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Generalized Gaussian Measures and a "Functional Equation" : III. Measures on R^n

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1. INTRODUCTION

The Gaussian distribution can be characterized as the sole probability distribution which is "invariant under rotations in $\mathbf{R}^2$." More precisely, let $U_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ rotate the plane through an angle θ which is not a multiple of $\pi/2$. Given a probability distribution μ on $\mathbf{R}$, define σ to be the distribution on $\mathbf{R}^2$ given by $\sigma(E) = \mu \times \mu(U_\theta(E))$. Then if $\sigma = \nu \times \nu$ for some probability distribution ν, μ is Gaussian. The first theorem along these lines seems to have been due to Kac [14]; see Feller, [3, pp. 77–8], for a more complete account.

In this paper, we prove similar results for more general σ-additive set functions on $\mathbf{R}^n$. That is, μ is required only to be a (complex) linear combination of regular measures on $\mathbf{R}^n$. The function μ itself need not be a measure, since it may be undefined on some Borel sets in $\mathbf{R}^n$. (For instance, if μ were Lebesgue measure on $[0, \infty)$ and the negative of Lebesgue measure on $(-\infty, 0)$, then μ would be covered by the theorem, although $\mu(\mathbf{R})$ is not defined.) For brevity, we shall refer to μ as a measure.

The first main result of this paper is the following:

THEOREM 1.1. *Let $A : \mathbf{R}^n \to \mathbf{R}^n$ be an invertible self-adjoint operator. Define $\xi_A : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n$* by

$$\xi_A(x, y) = (x + Ay, Ax - y).$$

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Suppose that μ, ν *are measures on* $\mathbf{R}^n$ *such that for all measurable sets* $E \subseteq \mathbf{R}^n \times \mathbf{R}^n$,

$$\mu \times \mu(E) = \nu \times \nu(\xi_A(E)). \tag{1.1}$$

Suppose further that for some positive real number a, *the function* $\exp(a\langle x, x\rangle)$ *is* μ*-integrable.* ($\langle , \rangle$ *is the usual inner product on* R^n.) *Then* μ *is concentrated on a subspace of* $\mathbf{R}^n$, *which we may assume is* $\mathbf{R}^m$. *Furthermore,* $\exists a$ *constant* c *and a symmetric complex* $n \times n$ *matrix* B *on* $\mathbf{R}^m$ *such that*

$$d\mu(x) = c \exp(-\langle \pi Bx, x\rangle)\, dx. \tag{1.2}$$

The hypothesis that $\exp(-a\langle x, x\rangle)$ be integrable is not very aesthetic; in at least one case, it is unnecessary.

THEOREM 1.2. *Suppose in Theorem* 1.1 *that* $A = I$ (*i.e.*, $\xi(x, y) = (x + y, x - y)$.) *Then the conclusion of Theorem* 1.1 *holds without the hypothesis that* $\exp(-a\langle x, x\rangle)$ *is* μ*-integrable.*

A theorem like Theorem 1.2 should hold for a wider class of operators A, for instance, for $A = \lambda I$ ($\lambda \neq 0$). With the methods used in this paper, however, one runs into rather messy technical difficulties. It is quite possible, too, that Theorem 1.1 holds for a wider class of operators A. For instance, one may need only to assume that A and $I + A^2$ are invertible. The only obstacle to proving this more general result is Lemma 3.2.

The work in proving Theorem 1.1 occurs in the case where μ is finite; this case is discussed in Section 3 of this paper. Section 2 is concerned with some technical lemmas about matrices, and the theorems are proved in Section 4.

Theorem 1.2 has some obvious applications to the generalized Gaussian measures discussed in [2]. A brief discussion of these results is given in Section 5. Section 6 examines the more general case of measures μ, ν_1, ν_2 satisfying

$$\mu \times \mu(\xi_A(E) = \nu_1 \times \nu_2(E)). \tag{1.3}$$

2. SOME LEMMAS

In what follows, A is the operator defined in Theorem 1.1, and $D = (I + A^2)^{-1}$. The Fourier transform of f is given by $f(x) =$

$\int f(a)\, e^{2\pi i\langle a,x\rangle}\, da$; thus Lebesgue measure is its own dual for Fourier inversion. If $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are n-tuples of complex numbers $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$.

