ADVANCES IN MATHEMATICS **21**, 293–329 (1976)

Curvatures of Left Invariant Metrics on Lie Groups

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This article outlines what is known to the author about the Riemannian geometry of a Lie group which has been provided with a Riemannian metric invariant under left translation.

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INTRODUCTION

When studying relationships between curvature of a complete Riemannian manifold and other topological or geometric properties, it is useful to have many examples. This paper will describe the rich collection of examples which are obtained by providing an arbitrary Lie group G with a Riemannian metric invariant under left translations. (This class of examples can be enlarged substantially, with no extra work, as follows. If Γ is any discrete subgroup of G, then a left invariant metric on G gives rise to a metric on the quotient space $\Gamma\backslash G$, with identical curvature properties. The case where $\Gamma\backslash G$ is compact is of particular interest. Compare 4.9 and 6.2.)

The first four sections will survey the subject, giving some old results and some new results. In the 3-dimensional case the theory is essentially complete (Section 4), but in higher dimensions there remain many unsolved problems. Most proofs will be deferred until the last three sections.

The author is indebted to Nolan Wallach for extremely helpful suggestions.

1. Sectional Curvature

Let G be an n-dimensional Lie group, and let $\mathfrak{g}$ be the associated Lie algebra, consisting of all smooth vector fields on G which are invariant under left translations. (See for example [3, 7, 17, 20].) Choosing some basis $e_1, ..., e_n$ for the vector space $\mathfrak{g}$, it is easy to check that there is one and only one Riemannian metric on G so that these vector fields $e_1, ..., e_n$ are everywhere orthonormal. More generally, given any $n \times n$ positive definite symmetric matrix (β_{ij}) of real numbers, there is one and only one Riemannian metric so that the Riemannian inner product $\langle e_i, e_j \rangle$ is everywhere equal to the constant function β_{ij}. Evidently this construction provides the most general Riemannian metric on G which is left invariant (i.e., invariant under left translations of G). *Thus each n-dimensional Lie group possesses a* $\frac{1}{2}n(n+1)$*-dimensional family of distinct left invariant metrics.* We will see that different metrics on the same Lie group may exhibit drastically different curvature properties.

Choosing some fixed left invariant metric on G, note that the resulting Riemannian manifold is *homogeneous*. That is, there exists an isometry carrying any point to any other point. It follows easily that G is complete. In fact, choosing $\epsilon > 0$ so that the closed ϵ-ball about the identity is compact, it follows that every ball of radius ϵ is compact, hence every Cauchy sequence lies eventually within a compact set. (Compare [11, p. 176].)

The curvature of a Riemannian manifold at a point can be described most easily by the bi-quadratic *curvature function*

$$\kappa(x, y) = \langle R_{xy}(x), y \rangle.$$

(Compare Section 5.) Here x and y range over all tangent vectors at the given point. A given function $\kappa(x, y)$ can occur as curvature function for some Riemannian metric if and only if it is symmetric and bi-quadratic as a function of x and y, and vanishes whenever $x = y$. (I know of no reference for this elementary fact.) The collection of all such symmetric, bi-quadratic functions with $\kappa(x, x) \equiv 0$ forms a real vector space of dimension $n^2(n^2 - 1)/12$. In other words one must prescribe $n^2(n^2 - 1)/12$ real numbers in order to describe the Riemannian curvature of an n-dimensional manifold at a single point.

If u and v are orthogonal unit vectors (or more generally if the determinant $\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2$ is equal to 1), then the real number $K = \kappa(u, v)$ is called the *sectional curvature* of the tangential 2-plane spanned by u and v. Geometrically, K can be described as the Gaussian curvature, at the point, of the surface swept out by all geodesics having a linear combination of u and v as tangent vector.

In order to study a Lie group with left invariant metric, it is best to choose an orthonormal basis $e_1, \ldots, e_n$ for the left invariant vector fields. The Lie algebra structure can then be described by an $n \times n \times n$ array of *structure constants* α_{ijk} where

$$[e_i, e_j] = \sum_k a_{ijk} e_k$$

or equivalently

$$\alpha_{ijk} = \langle [e_i, e_j], e_k \rangle.$$

This array is skew-symmetric in the first two indices. The curvature function κ can then be expressed as a complicated quadratic function of the α_{ijk}. The explicit formula is usually not too useful. However, we will write down the following just to show that it can be done.

LEMMA 1.1. *With structure constants* α_{ijk} *as above, the sectional curvature* $\kappa(e_1, e_2)$ *is given by the formula*

$$\kappa(e_1, e_2) = \sum \left(\tfrac{1}{2}\alpha_{12k}(-\alpha_{12k} + \alpha_{2k1} + \alpha_{k12}) - \tfrac{1}{4}(\alpha_{12k} - \alpha_{2k1} + \alpha_{k12})(\alpha_{12k} + \alpha_{2k1} - \alpha_{k12}) - \alpha_{k11}\alpha_{k22}\right),$$

to be summed over k.

The proof, and a more useful expression for curvature, will be found in Section 5.

At least this explicit expression shows that the curvature can be computed completely from information about the Lie algebra, together with its metric. Furthermore the curvature depends continuously on the structure constants α_{ijk} and vanishes whenever they vanish.

In some cases of interest, there is a great deal of cancellation in (1.1) so that we obtain a more useful formula.

Recall that the *adjoint* L^* of a linear transformation L between metric vector spaces is defined by the formula

$$\langle Lx, y \rangle = \langle x, L^* y \rangle.$$

The transformation L is *skew-adjoint* if $L^* = -L$. For any element x in a Lie algebra $\mathfrak{g}$ the linear transformation

$$y \mapsto [x, y]$$

from $\mathfrak{g}$ to itself is called $\mathrm{ad}(x)$.

Given G with left invariant metric, let u be a vector in the associated Lie algebra.

LEMMA 1.2. *If the linear transformation* $\mathrm{ad}(u)$ *is skew-adjoint, then*

$$\kappa(u, v) \geqslant 0$$

for all v, *where equality holds if and only if* u *is orthogonal to the image* $[v, \mathfrak{g}]$.

Proof. We may assume without loss of generality that u and v are orthonormal. Choosing an orthonormal basis $e_1, \dots, e_n$ with $e_1 = u$, $e_2 = v$, the statement that $\mathrm{ad}(e_1)$ is skew-adjoint means that the array α_{ijk} is skew in the last two indices for $i = 1$. Inspection then shows that the formula 1.1 reduces to

$$\kappa(e_1, e_2) = \sum_k (\alpha_{2k1})^2/4.$$

Thus $\kappa(e_1, e_2) \geqslant 0$, as asserted. ▮

The hypothesis of 1.2 depends of course on a particular choice of metric.

COROLLARY 1.3. *If* u *belongs to the center of the Lie algebra* $\mathfrak{g}$, *then for any left invariant metric the inequality* $\kappa(u, v) \geqslant 0$ *is satisfied for all* v.

