Asymptotic behavior of supercuspidal representations and Sato-Tate equidistribution for families

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We establish properties of families of automorphic representations as we vary prescribed supercuspidal representations at a given finite set of primes. For the tame supercuspidals constructed by J.-K. Yu we prove the limit multiplicity property with error terms. Thereby we obtain a Sato-Tate equidistribution for the Hecke eigenvalues of these families. The main new ingredient is to show that the orbital integrals of matrix coefficients of tame supercuspidal representations with increasing formal degree on a connected reductive p-adic group tend to zero uniformly for every noncentral semisimple element.

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1. Introduction

1.1. Limit multiplicity

We begin this introduction by explaining our results on counting discrete automorphic representations. Let G be a semisimple group over a totally real field F. Write $F_\infty :=
F \otimes_\mathbb{Q} \mathbb{R}$. Consider a sequence of lattices $\{\Gamma_j\}_{j\geq 1}$ in $G(F_\infty)$ whose covolumes tend to infinity as $j \to \infty$. For an irreducible unitary representation π_∞ denote by $m(\pi_\infty, \Gamma_j)$ the multiplicity of π_∞ occurring in the discrete spectrum of $L^2(\Gamma_j \backslash G(F_\infty))$. DeGeorge-Wallach [24,25] in the compact case and Rohlfs-Speh [59] and Savin [63] in the arithmetic non-compact case proved that if $\{\Gamma_j\}$ is a normal series whose intersection is the identity, then

$$\lim_{j\to\infty} \frac{m(\pi_\infty, \Gamma_j)}{\mathrm{vol}(\Gamma_j \backslash G)} = \deg(\pi_\infty). \tag{1.1}$$

Here $\deg(\pi_\infty)$ is the formal degree which by convention is non-zero if and only if π_∞ is square-integrable. By different methods it is shown in [1] that (1.1) holds if $\{\Gamma_j\}$ is Benjamini-Schramm convergent and uniformly discrete (which recovers the compact case but not the non-compact case).

Our goal is to investigate refinements where instead of the lattice Γ_j we impose a prescribed supercuspidal representation σ_j. Let u be a finite place, and consider a sequence $\{\sigma_j\}_{j\geq 1}$ of tame supercuspidal representations of $G(F_u)$ whose formal degrees tend to infinity as $j \to \infty$. For an irreducible algebraic representation ξ of $G(F_\infty)$ with regular highest weight, let $\Pi_\infty(\xi)$ denote the L-packet of square-integrable representations of $G(F_\infty)$ whose infinitesimal and central characters are dual to those of ξ. Write $m(\xi, \sigma_j)$ for the number of discrete automorphic representations π in $L^2(G(F)\backslash G(\mathbb{A}_F))$, counted with automorphic multiplicity $m(\pi)$, such that $\pi_\infty \in \Pi_\infty(\xi)$, $\pi_u \simeq \sigma_j$, and π is unramified at all finite places away from u. By results of Harish-Chandra the cardinality $|\Pi_\infty(\xi)|$ is equal to the order of the Weyl group of $G(F_\infty)$ divided by the order of the Weyl group of a maximal compact subgroup of $G(F_\infty)$. Informally one of our main results (Corollary 5.8, cf. (1.4) below) states, provided that the residue characteristic of F_u is sufficiently large, that

$$\lim_{j \to \infty} \frac{m(\xi, \sigma_j)}{\deg(\sigma_j)} = c|\Pi_\infty(\xi)| \dim \xi \tag{1.2}$$

for a positive constant c independent of ξ. We refer to §5.2 for the determination of c which is related to the Tamagawa number of G.

Note that the integer $|\Pi_\infty(\xi)| \dim \xi$ is equal to $\sum_{\pi_\infty \in \Pi_\infty(\xi)} \deg(\pi_\infty)$ up to a multiplicative constant depending only on the Haar measure on $G(F_\infty)$. Compared with (1.1), we are averaging over π_∞ ranging in the L-packet $\Pi_\infty(\xi)$ for technical simplicity in the trace formula; this simplification does not interfere with the new phenomena at the finite prime u that we are concentrating on.

Example 1.1. For the group $\mathrm{PGL}(2)$ consider discrete automorphic representations that ramify above a single prime q and are unramified elsewhere. Let $\mathcal{D}_k$ be the discrete series representation of $\mathrm{PGL}(2, \mathbb{R})$ of weight $k \geq 2$ (necessarily even). For each simple supercuspidal representation σ of $\mathrm{PGL}(2, \mathbb{Q}_q)$ with $q > 3$, there is an *exact* multiplicity formula

$$m(\mathcal{D}_k, \sigma) + m(\mathcal{D}_k, \sigma') = \frac{k-1}{12}(q^2 - 1), \tag{1.3}$$

where σ' is the other representation with the same affine generic character as σ, see [38]. (The assumptions in [38] differ slightly, but one can verify that the same argument applies, the key point being that $\mathrm{PGL}(2, \mathbb{Q})$ has no q-torsion.)

Simple supercuspidal representations for $\mathrm{PGL}_2(\mathbb{Q}_q)$ coincide with the representations of $\mathrm{GL}_2(\mathbb{Q}_q)$ of conductor q^3 and trivial central character. There are $2(q - 1)$ distinct simple supercuspidal σ's, partitioned into $q - 1$ pairs $\{\sigma, \sigma'\}$, thus (1.3) leads to the observation [38] that for any even integer $k \geq 2$ and any prime $q > 3$, the dimension of the space $S_k(q^3)^{\mathrm{new}}$ of newforms of weight k and level $\Gamma_0(q^3)$ is[1]

$$m(\mathcal{D}_k, \Gamma_0(q^3))^{\mathrm{new}} = \dim S_k(q^3)^{\mathrm{new}} = \frac{k-1}{12}(q+1)(q-1)^2.$$

Since the formal degree of $\mathcal{D}_k$ is $\frac{k-1}{12}$, this is a strong form of the limit multiplicity property (1.1) as $q \to \infty$.

On the other hand, we establish by the same method of proof as (1.2), see also [73], the following asymptotic

$$m(\mathcal{D}_k, \sigma) \sim \frac{k-1}{24}(q^2 - 1), \quad \text{as } k, q \to \infty,$$

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[1] This formula can also be established from

$$\dim S_k(q^3)^{\mathrm{new}} = \dim S_k(q^3) - 2\dim S_k(q^2) + \dim S_k(q),$$

and the dimension formulas of $S_k(N)$ derived from Riemann-Roch [52].

because the formal degree of simple supercuspidals is $\deg(\sigma) = \deg(\sigma') = \frac{q^2-1}{2}$. Thus, the asymptotic (1.2) and the exact formula (1.3) are consistent, although none implies the other.

Depth aspect

Recall the notion of depth of an admissible representation [55]. As a special case of Conjecture 3.15 below, we conjecture that for a sequence $(\sigma_j)_{j \geq 1}$ of supercuspidal representations of a p-adic group, the condition $\deg(\sigma_j) \to \infty$ is equivalent to $\operatorname{depth}(\sigma_j) \to \infty$. In our present context of tame supercuspidals, this is easy to verify (see, e.g., the proof of Proposition 3.13).

We may refer to the asymptotic (1.2) as a limit multiplicity result in the depth aspect. Indeed we view it as an analogue at finite places of the limit multiplicity in the "weight aspect" [69,70]. Indeed in the weight aspect, the roles of σ_j and ξ are interchanged, namely σ_j remains fixed and $\xi \to \infty$ (in the sense that the highest weight for ξ gets arbitrarily far from the walls), whereas the above families have fixed ξ and $\deg(\sigma_j) \to \infty$. In fact, we also establish hybrid results in this paper, where both ξ and σ_j tend to infinity. For example our results below allow us to obtain an error bound for (1.2), saving powers for both $\dim(\xi)$ and $\deg(\sigma_j)$ (Corollary 5.10 below).

1.2. Quantitative equidistribution for a family

In the same context as before, for simplicity, let G be a split semisimple group over a totally real field F with trivial center. (In the main text G need not be either split or semisimple with trivial center.) Let each of S_0 and S be a finite set of finite places of F such that $S \neq \emptyset$ (but S_0 could be empty) and $S_0 \cap S = \emptyset$. Denote by S_∞ the set of infinite places of F and put $\mathfrak{S} := S_\infty \cup S_0 \cup S$. Set $F_{S_0} := \prod_{v \in S_0} F_v$ and $\mathbb{A}_F^S := \prod'_{v \notin S} F_v$. Let K_{S_0} be an open compact subgroup of $G(F_{S_0})$, and $K^{\mathfrak{S}}$ an open compact subgroup of $G(\mathbb{A}_F^{\mathfrak{S}})$ which is the product of hyperspecial subgroups over $v \notin \mathfrak{S}$ (which arise from a global Chevalley group for G over $\mathbb{Z}$). We will consider

- irreducible algebraic representations ξ of $G(F_\infty)$ with regular highest weight,
- irreducible supercuspidal representations σ of $G(F_S)$.

For a technical reason we will impose the condition that $\xi \in \operatorname{Irr}_C^{\mathrm{reg}}(G(F_\infty))$ for a fixed constant $C \geq 1$ (see §5.4 for details; the error bound in the theorem depends on C).

Let $\mathcal{F}(\xi, \sigma, K_{S_0})$ be the multi-set of discrete automorphic representations π, counted with multiplicity $m(\pi) \dim(\pi_{S_0})^{K_{S_0}}$ (a number occurring naturally in the limit multiplicity problem), such that $\pi_\infty \in \Pi_\infty(\xi)$, $\pi_S \simeq \sigma$, and $(\pi^{\mathfrak{S}})^{K^{\mathfrak{S}}} \neq 0$. We let both ξ and σ vary, which puts discrete series representations at infinite places (grouped in L-packets) and supercuspidal representations at finite places on an equal footing. Write $m(\xi, \sigma, K_{S_0}) := |\mathcal{F}(\xi, \sigma, K_{S_0})|$. Fix a Haar measure on $G(\mathbb{A}_F)$.

Our main result in a simpler form is the following Sato-Tate equidistribution for the family $\mathcal{F}$. See Theorem 5.4 and Corollaries 5.8 and 5.10 for precise statements. In the special case where σ is fixed, our result generalizes [70, Thm. 9.19].

Theorem 1.2. *Suppose that the residue characteristic of every $v \in S$ is sufficiently large (in a way depending on G). We have the limit multiplicity formula as $\dim(\xi), \deg(\sigma) \to \infty$*

$$m(\xi, \sigma, K_{S_0}) \sim c \cdot \dim(\xi) \deg(\sigma) \tag{1.4}$$

for an explicit constant $c > 0$. Moreover there exist $\nu, A > 0$ depending only on G such that for every ξ and σ as above, and for every function $\phi : G(\mathbb{A}_F^{\mathfrak{S}}) \to \mathbb{C}$ which is the characteristic function of a $K^{\mathfrak{S}}$-double coset, we have the asymptotic formula

$$\sum_{\pi \in \mathcal{F}(\xi, \sigma, K_{S_0})} \operatorname{tr} \pi^{\mathfrak{S}}(\phi) = m(\xi, \sigma, K_{S_0})\phi(e) + O(m(\xi, \sigma, K_{S_0})^{1-\nu}\|\phi\|_1^A). \tag{1.5}$$

The multiplicative constant depends on G, C, S, K_{S_0} but is independent of ξ, σ and ϕ.

Example 1.3. Suppose $G = \mathrm{PGL}(2)$ and $F = \mathbb{Q}$. We take $S_0 = \emptyset$ and S any non-empty finite set of sufficiently large finite primes. We are counting, for even integer weights $k \geq 2$, and irreducible supercuspidal representations σ of $\mathrm{PGL}(2, \mathbb{Q}_S)$, the number $m(\mathcal{D}_k, \sigma)$ of cusp forms $f \in \mathcal{F}(\mathcal{D}_k, \sigma)$ of weight k unramified outside of S and with local component σ at S. The limit multiplicity asymptotic (1.4) recovers [73] as in Example 1.1:

$$m(\mathcal{D}_k, \sigma) \sim \frac{k-1}{12} \deg(\sigma) \quad \text{as } k, \dim(\sigma) \to \infty.$$

The second assertion (1.5) on Sato-Tate equidistribution is new already in this case of PGL(2). For example, if the function ϕ is a Hecke operator T_n for some integer $n \geq 1$ not divisible by any prime in S, then:

$$\sum_{f \in \mathcal{F}(\mathcal{D}_k, \sigma)} \frac{a_n(f)}{\sqrt{n}} = m(\mathcal{D}_k, \sigma)\delta_{n=\square} + O(n), \tag{1.6}$$

where $\delta_{n=\square}$ is one if n is a perfect square and zero otherwise. Note that we normalize $a_n(f)$ in such a way that Deligne's bound reads $|a_p(f)| \leq 2$. The above precise error term (corresponding to $\nu = 1$ and $A = 0$) is derived from the Sally-Shalika character formula [46, App. A]. Similarly as in §1.1, the equidistribution (1.6) is a refinement of an earlier result. Namely, Serre [65] considered the trace of the Hecke operator T_n on the space $S_k(N)$ of cusp forms f of weight k and level $\Gamma_0(N)$, and established for $n \geq 1$ coprime with N,

$$\sum_{f \in \mathcal{F}(\mathcal{D}_k, \Gamma_0(N))} \frac{a_n(f)}{\sqrt{n}} = m(\mathcal{D}_k, \Gamma_0(N))\delta_{n=\square} + O(n).$$

Remark

Interestingly, the depth aspect families studied in this paper are rather thin compared to the families formed by varying a lattice subgroup. In favorable situations, and assuming that $S_0 = \emptyset$, the global root number of $\pi \in \mathcal{F}(\xi, \sigma, \emptyset)$ depends only on ξ and σ. This almost never happens for thicker families arising from limit multiplicity problems where the whole lattice subgroup Γ_j varies.

Conductor vs depth

We view the sets $\mathcal{F}(\xi, \sigma, K_{S_0})$ with varying tame supercuspidal representations σ of $G(F_S)$ as forming a harmonic family in the sense of [62]. Then Theorem 1.2 essentially gives us the Sato–Tate equidistribution for families stated as Conjecture 1 in [62]. One difference is that the formulation of [62, Conj. 1] involves analytic conductors whereas our results are expressed in terms of formal degrees. The relation between formal degree and conductor is not yet established in general, this is a problem closely related to that of the depth preservation in the local Langlands correspondence [75].

1.3. Bounds towards Ramanujan

We can deduce from Theorem 1.2 an average bound towards Ramanujan. For every place $v \notin \mathfrak{S}$ and every $\theta > 0$, there is $\varrho > 0$ such that

$$\#\{\pi \in \mathcal{F}(\xi, \sigma, K_{S_0}), \ \log_v |\alpha(\pi_v)| > \theta\} \ll m(\xi, \sigma, K_{S_0})^{1-\varrho}, \tag{1.7}$$

where $\alpha(\pi_v)$ is the Satake parameter of the unramified representation π_v. It is unitarily normalized so that $|\alpha(\pi_v)| = 1$ if and only if π_v is tempered. The multiplicative constant and the exponent ϱ are independent of ξ and σ (they depend only on G, C, S, K_{S_0}, θ, v). The proof proceeds in the same way as for [53, Cor. 1.8]. Namely we first construct a function ϕ_1 which is a bi-K_v-invariant function on $G(F_v)$ such that $\operatorname{tr} \pi_v(\phi_1)$ is uniformly small for $|\alpha(\pi_v)| \leq 1$ and uniformly large for $|\alpha(\pi_v)| \geq q_v^\theta$. Then we apply Theorem 1.2 for $\phi := (\phi_1 * \phi_1^\vee)^{*k}$ with the integer $k \geq 1$ chosen proportional to $\log m(\xi, \sigma, K_{S_0})$, see [53, §3].

The estimate (1.7) shows that exceptions to the Ramanujan bound are sparse. For quasi-split classical groups the Ramanujan bound may be reduced to the self-dual or conjugate self-dual case of general linear groups via work of Arthur [12] and Mok [54]. The latter case is settled when cuspidal automorphic representations are cohomological (over totally real fields in the self-dual case; over CM fields in the conjugate self-dual case) by [18,50,68,19,15]. In particular the Ramanujan conjecture is known for the representations $\pi \in \mathcal{F}(\xi, \sigma, K_{S_0})$ if G is a split classical group.[2] For exceptional groups G very little is known and even a formulation of the Ramanujan conjecture is delicate, see [61] and [67] for recent treatments.

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[2] Here we use the fact that ξ has *regular* highest weight. This forces the representation of Arthur's SL(2) in the global Arthur parameter to be trivial by examining the infinitesimal character at infinite places.

1.4. Trace formula and tame supercuspidal coefficients

We find that the limit multiplicity and quantitative equidistribution described above are related to asymptotic properties of orbital integrals. The first step in the proof of Theorem 1.2 is to express the left-hand side of (1.5) as the spectral side of the trace formula for a suitably chosen test function. Since the weight ξ is regular and σ is supercuspidal we can use the simple trace formula.

There exist test functions f_σ that single out the given supercuspidal representation σ in the trace formula, obtained by forming matrix coefficients.[3] In our situation J.-K. Yu's construction gives σ as compactly induced from a finite dimensional representation on a compact open modulo center subgroup of $G(F_S)$. (Every σ arises in this way if the residue characteristics of places in S are sufficiently large by the exhaustion theorem [45].) This provides an explicit f_σ which is essential for our purpose.

We can now explain in more details the geometric side in the application of the trace formula. The geometric side is a sum over conjugacy classes of semisimple elements $\gamma \in G(F)$ of a volume term times a global orbital integral. The global orbital integral is a product of orbital integrals at ramified places in S, for which the main contribution is $\mathcal{O}_\gamma(f_\sigma)$, and orbital integrals at unramified places.

Here we are varying the supercuspidal coefficient f_σ which is unlike the usual applications of the trace formula where it is fixed. A general approach to this situation appears in [70] in the weight aspect and we can use the results of [70] to estimate most of the terms in the geometric side of the trace formula, except for $\mathcal{O}_\gamma(f_\sigma)$ which is new.

For the proof of (1.2) we establish that $|\mathcal{O}_\gamma(f_\sigma)| = o(\deg(\sigma))$ as $\deg(\sigma) \to \infty$, and for any fixed γ. The proof of Theorem 1.2 is much more difficult due to the uniformity in ϕ. As in [70] the number of terms in the geometric side is unbounded, and uniform estimates for orbital integrals are needed. Moreover the estimate for $\mathcal{O}_\gamma(f_\sigma)$ has to be made quantitative and uniform in γ which we discuss in the next subsection.

1.5. Asymptotic behavior of orbital integrals

We have seen in the previous subsection that our approach leads to the problem of establishing uniform bounds for orbital integrals of supercuspidal coefficients. In general it would be desirable to develop a quantitative theory of orbital integrals. This is for example advocated in the introduction of [29]. Our present problem of establishing uniform bounds for $\mathcal{O}_\gamma(f_\sigma)$ goes in this direction.

Theorem 3.11 below states that there exists a constant $\eta < 1$ depending only on the group $G(F_S)$ such that for all noncentral elements γ and all tame supercuspidal representation σ of $G(F_S)$, we have

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[3] In general f_σ is compactly supported only modulo center of G, but the center is finite as G is semisimple. In the main text we work with reductive groups with compact center but see Remark 3.14.

$$D(\gamma)^{\frac{1}{2}} \, |\mathcal{O}_\gamma(f_\sigma)| \ll \deg(\sigma)^\eta, \qquad (1.8)$$

where $D(\gamma)$ is the discriminant of γ. This result is the technical heart of the paper.

The properties of $\mathcal{O}_\gamma(f_\sigma)$ are related to the trace character $\Theta_\sigma(\gamma)$. In fact the two quantities agree if γ is regular elliptic, and we derive some consequences in §3.5. However it should also be noted that for our application it is essential to include the case where γ is *non-regular* elliptic (in which case $\Theta_\sigma(\gamma)$ is undefined). For explicit computations of $\Theta_\sigma(\gamma)$ for regular semisimple γ, we refer to [6,30,43].

In some special cases $\mathcal{O}_\gamma(f_\sigma)$ can be computed exactly, especially if one allows an additional average of σ (over an L-packet). In fact one could allow σ to be not only supercuspidal but also discrete series representations. Notably if σ is the Steinberg representation, then Kottwitz [49] constructed an Euler-Poincaré function f^{EP} which is a pseudo-coefficient for σ. In this case (1.8) holds with $f_\sigma = f^{\mathrm{EP}}$ in the horizontal aspect as the residue characteristics of places in S grow to infinity, see Section 6.

Though exact formulas for orbital integrals and for trace characters are extremely difficult to obtain beyond some special cases, we manage to prove the desired asymptotic (1.8). We indicate an outline of our proof. It follows from Yu's construction that the function f_σ can be chosen as a matrix coefficient and is supported on an explicit open compact subgroup $J \subset G(F_S)$. We recall in Section 2 how J is constructed from a generic G-datum. From this we reduce the estimate to the orbital integral of the characteristic function of a larger compact open subgroup L_s which is generated by a principal congruence subgroup and a parahoric subgroup of a proper twisted Levi subgroup. We conclude the proof in Section 4 based on a detailed analysis of Moy-Prasad subgroups.

1.6. Prescribed Steinberg representations

In a direction somewhat orthogonal to our main results described above, we have developed the case of families with prescribed Steinberg representations. We let the group G and the finite sets S, S_0 of finite places be as before (§1.2). Let St_S be the Steinberg representation of $G(F_S)$. We consider the multi-set $\mathcal{F}(\xi, \mathrm{St}_S, K_{S_0})$ of discrete automorphic representations π, counted with multiplicity $m(\pi) \dim(\pi_{S_0})^{K_{S_0}}$ such that $\pi_\infty \in \Pi_\infty(\xi)$, $\pi_S \simeq \mathrm{St}_S$, and $(\pi^{\mathfrak{S}})^{K^{\mathfrak{S}}} \neq 0$. We let S vary and refer to $\mathcal{F}(\xi, \mathrm{St}_S, K_{S_0})$ as an horizontal family. This is to be compared with the previous vertical families $\mathcal{F}(\xi, \sigma, K_{S_0})$ where S was fixed and σ was a varying supercuspidal representation of $G(F_S)$.

In this case the needed estimates for orbital integrals can be deduced from results of Kottwitz on Euler-Poincaré functions [49]. We establish the Sato-Tate equidistribution for these horizontal families. The main point is that our method [70] described above for vertical families applies almost without change to these horizontal families, but with the simplification that the rather subtle bound on orbital integrals from Sections 3 and 4 is replaced with easier bounds such as (6.5) below. In the following example we explain the significance of the result for classical modular forms and refer to Theorem 6.4 for the precise statement in general.

Example 1.4. Consider again the group PGL(2). The Steinberg representation St_q and the quadratic twist of the Steinberg representation $\mathrm{St}_q \otimes \chi_q$ are the two representations of $\mathrm{PGL}(2, \mathbb{Q}_q)$ of level $\Gamma_0(q)$, thus

$$m(\mathcal{D}_k, \mathrm{St_q}) + m(\mathcal{D}_k, \mathrm{St_q} \otimes \chi_q) = \dim S_k(q)^{\mathrm{new}} = \dim S_k(q) - 2\dim S_k(1).$$

We note that $\mathcal{F}(\mathcal{D}_k, \mathrm{St}_q)$ (resp. $\mathcal{F}(\mathcal{D}_k, \mathrm{St}_q \otimes \chi_q)$) is the set of cuspidal modular forms of weight k, level $\Gamma_0(q)$ with global root number 1 (resp. -1), see e.g. [21]. Iwaniec–Luo–Sarnak [42, Cor. 2.14] proved that the following asymptotic holds:

$$m(\mathcal{D}_k, \mathrm{St_q}) \sim \frac{k-1}{24}q, \quad \text{as } k, q \to \infty$$

and similarly for $m(\mathcal{D}_k, \mathrm{St_q} \otimes \chi_q)$. We shall discuss this in a more general context in Section 6 and revisit the PGL(2) case in Example 6.6.

1.7. Notation

Let F be a number field, S (any) finite set of places of F, and S_∞ the set of all infinite places of F. Then set $F_S := \prod_{v \in S} F_v$, $F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R}$, $\mathbb{A}_F^S := \prod'_{v \notin S} F_v$, and $\mathbb{A}_F^{S,\infty} := \prod'_{v \notin S \cup S_\infty} F_v$, where $\prod'$ denotes the restricted product over all places v under the given constraint. Now let G be a connected reductive group over F. The center of G is denoted $Z(G)$, the maximal $\mathbb{Q}$-split torus in the center of $\mathrm{Res}_{F/\mathbb{Q}}G$ is A_G, and $A_{G,\infty} := A_G(\mathbb{R})^0$. Write G_∞ for $\mathrm{Res}_{F_\infty/\mathbb{R}}(G \times_F F_\infty)$.

Let $\mathcal{H}(G(\mathbb{A}_F^S)) = C_c^\infty(G(\mathbb{A}_F^S))$ denote the space of locally constant compactly supported $\mathbb{C}$-valued functions on $G(\mathbb{A}_F^S)$. Similarly $\mathcal{H}(G(F_S))$ is defined. The unitary dual of $G(F_S)$ is denoted $G(F_S)^\wedge$. Its Plancherel measure is written as $\widehat{\mu}_S^{\mathrm{pl}}$. We typically write ϕ_S for an element of $\mathcal{H}(G(F_S))$ and $\widehat{\phi}_S$ for the associated function $\pi_S \mapsto \mathrm{tr}\,\pi_S(\phi_S)$ on $G(F_S)^\wedge$.

When π is an admissible representation of a p-adic group G, write Θ_π for its Harish-Chandra character. We write $[g, h] := ghg^{-1}h^{-1}$ for $g, h \in G$.

1.8. Acknowledgment

2. Yu's construction of supercuspidal representations

In this section we review the construction of supercuspidal representations of a p-adic reductive group from the so-called generic data due to Jiu-Kang Yu and recall

from [45] that his construction exhausts all supercuspidal representations provided the residue characteristic of the base field is sufficiently large. The construction yields a supercuspidal representation concretely as a compactly induced representation, and this will be an important input in the next section.

