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Module categories over graded fusion categories

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ABSTRACT

Let $\mathcal{C}$ be a fusion category which is an extension of a fusion category $\mathcal{D}$ by a finite group G. We classify module categories over $\mathcal{C}$ in terms of module categories over $\mathcal{D}$ and the extension data (c, M, α) of $\mathcal{C}$. We also describe functor categories over $\mathcal{C}$ (and in particular the dual categories of $\mathcal{C}$). We use this in order to classify module categories over the Tambara Yamagami fusion categories, and their duals.

1. Introduction

Let $\mathcal{C}$ be a fusion category. We say that $\mathcal{C}$ is an extension of the fusion category $\mathcal{D}$ by a finite group G if $\mathcal{C}$ is faithfully graded by the group G in such a way that $\mathcal{C}_e = \mathcal{D}$. In [3], Etingof et al. classified extensions of a given fusion category $\mathcal{D}$ by a given finite group G. By their classification, an extension is given by a triple (c, M, α), where $c : G \to \mathrm{Pic}(\mathcal{D})$ is a homomorphism, M belongs to a torsor over $H^2(G, \mathrm{inv}(\mathcal{Z}(\mathcal{D})))$, and α belongs to a torsor over $H^3(G, k^*)$. The group $\mathrm{Pic}(\mathcal{D})$ is the group of invertible $\mathcal{D}$-bimodule categories (up to equivalence), and the group $\mathrm{inv}(\mathcal{Z}(\mathcal{D}))$ is the group of (isomorphism classes of) invertible objects in the center $\mathcal{Z}(\mathcal{D})$ of $\mathcal{D}$.

Let us recall briefly the construction and the relevant notions from [3]. If $\mathcal{D}$ is a fusion category, a $\mathcal{D}$-bimodule category will be a category $\mathcal{N}$ which is both a left and a right module category over $\mathcal{D}$ in a compatible way. In other words, for any two objects $X, Y \in \mathcal{D}$ and $N \in \mathcal{N}$ we will have two isomorphic objects of $\mathcal{N}$: $(X \otimes N) \otimes Y \cong X \otimes (N \otimes Y)$. In Section 3 of [3] the notion of tensor product (over $\mathcal{D}$) of $\mathcal{D}$-bimodule categories is defined. It is shown that for any two $\mathcal{D}$-bimodule categories $\mathcal{N}_1$, $\mathcal{N}_2$, the tensor product $\mathcal{N}_1 \boxtimes_{\mathcal{D}} \mathcal{N}_2$ always exists and satisfies a universal mapping property.

A $\mathcal{D}$-bimodule category $\mathcal{N}$ will be called invertible if there is another $\mathcal{D}$-bimodule category $\mathcal{N}'$ such that $\mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{N}' \cong \mathcal{N}' \boxtimes_{\mathcal{D}} \mathcal{N} \cong \mathcal{D}$ as $\mathcal{D}$-bimodule categories. The set of equivalence classes of all invertible $\mathcal{D}$-bimodule categories forms a group with respect to tensor product over $\mathcal{D}$. This group is called the Picard group of $\mathcal{D}$, and is denoted by $\mathrm{Pic}(\mathcal{D})$.

Suppose then that we are given a classification data (c, M, α). The corresponding category $\mathcal{C}$ will be $\bigoplus_{g \in G} c(g)$ as a $\mathcal{D}$-bimodule category. If we choose arbitrary isomorphisms $c(g) \boxtimes_{\mathcal{D}} c(h) \to c(gh)$ for the tensor product in $\mathcal{C}$, the multiplication will not necessarily be associative. This non associativity is encoded in a cohomological obstruction $O_3(c) \in Z^3(G, \mathrm{inv}(\mathcal{Z}(\mathcal{D})))$. The element M belongs to $C^2(G, inv(\mathcal{Z}(\mathcal{D})))$, and should satisfy $\partial M = O_3(c)$ (that is, it should be a "solution" to the obstruction $O_3(c)$). If we change M by a coboundary, we get an equivalent solution. Therefore, the choice of M is equivalent to choosing an element from a torsor over $H^2(G, \mathrm{inv}(\mathcal{Z}(\mathcal{D})))$. Given c and M, we still have one more obstruction in order to furnish $\mathcal{C}$ with a structure of a fusion category. This obstruction is the commutativity of the pentagon diagram, and is given by a four cocycle $O_4(c, M) \in Z^4(G, k^*)$. The element α belongs to $C^3(G, k^*)$, and should satisfy $\partial\alpha = O_4(c, M)$. We think of α as a solution to the obstruction $O_4(c, M)$. Again, if we change α by a coboundary, we will get an equivalent solution. Therefore, the choice of α can be seen as a choice from a torsor over $H^3(G, k^*)$.

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We shall write $\mathcal{C} = \mathcal{D}(G, c, M, \alpha)$ to indicate the fact that $\mathcal{C}$ is an extension of $\mathcal{D}$ by G given by the extension data (c, M, α), and we shall assume from now on that $\mathcal{C} = \mathcal{D}(G, c, M, \alpha)$.

In this paper we shall classify module categories over $\mathcal{C}$ in terms of module categories over $\mathcal{D}$ and the extension data (c, M, α).

Our classification of module categories will follow the lines of the classification of [3]. We will begin by proving the following structure theorem for module categories over $\mathcal{C}$.

Theorem 1. *Let $\mathcal{C}$ be a G-extension of the fusion category $\mathcal{D}$, and let $\mathcal{L}$ be an indecomposable module category over $\mathcal{C}$. There is a subgroup $H < G$, and an indecomposable $\mathcal{C}_H = \bigoplus_{a\in H} \mathcal{C}_a$ module category $\mathcal{N}$ that remains indecomposable over $\mathcal{D}$ such that $\mathcal{L} \cong Ind^{\mathcal{C}}_{\mathcal{C}_H}(\mathcal{N}) \triangleq \mathcal{C} \boxtimes_{\mathcal{C}_H} \mathcal{N}$.*

This theorem enables us to reduce the classification of $\mathcal{C}$-module categories to the classification of $\mathcal{C}_H$-module categories that remain indecomposable over $\mathcal{D}$, where H varies over subgroups of G.

In order to classify such categories we will go, in some sense, the other way around. We will begin with an indecomposable $\mathcal{D}$-module category $\mathcal{N}$, and we will ask how can we equip $\mathcal{N}$ with a structure of a $\mathcal{C}_H$-module category.

As in the classification in [3], the answer will also be based upon choosing solutions to certain obstruction (in case it is possible). We will begin with the observation, in Section 3, that we have a natural action of G on the set of (equivalence classes of) indecomposable $\mathcal{D}$-module categories. This action is given by the following formula

$$g \cdot \mathcal{N} = \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}.$$

If $\mathcal{N}$ has a structure of a $\mathcal{C}_H$-module category, then the action of $\mathcal{C}_H$ on $\mathcal{N}$ will give an equivalence of $\mathcal{D}$-module categories $h \cdot \mathcal{N} \cong \mathcal{N}$ for every $h \in H$. In other words- $\mathcal{N}$ will be H-invariant. We may think of the fact that $\mathcal{N}$ should be H-invariant as the "zeroth obstruction" we have in order to equip $\mathcal{N}$ with a structure of a $\mathcal{C}_H$-module category.

In case $\mathcal{N}$ is H-invariant, we choose equivalences $\psi_a : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}$ for every $a \in H$. We would like these equivalences to give us a structure of a $\mathcal{C}_H$-module category on $\mathcal{N}$. As one might expect, not every choice of equivalences will do that. If $\mathcal{N}$ has a structure of a $\mathcal{C}_H$-module category, we will see in Section 4 that we have a natural action of H on the group $\Gamma = \mathrm{Aut}_{\mathcal{D}}(\mathcal{N})$. In case we only know that $\mathcal{N}$ is H-invariant, we only have an *outer* action of H on Γ (i.e. a homomorphism $\rho : H \to \mathrm{Out}(\Gamma)$). The first obstruction will thus be the possibility to lift this outer action to a proper action.

Once we overcome this obstruction (and choose a lifting Φ for the outer action), our second obstruction will be the fact that the two functors

$$F_1, F_2 : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \boxtimes \mathcal{N} \to \mathcal{N}$$

defined by

$$F_1(X \boxtimes Y \boxtimes N) = (X \otimes Y) \otimes N$$

and

$$F_2(X \boxtimes Y \boxtimes N) = X \otimes (Y \otimes N)$$

should be isomorphic. We will see that this obstruction is given by a certain two cocycle $O_2(\mathcal{N}, c, H, M, \Phi) \in Z^2(H, \mathcal{Z}(\mathrm{Aut}_{\mathcal{D}}(\mathcal{N})))$. A solution for this obstruction is an element $v \in C^1(H, \mathcal{Z}(\mathrm{Aut}_{\mathcal{D}}(\mathcal{N})))$ that should satisfy $\partial v = O_2(\mathcal{N}, c, H, M, \Phi)$.

Our last obstruction will be the fact that the above functors should be not only isomorphic, but they should be isomorphic in a way which will make the pentagon diagram commutative. This obstruction is encoded by a three cocycle $O_3(\mathcal{N}, c, H, M, \Phi, v, \alpha) \in Z^3(H, k^*)$. A solution β for this obstruction will be an element of $C^2(H, k^*)$ such that $\partial\beta = O_3(\mathcal{N}, c, H, M, \Phi, v, \alpha)$.

We can summarize our main result in the following theorem:

Theorem 2. *An indecomposable module category over $\mathcal{C}$ is given by a tuple $(\mathcal{N}, H, \Phi, v, \beta)$, where $\mathcal{N}$ is an indecomposable module category over $\mathcal{D}$, H is a subgroup of G which acts trivially on $\mathcal{N}$, $\Phi : H \to Aut(Aut_{\mathcal{D}}(\mathcal{N}))$ is a homomorphism, v belongs to a torsor over $H^1(H, \mathcal{Z}(Aut_{\mathcal{D}}(\mathcal{N})))$, and β belongs to a torsor over $H^2(H, k^*)$.*

We shall denote the indecomposable module category which corresponds to the tuple $(\mathcal{N}, H, \Phi, v, \beta)$ by $\mathcal{M}(\mathcal{N}, H, \Phi, v, \beta)$. In order to classify module categories, we need to give not only a list of all indecomposable module categories, but also to explain when two elements in the list define equivalent module categories. We will see in Section 6 that if $\mathcal{M}(\mathcal{N}, H, \Phi, v, \beta)$ is any indecomposable module category, $g \in G$ is an arbitrary element and $F : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}'$ is an equivalence of $\mathcal{D}$-module categories (where $\mathcal{N}'$ is another indecomposable $\mathcal{D}$-module category), then F gives rise to a tuple $(\mathcal{N}', gHg^{-1}, \Phi', v', \beta')$ which defines an equivalent $\mathcal{C}$-module category. Our second main result is that this is the only way in which we can get equivalent module categories:

Theorem 3. *Two tuples $(\mathcal{N}, H, \Phi, v, \beta)$ and $(\mathcal{N}', H', \Phi', v', \beta')$ determine equivalent $\mathcal{C}$-module categories if and only if there exists an element $g \in G$ and an equivalence $F : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}'$ such that $H' = gHg^{-1}$, and the second tuple arise from the first tuple, g and F in the way described in Section 6.*

We shall prove Theorem 3 in Section 6. We will also decompose this condition into a few simpler ones: we will see, for example, by considering the case $g = 1$, that we can change Φ to be $t\Phi t^{-1}$, where t is any conjugation automorphism of $\mathrm{Aut}_{\mathcal{D}}(\mathcal{N})$.

In Section 7 we will describe the category of functors $\mathrm{Fun}_{\mathcal{C}}(\mathcal{N}, \mathcal{M})$ where $\mathcal{N}$ and $\mathcal{M}$ are two module categories over $\mathcal{C}$. We will prove a Mackey type decomposition theorem, and we will also see that we can view this category as the equivariantization of the category $\mathrm{Fun}_{\mathcal{D}}(\mathcal{N}, \mathcal{M})$ with respect to an action of G (the equivariantization of a category $\mathcal{C}$ with respect to an action of a group G is the category of objects in $\mathcal{C}$ which are invariant under the action of G in a compatible way. Exact definition will be given in Section 7).

Recall that a category $\mathcal{C}$ is called group theoretical if it is Morita equivalence to a pointed category (see Section 8.8 of [2] for an introduction to group theoretical categories). Group theoretical categories play an important role in the general theory of fusion categories. In Section 7 we will prove the following criterion for $\mathcal{C}$ to be group theoretical: $\mathcal{C}$ is group theoretical if and only if there is a pointed $\mathcal{D}$-module category $\mathcal{N}$ (i.e., $\mathcal{D}^*_{\mathcal{N}}$ is pointed), stable under the G-action, i.e., for every $g \in G$, $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \cong \mathcal{N}$ as $\mathcal{D}$-module categories. We shall also explain why this is a reformulation of the criterion which appears in Corollary 3.10 of [6].

A theorem of Ostrik (Theorem 1 in [8]) says that any indecomposable module category over a fusion category $\mathcal{D}$ is equivalent to a category of the form $\mathrm{Mod}_{\mathcal{D}} - A$, of right A-modules in the category $\mathcal{D}$, where A is some semisimple indecomposable algebra in the category $\mathcal{D}$. In other words, any module category has a description by objects which lie inside the fusion category $\mathcal{D}$. In Section 8 we will explain how we can understand the obstructions and their solutions, and also the functor categories, by intrinsic description; that is, by considering algebras and modules inside the categories $\mathcal{D}$ and $\mathcal{C}$.

This description will be much more convenient for calculations. It will also enable us to view the first and the second obstruction in a unified way. Indeed, in Section 8 we will show that we have a natural short exact sequence

$$1 \to \Gamma \to \Lambda \to H \to 1$$

where $\Gamma = \mathrm{Aut}_{\mathcal{D}}(\mathcal{N})$ and Λ is the group of invertible $A-A$-bimodules in $\mathcal{C}_H$. We will show that a solution for the first two obstructions is equivalent to a choice of a splitting of this sequence (and therefore, we can solve the first two obstructions if and only if this sequence splits). We will also show, following the results of Section 8, that two splittings which differ by conjugation by an element of Γ will give us equivalent module categories.