LEMMA 2.1. $\| D \| < 1$, $\| AD \| < 1$, $\| A^2 \| < 1$.

Proof. For $\| x \| = 1$, set $x = (I + A^2)z$. Then if $z \neq 0$,

$$\langle ADx, ADx\rangle = \langle Az, Az\rangle < \langle z, z\rangle + 2\langle Az, Az\rangle + \langle A^2 z, A^2 z\rangle$$
$$= \langle (A^2 + I)z, (A^2 + I)z\rangle = \langle x, x\rangle = 1.$$

Because the unit ball of $\mathbf{R}^n$ is compact, $\| AD \| < 1$. The other parts are similar.

LEMMA 2.2. *Suppose B is a symmetric $n \times n$ complex matrix whose real part is positive definite. Then B is invertible, and B^{-1} is symmetric and its real part is positive definite. If $f(x) = \sqrt{\det B}\, \exp(-\pi\langle Bx, x\rangle)$, then $\hat{f}(x) = \exp(-\pi\langle B^{-1}x, x\rangle)$, when the proper sign is chosen for the square root.*

Proof. This is essentially the content of Lemmas 4.1 and 4.2 of [2].

LEMMA 2.3. *Let C be an invertible symmetric $n \times n$ complex matrix; say $C = C_1 + iC_2$, where the C_j are real. Suppose that C_1 is positive semidefinite. Then $\exists \delta > 0$ such that $\delta C_2^2 - \delta C_1 + C_1$ and $\delta^{-1} I - C_1$ are both positive definite.*

Proof. Pick δ so that $1/\delta >$ largest eigenvalue of C_1. Then $\delta^{-1}I - C_1$ is positive definite. If $x \in \mathbf{R}^n$, write $x = x_1 + x_2$, with $x_1 \in \ker C_1$ and $x_2 \perp \ker C_1$; then

$$\langle (\delta C_2^2 - \delta C_1^2 + C)(x_1 + x_2), x_1 + x_2\rangle$$
$$= \langle \delta C_2(x_1 + x_2), C_2 x_1 + x_2\rangle + \langle C_1(I - \delta C_1)\, x_2, x_2\rangle.$$

On $(\ker C_2)^{\perp}$, C has a square root; therefore the second term is positive if $x_2 \neq 0$. If $x_2 = 0$, but $x_1 \neq 0$, then the first term is >0, as $C_2 \neq 0$ on $\ker C_1$ (since C is invertible.)

This proves the lemma.

3. Proof of the Theorem for Finite Measures

We now prove Theorem 1.1 under the added hypotheses that μ (and hence also ν) are finite and that $\mu(\mathbf{R}^n) \neq 0$. The rest of the proof will consist of reducing the general situation to this more special one.

Let f, g be the Fourier–Stieltjes transforms of μ, ν, respectively. Then $f(0) \neq 0$.

Lemma 3.1. $f(x) f(y) = g(D(x + Ay))\, g(D(Ax - y))$.

Proof.

$$\begin{aligned} f(x) f(y) &= \iint \exp(-2\pi i\langle\alpha, x\rangle) \exp(-2\pi i\langle\beta, y\rangle)\, d\mu(\alpha)\, d\mu(\beta) \\ &= \iint_{R^n \times R^n} \exp(-2\pi i(\langle\alpha + A\beta, D(x + Ay)\rangle + \langle A\alpha - \beta, D(Ax - y)\rangle) \\ &\quad \times d(\mu \times \mu)(\alpha, \beta) \\ &= \iint_{R^n \times R^n} \exp(-2\pi i(\langle\alpha + A\beta, D(x + Ay)\rangle + \langle A\alpha - \beta, D(Ax - y)\rangle) \\ &\quad \times d(\nu \times \nu)(\alpha + A\beta, A\alpha - \beta) \\ &= \iint_{R^n \times R^n} \exp(-2\pi i\langle\alpha, D(x + Ay)\rangle) \exp(-2\pi i\langle\beta, D(Ax - y)\rangle) \\ &\quad \times d\nu(\alpha)\, d\nu(\beta) \\ &= g(D(x + Ay))\, g(D(Ax - y)). \end{aligned}$$

Note that Lemma 3.1 implies

$$g(x)\, g(y) = f(x + Ay) f(Ax - y). \tag{3.1}$$

Lemma 3.2. $f(x) \neq 0$, $g(x) \neq 0$ *for all* x.

