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The Derived Module of a Homomorphism*

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1. INTRODUCTION

The importance of knot theory as a mathematical discipline is due primarily to its intersection with other branches of mathematics. Notable among these are the subject of infinite discrete non-Abelian groups, topics in the homology of groups and the theory of Noetherian modules, and the study of covering spaces. One construction, which is basic to knot theory and which is discussed at length in Ref. [4], is that of the Alexander matrix. This matrix has led to the definition of a corresponding module, sometimes called the Alexander module, and thence to the link module sequence studied in Refs. [3] and [5].

In spite of references in the literature, neither the connection between the Alexander matrix and the module sequence nor the fundamental geometric significance of these ideas is readily available or familiar to beginning students of knot theory. The purpose of this paper is to provide a good foundation. In this approach the basic definition, that of the derived module of a homomorphism, is simple and elegant. It is computationally useful, as we show in the last section, since it is conceptually very close to the definition by Fox of the Alexander matrix of a group presentation based on his free differential calculus [7]. It leads nicely to the algebraic treatments of group and module sequences in Refs. [2] and [9]. Moreover, it quickly implies the important geometric description in terms of the homology of covering spaces, as discussed, for example, by Milnor in Ref. [11].

2. THE DERIVED MODULE

If G is any multiplicative group (generally non-Abelian), we denote its integral group ring by $Z(G)$. By a left G module we mean a left

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$Z(G)$ module, and a morphism $f: A \to A'$ of left G modules will be called simply a G morphism. If A is any left G module, then a mapping $\partial : G \to A$ is a *crossed homomorphism* if, for all $g_1, g_2 \in G$,

$$\partial(g_1 g_2) = \partial(g_1) + g_1 \partial(g_2).$$

In the remainder of this section we consider an arbitrary homomorphism of multiplicative groups

$$\varphi : G \to H.$$

There exists a unique extension of φ to a ring homomorphism of the integral group rings, which we shall denote by the same letter, $\varphi : Z(G) \to Z(H)$. Every left H module A is also a left G module relative to φ, since, for each $u \in Z(G)$ and $a \in A$, we define

$$ua = \varphi(u)a.$$

A crossed homomorphism $\partial : G \to A$ into a left H module is therefore a mapping which satisfies the equation

$$\partial(g_1 g_2) = \partial(g_1) + \varphi(g_1)\, \partial(g_2) \qquad \text{for all} \quad g_1, g_2 \in G. \tag{1}$$

It follows easily that

(2.1) *If A is a left H module and $\partial : G \to A$ a crossed homomorphism, then*

(i) $\partial(1) = 0$;

(ii) $\partial(g^{-1}) = -\varphi(g)^{-1}\partial(g)$, *for any* $g \in G$.

Proof. We have

$$\partial(1) = \partial(1 \cdot 1) = \partial(1) + \varphi(1)\, \partial(1) = \partial(1) + \partial(1),$$

which implies (i). Hence,

$$0 = \partial(g^{-1}g) = \partial(g^{-1}) + \varphi(g^{-1})\, \partial(g) = \partial(g^{-1}) + \varphi(g)^{-1}\, \partial(g),$$

which yields (ii). ∎

A *derived module* of the group homomorphism $\varphi : G \to H$ consists of a left H module A_φ and a crossed homomorphism $\partial : G \to A_\varphi$ such that, for any left H module A and crossed homomorphism $\partial' : G \to A$, there exists a unique H morphism $\lambda : A_\varphi \to A$ such that $\lambda\partial = \partial'$, i.e., the following diagram is consistent:

$$\begin{tikzcd} G \arrow[r, "\partial"] \arrow[dr, "\partial'"'] & A_\varphi \arrow[d, "\lambda"] \\ & A \end{tikzcd}$$

(2.2) *A derived module A_φ and crossed homomorphism $\partial : G \to A_\varphi$ exist, and A_φ is unique up to H isomorphism.*

Proof. Uniqueness up to isomorphism is obtained by the standard argument. The proof of existence follows MacLane, Ref. [10, p. 120]. Let X be a free left H module having a basis in one-one correspondence with the group G. That is, there exists an injection $i : G \to X$ whose image is a basis for X. Let Y be the submodule of X generated by all elements of the form

$$i(g_1 g_2) - i(g_1) - \varphi(g_1)\, i(g_2) \qquad \text{for all} \quad g_1, g_2 \in G.$$

Set $X/Y = A_\varphi$, and denote the quotient morphism by $\gamma : X \to A_\varphi$. We define

$$\partial : G \to A_\varphi$$

to be the composition $\partial = \gamma i$. For any $g_1, g_2 \in G$,

$$\begin{aligned} \partial(g_1 g_2) &= \gamma i(g_1 g_2) = \gamma[i(g_1) + \varphi(g_1)\, i(g_2)] \\ &= \gamma i(g_1) + \varphi(g_1)\, \gamma i(g_2) = \partial(g_1) + \varphi(g_1)\, \partial(g_2). \end{aligned}$$

Hence, ∂ is a crossed homomorphism.

Next, consider an arbitrary left H module A and crossed homomorphism $\partial' : G \to A$. Since $i(G)$ is a basis for X, there exists a unique H morphism $\mu : X \to A$ such that $\partial' = \mu i$. Moreover,

$$\mu[i(g_1 g_2) - i(g_1) - \varphi(g_1)\, i(g_2)] = \partial'(g_1 g_2) - \partial'(g_1) - \varphi(g_1)\, \partial'(g_2) = 0.$$

Hence, there exists a unique H morphism $\lambda : A_\varphi \to A$ such that $\lambda\gamma = \mu$;

$$\begin{tikzcd}
& & A_\varphi \arrow[dd, "\lambda"] \\
G \arrow[urr, "\partial"] \arrow[r, "i"] \arrow[drr, "\partial'"'] & X \arrow[ur, "\gamma"] \arrow[dr, "\mu"'] & \\
& & A
\end{tikzcd}$$

Clearly,

$$\lambda\partial = \lambda(\gamma i) = (\lambda\gamma)i = \mu i = \partial'.$$

Finally, suppose that $\lambda' : A_\varphi \to A$ is also an H morphism such that $\lambda'\partial = \partial'$. If we set $\mu' = \lambda'\gamma$, then

$$\mu' i = (\lambda'\gamma)i = \lambda'(\gamma i) = \lambda'\partial = \partial'.$$

From the uniqueness of μ it follows that $\mu = \mu'$. Hence, $\lambda'\gamma = \mu' = \mu = \lambda\gamma$, and, since γ is an epimorphism, we conclude that $\lambda = \lambda'$. This completes the proof of (2.2). ▌

Following common practice concerning objects defined by universal properties, we shall henceforth speak of *the* derived module A_φ of a group homomorphism $\varphi : G \to H$. In the special case that H is the commutator quotient group of G and φ is the canonical Abelianizing homomorphism, A_φ will be called the *Alexander module* of the group G.