2.1. Notation and definitions

The following local notation will be in use until Section 4. Let p be a prime. Let k be a finite extension of $\mathbb{Q}_p$. Denote by q the cardinality of the residue field of k. Let $\mathbf{G}$ be a connected reductive group over k, whose Lie algebra is denoted $\mathfrak{g}$. Denote the center of $\mathbf{G}$ by $\mathbf{Z_G}$. Write G and $\mathfrak{g}$ for $\mathbf{G}(k)$ and $\mathfrak{g}(k)$, respectively. For a tamely ramified extension E of k, denote by $\mathcal{B}(\mathbf{G}, E)$ the extended building of $\mathbf{G}$ over E. If $\mathbf{T}$ is a maximal E-split k-torus, let $\mathcal{A}(\mathbf{T}, \mathbf{G}, E)$ denote the apartment associated to $\mathbf{T}$ in $\mathcal{B}(\mathbf{G}, E)$. It is known that for any tamely ramified Galois extension E' of E, $\mathcal{A}(\mathbf{T}, \mathbf{G}, E)$ can be identified with the set of all $\mathrm{Gal}(E'/E)$-fixed points in $\mathcal{A}(\mathbf{T}, \mathbf{G}, E')$. Likewise, $\mathcal{B}(\mathbf{G}, E)$ can be embedded into $\mathcal{B}(\mathbf{G}, E')$ and its image is equal to the set of the Galois fixed points in $\mathcal{B}(\mathbf{G}, E')$ [60,57].

For $(x, r) \in \mathcal{B}(\mathbf{G}, E) \times \mathbb{R}$, there is a filtration lattice $\mathfrak{g}(E)_{x,r}$ and a subgroup $\mathbf{G}(E)_{x,r}$ if $r \geq 0$ defined by Moy–Prasad [55]. We shall normalize the valuation of E to extend the valuation of k, which is a different convention than in [55] (where the valued group is normalized to be $\mathbb{Z}$). Our convention is so that for a tamely ramified Galois extension E' of E and $x \in \mathcal{B}(\mathbf{G}, E) \subset \mathcal{B}(\mathbf{G}, E')$, we have [2, Prop. 1.4.1]:

$$\mathfrak{g}(E)_{x,r} = \mathfrak{g}(E')_{x,r} \cap \mathfrak{g}(E).$$

If $r > 0$, we also have

$$\mathbf{G}(E)_{x,r} = \mathbf{G}(E')_{x,r} \cap \mathbf{G}(E).$$

For simplicity, we put $\mathfrak{g}_{x,r} := \mathfrak{g}(k)_{x,r}$, etc., and $\mathcal{B}(G) := \mathcal{B}(\mathbf{G}, k)$. For $r \in \mathbb{R}$ and $x \in \mathcal{B}(G)$ we will also use the following notation:

- $\mathfrak{g}_{x,r+} := \cup_{s>r} \mathfrak{g}_{x,s}$, and if $r \geq 0$, $G_{x,r+} := \cup_{s>r} G_{x,s}$.
- $\mathfrak{g}_{x,r}^* := \left\{ \chi \in \mathfrak{g}^* \mid \chi(\mathfrak{g}_{x,(-r)+}) \subset \mathfrak{p}_k \right\}$, where $\mathfrak{p}_k$ is the maximal ideal of the ring of integers of k.
- $\mathfrak{g}_r := \cup_{y \in \mathcal{B}(G)} \mathfrak{g}_{y,r}$ and $\mathfrak{g}_{r+} := \cup_{s>r} \mathfrak{g}_s$.
- $G_r := \cup_{y \in \mathcal{B}(G)} G_{y,r}$ and $G_{r+} := \cup_{s>r} G_s$ for $r \geq 0$.

Lastly, for $x \in \mathcal{B}(G)$, we denote the stabilizer of x in G by $G_{[x]}$.

2.2. Generic G-datum

Yu's construction of supercuspidal representations starts with a *generic G-datum*, which we recall. The reader is referred to [74] for further details and any notions undefined here.

Definition 2.1. A *generic G-datum* is a quintuple $\Sigma = (\vec{\mathbf{G}}, x, \vec{r}, \vec{\phi}, \rho)$ satisfying the following:

D1. $\vec{\mathbf{G}} = (\mathbf{G}^0, \mathbf{G}^1, \cdots, \mathbf{G}^d = \mathbf{G})$ is a tamely ramified twisted Levi sequence such that $\mathbf{Z}_{\mathbf{G}^0}/\mathbf{Z}_{\mathbf{G}}$ is anisotropic.

D2. $x \in \mathcal{B}(G^0) = \mathcal{B}(\mathbf{G}^0, k)$.

D3. $\vec{r} = (r_0, r_1, \cdots, r_{d-1}, r_d)$ is a sequence of positive real numbers with $0 < r_0 < \cdots < r_{d-2} < r_{d-1} \leq r_d$ if $d > 0$, $0 \leq r_0$ if $d = 0$.

D4. $\vec{\phi} = (\phi_0, \cdots, \phi_d)$ is a sequence of quasi-characters, where ϕ_i is a generic quasi-character of G^i (see [74, §9] for the definition of generic quasi-characters). When $d = 0$, ϕ_0 is trivial on G_{x,r_0^+}, but, nontrivial on G_{x,r_0}. When $d \geq 1$, ϕ_i is trivial on G^i_{x,r_i^+}, but non-trivial on G^i_{x,r_i} for $0 \leq i \leq d-1$. If $r_{d-1} < r_d$, ϕ_d is nontrivial on G^d_{x,r_d} and trivial on G^d_{x,r_d^+}; otherwise, $\phi_d = 1$.

D5. ρ is an irreducible representation of $G^0_{[x]}$, the stabilizer in G^0 of the image $[x]$ of x in the reduced building of $\mathbf{G}^0$, such that $\rho|G^0_{x,0^+}$ is isotrivial and $c\text{-Ind}^{G^0}_{G^0_{[x]}} \rho$ is irreducible and supercuspidal.

In **D5**, note that G^0_x is compact while $G^0_{[x]}$ is only compact mod center. Recall from [74, p.585] that there is a canonical sequence of embeddings

$$\mathcal{B}(\mathbf{G}^0, E) \hookrightarrow \mathcal{B}(\mathbf{G}^1, E) \hookrightarrow \cdots \hookrightarrow \mathcal{B}(\mathbf{G}^d, E).$$

Hence, x can be regarded as a point of each of $\mathcal{B}(G^i) = \mathcal{B}(\mathbf{G}^i, k)$.

Also **D5** implies that x is rational as a building point of $\mathcal{B}(G)$ because it is a vertex of $\mathcal{B}(G^0)$. This will become important in Hypothesis $(\mathscr{E})$.(ii) below.

Given a generic G-datum $\Sigma = (\vec{\mathbf{G}}, x, \vec{r}, \vec{\phi}, \rho)$, we introduce an open compact-mod-center subgroup of G

$$J_\Sigma := G^0_{[x]} G^1_{x,s_0} \cdots G^{d-1}_{x,s_{d-2}} G^d_{x,s_{d-1}},$$

where we set $s_i := r_i/2$ for each i. Yu constructs a finite dimensional representation ρ_Σ of J_Σ from the datum. His key result is that

Theorem 2.2 (Yu). $\pi_\Sigma = c\text{-Ind}^G_{J_\Sigma} \rho_\Sigma$ *is irreducible and thus supercuspidal.*

Fix a positive Haar measure vol_G on G and denote the formal degree of π_Σ by $\deg(\pi_\Sigma)$. It is well-known (see for example [22] or [46, Lem 2.9.(i)]) that we have

$$\deg(\pi_\Sigma) = \frac{\dim(\rho_\Sigma)}{\mathrm{vol}_{G/Z}(J_\Sigma/Z)}. \tag{2.1}$$

The construction of ρ_Σ is complicated, but in what follows we shall only need the inequality $\dim(\rho_\Sigma) \le q^{\dim G}$, see [46, Lem 2.9.(ii)] for a proof. For later reference, we write $s_\Sigma := s_{d-1}$.

2.3. Supercuspidal representations via compact induction

Denote by $\mathrm{Irr}(G)$ the set of (isomorphism classes of) irreducible smooth representations of G. Write $\mathrm{Irr}^2(G)$ (resp. $\mathrm{Irr}^{\mathrm{sc}}(G)$) for the subset of square-integrable (resp. supercuspidal) members. Define $\mathrm{Irr}^{\mathrm{Yu}}(G) \subset \mathrm{Irr}^{\mathrm{sc}}(G)$ to be the subset of all supercuspidal representations which are of the form π_Σ as above. Write $\mathrm{Irr}^{\mathrm{c\text{-}ind}}(G) \subset \mathrm{Irr}^{\mathrm{sc}}(G)$ for the set of π compactly induced from a representation on an open compact-mod-center subgroup of G. We have that

$$\mathrm{Irr}^{\mathrm{Yu}}(G) \subset \mathrm{Irr}^{\mathrm{c\text{-}ind}}(G) \subset \mathrm{Irr}^{\mathrm{sc}}(G),$$

where the first inclusion comes from Theorem 2.2. The second inclusion is expected to be an equality but not known in general; see [46, §2.6] for references to partial results by Bushnell, Kutzko, Stevens and others in this direction. The main result of [45] says that the above inclusions are equalities under a rather explicit set of four hypotheses (namely (Hk), (HB), (HGT), and (H$\mathcal{N}$) in [45, §3.4]); in particular the equalities hold when p is greater than some lower bound depending only on the absolute root datum of G and the absolute ramification index of k.[4]

3. Orbital integrals of pseudo-coefficients

We keep the notation from the last section and assume that $G = \mathbf{G}(k)$ has *compact* center throughout this section and the next section. (We will briefly explain how to carry over the results of the current section to the non compact center case in Remark 3.14 below.) For $\pi \in \mathrm{Irr}^{\mathrm{Yu}}(G)$ attached to a generic G-datum we will construct an explicit coefficient f_π of π and study the asymptotic behavior of the orbital integral of f_π on noncentral semisimple elements as $\deg(\pi) \to \infty$ (note that we use π instead of σ to denote a representation of G). The result admits an interpretation as an asymptotic formula for character values, cf. §3.5 below, and will be applied in §5 to obtain an equidistribution theorem for families of automorphic representations.

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[4] While this paper was under review, Fintzen [34] announced the proof that $\mathrm{Irr}^{\mathrm{Yu}}(G) = \mathrm{Irr}^{\mathrm{sc}}(G)$ only assuming that p does not divide the order of the Weyl group of G.

3.1. Pseudo-coefficients

As before we have $G = \mathbf{G}(k)$ and write Z for the center of G. Let us recall the definition of pseudo-coefficients, cf. [31, A.4].

Definition 3.1. Let $\pi \in \mathrm{Irr}^2(G)$. A function $f_\pi \in \mathcal{H}(G)$ is said to be a **pseudo-coefficient** of π if $\mathrm{tr}\,\pi'(f_\pi) = \delta_{\pi,\pi'}$ for every tempered $\pi' \in \mathrm{Irr}(G)$.

The existence of f_π follows from the trace Paley-Wiener theorem, cf. [17, Prop 1]. (For real groups this is due to Clozel and Delorme [20, Cor, p.213].) To make f_π explicit, one can employ Bruhat-Tits buildings as in [64, §III.4] for any $\pi \in \mathrm{Irr}^2(G)$ or proceed as in Lemma 3.3 below for $\pi \in \mathrm{Irr}^{\text{c-ind}}(G)$.

Although f_π is not unique, the orbital integrals of f_π and the trace values of f_π against irreducible admissible representations of G are uniquely determined by the condition of Definition 3.1. So, by [44, Thm. 0], its orbital integrals are uniquely determined. Note that f_π is a cuspidal function in the sense that the trace of every induced representation from a proper parabolic subgroup is zero against f_π. (This fact is built into the construction of [17, Prop 1].) Moreover the orbital integrals of f_π are well known to encode the elliptic character values of π (recall that $\gamma \in G$ is said to be elliptic if the centralizer $Z_G(\gamma)$ is compact). When γ is regular elliptic, we will use the Haar measure on the compact group G_γ assigning total volume 1 in the definition of the orbital integral below.

Proposition 3.2. *If $\gamma \in G$ is regular semisimple, we have*

$$\mathcal{O}_\gamma(f_\pi) = \begin{cases} \Theta_{\pi^\vee}(\gamma), & \gamma: \text{ elliptic}, \\ 0, & \gamma: \text{ non-elliptic}. \end{cases} \tag{3.1}$$

Moreover $\mathcal{O}_\gamma(f_\pi) = 0$ for every γ that is (non-regular) non-elliptic semisimple.

Proof. The first assertion can be derived from [10, Thm 5.1] specialized to the $M = G$ case. The last assertion is Lemma III.4.19 of [64] (noting that the Euler-Poincaré function in that lemma is a pseudo-coefficient in view of Proposition III.4.1 and Theorem III.4.6 of [64]). $\square$

3.2. Explicit supercuspidal coefficients

In the following lemma, we construct an explicit matrix coefficient (which is also a pseudo-coefficient) associated to a compactly induced supercuspidal representation.

Lemma 3.3. *Let ρ be a finite dimensional admissible representation of an open compact subgroup J of G. Suppose $\pi := c\text{-}\mathrm{ind}_J^G \rho$ is irreducible (thus supercuspidal). Let*

$$f_\pi(g) = \begin{cases} \frac{1}{\mathrm{vol}(J)}\Theta_{\rho^\vee}(g) & \text{if } g \in J \\ 0 & \text{otherwise} \end{cases}$$

where $\Theta_{\rho^\vee}$ is the character of $\rho^\vee$. Then, we have

(i) $\mathrm{tr}\,\pi'(f_\pi) = \delta_{\pi,\pi'}$ for every $\pi' \in \mathrm{Irr}(G)$,
(ii) $f_\pi(1) = \frac{\dim(\rho^\vee)}{\mathrm{vol}(J)} = \frac{\dim(\rho)}{\mathrm{vol}(J)} = \deg(\pi)$,
(iii) $\mathrm{Supp}(f_\pi) \subset J$.

In particular, f_π is a pseudo-coefficient and $\dfrac{|f_\pi|}{\deg(\pi)} \leq 1_J$, the characteristic function of J.

Proof. Assertion (iii) is immediate, and (ii) is [46, Lem 2.9]. Assertion (i) follows from the fact that f_π is a matrix coefficient of $\pi^\vee$ and from Frobenius reciprocity which implies $\mathrm{Hom}_J(\rho, \pi') = \delta_{\pi,\pi'}$. $\quad\square$

We will need the following hypotheses on the group $\mathbf{G}$ for Theorem 3.11. Hypothesis $(\mathscr{T})$ will be used in Lemma 4.1. Hypothesis $(\mathscr{E})$ will be used in the proof of Proposition 4.4 and Lemma 4.7. Observe that the hypotheses are inherited by tame twisted Levi subgroups.

3.3. Hypotheses

$(\mathscr{T})$ For any tame maximal torus $\mathbf{T}$, and $r > 0$, every nontrivial coset in T_r/T_{r+} contains a $\mathbf{G}$-good element.
$(\mathscr{E})$ There is a tamely ramified extension E of k such that
  (i) every k-torus in $\mathbf{G}$ splits over E;
  (ii) the k-order, in the sense of [58, §3.3], of every building point $x \in \mathcal{B}(G)$, which is a vertex in the building of some tamely ramified twisted Levi subgroup of $\mathbf{G}$ divides the ramification index of E over k;
  (iii) for every $r \geq 1$, the exponential map induces a homeomorphism

$$\exp : \mathbf{g}(E)_{x,r} \to \mathbf{G}(E)_{x,r},$$

  and an abelian group homomorphism $\mathbf{G}(E)_{x,r}\big/\mathbf{G}(E)_{x,r+} \simeq \mathbf{g}(E)_{x,r}\big/\mathbf{g}(E)_{x,r+}$.

We recall the notion [5] of $\mathbf{G}$-*good* elements (we shall simply write *good* when there is no risk of confusion). Define the *depth* $\mathrm{d}_{\mathbf{T}}(\gamma)$ of a compact element $\gamma \in T_0$ as the unique $r \in \mathbb{R}_{\geq 0}$ such that $\gamma \in T_r \setminus T_{r+}$. A compact element $\gamma \in T_0$ of positive depth $r > 0$ is $\mathbf{G}$-good if for each $\alpha \in \Phi$, either $\alpha(\gamma) = 1$ or $\mathrm{ord}(\alpha(\gamma) - 1) = r$. A compact semisimple element γ of depth 0 is $\mathbf{G}$-good if it is absolutely semisimple.

Hypothesis ($\mathscr{E}$) may appear complicated at first. In fact, it achieves different purposes simultaneously:

- If the tame extension E of k satisfies ($\mathscr{E}$).(i), (resp. ($\mathscr{E}$).(ii)), and E'/E is a larger tame extension, then E' also satisfies ($\mathscr{E}$).(i), (resp. ($\mathscr{E}$).(ii)), because the ramification index of E' over k is divisible by the ramification index of E over k.
- On the other hand, the radius of convergence of the exponential map in ($\mathscr{E}$).(iii) decreases with the ramification index. In particular, Hypothesis ($\mathscr{E}$).(iii) may be satisfied for the extension E of k, and fail for larger extensions E'/E.

To show that Hypotheses ($\mathscr{T}$) and ($\mathscr{E}$) hold true for sufficiently large p, we begin with a few general group-theoretic lemmas.

Lemma 3.4. *Fix a root datum $\mathcal{R}$. Then there exists a constant $d_{\mathcal{R}} > 0$ such that for every field K_0 and every connected reductive group $\mathbf{G}$ over K_0 with absolute root datum $\mathcal{R}$, the group $\mathbf{G}$ splits over an extension of K_0 with degree at most $d_{\mathcal{R}}$. In fact every maximal torus of $\mathbf{G}$ defined over K_0 splits over an extension with degree at most $d_{\mathcal{R}}$.*

Proof. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ defined over K_0. Then $\mathbf{T}$ splits over a finite separable extension K/K_0 such that $\mathrm{Gal}(K/K_0)$ acts faithfully on $X^*(\mathbf{T})$. Choose a $\mathbb{Z}$-basis for $X^*(\mathbf{T})$ to identify $X^*(\mathbf{T}) \simeq \mathbb{Z}^r$ so as to obtain a group embedding

$$\mathrm{Gal}(K/K_0) \hookrightarrow \mathrm{GL}(r, \mathbb{Z}).$$

By Minkowski's lemma (see e.g. [66]), $\mathrm{GL}(r, \mathbb{Z})$ has only finitely many finite subgroups up to conjugacy (for a fixed r), hence $[K : K_0]$ admits a bound $d_r \in \mathbb{Z}_{>0}$ only in terms of r. The lemma is proved with $d_{\mathcal{R}} := d_r$, as r depends only on the root datum $\mathcal{R}$. $\square$

When K_0 is a local field, we have an explicit and relatively tight bound on $d_{\mathcal{R}}$ assuming that the residual characteristic is not too small.

Lemma 3.5. *Fix a root datum $\mathcal{R}$. Let $M_{\mathcal{R}}$ be the largest integer m such that $\varphi(m) \leq \mathrm{rank}(\mathcal{R})$, and $p_{\mathcal{R}} := \mathrm{rank}(\mathcal{R})$. Then for every local field K_0 of residual characteristic $p > p_{\mathcal{R}}$, every K_0-torus $\mathbf{T}$ in a connected reductive group $\mathbf{G}$ over K_0 with absolute root datum $\mathcal{R}$, splits over a tamely ramified extension of K_0 of degree $\leq M_{\mathcal{R}}^2$.*

Proof. We will deduce this from the argument for Lemma 3.4. In the proof of that lemma, retaining the same notation, it suffices that the order of the finite subgroup $\mathrm{Gal}(K/K_0) \hookrightarrow \mathrm{GL}(r, \mathbb{Z})$ be coprime to p for K/K_0 to be tame. By Minkowski's lemma [66], this holds if $p > r + 1$. Moreover, the exponent m of every finite subgroup of $\mathrm{GL}(r, \mathbb{Z})$ satisfies $\varphi(m) \leq r$, because the eigenvalues of an element of order m inside $\mathrm{GL}(r, \mathbb{Z})$ consist of all m-th roots of unity (see [72, App. B, claim (1) p.256] for a similar argument). So K/K_0 is a finite tamely ramified extension of exponent m with

$m \leq M_{\mathcal{R}}$. If K' denotes the maximal unramified subextension of K/K_0 then K/K' and K'/K_0 are cyclic so $[K : K']$ and $[K' : K_0]$ divide m. Therefore $[K : K_0] \leq m^2$. $\square$

Lemma 3.6. *Fix a root datum $\mathcal{R}$. Then there exists a constant $N_{\mathcal{R}} > 0$ such that for every field K_0 of characteristic zero and every connected reductive group $\mathbf{G}$ over K_0 with absolute root datum $\mathcal{R}$, there exists a K_0-embedding $\mathbf{G} \hookrightarrow GL(n)$ with $n \leq N_{\mathcal{R}}$.*

Proof. Denote by $\mathbf{G}_0$ the Chevalley group over $k_0 := \mathbb{Q}$ determined by $\mathcal{R}$. Fix a k_0-embedding $\mathbf{G}_0 \hookrightarrow GL_{k_0}(V_0)$ with V_0 a finite dimensional space over k_0 once and for all. We will prove the lemma with $N_{\mathcal{R}} := d_{\mathcal{R}} \dim_{k_0} V_0$ with $d_{\mathcal{R}}$ as in Lemma 3.4.

Let $\mathbf{G}$ and K_0 be as in the current lemma. By Lemma 3.4, there exists K/K_0 with $[K : K_0] \leq d_{\mathcal{R}}$ such that $\mathbf{G} \otimes_{K_0} K$ is a split group, so that $\mathbf{G}_0 \otimes_{k_0} K \simeq \mathbf{G} \otimes_{K_0} K$. By base change, we have a K-embedding $\mathbf{G}_0 \otimes_{k_0} K \hookrightarrow GL_K(V_0 \otimes_{k_0} K)$. Let Res_{K/K_0} denote the Weil restriction of scalars with respect to K/K_0. Using the obvious embedding $\mathbf{G} \hookrightarrow \mathrm{Res}_{K/K_0}(\mathbf{G} \otimes_{K_0} K)$, we obtain a chain of K_0-embeddings

$$\mathbf{G} \hookrightarrow \mathrm{Res}_{K/K_0}(\mathbf{G} \otimes_{K_0} K)$$
$$\simeq \mathrm{Res}_{K/K_0}(\mathbf{G}_0 \otimes_{k_0} K) \hookrightarrow \mathrm{Res}_{K/K_0} GL_K(V_0 \otimes_{k_0} K) \hookrightarrow GL_k(V_0 \otimes_{k_0} K).$$

The last embedding follows from viewing $V_0 \otimes_{k_0} K$ as a K_0-vector space. Since $\dim_{K_0}(V_0 \otimes_{k_0} K) = [K : K_0] \dim_{k_0} V_0 \leq d_{\mathcal{R}} \dim_{k_0} V_0 = N_{\mathcal{R}}$, the proof is complete. $\square$

Remark 3.7. For certain kinds of groups $\mathbf{G}$, one can proceed more directly as follows:

- If $\mathbf{G}$ is semisimple, then we have $\mathrm{Aut}(\mathbf{G}) \simeq \mathrm{Inn}(\mathbf{G}) \rtimes \mathrm{Aut}(\mathcal{R})$, and the action of $\mathrm{Gal}(\overline{K_0}/K_0)$ on $\mathbf{T}$ factors through $W \rtimes \mathrm{Aut}(\mathcal{R})$. Thus it is sufficient that $p_{\mathcal{R}}$ be larger than every prime factor of the Weyl group W and the automorphism group $\mathrm{Aut}(\mathcal{R})$.
- If $\mathbf{G}$ is already split over a tamely ramified extension of K_0, then pick one tame K_0-torus $\mathbf{T}$. To ensure that all the other K_0-tori are tame, it suffices that the image of $H^1(K_0, N_{\mathbf{G}}(\mathbf{T})) \to \mathrm{Hom}(\mathrm{Gal}(\overline{K_0}/K_0), W)$ consists of elements of order coprime to p, which happens if $p_{\mathcal{R}}$ is larger than every prime factor of W.

Proposition 3.8. *Hypotheses $(\mathscr{E})$ and $(\mathscr{T})$ are satisfied if p is sufficiently large, depending only on the absolute root datum of $\mathbf{G}$ and the absolute ramification index of k over $\mathbb{Q}_p$.*

Proof. Let $\mathcal{R}$ be the absolute root datum of $\mathbf{G}$. Assuming $p > p_{\mathcal{R}}$, choose a tamely ramified extension K/k with $[K : k] \leq M_{\mathcal{R}}^2$ as in Lemma 3.5. We shall construct E as an extension of K, thus $(\mathscr{E}).(\mathrm{i})$ will be automatically satisfied.

Concerning Hypothesis $(\mathscr{E}).(\mathrm{ii})$, the key is to observe that every $x \in \mathcal{B}(\mathbf{G}, k)$ contained in a generic datum is a *vertex* of the sub-building $\mathcal{B}(\mathbf{G}^0, k)$ attached to a tamely ramified twisted Levi subgroup $\mathbf{G}^0 \subset \mathbf{G}$. Let $\mathbf{T}^0$ be a maximal k-torus in $\mathbf{G}^0$ so that $x \in$

$\mathcal{B}(\mathbf{G}^0, k) \cap \mathcal{A}(\mathbf{G}, \mathbf{T}^0, K)$. In particular x is a $\mathrm{Gal}(K/k)$-fixed point of $\mathcal{A}(\mathbf{G}, \mathbf{T}^0, K)$. Let $\mathbf{S}^0 \subset \mathbf{T}^0$ be the maximal k-split subtorus of $\mathbf{T}^0$.