Proof. It suffices to prove either half. If the lemma is false, let x_0 be an element with minimal norm such that $f(x_0) = 0$. By hypothesis, $x_0 \neq 0$. But $f(x_0) = 0 \Rightarrow g(Dx_0)\, g(DAx_0) = 0$, and $g(y_0) = 0 \Rightarrow f(y_0) f(Ay_0) = 0$; hence $f(x_0) = 0 \Rightarrow f(Dx_0) f(DAx_0)^2 f(DA^2x_0) = 0$. Now Lemma 2.1 gives a contradiction. From Lemma 3.1, $f(0)^2 = g(0)^2$; by taking constant multiples of μ and ν, we may assume $f(0) = g(0) = 1$. Note that $f(x) = g(D(x))\, g(D(Ax))$ and that $f(y) = g(D(Ay))\, g(D(-y)$; set $y = x$ to show that $g(x) = g(-x)$. Similarly, f is even.

Set $h(x, y) = f(x + y)/f(x)f(y)$.

LEMMA 3.3. *$h(x, y)$ is bilinear in x and y.*

Proof. Since h is clearly symmetric in x and y, it suffices to show linearity in either variable. Hence it suffices to show that $h(x + z, Ay) = h(x, Ay)\, h(z, Ay)$; we may assume further that $z = (A^2 + I)w$.

(2.1) shows that $g(x) = f(x) f(Ax)$, and hence that

$$f(x) f(Ax) f(y) f(Ay) = f(x + Ay) f(Ax - y).$$

Therefore $h(x, Ay) = f(Ax) f(y)/f(Ax - y)$ and so

$$h(x + z, Ay) = h(x + A^2w + w, Ay) = \frac{f(x + w + A^2w + Ay)}{f(x + w + A^2w) f(Ay)}$$

$$= \frac{f(Ay + A^2w)\, h(x + w, Ay + A^2w)}{f(A^2w)\, h(x + w, A^2w) f(Ay)}$$

$$= \frac{f(Ay + A^2w) f(Ax + Aw) f(y + Aw)}{f(A^2w) f(Ay)\, h(x + w, A^2w) f(Ax - y)}$$

$$= \frac{f(Ay + A^2w)}{f(Ay) f(A^2w)} \cdot \frac{f(Aw + y)}{f(Aw)} \cdot \frac{f(Ax)}{f(Ax - y)}$$

$$= h(A^2w, Ay)\, h(Aw, y)\, h(x, Ay).$$

For $x = 0$, this says that $h(z, Ay) = h(A^2w, Ay)\, h(Aw, y)$; the lemma follows.

LEMMA 3.4. $f(x) = h(x, x/2)$.

Proof. $h(x, -x) = f(x)^{-1} f(-x)^{-1} = f(x)^{-2}$; hence $h(x, x) = 1/h(x, -x) = f(x)^2$. Furthermore, $f(2x) = h(x, x) f(x)^2 = f(x)^4$. Substitute $x/2$ for x; then $h(x, x/2) = h(x/2, x/2)^2 = f(x/2)^4 = f(x)$.

LEMMA 3.5. *Set $\sigma = \mu * \mu$, and define $\varphi : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n$ by $\varphi(x, y) = (x + y, y - x)$. Then $(\sigma \times \sigma(\varphi(E)) = \mu \times \mu(E)$ for all measurable $E \subseteq R^n \times R^n$.*

Proof. The Fourier–Stieltjes transform of σ is $F = f^2$. $F(x) F(y) = f(x + y) f(x - y)$, as a simple computation using Lemma 3.4 shows.

The formula $F(x)F(y) = f(x+y)f(x-y)$ means that

$$\int u(x, y)\, d\sigma(x)\, d\sigma(y) = \int u(x+y, x-y)\, d\mu(x)\, d\mu(y)$$

whenever u is a product of characters on $\mathbf{R}^n$. It therefore holds for all functions u which are uniform limits of sums of such characters, and hence for all C^∞ functions in $\mathbf{R}^n \times \mathbf{R}^n$ with compact support. Take limits again; then the formula holds for all continuous u with compact support. By the Riesz representation theorem, it also holds when $u = \chi_{\varphi E}$, the characteristic function of φE; then, since $(x+y, x-y) \in \varphi E \Leftrightarrow (x, y) \in E$, we get $\sigma \times \sigma(\varphi E) = \mu \times \mu(E)$, as desired.