(2.3) *If G is generated by $\{g_j\}$, then the derived module A_φ is generated by $\{\partial(g_j)\}$.*

Proof. Let A be the submodule of A_φ generated by $\{\partial(g_j)\}$. Since $\{g_j\}$ generates G, it is a consequence of the definition of a crossed homomorphism (1) and of (2.1) that $\partial(g) \in A$ for every $g \in G$. Hence, there exists a mapping $\partial' : G \to A$ defined by $\partial'(g) = \partial(g)$ for all $g \in G$. It is obvious that ∂' is a crossed homomorphism. Thus there exists a unique H morphism $\lambda : A_\varphi \to A$ such that $\lambda\partial = \partial'$. Denote the inclusion monomorphism by $\sigma : A \to A_\varphi$. For every $g \in G$, we have

$$\sigma\lambda\partial(g) = \sigma\partial'(g) = \sigma\partial(g) = \partial(g).$$

It follows by the uniqueness part of the universal property in the definition of A_φ that the composition $\sigma\lambda$ is the identity. Hence, σ is an epimorphism, which implies that $A = A_\varphi$. ▌

The mapping $\epsilon : Z(G) \to Z$ defined by setting $\epsilon(g) = 1$ for all $g \in G$ is a ring homomorphism called the *augmentation mapping*. Its kernel, denoted by $I(G)$, is the *augmentation ideal* of G. The structure of the ring $Z(G)$ as an additive group is extremely simple: $Z(G)$ is a free Abelian group with G a basis. This fact implies that

(2.4) *The augmentation ideal $I(G)$ is a free Z module which has the set $\{g - 1 \mid g \in G \text{ and } g \neq 1\}$ as a basis.*

Proof. Consider an arbitrary $u \in I(G)$. Then

$$u = \sum_{g \in G} n_g g, \qquad n_g \in Z,$$

with $n_g = 0$ except for at most finitely many elements $g \in G$. Set $G_* = G - \{1\}$. Since

$$0 = \epsilon(u) = \sum_{g \in G} n_g ,$$

we have

$$u = \sum_{g \in G} n_g(g - 1) = \sum_{g \in G_*} n_g(g - 1).$$

Thus, $\{g - 1 \mid g \in G \text{ and } g \neq 1\}$ is a set of generators. Suppose that

$$0 = \sum_{g \in G_*} n_g(g - 1).$$

Then

$$0 = \sum_{g \in G_*} n_g g - \sum_{g \in G_*} n_g .$$

Since $Z(G)$ is a free Z module with G a basis, it follows that $n_g = 0$ for all $g \in G_*$. This completes the proof. ∎

The ideal $I(G)$ is, of course, also a G module. We define the mapping $\kappa : G \to I(G)$ by setting $\kappa(g) = g - 1$ for all $g \in G$.

(2.5) *The mapping κ is a crossed homomorphism, and, for any left G module A and crossed homomorphism $\partial : G \to A$, there exists a unique G morphism $\mu : I(G) \to A$ such that $\mu\kappa = \partial$.*

Proof. For any $g_1, g_2 \in G$, we have

$$\begin{aligned}\kappa(g_1g_2) &= g_1g_2 - 1 = g_1 - 1 + g_1(g_2 - 1)\\ &= \kappa(g_1) + g_1\kappa(g_2).\end{aligned}$$

Thus, κ is a crossed homomorphism. Using (2.4), we define a Z morphism $\mu : I(G) \to A$ by setting

$$\mu(g - 1) = \partial(g) \qquad \text{for all} \quad g \in G - \{1\}$$

Observe that this equation also holds for $g = 1$, see (2.1). To verify that μ is also a G morphism, it suffices to check that $\mu(g_1(g_2 - 1)) = g_1\mu(g_2 - 1)$ for all $g_1, g_2 \in G$. We have

$$\begin{aligned}
\mu(g_1(g_2 - 1)) &= \mu(g_1 g_2 - 1 - (g_1 - 1)) \\
&= \mu(g_1 g_2 - 1) - \mu(g_1 - 1) \\
&= \partial(g_1 g_2) - \partial(g_1) \\
&= \partial(g_1) + g_1\partial(g_2) - \partial(g_1) = g_1\partial(g_2) \\
&= g_1\mu(g_2 - 1).
\end{aligned}$$

Since $\{g - 1 \mid g \in G\}$ generates $I(G)$, the uniqueness of μ follows immediately. This completes the proof. ▮

An equivalent formulation of (2.5) is the statement that $I(G)$ is the derived module of the identity homomorphism $1_G : G \to G$ and $\kappa : G \to I(G)$ is the accompanying crossed homomorphism. This fact also follows from the module sequence developed in Section 4.

3. Presentation of the Derived Module

Let $\varphi : G \to H$ be a homomorphism of multiplicative groups. Consider a group presentation

$$G = (\mathbf{x} : \mathbf{r})_\zeta, \tag{2}$$

and let M be the Jacobian matrix of the presentation at φ, as defined by Fox in Ref. [7] and also in Ref. [8, p. 125]. In this section we shall show that M is a relation matrix for the derived module A_φ. The discussion provides, incidentally, an alternative construction of the derived module. If we take the special case in which H is the commutator quotient group of G and φ is the canonical Abelianizing homomorphism, then M is the Alexander matrix of (2), and our conclusion is that the Alexander matrix is a relation matrix for the Alexander module of G. Although this material is basic to knot theory, especially for purposes of computation, it is not required for an understanding of Sections 4 and 5 of the paper.

We shall assume familiarity with group presentations and the free differential calculus of Fox. Since the notion of a presentation of a module may be less common, we give a résumé. Let R be a commutative ring with 1. A *presentation* of an R module A consists of an exact sequence of R morphisms

$$X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} A \longrightarrow 0$$

in which X_1 and X_2 are free R modules with distinguished bases (b_j) and (c_i), respectively. The matrix (m_{ij}) of elements of R defined by

$$d_2(c_i) = \sum_j m_{ij} b_j$$

for each c_i, is the *matrix of the presentation*, and is also called a *relation matrix* for A.

Returning to the group presentation (2), we denote by F the free group with basis $\mathbf{x} = (x_j)$. Then $\zeta : F \to G$ is an epimorphism whose kernel is the normal subgroup of F generated by $\mathbf{r} = (r_i)$, i.e., the kernel of ζ is the consequence of $\mathbf{r}$. The Jacobian matrix at φ of the presentation is the matrix $M = (m_{ij})$ of elements of the integral group ring $Z(H)$ defined by

$$m_{ij} = \varphi\zeta\left(\frac{\partial r_i}{\partial x_j}\right). \tag{3}$$

Thus, there is one row for each relator $r_i \in \mathbf{r}$, and one column for each generator $x_j \in \mathbf{x}$. Let X_1 be a free H module with basis (b_j) in one-one correspondence with $\mathbf{x} = (x_j)$, and X_2 a free H module with basis (c_i) in one-one correspondence with $\mathbf{r} = (r_i)$, i.e., we have $b_j \leftrightarrow x_j$ and $c_i \leftrightarrow r_i$. The H morphism $d_2 : X_2 \to X_1$ is defined by

$$d_2(c_i) = \sum_j m_{ij} b_j .$$

Denote the quotient module $X_1/\text{image}(d_2)$ by A_0, and the canonical quotient epimorphism by $d_1 : X_1 \to A_0$. Hence, we have constructed the following exact sequence of H-morphisms:

$$X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} A_0 \longrightarrow 0. \tag{4}$$

We shall show that A_0 is the derived module A_φ.

Consider the mapping $\delta : F \to X_1$ defined by setting

$$\delta(u) = \sum_j \varphi\zeta\left(\frac{\partial u}{\partial x_j}\right) b_j \qquad \text{for every} \quad u \in F.$$

(3.1) *δ is a crossed homomorphism and $\delta(x_j) = b_j$ for every $x_j \in \mathbf{x}$.*

Proof. The properties of the free differential calculus [4, pp. 96, 98] imply at once that $\delta(x_j) = b_j$ and also that

$$\frac{\partial(u_1 u_2)}{\partial x_j} = \frac{\partial u_1}{\partial x_j} + u_1 \frac{\partial u_2}{\partial x_j} \qquad \text{for all} \quad u_1, u_2 \in F.$$

Hence,

$$\begin{aligned}
\delta(u_1 u_2) &= \sum_j \varphi\zeta\left(\frac{\partial u_1 u_2}{\partial x_j}\right) b_j \\
&= \sum_j \varphi\zeta\left(\frac{\partial u_1}{\partial x_j}\right) b_j + \varphi\zeta(u_1) \sum_j \varphi\zeta\left(\frac{\partial u_2}{\partial x_j}\right) b_j \\
&= \delta(u_1) + \varphi\zeta(u_1)\, \delta(u_2). \quad \blacksquare
\end{aligned}$$

It follows immediately that the composition $d_1\delta$ is also a crossed homomorphism.