In Section 9 we shall give a detailed example. We will consider the Tambara Yamagami fusion categories, $\mathcal{C} = \mathcal{TY}(A, \chi, \tau)$. In this case $\mathcal{C}$ is an extension of the category Vec_A, where A is an abelian group, by the group $\mathbb{Z}_2$.

Remark. During the final stages of the writing of this paper it came to our attention that César Galindo is working on a paper with similar results (see [5]). We would like to remark that our results and his were obtained independently.

2. Preliminaries

In this section, $\mathcal{C}$ will be a general fusion category and $\mathcal{D}$ a fusion subcategory of $\mathcal{C}$. We recall some basic facts about module categories over $\mathcal{C}$ and $\mathcal{D}$. For a more detailed discussion on these notions, we refer the reader to [8] and to [2]. Let $\mathcal{N}$ be a module category over $\mathcal{C}$. If $X, Y \in \mathrm{Ob}\,\mathcal{N}$, then the *internal hom* of X and Y is the unique object of $\mathcal{C}$ which satisfies the formula

$$\mathrm{Hom}_{\mathcal{C}}(W, \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)) = \mathrm{Hom}_{\mathcal{N}}(W \otimes X, Y)$$

for every $W \in \mathrm{Ob}\mathcal{C}$. For every $X \in \mathrm{Ob}\mathcal{N}$ the object $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$ has a canonical algebra structure. We say that X *generates* $\mathcal{N}$ (over $\mathcal{C}$) if $\mathcal{N}$ is the smallest $\mathcal{C}$-module subcategory of $\mathcal{N}$ which contains X. For every algebra A in $\mathcal{C}$, $\mathrm{Mod}_{\mathcal{C}} - A$, the category of right A-modules in $\mathcal{C}$, has a structure of a left $\mathcal{C}$-module category.

A theorem of Ostrik says that all module categories are of this form:

Theorem 4 (*See [8], Theorem 1*)**.** *Let $\mathcal{N}$ be a module category, and let X be a generator of $\mathcal{N}$ over $\mathcal{C}$. We have an equivalence of $\mathcal{C}$-module categories $\mathcal{N} \cong Mod_{\mathcal{C}} - \underline{Hom}(X, X)$ given by $F(Y) = \underline{Hom}(X, Y)$.*

Next, we recall the definition of the induced module category:

Definition 5. Let $\mathcal{D}$ be a fusion subcategory of $\mathcal{C}$, and let $\mathcal{N}$ be a $\mathcal{D}$-module category. The induced module category, $\mathrm{Ind}_{\mathcal{D}}^{\mathcal{C}}(\mathcal{N})$ is a module category over $\mathcal{C}$ which satisfies the Frobenius reciprocity law. This means that for every $\mathcal{C}$-module category $\mathcal{R}$ we have an equivalence of categories

$$\mathrm{Fun}_{\mathcal{C}}(\mathrm{Ind}_{\mathcal{D}}^{\mathcal{C}}(\mathcal{N}), \mathcal{R}) \cong \mathrm{Fun}_{\mathcal{D}}(\mathcal{N}, \mathcal{R}).$$

Notice that it is not clear from the definition that an induced module category always exists, but it is quite easy to prove that if it exists, then it is unique. Indeed, we can use general category theory arguments to show that if $\mathcal{N}_1$ and $\mathcal{N}_2$ are two $\mathcal{C}$-module categories which satisfy the Frobenius reciprocity law, then they must be equivalent.

The next lemma proves that the induced module category always exists. It will also gives us some idea about how the induced module category "looks like".

Lemma 6. *Suppose that $\mathcal{R} \cong Mod_{\mathcal{D}} - A$ for some algebra $A \in Ob\,\mathcal{D}$. Then A can also be considered as an algebra in $\mathcal{C}$, and $Ind_{\mathcal{D}}^{\mathcal{C}}(\mathcal{N}) \cong Mod_{\mathcal{C}} - A$.*

Proof. Let us prove that Frobenius reciprocity holds. For this, we first need to represent $\mathcal{R}$ in an appropriate way. We choose a generator X of $\mathcal{R}$ **over** $\mathcal{D}$. It is easy to see that X is also a generator over $\mathcal{C}$. Then, by Ostrik's Theorem we have that $\mathcal{R} \cong \mathrm{Mod}_{\mathcal{C}} - \underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$ over $\mathcal{C}$, and $\mathcal{R} \cong \mathrm{Mod}_{\mathcal{D}} - \underline{\mathrm{Hom}}_{\mathcal{D}}(X, X)$ over $\mathcal{D}$. If we denote $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$ by B, then it is easy to see by the definition of $\underline{\mathrm{Hom}}$ that $\underline{\mathrm{Hom}}_{\mathcal{D}}(X, X) \cong B_{\mathcal{D}}$, where $B_{\mathcal{D}}$ is the largest subobject of B which is also an object of $\mathcal{D}$ (since $\mathcal{D}$ is a fusion subcategory of $\mathcal{C}$, this is also a subalgebra of B). By another result of Ostrik (see the paragraph after Proposition 2.1 in [8]), we know that $\mathrm{Fun}_{\mathcal{C}}(\mathrm{Mod}_{\mathcal{C}} - A, \mathrm{Mod}_{\mathcal{C}} - B) \cong \mathrm{Bimod}_{\mathcal{C}} - A - B$. Using the theorem of Ostrik again, we see that $\mathrm{Fun}_{\mathcal{D}}(\mathcal{N}, \mathcal{R}) \cong \mathrm{Bimod}_{\mathcal{D}} - A - B_{\mathcal{D}}$. By the next lemma, these two categories are equivalent. □

Lemma 7. *The categories $Bimod_{\mathcal{C}} - A - B$ and $Bimod_{\mathcal{D}} - A - B_{\mathcal{D}}$ are equivalent.*

Proof. We shall define two functors, and prove that they are quasi inverse to each other. Let $F : \mathrm{Bimod}_{\mathcal{C}} - A - B \to \mathrm{Bimod}_{\mathcal{D}} - A - B_{\mathcal{D}}$ be given by $F(X) = X \cap \mathcal{D}$, the largest subobject of X which is also an object of the subcategory $\mathcal{D}$, and let $G : \mathrm{Bimod}_{\mathcal{D}} - A - B_{\mathcal{D}} \to \mathrm{Bimod}_{\mathcal{C}} - A - B$ be given by $G(Y) = Y \otimes_{B_{\mathcal{D}}} B$. Notice that we have natural maps $\alpha : Id \to FG$ and $\beta : GF \to Id$. The map α is given by the natural inclusion $\alpha_X : X \to FG(X) = (X \otimes_{B_{\mathcal{D}}} B) \cap \mathcal{D}$ and the map β is defined in a similar way.

We know that we have an equivalence of categories $\mathrm{Mod}_{\mathcal{D}} - B_{\mathcal{D}} \cong \mathcal{R} \cong \mathrm{Mod}_{\mathcal{C}} - B$. By considering carefully the proof of Theorem 1 of [8], we see that this equivalence is given in one direction by induction $X \mapsto X \otimes_{B_{\mathcal{D}}} B$ and in the other direction by $X \mapsto X \cap \mathcal{D}$. This means that for every $X \in \mathrm{Bimod}_{\mathcal{D}} - A - B_{\mathcal{D}}$ and every $Y \in \mathrm{Bimod}_{\mathcal{C}} - A - B$, α_X and β_Y will be isomorphisms when we consider them as maps of $B_{\mathcal{D}}$-modules and B-modules, but this implies that α_X and β_Y are isomorphisms, which finishes the proof of the claim. □

Remark 8. It is quite easy to show, using the universal property of tensor products (see Definition 3.3 in [3]), that the $\mathcal{C}$-module category $\mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{N}$ satisfies the Frobenius reciprocity law. Therefore, the induced module of $\mathcal{N}$ is isomorphic to $\mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{N}$. If we write $\mathcal{N}$ as $\mathrm{Mod}_{\mathcal{D}} - A$, where $A = \underline{\mathrm{Hom}}(X, X)$ for a generator X of $\mathcal{N}$ over $\mathcal{D}$, then the algebra $\underline{\mathrm{Hom}}(1 \boxtimes X, 1 \boxtimes X)$ in $\mathcal{C}$ is exactly A again. This is another way to prove Lemma 6. By using the isomorphism which appears in Ostrik's Theorem, we see that the equivalence $\mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathrm{Mod}_{\mathcal{C}} - A$ is given by $X \boxtimes V \mapsto X \otimes V$.

In particular, we have the following:

Corollary 9. *Let $\mathcal{C}$ be a fusion category and let $\mathcal{D}$ be a fusion subcategory of $\mathcal{C}$. Let $\mathcal{N}$ be a module category over $\mathcal{C}$. Suppose that X is a generator of $\mathcal{N}$ over $\mathcal{C}$, and that the algebra $A = \underline{Hom}(X, X)$ is supported on $\mathcal{D}$. Then $\mathcal{N} \cong Ind_{\mathcal{D}}^{\mathcal{C}}(Mod_{\mathcal{D}} - A)$.*

3. Decomposition of the module category over the trivial component subcategory. The zeroth obstruction

Let G be a finite group, and let $\mathcal{C} = \oplus_{g \in G} \mathcal{C}_g$ be a G-extension of $\mathcal{D} = \mathcal{C}_1$. We begin by considering the action of G on $\mathcal{D}$-module categories. For every $g \in G$, $\mathcal{C}_g$ is an invertible $\mathcal{D}$-bimodule category. Therefore, if $\mathcal{N}$ is an indecomposable $\mathcal{D}$-module category, the category $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ is also indecomposable. It is easy to see that we get in this way an action of G on the set of (equivalence classes of) indecomposable $\mathcal{D}$-module categories. Let now $\mathcal{L}$ be an indecomposable $\mathcal{C}$-module category. We can consider $\mathcal{L}$ also as a module category over $\mathcal{D}$. We claim the following:

Lemma 10. *As a $\mathcal{D}$-module category, $\mathcal{L}$ is G-invariant. Moreover, the action functor $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{L} \to \mathcal{L}$ given by $X \boxtimes Y \mapsto X \otimes Y$ is an equivalence of categories.*

Remark 11. For this lemma, we do not need to assume that $\mathcal{L}$ is indecomposable.

Proof. We have the following equivalences of $\mathcal{D}$-module categories

$$\begin{aligned}
&\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{L} \cong \mathcal{C}_g \boxtimes_{\mathcal{D}} (\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{L}) \cong \\
&(\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}) \boxtimes_{\mathcal{C}} \mathcal{L} \cong (\mathcal{C}_g \boxtimes_{\mathcal{D}} \oplus_{a \in G} \mathcal{C}_a) \boxtimes_{\mathcal{C}} \mathcal{L} \cong \\
&(\oplus_{a \in G} \mathcal{C}_{ga}) \boxtimes_{\mathcal{C}} \mathcal{L} \cong \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{L} \cong \mathcal{L}.
\end{aligned}$$

This proves the claim. We have used some general facts about the tensor product of bimodule categories in this calculation: the fact that the product is associative, the fact that tensor product with $\mathcal{C}$ over $\mathcal{C}$ is the identity, and the fact that for $g, a \in G$ we have that the tensor multiplication in $\mathcal{C}$ gives us an isomorphism of $\mathcal{D}$-bimodule categories $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_a \to \mathcal{C}_{ga}$ (see Section 3 and Theorem 6.1 of [3]). Also, notice that, by considering the image of $X \boxtimes Y$ under this equivalence, we see that the equivalence is given by $X \boxtimes Y \mapsto X \otimes Y$. □

If H is a subgroup of G, we have the subcategory $\mathcal{C}_H = \bigoplus_{h \in H} \mathcal{C}_h$ of $\mathcal{C}$, which is an extension of $\mathcal{D}$ by H. We claim the following:

Proposition 12. *There is a subgroup $H < G$, and an indecomposable $\mathcal{C}_H$-module category $\mathcal{N}$ that remains indecomposable over $\mathcal{D}$ such that $\mathcal{L} \cong Ind_{\mathcal{C}_H}^{\mathcal{C}}(\mathcal{N})$.*

Proof. Suppose that $\mathcal{L}$ decomposes over $\mathcal{D}$ as

$$\mathcal{L} = \bigoplus_{i=1}^{n} \mathcal{L}_i.$$

For every $g \in G$, we have seen that the action functor defines an equivalence of categories $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{L} \cong \mathcal{L}$. Since

$$\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{L} \cong \bigoplus_{i=1}^{n} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{L}_i,$$

we see that G permutes the index set $\{1, \ldots, n\}$. This action is transitive, as otherwise $\mathcal{L}$ would not have been indecomposable over $\mathcal{C}$. Let $H < G$ be the stabilizer of $\mathcal{L}_1$. Then $\mathcal{N} = \mathcal{L}_1$ is a $\mathcal{C}_H$-module category that remains indecomposable over $\mathcal{D}$. Let $X \in \mathrm{Ob}\mathcal{L}_1$ be a generator of $\mathcal{L}$ over $\mathcal{C}$ (any nonzero object would be a generator, as $\mathcal{L}$ is indecomposable over $\mathcal{C}$). By the fact that the stabilizer of $\mathcal{L}_1$ is H, it is easy to see that $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$ is contained in $\mathcal{C}_H$. The rest of the lemma now follows from Corollary 9. $\square$

So in order to classify indecomposable module categories over $\mathcal{C}$, we need to classify, for every $H < G$, the indecomposable module categories over $\mathcal{C}_H$ that remain indecomposable over $\mathcal{D}$. For every indecomposable module category $\mathcal{L}$ over $\mathcal{C}$, we have attached a subgroup H of G and an indecomposable $\mathcal{C}_H$-module category $\mathcal{L}_1$ that remains indecomposable over $\mathcal{D}$. The subgroup H and the module category $\mathcal{L}_1$ will be the first two components of the tuple which corresponds to $\mathcal{L}$. Notice that we could have chosen any conjugate of H as well.