Similarly as in [58, §2.6], if we fix an alcove and consider the finitely many affine roots ψ_i whose zero loci bound an alcove and which are positive valued on the alcove, then there is a linear relation $\sum_i b_i \psi_i = 1$ with $b_i \in \mathbb{Z}_{>0}$. Let ℓ_{ψ_i} be defined as in [55, 3.1]. Put $b_{\mathbf{G}} := \mathrm{l.c.m}(\ell_{\psi_i} b_i)$. Since x is a vertex of $\mathcal{B}(\mathbf{G}^0, k)$, we have $\psi(x) \in \frac{1}{b_{\mathbf{G}}}\mathbb{Z}$ for every affine k-root $\psi \in \Psi(\mathbf{G}^0, \mathbf{S}^0, k)$. We view $\Psi(\mathbf{G}^0, \mathbf{S}^0, k)$, resp. $\Psi(\mathbf{G}, \mathbf{T}^0, K)$, as affine linear functions on $X_*(\mathbf{S}^0, k) \otimes_{\mathbb{Z}} \mathbb{R}$, resp. $X_*(\mathbf{T}^0, K) \otimes_{\mathbb{Z}} \mathbb{R}$. We need to recall the details of how $\mathcal{B}(\mathbf{G}^0, k)$ injects in $\mathcal{B}(\mathbf{G}, k)$, see p.585, Rem.2.11, Rem.3.4 of [74]. The affine isomorphism $\mathcal{A}(\mathbf{G}^0, \mathbf{T}^0, K) \simeq \mathcal{A}(\mathbf{G}, \mathbf{T}^0, K)$ is canonically defined only up to translation by $X_*(\mathbf{Z}_{\mathbf{G}^0}, K) \otimes_{\mathbb{Z}} \mathbb{R}$. The choice in [74] is made essentially in such a way that the isomorphism be $\mathrm{Gal}(K/k)$-equivariant. Recall that $\mathbf{Z}_{\mathbf{G}^0}$ is anisotropic because $\mathbf{Z}_{\mathbf{G}}$ is anisotropic. Hence the origin is the only $\mathrm{Gal}(K/k)$-fixed point in $X_*(\mathbf{Z}_{\mathbf{G}^0}, K) \otimes_{\mathbb{Z}} \mathbb{R}$, and the isomorphism is uniquely determined. To rigidify the situation, we identify the apartments $\mathcal{A}(\mathbf{G}^0, \mathbf{S}^0, k)$ and $\mathcal{A}(\mathbf{G}, \mathbf{T}^0, K)$ with $X_*(\mathbf{S}^0, k) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X_*(\mathbf{T}^0, K) \otimes_{\mathbb{Z}} \mathbb{R}$, respectively. Since the center $\mathbf{Z}_{\mathbf{G}^0}$ is anisotropic, we have that the $\mathbb{Q}$-span of $\Psi(\mathbf{G}^0, \mathbf{S}^0, k)$ is equal to the space of all affine linear functions on $X_*(\mathbf{S}^0, k) \otimes_{\mathbb{Z}} \mathbb{Q}$. There is a positive integer e such that the $\mathbb{Z}$-span of $\frac{1}{e}\Psi(\mathbf{G}^0, \mathbf{S}^0, k)$ contains the image of $\Psi(\mathbf{G}, \mathbf{T}^0, K)$ under restriction to $X_*(\mathbf{S}^0, k) \otimes_{\mathbb{Z}} \mathbb{Q}$. We have $\psi(x) \in \frac{1}{b_{\mathbf{G}} e}\mathbb{Z}$ for every affine root $\psi \in \Psi(\mathbf{G}, \mathbf{T}^0, K)$. Since the integers $b_{\mathbf{G}}$ and e depend only on the relative root system attached to $(\mathbf{G}^0, \mathbf{S}^0, k)$, the absolute root system $\mathcal{R}$ attached to $(\mathbf{G}, \mathbf{T}^0)$, and the action of $\mathrm{Gal}(K/k)$, and since there are finitely many root systems of given rank (possibly non-reduced, and possibly reducible), we have $b_{\mathbf{G}} e | e_{\mathcal{R}}$ for some positive integer constant $e_{\mathcal{R}}$ depending only on $\mathcal{R}$. If $p \nmid e_{\mathcal{R}}$, then a totally tamely ramified extension E/K of degree $e_{\mathcal{R}}$ will satisfy $(\mathscr{E})$.(ii).

For $(\mathscr{E})$.(iii), one needs to show that for $X, Y \in \mathfrak{g}(E)_{x,r}$,

$$\log(\exp(X)\exp(Y)) \equiv X + Y \pmod{\mathfrak{g}(E)_{x,r^+}}. \qquad (*)$$

The Campbell-Hausdorff formula in the form given by Dynkin is

$$\log(\exp(X)\exp(Y)) = \sum_{d=1}^{\infty} \frac{(-1)^{d+1}}{d} Z_d = X + Y + \sum_{d=2}^{\infty} \frac{(-1)^{d+1}}{d} Z_d,$$

with

$$Z_d = \sum_{\substack{s_i + t_i \geq 1 \\ s_d + t_d = 1}} \frac{\mathrm{ad}(X)^{s_1}\mathrm{ad}(Y)^{t_1} \cdots \mathrm{ad}(X)^{s_{d-1}}\mathrm{ad}(Y)^{t_{d-1}}(Y)(X^{s_d}Y^{t_d})}{\sum_{i=1}^{d}(s_i + t_i) \cdot \prod_{i=1}^{d} r_i! s_i!},$$

where the convention is that $X^{s_d}Y^{t_d}$ is equal to X if $s_d = 1, t_d = 0$, and is equal to Y if $s_d = 0, t_d = 1$.

Since $[\mathfrak{g}(E)_{x,a}, \mathfrak{g}(E)_{x,b}] \subset \mathfrak{g}(E)_{x,a+b}$, each summand in Z_d is in $\mathfrak{g}(E)_{x,r'}$ where

$$r' = r \sum_i (s_i + t_i) - \mathrm{ord}_E(d) - \mathrm{ord}_E\left(\sum_i (s_i + t_i)\right) - \sum_i \mathrm{ord}_E(s_i!) - \sum_i \mathrm{ord}_E(t_i!).$$

By Legendre formula

$$\mathrm{ord}_E(s_i!) = \frac{\mathrm{ord}_E(p)}{p-1}(s_i - \sigma) < \frac{\mathrm{ord}_E(p)s_i}{p-1},$$

where σ is the sum of the base-p expansion digits of s_i, we have

$$r' > r\sum_i (s_i + t_i) - \mathrm{ord}_E(p)\left(\log_p(d) + \log_p\left(\sum_i (s_i + t_i)\right) + \frac{\sum_i (s_i + t_i)}{p-1}\right).$$

If $r > \dfrac{\mathrm{ord}_E(p)}{p-1}$, then we have $r' \to \infty$ as $d \to \infty$, because $\sum_{i=1}^d (s_i + t_i) \geq d$. Hence the infinite sum converges. Since $\log_p(x) \leq \dfrac{x}{2\log p}$ for every $x \geq 2$,

$$r' > \left(r - \frac{\mathrm{ord}_E(p)}{\log p} - \frac{1}{p-1}\right)\sum_i (s_i + t_i).$$

Thus $\dfrac{\mathrm{ord}_E(p)}{\log p} + \dfrac{1}{p-1} \leq \dfrac{1}{2} \leq \dfrac{r}{2}$ implies $r' > r$, and that $(*)$ is satisfied.

Consider again a totally tamely ramified extension E/K of degree $e_\mathcal{R}$. Then $\mathrm{ord}_E(p) = e_\mathcal{R}\mathrm{ord}_K(p) \leq e_\mathcal{R} M_\mathcal{R}\mathrm{ord}_k(p)$. The conclusion is that if

$$p > p_\mathcal{R}, \quad p \nmid e_\mathcal{R}, \quad \text{and} \quad \frac{e_\mathcal{R} M_\mathcal{R}\mathrm{ord}_k(p)}{\log p} + \frac{1}{p-1} \leq \frac{1}{2}, \tag{$**$}$$

then the extension E satisfies the second statement of Hypothesis $(\mathscr{E})$.(iii). We observe that the above calculation shows that $\exp(\mathbf{g}(E)_{x,r})$ is a group when p satisfies $(**)$.

We finally need to verify the first statement of Hypothesis $(\mathscr{E})$.(iii). Under the condition that $p > \dim(G)\mathrm{ord}_E(p) + 1$, the map $\exp: \mathbf{g}(E)_{0+} \to \mathbf{G}(E)_{0+}$ is a homeomorphism by [72, App.B] (An alternative approach would be to consider a faithful representation $\mathbf{G} \hookrightarrow \mathrm{GL}_n$ for some n, see also [28, App.B]. Note that $\mathbf{g}(E)_{0+}$ (resp. $\mathbf{G}(E)_{0+}$) are the set of topologically nilpotent (resp. unipotent) elements since $p > \dim(G)$.) Hence, $\exp_r = \exp|_{\mathbf{g}(E)_{x,r}}$ is injective. We also need to show that $\exp_r$ maps onto $\mathbf{G}(E)_{x,r}$. To see this, write $\mathbf{g}(E) = \mathbf{t}^0(E) \oplus (\oplus_\phi \mathbf{g}_\phi(E))$ where $\mathbf{g}_\phi$ is the root space of $\phi \in \Phi(\mathbf{T}^0, \mathbf{G}, E)$. Upon fixing a pinning, $\mathbf{g}(E)_{x,r}$ is the $\mathcal{O}_E$-span of $\mathbf{g}_\phi(\mathfrak{p}_E^n)$ with $\phi(x) + n/e_{E/k} \geq r$ where $e_{E/k}$ is the ramification index of E over k. Likewise, $\mathbf{G}(E)_{x,r}$ is generated by $\mathbf{T}^0(E)_r$ and $\mathbf{U}_\phi(\mathfrak{p}_E^n)$ where $\mathbf{U}_\phi$ is the root subgroup of ϕ. Then, $\exp_r$ takes $\mathbf{g}_\phi(\mathfrak{p}_E^n)$ onto $\mathbf{U}_\phi(\mathfrak{p}_E^n)$, and the proof of assertion (4) on p.258 of [72, App.B] shows that $\mathbf{t}^0(E)_r$ is exponentiated onto $\mathbf{T}^0(E)_r$. Now, if p satisfies $(**)$, then $\exp(\mathbf{g}(E)_{x,r})$ is a group and hence equals $\mathbf{G}(E)_{x,r}$. In sum, Hypothesis $(\mathscr{E})$ holds when p satisfies $(**)$ and $p > \dim(G)e_\mathcal{R} M_\mathcal{R}\mathrm{ord}_k(p) + 1$.

The assertion that $(\mathscr{T})$ holds for p sufficiently large goes back to [4, §5], which treated the Lie algebra version. A recent treatment of the existence of good elements for the group case is [33, Thm 3.6], with the sharp result that Hypothesis $(\mathscr{T})$ holds when $\mathbf{G}$ splits

over a tamely ramified extension and p does not divide the order of the Weyl group of **G**. $\quad\square$

Remark 3.9.

- By definition, a finite extension E of k is tame if the residue characteristic p is coprime to the ramification index of E over k. Thus, Hypothesis $(\mathscr{E}).(ii)$ implies p is coprime to the order of every x contained in a generic datum. As we have seen in the proof, this is satisfied if $p > h$, the Coxeter number of **G**, and **G** splits over a tamely ramified extension of k.
- In Hypothesis $(\mathscr{E}).(iii)$, if **G** is a classical group and $p \neq 2$, one can use a Cayley map instead of the exponential map. When p is very good ([13, (8.9)]), and **G** is semisimple and simply connected, a quasi-logarithmic map satisfying $(\mathscr{E}).(iii)$ is constructed in [13, Lem C.4]. In general the mock exponential map introduced by Adler could be used, compare also with [27, Hyp. 3.2.1].

For convenience, we shall work under Hypotheses $(\mathscr{T})$ and $(\mathscr{E})$ in Section 4 below, which is devoted to prove Proposition 3.13. However, inspecting the recursive argument in Proposition 4.4, it is sufficient to have Hypothesis $(\mathscr{E}).(iii)$ for large enough r, and this is always satisfied by the same argument. Also we shall need in Lemma 4.8 below that the isomorphisms in $(\mathscr{E}).(iii)$ be G-equivariant, which is also satisfied for large enough r. Hypothesis $(\mathscr{E}).(ii)$ is automatically satisfied for an extension of large enough degree. Hence a natural assumption for our setting is:

$(\mathscr{T}')$ Every k-torus **T** in **G** is tame, and for every large enough r, every nontrivial coset in T_r/T_{r+} contains a **G**-good element.

More precisely, the preceding paragraph explains that the arguments in Section 4 prove Proposition 3.13 under Hypothesis $(\mathscr{T}')$, instead of Hypotheses $(\mathscr{T})$ and $(\mathscr{E})$. This in turn implies Theorems 3.11 and 3.16 under Hypothesis $(\mathscr{T}')$.

Remark 3.10. To reach an optimal assumption, one could also factor out the center $\mathbf{Z_G}$, which shouldn't play a role in estimating orbital integrals, and therefore it shouldn't be necessary to assume that $\mathbf{Z_G}$ is tame.

3.4. A uniform bound on orbital integrals of supercuspidal coefficients

For a semisimple element $\gamma \in G_{ss}$ let $\mathfrak{g}_\gamma$ denote the Lie algebra of the connected centralizer of γ in G. Define

$$D(\gamma) = D^G(\gamma) := \left| \det(1 - \mathrm{Ad}(\gamma)|_{\mathfrak{g}/\mathfrak{g}_\gamma}) \right| \in \mathbb{R}_{>0}.$$

Note that it is unnecessary to assume γ to be regular. Given a generic G-datum Σ, Lemma 3.3 provides us with the pseudocoefficient $f_{\pi_\Sigma} \in \mathcal{H}(G)$ coming from J_Σ and ρ_Σ.

The following is a key local result of this paper.

Theorem 3.11. *Assume Hypothesis ($\mathscr{T}'$). There exist constants $C, \nu > 0$ depending only on G such that for every generic G-datum Σ,*

$$D(\gamma)^{1/2} |\mathcal{O}_\gamma(f_{\pi_\Sigma})| \leq C \cdot \deg(\pi_\Sigma)^{1-\nu}, \quad \forall \gamma \in G_{\mathrm{ss}} \backslash Z.$$

In fact we can take any $\nu < (\dim G)^{-1}$.

In particular the theorem implies that if $\gamma \notin Z$ then

$$\lim_{\deg(\pi) \to \infty} \frac{\mathcal{O}_\gamma(f_\pi)}{\mathcal{O}_1(f_\pi)} = \lim_{\deg(\pi) \to \infty} \frac{\mathcal{O}_\gamma(f_\pi)}{\deg(\pi)} = 0,$$

where $\pi = \pi_\Sigma$ varies in $\mathrm{Irr}^{\mathrm{Yu}}(G)$. If, on the contrary $\gamma \in Z$ then clearly $\mathcal{O}_\gamma(f_\pi)/\mathcal{O}_1(f_\pi) = \omega_\pi^{-1}(\gamma)$, where ω_π is the central character of π. So the above limit is never zero.

Remark 3.12. An interesting question is whether the above Theorem 3.11 remains valid if π is allowed more generally to run over $\mathrm{Irr}^2(G)$. We are inclined to believe that it is at least true for every sequence in $\mathrm{Irr}^{\mathrm{sc}}(G)$, possibly with a different value of $\nu \in \mathbb{R}_{>0}$.

Proof of Theorem 3.11. The orbital integral vanishes unless γ is elliptic, so we assume that γ is noncentral and elliptic semisimple from now on. Let $G_\gamma := Z(\gamma)$ and $\mathcal{O}(\gamma)$ the G-orbit of γ in G.

For $\Sigma = (\vec{\mathbf{G}}, x, \vec{r}, \vec{\phi}, \rho)$ as in Definition 2.1, we let

$$\mathbf{G}' := \mathbf{G}^{d-1}, \quad G' := G^{d-1}, \quad \text{and} \quad L_s := G_{x,s} G_x' \text{ for } s \in \mathbb{R}_{\geq 0}.$$

Since L_{s_Σ} is an open compact subgroup of G containing J_Σ, where we recall that $s_\Sigma = r_{d-1}/2$, it follows from Lemma 3.3 that we have the inequalities

$$\frac{|\mathcal{O}_\gamma(f_{\pi_\Sigma})|}{\deg(\pi_\Sigma)} \leq \mathrm{vol}_{\mathcal{O}(\gamma)}(J_\Sigma \cap \mathcal{O}(\gamma)) \leq \mathrm{vol}_{\mathcal{O}(\gamma)}(L_{s_\Sigma} \cap \mathcal{O}(\gamma)) = \mathcal{O}_\gamma(\mathbf{1}_{L_{s_\Sigma}}). \tag{3.2}$$

Our strategy is to study a power-saving upper bound for $\mathcal{O}_\gamma(\mathbf{1}_{L_s})$ as γ runs over semisimple elements of G and as $s \to \infty$. Indeed as Σ moves along a sequence of generic G-data such that $\deg(\pi_\Sigma) \to \infty$, up to conjugacy, there are only a finite number of choices for $(\mathbf{G}', x)$ appearing in Σ with $x \in \mathcal{B}(G^0)$. The crucial estimate is the following, whose proof is postponed to Section 4. (Regarding the hypothesis, see the discussion above Remark 3.10.)

Proposition 3.13. *Assume Hypothesis ($\mathscr{T}'$). There exists a constant $C_1 > 0$ depending only on G such that for all generic G-datum Σ, $s \in \mathbb{R}_{>0}$ and all noncentral semisimple $\gamma \in G$,*

$$\mathcal{O}_\gamma(\mathbf{1}_{L_s}) \leq C_1 \cdot (s+1) \cdot q^{-s} \cdot D(\gamma)^{-1/2}.$$

To relate $\deg(\pi_\Sigma)$ and s_Σ, we deduce the following from (2.1) and the fact that $J_\Sigma \supset G_{x,s_\Sigma}$:

$$\deg(\pi_\Sigma) \leq q^{\dim G} \mathrm{vol}(J_\Sigma)^{-1} \leq q^{\dim G} \mathrm{vol}(G_{x,s_\Sigma})^{-1}.$$

Since $G_{x,r}/G_{x,r+1}$ is in bijection with $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+1}$ for $r > 0$ by hypothesis $(\mathscr{E})$ and $\varpi\mathfrak{g}_{x,r} = \mathfrak{g}_{x,r+1}$, we have $[G_{x,r} : G_{x,r+1}] = q^{\dim G}$. Thus for $s \in \mathbb{R}_{\geq 0}$,

$$\mathrm{vol}(G_{x,s}) \geq q^{-\lceil s \rceil \dim G} \mathrm{vol}(G_{x,1}).$$

Hence $\mathrm{vol}(G_{x,1}) \deg(\pi_\Sigma) \leq q^{(s_\Sigma+2)\dim G}$.

We deduce from this, the inequality (3.2), and Proposition 3.13 above that

$$\frac{|\mathcal{O}_\gamma(f_{\pi_\Sigma})|}{\deg(\pi_\Sigma)} \leq q^2 C_1 \cdot (s_\Sigma + 1) \cdot (\mathrm{vol}(G_{x,1})\deg(\pi_\Sigma))^{-\frac{1}{\dim G}} D(\gamma)^{-1/2}.$$

The proof of Theorem 3.11 is complete. $\quad\square$

Remark 3.14. Let us explain what can be done when the center Z is not compact. Theorem 3.11 and Proposition 3.13 remain valid as stated. The only modification to the proof in Section 4 is that we need to use the subgroups $L_s := G_{x,s}G'_{[x]}$ which are only compact-mod-center, in place of $G_{x,s}G'_x$.

Lemma 3.3 is still true in this case, if $\mathrm{vol}(J)$ is replaced with the volume of J/Z in G/Z, and verified by the same argument. It may be troubling at first that the support of f_π is compact only modulo center but equality (3.1) remains valid.[5]

Conjecture 3.15 below remains the same, even when the center Z is not compact. One only needs to remember that in our convention $\mathrm{Irr}^{sc}(G)$ consist of *unitary* supercuspidal representations, so the central character of π is also unitary.

Finally, in the global application we appeal to the simple trace formula with fixed character on a closed central subgroup so as to allow f_π as the local component of a test function. Alternatively one could work with a truncated (pseudo-)coefficient (cf. [40, 1.9]), which is compactly supported, in place of f_π.

3.5. Asymptotic behavior of supercuspidal characters

The main result, Theorem 3.16, of this section may be rephrased as a uniform upper bound for the characters of supercuspidal representations constructed by Yu on elliptic regular elements via Proposition 3.2.

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[5] We recall that (3.1) follows from [10, Thm 5.1], based on Arthur's local trace formula [10, Thm 4.2]. In the case Z is not compact, [11, Prop 6.1] gives the local trace formula with fixed central character and similarly implies the analogue of (3.1).

Let us recall the context of the problem. Even though the precise character formulas for supercuspidal (and discrete series) representations of a p-adic group remain largely mysterious, we conjectured in [46, Conjecture 4.1] that the character values behave in a controlled manner as the formal degree tends to infinity. Recall that $\mathrm{Irr}^{\mathrm{sc}}(G)$ denote the set of isomorphism classes of supercuspidal representations. Write G_{ell} (resp. G_{rs}) for the set of elliptic (resp. regular semisimple) elements in G.

Conjecture 3.15. *(i) For every $\gamma \in G_{\mathrm{rs}}$,*

$$\lim_{\substack{\pi \in \mathrm{Irr}^{\mathrm{sc}}(G) \\ \deg(\pi) \to \infty}} \frac{\Theta_\pi(\gamma)}{\deg(\pi)} = 0.$$

In other words, for every $\gamma \in G_{\mathrm{rs}}$ and every $\epsilon > 0$ there exists $d_{\gamma,\epsilon} > 0$ such that $|\Theta_\pi(\gamma)| < \epsilon \deg(\pi)$ for every $\pi \in \mathrm{Irr}^{\mathrm{sc}}(G)$ with $\deg(\pi) > d_{\gamma,\epsilon}$.

(ii) Let $\mathcal{B} \subset G$ be a bounded subset. Then there exist constants $\nu > 0$ depending only on G and $C_{\mathcal{B}} > 0$ depending only on G and $\mathcal{B}$ such that

$$D(\gamma)^{1/2}|\Theta_\pi(\gamma)| \leq C_{\mathcal{B}} \cdot \deg(\pi)^{1-\nu}, \quad \forall \pi \in \mathrm{Irr}^{\mathrm{sc}}(G), \ \forall \gamma \in G_{\mathrm{rs}} \cap \mathcal{B}.$$

(iii) There exist constants $\nu > 0$ and $C_{\mathrm{ell}} > 0$ depending only on G such that

$$D(\gamma)^{1/2}|\Theta_\pi(\gamma)| \leq C_{\mathrm{ell}} \cdot \deg(\pi)^{1-\nu}, \quad \forall \pi \in \mathrm{Irr}^{\mathrm{sc}}(G), \ \forall \gamma \in G_{\mathrm{rs}} \cap G_{\mathrm{ell}}.$$

Note that (ii) implies (i). Since G has finitely many (elliptic) maximal tori up to conjugacy ([56, p.320, Cor 3]), (ii) also implies (iii). In [46, Thm 4.2], we have proved a result for (ii), provided that the residue characteristic of k is sufficiently large and that γ runs over the set G_{0+} of topologically unipotent elements (which are not necessarily elliptic). The argument of that paper is based on an estimate of the number of fixed points of γ acting on certain coset spaces (without estimating orbital integrals).

Theorem 3.11 restricted to regular elements may be interpreted in terms of character values to establish (iii) of the conjecture provided that the residue characteristic of k is large enough. Note that the proof of Theorem 3.11 is independent of what is done in [46] and relies on quite a different method.

Theorem 3.16. *Under Hypothesis $(\mathscr{T}')$, part (iii) of Conjecture 3.15 holds true if $\pi \in \mathrm{Irr}^{\mathrm{Yu}}(G)$. In particular, part (iii) of the conjecture is true if the hypotheses of [45, 3.4] are also met (cf. the last paragraph of §2.3).*

Proof. The first assertion follows from equality (3.1) and Theorem 3.11. The second is deduced from the first assertion and the exhaustion theorem of [45]. $\square$

It is natural to wonder whether the method of this paper may be pushed further to cover non-elliptic elements. Proposition 3.2 is only a special case of Arthur's formula

[7] relating supercuspidal character values on non-elliptic regular elements to weighted orbital integrals of supercuspidal coefficients. (This extends to cover general discrete series via the local trace formula [10].) So the problem is to bound such weighted orbital integrals.

A different question concerning trace characters is whether the constant $\nu > 0$ can be found independent of the field k and the residue characteristic p. In this direction we observe that there exist analogues for finite groups, notably a general estimate by Gluck [35].

It is natural to ask for a common generalization of the families in the depth aspect and in the "level aspect". Here, level aspect means that $q_S \to \infty$. We would consider the multi-set $\mathcal{F}(\xi, \sigma, K_{S_0})$ where both the finite set of places S and the discrete series representation σ of $G(F_S)$ are varying. In any non-trivial sequence, the formal degree $\deg(\sigma)$ with respect to the canonical measure (see [36] or [70, §6.6] for definition) goes to infinity, either because the depth of σ goes to infinity or because the residue characteristic q_S goes to infinity. Theorem 1.2 above corresponds to families in the depth aspect for which S is fixed. Theorem 6.4 below corresponds to a refinement of the families in the level aspect where $\sigma = \mathrm{St}_S$. To establish such a common generalization one would need to address the above question of uniformity of the constant $\nu > 0$, and one also would need to keep track of the polynomial dependence in the constants C_{ell} in Conjecture 3.15.

4. Proof of Proposition 3.13

In this section we work with a pair $(\mathbf{G}, \mathbf{G}')$ where $\mathbf{G}'$ is a tamely ramified twisted Levi subgroup with $\mathbf{G}' \neq \mathbf{G}$. We shall also fix $x \in \mathcal{B}(\mathbf{G}') = \mathcal{B}(\mathbf{G}', k)$ and recall that $L_s = G_{x,s} G'_x$.

We start by recalling some basic definitions. For a maximal k-torus $\mathbf{T}$ and $\gamma \in T \setminus Z$, the *singular depth* of γ is defined as

$$sd(\gamma) := \max\{\mathrm{ord}(\alpha(\gamma) - 1) \mid \alpha \in \Phi, \ \alpha(\gamma) \neq 1\},$$

where Φ is the set of $\mathbf{T}$-roots (see [3]). We also define the *minimal depth* of γ as

$$md(\gamma) := \min\{\mathrm{ord}(\alpha(\gamma) - 1) \mid \alpha \in \Phi, \ \alpha(\gamma) \neq 1\}.$$

Both $sd(\gamma)$ and $md(\gamma)$ are independent of the choice of $\mathbf{T}$ containing γ, thus are well-defined for every semisimple $\gamma \in G \setminus Z$. Moreover, we have $md(\gamma) = md(g\gamma g^{-1})$ and $sd(\gamma) = sd(g\gamma g^{-1})$ for any $g \in G$.

The following is a special case of the decomposition theorem in [5]. Compared to [5], our situation is simpler and hypothesis (C) in [5] is not needed because we do not keep track of centralizers of good elements here. We include a proof for completeness.

Lemma 4.1. *Suppose Hypothesis* $(\mathscr{T})$ *holds. Let* $\mathbf{T}$ *be a tame maximal* k-torus, $\gamma \in T_0 \setminus Z_0$ *a compact element, and* $r = md(\gamma)$. *There exist elements* z, γ', γ_+, *such that* $\gamma = z\gamma'\gamma_+$, *and*

(i) *either* $z \in Z_0 \setminus Z_r$, *or* $z = 1$;
(ii) $\gamma' \in T_r \setminus Z_r T_{r+}$ *is* $\mathbf{G}$-*good of depth* r;
(iii) $\gamma_+ \in T_{r+}$.

