forall\, x \in X, \quad \text{and}$$
$$u \text{ is } \mathbb{G}\text{-strict} \iff D_x^2 u \in \mathrm{Int}\mathcal{P}(\mathbb{G}) \quad \forall\, x \in X.$$

The next result justifies the terminology. We shall say that a function u is *subharmonic on an affine subspace* $\mathbf{W}$ if $\Delta_{\mathbf{W}}\left(u\big|_{\mathbf{W}\cap X}\right) \geq 0$ where $\Delta_{\mathbf{W}}$ is the euclidean Laplacian on $\mathbf{W}$. A p-dimensional affine subspace $\mathbf{W}$ is called an *affine* $\mathbb{G}$-plane if its corresponding vector subspace W is a $\mathbb{G}$-plane.

Proposition 2.2. *A function $u \in C^\infty(X)$ is $\mathbb{G}$-plurisubharmonic if and only if the restriction $u\big|_{\mathbf{W}\cap X}$ is subharmonic for all affine $\mathbb{G}$-planes $\mathbf{W} \subset \mathbf{R}^n$.*

Proof. This is obvious from Condition (2) since with $v = u\big|_{\mathbf{W}\cap X}$, we have $\mathrm{tr}_W D^2 u = \Delta_{\mathbf{W}} v$ on $\mathbf{W} \cap X$. □

2.2. *Riemannian manifolds*

Suppose X is an n-dimensional riemannian manifold. Then the euclidean notions above carry over with $V = T_x X$ and the ordinary second derivative of a smooth function replaced by the *riemannian hessian*. Now the set $\mathbb{G}$ will be an arbitrary closed subset of the Grassmann bundle $\pi : G(p, TX) \to X$. For $u \in C^\infty(X)$ there is a well defined section of the bundle $\mathrm{Sym}^2(TX)$ given on tangent vector fields V, W by

$$(\mathrm{Hess}\, u)(V, W) = VWu - (\nabla_V W)u, \tag{2.4}$$

where ∇ denotes the Levi–Civita connection. Note that under composition with a smooth function $\varphi : \mathbf{R} \to \mathbf{R}$,

$$\mathrm{Hess}\, \varphi(u) = \varphi'(u)\mathrm{Hess}\, u + \varphi''(u)\nabla u \circ \nabla u. \tag{2.5}$$

Definition 2.1′. A smooth function u on X is said to be $\mathbb{G}$-*plurisubharmonic* if $\mathrm{Hess}_x u$ is $\mathbb{G}_x$-positive (where $\mathbb{G}_x = \mathbb{G} \cap \pi^{-1}(x)$) at each point $x \in X$, i.e.,

$$\mathrm{tr}_W \mathrm{Hess}_x u \geq 0 \quad \forall W \in \mathbb{G}_x \quad \text{and} \quad \forall x \in X. \tag{2.3$'$}$$

Again let $\mathrm{PSH}^\infty_{\mathbb{G}}(X)$ denote the set of all smooth $\mathbb{G}$-plurisubharmonic functions on X, and let $\mathcal{P}(\mathbb{G})$ denote the subset of $\mathrm{Sym}^2(TX)$ with fibers $\mathcal{P}(\mathbb{G}_x)$, the set of $\mathbb{G}_x$-positive elements in $\mathrm{Sym}^2(T_xX)$. If the inequalities in (2.3′) are all strict at x, then we say that u is *strictly* $\mathbb{G}$-*plurisubharmonic at* x.

Exercise 2.1 (*Convex Composition Property*)**.** If $\varphi \in C^\infty(\mathbf{R})$ is convex and increasing, then $u \in \mathrm{PSH}^\infty_{\mathbb{G}}(X) \Rightarrow \varphi \circ u \in \mathrm{PSH}^\infty_{\mathbb{G}}(X)$. If, furthermore φ is strictly increasing and convex, then u strictly $\mathbb{G}$-psh $\Rightarrow \varphi \circ u$ strictly $\mathbb{G}$-psh.

Exercise 2.2. Show that if $u \in C^\infty(X)$ is strictly $\mathbb{G}$-psh at a point $x \in X$, then u is strictly $\mathbb{G}$-psh in a neighborhood of x. (See Claim 1 in the proof of Lemma A.3.)

Exercise 2.3. Take $X \equiv \mathbf{R}$ and let $\mathbb{G} \subset G(1, TX) = X \times G(1, \mathbf{R})$ be defined by setting $\mathbb{G}_x = G(1, \mathbf{R})$ if $x \geq 0$ and $\mathbb{G}_x = \emptyset$ if $x < 0$. Show that $\mathcal{P}(\mathbb{G}) \subset X \times \mathrm{Sym}^2(\mathbf{R}) = \mathbf{R}^2$ has fibers $\mathbf{R}$ if $x < 0$ and $\mathbf{R}^+ = [0, \infty)$ if $x \geq 0$. In particular, note that $\mathcal{P}(\mathbb{G})$ is not a closed set even though $\mathbb{G}$ is closed.

3. $\mathbb{G}$-submanifolds and restriction

The appropriate geometric objects (in a sense dual to the $\mathbb{G}$-plurisubharmonic functions) are the minimal $\mathbb{G}$-submanifolds. In the euclidean case this enlarges the family of affine $\mathbb{G}$-planes used in Proposition 2.2.

Definition 3.1. If M is a p-dimensional submanifold of X with $T_xM \in \mathbb{G}_x$ for all $x \in M$, then M is said to be a $\mathbb{G}$-*submanifold.*

Restriction holds as follows.

Theorem 3.2. *If a function $u \in C^\infty(X)$ is $\mathbb{G}$-plurisubharmonic, then the restriction of u to every minimal $\mathbb{G}$-submanifold M is subharmonic in the induced riemannian structure on M.*

Remark 3.3. If $\mathbb{G}$ is determined by a calibration ϕ, i.e., $\mathbb{G}$ consists of the p-planes calibrated by ϕ (with the orientation dropped), then $\mathbb{G}$-submanifolds are automatically minimal. Recently, Robles [17] has shown that if the calibration is parallel, then this remains true for any critical set $\mathbb{G}$ corresponding to a non-zero critical value of the calibration.

Proof. Suppose $M \subset X$ is any p-dimensional submanifold, and let H_M denote its mean curvature vector field. Then

$$\Delta_M \left(u\big|_M\right) = \mathrm{tr}_{TM} \mathrm{Hess}\, u - H_M u.$$

In particular, if M is minimal, then

$$\Delta_M \left(u\big|_M\right) = \mathrm{tr}_{TM} \mathrm{Hess}\, u. \tag{3.1}$$

If M is a $\mathbb{G}$-submanifold, then $\mathrm{tr}_{TM} \mathrm{Hess}\, u \geq 0$ and the result follows. □

Remark 3.4. If for every point $x \in X$ and every p-plane $W \in \mathbb{G}_x$, there exists a minimal submanifold M with $T_xM = W$, then the converse to Theorem 3.2 is true (use the formula (3.1)).

4. $\mathbb{G}$-convexity and the core

We will answer four questions concerning the existence of $\mathbb{G}$-plurisubharmonic functions.

(1) When does there exist $u \in \mathrm{PSH}^\infty_{\mathbb{G}}(X)$ which is everywhere strict?
(2) When does there exist $u \in \mathrm{PSH}^\infty_{\mathbb{G}}(X)$ which is a proper exhaustion for X?
(3) When does there exist $u \in \mathrm{PSH}^\infty_{\mathbb{G}}(X)$ which is both strict and an exhaustion?
(4) When does there exist $u \in \mathrm{PSH}^\infty_{\mathbb{G}}(X)$ which is an exhaustion and strict near ∞?

The answers illustrate some of the flexibility available in constructing $\mathbb{G}$-plurisubharmonic functions.

First we characterize those manifolds X which admit a smooth strictly $\mathbb{G}$-plurisubharmonic function.

Definition 4.1 (*The Core*)**.** The $\mathbb{G}$-*core* of X is defined to be the subset

$$\mathrm{Core}_{\mathbb{G}}(X) = \{x \in X : \text{no } u \in \mathrm{PSH}^\infty_{\mathbb{G}}(X) \text{ is strict at } x\}.$$

Note that the core is the intersection over $u \in \mathrm{PSH}^\infty_{\mathbb{G}}(X)$ of the closed sets where the given u is not strict, and as such is a closed subset of X (see Exercise 2.2).

Theorem 4.2. *The manifold X admits a smooth strictly $\mathbb{G}$-plurisubharmonic function $\iff$ $\mathrm{Core}_{\mathbb{G}}(X) = \emptyset$. In fact, there exists a function $\psi \in \mathrm{PSH}^\infty_{\mathbb{G}}(X)$ which is $\mathbb{G}$-strict at each point $x \notin \mathrm{Core}_{\mathbb{G}}(X)$.*

Proof. The implication $\Rightarrow$ is clear from the definition. For the converse choose an exhaustion of X by compact subsets $K_1 \subset K_2 \subset \cdots$. Given any sequence of smooth functions $u_j \in C^\infty(X)$ and numbers $\epsilon_j > 0$, $j \geq 1$ with $\sum \epsilon_j < \infty$, if we choose numbers $\delta_j > 0$ sufficiently small that the semi-norms

$$\|w\|_{K,j} \equiv \sup_K \sum_{|\alpha| \leq j} |D^\alpha u_j| < \epsilon_j$$

satisfy

$$\delta_j \|u_j\|_{K_j,j} \leq \epsilon_j,$$

then $u = \sum_j \delta_j u_j$ converges in the C^∞-topology to $u \in C^\infty(X)$.

If v is $\mathbb{G}$-strict at a point x, then v is $\mathbb{G}$-strict in a neighborhood of x (Exercise 2.2). Therefore, if K is a compact set disjoint from $\mathrm{Core}_{\mathbb{G}}(X)$, then we can find $v \in \mathrm{PSH}^\infty_{\mathbb{G}}(X)$ which is $\mathbb{G}$-strict at each point of K. Hence, we may choose $u_j \in \mathrm{PSH}^\infty_{\mathbb{G}}(X)$ with u_j strict at each point of K_j of distance $\geq 1/j$ from $\mathrm{Core}_{\mathbb{G}}(X)$. Take $\psi \equiv \sum \delta_j u_j$ as above. $\square$

Remark. Essentially the same argument proves that there exists $\psi \in \mathrm{PSH}^\infty_{\mathbb{G}}(X)$ such that $\mathrm{tr}_W \mathrm{Hess}\, \psi > 0$ for all $\mathbb{G}$-planes W which do not lie in the *tangential core* (see [5]).

Definition 4.3 (*The $\mathbb{G}$-Convex Hull*)**.** Given a subset $K \subset X$, the $\mathbb{G}$-convex *hull of* K is the set

$$\widehat{K} = \left\{x \in X : u(x) \leq \sup_K u \ \forall u \in \mathrm{PSH}^\infty_{\mathbb{G}}(X)\right\}.$$

Note that $\widehat{\widehat{K}} = \widehat{K}$ and that $\widehat{K}$ is closed.

Theorem 4.4 ($\mathbb{G}$-*Convexity and Exhaustion*)**.** *The following three conditions are equivalent.*

(1) *If* $K \subset\subset X$*, then* $\widehat{K} \subset\subset X$.
(2) X *admits a smooth* $\mathbb{G}$*-plurisubharmonic proper exhaustion function* u.
(3) *For some neighborhood of* ∞, $X - K$ *with* K *compact, there exists* $u \in \mathrm{PSH}^{\infty}_{\mathbb{G}}(X - K)$ *with* $\lim_{x\to\infty} u(x) = +\infty$.

Condition (3) is a weakening of condition (2) to a local condition at ∞ in the one-point compactification $\overline{X} = X \cup \{\infty\}$.

Definition 4.5. We say that X *is* $\mathbb{G}$-convex if one of the equivalent condition in Theorem 4.4 holds.

The implication (3) ⇒ (2) is immediate from the next (stronger) result. Here K is a compact subset of X.

Lemma 4.6. *Given* $v \in \mathrm{PSH}^{\infty}_{\mathbb{G}}(X - K)$ *with* $\lim_{x\to\infty} v(x) = +\infty$, *there exists* $u \in \mathrm{PSH}^{\infty}_{\mathbb{G}}(X)$ *such that* $u = v$ *in a neighborhood of* ∞.