For if u is central then $\mathrm{ad}(u) = 0$, and the zero transformation is certainly skew-adjoint. ▮

It may be conjectured that central elements are the only ones with this property. (Compare Section 2.5.)

Some Lie groups may possess a metric which is invariant not only under left translation but also under right translation. The basic facts about such *bi-invariant* metrics can be summarized as follows.

LEMMA. *A left invariant metric on a connected Lie group is also right invariant if and only if* $\mathrm{ad}(x)$ *is skew-adjoint for every* $x \in \mathfrak{g}$. *A* *connected Lie group admits such a bi-invariant metric if and only if it is isomorphic to the cartesian product of a compact group and a commutative group.*

This will be proved in 7.2 and 7.5 below.

COROLLARY 1.4. *Every compact Lie group admits a left invariant (and in fact a bi-invariant) metric so that all sectional curvatures satisfy* $K \geqslant 0$.

Proof. This follows from the Lemma just stated, together with 1.2. ∎

In fact we will see in 7.3 that sectional curvatures associated with a bi-invariant metric can be computed by the explicit formula

$$\kappa(u, v) = \tfrac{1}{4}\langle [u, v], [u, v]\rangle.$$

This will give an alternative proof that $\kappa(u, v) \geqslant 0$.

More generally if G is the semi-direct product AB of a commutative normal subgroup A and a subgroup B with bi-invariant metric which operates orthogonally on A, then G possesses a left invariant metric with all sectional curvatures $K \geqslant 0$. (Compare 7.8.) It would be of interest to know whether these are the only groups which admit such a metric.

If we sharpen the inequality and require that $K > 0$, then examples become very scarce indeed.

THEOREM OF WALLACH. *The 3-sphere group* $SU(2)$, *consisting of* 2×2 *unitary matrices of determinant* 1, *is the only simply connected Lie group which admits a left invariant metric of strictly positive sectional curvature.*

(Compare 4.5.) Thus there are no examples at all in higher dimensions. For the proof, the reader is referred to [19].

The easiest Riemannian manifolds to understand are those which are *flat* in the sense that the sectional curvature K is identically zero. A classical theorem asserts that a complete Riemannian manifold is flat if and only if its universal covering manifold is isometric to Euclidean space. In the case of a left invariant metric, the precise criterion for flatness can be stated as follows (Section 7). Recall that a Lie algebra is called *commutative* if the bracket product $[x, y]$ is identically zero. If $\mathfrak{g}$ is commutative, then it follows from 1.1 that *every* left invariant metric is flat.

THEOREM 1.5. *A Lie group with left invariant metric is flat if and only if the associated Lie algebra splits as an orthogonal direct sum* $\mathfrak{b} \oplus \mathfrak{u}$ *where* $\mathfrak{b}$ *is a commutative subalgebra,* $\mathfrak{u}$ *is a commutative ideal, and where the linear transformation* $\mathrm{ad}(b)$ *is skew-adjoint for every* $b \in \mathfrak{b}$.

Thus there exist noncommutative Lie groups with flat left invariant metric, but they are all solvable of a very sharply restricted form. The simplest example is provided by the group $E(2)$ of rigid motions of the Euclidean plane (Section 4.8).

Those Lie groups which admit left invariant metrics of strictly negative sectional curvature have been classified by Heintze. The necessary and sufficient condition is that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathbf{R}x$ for some x such that all eigenvalues of $\mathrm{ad}(x) \mid [\mathfrak{g}, \mathfrak{g}]$ have positive real part. It follows for example that no cartesian product admits a left invariant metric of strictly negative curvature.

Those with $K \leqslant 0$ have been classified by Azencott and Wilson [1]. Since the statements are complicated we will content ourselves with the following qualitative result. Recall that a Lie group is called *unimodular* if its left invariant Haar measure is also right invariant (Section 6).

THEOREM 1.6. *If a connected Lie group G has a left invariant metric with all sectional curvatures* $K \leqslant 0$, *then it is solvable. If G is unimodular, then any such metric with* $K \leqslant 0$ *must actually be flat* ($K \equiv 0$).

In other words, every left invariant metric on a unimodular Lie group must possess some strictly positive sectional curvature unless it is completely flat as in 1.5.

Proof. This follows from [1, Sections 5.2, 6.2, 4.4], using 6.3 below. ∎

Here is an explicit (although rather exceptional) example of a left invariant metric with sectional curvatures $K < 0$.

SPECIAL EXAMPLE 1.7. Suppose that the Lie algebra $\mathfrak{g}$ has the property that the bracket product $[x, y]$ is always equal to a linear combination of x and y. Assume that $\dim \mathfrak{g} \geqslant 2$, then in fact

$$[x, y] = l(x)y - l(y)x$$

where l is a well defined linear mapping from $\mathfrak{g}$ to the real numbers.

Choosing any positive definite metric, the sectional curvatures are constant:

$$K = -\| l \|^2.$$

Thus, in the noncommutative case $l \neq 0$, every possible metric on $\mathfrak{g}$ has constant negative sectional curvature.

Here $\| l \|$ denotes the norm of the linear operator l. The proof will be given in Section 5.

We will see in 2.5 that these special examples can also be characterized as the only Lie algebras such that every metric has sectional curvatures of constant sign.

For other examples of Lie algebras with metrics of constant or non-constant negative curvature see 4.11.

2. Ricci Curvature

A somewhat cruder description of the curvature of a Riemannian manifold at a point is provided by the *Ricci quadratic form* $r(x)$. This is a real valued quadratic function of the tangent vector x, defined by the formula

$$r(x) = \sum_i \kappa(x, e_i) = \sum_i \langle R_{xe_i}(x), e_i \rangle$$

where κ is the biquadratic curvature function of Section 1, and where $e_1, \ldots, e_n$ can be any orthonormal basis for the tangent space. If u is a unit vector, then $r(u)$ is called the *Ricci curvature* in the direction u. It is equal to $n - 1$ times the average of the sectional curvatures of all tangential 2-planes containing u.

For computational purposes it may be more convenient to work with the self adjoint *Ricci transformation* $\hat{r}$ defined by

$$\hat{r}(x) = \sum R_{e_i x}(e_i).$$

This is related to the quadratic form r by the identity

$$r(x) = \langle \hat{r}(x), x \rangle.$$

The eigenvalues of $\hat{r}$ are called the *principal Ricci curvatures*. If we choose an orthonormal basis $e_1, \ldots, e_n$ consisting of eigenvectors, note that the quadratic form is then diagonalized

$$r(\xi_1 e_1 + \cdots + \xi_n e_n) = \sum r(e_i)\, \xi_i^2.$$

In particular the numbers $r(e_i)$ can be identified with the principal Ricci curvatures, and the collection of signs $\{\operatorname{sgn} r(e_1),\ldots, \operatorname{sgn} r(e_n)\}$ can be identified with the *signature* of the quadratic form r.