Proof. We first consider the case $r > 0$. Let $b_1 = \mathrm{d}_{\mathbf{T}}(\gamma)$ so that $\gamma \in T_{b_1} \setminus T_{b_1^+}$. By Hypothesis $(\mathscr{T})$, there is a good element $\gamma_{b_1} \in T_{b_1}$ with $\gamma_{b_1} T_{b_1^+} = \gamma T_{b_1^+}$, thus $\gamma = \gamma_{b_1} \gamma_+$ for some $\gamma_+ \in T_{b_1^+}$. If $b_1 = md(\gamma)$, take $z = 1$, $\gamma' = \gamma_{b_1}$.

Otherwise $\gamma_{b_1} \in Z$, and we then let $b_2 = \mathrm{d}_{\mathbf{T}}(\gamma\gamma_{b_1}^{-1})$ so that $b_2 > b_1$ and $\gamma\gamma_{b_1}^{-1} \in T_{b_2} \setminus T_{b_2^+}$. By Hypothesis $(\mathscr{T})$, there is a good element γ_{b_2} such that $\gamma\gamma_{b_1}^{-1}\gamma_{b_2}^{-1} \in T_{b_2^+}$, thus $\gamma = \gamma_{b_1}\gamma_{b_2}\gamma_+$ for some $\gamma_+ \in T_{b_2^+}$. Repeating the process, one can write $\gamma = \gamma_{b_1}\gamma_{b_2}\cdots\gamma_{b_k}\gamma_+$ where $b_1 < b_2 < \cdots < b_k = md(\gamma)$, $\gamma_{b_i} \in Z$, $i = 1, \cdots, k-1$ and $\gamma_+ \in T_{b_k^+}$. Then, set $z = \gamma_{b_1}\gamma_{b_2}\cdots\gamma_{b_{k-1}}$, $\gamma' = \gamma_{b_k}$.

Now suppose $r = 0$. By [71, Prop 2.36], there is an absolutely semisimple element $\gamma_{as} \in T_0$ such that $\gamma = \gamma_{as}\gamma_{tu}$ where $\gamma_{tu} \in T_{0^+}$. If $\gamma_{as} \notin Z$, take $z = 1$, $\gamma' = \gamma_{as}$ and $\gamma_+ = \gamma_{tu}$. If $\gamma_{as} \in Z$, one proceeds as in the first case for γ_{tu} to reach the conclusion. $\square$

Corollary 4.2. *Suppose Hypothesis* $(\mathscr{T})$ *holds. Let* $\mathbf{T}$ *be a tame maximal* k-torus *and* $\gamma \in T_0 \setminus Z_0$. *Then there exists* $z \in Z_0$ *and* $\gamma_1 \in T_0 \setminus Z_0$ *such that* $\gamma = z\gamma_1$ *and*

$$\mathrm{d}_{\mathbf{T}}(\gamma_1) = md(\gamma).$$

Proof. If $md(\gamma) = 0$, then we can choose $z = 1$ and $\gamma_1 = \gamma$. Suppose $md(\gamma) > 0$. The assertion follows from Lemma 4.1 by setting $\gamma_1 = \gamma'\gamma_+$: Indeed $\gamma' \in T \setminus Z$ is $\mathbf{G}$-good of positive depth $\mathrm{d}_{\mathbf{T}}(\gamma') = md(\gamma') = md(\gamma)$. Moreover $\mathrm{d}_{\mathbf{T}}(\gamma') = \mathrm{d}_{\mathbf{T}}(\gamma_1)$, which concludes the proof. $\square$

We will prove Proposition 3.13 at the end of this section by induction based on the following two propositions. The first, Proposition 4.3, which is at the base of the induction, is concerned with orbital integrals of a fixed test function and proved by means of Shalika germ expansions. The second, Proposition 4.4, allows us to proceed inductively in the parameter $s \in \mathbb{R}_{\geq 0}$.

Proposition 4.3. *For each test function* $f \in \mathcal{H}(G)$ *there exists a constant* $c(f) > 0$ *such that for every semisimple* $\gamma \in G$, $|\mathcal{O}_\gamma(f)| \leq c(f)D(\gamma)^{-1/2}$.

Proof. [70, Thm A.1]. $\square$

Recall that $L_s = G_{x,s}G'_x$. The following power-saving bounds are essential ingredients in the proof of Proposition 3.13. Proposition 4.4 can be understood in comparison with

its analogue Corollary 4.9 below, which is easier to grasp and gives a bound for the function $\mathbf{1}_{G_{x,s}}$ in place of $\mathbf{1}_{L_s}$.

Proposition 4.4. *Suppose Hypotheses $(\mathscr{T})$ and $(\mathscr{E})$ hold. Let $\gamma \in G \setminus Z$ be semisimple, and $s \in \mathbb{Z}_{\geq 2}$.*

(i) *If $md(\gamma) \leq s - 2$, then $\mathcal{O}_\gamma(\mathbf{1}_{L_s}) \leq \frac{1}{q} \mathcal{O}_\gamma(\mathbf{1}_{L_{s-1}})$.*
(ii) *If $md(\gamma) \geq s + 1$, then $\mathcal{O}_\gamma(\mathbf{1}_{L_s}) \leq \frac{1}{q} \mathcal{O}_\gamma(\mathbf{1}_{L_{s-1}}) + \mathcal{O}_\gamma(\mathbf{1}_{G'_{x,s-2}G_{x,s}})$.*

The proof of Proposition 4.4 is postponed until we establish a handful of technical lemmas. Lemma 4.5 is used in the proof of Lemma 4.7.

Lemma 4.5. *Let G be a connected reductive group over a perfect field $\mathbb{F}$. Suppose M is a twisted Levi subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{m}$ be the Lie algebras of $\mathsf{G}(\mathbb{F})$ and $\mathsf{M}(\mathbb{F})$ respectively.*

(i) *For $X \in \mathfrak{m}$, let $\mathfrak{g}_X := \{Y \in \mathfrak{g} \mid [X, Y] = 0\}$. If $X \notin Z_{\mathfrak{g}}$, then $\mathfrak{m} + \mathfrak{g}_X$ is a proper $\mathbb{F}$-subspace of $\mathfrak{g}$.*
(ii) *For $\delta \in \mathsf{M}$, let $\mathfrak{g}_\delta := \{Y \in \mathfrak{g} \mid \mathrm{Ad}(\delta)(Y) = Y\}$. If $\delta \notin Z_{\mathsf{G}}$, then $\mathfrak{m} + \mathfrak{g}_\delta$ is a proper $\mathbb{F}$-subspace of $\mathfrak{g}$.*

Proof. (i) Let $X = X_{ss} + X_n$ be the Jordan decomposition of X with $[X_{ss}, X_n] = 0$ and X_{ss} (resp. X_n) semisimple (resp. X_n nilpotent). Since the lemma may be proved after taking a finite extension of $\mathbb{F}$, we may assume that M (thus also G) and X_{ss} split over $\mathbb{F}$ and that M is a maximal proper Levi subgroup of G. Thus $\mathfrak{m}$ is a maximal proper Levi subalgebra of $\mathfrak{g}$. Without loss of generality, we may assume that the Dynkin diagram of G is connected.

Let T be a maximal $\mathbb{F}$-split torus in M whose Lie algebra contains X_{ss}. Let Φ be the set of T-roots. Let Δ (resp. Φ^+) be the set of simple roots (resp. the set of positive roots) associated with T such that $X_n \in \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space of α in $\mathfrak{g}$. Let Δ_{M} (resp. Φ_{M}^+) be the subset of Δ (resp. Φ^+) associated with M. Let U and U^- be the unipotent and the opposite unipotent subgroup respectively. Let $\beta \in \Delta$ such that $\Delta = \Delta_{\mathsf{M}} \cup \{\beta\}$.

We will prove assertion (i) by showing its contrapositive that $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}_X$ implies that $X \in Z_{\mathfrak{g}}$. Assuming $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}_X$, we have that $\mathfrak{g}_\beta \subset \mathfrak{u} \subset \mathfrak{g}_X$. We proceed in two steps, first showing that X_{ss} is central, and then that $X_n = 0$.

We have $\mathfrak{u} \subset \mathfrak{g}_{X_{ss}}$ since $[X_{ss}, X_n] = 0$ and X_n is nilpotent. In particular, $\beta(X_{ss}) = 0$. Let $\alpha_1 \in \Delta_{\mathsf{M}}$ adjacent to β in the Dynkin diagram. Then, $\beta + \alpha_1 \in \Phi^+ \setminus \Phi_{\mathsf{M}}^+$ by [41, Prop 8.4] (this follows from the results about root strings) and $0 \neq \mathfrak{g}_{\alpha_1+\beta} \subset \mathfrak{u}$. Then, $(\beta + \alpha_1)(X_{ss}) = 0$, hence $\alpha_1(X_{ss}) = 0$ and $\mathfrak{g}_{\alpha_1} \subset \mathfrak{g}_{X_{ss}}$. Similarly if $\alpha_2 \neq \beta$ is adjacent to α_1, $\beta + \alpha_1 + \alpha_2 \in \Phi^+ \setminus \Phi_{\mathsf{M}}^+$ and $\mathfrak{g}_{\alpha_2} \subset \mathfrak{g}_{X_{ss}}$. Since the Dynkin diagram is connected, inductively, we conclude $\mathfrak{g}_\alpha \subset \mathfrak{g}_{X_{ss}}$ for all $\alpha \in \Delta$. Thus, $\mathfrak{m} \subset \mathfrak{g}_{X_{ss}}$ and $\mathfrak{g}_{X_{ss}} = \mathfrak{g}$, therefore X_{ss} is central.

Without loss of generality, we may assume that $X_{ss} = 0$, thus $X = X_n$. Write $X = \sum_{\alpha \in \Phi^+} X_\alpha$ with $X_\alpha \in \mathsf{g}_\alpha$. Fix $\alpha' \in \Phi$. Let Δ be a simple root system with $\alpha' \in \Delta$. Let Δ_M, β, U and U^- be as in the previous case.

Let $\alpha_1, \cdots, \alpha_j = \alpha'$ be distinct simple roots in Δ_M such that β and α_1 (resp. α_i and α_{i+1}) are adjacent to each other in the Dynkin diagram. Then, we have the following:

(a) $\beta + \alpha_1 + \cdots + \alpha_i \in \Phi^+ \setminus \Phi_\mathsf{M}^+$ for $i \le j$. This follows from $\beta \notin \Delta_\mathsf{M}$ and by inductively applying [41, Prop 8.4] to the root string.

(b) $[\mathsf{g}_{\beta + \alpha_1 + \cdots + \alpha_i}, X] = 0$ for $i \le j$. This follows from $\mathsf{g}_{\beta + \alpha_1 + \cdots + \alpha_i} \subset \mathsf{u} \subset \mathsf{g}_X$.

(c) $[\mathsf{g}_{\beta + \alpha_1 + \cdots + \alpha_{i-1}}, X] = 0$ implies that $[\mathsf{g}_{\beta + \alpha_1 + \cdots + \alpha_{i-1}}, X_{\alpha_i}] = 0$. Then, $X_{\alpha_i} = 0$ since $\beta + \alpha_1 + \cdots + \alpha_i \ne 0$. Note that we are using the assumption that p is large enough (e.g. p is large enough so that it does not divide any structural constants of g, which is necessary for $(\mathscr{E})$-(iii)).

By (a), (b) and (c), $X_{\alpha'} = 0$ since the Dynkin diagram is connected. As α' runs over Φ, we have $X = 0$, which is what we wanted, completing the proof of (i).

Assertion (ii) can be proved similarly as in (i) using the Jordan decomposition $\delta = \delta_{ss} \delta_n$ in M. $\quad\square$

We recall the notation $[g, h] := ghg^{-1}h^{-1}$ for $g, h \in G$. We will frequently use the fact [55, (2.6)] that for $a, b \in \mathbb{R}_{\ge 0}$,

$$\text{if} \quad g \in G_{x,a}, \ h \in G_{x,b} \quad \text{then} \quad [g, h] \in G_{x,a+b}. \tag{4.1}$$

It is also useful for us to introduce functions $\mathfrak{d}_x : G_{x,0} \setminus \{1\} \to \mathbb{R}_{\ge 0}$ and $\mathfrak{d}_x^Z : G_{x,0} \setminus Z_0 \to \mathbb{R}_{\ge 0}$ as follows:

$$\mathfrak{d}_x(g) := \max\{r \in \mathbb{R}_{\ge 0} \mid g \in G_{x,r}\},$$
$$\mathfrak{d}_x^Z(g) := \max\{r \in \mathbb{R}_{\ge 0} \mid g \in ZG_{x,r}\}.$$

Clearly $\mathfrak{d}_x(g) \le \mathfrak{d}_x^Z(g)$. For $s \in \mathbb{R}_{\ge 0}$ and $\delta \in L_s$, define

$$\mathrm{d}_{x,s}(\delta) := \max\{t \in [0, s] \mid \delta \in ZG'_{x,t}G_{x,s}\}.$$

Clearly $\mathfrak{d}_x(g) = \mathfrak{d}_x(g^{-1})$, $\mathfrak{d}_x^Z(g) = \mathfrak{d}_x^Z(g^{-1})$, and $\mathrm{d}_{x,s}(\delta) = \mathrm{d}_{x,s}(\delta^{-1})$. It is also clear that $\mathrm{d}_{x,s}(\delta) \le \mathfrak{d}_x^Z(\delta)$, since $G'_{x,t} \subset G_{x,t}$.

Lemma 4.6. *Suppose Hypotheses $(\mathscr{T})$ and $(\mathscr{E})$ hold. For every $s \in \mathbb{R}_{>0}$ and semisimple element $\delta \in L_s \cap G_0 \setminus Z_0$,*

$$\min(\mathfrak{d}_x^Z(\delta), s) \le \mathrm{d}_{x,s}(\delta) \le md(\delta).$$

In particular, for a semisimple element $\delta \in L_s \cap G_0 \setminus Z_0 G_{x,s^+}$, we have $\mathfrak{d}_x^Z(\delta) = \mathrm{d}_{x,s}(\delta)$.

Proof. Let $r \leq s$. Since $G'_x \cap G_{x,r} = G'_{x,r}$ (see e.g., [5, §4]), we have $G'_x G_{x,s} \cap G_{x,r} = G'_{x,r} G_{x,s}$. The condition $\delta \in G_{x,r}$ thus implies $\delta \in G'_{x,r} G_{x,s}$. Then $\delta \in Z G_{x,r}$ implies $\delta \in Z G'_{x,r} G_{x,s}$. In other words $\min(\mathfrak{d}_x^Z(\delta), s) \leq \mathrm{d}_{x,s}(\delta)$, which is the first inequality.

By Hypothesis $(\mathscr{E})$, there exists a tame maximal k-torus $\mathbf{T}$ containing δ. It follows from Corollary 4.2 that

$$\max\{r \in \mathbb{R}_{\geq 0} \mid \gamma \in Z T_r\} = md(\delta).$$

The second inequality then follows from the definitions since $G'_{x,t} G_{x,s} \subset G_{x,t}$, and $G_{x,t} \cap T \subset T_t$.

The last assertion is a corollary of the first inequality because $\delta \notin Z_0 G_{x,s+}$ is equivalent to $\mathfrak{d}_x^Z(\delta) \leq s$. □

Lemma 4.7. *Suppose Hypotheses $(\mathscr{T})$ and $(\mathscr{E})$ hold. Let $s \in \mathbb{Z}_{\geq 2}$ and $\delta \in L_s$ be such that* $\mathrm{d} := \mathrm{d}_{x,s}(\delta) < s - 1$. *Define the map*

$$C_\delta : G_{x,s-\mathrm{d}-1} / G_{x,s-\mathrm{d}} \to G_{x,s-1} / (G'_{x,s-1} G_{x,s})$$

by $C_\delta(g) := [\delta^{-1}, g] \pmod{G'_{x,s-1} G_{x,s}}$. Then the cardinality of the image of C_δ is at least q.

Proof. The last assertion of Lemma 4.6 implies that $\mathfrak{d}_x^Z(\delta) = \mathrm{d}$. Thus the map C_δ is well-defined in view of (4.1). Since C_δ is unchanged if δ is multiplied by a central element, we may assume without loss of generality that $\delta \in G_{x,\mathrm{d}}$.

Let E be a tamely ramified finite extension of k that satisfies Hypothesis $(\mathscr{E})$, and fix a uniformizer ϖ_E of E. Let m be the ramification index of E over k. Hence $\mathrm{ord}(\varpi_E) = 1/m$, in view of our normalization of valuation and filtration index in §2.1. Since the order of x divides m by $(\mathscr{E})$.(ii), we have $\mathbf{g}(E)_{x,t/m} = \varpi_E^t \mathbf{g}(E)_{x,0}$ for $t \in \mathbb{Z}$ and there is no break for non-integral values of t. Moreover, $\mathrm{d} \in \frac{1}{m}\mathbb{Z}$. Let $i \in \mathbb{Z}_{>0}$ be such that

$$\frac{i}{m} = s - \mathrm{d} - 1.$$

We now divide the proof into the following four steps (1-4):

(1) Write

$$\overline{\mathbf{G}(E)}_{x,i/m} := \mathbf{G}(E)_{x,i/m} / \mathbf{G}(E)_{x,(i+1)/m} \; ; \quad \overline{\mathbf{g}(E)}_{x,i/m} := \mathbf{g}(E)_{x,i/m} / \mathbf{g}(E)_{x,(i+1)/m} \; .$$

Similarly, we define $\overline{G}_{x,i/m}$, $\overline{\mathbf{g}}_{x,i/m}$, $\overline{\mathbf{G}'(E)}_{x,i/m}$, $\overline{\mathbf{g}'(E)}_{x,i/m}$, $\overline{G'}_{x,i/m}$ and $\overline{\mathbf{g}'}_{x,i/m}$. Since $i > 0$, we have abelian group isomorphisms

$$\overline{\mathbf{G}(E)}_{x,i/m} \simeq \overline{\mathbf{g}(E)}_{x,i/m} \quad \text{and} \quad \overline{\mathbf{G}'(E)}_{x,i/m} \simeq \overline{\mathbf{g}'(E)}_{x,i/m}$$

(2) $\overline{\mathbf{g}(E)}_{x,i/m}$ and $\overline{\mathbf{g}'(E)}_{x,i/m}$ are vector spaces over the residue field $\mathbb{E}$ of the maximal unramified extension E^u of k in E. Let $\mathbb{F} = \mathbb{F}_q$ be the residue field of k. Let G (resp. G') be a reductive group defined over $\mathbb{F}$ which is the reductive quotient at the building point x of $\mathbf{G}$ (resp. $\mathbf{G}'$) given by Bruhat-Tits theory. Write g and g' for the Lie algebras of $\mathsf{G}(\mathbb{F})$ and $\mathsf{G}'(\mathbb{F})$ respectively. We have $\overline{\mathbf{g}(E)}_{x,0} \simeq \mathsf{g}(\mathbb{E})$. We claim $\dim_E \mathbf{g}(E) = \dim_\mathbb{E} \mathsf{g}(\mathbb{E})$. This equality can be established from [26, §4.2]. More precisely, we have the isomorphisms of $\mathbb{E}$-vector spaces

$$\overline{\mathbf{g}(E)}_x := \oplus_{r \in \mathbb{R}/\frac{1}{m}\mathbb{Z}} \mathbf{g}(E)_{x,r}/\mathbf{g}(E)_{x,r+} = \overline{\mathbf{g}(E)}_{x,0},$$

where the first equality is a definition given in [26]. Note that the range $r \in \mathbb{R}/\frac{1}{m}\mathbb{Z}$ comes from our normalization which is such that $\mathrm{ord}(\varpi_E) = 1/m$, and hence $\varpi_E \mathbf{g}(E)_{x,r} = \mathbf{g}(E)_{x,r+\frac{1}{m}}$ as recalled above, so the second equality is clear since all breaks occur when $r \in \frac{1}{m}\mathbb{Z}$. Then, by the result in [26], we have $\dim_E \mathbf{g}(E) = \dim_E \overline{\mathbf{g}(E)}_x$, which establishes the claim.

(3) If $\mathrm{d} = 0$, then $\frac{i}{m} = s - 1$, and the map $g \mapsto [\delta^{-1}, g]$ induces $\overline{\mathbf{G}(E)}_{x,i/m} \to \overline{\mathbf{G}(E)}_{x,i/m}$ thus a linear map $\mathsf{g}(\mathbb{E}) \to \mathsf{g}(\mathbb{E})$. Moreover, it maps $\overline{G}_{x,i/m} \subset \overline{\mathbf{G}(E)}_{x,i/m}$ into itself since δ^{-1} lies in G_x (rather than $\mathbf{G}(E)_x$). Its composition with the projection $\mathsf{g}(\mathbb{E}) \to \mathsf{g}(\mathbb{E})/\mathsf{g}'(\mathbb{E})$ is seen to be equal to the map

$$\overline{C}_\delta : \mathsf{g}(\mathbb{E}) \to \mathsf{g}(\mathbb{E})/\mathsf{g}'(\mathbb{E}) \simeq \mathbf{g}(E)_{x,i/m}/(\mathbf{g}'(E)_{x,i/m} + \mathbf{g}(E)_{x,(i+1)/m})$$

induced by the map $Y \mapsto \mathrm{Ad}(\delta^{-1})(Y) - Y$ from $\mathbf{g}(E)_{x,i/m}$ to $\mathbf{g}(E)_{x,i/m}$. Equivalently, $\overline{C}_\delta$ is induced by the map $Y \mapsto \mathrm{Ad}(\overline{\delta}^{-1})(Y) - Y$ from $\mathsf{g}(\mathbb{E})$ to itself, where

$$\overline{\delta} \in G_{x,0}/G_{x,0+} \subset \overline{\mathbf{G}(E)}_{x,0} \simeq \mathsf{G}(\mathbb{E})$$

is the image of δ.

(4) If $\mathrm{d} > 0$, then $\frac{i}{m} < s - 1$. Write $X \in \mathfrak{g}_{x,\mathrm{d}}$ for the element whose exponential is δ. Then the map $g \mapsto \delta^{-1} g \delta g^{-1}$ induces a map $\overline{\mathbf{G}(E)}_{x,i/m} \to \overline{\mathbf{G}(E)}_{x,s-1}$ and thus a linear map $\mathsf{g}(\mathbb{E}) \to \mathsf{g}(\mathbb{E})$ via (2). Composed with the projection $\mathsf{g}(\mathbb{E}) \to \mathsf{g}(\mathbb{E})/\mathsf{g}'(\mathbb{E})$, this map is equal to the map

$$\overline{C}_\delta : \mathsf{g}(\mathbb{E}) \to \mathsf{g}(\mathbb{E})/\mathsf{g}'(\mathbb{E}) \simeq \mathbf{g}(E)_{x,i/m} \Big/ (\mathbf{g}'(E)_{x,s-1} + \mathbf{g}(E)_{x,s-1})$$

induced by the map $Y \mapsto [Y, X]$ from $\mathbf{g}(E)_{x,i/m}$ to $\mathbf{g}(E)_{x,s-1}$ (a proof of this assertion is similar to that of Proposition 3.8). Let $\overline{X}$ denote the image of X under the isomorphism $\overline{\mathbf{g}(E)}_{x,\mathrm{d}} \simeq \mathsf{g}(\mathbb{E})$ in (2). Then $\overline{X} \in \mathsf{g}(\mathbb{E})$. We see that $\overline{C}_\delta$ is given by $\overline{C}_\delta(\overline{Y}) = [\overline{Y}, \overline{X}]$ modulo $\mathsf{g}'(\mathbb{E})$ for each $Y \in \mathsf{g}(\mathbb{E})$.

In summary, we have a series of maps such that the following diagram commutes:

$$(\dagger)$$

Here the three horizontal isomorphisms on the right are chosen to make the diagram commute.

Consider the map $\overline{C}_\delta : \mathsf{g}(\mathbb{E}) \to \mathsf{g}(\mathbb{E})/\mathsf{g}'(\mathbb{E})$. Since $\delta \notin ZG_{x,\mathrm{d}+}$ (because $\mathfrak{d}_x^Z(\delta) = \mathrm{d}$), we see that $\overline{\delta}$ is not in the center of $\mathsf{G}(\mathbb{E})$ when $\mathrm{d} = 0$ and similarly $\overline{X}$ is not in the center of $\mathsf{g}(\mathbb{E})$ when $\mathrm{d} > 0$. (Indeed $\delta \notin \mathbf{Z}(E)\mathbf{G}(E)_{x,\mathrm{d}+}$, hence $X \notin \mathfrak{z}(E) + \mathfrak{g}(E)_{x,\mathrm{d}+}$ in view of Hypothesis $(\mathscr{E})$.(iii), therefore the image of X in $\overline{\mathbf{g}(E)}_{x,\mathrm{d}}$ is noncentral). We apply Lemma 4.5 with $\mathsf{M} = \mathsf{G}'$, which yields $\dim_\mathbb{E}\left(\mathrm{Im}(\overline{C}_\delta)\right) \geq 1$.

In both cases (either $\mathrm{d} = 0$ or $\mathrm{d} > 0$), the map $\overline{C}_\delta$ sends the rational subspace g into the rational subspace g/g'. Via identification given in the diagram $(\dagger)$, after taking Galois invariants of $\mathbb{E}$-vector spaces, we see that the dimension of $\mathrm{Im}(\overline{C}_\delta) \cap \mathsf{g}$ (mod $\mathsf{g}'(\mathbb{E})$) over $\mathbb{F}$ is at least one. Indeed for each component in the above diagram, we have $\overline{\mathbf{H}(k)}_{x,r} \hookrightarrow \overline{\mathbf{H}(E)}_{x,r}$ where $\mathbf{H} = \mathbf{G}, \mathbf{G}'$ and $r = \frac{i}{m}, s-1$, and observe also that the horizontal isomorphisms on the right can be also chosen such that $\overline{\mathbf{H}(k)}_{x,r} \hookrightarrow \overline{\mathbf{H}(E)}_{x,r}$ induces $\mathsf{h} \hookrightarrow \mathsf{h}(\mathbb{E})$ where $\mathsf{h} = \mathsf{g}, \mathsf{g}'$. Lifting this fact to C_δ, we have $\sharp\left(\mathrm{Im}(C_\delta)\right) \geq q$. $\square$

Proof of Proposition 4.4. Without loss of generality, we may assume that $\gamma \in L_s$. Let $\psi_\gamma : G \to G$ be given by $\psi_\gamma(g) := g\gamma g^{-1}$. Then, $\psi_\gamma(G) = \mathscr{O}(\gamma)$. Moreover $g \in \psi_\gamma^{-1}(L_s)$ (resp. $g \in \psi_\gamma^{-1}(L_{s-1})$) if and only if $[\gamma^{-1}, g] \in L_s$ (resp. $[\gamma^{-1}, g] \in L_{s-1}$).