Proof. For c sufficiently large, v is smooth and $\mathbb{G}$-plurisubharmonic outside the compact set $\{x \in X : v(x) \le c - 1\}$. Pick a convex increasing function $\varphi \in C^{\infty}(\mathbf{R})$ with $\varphi \equiv c$ on a neighborhood of $(-\infty, c-1]$ and $\varphi(t) = t$ on $(c+1, \infty)$. Then by Exercise 2.1, the composition $\varphi \circ v$ is smooth and $\mathbb{G}$-plurisubharmonic on all of X. Moreover, $u = v$ outside the compact set $\{x \in X : v(x) \le c + 1\}$. □

Proof that (2) ⇒ (1). If K is compact, then $c = \sup_K u < \infty$, and $\widehat{K}$ is contained in the compact set $\{u \le c\}$. □

The implication (1) ⇒ (2) is a construction using the next lemma.

Lemma 4.7. *Suppose* $K \subset X$ *is compact. If* $x \notin \widehat{K}$, *then there exists* $u \in \mathrm{PSH}^{\infty}_{\mathbb{G}}(X)$ *satisfying:*

(a) $u \equiv 0$ *on a neighborhood of* K,
(b) $u(x) > 0$, *and*
(c) u *is strict at* x *if* $x \notin \mathrm{Core}_{\mathbb{G}}(X)$.

Proof. Suppose $x \notin \widehat{K}$. Then there exists $v \in \mathrm{PSH}^{\infty}_{\mathbb{G}}(X)$ with $\sup_K v < 0 < v(x)$. Pick $\varphi \in C^{\infty}(\mathbf{R})$ with $\varphi \equiv 0$ on $(-\infty, 0]$ and with $\varphi > 0$ and convex increasing on $(0, \infty)$. Then $u = \varphi \circ v$ satisfies the required conditions. Furthermore, assume $h \in \mathrm{PSH}^{\infty}_{\mathbb{G}}(X)$ is strict at x. Then take $\overline{v} = v + \epsilon h$. For small enough ϵ, $\sup_K \overline{v} < 0 < \overline{v}(x)$. If φ is also strictly increasing on $(0, \infty)$, then $u = \varphi \circ \overline{v}$ is strict at x. □

Proof that (1) ⇒ (2). A $\mathbb{G}$-plurisubharmonic proper exhaustion function on X is constructed as follows. Choose an exhaustion of X by compact $\mathbb{G}$-convex subsets $K_1 \subset K_2 \subset K_3 \subset \cdots$ with $K_m \subset K^0_{m+1}$ for all m. By Lemma 4.7 and the compactness of $K_{m+2} - K^0_{m+1}$, there exists a $\mathbb{G}$-plurisubharmonic function $f_m \ge 0$ on X with f_m identically zero on a neighborhood of K_m and $f_m > 0$ on $K_{m+2} - K^0_{m+1}$. By re-scaling we may assume $f_m > m$ on $K_{m+2} - K^0_{m+1}$. The locally finite sum $f = \sum_{m=1}^{\infty} f_m$ satisfies (2). □

Next we characterize the existence of a strict exhaustion function.

Theorem 4.8 (*Strict $\mathbb{G}$-Convexity*)**.** *The following conditions are equivalent:*

(1) $\mathrm{Core}_{\mathbb{G}}(X) = \emptyset$, *and if* $K \subset\subset X$, *then* $\widehat{K} \subset\subset X$,
(2) *X admits a smooth proper exhaustion function which is strictly $\mathbb{G}$-plurisubharmonic.*

Proof that (1) ⇒ (2). Since $\mathrm{Core}_{\mathbb{G}}(X) = \emptyset$, there exists a strictly $\mathbb{G}$-plurisubharmonic function v by Proposition 4.2. If u is a $\mathbb{G}$-plurisubharmonic exhaustion function given by Theorem 4.4, then $u + e^v$ is a strict exhaustion. □

Definition 4.9. We say that X is *strictly* $\mathbb{G}$-convex if one of the equivalent conditions of Theorem 4.8 holds.

Corollary 4.10. *Suppose that* $\mathrm{Core}_{\mathbb{G}}(X) = \emptyset$. *If X is $\mathbb{G}$-convex, then X is strictly $\mathbb{G}$-convex.*

Theorem 4.11 (*Strict $\mathbb{G}$-Convexity at Infinity*)**.** *The following conditions are equivalent:*

(1) $\mathrm{Core}_{\mathbb{G}}(X)$ *is compact, and if* $K \subset\subset X$, *then* $\widehat{K} \subset\subset X$,
(2) *X admits $u \in \mathrm{PSH}_{\mathbb{G}}(X)$ with $\lim_{x\to\infty} u(x) = \infty$ and u strict outside a compact subset.*
(3) $\mathrm{Core}_{\mathbb{G}}(X)$ *is compact, and X admits $u \in \mathrm{PSH}_{\mathbb{G}}(X - K)$, for some compact subset K, with* $\lim_{x\to\infty} u(x) = \infty$.

Proof that (3) ⇒ (2). Apply Lemma 4.6. □

Proof that (2) ⇒ (1) (*Straightforward*)**.** □

Proof that (1) ⇒ (3). $\mathrm{Core}_{\mathbb{G}}(X) \equiv K$ is compact $\Rightarrow \mathrm{Core}_{\mathbb{G}}(X - K) = \emptyset$. □

Definition 4.12. We say that X *is strictly $\mathbb{G}$-convex at infinity* if one of the equivalent condition in Theorem 4.11 holds.

Some of the previous results can be summarized as follows.

Corollary 4.13. *Suppose* $\mathrm{Core}_{\mathbb{G}}(X) = \emptyset$. *Then the following are equivalent.*

(1) *X is $\mathbb{G}$-convex.*
(2) *X is strictly $\mathbb{G}$-convex.*
(3) *X is strictly $\mathbb{G}$-convex at infinity.*

Proof. Use Theorems 4.4 and 4.11. □

Proposition 4.14. *Suppose $(M, \partial M)$ is a compact connected $\mathbb{G}$-submanifold-with-boundary in X. If M is minimal (stationary), then*

(1) *If $\partial M = \emptyset$, then $M \subset \mathrm{Core}_{\mathbb{G}}(X)$.*
(2) *If $\partial M \neq \emptyset$, then $M \subset \widehat{\partial M}$.*

Proof. Since the restriction of any $u \in \mathrm{PSH}^{\infty}_{\mathbb{G}}(X)$ to M is subharmonic on M, the maximum principle applies to $u\big|_M$. □

This proposition provides an analogue of the support Lemma 3.2 in [6]:

If M is a minimal $\mathbb{G}$ submanifold, then $M \subset \widehat{\partial M} \cup \mathrm{Core}_{\mathbb{G}}(X)$.

The existence question for strictly $\mathbb{G}$-convex manifolds has two sides. We briefly mention these results from both [5,8].

Definition 4.15 ($\mathbb{G}$-*Free*)**.** A subspace $V \subset T_X$ is said to be $\mathbb{G}$-*free* if there are no $\mathbb{G}$-planes contained in V. The maximal dimension of such a free subspace, taken over all points $x \in X$, is called the *free dimension* of $\mathbb{G}$ and is denoted freedim($\mathbb{G}$). A submanifold M of X is $\mathbb{G}$-*free* if $T_x M$ is $\mathbb{G}$-free for each $x \in M$.

Strict $\mathbb{G}$-convexity of X imposes conditions on the topology of X.

Theorem 4.16. *A strictly $\mathbb{G}$-convex manifold has the homotopy type of a CW complex of dimension $\leq$ freedim* ($\mathbb{G}$).

The free dimension of G is computed in many examples in [5] and summarized in [8].

On the other hand, the existence of many strictly $\mathbb{G}$-convex manifolds is guaranteed by another result (see Theorem 6.6 in [5]).

Theorem 4.17. *Suppose M is a $\mathbb{G}$-free submanifold of X. Then M has a fundamental neighborhood system in X consisting of strictly $\mathbb{G}$-convex manifolds, each of which has M as a deformation retract.*

5. Boundary convexity

Suppose that $\Omega \subset X$ is an open connected set with smooth non-empty boundary $\partial\Omega$ contained in an oriented riemannian manifold. Fix a closed subset $\mathbb{G} \subset G(p, TX)$.

Definition 5.1. A p-plane $W \in \mathbb{G}_x$ at $x \in \partial\Omega$ is called a *tangential $\mathbb{G}$-plane at x* if $W \subset T_x(\partial\Omega)$.

Denote by $II = II_{\partial\Omega}$ the second fundamental form of the boundary with respect to the *inward pointing* normal n. This is a symmetric bilinear form on each tangent space $T_x(\partial\Omega)$ defined by

$$II(v, w) = -\langle \nabla_v n, w\rangle = \langle n, \nabla_v W\rangle$$

where W is any vector field tangent to $\partial\Omega$ with $W_x = w$.

Definition 5.2. The boundary $\partial\Omega$ is $\mathbb{G}$-*convex at a point x* if $\mathrm{tr}_W II_x \geq 0$ for all tangential $\mathbb{G}$-planes W at x. If this inequality is strict, then we say that $\partial\Omega$ is *strictly $\mathbb{G}$-convex at x*.

Definition 5.3 (*Local Defining Functions*)**.** Suppose ρ is a smooth function on a neighborhood B of a point $x \in \partial\Omega$ with $\partial\Omega \cap B = \{\rho = 0\}$ and $\Omega \cap B = \{\rho < 0\}$. If $d\rho$ is non-zero on $\partial\Omega \cap B$, then ρ is called a *local defining function for $\partial\Omega$.*

Lemma 5.4. *If ρ is a local defining function for $\partial\Omega$, then for all $x \in \partial\Omega \cap B$,*

$$\mathrm{Hess}_x \rho\big|_{T_x(\partial\Omega)} = |\nabla\rho(x)| II_x.$$

Proof. Suppose that e is a vector field on B tangent to $\partial\Omega$ along $\partial\Omega$, and note that $II(e,e) = \langle n, \nabla_e e\rangle = -\frac{1}{|\nabla\rho|}\langle\nabla\rho, \nabla_e e\rangle$ and $-\langle\nabla\rho, \nabla_e e\rangle = -(\nabla_e e)(\rho) = e(e\rho) - (\nabla_e e)(\rho) = (\mathrm{Hess}\,\rho)(e,e)$. $\square$

Corollary 5.5. *The boundary $\partial\Omega$ is $\mathbb{G}$-convex at a point x if and only if*

$$\mathrm{tr}_W \mathrm{Hess}\,\rho \;\geq\; 0 \quad \textit{for all } \mathbb{G}\textit{- planes } W \textit{ tangent to } \partial\Omega \textit{ at } x \tag{5.1}$$

where ρ is a local defining function for $\partial\Omega$. In particular the condition (5.1) is independent of the choice of local defining function ρ. Moreover, the boundary is strictly $\mathbb{G}$-convex at a point x if and only if the inequalities in (5.1) are all strict, again with independence of the choice of ρ.

Remark 5.6. If $\partial\Omega$ is $\mathbb{G}$-free at a point $x \in \partial\Omega$ (see Definition 4.15), then $\partial\Omega$ is automatically strictly $\mathbb{G}$-convex at x since there are no tangential $\mathbb{G}$-planes W to consider. For example, in the extreme case $p = n$ (the Laplacian subequation) all boundaries $\partial\Omega$ are strict at each point since all hyperplanes in T_xX are $\mathbb{G}$-free.