LEMMA 3.6. *The support of μ is a subspace of $\mathbf{R}^n$.*

Proof. This proof is an easy modification of the proof of Lemma 4.1 in [1].

In view of this result, we may as well assume that μ is defined on all of $\mathbf{R}^n$.

We now proceed to prove the theorem for μ. From Lemma 3.3, we may assume that $f = \mu$ is a quadratic character. Let $e_1, \ldots, e_n$ be the usual basis for $\mathbf{R}^n$. For $a, b \in \mathbf{R}$, set $h_j(a, b) = f((a+b)e_j)/f(ae_j)f(be_j)$. Then h_j is bilinear, and so $\exists c_{jj} \in C$ with $h_j(a, b) = \exp(-2c_{jj}ab)$. Similarly, let $h_{ij}(a, b) = f(ae_i + be_j)/f(ae_i)f(be_j)$ for $i \neq j$; then $h_{ij}(a, b) \exp(-2\pi c_{ij}ab)$ for some $c_{ij} \in C$. Moreover, $c_{ij} = c_{ji}$. Let $C = (c_{ij})$; we show that $f(x) = \exp(\langle -\pi Cx, x\rangle)$. To do this, we write $x = \sum_{i=1}^n x_i l_i$ and induct on the number of nonzero x_i. If $x = x_i e_i$, then Lemma 3.4 shows that the formula holds; it is also easy to check the formula for $x = x_i e_i + x_j e_j$. The induction step follows from the formulas

$$\begin{aligned} f(x_1 + x_2 + x_3) &= f(x_1 + x_2) f(x_3)\, h(x_1 + x_2, x_3) \\ &= f(x_1 + x_2) f(x_3)\, h(x_1, x_3)\, h(x_2, x_3) \\ &= \frac{f(x_1 + x_2) f(x_1 + x_3) f(x_2 + x_3)}{f(x_1) f(x_2) f(x_3)}, \end{aligned}$$

$$\begin{aligned} &\langle C(x_1 + x_2 + x_3), x_1 + x_2 + x_3\rangle \\ &\quad = \langle C(x_1 + x_2), x_1 + x_2\rangle + \langle C(x_1 + x_3), x_1 + x_3\rangle \\ &\qquad + \langle C(x_2 + x_3), x_2 + x_3)\rangle - \langle C(x_1), x_1\rangle - \langle C(x_2), x_2\rangle - \langle C(x_3), x_3\rangle. \end{aligned}$$

$\operatorname{Re}(C)$ is at least positive semidefinite, since f is bounded. If $\operatorname{Re}(C)$ is positive definite, then Lemma 2.2 and Fourier inversion show that $d\mu(x) = \sqrt{\det B} \exp\langle -\pi Bx, x\rangle\, dx$, where $B = C^{-1}$. In that case we are done. We now show that $\operatorname{Re}(C) = C_1$ is positive definite. Otherwise, C_1 is positive semidefinite, but not positive definite. Set $C_2 = \operatorname{Im}(C)$. C is invertible, since otherwise f is constant on cosets of $\ker C$ and hence $\operatorname{supp} \mu \neq \mathbf{R}^n$ (by [5, Theorem 2.7.1]). By Lemma 2.3, $\exists \alpha$ such that $\alpha C_2{}^2 - \alpha C_1{}^2 + C_1$ and $\alpha^{-1}I - C_1$ are both positive definite. Then $\operatorname{Re}(\alpha^{-1}I - C)$ and $\operatorname{Re}(C - \alpha C^2)$ are positive definite; hence $\alpha^{-1}I - C$ and $C - \alpha C^2$ are invertible. Let their inverses be B_1, B_2, respectively, and let $\hat{g}(x) = \exp(\langle -\pi(C - \alpha C^2)\, x_1 x\rangle)$, $\hat{h}(x) = \exp(\langle -\pi(\alpha^{-1}I - C)\, x_1 x\rangle)$. Then $\hat{g}$, $\hat{h}$ are the Fourier transforms of $g(x) = \sqrt{\det B_2} \exp(\langle -\pi B_2 x_1 x\rangle)$ and $h(x) = \sqrt{\det B_1} \exp(\langle -\pi B_1 x_1 x\rangle)$, respectively. Moreover, $(\hat{\mu}_1 * \hat{h}) = \hat{g}$. However,