We consider the disjoint decomposition

$$L_s \cap \mathscr{O}(\gamma) = \bigsqcup_{i=1}^n \delta_i G'_{x,s-1} G_{x,s} \cap \mathscr{O}(\gamma)$$

for some $\delta_1, \cdots, \delta_n \in L_s \cap \mathscr{O}(\gamma)$. Set $\mathrm{d}_i := \mathrm{d}_{x,s}(\delta_i) = \mathrm{d}_{x,s}(\delta_i^{-1})$.

Case (i). $md(\gamma) \leq s - 2$. Then $\mathrm{d}_i \leq s - 2$ by the second inequality of Lemma 4.6. Furthermore, by the first inequality of Lemma 4.6, $\delta_i, \delta_i^{-1} \in ZG_{x,\mathrm{d}_i}$.

Put $V_{i,s} := \psi_\gamma^{-1}(\delta_i G'_{x,s-1} G_{x,s})$. We have that $V_{i,s} \subset \psi_\gamma^{-1}(L_s) \subset \psi_\gamma^{-1}(L_{s-1})$. We claim that for any $u \in G_{x,s-\mathrm{d}_i-1}$,

$$uV_{i,s} \subset \psi_\gamma^{-1}(L_{s-1}).$$

Indeed for $v \in V_{i,s}$ we have

$$\widetilde{v} := \psi_\gamma(v) = v\gamma v^{-1} \in \delta_i G'_{x,s-1} G_{x,s},$$

so in particular $\widetilde{v} \in L_s \cap ZG_{x,d_i}$. Hence $[\widetilde{v}^{-1}, u] \in G_{x,s-1}$ by (4.1), and therefore

$$\psi_\gamma(uv) = uv\gamma v^{-1} u^{-1} = \widetilde{v}[\widetilde{v}^{-1}, u] \in L_{s-1},$$

verifying the claim.

Next we want to show that the sets $G_{x,s-d_i-1} V_{i,s}$ are disjoint for $1 \leq i \leq n$. Indeed suppose that there exist $u \in G_{x,s-d_i-1}$, $u' \in G_{x,s-d_{i'}-1}$, $v \in V_{i,s}$ and $v' \in V_{i',s}$ such that $uv = u'v'$. As before $\widetilde{v} := v\gamma v^{-1} \in \delta_i G'_{x,s-1} G_{x,s}$ so we may write $\widetilde{v} = \delta_i g$ for $g \in G'_{x,s-1} G_{x,s} \subset G_{x,s-1}$. Then

$$uv\gamma(uv)^{-1} = (u\delta_i u^{-1})(ugu^{-1}) = (\delta_i[\delta_i^{-1}, u])(g[g^{-1}, u]).$$

Again $[\delta_i^{-1}, u], [g^{-1}, u] \in G_{x,s-1}$. Hence

$$\psi_\gamma(uv) = uv\gamma(uv)^{-1} \in \delta_i G_{x,s-1}.$$

The same reasoning shows that $u'v'\gamma(u'v')^{-1} \in \delta_{i'} G_{x,s-1}$. Since $uv = u'v'$, it implies that $\delta_i \equiv \delta_{i'} \pmod{G_{x,s-1}}$. This is promoted to $\delta_i \equiv \delta_{i'} \pmod{G'_{x,s-1} G_{x,s}}$ thanks to the fact that $\delta_i, \delta_{i'} \in L_s$, thus $i = i'$, verifying the disjointness.

Define a map C_{δ_i} as follows:

$$C_{\delta_i} : G_{x,s-d_i-1} / G_{x,s-d_i} \to G_{x,s-1} / (G'_{x,s-1} G_{x,s})$$

given by $C_{\delta_i}(g) := [\delta_i^{-1}, g] \pmod{G'_{x,s-1} G_{x,s}}$. By Lemma 4.7, for each i there exist q elements $u_{i1}, u_{i2}, \cdots, u_{iq} \in G_{x,s-d_i-1}$ such that $C_{\delta_i}(u_{ij})$ are distinct, $j = 1, \cdots, q$.

As consequence of the above claim we have

$$\psi_\gamma^{-1}(L_{s-1}) \supset \bigcup_{i=1}^{n} \bigcup_{j=1}^{q} u_{ij} V_{i,s}.$$

To finish the proof of (i) it is enough to prove that the terms on the right hand side are mutually disjoint. Indeed, if for each open compact subset $U \subset G$ we write $\mathrm{vol}_{G/G_\gamma}(U)$ to denote the volume of the image of U in G/G_γ then we will have

$$\mathcal{O}_\gamma(\mathbf{1}_{L_{s-1}}) = \mathrm{vol}_{G/G_\gamma}(\psi_\gamma^{-1}(L_{s-1})) \geq \sum_{i=1}^{n} \sum_{j=1}^{q} \mathrm{vol}_{G/G_\gamma}(u_{ij} V_{i,s})$$

$$= q \sum_{i=1}^{n} \mathrm{vol}_{G/G_\gamma}(V_{i,s}) = q\mathcal{O}_\gamma(\mathbf{1}_{L_s}).$$

Since the sets $G_{x,s-d_i-1}V_{i,s}$ are disjoint, it only remains to show that $u_{ij}V_{i,s}$ and $u_{ij'}V_{i,s}$ are disjoint for $j \neq j'$ and $1 \leq i \leq n$. Suppose $u_{ij}V_{i,s} \cap u_{ij'}V_{i,s} \neq \emptyset$ for some i, j, j'. There are $v, v' \in V_{i,s}$ such that $u_{ij}v = u_{ij'}v'$. As before, $v\gamma v^{-1} = \delta_i g$ for some $g \in G'_{x,s-1}G_{x,s}$. Hence,

$$u_{ij}v\gamma(u_{ij}v)^{-1} = (u_{ij}\delta_i u_{ij}^{-1})(u_{ij}gu_{ij}^{-1}) = (\delta_i[\delta_i^{-1}, u_{ij}])(g[g^{-1}, u_{ij}]).$$

Since $d_i \leq s-2$, we apply (4.1) to obtain $[g^{-1}, u_{ij}] \in G_{x,2s-d_i-2} \subset G_{x,s}$. Thus the term $u_{ij}v\gamma(u_{ij}v)^{-1}$ above belongs to $\delta_i C_{\delta_i}(u_{ij})G'_{x,s-1}G_{x,s}$. Since $u_{ij}v = u_{ij'}v'$, we deduce similarly that it also belongs to $\delta_i C_{\delta_i}(u_{ij'})G'_{x,s-1}G_{x,s}$. This implies $C_{\delta_i}(u_{ij}) = C_{\delta_i}(u_{ij'})$, hence $j = j'$.

Case (ii). $md(\gamma) \geq s+1$.

Observe that $\mathscr{O}(\gamma) \cap ZG_{x,s} \neq \emptyset$. This follows from combining Corollary 4.2 and the fact that for any torus $\mathbf{T}$, there exists $g \in G$ with $ZT_{s+1} \subset ZG_{gx,s}$. Hence, one can assume $\gamma \in ZG_{x,s}$. Without loss of generality, we assume that $\gamma \in G_{x,s}$.

Write $\gamma = z\gamma_1$ with $z \in Z$ and $\gamma_1 \in T_0 \setminus Z_0$ with $d_{\mathbf{T}}(\gamma_1) = md(\gamma)$ as in Corollary 4.2. Then, since

$$\min(d_{\mathbf{T}}(z), md(\gamma_1)) = d_{\mathbf{T}}(\gamma) \geq \mathfrak{d}_x(\gamma) \geq s,$$

we have $d_{\mathbf{T}}(z) \geq s$, hence $z \in Z_s \subset G_{x,s}$.

If $d_i = d_{x,s}(\delta_i) \geq s-2$, we have $\delta_i \in ZG'_{x,s-2}G_{x,s}$. We claim that $\delta_i \in G'_{x,s-2}G_{x,s}$. To prove this claim, write $\delta_i = z^{h}\gamma_1$ for some $h \in G$. Since $z \in G_{x,s}$, it is enough to show that ${}^{h}\gamma_1 \in G'_{x,s-2}G_{x,s}$. Suppose ${}^{h}\gamma_1 \notin G'_{x,s-2}G_{x,s}$, that is, $\mathfrak{d}_x({}^{h}\gamma_1) < s-2$. Then, since $d_{x,s}({}^{h}\gamma_1) \geq s-2$, there is $z' \in Z$ of depth $\mathfrak{d}_x({}^{h}\gamma_1)$ such that $z'{}^{h}\gamma_1 \in G'_{x,s-2}G_{x,s}$. However, $d_{h\mathbf{T}}(z'{}^{h}\gamma_1) = d_{h\mathbf{T}}(z') = \mathfrak{d}_x({}^{h}\gamma_1) < s-2$, hence $z'{}^{h}\gamma_1 \notin G_{x,s-2}$, which is a contradiction. Hence, the claim follows.

Now we can arrange the decomposition such that $d_i < s-2$ for $1 \leq i \leq n'$ and $\delta_i \in G'_{x,s-2}G_{x,s}$ for $n'+1 \leq i \leq n$:

$$L_s \cap \mathscr{O}(\gamma) = \bigsqcup_{i=1}^{n'} \left(\delta_i G'_{x,s-1}G_{x,s} \cap \mathscr{O}(\gamma) \right) \bigcup \left(G'_{x,s-2}G_{x,s} \cap \mathscr{O}(\gamma) \right).$$

For $1 \leq i \leq n'$, choose u_i and define $V_{i,s}$ as in Case (i). Then, $\psi_\gamma^{-1}(L_{s-1})$ contains

$$\bigcup_{i=1}^{n'} \bigcup_{j=1}^{q} u_{ij}V_{i,s}$$

and the summands are mutually disjoint by a similar argument to Case (i). Arguing as in Case (i) but keeping in mind that $G'_{x,s-2}G_{x,s} \cap \mathscr{O}(\gamma)$ accounts for $\mathcal{O}_\gamma(\mathbf{1}_{G'_{x,s-2}G_{x,s}})$, we complete the proof of Case (ii) as follows.

$$\mathcal{O}_\gamma(\mathbf{1}_{L_{s-1}}) \geq q \sum_{i=1}^{n'} \mathrm{vol}_{G/G_\gamma}(V_{i,s}) \geq q\left(\mathcal{O}_\gamma(\mathbf{1}_{L_s}) - \mathcal{O}_\gamma(\mathbf{1}_{G'_{x,s-2}G_{x,s}})\right). \quad \square$$

In preparation for the end of the proof of Proposition 3.13 we shall need a final bound from Corollary 4.9 below.

Lemma 4.8. *Assume that the isomorphisms of Hypothesis ($\mathscr{E}$).(iii) are G-equivariant. Let $s \in \mathbb{R}_{\geq 1}$, and $\gamma \in G_{x,s}$. Write $X := \exp^{-1}(\gamma)$. Then*

$$\mathcal{O}_\gamma^G(\mathbf{1}_{G_{x,s}}) = \mathcal{O}_X^{\mathfrak{g}}(\mathbf{1}_{\mathfrak{g}_{x,s}}).$$

Proof. We are going to follow the idea as in the proof of Theorem 3.2.3 of [32]. Let G_γ (resp. G_X) denote the connected centralizer of γ (resp. X) in G. Write $K := G_{x,s}$ and $\mathfrak{k} := \mathfrak{g}_{x,s}$. By the definition of orbital integrals,

$$\mathcal{O}_\gamma^G(\mathbf{1}_K) = \sum_{g \in G_\gamma \backslash G/K} \mathrm{vol}(G_\gamma \backslash G_\gamma g K)\mathbf{1}_K(g^{-1}\gamma g),$$

$$\mathcal{O}_X^{\mathfrak{g}}(\mathbf{1}_{\mathfrak{k}}) = \sum_{g \in G_X \backslash G/K} \mathrm{vol}(G_X \backslash G_X g K)\mathbf{1}_{\mathfrak{k}}(g^{-1}Xg).$$

Since $G_\gamma = G_X$ and $\mathbf{1}_K(g^{-1}\gamma g) = \mathbf{1}_{\mathfrak{k}}(g^{-1}Xg)$, the equality in the lemma holds. $\quad \square$

Corollary 4.9. *Assume that the isomorphisms of Hypothesis ($\mathscr{E}$).(iii) are G-equivariant. There is a constant $C > 0$ depending only on G such that the following holds: for every $s \in \mathbb{R}_{\geq 0}$, and every semisimple $\gamma \in G \setminus Z$,*

$$|\mathcal{O}_\gamma(\mathbf{1}_{G_{x,s}})| \leq C \cdot q^{-s} \cdot D(\gamma)^{-1/2}.$$

Proof. In view of Proposition 4.3, we may assume without loss of generality that $s \in \mathbb{R}_{\geq 2}$. It is enough to prove the bound for $s \in \mathbb{Z}_{\geq 2}$. Indeed, if the corollary is known for $s \in \mathbb{Z}_{\geq 2}$ then the corollary holds for $s \leq t < s + 1$ at the expense of increasing the constant:

$$|\mathcal{O}_\gamma(\mathbf{1}_{G_{x,t}})| \leq |\mathcal{O}_\gamma(\mathbf{1}_{G_{x,s}})| \leq C \cdot q^{-s}D(\gamma)^{-1/2} \leq (C \cdot q)q^{-t}D(\gamma)^{-1/2}.$$

For the rest of this proof, $s \in \mathbb{Z}_{\geq 2}$ and $\gamma = \exp(X) \in G_{x,s}$. Then

$$\mathcal{O}_\gamma^G(\mathbf{1}_{G_{x,s}}) = \mathcal{O}_X^{\mathfrak{g}}(\mathbf{1}_{\mathfrak{g}_{x,s}}) = \mathcal{O}_{\varpi^{1-s}X}^{\mathfrak{g}}(\mathbf{1}_{\mathfrak{g}_{x,1}}) = \mathcal{O}_{\exp(\varpi^{1-s}X)}^G(\mathbf{1}_{G_{x,1}}).$$

Here the first and third equalities are from Lemma 4.8 and the second follows from a direct computation. Applying Proposition 4.3 to $f = \mathbf{1}_{G_{x,1}}$ we obtain

$$\mathcal{O}_{\exp(\varpi^{1-s}X)}^G(\mathbf{1}_{G_{x,1}}) \leq C \cdot D(\exp(\varpi^{1-s}X))^{-1/2}$$

$$= C \cdot q^{\frac{s-1}{2}(\dim G_\gamma - \dim G)}D(\gamma)^{-\frac{1}{2}} \leq C \cdot q^{1-s}D(\gamma)^{-\frac{1}{2}},$$

where the constant $C = c(\mathbf{1}_{G_{x,1}}) \in \mathbb{R}_{>0}$ depends only on G and x which had been fixed since the beginning of this section.

We can arrange that $c(\mathbf{1}_{G_{x,1}})$ depend only on the G-orbit of the facet containing x: If $x' = gx$ with $g \in G$ then $G_{x',1} = gG_{x,1}g^{-1}$ so $\mathbf{1}_{G_{x,1}}$ and $\mathbf{1}_{G_{x',1}}$ have the same orbital integral on each conjugacy class. There are only finitely many G-orbits of facets, so the lemma holds true for a constant $C > 0$ depending only on G. $\square$

Proof of Proposition 3.13. As in the proof of Corollary 4.9, once we prove the proposition for a fixed x, the proposition also holds when x varies by a similar finiteness argument. So we may and will keep x fixed in this proof.

Let $a_s := D(\gamma)^{\frac{1}{2}} \mathcal{O}_\gamma(\mathbf{1}_{L_s})$, which is a decreasing function of $s \in \mathbb{R}_{\geq 0}$. We want to show that $a_s \leq C_1 \cdot q^{-s}$. It is sufficient to verify the inequality for $s \in \mathbb{Z}_{\geq 0}$ at the expense of replacing C_1 by qC_1. Applying Proposition 4.3 to $f = \mathbf{1}_{L_0}$ we have that $a_1 \leq a_0 \leq c(\mathbf{1}_{L_0})$.

Set m to be the largest integer such that $m \leq md(\gamma)$. Proposition 4.4.(ii) combined with Corollary 4.9 provides us with the recursive inequality (replacing C by $q^2 C$)

$$a_s \leq \frac{1}{q} a_{s-1} + C q^{-s}, \quad 2 \leq s \leq m - 1.$$

This implies that for any $2 \leq s \leq m - 1$,

$$a_s \leq q^{1-s} a_1 + C \left(q^{-s} + \frac{q^{1-s}}{q} + \cdots + \frac{q^{-1}}{q^{s-1}} \right) \leq q^{1-s} c(\mathbf{1}_{L_0}) + C s q^{-s}.$$

We have in particular

$$a_{m+1} \leq a_m \leq a_{m-1} \leq (c(\mathbf{1}_{L_0})q + Cm) \, q^{1-m}.$$

Next, Proposition 4.4.(i) shows that for $s \geq m + 2$, we have the inequality $a_s \leq \frac{1}{q} a_{s-1}$. Hence

$$a_s \leq q^{m+1-s} a_{m+1} \leq (c(\mathbf{1}_{L_0})q + Cm) \, q^{2-s}.$$

Proposition 3.13 is verified with $C_1 := (c(\mathbf{1}_{L_0})q + C)q^2$, which is indeed a constant depending only on G. $\square$

5. Automorphic Plancherel equidistribution with error terms

In this section our asymptotic formula for supercuspidal characters and orbital integrals is applied to produce an equidistribution theorem for a family of automorphic representations. The theorem can be informally summarized as follows: Consider the set of L^2-discrete automorphic representations with supercuspidals at a fixed finite place (suitably weighted). As the formal degree of the supercuspidal at the fixed place moves toward infinity, the local components (away from the fixed place) of the automorphic

representations are equidistributed with respect to the Plancherel measure. We prove the theorem in the case where a suitable condition at infinite places simplifies the trace formula so that the technical difficulties with general terms in the trace formula do not blur the close relationship between the asymptotic formula for supercuspidal characters and the equidistribution.

5.1. Preliminaries

In the rest of the article the following global setup will be in effect. Let G be a connected reductive group over a totally real number field F. (We used the symbol G differently in the preceding sections.) Put $F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R}$. Write $\mathcal{A}_{\mathrm{disc}}(G)$ for the set of discrete automorphic representations of $G(\mathbb{A}_F)$ up to isomorphism (i.e. without multiplicity). The automorphic multiplicity of $\pi \in \mathcal{A}_{\mathrm{disc}}(G)$ is denoted $m_{\mathrm{disc}}(\pi)$. Let S be a nonempty finite set of finite places of F. Fix a Haar measure μ_S on $G(F_S)$. Recall that the unitary dual $G(F_S)^\wedge$ is equipped with a positive Borel measure $\widehat{\mu}_S^{\mathrm{pl}}$, the Plancherel measure.

Write $\mathrm{Ram}(G)$ for the set of finite places v of F such that G is ramified over F_v. For each finite place $v \notin \mathrm{Ram}(G)$ let K_v^{hs} be a hyperspecial subgroup of $G(F_v)$. We choose K_v^{hs} such that at all but finitely many $v \notin \mathrm{Ram}(G)$, the group K_v^{hs} consists of the $\mathcal{O}_{F_v}$-points of some reductive integral model of G over $\mathcal{O}_F[1/N]$ for a sufficiently large integer N.

Let v be a place of F. Write μ_v^{can} for the canonical measure on $G(F_v)$ (denoted by $L(M^\vee(1)) \cdot |\omega_G|$ in [36]), and if $G(F_v)$ has compact center, denote by μ_v^{EP} the Euler-Poincaré measure on $G(F_v)$, cf. [36, §5, §7]. Assuming $G(F_\infty)$ has compact center, put $\mu_\infty^{\mathrm{EP}} := \prod_{v|\infty} \mu_v^{\mathrm{EP}}$. Similarly $\mu_S^{\mathrm{EP}} := \prod_{v \in S} \mu_v^{\mathrm{EP}}$ and $\mu^{\mathrm{can},\Sigma} := \prod_{v \notin \Sigma} \mu_v^{\mathrm{can}}$. When G is unramified over F_v it is known that μ_v^{can} assigns volume 1 to hyperspecial subgroups. From §5.3 on we will fix a finite set of places S and consider the (possibly negative) measure

$$
\mu^{\mathrm{can},\mathrm{EP}} := \left(\prod_{v \notin S \cup S_\infty} \mu_v^{\mathrm{can}} \right) \mu_S^{\mathrm{EP}} \mu_\infty^{\mathrm{EP}}.
$$

(This is different from the convention of [70]; there we used μ_v^{can} at all finite places. Also note that $\mu^{\mathrm{can},\mathrm{EP}}$ depends on the set S.) Define the volume of the adelic quotient

$$
\tau'(G, S) := \mu^{\mathrm{can},\mathrm{EP}}(G(F) \backslash G(\mathbb{A}_F)),
$$

relative to the counting measure on the discrete subgroup $G(F)$. This volume is finite if $G(F_\infty)$ has compact center, which will always be the case by part (iii) of the assumptions

made in the next subsection.[6] The Tamagawa volume of $G(F)\backslash G(\mathbb{A}_F)$ is denoted by $\tau(G)$.

5.2. The simple trace formula

As the trace formula is going to play a central role in the proof, we recall some basic facts. Let $T_{\mathrm{ell}}, T_{\mathrm{disc}} : \mathcal{H}(G(\mathbb{A}_F)) \to \mathbb{C}$ designate the invariant distributions consisting of contributions from the elliptic conjugacy classes and the discrete automorphic spectrum, respectively. Arthur's trace formula is the equality of the two invariant distributions I_{geom} and I_{spec} on the geometric and spectral sides. In general I_{geom} (resp. I_{spec}) is the sum of T_{ell} (resp. T_{disc}) and other very complicated terms, but we will always be in the situation where the simple trace formula applies, i.e. $T_{\mathrm{ell}} = I_{\mathrm{geom}} = I_{\mathrm{spec}} = T_{\mathrm{disc}}$.

Suppose that $\phi \in \mathcal{H}(G(\mathbb{A}_F))$ admits a decomposition $\phi = \phi^{S \cup S_\infty} \phi_S \phi_\infty$ according to $G(\mathbb{A}_F) = G(\mathbb{A}_F^{S \cup S_\infty}) G(F_S) G(F_\infty)$ such that ϕ_∞ is an Euler-Poincaré function on $G(F_\infty)$ as in [20, Thm 3.(ii)] up to a nonzero scalar. For the rest of the paper we make the following overarching assumptions.

(i) $G(F_S)$ has compact center,
(ii) the function ϕ_S is cuspidal in the sense that orbital integrals vanish on non-elliptic regular semisimple conjugacy classes of $G(F_S)$, and
(iii) $G(F_\infty)$ contains a compact maximal torus.

By (iii), the real group $G(F_\infty)$ admits discrete series spectrum and the function ϕ_∞ is nonzero. Condition (ii) is equivalent to the condition that the trace of any (fully) induced representation from any proper parabolic subgroups vanishes against ϕ_S. Typical examples of such ϕ_S are matrix coefficients of supercuspidal representations (§3) and Kottwitz's Euler-Poincaré functions, cf. §6 below.

For a semisimple $\gamma \in G(F)$ write G_γ for its centralizer and I_γ for the neutral component of G_γ. Put $\iota(\gamma) := [G_\gamma(F) : I_\gamma(F)] \in \mathbb{Z}_{\geq 1}$. Let $\mu_{G(\mathbb{A}_F)}$ (resp. $\mu_{I_\gamma(\mathbb{A}_F)}$) denote a Haar measure on $G(\mathbb{A}_F)$ and $I_\gamma(\mathbb{A}_F)$, respectively. The elliptic part of the trace formula is the expansion

$$T_{\mathrm{ell}}(\phi, \mu_{G(\mathbb{A}_F)}) := \sum_{\substack{\gamma \in G(F)/\sim \\ \mathrm{elliptic}}} \iota(\gamma)^{-1} \mu_{I_\gamma}(I_\gamma) \mathcal{O}_\gamma(\phi, \mu_{G(\mathbb{A}_F)}/\mu_{I_\gamma(\mathbb{A}_F)}), \qquad (5.1)$$

where the sum runs over the set of F-elliptic conjugacy classes in $G(F)$, and $\mu_{I_\gamma}(I_\gamma)$ is the volume of $I_\gamma(F)\backslash I_\gamma(\mathbb{A}_F)$ for the quotient measure of $\mu_{I_\gamma(\mathbb{A}_F)}$. (Note that $I_\gamma(F_\infty)$ has compact center by ellipticity of γ and assumption (iii) above.) The discrete part of the trace formula is

---

[6] To have finite volume in general, one has to take a further quotient of $G(F)\backslash G(\mathbb{A}_F)$ by the $\mathbb{R}$-split part of the center of $G(F_\infty)$.

$$T_{\mathrm{disc}}(\phi, \mu_{G(\mathbb{A}_F)}) := \sum_{\pi \in \mathcal{A}_{\mathrm{disc}}(G)} m_{\mathrm{disc}}(\pi) \operatorname{tr} \pi(\phi, \mu_{G(\mathbb{A}_F)}), \qquad (5.2)$$

where $m_{\mathrm{disc}}(\pi)$ denotes the multiplicity of π in the discrete automorphic spectrum. Under the above hypotheses Arthur [8, Cor 7.3, Cor 7.4] provides us with the simple trace formula

$$T_{\mathrm{ell}}(\phi, \mu_{G(\mathbb{A}_F)}) = T_{\mathrm{disc}}(\phi, \mu_{G(\mathbb{A}_F)}). \qquad (5.3)$$

Indeed the assumption at v_1 (resp. at v_1 and v_2) in Corollary 7.3 (resp. 7.4) of that paper is satisfied by any $v_1 \in S$ (resp. any $v_1 \in S$ and any $v_2 \in S_\infty$) by (ii) and (iii) above. Here we use the property of Euler-Poincaré functions [9, p.270, p.281] that their orbital integrals vanish outside elliptic conjugacy classes.

5.3. Counting measures for automorphic representations

Let G be a connected reductive group over a totally real field F as in the preceding subsection. Let S_0, S, $\mathfrak{S}$, ξ, $\Pi_\infty(\xi)$, K_{S_0}, and $K^{\mathfrak{S}}$ be as in the introduction. (We allow S_0 to be empty.) Throughout this section G is assumed to be unramified away from the finite set of places $\mathfrak{S}$. This is always ensured by increasing the set S_0 if necessary. We make the following additional hypothesis, which is technically helpful as it was in [70].