Theorem 5.7. *Suppose that $\partial\Omega$ is strictly $\mathbb{G}$-convex. Then there exists a global $\mathbb{G}$-plurisubharmonic defining function $\rho \in C^\infty(\overline{\Omega})$ which is strict on a collar $\{-\epsilon \le \rho \le 0\}$. If* $\mathrm{Core}(\Omega) = \emptyset$*, then ρ can be chosen to be strict on all of $\overline{\Omega}$.*

Corollary 5.8. *If $\partial\Omega$ is strictly $\mathbb{G}$-convex, then Ω is strictly $\mathbb{G}$-convex at ∞; and if* $\mathrm{Core}(\Omega) = \emptyset$*, then Ω is strictly $\mathbb{G}$-convex.*

Proof of Corollary. Suppose that $\rho \in C^\infty(\overline{\Omega})$ is a defining function for $\partial\Omega$. Then $-\log(-\rho)$ is an exhaustion function for Ω. Since the function $\psi : (-\infty, 0) \to (-\infty, \infty)$ defined by $\psi(t) = -\log(-t)$ is strictly convex and increasing,

$$
\begin{aligned}
&-\log(-\rho) \text{ is strictly } \mathbb{G}\text{-plurisubharmonic at points in } \Omega \\
&\text{where } \rho \text{ is strictly } \mathbb{G}\text{-plurisubharmonic}
\end{aligned}
\tag{5.2}
$$

(see Exercise 2.1). □

Proof of Theorem. Start with an arbitrary defining function $\rho \in C^\infty(\overline{\Omega})$ for $\partial\Omega$. Set $\widetilde{\rho} \equiv \rho + \frac{\lambda}{2}\rho^2$ with $\lambda > 0$. Then at points in $\partial\Omega$

$$
\mathrm{Hess}\,\widetilde{\rho} = (1 + \lambda\rho)\mathrm{Hess}\,\rho + \lambda\nabla\rho \circ \nabla\rho = \mathrm{Hess}\,\rho + \lambda\nabla\rho \circ \nabla\rho. \tag{5.3}
$$

We will show that:

$$
\begin{aligned}
&\text{For } \lambda \text{ sufficiently large, } \widetilde{\rho} = \rho + \frac{\lambda}{2}\rho^2 \text{ is strictly } \mathbb{G}\text{-plurisubharmonic} \\
&\text{at every boundary point } x \in \partial\Omega.
\end{aligned}
\tag{5.4}
$$

It then follows that $\widetilde{\rho}$ is strictly $\mathbb{G}$-plurisubharmonic in a neighborhood of $\partial\Omega$ in X, and hence on some collar $\{-\epsilon \le \widetilde{\rho} \le 0\}$ with $\epsilon > 0$. Choose $\psi(t)$ convex and increasing with $\psi(t) \equiv -\epsilon$ if $t \le -\epsilon$, and $\psi(t) = t$ if $t \ge -\frac{\epsilon}{2}$. Then $\psi(\widetilde{\rho})$ is $\mathbb{G}$-plurisubharmonic on Ω and equal to $\widetilde{\rho}$ on the collar $\{-\epsilon \le \widetilde{\rho} \le 0\}$, thereby providing the required defining function. If $\mathrm{Core}(\Omega)$ is empty, then add the global strictly $\mathbb{G}$-plurisubharmonic function, provided by Theorem 4.2, to $\psi(\widetilde{\rho})$.

It remains to prove (5.4). Each p-plane $V \in G(p, T_xX)$ can be put in canonical form with respect to $T_x\partial\Omega$. Let n denote a unit normal to $T_x\partial\Omega$ in T_xX. Choose an orthonormal basis $e_1, \dots, e_p$ for V such that $e_2, \dots, e_p$ is an orthonormal basis for $V \cap (T_x\partial\Omega)$. Then $e_1 = \cos\theta_V n + \sin\theta_V e$ defines an angle θ_V mod π and a unit vector $e \in T_x\partial\Omega$. Now by (5.3) we have

$$
\mathrm{tr}_V \mathrm{Hess}\,\widetilde{\rho} = \mathrm{tr}_V \mathrm{Hess}\,\rho + \lambda\cos^2\theta_V |\nabla\rho|^2. \tag{5.5}
$$

The inequality $|\cos\theta_V| < \delta$ defines a fundamental neighborhood system for $G(p, T\partial\Omega)$ as a subset of the bundle $G(p, TX)\big|_{\partial\Omega}$. Intersecting with $\mathbb{G}\big|_{\partial\Omega}$ we see that $\mathbb{G} \cap G(p, T\partial\Omega)$ has a fundamental neighborhood system in $\mathbb{G}\big|_{\partial\Omega}$ given by $\mathcal{N}_\delta \equiv \{V \in \mathbb{G}_x : x \in \partial\Omega \text{ and } |\cos\theta_V| < \delta\}$.

Since $\partial\Omega$ is strictly $\mathbb{G}$-convex, there exists $\eta > 0$ such that $\mathrm{tr}_W \mathrm{Hess}\,\rho \geq 2\eta$ for all $W \in \mathbb{G} \cap G(p, T\partial\Omega)$. Hence for δ small, $\mathrm{tr}_V \mathrm{Hess}\,\rho \geq \eta$ for all $V \in \mathcal{N}_\delta$. Choose a lower bound $-M$ for $\mathrm{tr}_V \mathrm{Hess}\,\rho$ over all $V \in \mathbb{G}\big|_{\partial\Omega}$.

Assume $V \in \mathbb{G}_x$, $x \in \partial\Omega$. For $|\cos\theta_V| < \delta$, $\mathrm{tr}_V \mathrm{Hess}\widetilde{\rho} \geq \eta + \lambda\cos^2\theta_V |\nabla\rho|^2 \geq \eta$. For $|\cos\theta_V| \geq \delta$, $\mathrm{tr}_V \mathrm{Hess}\,\widetilde{\rho} \geq -M + \lambda\delta^2|\nabla\rho|^2$ which is $\geq\eta$ if λ is chosen large. This proves (5.4). □

Remark 5.9. Simple examples show that strict $\mathbb{G}$-convexity of $\partial\Omega$ does not imply that every defining function ρ for $\partial\Omega$ is strictly $\mathbb{G}$-plurisubharmonic at points of $\partial\Omega$. However, the exhaustion $-\log(-\rho)$ is always strictly $\mathbb{G}$-plurisubharmonic on a small enough collar of $\partial\Omega$. For the proof of this, compute $\mathrm{Hess}(-\log(-\rho))$ and mimic the proof of Theorem 5.7 on the hypersurfaces $\{\rho = \epsilon\}$ (see the proof of Theorem 5.6 in [5]).

Remark 5.10 (*Signed Distance*)**.** Recall that a defining function ρ for Ω satisfies $|\nabla\rho| \equiv 1$ in a neighborhood of $\partial\Omega$ if and only if ρ is the signed distance to $\partial\Omega$ (<0 in Ω and >0 outside of Ω). In fact any function ρ with $|\nabla\rho| \equiv 1$ in a riemannian manifold is, up to an additive constant, the distance function to (any) one of its level sets. In this case it is easy to see that

$$\mathrm{Hess}\,\rho = \begin{pmatrix} 0 & 0 \\ 0 & II \end{pmatrix} \tag{5.6}$$

where II denotes the second fundamental form of the hypersurface $H = \{\rho = \rho(x)\}$ with respect to the normal $n = -\nabla\rho$ and the blocking in (5.6) is with respect to the splitting $T_xX = N_xH \oplus T_xH$. For example let $\rho(x) = \|x\| \equiv r$ in $\mathbf{R}^n$. Then direct calculation shows that $\mathrm{Hess}\,\rho = \frac{1}{r}(I - \hat{x}\circ\hat{x})$ where $\hat{x} = x/r$. Moreover,

$$\mathrm{Hess}(\rho + \lambda\rho^2) = \begin{pmatrix} 2\lambda & 0 \\ 0 & II \end{pmatrix} \tag{5.7}$$

simplifying the proof of (5.3). Moreover, setting $\delta = -\rho \geq 0$, the actual distance to $\partial\Omega$ in Ω, we have

$$\mathrm{Hess}(-\log\delta) = \frac{1}{\delta}\begin{pmatrix} \frac{1}{\delta} & 0 \\ 0 & II \end{pmatrix} \tag{5.8}$$

giving an easy proof of Remark 5.9 for this ρ. Namely, with $\delta(x) \equiv \mathrm{dist}(x, \partial\Omega)$ we have that

$$\partial\Omega \text{ strictly } \mathbb{G}\text{-convex} \quad\Rightarrow\quad -\log\delta \text{ is strictly } \mathbb{G}\text{-psh in a collar.} \tag{5.9}$$

Remark 5.11 ($\mathbb{G}$-*Parallel*)**.** If $\mathbb{G}$ is parallel as a subset of $G(p, TX) \subset \mathrm{Sym}^2(TX)$, then a weakened form of the converse to (5.9) is true. Namely,

If $-\log\delta$ is $\mathbb{G}$-plurisubharmonic in a collar, then $\partial\Omega$ is $\mathbb{G}$-convex at each point.

Proof. If $\partial\Omega$ is not $\mathbb{G}$-convex at $x \in \partial\Omega$, then with $\rho \equiv -\delta$, $\mathrm{tr}_W \mathrm{Hess}_x\rho < 0$ for some $W \in \mathbb{G}_x$ tangential to $\partial\Omega$ at x let γ denote the geodesic segment in Ω which emanates orthogonally from $\partial\Omega$ at x. Since δ is the distance function to $\partial\Omega$, γ is an integral curve of $\nabla\delta$. Let W_y denote the parallel translate of W along γ to y. Then $W_y \in \mathbb{G}_y$ and $(\nabla\delta)_y \perp W_y$. Therefore by (5.8), $\mathrm{tr}_{W_y}\mathrm{Hess}_y(-\log\delta) = \frac{1}{\delta}\mathrm{tr}_{W_y}\mathrm{Hess}_y(\rho) < 0$ for y sufficiently close to x. Hence $-\log\delta$ is not $\mathbb{G}$-plurisubharmonic near $\partial\Omega$. □

5.1. *Local convexity of a domain $\Omega \subset X$*

For simplicity assume that $\mathrm{Core}_{\mathbb{G}}(X)$ is empty. Then for each open subset $Y \subset X$ the three notions of convexity, namely $\mathbb{G}$-convexity, strict $\mathbb{G}$-convexity, and strict $\mathbb{G}$-convexity at infinity, are all equivalent.

Definition 5.12. A domain $\Omega \subset X$ is *locally* $\mathbb{G}$*-convex* if each point $x \in \partial\Omega$ has a neighborhood U in X such that $U \cap \Omega$ is $\mathbb{G}$-convex.

Small balls are $\mathbb{G}$-convex and the intersection of two $\mathbb{G}$-convex domains is again $\mathbb{G}$-convex. Therefore:

$$\text{If } \Omega \text{ is } \mathbb{G}\text{-convex, then } \Omega \text{ is locally } \mathbb{G}\text{-convex.} \tag{5.10}$$

Using terminology from complex analysis, we formulate the "Levi Problem": For which pairs $X, \mathbb{G}$ does

$$\Omega \text{ locally } \mathbb{G}\text{-convex} \;\Rightarrow\; \Omega \text{ is } \mathbb{G}\text{-convex?} \tag{5.11}$$

Even in the euclidean case this is not always true. Here is a counterexample.

Example 5.13 (*Horizontal Convexity in* $\mathbf{R}^2$)**.** Take $\mathbb{G} = \{\mathbf{R} \times \{0\}\} \subset G(1, \mathbf{R}^2)$ a singleton consisting of the x_1-axis. A domain is $\mathbb{G}$-convex if and only if all of its horizontal slices are connected. Choose $\Omega \subset\subset \mathbf{R}^2$ with the property that $\partial\Omega$ contains the interval $[-1, 1]$ on the x_1-axis, the lower half of the circle of radius 3 about the origin, and the points $(-2, 1)$, $(2, 1)$. This can be done with Ω locally $\mathbb{G}$-convex but not globally $\mathbb{G}$-convex. In addition, the boundary of Ω can be made $\mathbb{G}$-convex.

By contrast, one of the main results of [12] is the solution to the Levi Problem in euclidean space in the extreme case $\mathbb{G} = G(p, \mathbf{R}^n)$.

6. Upper semi-continuous $\mathbb{G}$-plurisubharmonic functions

Let X be a riemannian manifold, and assume that $\mathbb{G} \subset G(p, TX)$ is a closed subset. Denote by $\mathrm{USC}(X)$ the space of upper semi-continuous $[-\infty, \infty)$-valued functions on X. By a *test function* for $u \in \mathrm{USC}(X)$ at a point x we mean a C^2-function φ, defined near x, such that $u \le \varphi$ near x and $u(x) = \varphi(x)$.