$$
\begin{aligned}
\widehat{(h\mu)}(x) &= \int \exp(-\pi i\langle x, a\rangle)\, h(a)\, d\mu(a) \\
&= \int \exp(-\pi i\langle x, a\rangle) \left(\int \hat{h}(g) \exp(\pi i\langle a, z\rangle)\, dz\right) d\mu(a) \\
&= \int\int \hat{h}(z) \exp(\langle -\pi i a, x - z\rangle)\, d\mu(a)\, dg \\
&= \int \hat{h}(z)\, \hat{\mu}(x - z)\, dz = \hat{h} * \hat{\mu}(x).
\end{aligned}
$$

Thus $\hat{g} = \widehat{(h\mu)}$, or $g = h\mu$. Hence $d\mu(x) = g(x)/h(x)\, dx$ is absolutely continuous. It is also finite. Thus $\hat{\mu}(x)$ vanishes at ∞; it follows that C_1 is positive definite. This finishes the proof.

4. Proof of the Main Theorems

Given the results of the last section, it is not hard to prove Theorem 1.1. It suffices to find a positive self-adjoint matrix T, which commutes with A, such that $\int e^{-\langle Tx,x\rangle}\, d\mu(x)$ exists and is nonzero. For then $d\mu'(x) = e^{-\langle Tx,x\rangle}\, d\mu(x)$ and $d\nu'(x) = e^{-\langle DTx,x\rangle}\, d\nu(x)$ satisfy the hypotheses for Section 3.

By hypothesis, $e^{-\langle \alpha x,x\rangle}$ is μ-integrable, i.e., $d\mu(x) = e^{-\langle \alpha x,x\rangle}\, d\mu(x)$ is finite. For each positive self-adjoint matrix T which commutes with A, $f_T(x) = e^{-\langle Tx,x\rangle}$ is μ_1-integrable. If $\exists T$ such that $\int_{\mathbf{R}^n} f_T(x)\, d\mu_1(x) \neq 0$, we are done. Otherwise, the f_T span a dense set on the space of continuous functions on $\mathbf{R}^n$ which vanish at ∞ (by the Stone–Weierstrass theorem). Then $\int_{\mathbf{R}^n} f(x)\, d\mu_1(x) = 0$ for all continuous functions f which vanish at infinity, and (by the Riesz representation theorem) $\mu_1 = 0$. This contradicts the hypothesis, and Theorem 1.1 is proved.

To prove Theorem 1.2, it suffices to show that if μ, ν satisfy (1.1), then $\exists\alpha > 0$ such that $e^{-\alpha\langle x,x\rangle}$ is μ-integrable.
For $\alpha > 0$, define

$$S(\alpha) = \{x : \| x \| < \alpha\},$$

$$M(\alpha) = \sup_{\|f\|_\infty = 1} \left| \int_{S(\alpha)} f(x)\, d\mu(x) \right|,$$

$$N(\alpha) = \sup_{\|f\|_\infty = 1} \left| \int_{S(\alpha)} d(x)\, d\nu(x) \right|.$$

For any ϵ, $0 < \epsilon < \alpha$, $S(2\alpha + 3\epsilon/2) \times S(\epsilon/2) \subseteq \xi(S(\alpha + \epsilon) \times S(\alpha + \epsilon))$, as is easily checked.
Similarly,

$$S(\alpha + \epsilon) \times S\left(\frac{\epsilon}{2}\right) \subseteq \xi^{-1}\left(S\left(\alpha + \frac{3\epsilon}{2}\right) \times S\left(\alpha + \frac{\epsilon}{2}\right)\right).$$

From (1.1),

$$M(\alpha + \epsilon)^2 > N\left(2\alpha + \frac{3\epsilon}{2}\right) N\left(\frac{\epsilon}{2}\right),$$

$$N\left(\alpha + \frac{3\epsilon}{2}\right)^2 > M(\alpha + \epsilon)\, M\left(\frac{\epsilon}{2}\right);$$

hence if $c = N(\epsilon/2)^2 M(\epsilon/2)$,

$$M(\alpha + \epsilon)^4 > cM(2\alpha + \epsilon). \tag{4.1}$$

Since $\mu \neq 0$, $M \neq 0$. There is no loss of generality in assuming that $\exists\epsilon < 1$ with $M(\epsilon/2) > 0$. Moreover, we may multiply μ and ν by the same nonnegative constant without affecting the result; thus we may assume $c = 1$.