- The highest weight of ξ is regular.

Write $\mathrm{Irr}^{\mathrm{Yu}}(G(F_S))$ for the set of $\sigma_S = \otimes_{v \in S} \sigma_v$ such that $\sigma_v \in \mathrm{Irr}^{\mathrm{Yu}}(G(F_v))$. Given $\sigma_S \in \mathrm{Irr}^{\mathrm{Yu}}(G(F_S))$, define

$$\mathcal{F} = \mathcal{F}(\xi, \sigma_S, K_{S_0})$$

to be the multi-set of $\pi \in \mathcal{A}_{\mathrm{disc}}(G)$ whose multiplicity is zero unless $\pi^{\mathfrak{S}}$ is unramified, $\pi_S \simeq \sigma_S$, and $\pi_\infty \in \Pi_\infty(\xi)$, in which case the multiplicity of π is

$$a_{\mathcal{F}}(\pi) := m_{\mathrm{disc}}(\pi) \dim(\pi_{S_0})^{K_{S_0}}.$$

By Harish-Chandra's finiteness theorem, $a_{\mathcal{F}}(\pi) \neq 0$ only for finitely many π. We may replace $m_{\mathrm{disc}}(\pi)$ by the multiplicity in the cuspidal spectrum since every automorphic representation with a supercuspidal component (or with π_∞ in discrete series) is cuspidal. For each $\sigma_\infty \in \Pi_\infty(\xi)$ let f_{σ_∞} be a pseudo-coefficient for σ_∞. Set

$$f_\xi := \sum_{\sigma_\infty \in \Pi_\infty(\xi)} f_{\sigma_\infty}.$$

Then $\operatorname{tr} \pi_\infty(f_\xi) \neq 0$ if and only if $\pi_\infty \in \Pi_\infty(\xi)$, in which case the trace equals 1. (The only if part follows from the results of Vogan–Zuckerman on Lie algebra cohomology.)

Moreover f_ξ is an Euler-Poincaré function up to a nonzero constant, cf. [50, Lem 3.2]. Write $f_{\sigma_S} \in C_c^\infty(G(F_S))$ for the product of explicit supercuspidal coefficients f_{σ_v} (see §3.2).

Lemma 5.1. *Let* $\phi^{\mathfrak{S}} \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{A}_F^{\mathfrak{S}}))$. *Put* $\phi := \phi^{\mathfrak{S}} f_{\sigma_S} \mathbf{1}_{K_{S_0}} f_\xi$. *Then*

$$\sum_{\pi \in \mathcal{F}(\xi, \sigma_S, K_{S_0})} \mathrm{tr}\, \pi^{\mathfrak{S}}\left(\phi^{\mathfrak{S}}, \mu^{\mathrm{can},\mathfrak{S}}\right) = \mu_{S_0}^{\mathrm{can}}(K_{S_0})^{-1} T_{\mathrm{ell}}(\phi, \mu^{\mathrm{can,EP}}).$$

Proof. It follows from the definition that the left hand side equals

$$\sum_{\pi \in \mathcal{A}_{\mathrm{disc}}(G)} m_{\mathrm{disc}}(\pi) \mathrm{tr}\, \pi^{\mathfrak{S}}(\phi^{\mathfrak{S}}) \mathrm{tr}\, \pi_S(f_{\sigma_S}) \frac{\mathrm{tr}\, \pi_{S_0}(\mathbf{1}_{K_{S_0}})}{\mu_{S_0}^{\mathrm{can}}(K_{S_0})} \mathrm{tr}\, \pi_\infty(f_\xi),$$

which is none other than $\mu_{S_0}^{\mathrm{can}}(K_{S_0})^{-1} T_{\mathrm{disc}}(\phi)$. We conclude by (5.3). $\square$

5.4. Bounds on the geometric side

Here we recollect various bounds on the terms appearing on the geometric side, mostly from [70]. Given each semisimple element $\gamma \in G(F)$, fix a maximal torus T_γ in G over $\overline{F}$ containing γ and write Φ_γ for the set of roots of T_γ in G outside I_γ, namely the set of roots α in G such that $\alpha(\gamma) \neq 1$. Thus Φ_γ is nonempty if and only if $\gamma \notin Z(F)$. Define S_γ for the following set of finite places of F:

$$S_\gamma := \{v \,:\, |1 - \alpha(\gamma)| \neq 1 \text{ for some } \alpha \in \Phi_\gamma\}.$$

Evidently S_γ is independent of the choice of T_γ. In the same way we defined $\mathrm{Ram}(G)$, we have the set $\mathrm{Ram}(I_\gamma)$. For each $v \notin \mathrm{Ram}(G)$, we have a maximal split torus A_v in $G \otimes_F F_v$ such that K_v^{hs} is in a good relative position to A_v.

Let $\Sigma \supset \mathrm{Ram}(G) \cup S_\infty$ be a finite set of places of F. Choose $\underline{\kappa} = (\kappa_v)_{v \notin \Sigma}$ with $\kappa_v \in \mathbb{Z}_{\geq 0}$ such that $\kappa_v \neq 0$ for only finitely many v. Define

$$Q := \prod_v q_v^{\min(1, \kappa_v)}, \qquad Q^{\underline{\kappa}} := \prod_v q_v^{\kappa_v},$$

where v runs over places of F outside Σ. For simplicity we will write $Q^{a+b\underline{\kappa}}$ to mean $Q^a(Q^{\underline{\kappa}})^b$. Put $U_v^{\leq \kappa_v} := \cup_{\|\lambda\| \leq \kappa_v} K_v^{\mathrm{hs}} \lambda(\varpi_v) K_v^{\mathrm{hs}}$, where $v \notin \Sigma$ and $\lambda \in X_*(A_v)$. Here $\|\cdot\|$ is an Ω_G-invariant norm on $X_*(A_v)$, which depends on the choice of an $\mathbb{R}$-basis of $X_*(A_v)_{\mathbb{R}}$.

Write $d(G_\infty)$ for the cardinality of $\Pi_\infty(\xi)$, which is independent of ξ. Let $q(G_\infty)$ denote the real dimension of $G(F_\infty)$ modulo (any) maximal compact subgroup. Given a constant $C \geq 1$ and $\star \in \{\mathrm{alg}, \mathrm{reg}\}$, we define a set

$$\mathrm{Irr}_C^\star(G(F_\infty)) := \{\xi \in \mathrm{Irr}^\star(G(F_\infty)) \,:\, \max(\xi)/\min(\xi) \leq C\},$$

where $\max(\xi)$ and $\min(\xi)$ are given as follows. Let T be a maximal torus in G over $\mathbb{C}$ and choose a Borel subgroup B containing T. Let $\lambda_\xi \in X^*(T)$ denote the B-dominant weight of ξ. Write Φ^+ for the set of B-positive roots of T in G, and ρ for the half sum of roots in Φ^+. Then $\max(\xi)$ (resp. $\min(\xi)$) is the maximum (resp. minimum) value of the natural pairing $\langle \alpha, \lambda_\xi + \rho \rangle$ as α runs over Φ^+. The value is independent of the choice of T and B. (We introduce $\mathrm{Irr}_C^\star(G(F_\infty))$ so that we can vary ξ in a controlled manner in that set.)

Proposition 5.2.

(i) *There exists $A_1 > 0$ depending only on G such that the following holds for any Σ and $\underline{\kappa} = (\kappa_v)$ as above: Choose U_Σ to be a compact subset of $G(F_\Sigma)$. Write $\mathcal{Y}(\underline{\kappa})$ for the set of $G(\mathbb{A}_F)$-conjugacy classes of $\gamma \in G(F)_{\mathrm{ss}}$ which meet $\prod_{v \notin \Sigma} U_v^{\leq \kappa_v} \times U_\Sigma$. Then $|\mathcal{Y}(\underline{\kappa})| = O(Q^{A_1 \underline{\kappa}})$.*

(ii) *Let $\xi \in \mathrm{Irr}^{\mathrm{alg}}(G(F_\infty))$. If $\gamma \notin G(F_\infty)_{\mathrm{ell}}$ then $\mathcal{O}_\gamma^{G(F_\infty)}(f_\xi) = 0$. Moreover,*

$$\mathcal{O}_z^{G(F_\infty)}(f_\xi, \mu_\infty^{\mathrm{EP}}) = (-1)^{q(G_\infty)} \omega_\xi(z) d(G_\infty) \dim \xi, \quad \text{if } z \in Z(F_\infty).$$

For every $C \geq 1$, $\xi \in \mathrm{Irr}_C^{\mathrm{alg}}(G(F_\infty))$ and $\gamma \in G(F_\infty)_{\mathrm{ss}}$ with $\gamma \notin Z(F_\infty)$, we have

$$D_\infty(\gamma)^{1/2} |\mathcal{O}_\gamma^{G(F_\infty)}(f_\xi, \mu_\infty^{\mathrm{EP}})| = O_C(\dim(\xi)^{1-\nu_\infty}),$$

where $\nu_\infty \in \mathbb{R}_{>0}$ depends only on $G(F_\infty)$ and the implicit constant is independent of γ and ξ. One can choose ν_∞ to be the minimum of $1 - (\dim_{\mathbb{R}} I_\gamma - \mathrm{rk}_{\mathbb{R}} I_\gamma)/(\dim_{\mathbb{R}} G_\infty - \mathrm{rk}_{\mathbb{R}} G_\infty)$ as γ runs over noncentral elements in $G(F_\infty)_{\mathrm{ss}}$.

(iii) *Let $v \notin S_\gamma \cup \mathrm{Ram}(G)$. Suppose that $\gamma \in G(F_v)_{\mathrm{ss}}$ is conjugate to an element of K_v^{hs}. Then I_γ is unramified over F_v, and $\mathcal{O}_\gamma(\mathbf{1}_{K_v^{\mathrm{hs}}}, \mu_v^{\mathrm{can}}) = 1$.*

(iv) *There exists a lower bound $p_0 > 0$ and $A_3, B_3 > 0$ depending only on G such that for every finite place v whose residue characteristic is greater than p_0, for every $\gamma \in G(F_v)_{\mathrm{ss}}$, and for every $\lambda \in X_*(A_v)$ with $\|\lambda\| \leq \kappa_v$,*

$$D(\gamma)^{1/2} \mathcal{O}_\gamma(\mathbf{1}_{K_v^{\mathrm{hs}} \lambda(\varpi_v) K_v^{\mathrm{hs}}}, \mu_v^{\mathrm{can}}) \leq q_v^{A_3 + B_3 \kappa_v}.$$

Proof. Part (i) follows from [70, Prop 8.7]. (Take S_0 and S_1 there to be our $\Sigma \backslash S_\infty$ and $\{v \notin \Sigma : \kappa_v \neq 0\}$, respectively. The proposition there assumes that the nonzero values of κ_v are all equal, but the same proof works when κ_v are different. Finally observe that A_3 can be absorbed into B_3 in that proposition.)

Let us prove (ii). It is a standard fact ([50, p.659]) for a discrete series representation π_∞ that $\mathcal{O}_\gamma(f_{\pi_\infty})$ vanishes unless γ is elliptic semisimple, in which case

$$\mathcal{O}_\gamma(f_{\pi_\infty}) = (-1)^{q(G_\infty)} \mathrm{tr}\, \xi(\gamma)$$

if the Euler-Poincaré measures are used on $G(F_\infty)$ and $I_\gamma(F_\infty)$. This implies everything but the last bound in (ii) as f_ξ is the sum of f_{σ_∞} over $\sigma_\infty \in \Pi_\infty(\xi)$. It remains to bound

$D_\infty^G(\gamma)^{1/2}|\operatorname{tr}\xi(\gamma)| = O(\dim(\xi)^{1-\nu_\infty})$ with ν_∞ described as in the proposition. We get the bound from [70, Lem 6.10.(ii)], observing that $m(\xi)$ there is equal to $\min(\xi)$ and that $\dim(\xi)/\min(\xi)^{|\Phi^+|}$ is bounded both above and below (in terms of C).

We get (iii) from [48, Prop 7.1, Cor 7.3]. Finally (iv) is proved by motivic integration in [70, Thm 14.1]. $\square$

Write $\operatorname{Mot}_{I_\gamma}$ for the Artin-Tate motive associated to I_γ by Gross [36]. We have the decomposition $\operatorname{Mot}_{I_\gamma} = \oplus_{d\in\mathbb{Z}_{\geq 1}}\operatorname{Mot}_{I_\gamma,d}(1-d)$, where $\operatorname{Mot}_{I_\gamma,d}$ is an Artin motive, and $(1-d)$ denotes the Tate twist. For any Artin-Tate motive M over F, denote by $L(\operatorname{Mot}_{I_\gamma})$ (resp. $L_v(\operatorname{Mot}_{I_\gamma})$) the global (resp. local) L-function evaluated at $s=0$. Write Ω_{I_γ} for the absolute Weyl group of I_γ. There is a certain local cohomological invariant $c_v(I_\gamma) \in \mathbb{Q}_{>0}$ defined in [36, (8.1)]. We do not need the definition but only the property that

$$c_v(I_\gamma) = |H^1(F_v, I_\gamma)| \geq 1 \text{ if } v \nmid \infty, \quad c_v(I_\gamma) \geq |\Omega_{I_\gamma}|^{-1}, \text{ if } v|\infty. \tag{5.4}$$

Let $\gamma \in G(F)_{\mathrm{ell}}$ so that I_γ has F-anisotropic center. (This ensures that $\mu_{I_\gamma}^{\mathrm{can,EP}}(I_\gamma)$ is finite, cf. [36, Prop 9.4].) We have the identity [36, Thm 9.9]

$$|\mu_{I_\gamma}^{\mathrm{can,EP}}(I_\gamma)| = \frac{|L(\operatorname{Mot}_{I_\gamma})|}{\prod_{v\in S}|L_v(\operatorname{Mot}_{I_\gamma})|}\frac{\tau(I_\gamma)\cdot|\Omega_{I_\gamma}|}{\prod_{v\in S\cup S_\infty}c_v(I_\gamma)}. \tag{5.5}$$

Lemma 5.3. *Let γ vary over the set $G(F)_{\mathrm{ell}}$ and retain the above notation. Let S_{iso} denote the set of finite places v of F such that Z contains a nontrivial F_v-split torus. (So $S \cap S_{\mathrm{iso}} = \emptyset$.)*

(i) There exist constants $0 < c_1 < c_2$ such that for all $\gamma \in G(F)_{\mathrm{ss}}$,

$$|L_v(\operatorname{Mot}_{I_\gamma})|^{-1} \leq c_2 q_v^{\frac{1}{2}(\dim I_\gamma - \mathrm{rk}I_\gamma)} \quad \forall v \notin S_{\mathrm{iso}} \cup S_\infty,$$

$$|L_v(\operatorname{Mot}_{I_\gamma})|^{-1} \geq c_1 q_v^{\frac{1}{2}(\dim I_\gamma - \mathrm{rk}I_\gamma)} \quad \forall v \notin S_{\mathrm{iso}} \cup S_\infty \text{ such that } I_\gamma \text{ is unramified at } v.$$

(ii) There exist constants $c_0, c_3, A_2 > 0$ depending only on G such that the following holds: for all S such that G is unramified at all places in S, we have

$$\frac{|\mu_{I_\gamma}^{\mathrm{can,EP}}(I_\gamma)|}{|\mu_G^{\mathrm{can,EP}}(G)|} \leq c_0 c_3^{|S|} \cdot q_S^{\frac{1}{2}(\dim I_\gamma - \mathrm{rk}I_\gamma) - \frac{1}{2}(\dim G - \mathrm{rk}G)} \prod_{v\in\mathrm{Ram}(I_\gamma)} q_v^{A_2}, \quad \forall\gamma \in G(F)_{\mathrm{ss}}.$$

Proof. (i) Observe that

$$|L_v(\operatorname{Mot}_{I_\gamma})|^{-1} = \prod_{d\geq 1}|L_v(\operatorname{Mot}_{I_\gamma,d}(1-d))|^{-1} = \prod_{d\geq 1}\prod_{i\in I_d}|1-a_{d,i}|,$$

where I_d is a finite index set, and $a_{d,i} \in \mathbb{C}$ has absolute value q^{d-1} for all i. (Since $v \notin S_{\mathrm{iso}}$ we always have $a_{d,i} \neq 1$.) For each d and i we have the obvious bounds $q^{d-1} - 1 \leq$

$|1 - a_{d,i}| \leq q^{d-1} + 1$. Part (i) is now easily deduced from the following facts [36, §1], cf. [70, Prop 6.3]:

- $|I_d| \leq \dim \mathrm{Mot}_{I_\gamma, d}$ with equality when I_γ is unramified at v,
- $\sum_{d \geq 1} |I_d| \leq \dim \mathrm{Mot}_{I_\gamma} = \mathrm{rk} I_\gamma$,
- $\sum_{d \geq 1} (d - 1) \dim \mathrm{Mot}_{I_\gamma, d} = \frac{1}{2}(\dim I_\gamma - \mathrm{rk} I_\gamma)$.

(ii) From (5.5) (applied one more time with $\gamma = 1$) we have the bound

$$\frac{|\mu_{I_\gamma}^{\mathrm{can,EP}}(I_\gamma)|}{|\mu_G^{\mathrm{can,EP}}(G)|} = O\left(|L(\mathrm{Mot}_{I_\gamma})| \prod_{v \in S} \frac{c_v(G)|L_v(\mathrm{Mot}_G)|}{|L_v(\mathrm{Mot}_{I_\gamma})|}\right).$$

Using (i) we bound

$$|L_v(\mathrm{Mot}_G)|/|L_v(\mathrm{Mot}_{I_\gamma})| = O\left(c_2 q_v^{\frac{1}{2}(\dim I_\gamma - \mathrm{rk} I_\gamma) - \frac{1}{2}(\dim G - \mathrm{rk} G)}\right).$$

Note that the implicit constants in both $O(\cdot)$ depend only on G, not on γ or S. (This is obvious in the latter. For the former, it is enough to observe that $|\Omega_{I_\gamma}|$ and $\tau(I_\gamma)$ are uniformly bounded as γ varies. The Weyl group is bounded by $|\Omega_{I_\gamma}| \leq |\Omega_G|$. The Tamagawa measure formula [49] tells us that $\tau(I_\gamma) \leq |Z(\widehat{I_\gamma})^{\mathrm{Gal}(\overline{F}/F)}|$, and the latter is uniformly bounded by [70, Cor 8.12].) From (5.4) and [48, Thm 1.2], we have

$$c_v(G) = |H^1(F_v, G)| = |\pi_0(Z(\widehat{G})^{\mathrm{Gal}(\overline{F}_v/F_v)})|, \qquad v \nmid \infty.$$

Since $\mathrm{Gal}(\overline{F}/F)$ acts on $Z(\widehat{G})$ through a finite quotient, say $\mathrm{Gal}(F'/F)$, if we take $c_2' \in \mathbb{Z}_{\geq 1}$ be the maximum of $|\pi_0(Z(\widehat{G})^H)|$ as H runs over all subgroups of $\mathrm{Gal}(F'/F)$, then clearly $c_v(G) \leq c_2'$. Note that c_2' is independent of v. Finally we have

$$|L(\mathrm{Mot}_{I_\gamma})| = O\left(\prod_{v \in \mathrm{Ram}(I_\gamma)} q_v^{A_2}\right)$$

for a uniform constant $A_2 > 0$, with an implicit constant of $O(\cdot)$ which is uniform for all γ, by [70, Cor 6.16]. Now the bound of (ii) follows by putting $c_3 := c_2 c_2'$. $\square$

5.5. Equidistribution results

Fix S, S_0, and K_{S_0}. We keep the notation from the previous subsection with $\mathfrak{S} = S_\infty \cup S_0 \cup S$. Throughout this subsection we suppose that

- the residue characteristic of every $v \in S$ is sufficiently large such that Theorem 3.11 applies to $G(F_v)$ at each $v \in S$,

- $S \cup S_0$ contains the places of F with small residue characteristics such that the result by Cluckers–Gordon–Halupczok on uniform bound on orbital integrals [70, Thm 14.1] applies to places outside $S \cup S_0$.

Note that the lower bound on the residue characteristic for Theorem 3.11 can be made effective, whereas the lower bound for [70, Thm 14.1] to hold is ineffective by the nature of its proof.

Our interest lies in statistics of the family $\mathcal{F} = \mathcal{F}(\xi, \sigma_S, K_{S_0})$ as ξ and σ_S vary. Write $K^{\mathfrak{S}} := \prod_{v \notin \mathfrak{S}} K_v^{\mathrm{hs}}$. Since we are assuming $Z(F_S)$ and $Z(F_\infty)$ are compact, the intersection $Z(F) \cap K_{S_0} K^{\mathfrak{S}}$ (taken in $G(\mathbb{A}_F)$) is finite. (The same is true with any compact $K^{\mathfrak{S}}$-bi-invariant subset of $G(\mathbb{A}_F^{\mathfrak{S}})$ in place of $K^{\mathfrak{S}}$.) As in [70, §2.3], define the truncated Hecke algebra $\mathcal{H}^{\mathrm{ur}, \leq \underline{\kappa}}(G(\mathbb{A}_F^{\mathfrak{S}}))$ as the space of locally constant bi-$K^{\mathfrak{S}}$-invariant functions on $G(\mathbb{A}_F^{\mathfrak{S}})$ whose support is inside the compact subset $\prod_{v \notin \mathfrak{S}} U_v^{\leq \kappa_v}$.

Theorem 5.4. *There exist constants $\nu_S, \nu_\infty > 0$ and $A > 0$ such that for every $\xi \in \mathrm{Irr}_C^{\mathrm{reg}}(G(F_\infty))$, for every $\sigma_S \in \mathrm{Irr}^{\mathrm{Yu}}(G(F_S))$, for every $\underline{\kappa} = (\kappa_v)_{v \notin \mathfrak{S}}$, and for every $\phi^{\mathfrak{S}} \in \mathcal{H}^{\mathrm{ur}, \leq \underline{\kappa}}(G(\mathbb{A}_F^{\mathfrak{S}}))$ which is the characteristic function of a bi-$K^{\mathfrak{S}}$-invariant compact subset,*

$$
\sum_{\pi \in \mathcal{F}} \mathrm{tr}\, \pi^{\mathfrak{S}}(\phi^{\mathfrak{S}}) = (-1)^{q(G_\infty)} d(G_\infty) \dim(\xi) \deg(\sigma_S) \frac{\tau'(G, S)}{\mu_{S_0}^{\mathrm{can}}(K_{S_0})} \sum_{z \in Z(F) \cap K_{S_0}} \frac{\omega_\xi(z)}{\omega_{\sigma_S}(z)} \phi^{\mathfrak{S}}(z)
$$

$$
+ O(\dim(\xi)^{1-\nu_\infty} \deg(\sigma_S)^{1-\nu_S} Q^{A\underline{\kappa}}). \tag{5.6}
$$

The implicit constant in $O(\cdot)$ depends on G, S, S_0, K_{S_0}, and C (but is independent of ξ, σ_S, $\underline{\kappa}$, and $\phi^{\mathfrak{S}}$).

Remark 5.5. The proof shows that ν_∞ can be chosen to be as in Proposition 5.2. We have restricted to the set $\xi \in \mathrm{Irr}_C^{\mathrm{reg}}(G(F_\infty))$ to underline the analogy between the finite places S and the infinite places S_∞. Without the restriction the error bound could be stated in terms of $\min(\xi)$ as in [70, Thm 9.19]. The same remark applies to Theorem 6.4 below.

Remark 5.6. If we fix $\underline{\kappa}$ and $\phi^{\mathfrak{S}}$ (while allowing ξ and σ_S to vary) then the same proof shows the asymptotic formula with error bound $O(\dim(\xi)^{1-\nu_\infty} \deg(\sigma_S)^{1-\nu_S})$. This holds under a weaker assumption on S, namely that the residue characteristic of each $v \in S$ has to be large enough so that only Theorem 3.11, but not [70, Thm 14.1], applies. Hence the lower bound for the residue characteristic can be explicitly given. Thus an explicit lower bound is possible for Corollaries 5.8, 5.11, and 6.5.

Proof. By Lemma 5.1 the left hand side is equal to

$$\mu_{S_0}^{\mathrm{can}}(K_{S_0})^{-1} \sum_{\substack{\gamma \in G(F)/\sim \\ \mathrm{elliptic}}} \iota(\gamma)^{-1} \mu_{I_\gamma}^{\mathrm{can,EP}}(I_\gamma) \mathcal{O}_\gamma^{G(\mathbb{A}_F)}(\phi^{\mathfrak{S}} f_{\sigma_S} \mathbf{1}_{K_{S_0}} f_\xi).$$

For each $v \in S$, write U_v for the (finite) union of a set of representatives for $G(F_v)$-conjugacy classes of elliptic maximal tori in $G(F_v)$. Take U_v to be an elliptic maximal torus in $G(F_v)$ for infinite places $v \in S_\infty$. Thus U_v is compact in both cases. Clearly the summand in the preceding formula vanishes unless $\gamma \in \mathcal{Y}(\underline{\kappa})$, where $\mathcal{Y}(\underline{\kappa})$ is as in Proposition 5.2 taking $\Sigma = \mathfrak{S}$ and $U_\Sigma = K_{S_0} \times \prod_{v \in S \cup S_\infty} U_v$.

The contribution from central elements $z \in Z(F)$ is computed as in the first line on the right hand side of (5.6). For this it is enough to observe that $\mu_G^{\mathrm{can,EP}}(G) = \tau'(G, S)$ by definition, $f_{\sigma_S}(z) = \omega_{\sigma_S}^{-1}(z) \deg(\sigma_S)$ by Harish-Chandra's Plancherel theorem, and $f_\xi(z) = (-1)^{q(G_\infty)} \omega_\xi(z) \dim \xi$ by Proposition 5.2. This implies that the contribution of $\gamma \in Z(F)$ in the above sum equals the main term in the right hand side of (5.6). Hence it suffices to show that for some uniform constants $\nu_S, \nu_\infty, A > 0$,

$$\sum_{\substack{\gamma \in \mathcal{Y}(\underline{\kappa}) \\ \mathrm{s.t.} \ \gamma \notin Z(F)}} \left| \mu_{I_\gamma}^{\mathrm{can,EP}}(I_\gamma) \mathcal{O}_\gamma^{G(\mathbb{A}_F)}(\phi^{\mathfrak{S}} f_{\sigma_S} \mathbf{1}_{K_{S_0}} f_\xi) \right| = O(\dim \xi^{1-\nu_\infty} \deg(\sigma_S)^{1-\nu_S} Q^{A\underline{\kappa}}).$$

(5.7)

We will bound the summand for each $\gamma \in \mathcal{Y}(\underline{\kappa})$. Without loss of generality we assume that γ belongs to $\prod_{v \notin \mathfrak{S}} U_v^{\leq \kappa_v} \times K_{S_0} \times \prod_{v \in S \cup S_\infty} U_v$. Define Φ_γ and S_γ as in the last subsection (with $\mathfrak{S} = S_0 \cup S \cup S_\infty$). Write S'_γ for the set of $v \in S_\gamma$ with $v \notin \Sigma$. The subset of $v \in S'_\gamma$ with $\kappa_v = 0$ is written by $S'_{\gamma,0}$.