Definition 6.1. A function $u \in \mathrm{USC}(X)$ is $\mathbb{G}$*-plurisubharmonic* if for each $x \in X$ and each test function φ for u at x, the riemannian hessian $\mathrm{Hess}_x\varphi$ at x satisfies

$$\mathrm{tr}_W \mathrm{Hess}_x \varphi \ge 0 \quad \forall W \in \mathbb{G}_x$$

i.e., $\mathrm{Hess}_x\varphi \in \mathcal{P}(\mathbb{G}_x)$. The space of these functions is denoted by $\mathrm{PSH}_{\mathbb{G}}(X)$.

This definition is an extension of Definition 2.1′ because of the following.

Lemma 6.2. *Suppose $u \in C^2(X)$. Then for a point $x \in X$, the following are equivalent:*

$$\mathrm{tr}_W \mathrm{Hess}_x \varphi \ge 0 \quad \forall W \in \mathbb{G}_x \textit{ and all test functions } \varphi \text{ for } u \text{ at } x, \tag{6.1}$$

$$\mathrm{tr}_W \mathrm{Hess}_x u \ge 0 \quad \forall W \in \mathbb{G}_x. \tag{6.2}$$

Proof. Note that (6.1) ⇒ (6.2) because we can take $\varphi = u$ in (6.1). Assume (6.2) and that φ is a test function for u at x. Then $\psi \equiv \varphi - u \ge 0$ near x and vanishes at x. Hence x is a critical point for ψ, and the second derivative or hessian of ψ is a well defined non-negative element of $\mathrm{Sym}^2(T_xX)$, independent of any metric. In particular, $\mathrm{tr}_W \mathrm{Hess}_x\psi \geq 0$ for all $W \in G(p, T_xX)$. Since $\mathrm{Hess}_x\varphi = \mathrm{Hess}_x u + \mathrm{Hess}_x\psi$, taking the W-trace with $W \in \mathbb{G}_x$, we see that (6.2) $\Rightarrow$ (6.1). $\square$

Remark 6.3 (*Positivity*)**.** Let $\mathcal{P}_x \subset \mathrm{Sym}^2(T_xX)$ denote the subset of non-negative elements. Replacing $\mathcal{P}(\mathbb{G}) \subset \mathrm{Sym}^2(TX)$ with a general closed subset $F \subset \mathrm{Sym}^2(TX)$, the above (standard) proof shows that (6.2) implies (6.1), i.e., $\mathrm{Hess}_x u \in F_x \Rightarrow \mathrm{Hess}_x\varphi \in F_x$, provided that F satisfies the *positivity condition*:

$$F_x + \mathcal{P}_x \subset F_x \quad \text{for all } x \in X. \tag{P}$$

There are several equivalent ways of stating the condition (6.1). We record one that is particularly useful, and refer the reader to Appendix A in [10] for the proof as well as the statements of the other conditions.

Lemma 6.4. *Suppose $u \in \mathrm{USC}(X)$. Then $u \notin \mathrm{PSH}_{\mathbb{G}}(X)$ if and only if $\exists\, x_0 \in X, \alpha > 0$, and a smooth function φ defined near x_0 satisfying:*

$$u - \varphi \leq -\alpha|x - x_0|^2 \quad \text{near } x_0$$
$$u - \varphi = 0 \quad \text{at } x_0$$

but with $\mathrm{tr}_W \mathrm{Hess}_{x_0}\varphi < 0$ for some $W \in \mathbb{G}_{x_0}$.

6.1. Elementary properties

Even though $\mathbb{G} \subset \mathrm{Sym}^2(TX)$ is closed, the subset $\mathcal{P}(\mathbb{G}) \subset \mathrm{Sym}^2(TX)$ of $\mathbb{G}$-positive elements may not be closed (see Exercise 2.3). However, by Proposition A.6 below, $\mathcal{P}(\mathbb{G})$ is closed if and only if $\pi\big|_{\mathbb{G}}$ is a local surjection. We make this assumption unless the contrary is stated.

The following basic facts can be found for example in [10, Theorem 2.6]. In fact they hold with $\mathcal{P}(\mathbb{G})$ replaced by any subequation (see Definition A.2).

Theorem 6.5. (a) *(Maximum property) If $u, v \in \mathrm{PSH}_{\mathbb{G}}(X)$, then $w = \max\{u, v\} \in \mathrm{PSH}_{\mathbb{G}}(X)$.*
(b) *(Coherence property) If $u \in \mathrm{PSH}_{\mathbb{G}}(X)$ is twice differentiable at $x \in X$, then $\mathrm{Hess}_x u$ is $\mathbb{G}$-positive.*
(c) *(Decreasing sequence property) If $\{u_j\}$ is a decreasing ($u_j \geq u_{j+1}$) sequence of functions with all $u_j \in \mathrm{PSH}_{\mathbb{G}}(X)$, then the limit $u = \lim_{j\to\infty} u_j \in \mathrm{PSH}_{\mathbb{G}}(X)$.*
(d) *(Uniform limit property) Suppose $\{u_j\} \subset \mathrm{PSH}_{\mathbb{G}}(X)$ is a sequence which converges to u uniformly on compact subsets to X, then $u \in \mathrm{PSH}_{\mathbb{G}}(X)$.*
(e) *(Families locally bounded above) Suppose $\mathcal{F} \subset \mathrm{PSH}_{\mathbb{G}}(X)$ is a family of functions which are locally uniformly bounded above. Then the upper semi-continuous regularization v^* of the upper envelope*

$$v(x) = \sup_{u\in\mathcal{F}} u(x)$$

belongs to $\mathrm{PSH}_{\mathbb{G}}(X)$.

Example 6.6. The following examples show that Properties (c)–(e) require that the set $\mathcal{P}(\mathbb{G})$ be closed. Let $X = \mathbf{R}$ and $\mathbb{G}_x = \{T_x\mathbf{R}\} \in G(1, TX)$ if $x \geq 0$ and $\mathbb{G}_x = \emptyset$ for $x < 0$. Note that $\mathbb{G}$ is a closed set. Then $\mathcal{P}(\mathbb{G}_x) = \mathrm{Sym}^2(T_x X) \cong \mathbf{R}$ for $x < 0$ and $\mathcal{P}(\mathbb{G}_x) = \{A \in \mathrm{Sym}^2(T_x X) : A \geq 0\}$ for $x \geq 0$. Note that $\mathcal{P}(\mathbb{G})$ is not closed in $\mathbf{R} \times \mathbf{R}$. This subequation is simply the requirement that

$$u''(x) \geq 0 \quad \text{for all } x \geq 0.$$

Fix a constant $a > 0$ and set

$$u(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ x(a-x) & \text{if } x \leq 0. \end{cases}$$

This function fails to be $\mathbb{G}$-plurisubharmonic at 0. To see this note that $\varphi(x) = x(a-x)$ is a test function for u at 0 and $\varphi''(0) < 0$.

For each $\delta > 0$ set $v_\delta(x) = u(x+\delta) + \delta$. Note that $\mathrm{graph}(v_\delta) = \mathrm{graph}(u) + (-\delta, \delta)$. Then each v_δ is $\mathbb{G}$-plurisubharmonic and $v_\delta \downarrow u$ as $\delta \to 0$. Hence condition (c) fails.

Now for each $\epsilon > 0$, define $u_\epsilon \equiv \min\{u, -\epsilon\}$. Then u_ϵ is $\mathbb{G}$-plurisubharmonic for all ϵ and $u_\epsilon \uparrow u$ as $\epsilon \to 0$. Hence conditions (d) and (e) also fail.

6.2. *Restriction*

Throughout this subsection we assume that $\mathbb{G} \subset G(p, TX)$ is a closed set admitting a smooth neighborhood retraction preserving the fibers of the projection $\pi : G(p, TX) \to X$. The terminology $\mathbb{G}$-plurisubharmonic for $u \in \mathrm{USC}(X)$ is justified by the next result, which extends Theorem 3.2.

Theorem 6.7. *If $u \in \mathrm{PSH}_{\mathbb{G}}(X)$, then for every minimal $\mathbb{G}$-submanifold M, the restriction $u\big|_M$ is Δ-subharmonic where Δ is the Laplace–Beltrami operator in the induced riemannian metric on M.*

This result can be extended to submanifolds M of dimension larger that p. Let $\mathbb{G}_M \equiv \{W \in G(p, TM) : W \in \mathbb{G}\}$ denote the set of *tangential $\mathbb{G}$-planes to M*. This set $\mathbb{G}_M$ defines a notion of $\mathbb{G}_M$-plurisubharmonicity for functions $w \in \mathrm{USC}(M)$.

Definition 6.8. We say that $\mathbb{G}$ is *regular* if at every point $x_0 \in X$, each element $W_0 \in \mathbb{G}_{x_0}$ has a local smooth extension to a section $W(x)$ of $\mathbb{G}$.

Definition 6.9. A submanifold M of X is $\mathbb{G}$-*flat* if the second fundamental form B of M satisfies

$$\mathrm{tr}\left(B\big|_W\right) = 0 \quad \text{for all tangential } \mathbb{G} \text{ planes } W \in \mathbb{G}_M. \tag{6.3}$$

Theorem 6.10. *Suppose M is a $\mathbb{G}$-flat submanifold of X and that the subset $\mathbb{G}_M \subset G(p, TM)$ is regular on M. If $u \in \mathrm{PSH}_{\mathbb{G}}(X)$, then $u\big|_M \in \mathrm{PSH}_{\mathbb{G}_M}(M)$.*

See Section 8 of [11] for a more complete discussion, including Example 8.4, which shows that $\mathbb{G}_M$ being regular is necessary in Theorem 6.10. The proof uses Lemma 8.3 in [11] which is stated in this paper as Proposition 8.4 below.

7. $\mathbb{G}$-harmonic functions and the Dirichlet problem

In this section we discuss the Dirichlet problem for $\mathbb{G}$-*harmonic* functions (sometimes referred to as *extremal* or *maximal* functions). These are natural generalizations of solutions of the classical homogeneous Monge–Ampère problem, in both the real and complex cases, and they constitute a very special case of the general F-harmonic functions treated in [10]. To do this we must introduce the *Dirichlet dual*.

7.1. Dually $\mathbb{G}$-plurisubharmonic functions

We first define the *Dirichlet dual* of the subset $F \equiv \mathcal{P}(\mathbb{G}) \subset \mathrm{Sym}^2(TX)$, to be the subset $\widetilde{F} \equiv \widetilde{\mathcal{P}(\mathbb{G})} \subset \mathrm{Sym}^2(TX)$ whose fibers are given by

$$\widetilde{F_x} = -(\sim \mathrm{Int} F_x) = \sim(-\mathrm{Int} F_x). \tag{7.1}$$

Since

$$A \in \mathrm{Int} F_x \iff \mathrm{tr}_W A > 0 \quad \text{for all } W \in \mathbb{G}_x, \tag{7.2}$$

it is easy to see that

$$A \in \widetilde{F_x} \iff \mathrm{tr}_W A \geq 0 \quad \text{for some } W \in \mathbb{G}_x. \tag{7.3}$$

Definition 7.1. A smooth function u on X is said to be *dually* $\mathbb{G}$*-plurisubharmonic* if at each point $x \in X$

$$\exists W \in \mathbb{G}_x \quad \text{with } \mathrm{tr}_W \mathrm{Hess}_x u \ \geq\ 0, \quad \text{or equivalently} \quad \mathrm{Hess}_x u \in \widetilde{\mathcal{P}(\mathbb{G})}.$$

More generally a function $u \in \mathrm{USC}(X)$ is *dually* $\mathbb{G}$*-plurisubharmonic* if for each point $x \in X$ and each test function φ for u at x,

$$\exists W \in \mathbb{G}_x \quad \text{with } \mathrm{tr}_W \mathrm{Hess}_x \varphi \ \geq\ 0, \quad \text{or equivalently} \quad \mathrm{Hess}_x \varphi \in \widetilde{\mathcal{P}(\mathbb{G})}.$$

The set of all such functions is denoted $\widetilde{\mathrm{PSH}}_{\mathbb{G}}(X)$.