(3.2) *If $\zeta(u) = 1$, then $d_1\delta(u) = 0$.*

Proof. Consider an arbitrary relator $r_i \in \mathbf{r}$.

$$\delta(r_i) = \sum_j \varphi\zeta\left(\frac{\partial r_i}{\partial x_j}\right) b_j = \sum_j m_{ij} b_j = d_2(c_i).$$

Thus, $d_1\delta(r_i) = d_1 d_2(c_i) = 0$. Next, consider a conjugate $u r_i u^{-1}$, where $u \in F$. It is a consequence of the preceding, as well as of (1) and (2.1), that

$$\begin{aligned}
\delta(u r_i u^{-1}) &= \delta(u) + \varphi\zeta(u)\, \delta(r_i) - \varphi\zeta(u r_i u^{-1})\, \delta(u) \\
&= \delta(u) + \varphi\zeta(u)\, \delta(r_i) - \delta(u) \\
&= \varphi\zeta(u)\, \delta(r_i).
\end{aligned}$$

Hence, $d_1\delta(u r_i u^{-1}) = \varphi\zeta(u)\, d_1\delta(r_i) = 0$. Finally, it is clear that any product of elements $u r_i u^{-1}$ and their inverses will also be mapped onto 0 by $d_1\delta$. This completes the proof. $\blacksquare$

Suppose that $\zeta(u_1) = \zeta(u_2)$. Then $\zeta(u_1 u_2^{-1}) = 1$. From (3.2) and the fact that $d_1\delta$ is a crossed homomorphism, we obtain

$$\begin{aligned} 0 = d_1\delta(u_1u_2^{-1}) &= d_1\delta(u_1) - \varphi\zeta(u_1u_2^{-1})\, d_1\delta(u_2) \\ &= d_1\delta(u_1) - d_1\delta(u_2). \end{aligned}$$

Thus, $d_1\delta(u_1) = d_1\delta(u_2)$. We conclude that a mapping $\partial : G \to A_0$ is well-defined by the equation

$$\partial\zeta(u) = d_1\delta(u) \qquad \text{for every} \quad u \in F.$$

(3.3) *∂ is a crossed homomorphism.*

Proof. Consider $g_1, g_2 \in G$, and choose $u_1, u_2 \in F$ such that $\zeta(u_1) = g_1$ and $\zeta(u_2) = g_2$. By the definition of ∂ and the fact that $d_1\delta$ is a crossed homomorphism, we have

$$\begin{aligned} \partial(g_1g_2) = \partial\zeta(u_1u_2) &= d_1\delta(u_1u_2) \\ &= d_1\delta(u_1) + \varphi\zeta(u_1)\, d_1\delta(u_2) \\ &= \partial\zeta(u_1) + \varphi\zeta(u_1)\, \partial\zeta(u_2) \\ &= \partial(g_1) + \varphi(g_1)\, \partial(g_2). \quad \blacksquare \end{aligned}$$

Thus, we have constructed the following consistent diagram in which the bottom row is the exact sequence (4) and in which the vertical mappings are crossed homomorphisms:

$$\begin{tikzcd} & F \arrow[r, "\zeta"] \arrow[d, "\delta"] & G \arrow[d, "\partial"] & \\ X_2 \arrow[r, "d_2"] & X_1 \arrow[r, "d_1"] & A_0 \arrow[r] & 0 \end{tikzcd} \qquad d_1(b_j) = \partial\zeta(x_j) \tag{5}$$

(3.4) *The module A_0 is the derived module A_φ, and ∂ is the accompanying crossed homomorphism.*

Proof. Let A be an arbitrary left H module and $\partial' : G \to A$ a crossed homomorphism. We define the H morphism $\rho : X_1 \to A$ by setting $\rho(b_j) = \partial'\zeta(x_j)$. The kernel of d_1 is generated by the elements $d_2(c_i)$.

Thus, to show that kernel$(d_1) \subset$ kernel(ρ), it suffices to prove that $\rho d_2(c_i) = 0$. To begin with, we have

$$\rho d_2(c_i) = \rho \left(\sum_j m_{ij} b_j\right) = \sum_j m_{ij} \partial' \zeta(x_j).$$

The H module A is also a left G module relative to the homomorphism $\varphi : G \to H$. Hence, by (2.5), there exists a G morphism μ such that $\mu\kappa = \partial'$. Recalling the definition of the action of $Z(G)$ on A, we obtain

$$\begin{aligned} \rho d_2(c_i) &= \sum_j m_{ij} \mu\kappa\zeta(x_j) = \sum_j \varphi\zeta\left(\frac{\partial r_i}{\partial x_j}\right) \mu(\zeta(x_j) - 1) \\ &= \mu\left(\sum_j \zeta\left(\frac{\partial r_i}{\partial x_j}\right)(\zeta(x_j) - 1)\right) \\ &= \mu\zeta\left(\sum_j \frac{\partial r_i}{\partial x_j}(x_j - 1)\right). \end{aligned}$$

According to the fundamental formula of the free differential calculus [4, p. 100],

$$r_i - 1 = \sum_j \frac{\partial r_i}{\partial x_j}(x_j - 1).$$

Hence,

$$\rho d_2(c_i) = \mu\zeta(r_i - 1) = \mu(0) = 0,$$

and it follows that kernel$(d_1) \subset$ kernel(ρ). This implies that there exist an H morphism $\lambda : A_0 \to A$ such that $\lambda d_1 = \rho$. In order to verify that $\partial' = \lambda\partial$, it is sufficient to check the equation on the generators $\zeta(x_j)$. We obtain

$$\lambda\partial\zeta(x_j) = \lambda d_1(b_j) = \rho(b_j) = \partial'\zeta(x_j).$$

Finally, the uniqueness of λ follows from the fact that, since A_0 is generated by the elements $d_1(b_j) = \partial\zeta(x_j)$, it is generated by image(∂). This completes the proof. ▮

It is a direct consequence of Theorem (3.4) that the Jacobian matrix at φ of the group presentation (2) is a relation matrix for the derived $A_\varphi = A_0$.

An an application of these ideas, let us take for φ the identity homomorphism $1_G : G \to G$. In this case, as noted at the end of the preceding section, the augmentation ideal $I(G)$ is the derived module, and the accompanying crossed homomorphism is the mapping $\kappa : G \to I(G)$ defined by $\kappa(g) = g - 1$. Hence, the sequence (4) becomes the exact sequence of G morphisms

$$X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} I(G) \longrightarrow 0, \tag{6}$$

where

$$d_2(c_i) = \sum_j \zeta\left(\frac{\partial r_i}{\partial x_j}\right) b_j ,$$

$$d_1(b_j) = \zeta(x_j) - 1.$$

The final morphism $I(G) \to 0$ in (6) may be replaced by the sequence $Z(G) \xrightarrow{\epsilon} Z \to 0$, where ϵ is the augmentation mapping. The result is the exact sequence

$$X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} Z(G) \xrightarrow{\epsilon} Z \longrightarrow 0,$$

which is an extremely useful free (and hence projective) G resolution of Z. Alternatively, we may consider the homomorphism φ again and form the tensor product of (6) with $Z(H) \otimes_G$ on the left. The exact sequence thus obtained is equivalent to (4), and it follows that

$$A_\varphi = Z(H) \otimes_G I(G).$$

Actually, this result together with the fact that $\partial(g) = 1 \otimes (g - 1)$, for all $g \in G$, is more easily proved directly from the universal definition of A_φ. The tensor product form of the derived module has been studied in both Refs. [2] and [9].