According to Lemma 5.3 (iii), the group I_γ is unramified at v if $v \notin \mathrm{Ram}(G)$, $v \notin S_\gamma$, and $\kappa_v = 0$. (The last condition ensures that the v-component of $\phi^{\mathfrak{S}}$ is supported on K_v^{hs} so that γ is conjugate to K_v^{hs}.) Hence

$$\mathrm{Ram}(I_\gamma) \subset \mathrm{Ram}(G) \cup S_\gamma \cup \{v \notin \Sigma : \kappa_v \neq 0\} \subset S \cup S_0 \cup S'_\gamma \cup \{v \notin \Sigma : \kappa_v \neq 0\}.$$

Now Lemma 5.3, Proposition 5.2, Theorem 3.11 and Proposition 4.3 tell us that there exist positive constants $c_0, c_S, c_{S_0}, c_\infty, A_2, A_3, B_3, \nu_S, \nu_\infty$ such that

$$|\mu_{I_\gamma}^{\mathrm{can,EP}}(I_\gamma)| \leq c_0 c_3^{|S|} \prod_{v \in \mathrm{Ram}(G) \cup S_\gamma \cup \{v \notin \Sigma : \kappa_v \neq 0\}} q_v^{A_2} \leq c_0 q_S^{A_2} q_{S_0}^{A_2} q_{S'_\gamma}^{A_2} Q^{A_2},$$

(5.8)

$$D^{\mathfrak{S}}(\gamma)^{1/2} |\mathcal{O}_\gamma(\phi^{\mathfrak{S}})| \leq q_{S'_{\gamma,0}}^{A_3} Q^{A_3 + B_3 \underline{\kappa}},$$

$$D_S(\gamma)^{1/2} |\mathcal{O}_\gamma(f_{\sigma_S})| \leq c_S \deg(\sigma_S)^{1-\nu_S},$$

(5.9)

$$D_{S_0}(\gamma)^{1/2} |\mathcal{O}_\gamma(\mathbf{1}_{K_{S_0}})| \leq c_{S_0},$$

$$D_\infty(\gamma)^{1/2} |\mathcal{O}_\gamma(f_\xi)| \leq c_\infty \dim(\xi)^{1-\nu_\infty}.$$

At this point, the main remaining task is to bound $q_{S'_\gamma}$ and $q_{S'_{\gamma,0}}$ independently of γ.

We introduce some invariants of the group G over F. Write d_G for the dimension of G, w_G for the order of the absolute Weyl group, and s_G for the minimal degree over F of an extension field over which G splits.

Define δ_∞ to be the supremum of $\prod_{v \in S_\infty} |1 - \alpha(\gamma_\infty)|_v$ as γ_∞ runs over $\prod_{v \in S_\infty} U_v$, and as α runs over the set of absolute roots for the compact maximal torus $\prod_{v \in S_\infty} U_v$ in $G(F_\infty)$. (It makes no difference if we impose $\alpha(\gamma_\infty) \neq 1$.) Then $\delta_\infty < \infty$ as the supremum of a continuous function on a compact set is finite. Likewise define

$$\delta_v := \sup_{\gamma_v, \alpha} |1 - \alpha(\gamma_v)|_v, \qquad v \in S \cup S_0, \tag{5.10}$$

as γ_v runs over the v-component of K_{S_0} if $v \in S_0$, and as γ_v runs over elliptic semisimple elements of $G(F_v)$ if $v \in S$, and α runs over the set of absolute roots of a maximal torus containing γ_v in G. By definition δ_v with $v \in S \cup S_0$ depends only on G, S, S_0, and K_{S_0}. We see that $\delta_v < \infty$ for $v \in S \cup S_0$ by continuity and compactness as for δ_∞. (For instance when $v \in S_0$, there are only finitely many $G(F_v)$-conjugacy classes of elliptic maximal tori, which are compact as $G(F_v)$ has compact center. For each compact maximal torus, the value $|1 - \alpha(\gamma)|_v$ is bounded as α varies over the set of absolute roots and γ on the maximal torus.) For our purpose below, we may and will arrange that $\delta_v, \delta_\infty \geq 1$ by redefining each of them to be 1 if smaller than 1.

Noting that $|1 - \alpha(\gamma)|_v = 1$ at $v \notin S_\gamma \cup S_\infty$ with $\alpha \in \Phi_\gamma$, we deduce from the product formula that

$$1 = \prod_v |1 - \alpha(\gamma)|_v \leq \delta_\infty \prod_{v \in S_\gamma} |1 - \alpha(\gamma)|_v, \quad \forall \alpha \in \Phi_\gamma. \tag{5.11}$$

By [70, Lem 2.18], for some $B_1 > 0$ which is independent of γ and $\underline{\kappa}$,

$$\prod_{v \in S'_\gamma \setminus S'_{\gamma,0}} |1 - \alpha(\gamma)|_v \leq Q^{B_1 \underline{\kappa}}, \quad \forall \alpha \in \Phi_\gamma. \tag{5.12}$$

For each $v \in S'_{\gamma,0}$, we have $|1 - \alpha(\gamma)|_v \leq 1$ for every $\alpha \in \Phi_\gamma$. By the definition of $S'_{\gamma,0}$, there exists $\alpha \in \Phi_\gamma$ such that $|1 - \alpha(\gamma)|_v < 1$. The argument as in the proof of [70, Prop 8.7] (also see the proof of Theorem 9.19 there) shows that a fortiori $|1 - \alpha(\gamma)|_v \leq q_v^{-\frac{1}{w_G s_G}}$. Hence

$$\prod_{\alpha \in \Phi_\gamma} |1 - \alpha(\gamma)|_v \leq q_v^{-\frac{1}{w_G s_G}}, \quad v \in S'_{\gamma,0}. \tag{5.13}$$

Taking the product of (5.11) over $\alpha \in \Phi_\gamma$ and applying the estimates (5.10) at $v \in S_\gamma \setminus S'_\gamma$ (which is contained in $S \cup S_0$), (5.12) at $v \in S'_\gamma \setminus S'_{\gamma,0}$, and (5.13) at $v \in S'_{\gamma,0}$, we see that

$$1 \leq q_{S'_{\gamma},0}^{-\frac{1}{w_G s_G}} \left(Q^{B_1 \underline{\kappa}} \delta_\infty \prod_{v \in S_\gamma \setminus S'_\gamma} \delta_v \right)^{|\Phi_\gamma|} \leq q_{S'_{\gamma},0}^{-\frac{1}{w_G s_G}} \left(Q^{B_1 \underline{\kappa}} \delta_\infty \prod_{v \in S \cup S_0} \delta_v \right)^{d_G}.$$

Therefore $q_{S'_{\gamma},0} = O(Q^{w_G s_G d_G B_1 \underline{\kappa}})$, implying that

$$q_{S'_\gamma} \leq Q \cdot q_{S'_\gamma,0} = O(Q^{1 + w_G s_G d_G B_1 \underline{\kappa}}).$$

To summarize so far, the absolute value of each summand in (5.7) is bounded by, if we set $C := c_0 c_3^{|S|} c_S c_{S_0} c_\infty (q_S q_{S_0})^{A_2}$, the following:

$$C q_{S_\gamma}^{A_2} q_{S'_\gamma,0}^{A_3} Q^{A_2 + A_3 + B_3 \underline{\kappa}} \deg(\sigma_S)^{1-\nu_S} \dim(\xi)^{1-\nu_\infty}.$$

Applying the above bounds on $q_{S'_\gamma}$ and $q_{S'_\gamma,0}$, the summand admits a bound of the form $O(Q^{A_4 + B_4 \underline{\kappa}} \deg(\sigma_S)^{1-\nu_S} \dim(\xi)^{1-\nu_\infty})$. The number of nonzero summands is bounded as $O(Q^{A_1 \underline{\kappa}})$ by Proposition 5.2. All in all, the absolute value of the left hand side of (5.7) is

$$O(Q^{A_4 + (A_1 + B_4)\underline{\kappa}} \deg(\sigma_S)^{1-\nu_S} \dim(\xi)^{1-\nu_\infty}).$$

The proof is complete by taking $A = A_4 + A_1 + B_4$. (Observe that $Q \leq Q^{\underline{\kappa}}$.) $\quad\square$

Remark 5.7. An affirmative answer to the question in Remark 3.12 would immediately improve Theorem 5.4 with the hypotheses relaxed accordingly, by exactly the same argument.

Consider the set of pairs $(\xi, \sigma_S) \in \mathrm{Irr}_C^{\mathrm{reg}}(G(F_\infty)) \times \mathrm{Irr}^{\mathrm{Yu}}(G(F_S))$. We partition the set into $\mathcal{P}_=$ and $\mathcal{P}_{\neq}$ according as whether $\omega_\xi = \omega_{\sigma_S}$ or not on $Z(F) \cap K_{S_0} K^{\mathfrak{S}}$. Recall that $Z(F) \cap K_{S_0} K^{\mathfrak{S}}$ is finite.

Corollary 5.8. *We have the limit multiplicity formulas*

$$\lim_{\substack{(\xi, \sigma_S) \in \mathcal{P}_{\neq} \\ \dim(\xi) |\deg(\sigma_S)| \to \infty}} \frac{m(\xi, \sigma_S, K_{S_0})}{d(G_\infty) \dim(\xi) \deg(\sigma_S)} = 0,$$

$$\lim_{\substack{(\xi, \sigma_S) \in \mathcal{P}_= \\ \dim(\xi) |\deg(\sigma_S)| \to \infty}} \frac{m(\xi, \sigma_S, K_{S_0})}{d(G_\infty) \dim(\xi) \deg(\sigma_S)} = (-1)^{q(G_\infty)} \frac{\tau'(G, S)}{\mu_{S_0}^{\mathrm{can}}(K_{S_0})} |Z(F) \cap K_{S_0} K^{\mathfrak{S}}|.$$

Remark 5.9. It is clear from the definition that $m(\xi, \sigma_S, K_{S_0}) \geq 0$. This is consistent with the signs in the second formula above. Indeed the sign of the measure $\mu_\infty^{\mathrm{EP}} \mu_S^{\mathrm{EP}}$ is the same as that of $\tau'(G, S)$ since the canonical measure μ_v^{can} is a positive measure for $v \notin S \cup S_\infty$. The signs of μ_∞^{EP} and μ_S^{EP} are the same as those of $(-1)^{q(G_\infty)}$ and $\deg(\sigma_S)$, respectively.

Proof. We start by claiming that $\dim(\xi)\deg(\sigma_S) \to \infty$ if and only if $\dim(\xi)^{\nu_\infty}\deg(\sigma_S)^{\nu_S}$ tends to infinity. To see this, we partition the set of all σ_S into two sets where $|\deg(\sigma_S)| \geq 1$ and $|\deg(\sigma_S)| < 1$. The claim is obvious in the first set. (Recall that $\nu_\infty, \nu_S > 0$ and note that $\dim(\xi) \geq 1$.) In the second set, the claim follows from the fact [39, Thm 7] that $\deg(\sigma_S)$ is bounded below by a positive constant. (More precisely $\deg(\sigma_S)$ is an integral multiple of a constant depending only on a Haar measure on $G(F_S)$.)

Now the corollary readily follows from the preceding theorem by plugging in $\kappa_v = 0$ for all $v \notin \mathfrak{S}$ and $\phi^{\mathfrak{S}} = \mathbf{1}_{K^{\mathfrak{S}}}$. □

We can restate Theorem 5.4 in terms of $m(\xi, \sigma_S, K_{S_0})$, assuming G is split and semisimple for simplicity. The L^1-norm of $\phi^{\mathfrak{S}} \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{A}_F^{\mathfrak{S}}))$ is given by $\|\phi^{\mathfrak{S}}\|_1 := \int_{G(\mathbb{A}_F^{\mathfrak{S}})} |\phi^{\mathfrak{S}}(g)|\, d\mu^{\mathrm{can},\mathfrak{S}}$.

Corollary 5.10. *Suppose that G is a split semisimple group over F. There exist constants $\epsilon, A > 0$ such that for every $(\xi, \sigma_S) \in \mathcal{P}_=$ and for every $\phi^{\mathfrak{S}} = \mathbf{1}_{K^{\mathfrak{S}}gK^{\mathfrak{S}}} \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{A}_F^{\mathfrak{S}}))$ with some $g \in G(\mathbb{A}_F^{\mathfrak{S}})$,*

$$\sum_{\pi \in \mathcal{F}(\xi, \sigma_S, K_{S_0})} \mathrm{tr}\, \pi^{\mathfrak{S}}(\phi^{\mathfrak{S}}) = \frac{m(\xi, \sigma_S, K_{S_0})}{|Z(F) \cap K_{S_0}K^{\mathfrak{S}}|} \sum_{z \in Z(F) \cap K_{S_0}} \phi^{\mathfrak{S}}(z)$$
$$+ O(m(\xi, \sigma_S, K_{S_0})^{1-\nu}\|\phi^{\mathfrak{S}}\|_1^A).$$

The implicit constant in $O(\cdot)$ depends on G, S, S_0, K_{S_0}, and C (but is independent of ξ, σ_S, and $\phi^{\mathfrak{S}}$).

Proof. Since G has finite center, the center is contained in every maximal compact subgroup. So $Z(F) \cap K_{S_0} = Z(F) \cap K_{S_0}K^{\mathfrak{S}}$ and $\omega_{\sigma_S} = \omega_\xi$ on $Z(F) \cap K_{S_0}$ for $(\xi, \sigma_S) \in \mathcal{P}_=$. Our task is to turn the right hand side of (5.6) to the right hand side as in the corollary. Let $\underline{\kappa} = (\kappa_v)_{v \notin \mathfrak{S}}$ be chosen such that $\phi^{\mathfrak{S}} \in \mathcal{H}^{\mathrm{ur}, \leq \underline{\kappa}}(G(\mathbb{A}_F^{\mathfrak{S}}))$. We know that

$$\left| m(\xi, \sigma_S, K_{S_0}) - (-1)^{q(G_\infty)}d(G_\infty)\dim(\xi)\deg(\sigma_S)\frac{\tau'(G,S)|Z(F) \cap K_{S_0}K^{\mathfrak{S}}|}{\mu_{S_0}^{\mathrm{can}}(K_{S_0})} \right|$$
$$= O(\dim(\xi)^{1-\nu_\infty}\deg(\sigma_S)^{1-\nu_S})$$

from Theorem 5.4, cf. the proof of Corollary 5.8. So it is enough to show that for some constants $A', B' > 0$ (whose independence is as in the corollary),

(i) $\sum_{z \in Z(F) \cap K_{S_0}} \phi^{\mathfrak{S}}(z) = O(Q^{B'\underline{\kappa}})$,
(ii) $Q^{A\underline{\kappa}} \leq \|\phi^{\mathfrak{S}}\|_1^{A'}$.

Part (i) is equivalent to $|Z(F) \cap K_{S_0}(K^{\mathfrak{S}}gK^{\mathfrak{S}})K_\infty| = O(Q^{A'\underline{\kappa}})$ for any maximal compact subgroup $K_\infty \subset G(F_\infty)$. This immediately follows from [70, Prop 8.7].

It remains to check (ii). We adopt the notation about truncated Hecke algebras from [70, §2.3]. By Cartan decomposition we may write $g = (g_v)_{v \notin \mathfrak{S}} \in G(\mathbb{A}_F^{\mathfrak{S}})$ with $g_v = \mu_v(\varpi_v)$ for the cocharacter T_v of a maximal F_v-split torus T_v of G and a uniformizer ϖ_v of F_v. We take $\underline{\kappa} = (\kappa_v)_{v \notin \mathfrak{S}}$ such that $\kappa_v = \|\mu_v\|_{\mathcal{B}}$ for a suitable $\mathbb{R}$-basis $\mathcal{B} = \{e_1, ..., e_r\}$ of $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, where $X_*(T)$ is the cocharacter group of a maximal torus T over $\overline{F}$. (Thus $r = \dim T = \mathrm{rk} G$.) Of course $\kappa_v = 0$ for all but finitely many v. Choose a Borel subgroup containing T so that we have a set of positive coroots $\Phi^{\vee,+}$ for T. We take $\mathcal{B}$ to consist of simple coroots in $\Phi^{\vee,+}$. Similarly we choose a Borel subgroup $B_v \supset T_v$ and a set of positive coroots $\Phi_v^{\vee,+}$ for T_v. Without loss of generality we assume that μ_v is B_v-dominant. Set $\rho_v^{\vee} := \sum_{\alpha^{\vee} \in \Phi_v^{\vee,+}} \alpha^{\vee}$. The equality $\kappa_v = \|\mu_v\|_{\mathcal{B}}$ means that $\kappa_v = \langle \mu_v, i\alpha^{\vee} \rangle$ for some $\alpha^{\vee} \in \Phi^{\vee,+}$, where i is an inner automorphism of G sending T to T_v. Hence for each $v \notin \mathfrak{S}$, we see that $\langle \mu_v, \beta_v^{\vee} \rangle = \kappa_v$ for some coroot $\beta_v \in \Phi_v^{\vee,+}$.

We claim that there exists a constant $c > 1$ such that for every $v \notin \mathfrak{S}$ we have

$$\|\mathbf{1}_{K_v g_v K_v}\|_1 \geq c^{-1} q^{\langle \mu_v, \rho^{\vee} \rangle}.$$

Indeed [37, Prop 7.4] tells us that $\|\mathbf{1}_{K_v g_v K_v}\|_1 / q^{\langle \mu_v, \rho^{\vee} \rangle}$ is equal to $|(G/P_{\mu_v})(\mathbb{F}_{q_v})| / q_v^{\dim(G/P_{\mu_v})}$ for a suitable parabolic subgroup P_{μ_v} of G. Of course there are finitely many parabolic subgroups (up to conjugation). We see that the quotient (which is a quotient of two polynomials in q_v) tends to one as $q_v \to \infty$. The claim follows.

For each v such that $\mu_v \neq 0$ (so $K_v g_v K_v \neq K_v$) we have $\|\mathbf{1}_{K_v g_v K_v}\|_1 \geq 2 = c^{\log 2/\log c}$. Setting $c' := \log c / \log 2 \in \mathbb{R}_{>0}$, we have (whether $\mu_v = 0$ or not)

$$\|\mathbf{1}_{K_v g_v K_v}\|_1^{1+c'} \geq q^{\langle \mu_v, \rho^{\vee} \rangle}, \quad \forall v \notin \mathfrak{S}.$$

Since μ_v is B_v-dominant, it is clear that $\langle \mu_v, \rho^{\vee} \rangle \geq \langle \mu_v, \beta_v^{\vee} \rangle = \kappa_v$. In conclusion

$$\|\phi^{\mathfrak{S}}\|_1^{1+c'} \geq Q^{\underline{\kappa}}.$$

The proof of (ii) is complete. □

Let us record a sample application to the existence of certain automorphic representations. For simplicity we assume that G is split over F. We fix a reductive model over the ring of integers $\mathcal{O}_F$, giving rise to hyperspecial subgroups K_v^{hs} at each finite place v.

Corollary 5.11. *Suppose that G is a split reductive group over F. Fix $\xi \in \mathrm{Irr}^{\mathrm{reg}}(G(F_{\infty}))$ and S a nonempty finite set of finite places. Suppose that the residue characteristic of each $v \in S$ is sufficiently large in the sense at the start of §5.5. Then there exists $d_0 > 0$ with the following property: For every $\pi_S^0 \in \mathrm{Irr}^{\mathrm{Yu}}(G(F_S))$ with $|\deg(\pi_S^0)| \geq d_0$ and $\omega_{\pi_S^0} = \omega_{\xi}$ on $Z(F) \cap K_{S_0} K^{\mathfrak{S}}$, there exists a cuspidal automorphic representation π of $G(\mathbb{A}_F)$ such that*

- $\pi_S \simeq \pi_S^0$,
- $\pi_\infty \in \Pi_\infty(\xi)$,
- $\pi^{S,\infty}$ *is unramified.*

Proof. In the preceding corollary, it is enough to take S_0 to be sufficiently large and contain all places of small residue characteristics, and set $K_{S_0} := \prod_{v \in S_0} K_v^{\mathrm{hs}}$. Then fix ξ and let $\deg(\sigma_S)$ go to infinity. $\quad\square$

According to Corollary 5.8, it is reasonable to restrict our attention to $(\xi, \sigma_S) \in \mathcal{P}_=$ when studying equidistribution problems on the following counting measure for the multiset $\mathcal{F} = \mathcal{F}(\xi, \sigma_S, K_{S_0})$, where $\delta_{\pi^{\mathfrak{S}}}$ denotes the Dirac delta measure supported at $\pi^{\mathfrak{S}}$ on the unramified unitary dual $G(\mathbb{A}_F^{\mathfrak{S}})^{\wedge,\mathrm{ur}} := \prod_{v \notin \mathfrak{S}} G(F_v)^{\wedge,\mathrm{ur}}$:

$$\widehat{\mu}_{\mathcal{F}}^{\mathrm{count}} := \frac{1}{|\mathcal{F}|} \sum_{\pi \in \mathcal{F}} \delta_{\pi^{\mathfrak{S}}}.$$

Of course this makes sense if $|\mathcal{F}| \neq 0$. To obtain a clean formula we will further assume that $Z(F) \cap K_{S_0} = Z(F) \cap K_{S_0} K^{\mathfrak{S}}$. (Alternatively we may instead restrict to the pairs $(\xi, \sigma_S) \in \mathcal{P}_=$ such that $\omega_\xi = \omega_{\sigma_S}$ on $Z(F) \cap K_{S_0}$.)

Given $z \in Z(F)$ let ω_z denote the function on $G(\mathbb{A}_F^{\mathfrak{S}})^{\wedge,\mathrm{ur}}$ whose value on each representation is its central character evaluated at z. Define a measure $\widehat{\mu}_z^{\mathrm{pl,ur},\mathfrak{S}}$ on $G(\mathbb{A}_F^{\mathfrak{S}})^{\wedge,\mathrm{ur}}$ to be $\omega_z \cdot \widehat{\mu}^{\mathrm{pl,ur},\mathfrak{S}}$, where $\widehat{\mu}^{\mathrm{pl,ur},\mathfrak{S}} := \prod_{v \notin \mathfrak{S}} \widehat{\mu}_v^{\mathrm{pl,ur}}$ is the product of the Plancherel measure $\widehat{\mu}_v^{\mathrm{pl,ur}}$ on the unramified unitary dual of $G(F_v)$.[7] Recall that $\phi^{\mathfrak{S}}$ defines a function $\widehat{\phi}^{\mathfrak{S}} : \pi \mapsto \mathrm{tr}\,\pi(\phi^{\mathfrak{S}})$ on $\prod_{v \notin \mathfrak{S}} G(F_v)^{\wedge,\mathrm{ur}}$. Note that we can integrate $\widehat{\phi}^{\mathfrak{S}}$ against the (possibly infinite) sum measure $\widehat{\mu}_{Z(F) \cap K_{S_0}}^{\mathrm{pl,ur},\mathfrak{S}} := \sum_{z \in Z(F) \cap K_{S_0}} \widehat{\mu}_z^{\mathrm{pl,ur},\mathfrak{S}}$. Indeed $\widehat{\mu}_z^{\mathrm{pl,ur},\mathfrak{S}}(\widehat{\phi}^{\mathfrak{S}})$ is nonzero for only finitely many $z \in Z(F) \cap K_{S_0}$ since $Z(F)$ intersects an open compact subset of $Z(\mathbb{A}_F)$ at only finitely many points.

Corollary 5.12. *Assume that* $Z(F) \cap K_{S_0} = Z(F) \cap K_{S_0} K^{\mathfrak{S}}$. *For every* $\phi^{\mathfrak{S}} \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{A}_F^{\mathfrak{S}}))$,

$$\lim_{\substack{\dim(\xi)\,\deg(\sigma_S) \to \infty \\ (\xi, \sigma_S) \in \mathcal{P}_=}} \widehat{\mu}_{\mathcal{F}(\xi, \sigma_S, K_{S_0})}^{\mathrm{count}}(\widehat{\phi}^{\mathfrak{S}}) = \frac{\widehat{\mu}_{Z(F) \cap K_{S_0}}^{\mathrm{pl,ur},\mathfrak{S}}(\widehat{\phi}^{\mathfrak{S}})}{|Z(F) \cap K_{S_0} K^{\mathfrak{S}}|}.$$

(The counting measure is defined when $\dim(\xi)|\deg(\sigma_S)| \gg 1$ *by Corollary 5.8.)*

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[7] Since we consider only those $\widehat{\phi}^{\mathfrak{S}}$ coming from $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{A}_F^{\mathfrak{S}}))$, the formulas remain valid if we use the Plancherel measure on the whole unitary dual.

Proof. Choose $\underline{\kappa} \geq 0$ such that $\phi^{\mathfrak{S}} \in \mathcal{H}^{\mathrm{ur}, \leq \underline{\kappa}}(G(\mathbb{A}_F^{\mathfrak{S}}))$. By (5.6), $\widehat{\mu}_{\mathcal{F}(\xi, \sigma_S, K_{S_0})}^{\mathrm{count}}(\widehat{\phi}^{\mathfrak{S}})$ equals

$$(-1)^{q(G_\infty)} \frac{\tau'(G, S) d(G_\infty) \dim(\xi) \deg(\sigma_S)}{\mu_{S_0}^{\mathrm{can}}(K_{S_0}) |\mathcal{F}(\xi, \sigma_S, K_{S_0})|} \sum_{z \in Z(F) \cap K_{S_0}} \phi^{\mathfrak{S}}(z)$$
$$+ O\left(\frac{\deg(\sigma_S)^{1-\nu_S} \dim \xi^{1-\nu_\infty}}{|\mathcal{F}(\xi, \sigma_S, K_{S_0})|} Q^{A_{\underline{\kappa}}} \right).$$

By Plancherel theorem, $\sum_{z \in Z(F) \cap K_{S_0}} \phi^{\mathfrak{S}}(z) = \widehat{\mu}_{Z(F) \cap K_{S_0}}^{\mathrm{pl,ur}, \mathfrak{S}}(\widehat{\phi}^{\mathfrak{S}})$. We apply Corollary 5.8 to finish the proof. $\square$

Remark 5.13. In particular if $Z(F) \cap K_{S_0} = \{1\}$ (e.g. if $Z(F)$ is trivial) then we have

$$\lim_{\substack{\dim(\xi) + \deg(\sigma_S) \to \infty \\ (\xi, \sigma_S) \in \mathcal{P}_=}} \widehat{\mu}_{\mathcal{F}(\xi, \sigma_S, K_{S_0})}^{\mathrm{count}}(\widehat{\phi}^{\mathfrak{S}}) = \widehat{\mu}^{\mathrm{pl,ur}, \mathfrak{S}}(\widehat{\phi}^{\mathfrak{S}}).$$

This confirms Conjecture 1 in [62], or more precisely its analogue as explained in the remark below it. In our case the limiting measure is the product of the unramified Plancherel measures so (i) and (ii) of the conjecture are true. Part (iii) is essentially [70, Prop 5.3], from which (iv) follows immediately.