Now let us return to the study of left invariant metrics. Here is a criterion for obtaining a direction of positive Ricci curvature.

LEMMA 2.1. *If the linear transformation* $\operatorname{ad}(u)$ *is skew-adjoint, then* $r(u) \geqslant 0$, *where equality holds if and only if* u *is orthogonal to the commutator ideal* $[\mathfrak{g}, \mathfrak{g}]$.

Proof. This follows immediately from Lemma 1.2. ▌

As an example, if u belongs to the center of $\mathfrak{g}$ then it certainly follows that $r(u) \geqslant 0$.

The criterion for everywhere positive Ricci curvature is classical and elegant.

THEOREM 2.2. *A connected Lie group admits a left invariant metric with all Ricci curvatures strictly positive if and only if it is compact with finite fundamental group.*

Proof. In one direction this follows from the theorem of Myers which asserts that any complete Riemannian manifold with Ricci curvatures positive and bounded away from zero must be compact with finite fundamental group. (See for example [2] or [14].) In the other direction, if G is compact then we can choose a bi-invariant metric, so that each $\operatorname{ad}(x)$ is skew-adjoint. (Compare Section 1 or Section 7.) If G also has finite fundamental group, so that the universal covering group $\tilde{G}$ is compact, note that $\mathfrak{g}$ must be equal to its commutator ideal $[\mathfrak{g}, \mathfrak{g}]$. For otherwise there would exist a non-trivial Lie algebra homomorphism from $\mathfrak{g}$ to the commutative Lie algebra $\mathbf{R}$. This would induce a non-trivial homomorphism from $\tilde{G}$ to the additive Lie group $\mathbf{R}$, contradicting the hypothesis that $\tilde{G}$ is compact. Now, using Lemma 2.1, it follows that all Ricci curvatures are strictly positive. ▌

Remarks. We will see in Section 7.7 that this result can be sharpened: If G is compact with finite fundamental group, then it actually admits a bi-invariant metric with *constant* positive Ricci curvature. That is

$$r(u) = \text{constant} > 0$$

for all unit vectors u. Manifolds of constant Ricci curvature are often called *Einstein manifolds*. (Compare [11, p. 292].)

It would be of interest to characterize those connected Lie groups which admit left invariant metrics with all Ricci curvatures $\geqslant 0$. We will see in 6.4 that such a group G must necessarily be unimodular. It will follow from 3.1 that G cannot be solvable unless it is flat.

Similarly it would be of interest to know which groups admit left invariant metrics of Ricci curvature $\leqslant 0$. We will see in 4.7 that the simple group $SL(2, \mathbf{R})$ and the unimodular solvable group $E(1, 1)$ both admit non-flat left invariant metrics with Ricci curvature $\leqslant 0$. Both examples are in contrast with 1.6. It seems unlikely that any higher dimensional simple group admits such a metric.

Finally it would be interesting to know which groups admit left invariant metrics with Ricci curvature identically zero,[1] or with (constant or non-constant) strictly negative Ricci curvature.

Here is a criterion, completely analogous to 2.1, for obtaining a direction of negative Ricci curvature.

LEMMA 2.3. *If u is orthogonal to the commutator ideal* $[\mathfrak{g}, \mathfrak{g}]$, *then* $r(u) \leqslant 0$, *where equality holds if and only if* $\mathrm{ad}(u)$ *is skew-adjoint.*

This will be proved in Section 5. Combining 2.1 and 2.3 we obtain the following sharper version of a theorem of Wolf [21].

THEOREM 2.4. *Suppose that the Lie algebra of G is nilpotent but not commutative. Then for any left invariant metric there exists a direction of strictly negative Ricci curvature and a direction of strictly positive Ricci curvature.*

Proof. The statement that $\mathfrak{g}$ is nilpotent means that some term in the *lower central series*

$$\mathfrak{g} \supset [\mathfrak{g}, \mathfrak{g}] \supset [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supset \cdots$$

must be zero. Choosing a unit vector u in the last non-zero term of this sequence of ideals, it follows that u is central and contained in $[\mathfrak{g}, \mathfrak{g}]$, hence $r(u) > 0$ by 2.1.

Note that the vector space $\mathfrak{g}$ cannot be spanned by $[\mathfrak{g}, \mathfrak{g}]$ together with the center $\mathfrak{z}$. For if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{z}$ then $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}] + \mathfrak{z}] = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$,

[1] Note added in proof. See Alekseevskii and Kimel'fel'd, *Functional Anal. Appl.* **9** (1975), 97–102.

hence the lower central series would stabilize prematurely. Therefore there exists a unit vector v orthogonal to $[\mathfrak{g}, \mathfrak{g}]$ and not contained in $\mathfrak{z}$. The linear transformation ad(v), being nonzero and nilpotent, cannot be skew-adjoint. (For if a linear transformation L is skew-adjoint with $Lx \neq 0$, then assuming inductively that $L^k x \neq 0$ it follows that

$$\langle L^{2k}x, x\rangle = \pm\langle L^k x, L^k x\rangle \neq 0$$

hence $L^{2k}x \neq 0$. Thus L cannot be nilpotent.) Since v is orthogonal to $[\mathfrak{g}, \mathfrak{g}]$ and ad(v) is not skew-adjoint, it follows that $r(v) < 0$ by Lemma 2.3. ▮

More generally we will prove the following.

THEOREM 2.5. *If the Lie algebra of G contains linearly independent vectors x, y, z so that*

$$[x, y] = z,$$

then there exists a left invariant metric so that $r(x) < 0$ *and* $r(z) > 0$.

Thus almost any Lie group has a left invariant metric with both positive and negative Ricci curvatures. The only exceptions are the "special examples" of Section 1.7, characterized by the property that $[x, y]$ is always a linear combination of x and y.

Proof. Choose a fixed basis $b_1 ,..., b_n$ with $b_1 = x$, $b_2 = y$, $b_3 = z$. For any real number $\epsilon > 0$, consider an auxiliary basis $e_1 ,..., e_n$ defined by $e_1 = \epsilon b_1$, $e_2 = \epsilon b_2$, and $e_i = \epsilon^2 b_i$ for $i \geqslant 3$. Define a left invariant metric by requiring that $e_1 ,..., e_n$ should be orthonormal. Let $\mathfrak{g}_\epsilon$ denote the Lie algebra $\mathfrak{g}$ provided with this particular metric and this particular orthonormal basis. Setting $[e_i , e_j] = \sum \alpha_{ijk} e_k$, the structure constants α_{ijk} are clearly functions of ϵ. Now consider the limit as $\epsilon \to 0$. Inspection shows that each α_{ijk} tends to a well defined limit. Thus we obtain a limit Lie algebra $\mathfrak{g}_0$ with prescribed metric and prescribed orthonormal basis. Furthermore the bracket product in $\mathfrak{g}_0$ is given by

$$[e_1 , e_2] = -[e_2 , e_1] = e_3 ,$$

with $[e_i , e_j] = 0$ otherwise. Applying 2.1 and 2.3 it follows that the inequalities $r(e_1) < 0 < r(e_3)$ are satisfied in $\mathfrak{g}_0$. (For more explicit computations, see 4.6.) But these Ricci curvatures must vary continuously as we vary the structure constants, so it follows that $r(e_1) < 0 < r(e_3)$ whenever ϵ is sufficiently close to zero. ▮

3. Scalar Curvature

Choosing any orthonormal basis $e_1, \ldots, e_n$ for the tangent vectors at a point of a Riemannian manifold, the real number

$$\rho = r(e_1) + \cdots + r(e_n) = 2 \sum_{i<j} \kappa(e_i, e_j)$$

is called the *scalar curvature* at the point. Alternatively ρ can be described as $n(n-1)$ times the average of all sectional curvatures at the point.