4. The Module Sequence

Consider an arbitrary short exact sequence of homomorphisms of multiplicative groups

$$1 \longrightarrow K \xrightarrow{\theta} G \xrightarrow{\varphi} H \longrightarrow 1. \tag{7}$$

In this section we shall construct from (7) an exact sequence of morphisms of left H modules

$$0 \longrightarrow B \xrightarrow{\theta_*} A_\varphi \xrightarrow{\varphi_*} I(H) \longrightarrow 0 \tag{8}$$

called the *module sequence* of (7). The middle term A_φ is the derived module of φ, and $I(H)$ is the augmentation ideal of H. This sequence is equivalent to the one described in Ref. [2] and in Ref. [10, p. 120]. The principal application of this construction in knot theory occurs where $G = \pi_1(S^3 - L)$ is the group of a link, K is the commutator subgroup of G, and H is the commutator quotient group. In this case (8) is the link module sequence studied in Refs. [3] and [5], and A_φ is the Alexander module.

The unique extension of $\varphi : G \to H$ to a ring homomorphism of the respective integral group rings has been denoted also by φ. Where both augmentation mappings are denoted by ϵ, it is clear that the following diagram is consistent:

$$\begin{tikzcd} Z(G) \arrow[rr, "\varphi"] \arrow[dr, "\epsilon"'] & & Z(H) \arrow[dl, "\epsilon"] \\ & Z & \end{tikzcd}$$

Hence $\varphi(I(G)) \subset I(H)$, and we use the letter φ again for the obvious mapping $\varphi : I(G) \to I(H)$. We proved in (2.5) that the mapping $\kappa : G \to I(G)$ defined by $\kappa(g) = g - 1$, for all $g \in G$, is a crossed homomorphism. It follows immediately that the composition $\varphi\kappa$ is also a crossed homomorphism. By the universal property of the derived module, there exists a unique H morphism $\varphi_* : A_\varphi \to I(H)$ such that $\varphi_*\partial = \varphi\kappa$. Thus, the following diagram is consistent:

$$\begin{tikzcd} G \arrow[r, "\kappa"] \arrow[d, "\partial"'] & I(G) \arrow[d, "\varphi"] \\ A_\varphi \arrow[r, "\varphi_*"] & I(H) \end{tikzcd}$$

(4.1) *φ_* is an epimorphism.*

Proof. Consider an arbitrary element $u \in I(H)$. As a result of (2.4) we can write

$$u = \sum_{h \in H} n_h(h - 1), \qquad n_h \in Z,$$

with $n_h = 0$ except for at most a finite number of elements $h \in H$. Since $\varphi : G \to H$ is surjective, there exists a mapping $r : H \to G$ such that $\varphi r = 1_H$ (identity). Let

$$v = \sum_{h \in H} n_h \partial r(h).$$

Then

$$\varphi_*(v) = \sum_{h\in H} n_h \varphi_* \partial r(h) = \sum_{h\in H} n_h \varphi(r(h) - 1)$$

$$= \sum_{h\in H} n_h (h - 1) = u.$$

This completes the proof. ▮

The module B will be defined by first considering its structure as an additive Abelian group. Let K' be the commutator subgroup of K. The group B is then simply the quotient group K/K' written additively. Let $\alpha : K \to B$ be the canonical Abelianizing homomorphism. Then

(4.2) *There exists a unique group homomorphism* $\theta_* : B \to A_\varphi$ *such that* $\theta_*\alpha = \partial\theta$, *i.e., such that the following diagram is consistent*:

$$\begin{array}{ccc} K & \xrightarrow{\theta} & G \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\partial} \\ B & \xrightarrow{\theta_*} & A_\varphi \end{array}$$

Proof. Consider arbitrary elements $k_1, k_2 \in K$. Then

$$\partial\theta(k_1k_2) = \partial(\theta(k_1)\,\theta(k_2)) = \partial\theta(k_1) + \varphi\theta(k_1)\,\partial\theta(k_2)$$

$$= \partial\theta(k_1) + \partial\theta(k_2).$$

Thus, the composition $\partial\theta : K \to A_\varphi$ is a group homomorphism (multiplicative-to-additive). To complete the proof, one must therefore show that $\partial\theta(K') = 0$. Since K' is generated by the set of all commutators of elements of K, it suffices to show that $\partial\theta([k_1, k_2]) = 0$, for any $k_1, k_2 \in K$. Since $\partial\theta$ is a homomorphism, we obtain

$$\partial\theta([k_1, k_2]) = \partial\theta(k_1k_2k_1^{-1}k_2^{-1})$$

$$= \partial\theta(k_1) + \partial\theta(k_2) - \partial\theta(k_1) - \partial\theta(k_2) = 0. \quad ▮$$

To define the module structure on B, it is sufficient to describe the action of H. Specifically, let $f' : G \to \text{aut}(K)$ be the homomorphism defined by

$$\theta(f'(g)(k)) = g\theta(k)\,g^{-1}$$

for all $g \in G$ and $k \in K$. The Abelianizing mapping α induces a homomorphism $\Phi : \mathrm{aut}(K) \to \mathrm{aut}(B)$ in a natural way: For every $\xi \in \mathrm{aut}(K)$, there exists a unique $\Phi(\xi) \in \mathrm{aut}(B)$ such that the diagram

$$\begin{tikzcd} K \arrow[r, "\xi"] \arrow[d, "\alpha"'] & K \arrow[d, "\alpha"] \\ B \arrow[r, "\Phi(\xi)"] & B \end{tikzcd}$$

is consistent. It is straightforward to check that $\Phi f'\theta(K) = 1_B$ (identity). As a result, there exists a unique homomorphism f such that the following diagram is consistent:

$$\begin{tikzcd} G \arrow[r, "f'"] \arrow[d, "\varphi"'] & \mathrm{aut}(K) \arrow[d, "\Phi"] \\ H \arrow[r, "f"] & \mathrm{aut}(B) \end{tikzcd}$$

Thus, B is a left H module with the action of H defined by $hb = f(h)(b)$ for all $h \in H$ and $b \in B$. To compute hb, choose $g \in G$ and $k \in K$ such that $\varphi(g) = h$ and $\alpha(k) = b$. Then

$$g\theta(k)\, g^{-1} = \theta(k_1)$$

for some $k_1 \in K$, and

$$hb = \alpha(k_1).$$

(4.3) *The mapping* $\theta_* : B \to A_\varphi$ *is an* H *morphism.*

Proof. Using the preceding two equations and the definition of θ_*, we have

$$\theta_*(hb) = \theta_*\alpha(k_1) = \partial\theta(k_1) = \partial(g\theta(k)\, g^{-1}).$$

From Eq. (1) and also (2.1), we obtain

$$\begin{aligned} \partial(g\theta(k)\, g^{-1}) &= \partial(g) + \varphi(g)\, \partial\theta(k) - \varphi(g\theta(k)\, g^{-1})\, \partial(g) \\ &= \partial(g) + \varphi(g)\, \partial\theta(k) - \partial(g) \\ &= \varphi(g)\, \partial\theta(k) = h\theta_*\alpha(k) = h\theta_*(b). \end{aligned}$$

Hence, $\theta_*(hb) = h\theta_*(b)$, and the proof is complete. ∎

The module sequence (8) has now been constructed. It remains, of course, to prove that it is exact. We have shown that φ_* is an epimorphism. Moreover,

(4.4) $\varphi_*\theta_* = 0.$

Proof. Consider an arbitrary element $b \in B$, and choose $k \in K$ such that $\alpha(k) = b$. Then

$$\begin{aligned}\varphi_*\theta_*(b) &= \varphi_*\theta_*\alpha(k) = \varphi_*\partial\theta(k)\\ &= \varphi\kappa\theta(k) = \varphi(\theta(k) - 1) = 0. \quad \blacksquare\end{aligned}$$