First note that $\widetilde{\mathcal{P}(\mathbb{G})}$ satisfies the positivity condition (P), so that as noted in Remark 6.3, if a smooth function u satisfies $\mathrm{Hess}_x u \in \widetilde{\mathcal{P}(\mathbb{G})}$, then for each test function φ for u at x, we have $\mathrm{Hess}_x \varphi \in \widetilde{\mathcal{P}(\mathbb{G})}$, making the second definition an extension of the first definition. Second, assuming that $\pi\big|_{\mathbb{G}}$ is a local surjection as in Definition A.5, it then follows that not only $\mathcal{P}(\mathbb{G})$, but also $\widetilde{\mathcal{P}(\mathbb{G})}$ is closed. As a consequence,

$$\text{the set } \widetilde{\mathrm{PSH}}_{\mathbb{G}}(X) \text{satisfies all of the properties given in Theorem 6.5.} \tag{7.4}$$

In fact $\widetilde{\mathcal{P}(\mathbb{G})}$ is a subequation (Definition A.2).

By Theorem 4.2 if $\mathrm{Core}_{\mathbb{G}}(X) = \emptyset$, then X admits a smooth function ψ which is strictly $\mathbb{G}$-plurisubharmonic at each point. Of course, $\mathcal{P}(\mathbb{G}) \subset \widetilde{\mathcal{P}(\mathbb{G})}$, so that the dually $\mathbb{G}$-plurisubharmonic functions on X constitute a much larger class than the $\mathbb{G}$-plurisubharmonic functions. Again we assume that $\pi\big|_{\mathbb{G}}$ is a local surjection.

Theorem 7.2 (*The Maximum Principle for Dually* $\mathbb{G}$*-Plurisubharmonic Functions*)**.** *Suppose* $\mathrm{Core}_{\mathbb{G}}(X) = \emptyset$*. Then for each compact subset* $K \subset X$ *and each* $u \in \widetilde{\mathrm{PSH}}_{\mathbb{G}}(K) \equiv \mathrm{USC}(X) \cap \widetilde{\mathrm{PSH}}_{\mathbb{G}}(\mathrm{Int} K)$ *we have:*

$$\sup_K u \leq \sup_{\partial K} u.$$

The proof is classical and completely elementary. Moreover, one can easily see that this maximum principle is equivalent to the special case of comparison (Theorem 7.7 below) where u is smooth.

Proof. Suppose it fails. Then there exist a compact set K, a function $u \in \widetilde{\mathrm{PSH}}_{\mathbb{G}}(K)$ and a point $\bar{x} \in \mathrm{Int}K$ with $u(\bar{x}) > \sup_{\partial K} u$. Let ψ be a smooth strictly $\mathbb{G}$-psh function on X. Then for $\epsilon > 0$ sufficiently small, the function $u + \epsilon\psi$ will also have a maximum at some point $x \in \mathrm{Int}K$. Thus $-\epsilon\psi$ is a test function for u at x, and therefore $\mathrm{Hess}_x(-\epsilon\psi) \in \widetilde{\mathcal{P}_x(\mathbb{G})} = -({\sim}\mathrm{Int}\mathcal{P}_x(\mathbb{G}))$, i.e., $\mathrm{Hess}_x(\psi) \notin \mathrm{Int}\mathcal{P}_x(\mathbb{G})$ contradicting the strictness of ψ at x. □

The Convex-Increasing Composition Property in Exercise 2.1 not only extends to the upper semi-continuous case, but also to the much larger class of dually $\mathbb{G}$-plurisubharmonic functions.

Lemma 7.3 (*Composition Property*)**.** *Suppose $\varphi : \mathbf{R} \to \mathbf{R}$ is both convex and increasing (i.e., non-decreasing). Then*

$$u \in \widetilde{\mathrm{PSH}_{\mathbb{G}}}(X) \;\Rightarrow\; \varphi \circ u \in \widetilde{\mathrm{PSH}_{\mathbb{G}}}(X). \tag{a}$$

If φ is also strictly increasing, then in addition to (a) we have that

$$u \text{ is } \mathbb{G} \text{ strict} \;\Rightarrow\; \varphi \circ u \text{ is } \mathbb{G} \text{ strict} \tag{b}$$

where we refer ahead to Definition 7.7 for the notion of strictness.

Proof. We can assume that φ is smooth since it can be approximated by a decreasing sequence φ_ϵ via convolution. Observe now that:

$$\psi \textit{ is a test function for } u \textit{ at } x \iff \varphi \circ \psi \textit{ is a test function for } \varphi \circ u \textit{ at } x.$$

This reduces the proof to the case where φ and u are both smooth, and formula (2.5) applies with both coefficients $\varphi'(u(x))$ and $\varphi''(u(x)) \geq 0$. □

7.2. $\mathbb{G}$-harmonics

To understand the next definition note that

$$\partial\mathcal{P}(\mathbb{G}) = \mathcal{P}(\mathbb{G}) \cap (-\widetilde{\mathcal{P}(\mathbb{G})}). \tag{7.5}$$

Definition 7.4. A function u on X is said to be $\mathbb{G}$-*harmonic* if

$$u \in \mathrm{PSH}_{\mathbb{G}}(X) \quad \text{and} \quad -u \in \widetilde{\mathrm{PSH}}_{\mathbb{G}}(X).$$

By (7.5) we see that a C^2-function u on X is $\mathbb{G}$-harmonic if and only if

$$\mathrm{Hess}_x u \in \partial\mathcal{P}(\mathbb{G}_x) \quad \text{for all } x \in X.$$

In order to solve the Dirichlet Problem for $\mathbb{G}$-harmonic functions on domains $\Omega \subset X$, we restrict $\mathbb{G} \subset G(p, TX)$ to be modeled on a "constant coefficient" case $\mathbb{G}_0 \subset G(p, \mathbf{R}^n)$.

Definition 7.5. A closed subset $\mathbb{G} \subset G(p, TX)$ is *locally trivial with fiber* $\mathbb{G}_0 \subset G(p, \mathbf{R}^n)$, if in a neighborhood each point $x \in X$ there exists a local tangent frame field so that under the associated trivialization $\phi : G(p, TU) \xrightarrow{\cong} U \times G(p, \mathbf{R}^n)$ we have

$$\phi : \mathbb{G}\big|_U \xrightarrow{\cong} U \times \mathbb{G}_0.$$

This can be formulated somewhat differently. Let $\mathrm{Aut}(\mathbb{G}_0) = \{g \in \mathrm{GL}_n : g(\mathbb{G}_0) = \mathbb{G}_0\}$. Then given a closed subset $\mathbb{G} \subset G(p, TX)$ which is locally trivial with fiber $\mathbb{G}_0$, the local tangent frame fields in Definition 7.5 provide X with a topological $\mathrm{Aut}(\mathbb{G}_0)$-structure (see Section 5 in [10]). Conversely, if X admits a topological $\mathrm{Aut}(\mathbb{G}_0)$-structure, then the euclidean model $\mathbb{G}_0 \subset G(p, \mathbf{R}^n)$ determines a canonical closed subset $\mathbb{G} \subset G(p, TX)$ which is locally trivial with fiber $\mathbb{G}_0$. In other words, a euclidean model can be transplanted to any manifold with a topological $\mathrm{Aut}(\mathbb{G}_0)$-structure (again see Section 5 in [10]).

In the language of [10, Section 6]: "$\mathbb{G}$ is locally trivial with fiber $\mathbb{G}_0$" means that the subequation $\mathcal{P}(\mathbb{G})$ is locally *jet equivalent* to the constant coefficient subequation $\mathcal{P}(\mathbb{G}_0)$.

In the next two theorems X is a riemannian manifold and $\mathbb{G} \subset G(p, TX)$ is a closed, locally trivial set with non-empty fiber.

Theorem 7.6 (*The Dirichlet Problem*)**.** *Suppose that $\Omega \subset\subset X$ is a domain with a smooth, strictly $\mathbb{G}$-convex boundary $\partial\Omega$ and* $\mathrm{Core}_{\mathbb{G}}(\Omega) = \emptyset$. *Then the Dirichlet problem for $\mathbb{G}$-harmonic functions is uniquely solvable on Ω. That is, for each $\varphi \in C(\partial\Omega)$, there exists a unique $\mathbb{G}$-harmonic function $u \in C(\overline{\Omega})$ such that*

(i) $u\big|_{\Omega}$ *is $\mathbb{G}$-harmonic, and*
(ii) $u\big|_{\partial\Omega} = \varphi$.

This is the special case Theorems 16.1 of Theorem 13.1 in [10]. There are many interesting examples. See [10] for a long list.

Boundary convexity is not required for uniqueness, only an empty core for X. As usual uniqueness is immediate from comparison.

Theorem 7.7 (*Comparison*)**.** *Suppose that* $\mathrm{Core}_{\mathbb{G}}(X) = \emptyset$ *and $K \subset X$ is compact. If* $u \in \mathrm{PSH}_{\mathbb{G}}(K)$ *and* $v \in \widetilde{\mathrm{PSH}}_{\mathbb{G}}(K)$*, then the zero maximum principle holds, that is,*

$$u + v \le 0 \text{ on } \partial K \;\Rightarrow\; u + v \le 0 \quad \text{on } K. \tag{ZMP}$$

Outline of proof. By definition $u, v \in \mathrm{USC}(K)$ and on the interior $\mathrm{Int}K$, u is $\mathbb{G}$-plurisubharmonic and v is dually $\mathbb{G}$-plurisubharmonic. The appropriate notion of *strict* plurisubharmonicity for general upper semi-continuous functions plays a crucial role, and will be discussed below after outlining its importance. If (ZMP) holds for all compact $K \subset X$ under the additional assumption that u is $\mathbb{G}$-strict, we say that *weak comparison holds for* $\mathbb{G}$ *on* X. This weakened version of comparison has one big advantage, namely that *local implies global* (Theorem 8.3 in [10]). The proof of completed by showing two things. First,

$$\text{Weak comparison is true locally.} \tag{7.6}$$

This follows by a argument based on the "Theorem on Sums" — see Section 10 in [10]. Second, *strict approximation* holds. That is, since $\mathrm{Core}_{\mathbb{G}}(X) = \emptyset$, X supports a C^2 strictly $\mathbb{G}$-plurisubharmonic function ψ, and

$$\begin{aligned}&\text{If } u \text{ is } \mathbb{G}\text{-plurisubharmonic,}\\ &\text{then } u + \epsilon\psi \text{ is strictly } \mathbb{G}\text{-plurisubharmonic,} \quad \text{for each } \epsilon > 0.\end{aligned} \tag{7.7}$$

This follows easily from the definition of strictness. Using weak comparison and strict approximation, one shows that in the limit comparison holds. $\square$

7.3. Strictness

Definition 7.8. A function $u \in \mathrm{USC}(X)$ is *strictly* $\mathbb{G}$*-plurisubharmonic* if each point in X has a neighborhood U along with a constant $c > 0$ such that for each point $x \in U$ and each test function φ for u at x

$$\mathrm{tr}_W \mathrm{Hess}_x \varphi \geq c \quad \text{for all } W \in \mathbb{G}_x. \tag{7.8}$$

To see that this definition of strict agrees with the one given in [10, Definition 7.4] one must compare (7.8) with distance in $\mathrm{Sym}^2(T_xX)$. For this first note that for $W \in G(p, T_xX)$ the (signed) distance of a point $A \in \mathrm{Sym}^2(T_xX)$ to the boundary of the positive half-space defined by the unit normal $\frac{1}{p}P_W$ is simply $\left\langle A, \frac{1}{p}P_W\right\rangle$. Consequently, the distance from $A \in \mathcal{P}(\mathbb{G}_x)$ to $\sim\mathcal{P}(\mathbb{G}_x)$ is given by

$$\mathrm{dist}(A, \sim\mathcal{P}(\mathbb{G}_x)) = \inf_{W\in\mathbb{G}_x}\left\langle A, \frac{1}{p}P_W\right\rangle = \inf_{W\in\mathbb{G}_x}\frac{1}{p}\mathrm{tr}_W A. \tag{7.9}$$

For each fixed $c > 0$, c-strictness is a subequation. Therefore, all the properties in Theorem 6.5 hold for c-strict $\mathbb{G}$-plurisubharmonic functions. Moreover, as noted in Lemma 7.3, if φ is convex and strictly increasing, the composition property holds. Finally, strictness is “stable”.

Lemma 7.9 (C^∞*-Stability Property*)**.** *Suppose u is strictly $\mathbb{G}$-plurisubharmonic and $\psi \in C^\infty(X)$ with compact support. Then $u + \epsilon\psi$ is strictly $\mathbb{G}$-plurisubharmonic for all ϵ sufficiently small.*

Proof. This is Corollary 7.6 in [10]. $\square$

8. Geometric subequations involving all the variables

This is a concept which distinguishes, for example, the full Laplacian on $\mathbf{R}^n$, which involves all the variables, from the pth partial Laplacian Δ_p, which does not. We shall first treat the euclidean case (see Section 2 of [8]). The results will then be carried over to a general riemannian manifold X.