4. The first two obstructions

Let $\mathcal{L}$, $\mathcal{N} = \mathcal{L}_1$ and H be as in the previous section. For every $a \in H$ we have an equivalence of $\mathcal{D}$-module categories $\psi_a : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \cong \mathcal{N}$ given by the action of $\mathcal{C}_H$ on $\mathcal{N}$. Suppose on the other hand that we are given an H-invariant indecomposable module category $\mathcal{N}$ over $\mathcal{D}$. Let us fix a family of equivalences $\{\psi_a\}_{a \in H}$, where $\psi_a : \mathcal{C}_a \boxtimes \mathcal{N} \to \mathcal{N}$. Let us see when does this family come from an action of $\mathcal{C}_H$ on $\mathcal{N}$.

We know that the two functors

$$\mathcal{C}_H \boxtimes \mathcal{C}_H \boxtimes \mathcal{N} \xrightarrow{m \boxtimes 1_{\mathcal{N}}} \mathcal{C}_H \boxtimes \mathcal{N} \xrightarrow{\cdot} \mathcal{N}$$

and

$$\mathcal{C}_H \boxtimes \mathcal{C}_H \boxtimes \mathcal{N} \xrightarrow{1_{\mathcal{C}_H} \boxtimes (\cdot)} \mathcal{C}_H \boxtimes \mathcal{N} \xrightarrow{\cdot} \mathcal{N}$$

should be isomorphic.

Since the action of $\mathcal{C}_H$ on $\mathcal{N}$ is given by the action of $\mathcal{D}$ together with the ψ_a's, this condition translates to the fact that for every $a, b \in H$ the two functors

$$\mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{M_{a,b} \boxtimes 1_{\mathcal{N}}} \mathcal{C}_{ab} \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{\psi_{ab}} \mathcal{N}$$

and

$$\mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{1_{\mathcal{C}_a} \boxtimes \psi_b} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{\psi_a} \mathcal{N}$$

should be isomorphic. We can express this condition in the following equivalent way, for every $a, b \in H$, the autoequivalence of $\mathcal{N}$ as a $\mathcal{D}$-module category

$$Y_{a,b} = \mathcal{N} \xrightarrow{\psi_a^{-1}} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{1_{\mathcal{C}_a} \boxtimes \psi_b^{-1}} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N}$$
$$\xrightarrow{M_{a,b} \boxtimes 1_{\mathcal{N}}} \mathcal{C}_{ab} \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{\psi_{ab}^{-1}} \mathcal{N}$$

should be isomorphic to the identity autoequivalence. We shall decompose this condition into two simpler ones.

Consider the group $\Gamma = \mathrm{Aut}_{\mathcal{D}}(\mathcal{N})$, where by $\mathrm{Aut}_{\mathcal{D}}$ we mean the group of $\mathcal{D}$-autoequivalences (up to isomorphism) of $\mathcal{N}$. For $a \in H$ and $F \in \Gamma$ define $a \cdot F \in \Gamma$ as the composition

$$\mathcal{N} \xrightarrow{\psi_a^{-1}} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{1_{\mathcal{C}_a} \boxtimes F} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{\psi_a} \mathcal{N}.$$

We get a map $\Phi : H \to \mathrm{Aut}(\Gamma)$ given by $\Phi(h)(F) = h \cdot F$. This map depends on the choice of the ψ_a's and is not necessarily a group homomorphism. However, the following equation does hold for every $a, b \in H$:

$$\Phi(a)\Phi(b) = \Phi(ab)C_{Y_{a,b}}, \tag{4.1}$$

where we write C_x for conjugation by $x \in \Gamma$.

Notice that ψ_a is determined up to composition with an element in Γ, and that by changing ψ_a to be $\psi_a' = \gamma\psi_a$, for $\gamma \in \Gamma$, we change $\Phi(a)$ to be $\Phi(a)C_\gamma$. Eq. (4.1) shows that the composition $\rho = \pi\Phi$, where π is the quotient map $\pi : \mathrm{Aut}(\Gamma) \to \mathrm{Out}(\Gamma)$ does give a group homomorphism. Notice that by the observation above, ρ does not depend on the choice of the ψ_a's, but only on the homomorphism c, the module category $\mathcal{N}$ and the subgroup H. We have the following

Lemma 13. *Let $\mathcal{N}$ be an H-invariant $\mathcal{D}$-module category. There is a well defined group homomorphism $\rho : H \to Out(\Gamma)$. If the ψ_a's arise from an action of $\mathcal{C}_H$ on $\mathcal{N}$, then the map Φ described above is a group homomorphism.*

Proof. This follows from the fact that by the discussion above, if the ψ_a's arise from an action of $\mathcal{C}_H$ on $\mathcal{N}$, then $Y_{a,b}$ is trivial for every $a, b \in H$, and by Eq. (4.1) we see that Φ is a group homomorphism. $\square$

So the homomorphism c, the module category $\mathcal{N}$ and the subgroup H determine a homomorphism $\rho : H \to \mathrm{Out}(\Gamma)$. We thus see that in order to give $\mathcal{N}$ a structure of a $\mathcal{C}_H$-module category, we need to give a lifting of ρ to a homomorphism to $\mathrm{Aut}(\Gamma)$. The first obstruction is thus the possibility to lift ρ in such a way.

Suppose then that we have a lifting, that is- a homomorphism $\Phi : H \to \mathrm{Aut}(\Gamma)$ such that $\pi\Phi = \rho$. To say that Φ is a homomorphism is equivalent to say that we have chosen the ψ_a's in such a way that $C_{Y_{a,b}} = Id$, or in other words, in such a way that for every $a, b \in H$, $Y_{a,b}$ is in $Z(\Gamma)$, the center of Γ.

Notice that after choosing Φ, we still have some liberty in changing the ψ_a's. Indeed, if we choose $\psi'_a = \gamma_a\psi_a$, where $\gamma_a \in Z(\Gamma)$ for every $a \in H$, we still get the same Φ, and it is easy to see that every ψ'_a that will give us the same Φ is of this form.

In order to furnish a structure of a $\mathcal{C}_H$-module category on $\mathcal{N}$, we need $Y_{a,b}$ to be not only central, but trivial. A straightforward calculation shows now that the function $H \times H \to Z(\Gamma)$ given by $(a, b) \mapsto Y_{a,b}$ is a two cocycle. If we choose a different set of isomorphisms $\psi'_a = \gamma_a\psi_a$ where $\gamma_a \in Z(\Gamma)$, we will get a cocycle Y' which is cohomologous to Y. So the second obstruction is the cohomology class of the two cocycle $(a, b) \mapsto Y_{a,b}$. We shall denote this obstruction by $O_2(\mathcal{N}, c, H, M, \Phi) \in Z^2(H, Z(\Gamma))$. Notice that this obstruction depends linearly on M in the following sense: we have a natural homomorphism of groups $\xi : \mathrm{inv}(\mathcal{Z}(\mathcal{D})) \to \Gamma$, given by the formula

$$\xi(T)(N) = T \otimes N$$

(that is- $\xi(T)$ is just the autoequivalence given by the action of T) It can be seen that if we had chosen $M' = M\zeta$, where $\zeta \in Z^2(G, \mathrm{inv}(\mathcal{Z}(\mathcal{D})))$, then we would have changed O_2 to be $O_2\mathrm{res}^G_H(\xi_*(\zeta))$.

In conclusion, we saw that if $\mathcal{N}$ is a $\mathcal{D}$-module category upon which H acts trivially, then we have an induced homomorphism $\rho : H \to \mathrm{Out}(\Gamma)$. The first obstruction to define on $\mathcal{N}$ a structure of a $\mathcal{C}_H$-module category is the fact that ρ should be of the form $\pi\Phi$ where $\Phi : H \to \mathrm{Aut}(\Gamma)$ is a homomorphism. After choosing such a lifting Φ we get the second obstruction, which is a two cocycle $O_2(\mathcal{N}, c, H, M, \Phi) \in Z^2(H, Z(\Gamma))$. A solution to this obstruction will be an element $v \in C^1(H, Z(\Gamma))$ which satisfies

$$\partial v = O_2(\mathcal{N}, c, H, M, \Phi).$$

We will see later, in Section 8, that to find a solution to the first and to the second obstruction is the same thing as to find a splitting for a certain short exact sequence. We will also see why two solutions v and v' which differ by a coboundary give equivalent module categories (and therefore we can view the set of possible solutions, in case it is not empty, as a torsor over $H^1(H, Z(\Gamma))$.

5. The third obstruction

So far we have almost defined a $\mathcal{C}_H$-action on $\mathcal{N}$, by means of the equivalences $\psi_a : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}$. The solutions to the first and to the second obstruction ensures us that for every $a, b \in H$ the two functors

$$F_1 : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N} \overset{M_{a,b}\boxtimes 1_{\mathcal{N}}}{\to} \mathcal{C}_{ab} \boxtimes_{\mathcal{D}} \mathcal{N} \overset{\psi_{ab}}{\to} \mathcal{N}$$

and

$$F_2 : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N} \overset{1_{\mathcal{C}_a}\boxtimes\psi_b}{\to} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \overset{\psi_a}{\to} \mathcal{N}$$

are isomorphic.

For every $a, b \in H$, let us fix an isomorphism $\eta(a, b) : F_1 \to F_2$ between the two functors. In other words, for every $X \in \mathcal{C}_a$, $Y \in \mathcal{C}_b$ and $N \in \mathcal{N}$ we have a natural isomorphism

$$\eta(a, b)_{X,Y,N} : (X \otimes Y) \otimes N \to X \otimes (Y \otimes N).$$

Since F_1 and F_2 are simple as objects in the relevant functor category (they are equivalences), the choice of the isomorphism $\eta(a, b)$ is unique up to a scalar, for every $a, b \in H$.

The final condition for $\mathcal{N}$ to be a $\mathcal{C}_H$-module category is the commutativity of the pentagonal diagram. In other words, for every $a, b, d \in H$, and every $X \in \mathcal{C}_a$, $Y \in \mathcal{C}_b$, $Z \in \mathcal{C}_d$ and $N \in \mathcal{N}$, the following diagram should commute:

$$\begin{tikzcd}
(X \otimes (Y \otimes Z)) \otimes N \arrow[rr, "\eta(a,bd)_{X,Y\otimes Z,N}"] & & X \otimes ((Y \otimes Z) \otimes N) \arrow[d, "\eta(b,d)_{Y,Z,N}"] \\
((X \otimes Y) \otimes Z) \otimes N \arrow[u, "\alpha_{X,Y,Z}"] \arrow[dr, "\eta(ab,d)_{X\otimes Y,Z,N}"] & & X \otimes (Y \otimes (Z \otimes N)) \\
& (X \otimes Y) \otimes (Z \otimes N) \arrow[ur, "\eta(a,b)_{X,Y,Z\otimes N}"] &
\end{tikzcd}$$

This diagram will always be commutative up to a scalar $O_3(a, b, d)$ which depends only on a, b and d, and not on the particular objects X, Y, Z and N. One can also see that the function $(a, b, d) \mapsto O_3(a, b, d)$ is a three cocycle on H with values in k^*, and that choosing different $\eta(a, b)$'s will change O_3 by a coboundary. We call $O_3 = O_3(\mathcal{N}, c, H, M, \Phi, \upsilon, \alpha) \in Z^3(H, k^*)$ the third obstruction. A solution to this obstruction is equivalent to giving a set of $\eta(a, b)$'s such that the pentagon diagram will be commutative. We will see in the next section that by altering η by a coboundary we will get equivalent module categories. Thus, we see that the set of solutions for this obstruction will be a torsor over the group $H^2(H, k^*)$ (in case a solution exists). Notice that this obstruction depends "linearly" on α, in the sense that if we had changed α to be $\alpha\zeta$ where $\zeta \in H^3(G, k^*)$, then we would have changed the obstruction by ζ. In other words:

$$O_3(\mathcal{N}, c, H, M, \Phi, \upsilon, \alpha\zeta) = O_3(\mathcal{N}, c, H, M, \Phi, \upsilon, \alpha)\mathrm{res}^G_H(\zeta).$$

This ends the proof of Theorem 2.

6. The isomorphism condition

In this section we answer the question of when the $\mathcal{C}$-module categories $\mathcal{M}(\mathcal{N}, H, \Phi, \upsilon, \beta)$ and $\mathcal{M}(\mathcal{N}', H', \Phi', \upsilon', \beta')$ are equivalent.

Assume then that we have an equivalence of $\mathcal{C}$-module categories

$$F : \mathcal{M}(\mathcal{N}, H, \Phi, \upsilon, \beta) \to \mathcal{M}(\mathcal{N}', H', \Phi', \upsilon', \beta').$$

Let us denote these categories by $\mathcal{M}$ and $\mathcal{M}'$ respectively. Then F is also an equivalence of $\mathcal{D}$-module categories. Recall that as $\mathcal{D}$-module categories, $\mathcal{M}$ splits as

$$\bigoplus_{g \in G/H} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}.$$

A similar decomposition holds for $\mathcal{M}'$.

By considering these decompositions, it is easy to see that F induces an equivalence of $\mathcal{D}$-module categories between $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ and $\mathcal{N}'$ for some $g \in G$. Let us denote the restriction of F to $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ as a functor of $\mathcal{D}$-module categories by t_F. We can reconstruct the tuple $(\mathcal{N}', H', \Phi', \upsilon', \beta')$ from t_F in the following way: we have already seen that $\mathcal{N}'$ is equivalent to $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ and that the stabilizer subgroup of the category $\mathcal{N}'$ will be $H' = gHg^{-1}$.