The rest of the proof of exactness is motivated by the outline in Ref. [10, p. 120]. As in the proof of (4.1), consider a mapping $r : H \to G$ such that $\varphi r = 1_H$. In addition, we define $r(1) = 1$. A convenient notation, which we shall adopt, is to write $r(h) = [h]$ for every $h \in H$. Observe that

$$\varphi([h]\, g[h\varphi(g)]^{-1}) = h\varphi(g)(h\varphi(g))^{-1} = 1,$$

for all $h \in H$ and $g \in G$. Hence, since the original sequence (7) of group homomorphisms is exact, there exists in K an element, which we denote by $h \times g$, uniquely determined by the equation

$$\theta(h \times g) = [h]\, g[h\varphi(g)]^{-1}.$$

A useful lemma is

(4.5) $h \times g_1g_2 = (h \times g_1)(h\varphi(g_1) \times g_2).$

Proof. It suffices to show that both sides of the equation have the same image under the monomorphism θ,

$$\begin{aligned}\theta((h \times g_1)(h\varphi(g_1) \times g_2)) &= [h]\, g_1[h\varphi(g_1)]^{-1}[h\varphi(g_1)]\, g_2[h\varphi(g_1)\, \varphi(g_2)]^{-1}\\ &= [h]\, g_1g_2[h\varphi(g_1g_2)]^{-1} = \theta(h \times g_1g_2). \quad \blacksquare\end{aligned}$$

We next recall the construction in (2.2) of the derived module A_φ as the quotient of the free H module X with basis $i(G)$, in which the crossed homomorphism ∂ is the composition of the mappings

$$G \xrightarrow{i} X \xrightarrow{\gamma} A_\varphi .$$

As an additive Abelian group, X is free with a basis equal to the set of all elements $hi(g)$ such that $h \in H$ and $g \in G$. Hence, where $\alpha : K \to B$

is the Abelianizing homomorphism, we may define the Z morphism $\tau : X \to B$ by setting

$$\tau(hi(g)) = \alpha(h \times g).$$

(4.6) *There exists a unique Z morphism $\eta : A_\varphi \to B$ such that $\eta\gamma = \tau$.*

Proof. The kernel of γ, regarded as a Z module, is generated by all elements

$$hi(g_1 g_2) - hi(g_1) - h\varphi(g_1)\, i(g_2)$$

for any $h \in H$ and $g_1, g_2 \in G$. Hence, it suffices to show that

$$\tau(hi(g_1 g_2)) = \tau(hi(g_1)) + \tau(h\varphi(g_1)\, i(g_2)).$$

Using (4.5), the definition of τ, and recalling the multiplicative-to-additive property of α, we obtain

$$\begin{aligned}\tau(hi(g_1 g_2)) &= \alpha(h \times g_1 g_2) = \alpha(h \times g_1) + \alpha(h\varphi(g_1) \times g_2)\\ &= \tau(hi(g_1)) + \tau(h\varphi(g_1)\, i(g_2)). \quad \blacksquare\end{aligned}$$

(4.7) *θ_* is a monomorphism, since $\eta\theta_* = 1_B$.*

Proof. Consider an arbitrary $b \in B$ and element $k \in K$ such that $\alpha(k) = b$. Then

$$\begin{aligned}\eta\theta_*(b) &= \eta\theta_*\alpha(k) = \eta\partial\theta(k) = \eta\gamma i\theta(k)\\ &= \tau i\theta(k) = \alpha(1 \times \theta(k)).\end{aligned}$$

Since $[1] = 1$, the defining equation of $1 \times \theta(k)$ yields

$$\begin{aligned}\theta(1 \times \theta(k)) &= [1]\, \theta(k)[1\varphi\theta(k)]^{-1}\\ &= [1]\, \theta(k)[1]^{-1} = \theta(k).\end{aligned}$$

Hence, $1 \times \theta(k) = k$, and so

$$\eta\theta_*(b) = \alpha(k) = b. \quad \blacksquare$$

It follows from (2.4) that the ideal $I(H)$ is a free Z module which has the set $\{h - 1 \mid h \in H \text{ and } h \neq 1\}$ as a basis. Hence, a Z morphism $\sigma : I(H) \to A_\varphi$ is defined by setting

$$\sigma(h - 1) = \partial([h]) \qquad \text{for every} \quad h \in H - \{1\}.$$

Notice that this equation also holds if $h = 1$, since $[1] = 1$ and $\partial(1) = 0$. The final lemma in the proof of exactness is the identity

(4.8) $\quad 1_{A_\varphi} = \theta_*\eta + \sigma\varphi_*$.

Proof. It is a consequence of (2.3) that, as a Z module, the derived module A_φ is generated by the set of all elements $h\partial(g)$, where $h \in H$ and $g \in G$. It suffices therefore to verify the identity on these generators. We have

$$\begin{aligned}
\theta_*\eta(h\partial(g)) &= \theta_*\eta(h\gamma i(g)) = \theta_*\eta\gamma(hi(g)) \\
&= \theta_*\tau(hi(g)) = \theta_*\alpha(h \times g) \\
&= \partial\theta(h \times g) = \partial([h]\, g[h\varphi(g)]^{-1}) \\
&= \partial([h]) + h\partial(g) - \partial([h\varphi(g)]) \\
&= h\partial(g) + \sigma(h - 1) - \sigma(h\varphi(g) - 1).
\end{aligned}$$

However,

$$\begin{aligned}
\sigma\varphi_*(h\partial(g)) &= \sigma(h\varphi_*\partial(g)) = \sigma(h\varphi\kappa(g)) \\
&= \sigma(h(\varphi(g) - 1)).
\end{aligned}$$

Since $h(\varphi(g) - 1) = (h\varphi(g) - 1) - (h - 1)$, we get

$$\sigma\varphi_*(h\partial(g)) = \sigma(h\varphi(g) - 1) - \sigma(h - 1).$$

Combining the results, we conclude that

$$\theta_*\eta(h\partial(g)) = h\partial(g) - \sigma\varphi_*(h\partial(g)),$$

or equivalently,

$$h\partial(g) = \theta_*\eta(h\partial(g)) + \sigma\varphi_*(h\partial(g)).$$

This completes the proof. ▮

The exactness of the module sequence (8) is now established. The only detail not yet proved explicitly is the inequality

$$\text{kernel}(\varphi_*) \subset \text{image}(\theta_*),$$

and this is a direct corollary of (4.8).

It is easy to prove in addition that $\varphi_*\sigma = 1_{I(H)}$ and that $\eta\sigma = 0$, since both equations are readily seen to hold for the generators $h - 1$ of $I(H)$. Thus, we have

$$\varphi_*\theta_* = 0, \qquad \eta\theta_* = 1_B,$$
$$\eta\sigma = 0, \qquad \varphi_*\sigma = 1_{I(H)},$$
$$1_{A_\varphi} = \theta_*\eta + \sigma\varphi_*.$$

It follows, see Ref. [12, p. 11], that these mappings constitute a complete representation, as Z modules, of A_φ as the direct sum of B and $I(H)$. However, it is essential to realize that the exact $Z(H)$ module sequence

$$0 \longrightarrow B \xrightarrow{\theta_*} A_\varphi \xrightarrow{\varphi_*} I(H) \longrightarrow 0$$

is generally *not* split exact. For example, it is shown in Ref. [3] that if this sequence is the link module sequence of an m-component link L, then it splits if and only if either $m = 1$ or $m = 2$ and the linking number of the two components is ± 1.

5. Covering Spaces

We shall now show that the module sequence developed in the last section is part of the homology sequence of a pair (X, F) consisting of a covering space and its fiber. This fact provides the basic geometric interpretation of the sequence. Moreover, in this context formula (1) for the crossed homomorphism into the derived module appears as an application of the path-lifting property of covering spaces to a product of elements of the fundamental group of the base space. Consider the short exact sequence of group homomorphisms

$$1 \longrightarrow K \xrightarrow{\theta} G \xrightarrow{\varphi} H \longrightarrow 1.$$

The corresponding exact sequence of morphisms of left H modules may be written as in (8), or equivalently, as

$$0 \longrightarrow B \xrightarrow{\theta_*} A_\varphi \xrightarrow{\varphi_*} Z(H) \xrightarrow{\epsilon} Z \longrightarrow 0. \tag{9}$$

It is this longer form which we consider here.