To prove that $e^{-a\langle x,x\rangle}$ is integrable, it suffices to show that it is integrable on the complement of $S(1)$. Let T_n be the complement of $S(2^{n-1})$ in $S(2^n)$.

On T_n, $e^{-a\langle x,x\rangle} \leqslant \exp(-a2^{2n-2})$; hence

$$\left| \int_{T_n} e^{-a<x,x>} \, d\mu(x) \right| \leqslant M(2^n) \exp(-a2^{2n-2})$$

$$\leqslant M(2^n + \epsilon) \exp\left(-\frac{a}{4} 2^{2n}\right)$$

$$< M(1 + \epsilon)\, 2^{2n} \exp\left(-\frac{a}{4} 2^{2n}\right) = (M(1 + \epsilon)\, e^{-a/4})\, 2^{2n}.$$

If a is chosen such that $e^{a/4} > M(1 + \epsilon)$, then $\sum_{n=1}^{\infty} | \int_{T_n} e^{-a\langle x,x\rangle} \, d\mu(x) |$ converges. The theorem follows.

It should be noted that Theorems 1.1 and 1.2 apply to ν as well as to μ. The simplest way to show this is to set $\mu_0(E) = \mu(DE)$; then $\nu \times \nu(E) = \mu_0 \times \mu_0(\xi_A(E))$, and the theorems can be applied directly.

5. Applications

In [1] the notion of a Gaussian measure was generalized to locally compact Abelian groups G for which $x \mapsto 2x$ is an automorphism: μ is symmetric Gaussian on G if $\exists$ a measure ν on G such that $\mu \times \mu(E) = \nu \times \nu(\xi(E))$ for all measurable $E \subseteq G \times G$ ($\xi : G \times G \to G \times G$ is defined by $\xi(x, y) = (x + y, x - y)$). Theorem 1.2, in effect, characterizes the Gaussian measures on $\mathbf{R}^n$.

We now characterize the symmetric Gaussian measures on groups of the form $G = \mathbf{R}^n \times C$, where C is a compact group such that $x \mapsto 2x$ is an automorphism. The first step is to reduce the problem to one with finite measures. For $(x, y) \in \mathbf{R}^n \times C$, define $\|(x, y)\| = \| x \|$.

Lemma 5.1. *If μ is symmetric Gaussian on $G = \mathbf{R}^n \times C$, then $\exists$ a real number a and a finite symmetric Gaussian measure μ_0 on G such that $d\mu(x) = e^{-a\|x\|^2} \, d\mu_0(x)$.*

The proof is exactly like that used to deduce Theorem 1.2 from Theorem 1.1.

Now let μ be a finite Gaussian measure on G. Then (as in Lemma 3.6) supp μ is a closed subgroup of G closed under $x \mapsto 2x$. Because of the structure theorem for Abelian groups [5, Theorem 2.4], there is no generality in supposing that supp $\mu = G$. The Fourier–Stieltjes transform, $\hat{\mu}$, is a multiple of a quadratic character (by [1, Lemma 5.2]), and hence is nonzero on a subgroup of G. That subgroup includes $\mathbf{R}^n$ (because $\mu(0) \neq 0$). Then the argument of [2, Theorem 2.1], shows that we may assume $\hat{\mu}$ is never zero on G.

Let G_0 be the dual of the discrete rationals. R can be imbedded in G_0 by identifying $x \in R$ with the character $a \mapsto e^{2\pi i x a}$. In this way, G can be imbedded in $G_1 = G_0{}^n \times C$. G_1 is a subgroup of G; if one regards μ as a measure on G_1, its Fourier–Stieltjes transform (on $\hat{G}_1$) is the restriction of μ to G_1.