According to Eliasson any smooth manifold of dimension $\geqslant 3$ admits a Riemannian metric of strictly negative scalar curvature. However, metrics of positive scalar curvature do not always exist. (Compare [8, 10, 13].) For left invariant metrics, the situation can be described as follows.

THEOREM 3.1. *If the Lie group G is solvable, then every left invariant metric on G is either flat (as in* 1.5*), or else has strictly negative scalar curvature.*

This will be proved in Section 5. We will see in Section 4.7 that the corresponding statement is true also for the 3-dimensional simple group $SL(2, \mathbf{R})$. *It may be conjectured that it is true for any Lie group whose universal covering space is homeomorphic to Euclidean space.*

The following is an immediate consequence.

COROLLARY 3.2. *If G is solvable and unimodular, then every left invariant metric on G is either flat, or has both positive and negative sectional curvatures.*

In the nilpotent case we obtained the sharper statement that there exist positive and negative Ricci curvatures (Section 2.4). However, a solvable unimodular group does not necessarily have any directions of positive Ricci curvature (Section 4.7).

Proof of 3.2. If G is unimodular and the metric is not flat, then there exist positive sectional curvatures by the Azencott, Wilson Theorem 1.6; while if G is solvable and the metric is not flat, then there exist negative sectional curvatures by 3.1. ∎

THEOREM 3.3. *If the Lie algebra of G is noncommutative, then G possesses a left invariant metric of strictly negative scalar curvature.*

Proof. First suppose that there exist linearly independent vectors x, y, z in the Lie algebra with $[x, y] = z$. As in the proof of 2.5, we can choose a basis $b_1, \ldots, b_n$ with $b_1 = x$, $b_2 = y$, $b_3 = z$, and for any $\epsilon > 0$ we can choose a metric so that the vectors

$$\epsilon b_1, \epsilon b_2, \epsilon^2 b_3, \ldots, \epsilon_2 b_n$$

are orthonormal. Denote this Lie algebra with prescribed metric and prescribed orthonormal basis by $\mathfrak{g}_\epsilon$. As ϵ tends to zero, $\mathfrak{g}_\epsilon$ tends to a well defined limit $\mathfrak{g}_0$ which is nilpotent but not commutative. Hence $\rho(\mathfrak{g}_0) < 0$ by 3.1 and 2.4, or by explicit computation as in 4.6. It follows by continuity that $\rho(\mathfrak{g}_\epsilon) < 0$ whenever ϵ is sufficiently close to zero.

On the other hand, if x, y and $[x, y]$ are always linearly dependent, then $\mathfrak{g}$ is isomorphic to the special example of Section 1.7, hence $\mathfrak{g}$ has strictly negative curvature for any choice of metric. ▌

There remains the question as to which Lie groups admit left invariant metrics of positive scalar curvature. The following result was communicated to the author by Nolan Wallach. (Compare [12].) Let G be a connected Lie group.

THEOREM 3.4 (Wallach). *If the universal covering of G is not homeomorphic to Euclidean space (or equivalently if G contains a compact non-commutative subgroup), then G admits a left invariant metric of strictly positive scalar curvature.*

This will be proved in Section 7. As noted earlier, it is conjectured that these are the only groups which admit a left invariant metric with $\rho > 0$.

4. THE 3-DIMENSIONAL CASE

In order to study 3-dimensional Lie algebras we will make use of the familiar Euclidean *cross product operation.* If u and v are elements of a 3-dimensional vector space which is provided with a positive definite metric and with a preferred orientation, then the cross product $u \times v$ is defined. This product is bilinear and skew symmetric as a function of u and v. The vector $u \times v$ is orthogonal to both u and v and has length equal to the square root of the determinant $\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2$. Its direction is determined by the requirement that the triple $u, v, u \times v$ is positively oriented whenever u and v are linearly independent.

Let G be a connected 3-dimensional Lie group with left invariant metric. Choose an orientation for the Lie algebra of G, so that the cross product is defined.

LEMMA 4.1. *The bracket product operation in this Lie algebra* $\mathfrak{g}$ *is related to the cross product operation by the formula*

$$[u, v] = L(u \times v)$$

where L *is a uniquely defined linear mapping from* $\mathfrak{g}$ *to itself. The Lie group* G *is unimodular if and only if this linear transformation* L *is self adjoint.*

All proofs will be given in Section 6.

Now let us specialize to the unimodular case. If L is self adjoint, then there exists an orthonormal basis e_1, e_2, e_3 consisting of eigenvectors, $Le_i = \lambda_i e_i$. Replacing e_1 by $-e_1$ if necessary, we may assume that the basis e_1, e_2, e_3 is positively oriented. The bracket product operation is then given by $[e_1, e_2] = L(e_1 \times e_2) = \lambda_3 e_3$, with similar expressions for the other $[e_i, e_j]$. Thus we obtain the following normal form,

$$[e_2, e_3] = \lambda_1 e_1, \qquad [e_3, e_1] = \lambda_2 e_2, \qquad [e_1, e_2] = \lambda_3 e_3, \tag{4.2}$$

for the bracket product operation in a 3-dimensional unimodular Lie algebra with metric.

The three eigenvalues λ_1, λ_2, λ_3 are apparently well defined up to order. However, the construction was based on a choice of orientation. If we reverse the orientation of $\mathfrak{g}$, then the cross product operation will change sign, hence L and its eigenvalues λ_1, λ_2, λ_3 will all change sign.

The curvature properties of the metric Lie algebra (4.2) can be described as follows. It is convenient to define numbers μ_1, μ_2, μ_3 by the formula

$$\mu_i = \tfrac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i$$

so that, for example, $\mu_1 + \mu_2 = \lambda_3$.