Let X be a connected and locally pathwise connected (lpc) covering space relative to a covering map $p : X \rightarrow B$, and suppose that there exist points $b_0 \in B$ and $x_0 \in p^{-1}(b_0)$ such that

$$G = \pi_1(B, b_0),$$
$$K = \pi_1(X, x_0),$$

and $\theta : K \rightarrow G$ is the monomorphism induced by p. We denote the fiber over b_0 by F, i.e., $F = p^{-1}(b_0)$. There is a well-known right action of G on F (the definition of which is reviewed in the proof of (5.2) below). The result of the action of a group element g on a point x of the fiber will be denoted by xg, and we have the characteristic properties

$$\left.\begin{aligned} x1 &= x, \\ x(g_1 g_2) &= (xg_1)\, g_2\,, \end{aligned}\right\} \qquad \text{for all} \quad x \in F \qquad \text{and} \qquad g_1\,, g_2 \in G.$$

The group H is canonically isomorphic to the group T of covering translations. It is convenient in this section to assume that $H = T$ and that $\varphi : G \rightarrow H$ is the epimorphism defined by the equation

$$\varphi(g)(x_0) = x_0 g \qquad \text{for all} \quad g \in G.$$

Using singular homology with integer coefficients, consider the following part of the homology sequence of the pair (X, F):

$$H_1(F) \rightarrow H_1(X) \rightarrow H_1(X, F) \rightarrow H_0(F) \rightarrow H_0(X) \rightarrow H_0(X, F).$$

Since X is connected and lpc, it is also pathwise connected. The fiber F is nonempty and we therefore conclude that $H_0(X, F) = 0$. From the well-known fact that F is a discrete subset of X it follows that $H_1(F) = 0$. Thus, we obtain the exact homology sequence

$$0 \longrightarrow H_1(X) \xrightarrow{j_*} H_1(X, F) \xrightarrow{\beta} H_0(F) \xrightarrow{i_*} H_0(X) \longrightarrow 0. \qquad (10)$$

The connecting homomorphism has been denoted by β because the commonly used symbol ∂ is reserved for crossed homomorphisms in this paper.

Each element $h \in H$ is a homeomorphism $h : X \rightarrow X$ such that $ph = p$. Since $h(F) = F$, there are induced homeomorphisms $h : F \rightarrow F$ and

$h : (X, F) \to (X, F)$. From the Eilenberg–Steenrod axioms we obtain the consistent diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(X) & \longrightarrow & H_1(X,F) & \longrightarrow & H_0(F) & \longrightarrow & H_0(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle h_*} & & \\ 0 & \longrightarrow & H_1(X) & \longrightarrow & H_1(X,F) & \longrightarrow & H_0(F) & \longrightarrow & H_0(X) & \longrightarrow & 0 \end{array} \tag{11}$$

Let A be any one of the homology groups in the sequence (10). There exists a mapping $H \to \operatorname{aut}(A)$ defined by $h \to h_*$. Since $(h_1h_2)_* = h_{1*}h_{2*}$ for all $h_1, h_2 \in H$, this mapping is a homomorphism. Hence, A is a left H module with the action of H defined by

$$ha = h_*(a) \qquad \text{for all} \quad h \in H \quad \text{and} \quad a \in A.$$

With this definition of the module structure, the fact that the diagram (11) is consistent is equivalent to the statement that the additive group homomorphisms represented by the horizontal arrows are H morphisms. Thus, we have shown that

(5.1) *The homology sequence* (10) *is an exact sequence of morphisms of left H modules.* ▌

The remainder of the section is devoted to the construction of an isomorphism ψ from the module sequence (9) onto the homology sequence (10). We shall adopt the notation of systematically denoting the homology class containing a give cycle c by $[c]$. We also propose to work from right to left, thus disposing of the simpler and less interesting end of the sequences first.

Let the infinite cyclic group $H_0(X)$ be generated by the homology class $[x_0]$ containing the point x_0. For every $h \in H$, we have $h(x_0) \sim x_0$, and so the induced isomorphism $h_* : H_0(X) \to H_0(X)$ is the identity. Thus, $H_0(X)$ is a trivial H module, and there exists an isomorphism $\psi_0 : Z \to H_0(X)$ with $\psi_0(1) = [x_0]$.

The group $H_0(F)$ is the direct sum $\bigoplus_{x \in F} H_0(x)$, in which each group $H_0(x)$ is infinite cyclic and generated by $[x]$. There exists a one-one correspondence $H \to F$ defined by $h \to h(x_0)$. Hence, there exists a Z isomorphism $\psi_1 : Z(H) \to H_0(F)$ defined by $\psi_1(h) = [h(x_0)]$ for every $h \in H$. Since

$$\begin{aligned} \psi_1(h_1h_2) &= [h_1h_2(x_0)] = h_{1*}([h_2(x_0)]) \\ &= h_1\psi_1(h_2), \end{aligned}$$

it follows that ψ_1 is an H isomorphism. The morphism i_*: $H_0(F) \to H_0(X)$ maps each class $[x]$ onto $[x_0]$, and we therefore obtain the consistent diagram of H morphisms

$$\begin{array}{ccccc} Z(H) & \xrightarrow{\epsilon} & Z & \longrightarrow & 0 \\ \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle\psi_0} & & \\ H_0(F) & \xrightarrow{i_*} & H_0(X) & \longrightarrow & 0 \end{array}$$

The mapping $\partial' : G \to H_1(X, F)$ is defined as follows: Let $g \in G$ be an arbitrary element and select a representative b_0-based loop $a \in g$. By the path-lifting property of covering spaces, there exists a unique path $a' : I \to X$ such that $pa' = a$ and $a'(0) = x_0$. Obviously a' is a relative 1-cycle, and its homology class in $H_1(X, F)$ is denoted by $[a']$. We define

$$\partial'(g) = [a']. \tag{12}$$

Since covering spaces have the covering homotopy property, it follows at once that the mapping $\partial' : G \to H_1(X, F)$ is well-defined by (12).

(5.2) *∂' is a crossed homomorphism.*

Proof. Consider arbitrary elements $g_1, g_2 \in G$ and representative b_0-based loops $a_1 \in g_1$ and $a_2 \in g_2$. Let $a_1', a_2' : I \to X$ be the unique paths such that $pa_i' = a_i$ and $a_i'(0) = x_0$ for $i = 1$ and 2. The definition of the right action of the group G on the fiber F yields

$$x_0 g_1 = a_1'(1).$$

The covering translation $\varphi(g_1)$ is defined by the equation $\varphi(g_1)(x_0) = x_0 g_1$, and so we have $\varphi(g_1)(x_0) = a_1'(1)$. Thus, the composition $\varphi(g_1)\, a_2'$ is a path which covers a_2 and has initial point

$$\varphi(g_1)\, a_2'(0) = \varphi(g_1)(x_0) = a_1'(1).$$

Hence, the product path $a_1'(\varphi(g_1)a_2')$ is defined. It covers the product $a_1 a_2$ and has initial point x_0. It therefore follows from the definition of ∂' in (12) that $\partial'(g_1 g_2) = [a_1'(\varphi(g_1)\, a_2')]$. Since the product of two paths is homologous to their sum, we obtain

$$\partial'(g_1 g_2) = [a_1'(\varphi(g_1)\, a_2')] = [a_1'] + [\varphi(g_1)\, a_2'].$$

The definition of the action of H on the homology groups then implies that

$$\begin{aligned}\partial'(g_1g_2) &= [a_1'] + \varphi(g_1)[a_2'] \\ &= \partial'(g_1) + \varphi(g_1)\,\partial'(g_2),\end{aligned}$$

and the proof is complete. ∎

It is a corollary of (5.2) and the universal property of the derived module A_φ that there exists a unique H morphism $\psi_2 : A_\varphi \to H_1(X, F)$ such that the following diagram is consistent:

$$\begin{tikzcd} G \arrow[r, "\partial"] \arrow[d, "\partial'"'] & A_\varphi \arrow[dl, "\psi_2"] \\ H_1(X,F) & \end{tikzcd}$$

(5.3) $\psi_1\varphi_* = \beta\psi_2$.