Let us denote by Γ' the group $\mathrm{Aut}_{\mathcal{D}}(N')$. We have a natural isomorphism $\nu : \Gamma \to \Gamma'$ given by the formula

$$\nu(t) : \mathcal{N}' \xrightarrow{F^{-1}} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{1\boxtimes t} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{F} \mathcal{N}'.$$

Using the functor t_F and the map ν we can see that the map

$$\rho' : gHg^{-1} \to \mathrm{Out}(\Gamma')$$

which appears in the construction of the second module category is the composition

$$gHg^{-1} \xrightarrow{c_g} H \xrightarrow{\rho} \mathrm{Out}(\Gamma) \to \mathrm{Out}(\Gamma'),$$

where the last morphism is induced by ν. The map Φ' which lifts ρ' will depend on Φ in a similar way. The same holds for the second obstruction and its solution.

For the third obstruction, the situation is a bit more delicate. Since F is a functor of $\mathcal{C}$-module categories, we have, for each $a \in H$, a natural isomorphism between the functors

$$\mathcal{C}_{gag^{-1}} \boxtimes_{\mathcal{D}} (\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}) \xrightarrow{1\boxtimes F} \mathcal{C}_{gag^{-1}} \boxtimes_{\mathcal{D}} \mathcal{N}' \xrightarrow{\cdot} \mathcal{N}'$$

and

$$\mathcal{C}_{gag^{-1}} \boxtimes_{\mathcal{D}} (\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}) \xrightarrow{\cdot} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{F} \mathcal{N}'.$$

For any $a \in H$, the choice of the natural isomorphism is unique up to a scalar. A direct calculation shows that if we change the natural isomorphisms by a set of scalars ζ_a, we will get an equivalence $\mathcal{M}(N, H, \Phi, \upsilon, \beta) \to \mathcal{M}(N', H', \Phi', \upsilon', \beta'')$ where $\beta'' = \beta'\partial\zeta$. This is the reason that cohomologous solutions for the third obstruction will give us equivalent module categories.

In conclusion, we have the following:

Proposition 14. *Assume that we have an isomorphism $F : \mathcal{M}(\mathcal{N}, H, \Phi, \upsilon, \beta) \to \mathcal{M}(\mathcal{N}', H', \Phi', \upsilon', \beta')$ Then there is a $g \in G$ such that F induces an equivalence of $\mathcal{D}$-module categories $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}'$, and the data $(\mathcal{N}', H', \Phi', \upsilon', \beta')$ can be reconstructed from t_F in the way described above (β' will be reconstructible only up to a coboundary).*

Notice that we do not have any restriction on t_F. In other words, given any $t_F : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}'$ we can always reconstruct the tuple $(\mathcal{N}', H', \Phi', \upsilon', \beta')$ in the way described above.

We would like now to "decompose" the equivalence in Proposition 14 into several steps. The first ingredient that we need in order to get an equivalence is an element $g \in G$ such that $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \cong \mathcal{N}'$.

Consider now the case where $g = 1$, so that $\mathcal{N} = \mathcal{N}'$ and $H = H'$. In that case t_F is an autoequivalence of the $\mathcal{D}$-module category $\mathcal{N}$. Let us denote by $\psi_a : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}$ and by $\psi'_a : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}$ the structural equivalences of the two categories (where $a \in H$). Since F is an equivalence of $\mathcal{C}$-module categories, we see that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[r, "\psi_a"] \arrow[d, "1\boxtimes t_F"] & \mathcal{N} \arrow[d, "t_F"] \\ \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[r, "\psi'_a"] & \mathcal{N} \end{tikzcd}$$

and a direct calculation shows that Φ and Φ' satisfy the following formula:

$$\Phi'(a)(V) = t_F \Phi(a)(t_F^{-1} V t_F) t_F^{-1} \tag{6.1}$$

where V is any element in Γ.

Another way to write Eq. (6.1) is $\Phi' = C_{t_F} \Phi C_{t_F}^{-1}$, where by C_{t_F} we mean the automorphism of Γ of conjugation by t_F. In other words, this shows that we have some freedom in choosing Φ, and if we change Φ in the above way, we will still get equivalent categories.

Consider now the case where also $\Phi = \Phi'$. This means that for every $a \in H$ the element $t_F \Phi(a)(t_F)^{-1}$ is central in Γ. A direct calculation shows that the function r defined by $r(a) = t_F \Phi(a)(t_F)^{-1}$ is a one cocycle with values in $Z(\Gamma)$, and that $\upsilon/\upsilon' = r$. Notice in particular that by choosing arbitrary $t_F \in Z(\Gamma)$ we see that cohomologous solutions to the second obstruction will give us equivalent categories. However, we see that more is true, and it might happen that non cohomologous υ and υ' will define equivalent categories.

Last, if the situation is that $t_F = \Phi(a)(t_F)$ for every $a \in H$, we will have the same $(\mathcal{N}, H, \Phi, \upsilon)$, but β might be different. We have seen that if β and β' are cohomologous they will define equivalent categories, but it might happen that noncohomologous β and β' will define equivalent categories as well.

7. Functor categories

In this section we are going to describe the category of functors between module categories over an extension in terms of module categories over the trivial component of the extension. We prove a categorical analogue of Mackey's Theorem and we give a criterion for an extension to be group theoretical. In addition, given that $\mathcal{C}$ is a G-extension of $\mathcal{D}$, we describe the category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ of $\mathcal{C}$-module functors as an equivariantization of the category $\mathrm{Fun}_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$ of $\mathcal{D}$-module functors with respect to G.

7.1. Mackey's Theorem for module categories

Let $\mathcal{C}$ be a G-extension of $\mathcal{D}$. For any subset $S \subseteq G$ denote the subcategory $\bigoplus_{g\in S} \mathcal{C}_g$ by $\mathcal{C}_S$. If S is a subgroup of G then $\mathcal{C}_S$ is a fusion subcategory. Let H and K be subgroups of G and let $\mathcal{N}$ be a $\mathcal{C}_K$-module category. We prove now a categorical version of Mackey's Theorem.

Theorem 15. *$(\mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N})_{|\mathcal{C}_H} \cong \bigoplus_{HgK} \mathcal{C}_H \boxtimes_{\mathcal{C}_{H^g}} \mathcal{N}^g$, where $H^g = H \cap gKg^{-1}$ and $\mathcal{N}^g = (\mathcal{C}_{gK} \boxtimes_{\mathcal{C}_K} \mathcal{N})_{|H^g}$ is $\mathcal{C}_{H^g}$-module category and the sum is taken over all the double cosets.*

Proof. First, consider the transitive $H \times K^{op}$-action on HgK. The stabilizer of g is $\{(gkg^{-1}, k^{-1}) | k \in K, gkg^{-1} \in H\}$. Hence, $\mathcal{C}_{HgK}$ is isomorphic to $\mathcal{C}_H \boxtimes_{\mathcal{C}_{H^g}} \mathcal{C}_{gK}$ as $(\mathcal{C}_H, \mathcal{C}_K)$-bimodule category. Next, $(\mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N})_{|\mathcal{C}_H} \cong \bigoplus_{HgK} \mathcal{C}_{HgK} \boxtimes_{\mathcal{C}_K} \mathcal{N}$ where the sum is over all the double cosets. Finally $\mathcal{C}_{HgK} \boxtimes_{\mathcal{C}_K} \mathcal{N} \cong (\mathcal{C}_H \boxtimes_{\mathcal{C}_{H^g}} \mathcal{C}_{gK}) \boxtimes_{\mathcal{C}_K} \mathcal{N} \cong \mathcal{C}_H \boxtimes_{\mathcal{C}_{H^g}} (\mathcal{C}_{gK} \boxtimes_{\mathcal{C}_K} \mathcal{N}) = \mathcal{C}_H \boxtimes_{\mathcal{C}_{H^g}} \mathcal{N}^g$. $\square$

Remark 16. The above theorem could be stated in the original Mackey's Theorem language, namely $\mathrm{res}^G_H \mathrm{Ind}^G_K(\mathcal{N}) \cong \bigoplus_{HgK} \mathrm{Ind}^H_{H^g} \mathrm{res}^K_{H^g}(\mathcal{N}^g)$. One notices that the proof of the theorem uses only basic consideration about double cosets.

7.2. Functor categories

Assume that we have two module categories $\mathcal{M}_1 = \mathcal{M}(\mathcal{N}, H, \Phi, \upsilon, \beta)$, and $\mathcal{M}_2 = \mathcal{M}(\mathcal{N}', H', \Phi', \upsilon', \beta')$. Let us denote H' by K. Our goal is to calculate $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ in terms of functor categories over $\mathcal{D}$. We have

$$\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2) = \mathrm{Fun}_{\mathcal{C}}(\mathcal{C} \boxtimes_{\mathcal{C}_H} \mathcal{N}, \mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N}').$$

By Frobenius reciprocity

$$\mathrm{Fun}_{\mathcal{C}}(\mathcal{C} \boxtimes_{\mathcal{C}_H} \mathcal{N}, \mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N}') \cong \mathrm{Fun}_{\mathcal{C}_H}(\mathcal{N}, (\mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N}')_{|\mathcal{C}_H}).$$

Since a module category is, by definition, a semisimple category, every functor has both a left adjoint and a right adjoint. Taking left adjoints (right adjoints) gives us an equivalence of the corresponding functor categories.

Thus we obtain the following equivalence by taking left adjoints

$$\mathrm{Fun}_{\mathcal{C}_H}(\mathcal{N}, (\mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N}')_{|\mathcal{C}_H}) \cong \mathrm{Fun}_{\mathcal{C}_H}((\mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N}')_{|\mathcal{C}_H}, \mathcal{N}).$$

By Mackey's Theorem for module categories we have

$$\mathrm{Fun}_{\mathcal{C}_H}((\mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N}')_{|\mathcal{C}_H}, \mathcal{N}) \cong \mathrm{Fun}_{\mathcal{C}_H}\left(\bigoplus_{HgK} \mathcal{C}_H \boxtimes_{\mathcal{C}_{H^g}} \mathcal{N}'^g, \mathcal{N}\right)$$

and

$$\mathrm{Fun}_{\mathcal{C}_H}\left(\bigoplus_{HgK} \mathcal{C}_H \boxtimes_{\mathcal{C}_{H^g}} \mathcal{N}'^g, \mathcal{N}\right) \cong \bigoplus_{HgK} \mathrm{Fun}_{\mathcal{C}_{H^g}}(\mathcal{N}'^g, \mathcal{N}_{|\mathcal{C}_{H^g}}).$$

Finally, by taking right adjoints, we end up with the following

Proposition 17. *In the above notations*

$$Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2) \cong \bigoplus_{HgK} Fun_{\mathcal{C}_{H^g}}(\mathcal{N}_{|\mathcal{C}_{H^g}}, \mathcal{N}'^g).$$

7.3. A criterion for an extension to be group theoretical

Let $\mathcal{C}$ be a fusion category. Recall that a $\mathcal{C}$-module category $\mathcal{M}$ is called *pointed* if $\mathcal{C}^*_{\mathcal{M}}$, the dual category with respect to $\mathcal{M}$, is pointed. We say that $\mathcal{C}$ is group theoretical in case $\mathcal{C}$ has a pointed module category. As can easily be seen, $\mathcal{C}$ is pointed if and only if it has an indecomposable module category $\mathcal{N}$ such that any simple $\mathcal{C}$-linear functor $F : \mathcal{N} \to \mathcal{N}$ is invertible.

We now prove a criterion for an extension category to be group theoretical.

Theorem 18. *Let $\mathcal{C}$ be a G-extension of $\mathcal{D}$. Then $\mathcal{C}$ is group theoretical if and only if $\mathcal{D}$ has a pointed module category $\mathcal{N}$ which is G-stable, namely, for every $g \in G$, $\mathcal{N} \cong \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$.*

Proof. Suppose $\mathcal{N}$ is a pointed G-stable $\mathcal{D}$-module category. Consider $\mathcal{M} = \mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{N}$. By Frobenius reciprocity we have

$$\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M}) \cong \oplus_{g\in G} \mathrm{Fun}_{\mathcal{D}}(\mathcal{N}, \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}).$$

Since for any $g \in G$, $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \cong \mathcal{N}$ and since all simple functors in $\mathrm{Fun}_{\mathcal{D}}(\mathcal{N}, \mathcal{N})$ are invertible, we see that the same happens in $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$, that is- $\mathcal{M}$ is pointed over $\mathcal{C}$ and $\mathcal{C}$ is group theoretical.

Conversely, suppose that $\mathcal{C}$ is group theoretical and suppose that $\mathcal{M}$ is an indecomposable pointed $\mathcal{C}$-module category. We thus know that any simple functor $F : \mathcal{M} \to \mathcal{M}$ is invertible. We also know that there is a subgroup $H < G$ and an indecomposable $\mathcal{C}_H$-module category $\mathcal{N}$ such that $\mathcal{M} \cong \mathcal{C} \boxtimes_{\mathcal{C}_H} \mathcal{N} = \oplus_{gH\in G/H} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ Since $\mathcal{M}$ is indecomposable, it is easy to see that for every $g \in G$ there is some simple $\mathcal{C}$-endofunctor $F : \mathcal{M} \to \mathcal{M}$ such that $F(\mathcal{N}) \subseteq \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$. But such a functor must be invertible, and it follows that F induces an equivalence of $\mathcal{D}$ module categories $\mathcal{N} \cong \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$. Thus $\mathcal{N}$ is G-invariant.