Theorem 5.1 of [2] shows that (as a measure on G_1) $\mu = \mu_1 * \mu_2$, where μ_1 is a large Gaussian measure on G_1 and μ_2 is a matricial Gaussian measure concentrated on a subgroup of G. The Fourier–Stieltjes transform μ_1 is a finite-valued quadratic character; it must therefore be constant on the cosets of $(\widehat{G_0{}^n}) = Q^n$. Hence μ_1 is concentrated on C. μ_1 is invertible. Regard it as a measure on $\mathbf{R}^n \times C$; then we may regard $\mu_2 = \mu^{-1} * \mu$ as a measure on $\mathbf{R} \times C$. We have proved

THEOREM 5.2. *A finite symmetric Gaussian measure μ whose support is $\mathbf{R}^n \times C$ and whose Fourier–Stieltjes transform never vanishes can be written as $\mu = \mu_1 * \mu_2$, where μ_1 is a "matricial" Gaussian measure on $\mathbf{R}^n \times C$ and μ_2 is a large Gaussian measure concentrated at finitely many points of C.*

The quotation marks are around "matricial" because the term was defined only for compact groups in [2]. The extended meaning should be clear.

A slight extension of this proof enable one to find all finite Gaussian measures on group G for which $x \mapsto 2x$ is an automorphism.

6. A Generalization of the Main Theorem

Theorem 1.1 has a generalization, obtained by using (1.3) as the "functional equation."

THEOREM 6.1. *Suppose that μ, ν_1, and ν_2 are nonzero measures on $\mathbf{R}^n$ which satisfy* (1.3), *and assume further that $\exists\ a \in \mathbf{R}$ such that $e^{-a\|x\|^2}$ is integrable with respect to all three measures. Then μ is a translate of a character times a measure satisfying* (1.1); *that is, $\exists$ a measure μ_0 and a character α on $\mathbf{R}^n$ such that μ_0 is of the form described in Theorem* 1.1 *and $\alpha(x)\, d\mu_0(x)$ is a translate of μ. The same applies to the ν_i.*

To begin the proof, note that (as in the proof of Theorem 1.1) we may assume that μ and the ν_i are finite measures and that $\mu(\mathbf{R}^n) \neq 0$. Then setting $f = \mu_0$, $g_i = \nu_i$ ($i = 1, 2$), one obtains (as in Lemma 3.1)

$$g_1(x)\, g_2(y) = f(x + Ay)\, f(Ax - y), \tag{6.1}$$

and it follows (as in Lemma 3.2) that f and the g_i never vanish. We may also assume that $f(0) = 1$, $g_i(0) = 1$ ($i = 1, 2$).

The meat of the rest of the proof if contained in an algebraic lemma.

LEMMA 6.2. *If f, g_1, and g_2 are nonvanishing functions satisfying* (6.1), *then* $\exists$ *a homomorphism* $s : \mathbf{R}^n \to \mathbf{C}^x$ *such that* $f_0 = fs$ *and* $g_0(x) = g_1(x)\, s(x)\, s(Ax)$ *satisfy* (3.1).

Proof. Set $r(x) = f(x)/f(-x)$. Then since $f(x)f(Ax) = g_1(x)$, $f(Ax)f(-x) = g_2(x)$, we have $r(x) = g_1(x)/g_2(x)$. Next,

$$\begin{aligned} r(Ax - Ay) &= \frac{f(Ax - Ay)}{f(Ay - Ax)} = \frac{f(Ax - Ay)\, f(A^2x + y)}{f(y + A^2x)\, f(Ay - Ax)} \\ &= \frac{g_1(Ax)\, g_2(-y)}{g_1(y)\, g_2(Ax)} = r(Ax)\, \frac{g_2(-y)}{g_1(y)}. \end{aligned}$$

For $x = 0$, we find that $r(Ay) = g_2(y)/g_1(-y)$; hence $r(Ax - Ay) = r(Ax)\, r(-Ay)$. Since A is an isomorphism, r is a homomorphism. Let $s(x) = r(-x/2)$, $f_0 = fs_1$, $g_0(x) = g_1(x)\, s(x + Ax)$, $g_3(x) = g_2(x)\, s(Ax - x)$. Then $g_3(x) = g_0(x)(g_2(x)/g_1(x)\, s(-2x) = g_0(x)$, and so $f_0(x + Ay)\, f_0(Ax - y) = g_0(x)\, g_3(y) = g_0(x)\, g_0(y)$, as desired.