Proof. The derived module A_φ is generated by the set of all elements $\partial(g)$ such that $g \in G$, see (2.2). Hence, it suffices to verify the above equation on an arbitrary such generator. Recalling the fact that $\varphi_*\partial = \varphi\kappa$, as indicated in the mapping diagram which precedes (4.1), we have

$$\begin{aligned}\psi_1\varphi_*\partial(g) &= \psi_1\varphi\kappa(g) = \psi_1(\varphi(g) - 1) \\ &= \psi_1(\varphi(g)) - \psi_1(1) \\ &= [\varphi(g)(x_0)] - [x_0] = [x_0g] - [x_0].\end{aligned}$$

Choose a representative loop $a \in g$, and let $a' : I \to X$ be the unique path such that $pa' = a$ and $a'(0) = x_0$. Then $a'(1) = x_0g$, and a' is a relative 1-cycle. Using the crossed homomorphism ∂' and the definition of the connecting homomorphism β, we obtain

$$\beta\psi_2\partial(g) = \beta\partial'(g) = \beta([a']) = [x_0g] - [x_0].$$

Thus, $\psi_1\varphi_*\partial(g) = \beta\psi_2\partial(g)$, and the proof is complete. ∎

Since X is pathwise connected, there exists a (multiplicative-to-additive) group epimorphism

$$\alpha' : K = \pi_1(X, x_0) \to H_1(X)$$

with kernel(α') $= K'$ and defined, for any $k \in K$, by

$$\alpha'(k) = [c],$$

where $[c]$ is the homology class of any representative loop $c \in k$. Since $\alpha : K \to B$ has been defined to be the canonical Abelianizing epimorphism, it follows that there exists a unique group isomorphism $\psi_3 : B \to H_1(X)$ such that $\psi_3\alpha = \alpha'$.

(5.4) *ψ_3 is an H isomorphism.*

Proof. Consider arbitrary elements $h \in H$ and $b \in B$, and choose $g \in G$ and $k \in K$ such that $\varphi(g) = h$ and $\alpha(k) = b$. Referring to the two equations preceding (4.3), we have $g\theta(k)g^{-1} = \theta(k_1)$ for some $k_1 \in K$, and $hb = \alpha(k_1)$. Hence,

$$\psi_3(hb) = \psi_3\alpha(k_1) = \alpha'(k_1),$$
$$h\psi_3(b) = h\psi_3\alpha(k) = h\alpha'(k) = h_*(\alpha'(k)).$$

Let $c \in k$ be an x_0-based representative loop, and let a be a b_0-based representative loop such that $a \in g$. Denote by $a' : I \to X$ the unique path such that $pa' = a$ and $a'(0) = x_0$. Then

$$h(x_0) = \varphi(g)(x_0) = x_0 g = a'(1).$$

Thus, the product $a'(hc)(a')^{-1}$ is defined and is an x_0-based loop. It represents the group element k_1 since $\theta = p_*$ and

$$\begin{aligned} p(a'(hc)(a')^{-1}) &= a(phc)\, a^{-1} \\ &= a(pc)\, a^{-1} \in g\theta(k)\, g^{-1}. \end{aligned}$$

The loop $a'(hc)(a')^{-1}$ is homologous to hc. Hence,

$$\psi_3(hb) = \alpha'(k_1) = [hc].$$

Since c represents k, we have $\alpha'(k) = [c]$ and

$$h\psi_3(b) = h_*(\alpha'(k)) = h_*([c]) = [hc].$$

We conclude that $\psi_3(hb) = h\psi_3(b)$. ∎

The layout of the mappings under present consideration is shown in the following diagram:

$$\begin{tikzcd}
K \arrow[rrr, "\theta = p_*"] \arrow[dr, "\alpha"] \arrow[ddr, "\alpha'"'] & & & G \arrow[dl, "\partial"'] \arrow[ddl, "\partial'"] \\
& B \arrow[r, "\theta_*"] \arrow[d, "\psi_3"] & A_\varphi \arrow[d, "\psi_2"'] & \\
& H_1(X) \arrow[r, "j_*"'] & H_1(X,F) &
\end{tikzcd}$$

(5.5) $\psi_2\theta_* = j_*\psi_3$.

Proof. Consider an arbitrary element $b \in B$, and select $k \in K$ such that $\alpha(k) = b$. Then

$$\psi_2\theta_*(b) = \psi_2\theta_*\alpha(k) = \psi_2\partial\theta(k) = \partial'\theta(k)$$

and

$$j_*\psi_3(b) = j_*\psi_3\alpha(k) = j_*\alpha'(k).$$

Let $c \in k$ be a representative x_0-based loop in the covering space X. The composition pc is a b_0-based loop which represents the group element $p_*(k) = \theta(k)$. It follows from the definition of ∂' given in (12) that

$$\partial'\theta(k) = [c],$$

where $[c]$ is the homology class of c in $H_1(X, F)$. According to the definition of α', we have $\alpha'(k) = [c]$, where this time $[c]$ is the homology class of c in $H_1(X)$. Since the morphism j_* is induced by inclusion,

$$j_*\alpha'(k) = [c] \quad (\text{in } H_1(X, F)) = \partial'\theta(k).$$

Hence, $\psi_2\theta_*(b) = j_*\psi_3(b)$ for all $b \in B$. ∎

Combining the results of this section, we have established the following consistent diagram of morphisms of left H modules in which the rows are exact and in which ψ_0 , ψ_1 , and ψ_3 are isomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "\theta_*"] \arrow[d, "\psi_3"] & A_\varphi \arrow[r, "\varphi_*"] \arrow[d, "\psi_2"] & Z(H) \arrow[r, "\epsilon"] \arrow[d, "\psi_1"] & Z \arrow[r] \arrow[d, "\psi_0"] & 0 \\
0 \arrow[r] & H_1(X) \arrow[r, "j_*"] & H_1(X, F) \arrow[r, "\beta"] & H_0(F) \arrow[r, "i_*"] & H_0(X) \arrow[r] & 0
\end{tikzcd}$$

It is a consequence of the algebraic theorem known as the Five Lemma, see Ref. [6, p. 16], that the H morphism ψ_2 is also an isomorphism. (The necessary fifth morphism in this case is, of course, the trivial mapping $\psi_4 : 0 \to 0$.) Thus, the equivalence between the module sequence of Section 4 and the homology sequence is proved.

The derived module A_φ is defined only up to H isomorphism. Since ψ_2 is now known to be an H isomorphism, we may write $A_\varphi = H_1(X, F)$ and one should recognize that the mapping $\partial' : G \to H_1(X, F)$, defined in (12) by lifting paths into the covering space, is the accompanying crossed homomorphism $\partial : G \to A_\varphi$. This fact is the geometric explanation of the formula for a crossed homomorphism given in (1).