Next, we would like to prove that $\mathcal{D}^*_{\mathcal{N}}$ is pointed. By Frobenius reciprocity we have $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M}) \cong \oplus_{gH\in G/H} \mathrm{Fun}_{\mathcal{C}_H}(\mathcal{N}, \mathcal{C}_g \boxtimes \mathcal{N})$ Thus the category $\mathrm{Fun}_{\mathcal{C}_H}(\mathcal{N}, \mathcal{N})$ is a fusion subcategory of the pointed category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ and is therefore pointed. We have a forgetful functor $\mathrm{Fun}_{\mathcal{C}_H}(\mathcal{N}, \mathcal{N}) \to \mathrm{Fun}_{\mathcal{D}}(\mathcal{N}, \mathcal{N})$ which is known to be onto (see Proposition 5.3 of [2]). This implies that $\mathrm{Fun}_{\mathcal{D}}(\mathcal{N}, \mathcal{N})$ is pointed, as required. $\square$

Remark 19. The above criterion is actually equivalent to the one given in Corollary 3.10 of [6], namely, $\mathcal{C}$ is group theoretical if and only if $\mathcal{Z}(\mathcal{D})$ contains a G-stable Lagrangian subcategory. In order to explain why the two conditions are equivalent, recall first the definitions of a Lagrangian subcategory and of the action of G on $\mathcal{Z}(\mathcal{D})$. A Lagrangian subcategory of $\mathcal{Z}(\mathcal{D})$ is a subcategory $\mathcal{E}$ such that $\mathcal{E}' = \mathcal{E}$ (see Section 3.2 of [1] for the definition of the Müger Centralizer $\mathcal{E}'$). The action of G on $\mathcal{Z}(\mathcal{D})$ is defined as follows: the center $\mathcal{Z}(\mathcal{D})$ can be considered as $\mathrm{Fun}_{\mathcal{D}\boxtimes\mathcal{D}^{op}}(\mathcal{D}, \mathcal{D})$, the category of $\mathcal{D}$-bimodule endofunctors of $\mathcal{D}$. Given an element $g \in G$ and a $\mathcal{D}$-bimodule functor $F : \mathcal{D} \to \mathcal{D}$, the functor $g(F) : \mathcal{D} \to \mathcal{D}$ is defined via

$$\mathcal{D} \xrightarrow{\cong} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{D} \boxtimes_{\mathcal{D}} \mathcal{C}_{g^{-1}} \xrightarrow{1\boxtimes_{\mathcal{D}} F \boxtimes_{\mathcal{D}} 1} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{D} \boxtimes_{\mathcal{D}} \mathcal{C}_{g^{-1}} \xrightarrow{\cong} \mathcal{D}.$$

Now if $\mathcal{N}$ is a pointed $\mathcal{D}$-module category, then the category $\mathcal{D}^*_{\mathcal{N}}$ will be pointed, and the isomorphism $\mathcal{Z}(\mathcal{D}) \cong \mathcal{Z}(\mathcal{D}^*_{\mathcal{N}})$ will give a Lagrangian subcategory of $\mathcal{Z}(\mathcal{D})$. From the above definition of the G-action, one can see that this correspondence is G-equivariant. Thus, if $\mathcal{D}$ has a G-invariant pointed module category, it will have a G-invariant Lagrangian subcategory. By using Theorem 4.66 of [1] together with Theorem 3.1 of [4] it follows that we can also go the other way around, and construct a pointed module category out of a Lagrangian subcategory of the center. The equivalence of the two conditions for being group theoretical thus follows. From our criterion, it also follows that if the fusion category $\mathcal{C}$ is given by the extension data (c, M, α) then the question whether or not $\mathcal{C}$ is group theoretical depends only on the category $\mathcal{D} = \mathcal{C}_1$ and on the homomorphism c. This is because the action of G on $\mathcal{D}$-module categories depends only on c and not on M and α.

7.4. Functor categories as equivariantizations

In this subsection we shall describe the category of $\mathcal{C}$-module functors between the module categories $\mathcal{M}(\mathcal{N}, H, \Phi, \upsilon, \beta)$ and $\mathcal{M}(\mathcal{N}', H', \Phi', \upsilon', \beta')$. For simplicity we shall denote these categories as $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively.

Consider the category $\mathrm{Fun}_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$. This is a k-linear category which by Theorem 2.16 of [2] is semisimple.

Lemma 20. *There is a natural G-action on $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$ induced by the structure of $\mathcal{C}$-module categories on $\mathcal{M}_1$ and $\mathcal{M}_2$.*

Proof. There are equivalences of $\mathcal{D}$-module categories $\psi_g : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_1 \cong \mathcal{M}_1$ and $\phi_g : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_2 \cong \mathcal{M}_2$, for every $g \in G$, defined by the $\mathcal{C}$-module structure on $\mathcal{M}_1$ and $\mathcal{M}_2$. Let $F : \mathcal{M}_1 \to \mathcal{M}_2$ be a $\mathcal{D}$-module morphism, we define $g \cdot F$ to be the following functor

$$\mathcal{M}_1 \xrightarrow{\psi_g^{-1}} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_1 \xrightarrow{Id_{\mathcal{C}_g} \boxtimes_{\mathcal{D}} F} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_2 \xrightarrow{\phi_g} \mathcal{M}_2 .$$

One can easily check that this defines an action of the group G on the category $\mathrm{Fun}_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$ in the sense of Section 4.1 of [1]. $\square$

In order to describe the Functor category in a more explicit way, we first recall the following definition:

Definition 21 (*See Section 4.1 in [1]*)**.** Let G be a group acting on a category $\mathcal{R}$. For $g, h \in G$ and $R \in \mathcal{R}$, denote by $\gamma_{g,h}(R)$ the natural isomorphism $g(h(R)) \to (gh)(R)$. Then an object of the equivariantization category $\mathcal{R}^G$ is a pair $(R, \{T_g\}_{g\in G})$, where $T_g : g(R) \to R$ are isomorphisms which satisfy the compatibility condition

$$T_{gh}\gamma_{g,h}(R) = T_g g(T_h).$$

Morphisms in $\mathcal{R}^G$ are defined in the obvious way.

Let $F : \mathcal{M}_1 \to \mathcal{M}_2$ be a $\mathcal{D}$-module functor.

To give a $\mathcal{D}$-module functor a structure of a $\mathcal{C}$-module functor is the same thing as to give, for every $g \in G$, a natural isomorphism between the functors

$$\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_1 \to \mathcal{M}_1 \xrightarrow{F} \mathcal{M}_2 \quad \text{and}$$
$$\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_1 \xrightarrow{1\boxtimes F} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_2 \to \mathcal{M}_2$$

which will satisfy certain coherence conditions. By composing the two functors with the quasi invertible functor $\mathcal{M}_1 \to \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_1$, we see that this is the same thing as to give, for every $g \in G$, an isomorphism of functors $T_g : g \cdot F \to F$. The compatibility conditions mentioned above are translated directly to the condition that $(F, \{T_g\})$ will be an element of the equivariantization category.

Let us conclude this discussion by the following.

Proposition 22. *The category $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is equivalent to the equivariantization $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)^G$ of the category $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$ with respect to the aforementioned G-action.*

Remark 23. Let $\mathcal{M}$ be an indecomposable $\mathcal{C}$-module category. Although $\mathcal{C}^*_{\mathcal{M}} \triangleq \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is a fusion category, $\mathrm{Fun}_{\mathcal{D}}(\mathcal{M}, \mathcal{M})$ is, in general, only a multifusion category because $\mathcal{M}$ might be decomposable as a $\mathcal{D}$-module category. When $\mathrm{Fun}_{\mathcal{D}}(\mathcal{M}, \mathcal{M})$ is also a fusion category, $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ contains a subcategory equivalent to *Rep*(G). This is exactly the subcategory of objects with support in the trivial object. In case $\mathrm{Fun}_{\mathcal{D}}(\mathcal{M}, \mathcal{M})$ is only multi-fusion, we will not necessarily have this subcategory.

In the next section we will give an intrinsic description of the functor categories, as categories of bimodules.

8. An intrinsic description by algebras and modules

The goal of this section is to explain more concretely the action of the grading group on indecomposable module categories, the action of the grading group on $\mathrm{Aut}_{\mathcal{D}}(\mathcal{N})$, the obstructions and their solutions.

In Theorem 1 of [8] Ostrik showed that any indecomposable module category over a fusion category $\mathcal{C}$ is equivalent as a module category to the category $\mathrm{Mod}_{\mathcal{C}} - A$ for some semisimple indecomposable algebra A in $\mathcal{C}$. In this section we will realize all the objects described in the previous sections by using algebras and modules inside $\mathcal{C}$. As before, we assume that $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$, we denote $\mathcal{C}_1$ by $\mathcal{D}$ and $\mathrm{Aut}_{\mathcal{D}}(\mathcal{N})$ by Γ.

8.1. The action of G on indecomposable module categories

Assume that A is a semisimple indecomposable algebra inside $\mathcal{D}$. Let $\mathcal{N} = \mathrm{Mod}_{\mathcal{D}} - A$ be the category of right A-modules inside $\mathcal{D}$. We denote by $\mathrm{Mod}_{\mathcal{C}_g} - A$ the category of A-modules with support in $\mathcal{C}_g$. We claim the following:

Lemma 24. *We have an equivalence of $\mathcal{D}$-module categories $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \cong Mod_{\mathcal{C}_g} - A$.*

Proof. We have already seen in Remark 8 in Section 2 that we have an equivalence of $\mathcal{C}$-module categories

$$\mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{N} \cong \mathrm{Mod}_{\mathcal{C}}(A)$$

which is given by $X \boxtimes M \mapsto X \otimes M$. As a $\mathcal{D}$-module category, the left hand side category decomposes as $\bigoplus_{g\in G} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ and the right hand side category decomposes as $\bigoplus_{g\in G} \mathrm{Mod}_{\mathcal{C}_g} - A$. By evaluating the equivalence above on objects of $\mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{N}$, we see that it translates one decomposition into the other, and therefore the functor $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathrm{Mod}_{\mathcal{C}_g} - A$ given by $X \boxtimes M \mapsto X \otimes M$ is an equivalence of $\mathcal{D}$-module categories. $\square$

Next, we understand how we can describe functors by using bimodules.

Lemma 25. *Let $\mathcal{N} = Mod_{\mathcal{D}} - A$ and $\mathcal{N}' = Mod_{\mathcal{D}} - A'$, and let $g \in G$. Then every functor $F : \mathcal{N} \to \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}'$ is of the form $F(T) = T \otimes_A Y$ for some $A - A'$-bimodule Y with support in $\mathcal{C}_g$. Here we identify $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}'$ with $Mod_{\mathcal{C}_g} - A'$ as above.*

Proof. The proof follows the lines of the remark after Proposition 2.1 of [9]. We simply consider $F(A)$. The multiplication map $A \otimes A \to A$ gives us a map $A \otimes F(A) \to F(A)$, thus equipping $F(A)$ with a structure of a left A-module. We now see that $F(A)$ is indeed an $A - A'$-bimodule. Since the category $\mathcal{N}$ is semisimple the functor F is exact. Since every object in $\mathcal{N}$ is a quotient of an object of the form $X \otimes A$ for some $X \in \mathcal{C}$, (due to the fact that the A-module X is a quotient of $X \otimes A$ by the map action), we see that F is given by $F(T) = T \otimes_A F(A)$. $\square$

Remark. Notice that by applying the (2-)functor $\mathcal{C}_{g^{-1}} \boxtimes_{\mathcal{D}} -$ we see that every functor $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}' \to \mathcal{N}$ is given by tensoring with some $A' - A$-bimodule with support in $\mathcal{C}_{g^{-1}}$.

8.2. The outer action of H on the group $Aut_{\mathcal{D}}(\mathcal{N})$. The first two obstructions

Assume, as in the rest of the paper, that we have a subgroup $H < G$ and a module category $\mathcal{N} = \mathrm{Mod}_{\mathcal{D}} - A$, and assume that $F_h : \mathcal{N} \cong \mathcal{C}_h \boxtimes_{\mathcal{D}} \mathcal{N}$ for every $h \in H$. It follows from Lemma 25 that this equivalence is of the form $F_h(M) = M \otimes_A A_h$ for some $A - A$-bimodule A_h with support in $\mathcal{C}_h$. The fact that this functor is an equivalence simply means that the bimodule A_h is an invertible $A - A$-bimodule. In other words, there is another $A - A$-bimodule B_h (whose support will necessarily be in $\mathcal{C}_{h^{-1}}$) such that $A_h \otimes_A B_h \cong B_h \otimes_A A_h \cong A$. By Lemma 25 we can identify the group $\Gamma = \mathrm{Aut}_{\mathcal{D}}(\mathcal{N})$ with the group of isomorphisms classes of invertible $A - A$-bimodules with support in $\mathcal{D}$.

Denote by Λ the group of isomorphisms classes all invertible $A - A$-bimodules with support in $\mathcal{C}_H$. Since every invertible $A - A$-bimodule is supported on a single graded component, we have a map $p : \Lambda \to H$ which assigns to an invertible $A - A$ bimodule the graded component it is supported on. We thus have a short exact sequence

$$1 \to \Gamma \to \Lambda \to H \to 1. \tag{8.1}$$

Using this sequence, we can understand the outer action of H on $\mathrm{Aut}_{\mathcal{D}}(\mathcal{N})$, and the first and the second obstruction. The outer action is given in the following way: for $h \in H$, choose an invertible $A - A$-bimodule A_h with support in $\mathcal{C}_h$. Choose an inverse to A_h and denote it by A_h^{-1}. Then the action of $h \in H$ on some invertible bimodule M with support in $\mathcal{D}$ is the following conjugation:

$$h \cdot M = A_h \otimes_A M \otimes_A A_h^{-1}.$$

This action depends on the choice we made of the invertible bimodule A_h.

The first obstruction is the possibility to lift this outer action to a proper action. In other words, it says that we can choose the A_h's in such a way that conjugation by $A_h \otimes A_{h'}$ is the same as conjugation by $A_{hh'}$, or in other words, in such a way that for every $h, h' \in H$, the invertible bimodule

$$B_{h,h'} = A_h \otimes_A A_{h'} \otimes_A A_{hh'}^{-1}$$

will be in the center of Γ (again, we identify Γ with the group of invertible bimodules with support in $\mathcal{D}$). A solution for the first obstruction will be a choice of a set of such bimodules A_h.

The second obstruction says that the cocycle $(h, h') \mapsto B_{h,h'}$ is trivial in $H^2(H, Z(\mathrm{Aut}_{\mathcal{D}}(\mathcal{N}))$. This simply says that we can change A_h to be $A_h \otimes_A D_h$ for some $D_h \in Z(\mathrm{Aut}_{\mathcal{D}}(\mathcal{N}))$, in such a way that

$$(A_h \otimes_A D_h) \otimes_A (A_{h'} \otimes_A D_{h'}) \otimes_A (A_{hh'} \otimes_A D_{hh'})^{-1} \cong A$$

as A-bimodules. A solution for the second obstruction will be a choice of such a set D_h of bimodules.