THEOREM 4.3. *The orthonormal basis* e_1, e_2, e_3, *chosen as above,* *diagonalizes the Ricci quadratic form, the principal Ricci curvatures being given by*

$$r(e_1) = 2\mu_2\mu_3\,, \qquad r(e_2) = 2\mu_1\mu_3\,, \qquad r(e_3) = 2\mu_1\mu_2\,.$$

In particular, it follows that scalar curvature is given by the formula

$$\rho = 2(\mu_2\mu_3 + \mu_1\mu_3 + \mu_1\mu_2).$$

Using this description of Ricci curvature, the sectional curvatures can easily be computed. In fact, at a point of any 3-dimensional manifold the explicit formula

$$\kappa(u, v) = \| u \times v \|^2 \rho/2 - r(u \times v)$$

is not difficult to verify. (There can be no such formula in dimensions $n > 3$ since the $\frac{1}{2}n(n+1)$ parameters needed to describe Ricci curvature can not suffice to determine the $\frac{1}{12}n^2(n^2-1)$ parameters needed to describe sectional curvature.)

COROLLARY 4.4. *In the* 3-*dimensional unimodular case, the determinant $r(e_1)\,r(e_2)\,r(e_3)$ of the Ricci quadratic form is always nonnegative. If this determinant is zero, then at least two of the principal Ricci curvatures must be zero.*

Proof. This follows immediately from 4.3. ∎

If this determinant $r(e_1)\,r(e_2)\,r(e_3)$ is non-zero, then it is easy to sove for μ_1, μ_2, μ_3, and hence for the structure constants λ_1, λ_2, λ_3, as functions of the principal Ricci curvatures, well defined up to simultaneous change of sign.

Now suppose that we alter the metric, keeping the bracket product operation fixed. If we choose a new metric so that the basis

$$\eta\zeta e_1\,, \; \xi\zeta e_2\,, \; \xi\eta e_3$$

is orthonormal, then the new structure constants in formula (4.2) will clearly be

$$\xi^2\lambda_1\,, \; \eta^2\lambda_2\,, \; \zeta^2\lambda_3\,.$$

Thus we can multiply λ_1, λ_2, λ_3 by arbitrary positive numbers without changing the underlying Lie algebra. There are now just six distinct cases, which we tabulate as follows. By changing signs if necessary, we may assume that at most one of the structure constants λ_1, λ_2, λ_3 is negative.

Signs of λ_1, λ_2, λ_3	Associated Lie group	Description
$+, +, +$	$SU(2)$ or $SO(3)$	compact, simple
$+, +, -$	$SL(2, \mathbf{R})$ or $O(1, 2)$	noncompact, simple
$+, +, 0$	$E(2)$	solvable
$+, -, 0$	$E(1, 1)$	solvable
$+, 0, 0$	Heisenberg group	nilpotent
$0, 0, 0$	$\mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R}$	commutative

It is not difficult to show that these six possibilities do really give rise to nonisomorphic Lie algebras. They can be distinguished for example by computing the signature of the Killing form

$$\beta(x, y) = \mathrm{trace}(\mathrm{ad}(x)\,\mathrm{ad}(y))$$

in each case. Here is a glossary.

$SU(2)$: group of 2×2 unitary matrices of determinant 1; homeomorphic to the unit 3-sphere.

$SO(3)$: rotation group of 3-space, isomorphic to $SU(2)/\{\pm I\}$.

$SL(2, \mathbf{R})$: group of 2×2 real matrices of determinant 1.

$O(1, 2)$: Lorentz group consisting of linear transformations preserving the quadratic form $t^2 - x^2 - y^2$. Its identity component is isomorphic to $SL(2, \mathbf{R})/\{\pm I\}$, or to the group of rigid motions of hyperbolic 2-space.

$E(2)$: group of rigid motions of Euclidean 2-space.

$E(1, 1)$: group of rigid motions of Minkowski 2-space. This group is a semi-direct product of subgroups isomorphic to $\mathbf{R} \oplus \mathbf{R}$ and to $\mathbf{R}$, where each $\tau \in \mathbf{R}$ acts on $\mathbf{R} \oplus \mathbf{R}$ by the matrix $\left[\begin{smallmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{smallmatrix}\right]$.

Finally, the Heisenberg group can be described as the group of all 3×3 real matrices of the form $\left[\begin{smallmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{smallmatrix}\right]$.

Let us study the extent to which curvature can be altered by a change of metric. The greatest freedom to alter curvature properties occurs in the compact case. (Compare [8, Section 3.3].)

COROLLARY 4.5. *Depending on the choice of left invariant metric, the Ricci quadratic form for the 3-sphere group $SU(2)$ can have signature either $(+, +, +)$ or $(+, 0, 0)$ or $(+, -, -)$; and the scalar curvature can be either positive, negative, or zero.*

Proof. This follows easily from 4.3. (Of course many parts of this corollary follow from theorems stated earlier, such as 2.2 and 3.3.) ∎

By way of contrast, in the commutative case where all left invariant metrics are flat, and also in the nilpotent case, curvature properties are essentially independent of the metric.

COROLLARY 4.6. *For any left invariant metric on the Heisenberg group, the Ricci quadratic form has signature $(+, -, -)$ and the scalar curvature ρ is strictly negative. Furthermore the principal Ricci curvatures satisfy*

$$|r(e_1)| = |r(e_2)| = |r(e_3)| = |\rho|.$$

Proof. Taking $\lambda_2 = \lambda_3 = 0$ formula 4.3 shows that

$$r(e_1) = -r(e_2) = -r(e_3) = -\rho$$

is equal to $\lambda_1{}^2/2$. ∎

The simple group $SL(2, \mathbf{R})$ and the solvable group $E(1, 1)$ are difficult to distinguish by curvature properties.

COROLLARY 4.7. *Let G be either $SL(2, \mathbf{R})$ or $E(1, 1)$. Then depending on the choice of left invariant metric the signature of the Ricci form can be either $(+, -, -)$ or $(0, 0, -)$. However, the scalar curvature ρ must always be strictly negative.*

Proof. If $\lambda_1 = 0$ while λ_2 and λ_3 have opposite sign, then the computation

$$\rho = -\tfrac{1}{2}(\lambda_2 - \lambda_3)^2$$

shows that $\rho < 0$. If the λ_i are all nonzero with say $\lambda_1 < 0 < \lambda_2, \lambda_3$, then the computation $\partial\rho/\partial\lambda_1 = -\lambda_1 + \lambda_2 + \lambda_3$ shows that ρ is strictly monotone as a function of λ_1 (keeping λ_2, λ_3 fixed) for $\lambda_1 \leqslant 0$. Therefore

$$\rho(\lambda_1, \lambda_2, \lambda_3) < \rho(0, \lambda_2, \lambda_3) = -\tfrac{1}{2}(\lambda_2 - \lambda_3)^2 \leqslant 0.$$

Further details will be left to the reader. ∎

COROLLARY 4.8. *The Euclidean group $E(2)$ is non-commutative, but admits a flat left invariant metric. Every nonflat left invariant metric has Ricci form of signature $(+, -, -)$, with scalar curvature $\rho < 0$.*

The proof is easily supplied. ∎

Remark 4.9. It is interesting that each of these 3-dimensional unimodular groups G possesses a discrete subgroup Γ so that the quotient $\Gamma\backslash G$ is compact. (Compare 6.2.) For a precise description of the possible compact quotient manifolds, the reader is referred to [16]. In the four solvable cases it can be shown that $\Gamma\backslash G$ is always a torus bundle over a circle. In the $SL(2, \mathbf{R})$ case, $\Gamma\backslash G$ always has a finite covering space which is a non-trivial circle bundle over a surface of genus $\geqslant 2$. The topology of $\Gamma\backslash G$ determines the associated Lie algebra uniquely except in one exceptional case: Both the Euclidean group $E(2)$ and the commutative group $\mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R}$ admit a discrete subgroup so that the quotient is topologically a torus $S^1 \times S^1 \times S^1$.