6. Application to Knot Theory

The crossed homomorphism $\partial : G \to A_\varphi$ into the Alexander module can be used effectively in conjunction with the Alexander matrix and the link module sequence to show that certain links are not boundary links. The definition, due to R. H. Fox (see Refs. [13] and [14]), asserts that an m-component link L in S^3 is a *boundary link* if there exist m pairwise-disjoint, connected, orientable, nonsingular surfaces $S_1, \ldots, S_m$ in S^3 such that $\partial(S_1), \ldots, \partial(S_m)$ are the components of L. In particular, every knot, i.e., one-component link, is a boundary link, since it is well-known that a knot always possesses such a spanning surface. A theorem of Fox states that *if L is a boundary link and if $G = \pi_1(S^3 - L)$, then every longitude of L lies in the second commutator subgroup of G.*

Consider an instance of the exact sequence (7) of group homomorphisms

$$1 \longrightarrow K \xrightarrow{\theta} G \xrightarrow{\varphi} H \longrightarrow 1,$$

in which $G = \pi_1(S^3 - L)$ is the group of a link, K is the commutator subgroup of G, the mapping θ is the inclusion, and φ is the canonical epimorphism onto the commutator quotient group H. In this case the derived module A_φ is the Alexander module. The second commutator subgroup of G is the group K', i.e., the commutator subgroup of K. It is a consequence of (4.2) that if $k \in K'$, then

$$\partial(k) = \partial\theta(k) = \theta_*\alpha(k) = 0.$$

Combining this result with Fox's theorem, we have

(6.1) *If l is a longitude of L and if $\partial(l) \neq 0$, then L is not a boundary link.*

Using (6.1), D. S. Cochran has shown as an example in Ref. [1] that the two-component link L pictured in Fig. 1 is not a boundary link. In the remainder of this section we give a modified version of his proof. The group $G = \pi_1(S^3 - L)$ of this link has a presentation

$$G = (a, b, c, d : c^{-1}ac = d^{-1}aca^{-1}d,\ c^{-1}bc = d^{-1}bcb^{-1}d,\ acac^{-1}a^{-1} = bcbc^{-1}b^{-1})_\zeta . \tag{13}$$

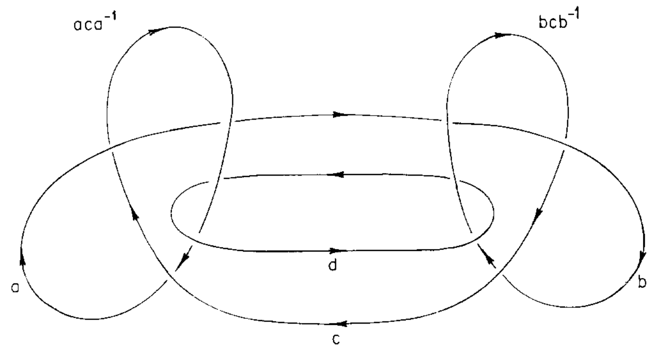

FIGURE 1

From this presentation of G we construct a presentation of A_φ,

$$X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} A_\varphi \longrightarrow 0,$$

as described in Section 2, whose matrix is the Alexander matrix. The correspondences between the generators and relations of the group presentation and the basis elements b_j and c_i of the free modules X_1 and X_2 will be given by

$$\begin{array}{ll} X_1 & \qquad X_2 \\ b_1 \leftrightarrow a, & c_1 \leftrightarrow c^{-1}ac = d^{-1}aca^{-1}d, \\ b_2 \leftrightarrow b, & c_2 \leftrightarrow c^{-1}bc = d^{-1}bcb^{-1}d, \\ b_3 \leftrightarrow c, & c_3 \leftrightarrow acac^{-1}a^{-1} = bcbc^{-1}b^{-1}. \\ b_4 \leftrightarrow d, & \end{array}$$

By Alexander duality, the commutator quotient group $H = H_1(S^3 - L)$ is free Abelian of rank two, and we write it multiplicatively. There exists a basis for H consisting of the two elements s and t such that

$$\zeta\varphi(a) = \zeta\varphi(b) = \zeta\varphi(c) = s,$$
$$\zeta\varphi(d) = t.$$

The Alexander matrix of the presentation (13) is directly computed to be

$$M = \begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline c_1 & s^{-1} - t^{-1} + st^{-1} & 0 & -s^{-1} + 1 - st^{-1} & t^{-1} - st^{-1} \\ c_2 & 0 & s^{-1} - t^{-1} + st^{-1} & -s^{-1} + 1 - st^{-1} & t^{-1} - st^{-1} \\ c_3 & 1 + s^2 - s & -1 - s^2 + s & 0 & 0 \end{array}$$

Reduction of M by standard methods [4] leads to an equivalent matrix

$$M' = \begin{array}{c|ccc} & b_2 - b_1 & b_3 & b_4 - b_1 \\ \hline c_1' & t - 1 & 0 & 0 \\ c_2' & s^2 - s + 1 & 0 & 0 \end{array}$$

The columns of M' are labeled so as to show the relation between the basis $(b_1, \ldots, b_4)$ of X_1 and the basis $(b_2 - b_1, b_3, b_4 - b_1)$ of the free submodule $X_1' \subset X_1$ of the equivalent presentation of A_φ associated with the matrix M'. Let us set

$$a_j = d_1(b_j), \qquad j = 1, \ldots, 4.$$

It follows that the Alexander module A_φ is generated by the three elements $a_2 - a_1$, a_3, and $a_4 - a_1$. The cyclic submodule generated by $a_2 - a_1$ is isomorphic to $Z(H)/I$, where I is the ideal generated by $s^2 - s + 1$ and $t - 1$. Each of a_3 and $a_4 - a_1$ generates a free submodule, and we have

$$A_\varphi = (Z(H)/I) \oplus Z(H) \oplus Z(H).$$

A longitude l of the link L shown in Fig. 1, which is a parallel of the unknotted component, is the element

$$l = \zeta(aca^{-1}bc^{-1}b^{-1}).$$

This expression for l is easily read from the picture. Using the mapping diagram (5), which appears before (3.4), and the definition of δ, we obtain

$$\begin{aligned} \partial(l) &= \partial\zeta(aca^{-1}bc^{-1}b^{-1}) = d_1\delta(aca^{-1}bc^{-1}b^{-1}) \\ &= d_1[(1 - s)\, b_1 + (s - 1)\, b_2 + (s - s)\, b_3 + 0b_4] \\ &= (1 - s)\, a_1 + (s - 1)\, a_2 = (s - 1)(a_2 - a_1). \end{aligned}$$

This result and the next-to-last sentence of the preceding paragraph imply that $\partial(l) = 0$ if and only if $s - 1$ belongs to the ideal I generated by $s^2 - s + 1$ and $t - 1$. To prove that $(s - 1) \notin I$, consider the ring epimorphism $\omega : Z(H) \to Z$ defined by setting $\omega(s) = -1$ and $\omega(t) = 1$. Since $\omega(s^2 - s + 1) = 3$ and $\omega(t - 1) = 0$, we know that $\omega(I) = 3Z$, i.e., the set of all integer multiples of three. On the other hand, $\omega(s - 1) = -2$, which is certainly not a multiple of three. It follows that $(s - 1) \notin I$ and, consequently, that $\partial(l) \neq 0$. We conclude from (6.1) that the link L shown in Fig. 1 is not a boundary link.

N. Smythe has observed that the group G of this link L can be mapped homomorphically onto a free group of rank two. A simple way to verify this assertion is to adjoin the relation $a = b$ to the presentation of G given in (13) above. The result yields a group G_0, which is certainly a homomorphic image of G and which is presented by

$$G_0 = (a, c, d : c^{-1}ac = d^{-1}aca^{-1}d).$$

If both sides of the relation $c^{-1}ac = d^{-1}aca^{-1}d$ are multiplied on the left by $a^{-1}d$ and on the right by $d^{-1}a$, one obtains the equivalent relation

$$a^{-1}dc^{-1}acd^{-1}a = c,$$

and the latter may be written as $c = (a^{-1}dc^{-1})\, a(a^{-1}dc^{-1})^{-1}$. Hence, if we let $a^{-1}dc^{-1} = x$, which is equivalent to $d = axc$, we have the equivalent presentation

$$G_0 = (a, c, d, x : c = xax^{-1}, d = axc).$$

Using Tietze operations [4], we get successively the presentations

$$G_0 = (a, c, x : c = xax^{-1}),$$

$$G_0 = (a, x, :).$$

Thus, G_0 is a free group of rank two. The fact that there exists a homomorphism of G onto a free group of rank two means that the link L of Fig. 1 is a member of a class of links, introduced by Smythe in Refs. [13] and [14] and called homology boundary links. This class includes boundary links. Hence, L is an homology boundary link, but not a boundary link.

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