It is easier to understand the first and the second obstruction together: we have one big obstruction, the sequence (8.1) should split, and we need to choose a splitting. First, if the sequence splits, then we can lift the outer action into a proper action, and we need to choose such a lifting. Then, the obstruction to the splitting *with the chosen action* is given by a two cocycle with values in the center of Γ. Thus, a solution for both the first and the second obstruction will be a choice of bimodules A_h for every $h \in H$ such that the support of A_h is in $\mathcal{C}_h$ and such that $A_h \otimes_A A_{h'} \cong A_{hh'}$ for every $h, h' \in H$. Following the line of Section 6, we see that we are interested in splittings only up to conjugation by an element of Γ.

8.3. The third obstruction

Assume then that we have a set of bimodules A_h as in the end of the previous subsection. We would like to understand now the third obstruction.

Recall that we are trying to equip $\mathcal{N}$ with a structure of a $\mathcal{C}_H$-module category. By Ostrik's Theorem (see [8]), there is an object $\mathcal{N} \in \mathcal{N}$ such that $A \cong \underline{\mathrm{Hom}}_{\mathcal{D}}(N, N)$ where by $\underline{\mathrm{Hom}}_{\mathcal{D}}$ we mean the internal Hom of $\mathcal{N}$, where we consider $\mathcal{N}$ as a $\mathcal{D}$-module category. So far we gave equivalences $F_h : \mathcal{N} \to \mathcal{C}_h \boxtimes_{\mathcal{D}} \mathcal{N}$. If $\mathcal{N}$ were a $\mathcal{C}_H$-module category via the choices of these equivalences, then the internal $\mathcal{C}_H$-Hom, $\tilde{A} = \underline{\mathrm{Hom}}_{\mathcal{C}_H}(N, N)$ would be

$$\tilde{A} = \bigoplus_{h \in H} A_h.$$

We thus see that to give on $\mathcal{N}$ a structure of a $\mathcal{C}_H$-module category is the same as to give on $\tilde{A}$ a structure of an associative algebra. For every $h, h' \in H$, choose an isomorphism of $A - A$-bimodules $A_h \otimes_A A_{h'} \to A_{hh'}$. Notice that since these are invertible $A - A$-bimodules, there is only one such isomorphism up to a scalar.

Now for every $h, h', h'' \in H$, we have two isomorphisms $(A_h \otimes_A A_{h'}) \otimes_A A_{h''} \to A_{hh'h''}$, namely

$$(A_h \otimes_A A_{h'}) \otimes_A A_{h''} \to A_{hh'} \otimes_A A_{h''} \to A_{hh'h''}$$

and

$$(A_h \otimes_A A_{h'}) \otimes_A A_{h''} \to A_h \otimes_A (A_{h'} \otimes_A A_{h''}) \to A_h \otimes_A A_{h'h''} \to A_{hh'h''}.$$

This two isomorphisms differ by a scalar $b(h, h', h'')$. The function $(h, h', h'') \mapsto b(h, h', h'')$ is a three cocycle which is the third obstruction. A solution to the third obstruction will thus be a choice of isomorphisms $A_h \otimes_A A_{h'} \to A_{hh'}$ which will make $\tilde{A}$ an associative algebra. Once we have such a choice, we can change it by some two cocycle to get another solution.

8.4. Functor categories

We end this section by giving an intrinsic description of functor categories. Assume that we have two module categories $\mathcal{M}_1 = \mathcal{M}(\mathcal{N}, H, \Phi, v, \beta)$, and $\mathcal{M}_2 = \mathcal{M}(\mathcal{N}', H', \Phi', v', \beta')$. Let us denote H' by K. As we have seen in the previous subsections, if $\mathcal{N} \cong \mathrm{Mod}_{\mathcal{D}} - A_1$ and $\mathcal{N}' \cong \mathrm{Mod}_{\mathcal{D}} - B_1$, then $\mathcal{M}_1 \cong \mathrm{Mod}_{\mathcal{C}} - A$ and $\mathcal{M}_2 \cong \mathrm{Mod}_{\mathcal{C}} - B$, where A is an algebra of the form $\oplus_{h \in H} A_h$, and a similar description holds for B.

The functor category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is equivalent to the category of $A - B$-bimodules in $\mathcal{C}$. Since A and B have a graded structure, we will be able to say something more concrete on this category.

Let X be an indecomposable $A - B$-bimodule in $\mathcal{C}$. It is easy to see that the support of X will be contained inside a double coset of the form HgK for some $g \in G$. Since the bimodules A_h and B_k are invertible, it is easy to see that the support will be exactly this double coset.

Consider now the g-component X_g of X. As can easily be seen, this is an $A_1 - B_1$-bimodule. Actually, more is true. Consider the category $\mathcal{C} \boxtimes \mathcal{C}^{op}$. Inside this category we have the algebra

$$(AB)_g = \oplus_{x \in H \cap gKg^{-1}} A_x \boxtimes B_{g^{-1}x^{-1}g}$$

with the multiplication defined by the restricting the multiplication from $A \boxtimes B \in \mathcal{C} \boxtimes \mathcal{C}^{op}$. The category $\mathcal{C}$ is a $\mathcal{C} \boxtimes \mathcal{C}^{op}$-module category in the obvious way, and we have a notion of an $(AB)_g$-module inside $\mathcal{C}$.

Lemma 26. *The category of $(AB)_g$-modules inside $\mathcal{C}$ is equivalent to the category of $A - B$-bimodules with support in the double coset HgK.*

Proof. If X is an $A - B$-bimodule with support in HgK, then X_g is an $(AB)_g$-module via restriction of the left A-action and the right B-action. Conversely, if V is an $(AB)_g$-module inside $\mathcal{C}$, we can consider the induced module

$$(A \boxtimes B) \otimes_{(AB)_g} V.$$

This is an $A - B$-bimodule, and one can see that the two constructions gives equivalences in both directions. □

Remark. This is a generalization of Proposition 3.1 of [9], where the same situation is considered for the special case that $\mathcal{C} = Vec_G^{\omega}$ and $\mathcal{D} = 1$. Also, notice that the decomposition to double cosets is the one which appears in Theorem 15

In conclusion, we have the following

Proposition 27. *The functor category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is equivalent to the category of $A-B$-bimodules. Each such simple bimodule is supported on a double coset of the form HgK, and the subcategory of bimodules with support in HgK is equivalent to the category of $(AB)_g$-modules inside $\mathcal{C}$.*

9. A detailed example: classification of module categories over the Tambara Yamagami fusion categories and their dual categories

As an example of our results, we shall now describe the module categories over the Tambara Yamagami fusion categories $\mathcal{C} = \mathcal{TY}(A, \chi, \tau)$ and the corresponding dual categories. Let A be a finite group. Let R_A be the fusion ring with basis $A \cup \{m\}$

whose multiplication is given by the following formulas:

$$g \cdot h = gh, \forall g, h \in A$$
$$g \cdot m = m \cdot g = m$$
$$m \cdot m = \sum_{g \in A} g$$

In [11] Tambara and Yamagami classified all fusion categories with the above fusion ring. They showed that if there is a fusion category $\mathcal{C}$ whose fusion ring is R_A then A must be abelian. They also showed that for a given A such fusion categories can be parameterized (up to equivalence) by pairs (χ, τ) where $\chi : A \times A \to k^*$ is a nondegenerate symmetric bicharacter, and τ is a square root (either positive or negative) of $\frac{1}{|A|}$. We denote the corresponding fusion category by $\mathcal{C} := \mathcal{T}\mathcal{Y}(A, \chi, \tau)$.

The category $\mathcal{C}$ is naturally graded by $\mathbb{Z}_2 = \langle\sigma\rangle$. The trivial component is Vec_A (with trivial associativity constraints) and the nontrivial component, which we shall denote by $\mathcal{M}$, has one simple object m. In Section 9 of [3] the authors described how a Tambara Yamagami fusion category corresponds to an extension data of Vec_A by the group $\mathbb{Z}_2$. We shall explain now the classification of module categories over $\mathcal{T}\mathcal{Y}(A, \chi, \tau)$ given by our parameterization.

Since A is an abelian group and the associativity constraints in Vec_A are trivial, module categories over Vec_A are parameterized by pairs (H, ψ) where $H < A$ is a subgroup and $\psi \in H^2(H, k^*)$. We shall denote the corresponding module category by $\mathcal{M}(H, \psi)$. As explained in Section 3, we have a natural action of $\mathbb{Z}_2 = \langle\sigma\rangle$ on the set of equivalence classes of module categories over Vec_A. We shall describe this action in Section 9.1.

Recall that the second component in the parameterization of a module category is a subgroup of the grading group. If this subgroup is the trivial subgroup, then we will just have a category which is induced from Vec_A. It is easy to see that such categories decompose over Vec_A to the direct sum of two indecomposable module categories. In that case, all the obstructions and solutions will be trivial. If this subgroup is $\mathbb{Z}_2$ itself, we will have a $\mathcal{C}$-module category structure on $\mathcal{M}(H, \psi)$ for some H and some ψ. In that case, it must hold that $\sigma(H, \psi) = (H, \psi)$, and we may have some nontrivial obstructions and solutions.

The rest of this section will be devoted to analyze the action of σ and the obstructions and their solutions (for the case in which we have obstructions). We will also describe the relations of our result with the result of Tambara on fiber functors on Tambara Yamagami categories, and also describe the dual categories.

We would like now to describe the main result of this section. We will split our main classification result into two proposition, according to the subgroup of $\mathbb{Z}_2$ which appears in the parameterization. Our first proposition follows in a straight forward way from the discussion in the previous sections

Proposition 28. *Module categories over $\mathcal{C}$ whose parameterization begins with $(\mathcal{M}(H, \psi), 1, \ldots)$ are the induced categories $Ind_{Vec_A}^{\mathcal{C}}(\mathcal{M}(H, \psi))$. We will have an equivalence of $\mathcal{C}$-module categories $Ind_{Vec_A}^{\mathcal{C}}(\mathcal{M}(H, \psi)) \cong Ind_{Vec_A}^{\mathcal{C}}(\mathcal{M}(H', \psi'))$ if and only if $(H, \psi) = (H', \psi')$ or if $(H, \psi) = \sigma(H'\psi')$.*

In order to describe the other case, we need some notations. Suppose that $H < A$ is a subgroup which contains $H^\perp$ (the subgroup perpendicular to H with respect to χ). If we denote by $\bar{H} := H/H^\perp$, then χ induces a non-degenerate symmetric bicharacter $\bar{\chi} : \bar{H} \times \bar{H} \to k^*$. If $\psi \in H^2(H, k^*)$ satisfies $\mathrm{Rad}(\psi) = H^\perp$ (the definition of $\mathrm{Rad}(\psi)$ is given in Section 9.1), then ψ is the inflation of a nondegenerate two cocycle $\bar{\psi}$ on $\bar{H}$. We will usually not distinguish between ψ and $\bar{\psi}$.

Proposition 29. *For $\mathcal{M}(H, \psi)$ to have a structure of a $\mathcal{C}$ module category, it is necessary that $\sigma(H, \psi) = (H, \psi)$. This implies that $Rad(\psi) = H^\perp < H$. If this holds, then $\mathcal{C}$-module categories structures on $\mathcal{M}(H, \psi)$ are parameterized by pairs (s, ν) where $s : H/H^\perp \to H/H^\perp$ is an involutive automorphism, and $\nu : H/H^\perp \to k^*$ is a function which satisfy for every $a, b \in H/H^\perp$*

$$\bar{\chi}(a, b) = \psi(s(a), b)/\psi(b, s(a))$$
$$\partial\nu(a, b) = \psi(a, b)/\psi(s(b), s(a))$$
$$\nu(a)\nu(s(a)) = 1$$
$$sign\left(\sum_{s(a)=a} \nu(a)\right) = sign(\tau)$$

Two such pairs (s, ν) and (s', ν') will give equivalent module category structures on $\mathcal{M}(H, \psi)$ if and only if $s = s'$ and there exist a character $\phi : H/H^\perp \to k^*$ such that $\nu(h)/\nu'(h) = \eta(h)/\eta(s(h))$.

9.1. The action of σ on indecomposable module categories and representations of twisted abelian group algebras

Recall that the Vec_A-module category $\mathcal{N} = \mathcal{M}(H, \psi)$ is the category of right modules over the algebra $k^\psi H$ inside Vec_A. We would like to understand the Vec_A-module category $\mathcal{M} \boxtimes_{Vec_A} \mathcal{N}$.

As explained in Section 8, this module category can be described as the category of right $k^\psi H$-modules with support in the category $\mathcal{M}$, the nontrivial grading component of $\mathcal{C}$. A $k^\psi H$-module with support in $\mathcal{M}$ is of the form $m \otimes V$ where V is a vector space which is a $k^\psi H$-module in the usual sense. So the category $\mathcal{M} \boxtimes_{Vec_A} \mathcal{N}$ is equivalent, at least as an abelian category, to the category of $k^\psi H$-modules in Vec.

We would like to describe $\mathcal{M} \boxtimes_{Vec_A} \mathcal{N}$ as a module category of the form $\mathcal{M}(H', \psi')$ for some $H' < A$ and some two cocycle $\psi' \in H^2(H', k^*)$. In order to do so, we begin by describing the simple $k^{\psi}H$-modules in Vec (they will correspond to the simple objects in $M \boxtimes_{Vec_A} \mathcal{N}$).

Let $k^{\psi}H = \oplus_{h\in H} U_h$. The multiplication in $k^{\psi}H$ is given by the rule $U_h U_k = \psi(h, k)U_{hk}$. Denote by $R = \mathrm{Rad}(\psi)$ the subgroup of all $h \in H$ such that U_h is central in $k^{\psi}H$.