It is interesting that many of these quotient manifolds $\Gamma\backslash G$ occur also in the study of algebraic singularities. Compare [4, 15, 16, 18].

Now let us turn to the nonunimodular case. The possible Lie algebras can be described as follows.

LEMMA 4.10. *If the connected 3-dimensional Lie group G is not unimodular, then its Lie algebra has a basis e_1, e_2, e_3 so that*

$$[e_1, e_2] = \alpha e_2 + \beta e_3$$

$$[e_1, e_3] = \gamma e_2 + \delta e_3$$

with $[e_2, e_3] = 0$, and so that the matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

has trace $\alpha + \delta = 2$. If we exclude the exceptional case where A is the identity matrix (compare Section 1.7), then the determinant $D = \alpha\delta - \beta\gamma$ provides a complete isomorphism invariant for this Lie algebra.

Curvature properties can be described as follows. Consider a Lie group as in 4.10 with $A \neq I$.

THEOREM 4.11. *If the determinant D is negative then every left invariant metric has Ricci quadratic form of signature $(+, -, -)$. But* *if $D \geqslant 0$ the signature* $(0, -, -)$ *is also possible, and if $D > 0$ the signature* $(-, -, -)$ *is also possible. In fact, for $D > 0$ there exists a left invariant metric of strictly negative sectional curvature and for $D > 1$ there exists a left invariant metric of constant negative curvature. In all cases the scalar curvature is strictly negative.*

Again proofs are deferred until Section 6.

In all of the non-unimodular cases, at least two of the principal Ricci curvatures are negative. *Comparing* 4.4, *we see that no* 3*-dimensional Lie group at all admits a left invariant metric with Ricci form of signature* $(+, +, -)$ *or* $(+, \pm, 0)$. I do not know whether there exist any analogous restrictions in higher dimensions.

5. Computations

This section contains proofs of several of the results stated earlier. Before making any actual computations it is necessary to define some basic concepts. For details the reader is referred to textbooks such as [2, 7, 11, 14].

First we consider the *Riemannian connection* ∇ associated with a Riemannian metric. This connection assigns to each pair of smooth vector fields x and y a smooth vector field $\nabla_x y$ called the *covariant derivative* of y in the direction x. For our purpose it suffices to know that ∇ is always uniquely defined, that $\nabla_x y$ is bilinear as a function of x and y, that it satisfies the "symmetry" condition

$$\nabla_x y - \nabla_y x = [x, y], \tag{5.1}$$

and that the identity

$$\langle \nabla_x y, z \rangle + \langle y, \nabla_x z \rangle = 0 \tag{5.2}$$

is satisfied whenever y and z are vector fields such that the Riemannian inner product $\langle y, z \rangle$ is a constant function. In particular, if y and z are left invariant vector fields on a Lie group with left invariant metric, this identity is certainly satisfied. If x is also left invariant, then $\nabla_x y$ is left invariant. *Thus, for each x in the Lie algebra, ∇_x is a skew-adjoint linear transformation from the Lie algebra to itself.*

If x, y, z are all left invariant vector fields, then combining 5.1 and 5.2 with the various identities obtained by permuting the variables, we can solve to obtain the following formula:

$$\langle \nabla_x y, z \rangle = \tfrac{1}{2}(\langle [x, y], z \rangle - \langle [y, z], x \rangle + \langle [z, x], y \rangle). \tag{5.3}$$

In particular, choosing an orthonormal basis $e_1, \ldots, e_n$ and setting $\alpha_{ijk} = \langle [e_i, e_j], e_k \rangle$ it follows that

$$\langle \nabla_{e_i} e_j , e_k \rangle = \tfrac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij})$$

or in other words

$$\nabla_{e_i} e_j = \sum_k \tfrac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij})\, e_k . \tag{5.4}$$

The *Riemann curvature tensor* R associates to each pair of smooth vector fields x and y the linear transformation

$$R_{xy} = \nabla_{[x,y]} - \nabla_x \nabla_y + \nabla_y \nabla_x$$

from smooth vector fields to smooth vector fields. Evidently $R_{xy} = -R_{yx}$. The transformation R_{xy} is always skew-adjoint. (If we consider only left invariant vector fields, this follows from the fact that each ∇_x is skew-adjoint.) If u and v are orthonormal recall that the number

$$K = \kappa(u, v) = \langle R_{uv}(u), v \rangle$$

is called the *sectional curvature* associated with u and v. The sectional curvatures depend symmetrically on u and v, and determine the Riemann tensor R uniquely. (See, for example, [11, Chapter 5].)

Caution: In the notation of Kobayashi, Nomizu, and of Helgason, our R would be denoted by $-R$, so that the sectional curvature K would equal $\langle -R_{uv}(u), v \rangle$.

We can now give an explicit formula for sectional curvature in terms of structure constants, as described in Section 1:

Proof of Lemma 1.1. If $e_1, \ldots, e_n$ is any orthonormal basis for the left invariant vector fields on G, then inserting formula 5.4 into the definition

$$\kappa(e_1, e_2) = \langle \nabla_{[e_1, e_2]} e_1 - \nabla_{e_1} \nabla_{e_2} e_1 + \nabla_{e_2} \nabla_{e_1} e_1 , e_2 \rangle$$

we easily obtain the required explicit formula. ∎

To illustrate these concepts consider the following important special case. *Suppose that the Lie algebra* $\mathfrak{g}$ *contains an ideal* $\mathfrak{u}$ *of codimension* 1. Choosing a unit vector b orthogonal to $\mathfrak{u}$, let

$$L : \mathfrak{u} \to \mathfrak{u}$$

denote the linear transformation ad(b) restricted to $\mathfrak{u}$, so that $L(u) = [b, u]$. Let L^* denote the adjoint transformation, and let $S = \frac{1}{2}(L + L^*)$ denote the self adjoint part of L.

We may also think of the ideal $\mathfrak{u}$ as a Lie algebra in its own right with the induced metric. Let $\bar{\nabla}$ denote the Riemannian connection for this metric Lie algebra $\mathfrak{u}$. The symbol ∇ will be reserved for the Riemannian connection of the original Lie algebra $\mathfrak{g}$.