It now follows that $f_0(x) = \exp(\langle -\pi Cx, x\rangle)$ for some symmetric matrix C, as in Theorem 1.1. Moreover, $s(-x) = \exp(-\pi\langle a_0 + 2ia_1, x\rangle)$, where a_0 and a_1 are n-tuples of real numbers. Hence

$$f(x) = \exp(-\pi\langle Cx + a_0 + 2ia_1, x\rangle).$$

Since f is bounded, $a_0 \in$ range (Re C). For let $a_0 = b_0 + c_0$, with $b_0 \in$ range (Re C), $b_1 \in$ range (Re C). Then $\mathrm{Re}(cb_1) = 0$, since Re C is self-adjoint. Hence $|f(-\lambda b_1)| = \exp(\langle -\pi a_0, -\lambda b_1\rangle) = \exp(\lambda \| b_1 \|^2)$ is bounded, and so $b_1 = 0$.

Pick x_0 with $2C(x_0) = a_0 + 2ia_2$. Then

$$f(x) = K \exp(2\pi i\langle a_2 - a_1, x - x_0\rangle) \exp(-\pi\langle C(x - x_0), x - x_0\rangle),$$

where K is a constant. Set $d\mu_0(x) = \exp(2\pi i\langle -x_0, x\rangle)\, d\mu(x + a_1 - a_2)$. Then $\mu_0(x) = K\exp(-\pi\langle cx, x\rangle)$, as desired. The result for ν_1 and ν_2 follows similarly, by expressing g_1 and g_2 in terms of g_0.

In the case where $A = I$, we can say a bit more.

THEOREM 6.3. *Suppose that $\mu_1, \mu_2, \nu_1, \nu_2$ are measures on $\mathbf{R}^n$ such that*

$$\mu_1 \times \mu_2(E) = \nu_1 \times \nu_2(\xi(\epsilon)) \tag{6.2}$$

for all measurable $E \subseteq \mathbf{R}^n \times \mathbf{R}^n$. Assume further that $\exists$ a real number a such that $\exp(-a\|x\|^2)$ *is integrable with respect to the μ_i, ν_i. Then the μ_i, ν_i are translates of characters times measures satisfying* (1.1), *as in Theorem* 6.1.

Proof. Most of the analysis is just like that of Theorem 6.1; what we need to prove is the analog of Lemma 6.2. To be precise, we need to show that if f_1, f_2, g_1, g_2 are nonvanishing functions, with $f_i(0) = g_i(0) = 1$, satisfying

$$g_1(x)\, g_2(y) = f_1(x + y) f_2(x - y), \tag{6.3}$$

then $\exists$ a homomorphism $u : \mathbf{R}^n \to \mathbf{C}^\times$ such that for $G_1 = g_1 u$, $G_2 = g_2 u$, $f_2(x + y) f_2(x - y) = G_1(x)\, G_2(y)$; moreover, $f_2 = f_1 u$. To show this set $t(x) = g_2(x)/g_2(-x) = f_1(x) f_2(-x)/f_1(-x) f_2(x)$; then

$$\begin{aligned} t(x+y) &= \frac{f_1(x+y) f_2(-x-y)}{f_1(-x-y) f_2(x+y)} \\ &= \frac{f_1(x+y) f_2(x-y) f_1(-y+x) f_2(-y-x)}{f_1(-y-x) f_2(-y+x) f_1(x-y) f_2(x+y)} \\ &= \frac{g_1(x)\, g_2(y)\, g_1(-y)\, g_2(x)}{g_1(-y)\, g_2(-x)\, g_1(x)\, g_2(-y)} = t(x)\, t(y). \end{aligned}$$

Now let $u(x) = t(-x/2)$. Then $G_1(x)\, G_2(y) = F_1(x + y) f_2(x - y)$ and $G_2(y) = G_2(-y)$. Hence $F_1(x + y) f_2(x - y) = F_2(x - y) f_2(x + y)$, or $F_1(x + y)/f_2(x + y) = F_1(x - y)/f_2(x - y)$, for all x, y. For $x = y$, we find $F_1/f_2 = F_1(0)/f_2(0) = 1$. This proves the assertion. The rest of the theorem now follows.

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