As the field k is algebraically closed of characteristic zero and H is abelian, the data that stored in the cocycle ψ is simply the way in which the U_h's commute. More precisely, let us define the following alternating form on H:

$$\xi_{\psi}(a, b) = \psi(a, b)/\psi(b, a).$$

It turns out (see Proposition 2.6 in [10]) that the assignment $\psi \mapsto \xi_{\psi}$ depends only on the cohomology class of ψ, and that it gives a bijection between $H^2(H, k^*)$ and the set of all alternating forms on H. The elements of R can be described as those $h \in H$ such that $\xi_{\psi}(h, -) = 1$. As can easily be seen, ξ_{ψ} is the inflation of an alternating form on H/R. It follows easily that ψ is the inflation of a two cocycle $\bar{\psi}$ on H/R.

It can also be seen that $\xi_{\bar{\psi}}$ is nondegenerate on H/R and that $k^{\bar{\psi}}H/R \cong M_n(k)$ where $n = \sqrt{|H/R|}$. It follows that $k^{\bar{\psi}}H/R$ has only one simple module (up to isomorphism) which we shall denote by V_1 (i.e, $\bar{\psi}$ is non degenerate on H/R). By inflation, V_1 is also a $k^{\psi}H$-module. Let ζ be a character of H, and let k^{ζ} be the corresponding one dimensional representation of H. Then $k^{\zeta} \otimes V_1$ is also a simple module of $k^{\psi}H$, where H acts diagonally. It turns out that these are all the simple modules of $k^{\psi}H$, and that $V_{\zeta_1} \cong V_{\zeta_2}$ if and only if the restrictions of ζ_1 and ζ_2 to R coincide.

The simple modules of $k^{\psi}H$ are thus parameterized by the characters of R (we use here the fact that the restriction from the character group of H to that of R is onto). For every character ζ of R, we denote by V_{ζ} the unique simple module of $k^{\psi}H$ upon which R acts via the character ζ. So the simple $k^{\psi}H$-modules with support in $\mathcal{M}$ are of the form $m \otimes V_{\zeta}$.

In order to understand the structure of $\mathcal{M} \boxtimes_{Vec_A} \mathcal{N}$ as a Vec_A-module category, let us describe $V_a \otimes (m \otimes V_{\zeta})$ for $a \in A$. It can easily be seen that this is also a simple module, so we just need to understand via which character R acts on it. Using the associativity constraints in $\mathcal{TY}(A, \chi, \tau)$, we see that for $v \in V_{\zeta}$ and $r \in R$ we have

$$(V_a \otimes m \otimes v) \cdot U_r = \chi(a, r)V_a \otimes (m \otimes v \cdot U_r) = \chi(a, r)\zeta(r)V_a \otimes m \otimes v.$$

This means that $V_a \otimes (m \otimes V_{\zeta}) = m \otimes V_{\zeta\chi(a,-)}$. So the stabilizer of V_{ζ} is the subgroup of all $a \in A$ such that $\chi(a, r) = 1$ for all $r \in R$, i.e., it is $R^{\perp}$. It follows that $\mathcal{M} \boxtimes_{Vec_A} \mathcal{N}$ is equivalent to a category of the form $\mathcal{M}(R^{\perp}, \tilde{\psi})$. Where $\tilde{\psi}$ is some two cocycle.

Let us figure out what is $\tilde{\psi}$. If $a \in R^{\perp}$, then the restriction of $\chi(a, -)$ to H is a character which vanishes on R. Therefore, there is a unique (up to multiplication by an element of R) element $t_a \in H$ such that $\xi_{\psi}(t_a, -) = \chi(a, -)$. It follows that there is an isomorphism $r_a : V_a \otimes (m \otimes V_1) \to m \otimes V_1$ which is given by the formula $V_a \otimes (m \otimes v) \mapsto m \otimes (v \cdot U_{t_a})$. Now for every $a, b \in R^{\perp}$, $\tilde{\psi}(a, b)$ should make the following diagram commute:

$$\begin{tikzcd}
(V_a \otimes V_b) \otimes (m \otimes V_1) \arrow[r] \arrow[d, "r_{ab}"] & V_a \otimes (V_b \otimes (m \otimes V_1)) \arrow[r, "r_b"] & V_a \otimes (m \otimes V_1) \arrow[d, "r_a"] \\
m \otimes V_1 \arrow[rr, "\tilde{\psi}(a,b)"] & & m \otimes V_1
\end{tikzcd}$$

An easy calculation shows that this means that $\tilde{\psi}(a, b) = \psi(t_b, t_a)$. We thus have the following result:

Lemma 30. *We have $\sigma \cdot \mathcal{M}(H, \psi) \equiv \mathcal{M}(R^{\perp}, \tilde{\psi})$ where R is the radical of ψ and $\tilde{\psi}$ is described above.*

Suppose now that $\sigma \cdot \mathcal{M}(H, \psi) \equiv \mathcal{M}(H, \psi)$. This implies that $\mathrm{Rad}(\psi) = H^{\perp}$. The bicharacter χ defines by restriction a pairing on $H \times H$, and by dividing out by $H^{\perp}$, we get a nondegenerate symmetric bicharacter $\bar{\chi} : H/H^{\perp} \times H/H^{\perp} \to k^*$. It is easy to see that the assignment $h \mapsto t_h$ that was described above induces an automorphism s of $H/H^{\perp}$ which satisfies

$$\bar{\chi}(a, b) = \xi_{\bar{\psi}}(s(a), b). \tag{9.1}$$

The fact that $\tilde{\psi} = \psi$ means that $\xi_{\bar{\psi}}(s(b), s(a)) = \xi_{\bar{\psi}}(a, b)$. Equivalently, this means that $\bar{\chi}(a, b) = \xi_{\bar{\psi}}(s(a), b) = \xi_{\bar{\psi}}(s(b), s^2(a)) = \bar{\chi}(b, s^2(a))$ and since $\bar{\chi}$ is nondegenerate, this is equivalent to the fact that $s^2 = Id$.

In summary:

Lemma 31. *We have $\sigma \cdot \mathcal{M}(H, \psi) \equiv \mathcal{M}(H, \psi)$ if and only if the following two conditions hold:*

1. $H^{\perp} < H$.
2. *There is an automorphism s of order 2 of $H/H^{\perp}$ such that $(a, b) \mapsto \bar{\chi}(s(a), b)$ is an alternating form, and the inflation of this alternating form to H is ξ_{ψ}.*

9.2. The vanishing of the first obstruction and invertible bimodules with support in Vec_A

Assume now that we have a module category $\mathcal{M}(H,\psi)$ such that $\sigma \cdot \mathcal{M}(H,\psi) \equiv \mathcal{M}(H,\psi)$. We would like to describe all module categories whose classification data begins with $(\mathcal{M}(H,\psi), \mathbb{Z}_2, \ldots)$. In other words, we would like to describe all possible ways (if any) to furnish a structure of a $\mathcal{C}$-module category on $\mathcal{M}(H,\psi)$.

So let s be an automorphism as in Lemma 31. In order to explain the first obstruction for furnishing a $\mathcal{TY}(A,\chi,\tau)$-module category structure on $\mathcal{M}(H,\psi)$, we need to consider the group of invertible $k^{\psi}H$-bimodules in $\mathcal{TY}(A,\chi,\tau)$. As we have seen in Section 8, such an invertible bimodule with support in Vec_A $(\mathcal{M})$ corresponds to a functor equivalence $F : \mathcal{N} \to \mathcal{N}$ $(F : \mathcal{M} \boxtimes_{Vec_A} \mathcal{N} \to \mathcal{N})$. The functor is given by tensoring with the invertible bimodule.

Let us first classify invertible $k^{\psi}H$-bimodules with support in Vec_A. Their description was given in Proposition 3.1 of Ostrik's paper [9]. We recall it briefly.

If $a \in A$ and λ is a character on H, we define the bimodule $M_{a,\lambda}$ to be

$$\oplus_{h\in H} V_{ah},$$

where the action of $k^{\psi}H$ is given by

$$U_h \cdot V_{ah'} \cdot U_{h''} = \psi(h,h')\lambda(h)\psi(hh',h'')V_{ahh'h''}.$$

Choose now coset representatives $a_1,\ldots,a_r$ of H in A. Proposition 3.1 of [9] tells us that the modules $M_{a_i,\lambda}$ where $i = 1,\ldots r$ and $\lambda \in \hat{H}$ are all the invertible $k^{\psi}H$-bimodules, and each invertible bimodule with support in Vec_A appears in this list exactly once.

By a more careful analysis we can get to the following description of the group of invertible bimodules: we have a homomorphism $\xi : H \to \hat{H}$ given by $h \mapsto \xi_{\psi}(h,-)$. Then the group E of all invertible bimodules with support in Vec_A can be described as the pushout which appears in the following diagram: (see Theorem 5.2 of [7] for a more general result)

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\xi"] & A \arrow[d] \\ \hat{H} \arrow[r] & E \end{tikzcd}$$

The group E is thus also isomorphic to the group $\mathrm{Aut}_{Vec_A}(\mathcal{M}(H,\psi))$. Notice that the group E is *abelian*. A solution to the first obstruction is a lifting of the natural map (see Section 4) $\mathbb{Z}_2 \to \mathrm{Out}(E)$ to a map $\mathbb{Z}_2 \to \mathrm{Aut}(E)$. But since E is abelian, $\mathrm{Out}(E) = \mathrm{Aut}(E)$, so this problem is trivial, and it has only one solution. So we have a proper (and not just outer) action of $\mathbb{Z}_2$ on E.

9.3. The group of all invertible bimodules and the second obstruction

Since $\mathcal{M}(H,\psi)$ is σ-invariant, we see by Section 8 that the group $\tilde{E}$ of (isomorphism classes of) invertible $k^{\psi}H$-bimodules in $\mathcal{C}$ is given as an extension

$$\Sigma : 1 \to E \to \tilde{E} \to \mathbb{Z}_2 \to 1.$$

Moreover, we have seen that the second obstruction is the cohomology class of this extension in $H^2(\mathbb{Z}_2, E)$, and that a solution to the second obstruction is a splitting of this sequence, up to conjugation by an element of E.

So our next goal is to understand if the sequence Σ splits. For this, we would like to understand the structure of the group $\tilde{E}$ better, and for this reason, we will describe now the invertible $k^{\psi}H$ bimodules with support in $\mathcal{M}$ (these are the elements of $\tilde{E}$ which goes to the nontrivial element in $\mathbb{Z}_2$). We begin by choosing such an invertible bimodule X explicitly. It should be of the form $X = m \otimes V$, where V is both a left and a right $k^{\psi}H$-module. The interaction between the left structure and the right structure follows from the associativity constraints and is given by the formula

$$(U_h \cdot v) \cdot U_{h'} = \chi(h,h')U_h \cdot (v \cdot U_{h'}). \tag{9.2}$$

The fact that X is invertible implies that V has to be simple as a left and as a right $k^{\psi}H$-module. Assume that V is V_{ϕ} from Section 9.1 as a right $k^{\psi}H$-module, where ϕ is some character of $H^{\perp}$. We need to define on V a structure of a left $k^{\psi}H$-module. By Eq. (9.1) we know that

$$(v \cdot U_{t_h}) \cdot U_{h'} = \chi(h,h')(v \cdot U_{h'})U_{t_h}.$$

By Eq. (9.2) and by the simplicity of V, we see that this means that we must have

$$U_h \cdot v = \nu(h)v \cdot U_{t_h} \tag{9.3}$$

for some set of scalars $\{\nu(h)\}_{h\in H}$. An easy calculation shows that these scalars should satisfy the equation

$$\nu(ab)\psi(a,b) = \nu(a)\nu(b)\psi(t_b,t_a)\phi(t_at_bt_{ab}^{-1})$$

for every $a, b \in H$. In other words-

$$\partial(\nu\phi(t_-)) = \psi(a, b)/\psi(t_b, t_a). \tag{9.4}$$

Since $\mathcal{N}$ is σ-invariant, we do know that the cocycles $\psi(a, b)$ and $\psi(t_b, t_a)$ are cohomologous, and therefore such a function ν exists. Notice that we have some freedom in choosing ν- we can change it to be $\nu\eta$ where η is some character on H. It is easy to see by this construction that the invertible $k^\psi H$-bimodules with support in $\mathcal{M}$ are parameterized by pairs (ϕ, ν) where ϕ is a character of $H^\perp$ by which it acts from the right on the module, and ν is a function which satisfy the equation

$$\partial\nu(a, b) = \psi(a, b)/\psi(t_a, t_b)\phi(t_{ab}t_a^{-1}t_b^{-1}).$$

We denote the corresponding invertible bimodule by $X(\phi, \nu)$. It is possible to choose ψ and t_h in such a way that will assure us that $\nu|_{H^\perp}$ is a character (for example, take ψ an inflation of a cocycle on $H/H^\perp$ and take $t_h = 1$ for $h \in H^\perp$. We will thus assume henceforth that this is the case.

We fix an invertible bimodule X for which $\phi = 1$, and for which the restriction of ν to $H^\perp$ is the trivial character (we use here the fact that we can alter ν by a character of H and the fact that any character of $H^\perp$ can be extended to a character of H). It is also easy to see that we can choose ϕ as we wish because for every choice of ϕ, Eq. (9.4) will have a solution. One last remark: notice that in that case, where $H^\perp$ acts trivially from the left and from the right, Eq. (9.3) implies that $\nu(h)$ depends only on the coset of h in $H^\perp$. We can thus consider ν also as a function from $H/H^\perp$ to k^*.

In conclusion, we have fixed an invertible bimodule X with support in $\mathcal{M}$ upon which $H^\perp$ acts trivially from the left and from the right. Any other invertible bimodule with support in $\mathcal{M}$ will be of the form $X \otimes_{k^\psi H} e$ for some $e \in E$. The action of the nontrivial element σ of $\mathbb{Z}_2$ on E will be conjugation by X, and the second obstruction is the possibility to choose an $e \in E$ such that

$$(X \otimes e) \otimes_{k^\psi H} (X \otimes e) \cong k^\psi H.$$

9.4. *The action of σ on E, and an explicit calculation of the second obstruction*

We would like to understand now the action of σ on E. This in turn will help us to understand the second obstruction.