Remark 5.14. The results above should carry over to the case where neither $G(F_S)$ nor $G(F_\infty)$ has compact center, at least if G is a cuspidal group in the sense that the center of $\mathrm{Res}_{F/\mathbb{Q}} G$ has the same split $\mathbb{Q}$-rank and split $\mathbb{R}$-rank. This requires some modification in the statements (e.g. pseudo-coefficients of a supercuspidal representation have compact support only modulo center) but would not lead to any significant change in the proof. Alternatively one could work with the trace formula with fixed central character (one could use [23, §6] for instance), in which case representations and test functions also have fixed central characters which are inverses of each other.

5.6. Possible generalizations

It is sensible to ask whether the method of this paper applies to non-supercuspidal discrete series representations π but there are difficulties. In that case we still have a somewhat explicit construction of pseudo-coefficients for π, cf. [64, §3.4] but they are not as simple as in §3.2 to be useful. In the trace formula, if we impose a pseudo-coefficient of π at a local place v then one still has the simple trace formula, but the spectral side picks up automorphic representations whose v-components are not only π but possibly nontempered representations in a finite list. This means that one has to control these spectral error terms. Alternatively one could allow more general test functions at v but then the trace formula will have more terms to be dealt with (unless the global reductive group is anisotropic modulo center).

6. Steinberg representations and horizontal families

The results in this section do not rely on Sections 2 through 4. However the reader will see a strong analogy both in the statements and proofs between the vertical families in the last section and the horizontal families in this one.

Let ξ, S, S_0, S_∞, and $\mathfrak{S}$ be as in §5.3, cf. §1.2. In particular the finite sets S, S_0, and S_∞ are mutually disjoint, and G is unramified outside $\mathfrak{S}$. We will fix S_0 and assume that

- $S \cap \mathrm{Ram}(G) = \emptyset$ (so that $\mathrm{Ram}(G) \subset S_0$),
- the residue characteristic of every $v \in S$ is sufficiently large.

Throughout this section we keep a large residue characteristic assumption as in §5.5:

- $S \cup S_0$ contains the places of F with small residue characteristics such that the result by Cluckers–Gordon–Halupczok on uniform bound on orbital integrals [70, Thm 14.1] applies to places outside $S \cup S_0$.

We also assume that (to use Lemma 6.2 relying on strong approximation)

- the simply connected cover of G^{der} has no F-simple factor that is F_v-anisotropic for every $v \in S$.

Let $\mathcal{F}(\xi, \mathrm{St}_S, K_{S_0})$ denote the multi-set of $\pi \in \mathcal{A}_{\mathrm{disc}}(G)$, with $m_{\mathrm{disc}}(\pi) \dim(\pi_{S_0})^{K_{S_0}}$ as the multiplicity of π, which satisfies the following conditions: first, $\pi_\infty \in \Pi_\infty(\xi)$; second, π_v is isomorphic to the Steinberg representation for all $v \in S$; third, π_v is unramified for all finite places $v \notin S \cup S_0$. In this section we study $\mathcal{F}(\xi, \mathrm{St}_S, K_{S_0})$ as we vary the set S. The situation is somewhat complementary to that in the previous section. We refer to $\mathcal{F}(\xi, \mathrm{St}_S, K_{S_0})$ as a *horizontal family*.[8]

Kottwitz [49] constructed Euler-Poincaré functions for p-adic groups. For any place $v \in S$, we denote it by $\phi_v^{\mathrm{EP}} \in C_c^\infty(G(F_v))$. We have that [49, Thm 2]

$$O_\gamma(\phi_v^{\mathrm{EP}}) = \begin{cases} 1, & \gamma \in G(F_v)_{\mathrm{ell}}, \\ 0, & \gamma \in G(F_v)_{\mathrm{ss}} \backslash G(F_v)_{\mathrm{ell}}. \end{cases} \tag{6.1}$$

We will assume that

- G is a simple[9] algebraic group, i.e. every proper normal subgroup of G over an algebraic closure of F is finite.

---

[8] Even though ξ is allowed to vary "vertically" (as in the last section), the main novel feature is to allow S to vary, so the family deserves the name.

[9] Such a group is often said to be absolutely almost simple, e.g. in [38].

In that case Casselman's theorem, cf. [16, Thm 2] and [49, Thm 2'], tells us that an irreducible unitary representation π_v with $\operatorname{tr} \pi_v(\phi_v^{\mathrm{EP}}) \neq 0$ can occur only in the following two cases (which may not be mutually exclusive):

(i) π_v is the trivial representation and $\operatorname{tr} \pi_v(\phi_v^{\mathrm{EP}}) = 1$, or
(ii) $\pi_v \simeq \mathrm{St}_v$ and $\operatorname{tr} \pi_v(\phi_v^{\mathrm{EP}}) = (-1)^{q(G_v)}$, where $q(G_v)$ is the semisimple rank of G_{F_v}.
     In particular $f_{\mathrm{St}_v} := (-1)^{q(G_v)} \phi_v^{\mathrm{EP}}$ is a pseudo-coefficient for St_v.

Set $f_{\mathrm{St}_S} := \prod_{v \in S} f_{\mathrm{St}_v}$, $q_S := \prod_{v \in S} q_v$, and $q(G_S) := \sum_{v \in S} q(G_v)$.

Remark 6.1. The formal degree of St_v is equal to $(-1)^{q(G_v)}$ for the Euler-Poincaré measure by the Plancherel theorem, (ii) above, and $\phi_v^{\mathrm{EP}}(1) = 1$ recalled above. See also [14].

We will state the analogue of Lemma 5.1 for $\mathcal{F}(\xi, \mathrm{St}_S, K_{S_0})$ after recalling the following well-known result on one-dimensional automorphic representations.

Lemma 6.2. *Let H be a connected reductive group over a number field E. Let v be a place of E, and assume that the simply connected cover of H^{der} has no E-simple factor that is E_v-anisotropic (in particular, $H(E_v)$ is not compact modulo center). If π is an automorphic representation of $H(\mathbb{A}_E)$ with $\dim \pi_v = 1$ then π is one-dimensional.*

Proof. Using a z-extension of H ([47, §1]) we can reduce to the case when the derived subgroup H^{der} of H is simply connected. Now the groups $H^{\mathrm{der}}(E)$ and $H^{\mathrm{der}}(E_v)$ are subgroups of $H(\mathbb{A}_E)$ so they act on π viewed as a subspace of the regular representation of $H(\mathbb{A}_E)$ on automorphic forms of $H(\mathbb{A}_E)$ (with a fixed central character). The one-dimensionality of π_v implies that $H^{\mathrm{der}}(E_v)$ acts trivially on π. Then any smooth vector of π, as a function $\phi : H(\mathbb{A}_E) \to \mathbb{C}$, has the property that $\phi(h_0 h h_v) = \phi(h)$ for $h_0 \in H^{\mathrm{der}}(E)$, $h \in H(\mathbb{A}_E)$, and $h_v \in H^{\mathrm{der}}(E_v)$. Hence $\phi(hh') = \phi(h)$ for $h' \in h^{-1} H^{\mathrm{der}}(E) h H^{\mathrm{der}}(E_v) = h^{-1} H^{\mathrm{der}}(E) H^{\mathrm{der}}(E_v) h$. By the strong approximation theorem, h' runs over a dense subset of $H^{\mathrm{der}}(\mathbb{A}_E)$. Therefore ϕ is constant on left $H^{\mathrm{der}}(\mathbb{A}_E)$-cosets, i.e. $H(\mathbb{A}_E)$ acts through an abelian quotient. Hence π is one-dimensional. $\square$

Lemma 6.3. *Let $\mathfrak{S} := S_\infty \cup S_0 \cup S$ and $\phi^{\mathfrak{S}} \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{A}_F^{\mathfrak{S}}))$. Put $\phi := \phi^{\mathfrak{S}} f_{\mathrm{St}_S} \mathbf{1}_{K_{S_0}} f_\xi$. Then*

$$\sum_{\pi \in \mathcal{F}(\xi, \mathrm{St}_S, K_{S_0})} \operatorname{tr} \pi^{\mathfrak{S}} \left(\phi^{\mathfrak{S}}, \mu^{\mathrm{can}, \mathfrak{S}} \right) = \mu_{S_0}^{\mathrm{can}}(K_{S_0})^{-1} T_{\mathrm{ell}}(\phi, \mu^{\mathrm{can}, \mathrm{EP}}).$$

Proof. We have the simple trace formula (5.3) for ϕ. Indeed f_{St_S} enjoys property (ii) in §5.2. It suffices to show that the left hand side equals $\mu_{S_0}^{\mathrm{can}}(K_{S_0})^{-1} T_{\mathrm{disc}}(\phi, \mu^{\mathrm{can}, \mathrm{EP}})$, which expands as

$$\sum_{\pi \in \mathcal{A}_{\mathrm{disc}}(G)} m_{\mathrm{disc}}(\pi) \operatorname{tr} \pi^{\mathfrak{S}}(\phi^{\mathfrak{S}}) \operatorname{tr} \pi_S(f_{\mathrm{St}_S}) \frac{\operatorname{tr} \pi_{S_0}(\mathbf{1}_{K_{S_0}})}{\mu_{S_0}^{\mathrm{can}}(K_{S_0})} \operatorname{tr} \pi_\infty(f_\xi). \tag{6.2}$$

Suppose that the summand for π is nonzero. Then π_∞ cannot be one-dimensional by regularity of ξ. Let $v \in S$ so that $\operatorname{tr} \pi_v(f_{\operatorname{St}_v}) \neq 0$. If $G(F_v)$ is compact modulo center then St_v is the trivial representation, so $\pi_v \simeq \operatorname{St}_v$ by Casselman's theorem above. If $G(F_v)$ is not compact modulo center then Lemma 6.2 tells us that π_v cannot be one-dimensional so $\pi_v \simeq \operatorname{St}_v$ by the same theorem. In either case $\operatorname{tr} \pi_v(f_{\operatorname{St}_v}) = 1$. We see that (6.2) equals the left hand side of the lemma. $\quad\square$

Theorem 6.4. *There exist real constants $\nu_\infty, A > 0$ and an integer $B \in \mathbb{Z}_{\geq 1}$ such that for every $\xi \in \operatorname{Irr}_C^{\operatorname{reg}}(G(F_\infty))$, for every $\underline{\kappa} = (\kappa_v)_{v \notin \mathfrak{S}}$, and for every $\phi^\mathfrak{S} \in \mathcal{H}^{\operatorname{ur}, \leq \underline{\kappa}}(G(\mathbb{A}_F^\mathfrak{S}))$ which is the characteristic function of a bi-$K^\mathfrak{S}$-invariant compact subset,*

$$\frac{1}{\tau'(G,S)} \sum_{\pi \in \mathcal{F}(\xi, \operatorname{St}_S, K_{S_0})} \operatorname{tr} \pi^\mathfrak{S}(\phi^\mathfrak{S}) = \frac{(-1)^{q(G_S)+q(G_\infty)} d(G_\infty) \dim \xi}{\mu_{S_0}^{\operatorname{can}}(K_{S_0})} \sum_{z \in Z(F) \cap K_{S_0}} \omega_\xi(z) \phi^\mathfrak{S}(z)$$
$$+ O(q_S^{-B} \dim(\xi)^{1-\nu_\infty} Q^{A\underline{\kappa}}).$$

The implicit constant in $O(\cdot)$ depends on G, S_0, K_{S_0}, and C (but is independent of ξ, S, $\underline{\kappa}$, and $\phi^\mathfrak{S}$).

Proof. The proof proceeds exactly as for Theorem 5.4. The main term coming from $\gamma \in Z(F)$ is computed similarly. (Note that $f_{\operatorname{St}_S}(z) = (-1)^{q(G_S)} = \deg(\operatorname{St}_S)$ for $z \in Z(F_S)$. So $\deg(\sigma_S)$ in (5.6) is replaced by $(-1)^{q(G_S)}$ here.) The issue is to bound the contribution from noncentral elements. To explain the mild modifications in the argument we freely use the notation from the proof there. It suffices to show the following analogue of (5.7) for uniform constants $\nu_\infty, A > 0$ and $B \in \mathbb{Z}_{\geq 1}$:

$$\frac{1}{\tau'(G,S)} \sum_{\substack{\gamma \in \mathcal{Y}(\underline{\kappa}) \\ \gamma \notin Z(F)}} \left| \mu_{I_\gamma}^{\operatorname{can}, \operatorname{EP}}(I_\gamma) \mathcal{O}_\gamma^{G(\mathbb{A}_F)}(\phi^\mathfrak{S} f_{\operatorname{St}_S} \mathbf{1}_{K_{S_0}} f_\xi) \right| = O(q_S^{-B} \dim(\xi)^{1-\nu_\infty} Q^{A\underline{\kappa}}). \quad (6.3)$$

The only nontrivial change is to replace (5.8) and (5.9) with the following inequalities for noncentral elements $\gamma \in \mathcal{Y}(\underline{\kappa})$ for suitable uniform constants $c_0, A_5 > 0$:

$$\frac{|\mu_{I_\gamma}^{\operatorname{can}, \operatorname{EP}}(I_\gamma)|}{\tau'(G,S)} \leq c_0 q_S^{-B} Q^{A_5 \underline{\kappa}}, \quad (6.4)$$

$$D_S(\gamma)^{1/2} |\mathcal{O}_\gamma(f_{\operatorname{St}_S})| \leq 1. \quad (6.5)$$

The bounds for orbital integrals away from S as well as the rest of the proof are exactly the same as in the proof of Theorem 5.4. So we content ourselves with justifying the two inequalities, beginning with (6.5). (Along the way we obtain a bound on q_{S_γ}. Evidently the same bound works for $q_{S'_{\gamma,0}}$. Together with (5.9), this is used to bound the orbital integral away from $\mathfrak{S}$.)

As we recalled in (6.1), $|\mathcal{O}_\gamma(f_{\mathrm{St}_S})|$ is bounded by 1 and nonzero only on γ that is elliptic in $G(F_v)$ for all $v \in S$. Then γ is contained in an F_v-anisotropic maximal torus T_v of $G(F_v)$ so $|\alpha(\gamma)|_v = 1$ for all absolute roots α of T_v in G by compactness, implying that $|1 - \alpha(\gamma)|_v \le 1$. Therefore $D_S(\gamma) \le 1$ and (6.5) holds.

It remains to check the uniform bound (6.4) for noncentral elements γ. If γ is not in the center of G, we have that $\frac{1}{2}(\dim I_\gamma - \mathrm{rk}I_\gamma) - \frac{1}{2}(\dim G - \mathrm{rk}G)$ is a negative integer. Take B to be the maximum of such integers as γ varies. Our initial assumption tells us that $\mathrm{Ram}(G) \subset S_0$. At each $v \in S_0$ we define $\delta_v < \infty$ as in (5.10) or $\delta_v = 1$, whichever is bigger. We define $d_G, s_G, w_G \in \mathbb{Z}_{>0}$ as in the proof of Theorem 5.4.

From Lemma 5.3 and the fact that

$$\mathrm{Ram}(I_\gamma) \subset \mathrm{Ram}(G) \cup S_\gamma \cup \{v \notin \mathfrak{S} : \kappa_v \ne 0\} \subset S_0 \cup S_\gamma \cup \{v \notin \mathfrak{S} : \kappa_v \ne 0\},$$

it follows that

$$\frac{|\mu_{I_\gamma}^{\mathrm{can,EP}}(I_\gamma)|}{\tau'(G,S)} \le c_0 q_S^{\frac{1}{2}(\dim I_\gamma - \mathrm{rk}I_\gamma) - \frac{1}{2}(\dim G - \mathrm{rk}G)} \prod_{v \in \mathrm{Ram}(I_\gamma)} q_v^{A_2}.$$

$$\le c_0 q_S^{-B} q_{S_0}^{A_2} q_{S_\gamma}^{A_2} Q^{A_2} = O(q_S^{-B} q_{S_\gamma}^{A_2} Q^{A_2}). \tag{6.6}$$

We will bound q_{S_γ} independently of γ, by considering the partition

$$S_\gamma = (S_\gamma \cap S) \coprod (S_\gamma \cap S_0) \coprod (S'_\gamma \backslash S'_{\gamma,0}) \coprod S'_{\gamma,0}.$$

When $v \in S_\gamma \cap S$, we have $|1 - \alpha(\gamma)|_v \le 1$ as explained above, and the inequality is strict for some $\alpha \in \Phi_\gamma$, thus as in (5.13), we have

$$\prod_{\alpha \in \Phi_\gamma} |1 - \alpha(\gamma)|_v \le q_v^{-\frac{1}{w_G s_G}}, \qquad v \in S_\gamma \cap S.$$

At $v \in S_\gamma \cap S_0$ we have $|1 - \alpha(\gamma)|_v \le \delta_v$. For $v \in S'_\gamma \backslash S'_{\gamma,0}$ and $v \in S'_{\gamma,0}$ the bounds of (5.12) and (5.13) remain valid without change. Applying these bounds to the product formula for $1 - \alpha(\gamma)$ with $\alpha \in \Phi_\gamma$ (compare with (5.11))

$$1 = \prod_{\alpha \in \Phi_\gamma} \prod_v |1 - \alpha(\gamma)|_v \le \delta_\infty^{|\Phi_\gamma|} \prod_{\alpha \in \Phi_\gamma} \prod_{v \in S_\gamma} |1 - \alpha(\gamma)|_v$$

$$\le (\delta_\infty \prod_{v \in S_0} \delta_v)^{d_G} q_{S_\gamma}^{-\frac{1}{w_G s_G}} \left(\prod_{v \in S'_\gamma \backslash S'_{\gamma,0}} q_v^{\frac{1}{w_G s_G}} \right) Q^{d_G B_1 \underline{\kappa}}.$$

Increasing B_1 if necessary, we can disregard the second bracketed term (the product over $v \in S'_\gamma \backslash S'_{\gamma,0}$). We deduce that $q_{S_\gamma} = O(Q^{d_G w_G s_G B_1 \underline{\kappa}})$. Using this bound in (6.6) we conclude:

$$\frac{|\mu_{I_\gamma}^{\mathrm{can,EP}}(I_\gamma)|}{\tau'(G,S)} = O\left(q_S^{-B} q_{S_\gamma}^{A_2} Q^{A_2}\right) = O\left(q_S^{-B} Q^{(d_G w_G s_G A_2 B_1 + A_2)_{\underline{\kappa}}}\right). \quad \square$$

Denote by Ξ_1 (resp. $\Xi_{\neq 1}$) the subset of $\xi \in \mathrm{Irr}_C^{\mathrm{alg}}(G(F_\infty))$ whose central character is trivial (resp. non-trivial).

Corollary 6.5. *We have the limit multiplicity formulas*

$$\lim_{\substack{q_S \dim(\xi) \to \infty \\ \xi \in \Xi_{\neq 1}}} \frac{m(\xi, \mathrm{St}_S, K_{S_0})}{d(G_\infty) \dim(\xi) \tau'(G,S)} = 0,$$

$$\lim_{\substack{q_S \dim(\xi) \to \infty \\ \xi \in \Xi_1}} \frac{(-1)^{q(G_S)} m(\xi, \mathrm{St}_S, K_{S_0})}{d(G_\infty) \dim(\xi) \tau'(G,S)} = \frac{(-1)^{q(G_\infty)} |Z(F) \cap K_{S_0} K^{\mathfrak{S}}|}{\mu_{S_0}^{\mathrm{can}}(K_{S_0})}. \quad (6.7)$$

More precisely, for each $\epsilon > 0$, there exists $\delta_\epsilon > 0$ with the following property: for every finite set of finite places S such that $S \cap \mathrm{Ram}(G) = \emptyset$ and for every $\xi \in \Xi_{\neq 1}$ such that $q_S \dim(\xi) > \delta_\epsilon$ (while S_0 is fixed), we have $|m(\xi, \mathrm{St}_S, K_{S_0})| < \epsilon \cdot d(G_\infty) \dim(\xi) \tau'(G,S)$. The second limit formula is interpreted in a similar way.

Proof. This follows from the preceding theorem exactly as Corollary 5.8 does from Theorem 5.4. $\square$

Example 6.6. Consider the case when $G = \mathrm{PGL}(2)$ over a totally real field F and $S_0 = \emptyset$. Then $d(G_\infty) = 1$, $q(G_\infty) = (-1)^{[F:\mathbb{Q}]}$, and the right hand side of (6.7) is $(-1)^{[F:\mathbb{Q}]}$. (When $F = \mathbb{Q}$, corresponding to classical holomorphic modular forms of even weight $k \in \mathbb{Z}_{\geq 2}$[10] with trivial Nebentypus character is the representation $\xi_k = \mathrm{Sym}^{k-2}$ of $\mathrm{PGL}(2)$ so that $\dim(\xi_k) = k - 1$. Similarly $\dim(\xi)$ is computed for general F.) Gross's motive for $\mathrm{PGL}(2)$ is $\mathbb{Q}(-1)$. We can easily compute $\tau(G) = 2$ and

$$\tau'(G,S) = \zeta_F(-1) 2^{1-[F:\mathbb{Q}]} \prod_{v \in S} \frac{1 - q_v}{2},$$

where ζ_F is the Dedekind zeta function (in particular $\zeta_{\mathbb{Q}}(-1) = -1/12$). Since $\dim G - \mathrm{rk} G = 2$, we can take $B = 1$ in Theorem 6.4 by its proof and $\nu_\infty = 1$ by Remark 5.5. So we have the asymptotic formula

$$m(\xi, \mathrm{St}_S, \emptyset) = |\zeta_F(-1)| 2^{1-[F:\mathbb{Q}]-|S|} \prod_{v \in S} (q_v - 1) + O(1). \quad (6.8)$$

Here the bound $O(1)$ comes from $q_S^{-1} \prod_{v \in S}(q_v - 1) = O(1)$. In the special case when $F = \mathbb{Q}$ and $\xi_k = \mathrm{Sym}^{k-2}$ is fixed, we have

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[10] The regularity condition on ξ excludes $k = 2$ but we can easily work out the case $k = 2$. The simple trace formula is still valid for the same test function ϕ. The only extra work is to bound the extra spectral terms from one-dimensional automorphic representations, cf. Lemma 6.2, which do not show up when $k > 2$.

$$m(\xi_k, \mathrm{St}_S, \emptyset) \sim \frac{(k-1)\phi(q_S)}{12 \cdot 2^{|S|}} \quad \text{as } q_S \to \infty,$$

where $\phi(\cdot)$ is Euler's phi-function. The difference from [42, Cor 2.14] by the factor $2^{|S|}$ is explained as follows. At each $v \in S$ there is another square-integrable representation St'_v such that St_v and St'_v differ as representations of $\mathrm{GL}_2(\mathbb{Q}_v)$ by the unique non-trivial unramified quadratic character of $\mathbb{Q}_v^\times$. The result in [42] can be interpreted as $\sum_{\sigma_S} m(\xi_k, \sigma_S, \emptyset)$, where the sum runs over σ_S such that $\sigma_v \in \{\mathrm{St}_v, \mathrm{St}'_v\}$. Thus their count is $2^{|S|}$ times ours. In this special case, observe that our $O(1)$ in (6.8) improves on the error bound $O((kq_S)^{2/3})$ obtained in [42].

Corollary 6.7. *Suppose that G is a split simple reductive group over F. There exist constants $\epsilon, A, B > 0$ such that for every $\xi \in \mathrm{Irr}_C^{\mathrm{reg}}(G(F_\infty))$ and for every $\phi^{\mathfrak{S}} = \mathbf{1}_{K^{\mathfrak{S}}{}_g K^{\mathfrak{S}}} \in \mathcal{H}^{\mathrm{ur}, \leq \underline{\kappa}}(G(\mathbb{A}_F^{\mathfrak{S}}))$ with some $g \in G(\mathbb{A}_F^{\mathfrak{S}})$,*

$$\sum_{\pi \in \mathcal{F}(\xi, \mathrm{St}_S, K_{S_0})} \mathrm{tr}\, \pi^{\mathfrak{S}}(\phi^{\mathfrak{S}}) = m(\xi, \mathrm{St}_S, K_{S_0}) \sum_{z \in Z(F) \cap K_{S_0}} \phi^{\mathfrak{S}}(z)$$
$$+ O(m(\xi, \mathrm{St}_S, K_{S_0})^{1-\nu} q_S^{-B} \|\phi^{\mathfrak{S}}\|_1^A).$$

The implicit constant in $O(\cdot)$ depends on G, S, S_0, K_{S_0}, and C (but is independent of ξ and $\phi^{\mathfrak{S}}$).

Proof. This is proved in the same way as Corollary 5.10. □

As an immediate consequence of Theorem 6.4, we deduce the existence of representations with very mild ramification (e.g. one can take S to be a singleton) and fixed weight.

Corollary 6.8. *Let G be a split simple reductive group over F. Fix $\xi \in \mathrm{Irr}^{\mathrm{reg}}(G(F_\infty))$. There exists a constant $q_0 > 0$ with the following property: for every finite set of places S such that G is unramified away from S, if $q_S > q_0$ then there exists a cuspidal automorphic representation π such that*

- $\pi_\infty \in \Pi_\infty(\xi)$ *(in particular it is a discrete series representation)*,
- π_v *is the Steinberg representation at each $v \in S$,*
- π *is unramified at every finite place $v \notin S$.*

Proof. We take $S_0 = \emptyset$. Theorem 6.4 implies that $\mathcal{F}(\xi, \mathrm{St}_S, \emptyset)$ is nonempty if q_S is sufficiently large since the main term in the right hand side of the theorem is nonzero. Any $\pi \in \mathcal{F}(\xi, \mathrm{St}_S, \emptyset)$ is (not only discrete but) cuspidal since it has a Steinberg component, cf. [51, Prop 4.5.4]. □

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