LEMMA 5.5. *With these notations, the covariant derivative operator* ∇_b *satisfies*

$$\nabla_b b = 0 \qquad \textit{and} \qquad \nabla_b u = \tfrac{1}{2}(L - L^*)u$$

for each $u \in \mathfrak{u}$. *Similarly the operator* ∇_u *satisfies*

$$\nabla_u b = -Su \qquad \textit{and} \qquad \nabla_u v = \bar{\nabla}_u v + \langle Su, v\rangle b$$

for each u *and* v *in* $\mathfrak{u}$.

Proof. To compute the component of $\nabla_u v$ in the b direction we use formula 5.3:

$$\begin{aligned}\langle \nabla_u v, b\rangle &= \tfrac{1}{2}(\langle [u, v], b\rangle - \langle [v, b], u\rangle + \langle [b, u], v\rangle) \\ &= \tfrac{1}{2}(0 + \langle Lv, u\rangle + \langle Lu, v\rangle) \\ &= \langle Su, v\rangle.\end{aligned}$$

To compute the component orthogonal to b we choose an arbitrary w in $\mathfrak{u}$ and use 5.3 to compute $\langle \nabla_u v, w\rangle$. Since the computation takes place completely within $\mathfrak{u}$, this inner product is equal to $\langle \bar{\nabla}_u v, w\rangle$. Hence $\nabla_u v$ is equal to $\bar{\nabla}_u v$ plus the normal component just computed. The other arguments are completely analogous. ∎

Here is a first application. (More important applications will follow.)

Verification of Example 1.7. For any elements x and y in the Lie algebra $\mathfrak{g}$ we suppose that $[x, y]$ is a linear combination of x and y. Fixing x, note that ad(x) induces a linear mapping from the quotient vector space $\mathfrak{g}/\mathbf{R}x$ to itself, with the property that every vector maps into some multiple of itself. It follows easily that this multiple must be a constant, depending only on x. Calling it $l(x)$, we have

$$[x, y] \equiv l(x)y \mod \mathbf{R}x.$$

The precise function $l(x)$ can be computed by noting that

$$\operatorname{trace} \operatorname{ad}(x) = (n-1)\, l(x).$$

Thus $l(x)$ depends linearly on x. Interchanging the roles of x and y, we can also compute $[x, y]$ modulo $\mathbf{R}y$. If x and y are linearly independent, these two computations combine to yield the precise formula

$$[x, y] = l(x)y - l(y)x.$$

Evidently this formula is also true, for trivial reasons, when x and y are linearly dependent.

Since the commutative case $l = 0$ is uninteresting, let us suppose that $l \neq 0$. Let $\mathfrak{u}$ be the kernel of the linear transformation l. Clearly $\mathfrak{u}$ is a commutative ideal. Choose a unit vector b orthogonal to $\mathfrak{u}$, and let $\lambda = l(b)$. (Evidently λ can be identified with the norm $\| l \|$ of the linear transformation l.) With notation as in 5.5, the linear transformation $L(u) = [b, u]$ is given by $L(u) = \lambda u$. Applying 5.5, it follows easily that ∇_b is the zero transformation and that

$$\nabla_u z = \lambda(b\langle u, z\rangle - u\langle b, z\rangle)$$

for any u in $\mathfrak{u}$ and any z in $\mathfrak{g}$. *We now assert that* R_{xy} *is given by the formula*

$$R_{xy}(z) = \lambda^2(x\langle y, z\rangle - y\langle x, z\rangle)$$

for any x, y, z *in the Lie algebra*. For example, if $x = b$ and $y = u \in \mathfrak{u}$ then $R_{bu} = \nabla_{[b,u]} - \nabla_b\nabla_u + \nabla_u\nabla_b = \lambda\nabla_u - 0 + 0$ equals the required expression; while if both x and y belong to $\mathfrak{u}$ then only a slightly longer computation is needed. The remaining cases follow by bilinearity.

Substituting this formula into the definition $\kappa(x, y) = \langle R_{xy}(x), y\rangle$ we see that

$$\kappa(x, y) = \lambda^2(\langle x, y\rangle^2 - \langle x, x\rangle\langle y, y\rangle).$$

Hence $\kappa(x, y)$ takes the constant value $-\lambda^2$ whenever x and y are orthonormal. In other words the metric has constant sectional curvature $K \equiv -\lambda^2 < 0$. ∎

Here is a further application of 5.5.

Proof of Lemma 2.3. The hypothesis of this lemma is that a unit vector (which we now call b) is orthogonal to the commutator ideal

$[\mathfrak{g}, \mathfrak{g}]$. It follows that the orthogonal complement of b contains $[\mathfrak{g}, \mathfrak{g}]$ and hence is an ideal, so that 5.5 applies. We must compute the Ricci curvature $r(b)$. By definition this equals $\kappa(b, u_1) + \cdots + \kappa(b, u_{n-1})$ where $u_1, \ldots, u_{n-1}$ is any orthonormal basis for $\mathfrak{u}$. It is easiest to work with an orthonormal basis consisting of eigenvectors

$$Su_i = \lambda_i u_i$$

of the self adjoint operator S. For any unit vector in $\mathfrak{u}$ the sectional curvature can be computed as

$$\begin{aligned}\kappa(b, u) &= \langle R_{bu}(b), u\rangle \\ &= \langle \nabla_{[b,u]}b, u\rangle - \langle \nabla_b \nabla_u b, u\rangle + \langle \nabla_u \nabla_b b, u\rangle \\ &= \langle -SLu, u\rangle + \langle \tfrac{1}{2}(L - L^*)\, Su, u\rangle + 0.\end{aligned}$$

Taking u to be an eigenvector as above, and noting that

$$\langle Lu_i , u_i\rangle = \langle u_i , L^* u_i\rangle = \lambda_i ,$$

this reduces to $\kappa(b, u_i) = -\lambda_i^2$. Hence

$$r(b) = -\lambda_1^2 - \cdots - \lambda_{n-1}^2 = -\mathrm{trace}(S^2).$$

Thus $r(b) \leqslant 0$, with equality if and only if $S = 0$ so that L is skew-adjoint. ∎

Caution: It is *not* asserted that $\kappa(b, u) \leqslant 0$ for all u. In fact, for some particular choice of u (not an eigenvector) it may well happen that $\kappa(b, u) > 0$ by 1.2. This happens, for example, in the case of the Heisenberg group.

Next let us compute the scalar curvature $\rho = \rho(\mathfrak{g})$. Thinking of $\mathfrak{u}$ as a Lie algebra in its own right with the induced metric, let $\rho(\mathfrak{u})$ denote its scalar curvature. With notations as in 5.5 we will prove the following.