As we have seen, a general element in E will be a bimodule of the form $U_{a_i,\lambda}$. So we would like to understand what is the bimodule $\sigma(U_{a_i,\lambda})$.

We have the equation

$$X \otimes_{k^\psi H} U_{a_i,\lambda} = \sigma(U_{a_i,\lambda}) \otimes_{k^\psi H} X.$$

A similar calculation to the calculations we had so far reveals the fact that if X is given by $(1, \nu)$ then $U_{a_i,\lambda} \otimes_{k^\psi H} X$ is given by $(\chi(a_i, -), \nu\lambda\chi^{-1}(a_i, t_-))$, while $X \otimes_{k^\psi H} U_{a_i,\lambda}$ is given by $(\lambda^{-1}, \nu\chi^{-1}(a_i, -)\lambda(t_-))$. From these two formulas we can derive an explicit formula for the action of σ on E. It follows that if $\sigma(U_{a_i,\lambda}) = U_{a_j,\mu}$ then j is the unique index which satisfies $\lambda^{-1} = \chi(a_j, -)$ on $H^\perp$, and μ is given by the formula $\mu = \chi^{-1}(a_i, -)\lambda(t_-)\chi(a_j, t_-)$.

Let us find now the second obstruction. For this, we just need to calculate $Q := X \otimes_{k^\psi H} X$. Consider first $X \otimes X$. It is isomorphic to $V \otimes V \otimes \bigoplus_{a\in A}(V_a)$. The bimodule Q is the quotient of $X \otimes X$ when we divide out the action of $k^\psi H$.

Let us divide out first by the action of $H^\perp$. If $h \in H^\perp$ we see that we divide $V \otimes V \otimes V_a$ by $v \otimes w - \chi(a, h)v \otimes w$. If $a \notin H$ then there is an $h \in H^\perp$ such that $\chi(a, h) \neq 1$. Therefore the support of $X \otimes_{k^\psi H} X$ will be Vec_H. Since V is simple as a left and as a right $k^\psi H$-module, it is easy to see that $V \otimes_{k^\psi H} V$ is one dimensional. We thus see that $X \otimes_{k^\psi H} X \cong U_{1,\lambda}$ for some character λ. A direct calculation shows that $\lambda(h) = \nu(h)\nu(t_h)$. This means that the second obstruction is the character λ, as an element of $H^2(\mathbb{Z}_2, E) = E^\sigma/im(1 + \sigma)$ (recall that $\hat{H}$ is a subgroup of E).

Suppose that the second obstruction does vanish, and suppose that we have a solution $X(\phi, \nu)$. In other words $X(\phi, \nu) \otimes_{k^\psi H} X(\phi, \nu) \cong k^\psi H$. A direct calculation similar to the one we had above shows that the restrictions of ϕ and ν to $H^\perp$ coincide. Recall from Section 8 that if $U_{a_i,\lambda}$ is any invertible $k^\psi H$-bimodule with support in Vec_A, then this solution is equivalent to the solution $U_{a_i,\lambda} \otimes_{k^\psi H} X(\phi, \nu) \otimes_{k^\psi H} U_{a_i,\lambda}^{-1}$. Extend the character $\nu_{H^\perp}$ to a character η of H. A direct calculation shows that $U_{1,\eta} \otimes_{k^\psi H} X(\phi, \nu) \otimes_{k^\psi H} U_{1,\eta}^{-1} = X(1, \nu')$. It follows that we can assume without loss of generality that $\phi = 1$.

As we have seen above, $X(1, \nu)^{\otimes 2} \cong k^\psi H$ if and only if $\nu(h)\nu(t_h) = 1$ for every $h \in H$. So the second obstruction vanishes if and only if there is a function ν which satisfies Eq. (9.4) and also the equation

$$\nu(h)\nu(t_h) = 1 \tag{9.5}$$

for every $h \in H$. It might happen, however, that we will have two different solutions ν and ν', that will be equivalent, that is, there will be an invertible $k^\psi H$-bimodule $U_{a_i,\lambda}$ such that $U_{a_i,\lambda} \otimes_{k^\psi H} X(1, \nu) \otimes_{k^\psi H} U_{a_i,\lambda}^{-1} \cong X(1, \nu')$. A careful analysis shows that this happen if and only if the following condition holds: there is a character η on H which vanishes on $H^\perp$, such that

$$\nu(h)/\nu'(h) = \eta(h)/\eta(t_h). \tag{9.6}$$

In conclusion, the second obstruction is the existence of a function $\nu : H \to H/H^\perp \to k^*$ which satisfy Eqs. (9.4) and (9.5) and two such functions ν and ν' give equivalent solutions if and only if there is a character η of H which vanishes on $H^\perp$ and which satisfies Eq. (9.6).

9.5. The third obstruction

As explained in Section 8, after solving the second obstruction, we can think about the third obstruction in the following way: we have an invertible $k^{\psi}H$-bimodule X with support in $\mathcal{M}$, and $X \otimes_{k^{\psi}H} X \cong k^{\psi}H$. We would like to turn $k^{\psi}H \oplus X$ into an algebra in $\mathcal{C}$. The only obstruction for that (and this is the third obstruction) is that the multiplication on $X \otimes X \otimes X$ might be associative only up to a scalar. This scalar is the third obstruction, considered as an element of $H^3(\mathbb{Z}_2, k^*) = \{1, -1\}$. Following the work of Tambara (see [10]), we see that this sign is the sign of the following expression

$$\sum_{a \in H/H^{\perp}} \nu(h)\tau.$$

If the third obstruction vanishes, we only have one possible solution, as $H^2(\mathbb{Z}_2, k^*) = 1$, since we have assumed that k is algebraically closed. This finishes the proof of Proposition 29

9.6. Relation to the Tambara's work

In [10], Tambara classified all fiber functors on $\mathcal{TY}(A, \chi, \tau)$. In the language of module categories, he classified all module categories over $\mathcal{TY}(A, \chi, \tau)$ of rank 1. In the language of our classification, he described all module categories whose parameterization begins with $(\mathcal{M}(A, \psi), \mathbb{Z}_2, \ldots)$ for some ψ.

There is a deeper connection between our result and the result of Tambara, as we will show now. Assume that we have a module category over $\mathcal{TY}(A, \chi, \tau)$ whose classification begins with $(\mathcal{M}(H, \psi), \mathbb{Z}_2, \ldots)$. Then, as we have seen, $H^{\perp} < H$, and χ induces a nondegenerate symmetric bicharacter $\bar{\chi}$ in $\bar{H} := H/H^{\perp}$. We thus have another Tambara Yamagami fusion category $\mathcal{D} := \mathcal{TY}(\bar{H}, \bar{\chi}, \bar{\tau})$, where $\bar{\tau}$ has the same sign as τ. In order to explain the connection, we first recall the following theorem of Tambara (Proposition 3.2 in [10])

Theorem 32. *Fiber functors on $\mathcal{D}$ correspond to triples (s, ψ, ν) which satisfies the following coherence conditions:*

$$\bar{\chi}(a, b) = \xi_{\psi}(s(a), b)$$
$$\partial\nu(a, b) = \psi(a, b)/\psi(s(a), s(b))$$
$$\nu(a)\nu(s(a)) = 1$$
$$sign\left(\sum_{s(a)=a} \nu(a)\right) = sign(\tau)$$

Two such triples (s, ψ, ν) and (s', ψ', ν') will give equivalent fiber functors if and only if $s = s'$ and there exist a function $\phi : H/H^{\perp} \to K$ such that $\psi = \partial\phi\psi'$ and $\nu(h)/\nu'(h) = \phi(h)/\phi(s(h))$

Remark 33. This is not exactly the original formulation in Tambara's paper, but it is equivalent.

The following lemma is now an easy corollary from Proposition 29 and the above theorem.

Lemma 34. *There is a one to one correspondence between equivalence classes of fiber functors on $\mathcal{D}$ which corresponds to triples which contains the two cocycle ψ and module categories over $\mathcal{C}$ whose parameterization begins with $(\mathcal{M}(H, \psi), \mathbb{Z}_2, \ldots)$.*

The Lemma says that we have a correspondence between fiber functors on one Tambara Yamagami category and some module categories over another Tambara Yamagami category. However, we do not know about a plausible explanation of *why* it happens.

We can now use the results of Tambara to obtain another description of our module categories. Indeed, in his paper Tambara gave several description of fiber functors of $\mathcal{D}$. Applying Theorem 3.5 from [10], we get the following

Corollary 35. *Let $\mathcal{TY}(A, \chi, \tau)$, $\mathcal{M}(H, \psi)$ be as above. Assume that $H^{\perp} < H$ and that $Rad(\psi) = H^{\perp}$. Then the different ways to put on $\mathcal{M}(H, \psi)$ a $\mathcal{TY}(A, \chi, \tau)$-module structure are parameterized by pairs (s, μ) where s is an involutive automorphism of $H/H^{\perp}$, and $\mu : \bar{H}^s/\bar{H}_s \to k^*$ satisfy*

$$\bar{\chi}(a, b) = \xi_{\psi}(s(a), b)$$
$$\mu(a)\mu(b)/\mu(ab) = \tilde{\chi}(a, b)$$
$$sign(\mu) = sign(\tau)$$

Here $\bar{H}^s$ is the subgroup of s-invariant elements, $\bar{H}_s$ is the subgroup of elements of the form $as(a)$, The map $\tilde{\chi}$ is the induced bilinear form on $\bar{H}^s/\bar{H}_s$ (one of Tambara's result is the fact that this is indeed well defined), and $sign(\mu)$ is the sign of μ as a quadratic map (It is quite easy to show that $\bar{H}^s/\bar{H}_s$ is a vector space over $\mathbb{Z}_2$ and therefore we can talk about this sign). See Tambara's paper [10] for more details.

9.7. Dual categories

In this subsection we shall give a general description of the dual categories of $\mathcal{TY}(A, \chi, \tau)$. First recall (see the remark after Proposition 2.1 in [9]) that if $\mathcal{L} \cong \mathrm{Mod}_{\mathcal{C}} - L$ is a module category over a fusion category $\mathcal{C}$, where L is an algebra in $\mathcal{C}$, then the dual category $\mathcal{C}^*_{\mathcal{L}}$ is equivalent as a fusion category to the category of L-bimodules in $\mathcal{C}$.

We begin with duals with respect to module categories of the form $\mathcal{L} = \mathcal{M}(\mathcal{N}, 1, \Phi, v, \beta)$. In this case, $\mathcal{L} \cong \mathrm{Mod}_{\mathcal{C}} - k^{\psi}H$ for some $H < A$ and some two cocycle ψ. We have described above the category of $k^{\psi}H$-bimodules with support in Vec_A. We have seen that it is a pointed category with an abelian group of invertible objects, which we have described in Section 9.2. Consider now the $k^{\psi}H$-bimodules with support in $\mathcal{M}$. Following previous calculations, we see that such a bimodule is given by a vector space V which is both a left and a right $k^{\psi}H$-module, and the interaction between the left and the right structure is given by the formula

$$(U_h \cdot v) \cdot U_{h'} = \chi(h, h')U_h \cdot (v \cdot U_{h'}). \tag{9.7}$$

We can think of such modules as $k^{\theta}[H \times H]$-modules, where θ is a suitable two cocycle. By this point of view, the isomorphism classes of indecomposable modules is in bijection with the characters of $\mathrm{Rad}(\theta) < H \times H$. Let us denote the indecomposable module which corresponds to a character ζ of $\mathrm{Rad}(\theta)$ by V_{ζ}. A routine and tedious calculation shows us that the group of invertible $k^{\psi}H$-bimodules with support in Vec_A acts on the modules with support in $\mathcal{M}$ via the following formulas:

$$U_{a_i,\lambda} \otimes_B V_{\zeta} = V_{(\lambda, \chi(a_i,-))\zeta}$$
$$V_{\zeta} \otimes_B U_{a_i,\lambda} = V_{(\chi^{-1}(a_i,-),\lambda^{-1})\zeta}$$

We know that the dual category is graded by $\mathbb{Z}_2$ in the obvious sense. We use this fact in order to conclude the following multiplication formula:

$$V_{\zeta} \otimes_B V_{\eta} = \bigoplus_{(\lambda,\chi(a_i,-))t^*(\eta)=\zeta} U_{a_i,\lambda}$$

where by $t^*(\eta)$ we mean the composition of η with the map $H \times H \to H \times H$ given by $(h_1, h_2) \mapsto (h_2, h_1)$. Notice that by the analysis done in Section 8 and by the observation that the group of invertible bimodules with support in Vec_A acts transitively on the set $\{V_{\zeta}\}$, we see that the dual is pointed if and only if the category $\mathcal{L}$ is σ-invariant.

We consider now module categories of the second type. By this we mean categories of the form $\mathcal{L} = \mathcal{M}(\mathcal{N}, \langle\sigma\rangle, \Phi, v, \beta)$. Assume that $\mathcal{N} = \mathcal{M}(H, \psi)$ as a Vec_A module category. Then $\sigma(H, \psi) = (H, \psi)$ and we have an action of σ on the abelian group E of invertible $k^{\psi}H$ bimodules with support in Vec_A. We have an equivalence of fusion categories $(Vec_A)^*_{\mathcal{N}} \cong Vec^{\omega}_E$ for some three cocycle $\omega \in H^3(E, k^*)$.

We have seen in Section 7 that the dual $\mathcal{C}^*_{\mathcal{L}}$ will be the equivariantization of this category with respect to the action of $\mathbb{Z}_2$. If, for example, we would have known that $\omega = 1$, then this equivariantization would have been equivalent to the representation category of the group $\mathbb{Z}_2 \ltimes \hat{E}$ In general, the description of this category is not much harder.

We conclude by observing that $\mathcal{TY}(A, \chi, \tau)$ is group theoretical if and only if there is a pair (H, ψ) such that $\sigma(H, \psi) = (H, \psi)$. This gives an alternative proof of the fact that $\mathcal{TY}(A, \chi, \tau)$ is group theoretical if and only if the metric group (A, χ) has a Lagrangian subgroup (see Corollary 4.9 of [6]).

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