Fix a finite dimensional inner product space V and suppose $\mathbb{G} \subset G(p, V)$ is a closed subset of the Grassmannian. Let $\mathrm{Span}\,\mathbb{G}$ denote the span in $\mathrm{Sym}^2(V)$ of the elements P_W with $W \in \mathbb{G}$, and let $\mathcal{P}_+(\mathbb{G})$ denote the convex cone on $\mathbb{G}$ with vertex at the origin in $\mathrm{Sym}^2(V)$. Examples show that $\mathrm{Span}\,\mathbb{G}$ is often a proper vector subspace of $\mathrm{Sym}^2(V)$, in which case $\mathcal{P}_+(\mathbb{G})$ will have no interior in $\mathrm{Sym}^2(V)$. However, considered as a subset of the vector space $\mathrm{Span}\,\mathbb{G}$, the interior of $\mathcal{P}_+(\mathbb{G})$ has closure equal to $\mathcal{P}_+(\mathbb{G})$. We define $\mathrm{Int}_0\mathcal{P}_+(\mathbb{G})$ to be the interior of $\mathcal{P}_+(\mathbb{G})$ in $\mathrm{Span}\,\mathbb{G}$ (not in $\mathrm{Sym}^2(V)$). In particular, $\mathrm{Int}_0\mathcal{P}_+(\mathbb{G})$ is never empty, and $\mathcal{P}_+(\mathbb{G}) = \overline{\mathrm{Int}_0\mathcal{P}_+(\mathbb{G})}$.

By Definition 2.1, $\mathcal{P}(\mathbb{G}) = \{B \in \mathrm{Sym}^2(V) : \langle B, P_W\rangle \geq 0 \text{ for all } W \in \mathbb{G}\}$. Hence, $\mathcal{P}(\mathbb{G}) \subset H(A)$ for each closed half-space $H(A) \equiv \{B \in \mathrm{Sym}^2(V) : \langle A, B\rangle \geq 0\}$ determined by a non-zero $A \in \mathcal{P}_+(\mathbb{G})$. This proves that

$$\mathcal{P}(\mathbb{G}) = \bigcap_{A\in\mathcal{P}_+(\mathbb{G})} H(A),$$

i.e., $\mathcal{P}(\mathbb{G})$ is the “polar” of $\mathcal{P}_+(\mathbb{G})$. (Therefore, by the Hahn–Banach/Bipolar Theorem $\mathcal{P}_+(\mathbb{G})$ is the polar of $\mathcal{P}(\mathbb{G})$.)

Since $\mathcal{P}_+(\mathbb{G}) = \overline{\mathrm{Int}_0\mathcal{P}_+(\mathbb{G})}$, this intersection can be taken over the smaller set of $A \in \mathrm{Int}_0\mathcal{P}_+(\mathbb{G})$. That is,

$$\mathcal{P}(\mathbb{G}) = \bigcap_{A\in\mathrm{Int}_0\mathcal{P}_+(\mathbb{G})} H(A). \tag{8.1}$$

This is what will be used below, since the involvement of all the variables in $\mathbb{G}$ insures that such A are positive definite, i.e., the linear operators $\langle A, D^2u\rangle$ are uniformly elliptic.

The linear operator $\Delta_A u \equiv \langle A, D^2u\rangle$ with $A \geq 0$ will be referred to as the *A-Laplacian*. Note that from our set theoretic point of view, the subequation $\Delta_A \subset \mathrm{Sym}^2(V)$ is precisely the closed half-space $H(A)$.

The following is a restatement of Proposition 2.8 in [8] (see also Remark 4.8, p. 874 of [15]).

Lemma 8.1. *The following are equivalent ways of defining the concept that $\mathbb{G}$ involves all the variables.*

(1) *The only vector $v \in \mathrm{Sym}^2(V)$ with $v \perp W$ for all $W \in \mathbb{G}$ is $v = 0$.*
(2) *For each unit vector $e \in V$, P_e is never orthogonal to* $\mathrm{Span}\mathbb{G}$.
(3) *There does not exist a hyperplane $W \subset V$ with $\mathbb{G} \subset \mathrm{Sym}^2(W) \subset \mathrm{Sym}^2(V)$.*
(4) $\mathrm{Int}_0\mathcal{P}_+(\mathbb{G}) \subset \mathrm{Int}\mathcal{P}$*, i.e., each $A \in \mathrm{Int}_0\mathcal{P}_+(\mathbb{G})$ is positive definite.*
(5) *There exists $A \in \mathrm{Span}\,\mathbb{G}$ with $A > 0$.*

In Section 2 of [8] such subsets $\mathbb{G}$ were called "elliptic".

We shall apply Lemma 8.1 to the case $V = T_xX$ on a riemannian manifold X. We say that a closed subset $\mathbb{G} \subset G(p, TX)$ *involves all the variables* if each fiber $\mathbb{G}_x \subset G(p, T_xX)$ involves all the variables in the vector space $V \equiv T_xX$. For any smooth section $A(x) \geq 0$ of $\mathrm{Sym}^2(TX)$ the linear operator

$$\Delta_A u \equiv \langle A(x), \mathrm{Hess}_x u\rangle$$

will again be referred to as the *A-Laplacian*.

Recall from Definition 6.8 that $\mathbb{G}$ *is regular* if each element $W_0 \in \mathbb{G}_x$ can be locally extended to a smooth section $W(y)$ of $\mathbb{G}$. This immediately implies that *each element $A_0 \in \mathcal{P}_+(\mathbb{G}_x)$ can be locally extended to a smooth section $A(y)$ with $A(y) \in \mathcal{P}_+(\mathbb{G}_y)$*, (since $A_0 = \sum_k t_k W_k$ for $t_k > 0$ and $W_k \in \mathbb{G}_x$). Furthermore, if $A(x) > 0$, then $A(y) > 0$ for y near x. This proves the following.

Lemma 8.2. *Suppose $\mathbb{G} \subset G(p, TX)$ is a closed subset involving all the variables and that $\mathbb{G}$ is regular. Then*

$$\mathcal{P}(\mathbb{G}_x) = \bigcap H(A(x)) \quad \text{for each } x \in X \tag{8.1$'$}$$

where the intersection is taken over all smooth $\mathcal{P}_+(\mathbb{G})$-valued section $A(y)$ where $A(y) > 0$ for y near x.

Theorem 8.3. *A function $u \in \mathrm{USC}(X)$ is $\mathbb{G}$-plurisubharmonic $\iff$ u is Δ_A-subharmonic for each smooth (local) section A of $\mathrm{Sym}^2(TX)$ with values in $\mathcal{P}_+(\mathbb{G})$ and $A > 0$.*

Proof. If A is a section of $\mathcal{P}_+(\mathbb{G})$, then $\mathcal{P}_+(\mathbb{G}) \subset \Delta_A$ over a neighborhood U of x, so that each $\mathbb{G}$-plurisubharmonic function on U is automatically Δ_A-subharmonic. Conversely, if u is Δ_A-subharmonic for each (local) section A of $\mathcal{P}_+(\mathbb{G})$ with $A > 0$, and if φ is a test function for u at x, then $\mathrm{Hess}_x\varphi \in H(A(x))$, and therefore by (8.1′), $\mathrm{Hess}_x\varphi \in \mathcal{P}(\mathbb{G}_x)$. $\square$

Note 8.4. The simple argument just given also shows the following. *Suppose F is a subequation on X which can be written as an intersection of subequations $F = \bigcap_\alpha F_\alpha$. Then for $u \in$* $\mathrm{USC}(X)$, *u is F-subharmonic if and only if u* is *F_α-subharmonic for all α.*

Theorem 8.3 has many consequences. We mention one.

Theorem 8.5 (*The Strong Maximum Principle for $\mathbb{G}$-Plurisubharmonic Functions*)**.** *Suppose $\mathbb{G} \subset G(p, TX)$ is regular and involves all the variables. Then for any compact subset K with* $\mathrm{Int}K$ *connected and $K = \overline{\mathrm{Int}K}$, if $u \in \mathrm{PSH}_{\mathbb{G}}(K)$ has an interior maximum point, then $u\big|_K$ is constant.*

Proof. Unlike the maximum principle, if the strong maximum principle is true locally, it is true globally. However, locally we have $\mathcal{P}(\mathbb{G}) \subset \Delta_A$ with $A > 0$, so the (SMP) for Δ_A implies the (SMP) for $\mathcal{P}(\mathbb{G})$. $\square$

We provide an example which shows that if the core is non-empty and the equation does not involve all the variables, then the (MP), and hence the (SMP) can fail.

Example 8.6. Let $X \subset \mathbf{R}^{n+1}$ be the unit sphere $S^n = \{(x_1, \dots, x_n, y) \in \mathbf{R}^n \times \mathbf{R} : x_1^2 + \cdots + x_n^2 + y^2 = 1\}$ with the points $y = \pm 1$ removed. Let $H = \ker\left(dy\big|_{TX}\right)$ be the field of "horizontal" $(n-1)$-planes on X tangent to the foliation by the latitudinal spheres $\{y = \text{constant}\}$, and set $\mathbb{G}_z = \{H(z)\}$ for $z \in S^n$ so that $\mathbb{G} \subset G(n-1, TX)$. Calculation shows that for a smooth function φ defined in a neighborhood of X,

$$(\mathrm{Hess}^X \varphi)(V, W) = (\mathrm{Hess}^{\mathbf{R}^{n+1}} \varphi)(V, W) - \langle V, W \rangle\, \nu \cdot \varphi$$

where ν is the outward-pointing unit normal to X.

Now let $\varphi = \frac{1}{2}(1 - y^2)$. Then for $V, W \in H(z)$ horizontal vector fields, the first term vanishes and the second term yields

$$(\mathrm{Hess}^X \varphi)(V, W) = y^2 \langle V, W \rangle .$$

Hence $\mathrm{tr}_W\{\mathrm{Hess}^X \varphi\} = (n-1)y^2$, proving that $\varphi \in \mathrm{PSH}^\infty_{\mathbb{G}}(X)$ and that it is $\mathbb{G}$-strict outside $y = 0$. Therefore, *the maximum principle fails for $\mathbb{G}$-plurisubharmonic functions on any domain $\Omega \subset\subset X$ which contains $S_0^{n-1} \equiv \{y = 0\}$ in its interior.* For any such domain,

$$S_0^{n-1} \subset \mathrm{Core}(\Omega)$$

because S_0^{n-1} is a compact minimal $\mathbb{G}$-submanifold and therefore any $\mathbb{G}$-plurisubharmonic function restricted to it must be constant (see Theorem 6.9).

Note that $\mathrm{tr}_H\{\mathrm{Hess}^X u\} \geq 0$ is a linear subequation of constant rank and therefore locally jet equivalent to the partial Laplacian Δ_n in local coordinates (Proposition B.3). Consequently, this subequation satisfies weak local comparison (see the discussion of the proof of Theorem 7.7). However, it does not satisfy comparison since it does not satisfy the maximum principle.

We note that the maximum principle also fails for the subequation consisting of all the p-dimensional linear subspaces of $\mathbb{G}$ (given above), for any p, $1 \leq p \leq n-1$.

9. Distributionally $\mathbb{G}$-plurisubharmonic functions

It is easy to see that for the pth partial Laplacian Δ_p on $V = \mathbf{R}^n$, $p < n$, there are lots of distributional subharmonics (i.e., distributions u with $\Delta_p u$ a non-negative measure) which are not represented by upper semi-continuous functions, and hence cannot be horizontally subharmonic. However, if a closed set $\mathbb{G} \subset G(p, V)$ involves all the variables, then the appropriate distributional definition of $\mathbb{G}$-plurisubharmonicity, although technically not equal, is, in a sense to be made precise, equivalent to Definition 6.1. This constant coefficient result was proved in Corollary 5.4 of [8]. In this section we extend the result to the variable coefficient case.

First we give the distributional definition.

Definition 9.1. A distribution $u \in \mathcal{C}'(X)$ on a riemannian manifold X is *distributionally* $\mathbb{G}$*-plurisubharmonic* if $\Delta_A u \geq 0$ (a non-negative measure) for every smooth section $A(x)$ of $\mathrm{Sym}^2(TX)$ taking values in $\mathcal{P}_+(\mathbb{G})$.

This distributional notion cannot be the "same" as $\mathbb{G}$-plurisubharmonicity, but it is equivalent in the following precise sense. We exclude the $\mathbb{G}$-plurisubharmonic functions which are $\equiv -\infty$ on any component of X. Let $L^1_{\mathrm{loc}}(X)$ denote the space of locally integrable functions on X.

Theorem 9.2. *Assume that $\mathbb{G} \subset G(p, TX)$ involves all the variables and is regular.*

(a) *Suppose u is $\mathbb{G}$-plurisubharmonic. Then $u \in L^1_{\mathrm{loc}}(X) \subset \mathcal{C}'(X)$, and u is distributionally $\mathbb{G}$-plurisubharmonic.*
(b) *Suppose $v \in \mathcal{C}'(X)$ is distributionally $\mathbb{G}$-plurisubharmonic. Then $v \in L^1_{\mathrm{loc}}(X)$, and there exists a unique upper semi-continuous representative u of the $L^1_{\mathrm{loc}}(X)$-class v which is $\mathbb{G}$-plurisubharmonic. In fact,*

$$u(x) = \operatorname{ess}\limsup_{y \to x} v(x)$$

is actually independent of $\mathbb{G}$.

Proof. Under the hypothesis of Theorem 9.2 we can use the next proposition along with Theorem 8.3 to reduce to proving the analogous result for A-Laplacians Δ_A where $A(x)$ is a smooth section of $\mathrm{Sym}^2(TX)$ having the additional property that $A(x) > 0$, i.e., Δ_A is uniformly elliptic. $\square$

Proposition 9.3. *A distribution $v \in \mathcal{D}'(X)$ is distributionally $\mathbb{G}$-plurisubharmonic $\iff$ v is distributionally Δ_A-subharmonic for each smooth (local) section A of* $\mathrm{Sym}^2(TX)$ *with values in $\mathcal{P}_+(\mathbb{G})$ and $A > 0$.*

Proof. Suppose A is a local smooth section of $\mathcal{P}_+(\mathbb{G})$ with $A(y) > 0$ as in Definition 8.1. Fix $x \in X$ and note that since $\mathbb{G}_x$ involves all the variables, there exists $S_0 \in \mathrm{Int}_0\mathcal{P}_+(\mathbb{G}_x)$ and $S_0 > 0$ (Lemma 8.1(4)). By the regularity of $\mathbb{G}$ there exists a local section $S(y)$ of $\mathcal{P}_+(\mathbb{G})$ extending S_0. Since $S_0 > 0$, we have that $S(y) > 0$ in a neighborhood U of x. Now for each $\epsilon > 0$, $(A + \epsilon S)(y) > 0$ on U. That is, locally any smooth section taking values in $\mathcal{P}_+(\mathbb{G})$ can be approximated by $\mathcal{P}_+(\mathbb{G})$-valued sections which are positive definite. Assuming $\Delta_{A+\epsilon S} u \geq 0$, this implies $\Delta_A u \geq 0$. $\square$

Completion of the Proof of Theorem 9.2. First note that this is a local result. Note that for each positive definite $\mathcal{P}_+(\mathbb{G})$-valued section $A(x)$, the A-Laplacian Δ_A is of the form

$$\Delta_A u = a(x) \cdot D_x^2 u + b(x) \cdot D_x u$$

where $a(x)$ is a positive definite $n \times n$ matrix and $b(x)$ is $\mathbf{R}^n$-valued. Now the analogue of Theorem 9.2, with $\mathbb{G}$-plurisubharmonicity replaced by Δ_A-subharmonicity, is true. Details can be found in Appendix A of [9]. An important point in the proof of Theorem 9.2(b) is that the upper semi-continuous representative u provided by Appendix A in [9] for a Δ_A-subharmonic distribution v is the same for all sections $A(x) > 0$, since it is the ess-limsup regularization of the L^1_{loc}-class v. $\square$

Remark 9.4. The Δ_A-harmonics are smooth, and the notion of Δ_A-subharmonicity is also equivalent to the self-defining notion "sub-the-Δ_A-harmonics" — again see Appendix A in [9].

The following gives an easily verified criterion for the regularity of $\mathbb{G}$.

Exercise 9.5. Suppose $\mathbb{G} \subset G(p, TX)$ is a closed subset which is a smooth fiber-wise neighborhood retract in $G(p, TX)$. Then $\mathbb{G}$ is regular.

Also note that $\mathbb{G}$ is a smooth fiber-wise neighborhood retract in $G(p, TX)$ if and only if it is a smooth fiber-wise neighborhood retract in $\mathrm{Sym}^2(TX)$.

9.1. Strictness

Recall that $\mathbb{G}$-strictness for $u \in \mathrm{USC}(X)$ was defined in Section 7. The requirement was that locally there exists $c > 0$ with uc-strict as defined by (7.8). Theorem 8.3 extends to c-strictness as follows.

Proposition 9.6. *A function $u \in \mathrm{USC}(X)$ is c-strictly $\mathbb{G}$-plurisubharmonic $\iff$ u is a c-strict Δ_A-subharmonic function for each smooth (local) section A of $\mathcal{P}_+(\mathbb{G})$ with $A > 0$ at each point.*

By *u is c-strict for Δ_A* we mean that at each point x and for each viscosity test function φ for u at x, we have $(\Delta_A \varphi)(x) \geq c$.

A distribution $v \in \mathcal{C}'(X)$ is said to be *c-strict for Δ_A* (an $A \geq 0$ Laplacian) if

$$\Delta_A v \geq c \quad \text{(as an inequality of measures).} \tag{9.1}$$

If this inequality is true for every smooth section A of $\mathcal{P}_+(\mathbb{G})$, then v is *c-strict as a $\mathbb{G}$-plurisubharmonic distribution*. Proposition 9.3 easily extends to

Proposition 9.7. *A distribution $v \in \mathcal{D}'(X)$ is c-strict for $\mathbb{G}$ $\iff$ v is c-strict for Δ_A for each smooth section A of $\mathcal{P}_+(\mathbb{G})$ which is positive definite.*

Since c-strictness for the A-Laplacian, when A is positive definite, can be show to be equivalent whether interpreted with viscosity test functions or distributional test functions, Theorem 9.2 has a obvious extension to the c-strict case ($c > 0$). The remainder of the proof is left to the reader, but here is the statement.

Theorem 9.8. *In either part (a) or part (b) of Theorem 9.2, if the function in the hypothesis is assumed to be c-strict, one has c-strictness in the conclusion.*

Finally we state a result, due to Richberg [16] in the complex case, which carries over to the $\mathbb{G}$-plurisubharmonic case, assuming the following local approximation is possible.

Definition 9.9. We say that $\mathbb{G}$ has the *local C^∞-approximation property* if each point $x \in X$ has a neighborhood U such that for all $u \in C(U) \cap \mathrm{PSH}_{\mathbb{G}}(U)$ which are c-strict, and all compact $K \subset U$ and $\epsilon > 0$, there exists $\widetilde{u} \in \mathrm{PSH}^\infty_{\mathbb{G}}(U)$ which is c-strict, with $u \leq \widetilde{u} \leq u + \epsilon$ on K.

Theorem 9.10. *Suppose $\mathbb{G}$ has the local C^∞ strict approximation property, and let $c, \epsilon \in C(X)$ be any given continuous functions satisfying $c > 0$ and $1 > \epsilon > 0$ on X. If $u \in C(X) \cap \mathrm{PSH}_{\mathbb{G}}(X)$ is c-strict, then there exists $\widetilde{u} \in \mathrm{PSH}^\infty_{\mathbb{G}}(X)$, which is $(1-\epsilon)c$-strict, with*

$$u \le \widetilde{u} \le u + \epsilon \quad \text{on } X.$$

The proof in Chapter I, Section 5 of [3], given in the complex case, carries over to this much more general case (see also [4]).

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Appendix A. Geometric subequations

Let X be a riemannian manifold and consider a closed subset

$$\mathbb{G} \subset G(p, TX)$$

of the Grassmannian of tangent p-planes. The natural candidate for a subequation $F = F(\mathbb{G})$ associated with $\mathbb{G}$ is defined by its fibers

$$F_x = \{A \in \mathrm{Sym}^2(T_x X) : \mathrm{tr}_W A \ge 0 \ \forall W \in \mathbb{G}_x\}. \tag{A.1}$$

For each $W \in \mathbb{G}_x$ the condition $\mathrm{tr}_W A \ge 0$ defines a closed half-space (with boundary a hyperplane through the origin). This is because under the natural inner product $\langle A, B\rangle = \mathrm{tr}\, AB$ on $\mathrm{Sym}^2(T_x X)$, one has $\mathrm{tr}_W A = \langle A, P_W\rangle$, where $P_W \in \mathrm{Sym}^2(T_x X)$ denotes orthogonal projection onto W. Consequently,

$$F_x \text{ is a closed convex cone with vertex at the origin, and} \tag{A.2}$$

$$\mathrm{Int} F_x = \{A \in \mathrm{Sym}^2(T_x X) : \mathrm{tr}_W A > 0 \ \forall W \in \mathbb{G}_x\}. \tag{A.3}$$

Let $\mathcal{P}_x$ denote the set of non-negative elements in $\mathrm{Sym}^2(T_x X)$. Since $\mathrm{tr}_W P \ge 0$ for all $W \in G(p, T_x X)$ when $P \in \mathcal{P}_x$, the fibers F_x defined by (A.1) satisfy the important *positivity condition*

$$F_x + \mathcal{P}_x \subset F_x. \tag{P}$$

Therefore the fibers F_x satisfy all of the properties of a constant coefficient (euclidean) pure second-order subequation.

Proposition A.1. (1) $F_x + \mathrm{Int}\mathcal{P}_x = \mathrm{Int} F_x$
(2) $F_x = \overline{\mathrm{Int} F_x}$
(3) $\mathrm{Int} F_x + \mathcal{P}_x = \mathrm{Int} F_x$
(4) $A \in \mathrm{Int} F_x \iff$ *there exists a neighborhood of A in F_x of the form $\mathcal{N}_\epsilon(A) \equiv A - \epsilon I + \mathrm{Int}\mathcal{P}_x$ for some $\epsilon > 0$.*

Proof. (4) Note that $\mathcal{N}_\epsilon(A)$ is an open set containing A, and that if $A - \epsilon I \in F_x$, then the positivity condition (P) implies that $\mathcal{N}_\epsilon(A) \subset F_x$.

(1) By positivity $F_x + \mathrm{Int}\mathcal{P}_x \subset F_x$, and it is open since it is the union over $A \in F_x$ of open sets. Hence it is contained in $\mathrm{Int} F_x$. Finally, $\mathrm{Int} F_x \subset F_x + \mathrm{Int}\mathcal{P}_x$ by (4).

(2) If $A \in F_x$, then by (1) we have $A + \epsilon I \in \mathrm{Int} F_x$ for all $\epsilon > 0$. Hence, $A = \lim_{\epsilon\to\infty}(A + \epsilon I) \in \overline{\mathrm{Int} F_x}$ proving that $F_x \subset \overline{\mathrm{Int} F_x}$. Since F_x is closed, we have equality.

(3) The containment "$\subset$" is proved as in the first half of (1). The containment "$\supset$" follows from $0 \in \mathcal{P}_x$. □

Recall the following definition from [10].

Definition A.2. A (general) subset $F \subset \mathrm{Sym}^2(TX)$ is called a *subequation* if it satisfies the *positivity condition:*

$$F_x + \mathcal{P}_x \subset F_x \quad \text{for all } x \in X \tag{P}$$

and the *three topological conditions:*

$$(\mathrm{T}_1)\ F = \overline{\mathrm{Int} F}, \qquad (\mathrm{T}_2)\ F_x = \overline{\mathrm{Int} F_x}, \qquad (\mathrm{T}_3)\ \mathrm{Int} F_x = (\mathrm{Int} F)_x.$$

(Here $\mathrm{Int} F_x$ means the interior relative to the fiber $\mathrm{Sym}^2(T_x X)$.)

Although $F(\mathbb{G})$ is not always closed (see Proposition A.6), we shall see that conditions (T_2) and (T_3) are always true. They will be a consequence of the following half of (T_3).

Lemma A.3. *The condition*

$$(\mathrm{T}_3)' \quad \mathrm{Int} F_x \subset (\mathrm{Int} F)_x$$

holds for any closed subset $\mathbb{G} \subset G(p, TX)$. Consequently, if a smooth function is $\mathbb{G}$-strict at a point, then it is $\mathbb{G}$ strict in a neighborhood of that point.

The proof is given at the end of this appendix.

Corollary A.4. *The set $F = F(\mathbb{G})$ satisfies*

$$(\mathrm{T}_1)'\ F \subset \overline{\mathrm{Int} F}, \qquad (\mathrm{T}_2)\ F_x = \overline{\mathrm{Int} F_x}, \qquad (\mathrm{T}_3)\ \mathrm{Int} F_x = (\mathrm{Int} F)_x.$$

Proof. Condition $(\mathrm{T}_3)'$ implies (T_3) since $(\mathrm{Int} F)_x$ is an open subset of F_x, and hence contained in $\mathrm{Int} F_x$. Property (T_2) is just condition (2) in Proposition A.1. Finally, by (T_2) and (T_3) we have $F_x = \overline{\mathrm{Int} F_x} = \overline{(\mathrm{Int} F)_x} \subset \overline{\mathrm{Int} F}$ which proves $(\mathrm{T}_1)'$. □

We can characterize the case where $F(\mathbb{G})$ is closed.

Definition A.5. The restricted projection $\pi : \mathbb{G} \to X$ is a *local surjection* if for each $W \in \mathbb{G}$ and each neighborhood U of W, the image $\pi(U \cap \mathbb{G})$ contains a neighborhood of $\pi(W)$. In this case we say that $\mathbb{G}$ has the *local surjection property*.

Proposition A.6. *$F(\mathbb{G})$ is closed $\iff$ $\pi : \mathbb{G} \to X$ is a local surjection.*

The proof is given at the end of this appendix.

Putting together the previous results yields, as an immediate consequence, our characterization of geometrically determined subequations.

Theorem A.7. *A closed subset $\mathbb{G} \subset G(p, TX)$ determines a subequation $F(\mathbb{G})$ via (A.1) if and only if $\pi : \mathbb{G} \to X$ is a local surjection.*

Consequently, we adopt the following definition.

Definition A.8. A subset $F \subset \mathrm{Sym}^2(TX)$ is a *geometrically determined subequation* if $F = F(\mathbb{G})$ with $\mathbb{G}$ a closed subset of $\mathrm{Sym}^2(TX)$ having the local surjection property.

A.1. Strictness

The concept of strictness given in Definition 7.8 plays an important role for upper semi-continuous functions, not just smooth functions (see Definition 2.1′) where the notion is unambiguous.

Definition A.9 (*c-Strict*)**.** For each $c > 0$ define $F^c = F^c(\mathbb{G})$ to be the subset of $\mathrm{Sym}^2(TX)$ with fibers

$$F_x^c \equiv \{A \in \mathrm{Sym}^2(T_xX) : \mathrm{tr}_W A \geq c \ \forall W \in \mathbb{G}_x\}. \tag{A.4}$$

The identity I is a well defined smooth section of $\mathrm{Sym}^2(TX)$, and $\mathrm{tr}_W I = p$ for all $W \in G(p, TX)$. Therefore,

$$F^c = F + \frac{c}{p} \cdot I \quad \text{(fibrewise sum)}. \tag{A.5}$$

Consequently, all of the previous results for F remain true for F^c ($c \geq 0$). In particular, we have:

Theorem A.10. *If $\mathbb{G} \subset G(p, TX)$ is a closed subset with the local surjection property, then for each $c \geq 0$ the set $F^c(\mathbb{G})$ is a subequation.*

A.2. Proofs

Proof of Lemma A.3. Assume we are working in a local trivialization $\mathrm{Sym}^2(T^*V) \cong V \times \mathrm{Sym}^2(\mathbf{R}^n)$ over an open subset $V \subset X$ containing x. Then each $A \in \mathrm{Sym}^2(T_xV)$ determines a smooth section (also denoted A) over V. It suffices to prove the following two claims. □

Claim 1. *Given $A \in \mathrm{Sym}^2(T_xV)$, there exists $c > 0$ such that*

$$A \in \mathrm{Int}F_x \ \Rightarrow \ A \in F_y^c \quad \text{for } y \text{ near } x.$$

Proof. If not, there exist sequences $\{y_j\}$ in U and $W_j \in \mathbb{G}_{y_j}$ such that

$$\lim_{j\to\infty} y_j = x \quad \text{and} \quad \lim_{j\to\infty} \mathrm{tr}_{W_j} A = 0.$$

By compactness we can assume that $W_j \to W \in \mathbb{G}_x$, and by continuity this gives $\mathrm{tr}_W A = 0$, contradicting our assumption that $A \in \mathrm{Int}F_x$ (see (A.3)). □

Claim 2. *If A is a continuous section of $\mathrm{Sym}^2(TV)$ and if for some $c > 0$, $A(y) \in F_y^c$ for all y near x, then $A(x) \in \mathrm{Int}F$.*

Proof. Since $A(y) \in F_y^c$, setting $\epsilon = \frac{c}{p}$, we have that $B(y) \equiv A(y) - \epsilon I \in F_y$ for all y near x. The set $\mathcal{N} \equiv B + \mathrm{Int}\mathcal{P}$, defined using fiber-wise sum, is the translation of the open subset $\mathrm{Int}\mathcal{P}$ of $V \times \mathrm{Sym}^2(\mathbf{R}^n)$ by a continuous section. Hence, $\mathcal{N}$ is open in $\mathrm{Sym}^2(TV)$. Since $B(y) \in F_y$ for all y, we have $\mathcal{N} \equiv B + \mathrm{Int}\mathcal{P} \subset F$. Hence, $\mathcal{N} \subset \mathrm{Int}F$. Finally, $A(x) = B(x) + \epsilon I \in \mathcal{N}$ by positivity. □

Proof of Proposition A.6. The assertion is local so we may assume that X is an open subset of $\mathbf{R}^n$ and $\pi : X \times G(p, \mathbf{R}^n) \to X$ is projection onto the first factor, with $G(p, \mathbf{R}^n) \subset \mathrm{Sym}^2(\mathbf{R}^n)$.

Suppose $\pi\big|_{\mathbb{G}}$ is locally surjective. Let $(x_j, A_j) \in F$ be a convergent sequence, $x_j \to x, A_j \to A$. Fix $W \in \mathbb{G}_x$. By hypothesis for each neighborhood $N_\delta(W)$ of W, $\pi\{(X \times$

$N_\delta(W)) \cap \mathbb{G}\}$ contains a neighborhood of x. Hence we may pick $W_j \in \mathbb{G}_{x_j}$ with $W_j \to W$. Since $\mathrm{tr}_{W_j} A_j \geq 0$ for all j we have $\mathrm{tr}_W A \geq 0$, and so $A \in F_x$.

For the converse, suppose $\pi\big|_{\mathbb{G}}$ is not locally surjective. Then there exists $(x, W) \in \mathbb{G}$ and a neighborhood $N(W)$ of W in $G(p, \mathbf{R}^n)$ so that $\pi\{(X \times N(W)) \cap \mathbb{G}\}$ does not contain a neighborhood of x. Hence there exists a sequence of points $x_j \to x$ in X, such that $\mathbb{G}_{x_j} \cap N(W) = \emptyset$ for all j.

If $\epsilon > 0$ is chosen small enough, then for all $V \in G(p, \mathbf{R}^n)$

$$\big\langle P_V, P_{W^\perp}\big\rangle < \epsilon p \;\Rightarrow\; V \in N(W). \tag{A.6}$$

Consequently, since $\langle P_V, -P_W\rangle \geq -p$, we have that

$$V \notin N(W) \;\Rightarrow\; \left\langle P_V, -P_W + \frac{1}{\epsilon} P_{W^\perp}\right\rangle \geq 0. \tag{A.7}$$

Since $\mathbb{G}_{x_j} \cap N(W) = \emptyset$, this proves that $A \equiv -P_W + \frac{1}{\epsilon} P_{W^\perp} \in F_{x_j}$. However, $\langle A, P_W\rangle = -1$ and $W \in \mathbb{G}_x$, and so $A \notin F_x$. We conclude that F is not closed. $\square$

Appendix B. The linear geometric case

In this appendix we consider the extreme geometric case where each $\mathbb{G}_x$ is a single point $W_x \in G(p, T_x X)$, or equivalently, each $\mathcal{P}(\mathbb{G}_x)$ is the half space in $\mathrm{Sym}^2(T_x X)$ with inward normal P_{W_x} (orthogonal projection onto W_x). Said differently, the subequation $\mathcal{P}(\mathbb{G})$ is linear and given by the *W-Laplacian*

$$(\Delta_W u)(x) = \big\langle P_{W_x}, \mathrm{Hess}_x u\big\rangle_{\text{riem.}} = \mathrm{tr}_{W_x} \mathrm{Hess}_x u. \tag{B.1}$$

It is more appropriate to refer to *W-subharmonic* functions, rather than $\mathbb{G}$-plurisubharmonic functions in this *linear-geometric case*.

Example B.1 (*The pth Horizontal Laplacian*)**.** In this example, choose a single p-plane $W \in G(p, \mathbf{R}^n)$, which might as well be the first coordinate p-plane $W \equiv \mathbf{R}^p \times \{0\} \subset \mathbf{R}^n$. Abbreviate $P_{\mathbf{R}^p \times \{0\}}$ to P. Then

$$\Delta_p u = \left\langle P, D^2 u\right\rangle = \mathrm{tr}_P D^2 u = \sum_{j=1}^{p} \frac{\partial^2 u}{\partial x_j^2} \tag{B.2}$$

is the pth *horizontal Laplacian*. The terminology "horizontally subharmonic" and "horizontally p-convex" is appropriate in this case.

Suppose h and H are smooth functions defined on an open subset on $\mathbf{R}^n$, with h taking values in $\mathrm{GL}_n(\mathbf{R})$ and with H taking values in $\mathrm{Hom}(\mathbf{R}^n, \mathrm{Sym}^2(\mathbf{R}^n))$.

Definition B.2. An equation of the form

$$Lu = \left\langle h^t P h, D^2 u\right\rangle + \big\langle H^t(P), Du\big\rangle \tag{B.3}$$

is said to be *jet equivalent to* Δ_p.

The linear-geometric case is jet equivalent to Δ_p in any local coordinate system.

Proposition B.3. *If W is a smooth section of the Grassmann bundle $G(p, TX)$ over X, then the W-Laplacian is jet equivalent to the pth horizontal Laplacian over any local coordinate chart.*

Proof. Choose a local orthonormal frame field $e_1, \dots, e_n$ for $\mathbf{R}^n$ with $e_1, \dots, e_p$ a frame for W. Define $h(x)$ with values in $\mathrm{GL}_n(\mathbf{R})$ by $e = h\frac{\partial}{\partial x}$. Then, in the given local coordinates,

$$\Delta_W u = \left\langle h^t P h, D^2 u\right\rangle - \left\langle \Gamma^t(h^t P h), Du\right\rangle \tag{B.4}$$

follows from Proposition 5.5 in [10]. □

Proposition B.4. *A subequation L is locally jet equivalent to the pth horizontal Laplacian Δ_p if and only if in any local coordinate system*

$$Lu = \left\langle E, D^2 u\right\rangle - \langle b, Du\rangle \tag{B.5}$$

where b and E are smooth and $E(x)$ has rank p at each point x.

Proof. First note that $E = h^t P h$ in (B.3) has constant rank p. Conversely, assume E in (B.5) has rank p at each point. Then $E(x)$ has a unique smooth square root $A(x)$ in $\mathrm{Sym}^2(\mathbf{R}^n)$. Let B denote orthogonal projection onto the null space of E. Then the inverse of $A + B$ conjugates E to P_W where $W^\perp \equiv \ker E$. Finally it is easy to (locally) conjugate P_W to P and find a smooth section H of $\mathrm{Hom}(\mathbf{R}^n, \mathrm{Sym}^2(\mathbf{R}^n))$ with $H^t(P) = b$. □

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