LEMMA 5.6. *The scalar curvature $\rho(\mathfrak{g})$ associated with the metric Lie algebra $\mathfrak{g}$ is equal to*

$$\rho(\mathfrak{u}) - \mathrm{trace}(S^2) - (\mathrm{trace}\ S)^2.$$

Proof. Given orthonormal vectors u, v in $\mathfrak{u}$, let us compare the sectional curvature

$$\kappa(u, v) = \langle \nabla_{[u,v]}u, v\rangle - \langle \nabla_u \nabla_v u, v\rangle + \langle \nabla_v \nabla_u u, v\rangle$$

as computed in $\mathfrak{g}$ with the sectional curvature

$$\bar{\kappa}(u, v) = \langle \bar{\nabla}_{[u,v]}u, v\rangle - \langle \bar{\nabla}_u \bar{\nabla}_v u, v\rangle + \langle \bar{\nabla}_v \bar{\nabla}_u u, v\rangle$$

as computed in the Lie algebra $\mathfrak{u}$. Using 5.5, inspection shows that

$$\kappa(u, v) = \bar{\kappa}(u, v) + \langle Su, v\rangle^2 - \langle Su, u\rangle\langle Sv, v\rangle.$$

Choosing an orthonormal basis consisting of eigenvectors, $Su_i = \lambda_i u_i$, it follows that

$$\kappa(u_i, u_j) = \bar{\kappa}(u_i, u_j) - \lambda_i \lambda_j$$

for $i \neq j$. Combining this with the formula

$$\kappa(b, u_i) = -\lambda_i^2,$$

we see that the Ricci curvature in the direction u_i is given by

$$r(u_i) = \bar{r}(u_i) - \lambda_i \operatorname{trace}(S).$$

Summing over i and adding $r(b) = -\operatorname{trace}(S^2)$, this gives the required formula

$$\rho(\mathfrak{g}) = \rho(\mathfrak{u}) - \operatorname{trace}(S)^2 - (\operatorname{trace} S^2). \quad \blacksquare$$

Caution: This computation does not provide a complete description of Ricci curvature, since the basis $b, u_1, \ldots, u_{n-1}$ may not diagonalize the Ricci quadratic form.

Proof of Theorem 3.1. If $\mathfrak{g}$ is solvable of dimension n, we will prove by induction on n that $\rho(\mathfrak{g}) \leqslant 0$. Certainly every solvable Lie algebra contains an ideal $\mathfrak{u}$ of codimension 1. Furthermore $\mathfrak{u}$ itself is solvable, so we may assume inductively that $\rho(\mathfrak{u}) \leqslant 0$. Therefore

$$\rho(\mathfrak{g}) \leqslant -\operatorname{trace}(S^2) - (\operatorname{trace} S)^2 \leqslant 0,$$

where equality holds only if both $S = 0$ and $\rho(\mathfrak{u}) = 0$.

If both of these conditions are satisfied, we must prove that $\mathfrak{g}$ is flat. Since $S = 0$, the formulas 5.5 imply that

$$\nabla_u v = \bar{\nabla}_u v \in \mathfrak{u}$$

for any u and v in the ideal $\mathfrak{u}$. (In other words the ideal $\mathfrak{u}$ is "totally geodesic" in $\mathfrak{g}$.) It follows immediately that

$$R_{uv}(w) = \bar{R}_{uv}(w)$$

for any u, v, w in the ideal. But we have assumed that the scalar curvature $\rho(\mathfrak{u})$ is zero, so it follows by induction that $\mathfrak{u}$ is flat. Thus $\bar{R} = 0$, hence $R_{uv}(w) = 0$.

Again applying 5.5 with $S = 0$, we have $\nabla_x b = 0$ for any x in the Lie algebra, hence $R_{xy}(b) = 0$ for any x and y. Using the symmetry property

$$\langle R_{xy}(b), z\rangle = \langle R_{bz}(x)\, y\rangle$$

of the Riemann tensor, it follows that $R_{bz} = 0$ for any z. Combining these statements with the fact that $R_{xy}(z)$ is trilinear as a function of x, y, z, it then follows easily that R is identically zero. ▮

Remark. Here is a sketch of an alternature argument. Let H be the subgroup of the universal covering $\tilde{G}$ which has Lie algebra $\mathfrak{u}$. Using the techniques of Section 7.2, one can prove that $S = 0$ if and only if the smooth mapping

$$(h, \tau) \mapsto h \exp(\tau b)$$

from $H \times \mathbf{R}$ to $\tilde{G}$ is a Riemannian isometry. Thus if H is flat and $S = 0$, it follows directly that $\tilde{G}$ is flat.

6. Unimodular and Non-unimodular Lie Groups

Recall that a Lie group G is called *unimodular* if its left invariant Haar measure is also right invariant. Here is a simple and classical criterion. Recall that each group element g determines an automorphism

$$h \mapsto ghg^{-1}$$

of the group G. The induced automorphism of the Lie algebra is called $\mathrm{Ad}(g)$.

Lemma 6.1. *The group G is unimodular if and only if the linear transformation* $\mathrm{Ad}(g)$ *has determinant* ± 1 *for every g in G.*

This is proved, for example, in [7, p. 366]. (A completely analogous argument is given in 7.1 below.) ▮

As an example, if G is compact or connected semisimple then the homomorphism

$$g \mapsto |\det \mathrm{Ad}(g)|$$

from G to the positive real numbers must certainly be trivial, hence G is unimodular. Here is another interesting criterion.

LEMMA 6.2. *If G admits a discrete subgroup Γ with compact quotient, then G is unimodular.*

Proof. It is not difficult to choose a compact fundamental domain D for the left action of Γ on G, that is a compact set $D \subset G$ so that the various left translates γD cover G and so that the intersections $\gamma D \cap \gamma' D$ have measure zero for $\gamma \neq \gamma'$. Choosing a left invariant Haar measure ω, note that the measure $\omega(D) \neq 0$ is independent of the choice of D. For if E is another fundamental domain then

$$\omega(E) = \sum \omega(\gamma D \cap E) = \sum \omega(D \cap \gamma^{-1} E) = \omega(D),$$

where both summations extend over all elements γ of Γ.

For any group element g, note that the right translate Dg is also a fundamental domain for the left action of Γ on G. Hence $\omega(D) = \omega(Dg)$, and it follows that the left invariant measure ω is also right invariant. ∎

Remark. Even if $\Gamma\backslash G$ is not compact, the left Haar measure on G induces a measure on $\Gamma\backslash G$. *Whenever $\Gamma\backslash G$ has finite measure, a similar argument shows that G is unimodular.*

In terms of the Lie algebra we have the following criterion.

LEMMA 6.3. *A connected Lie group is unimodular if and only if the linear transformation* $\mathrm{ad}(x)$ *has trace zero for every x in the associated Lie algebra.*

As an example, if $\mathfrak{g}$ is nilpotent, then every $\mathrm{ad}(x)$ is nilpotent, and hence has trace zero.

Proof. We will use two different "exponential mappings." If l is a linear transformation from a finite dimensional vector space to itself, let

$$e^l = \sum l^n/n!.$$

Using Jordan canonical form one sees that

$$\det(e^l) = e^{\mathrm{trace}\, l}.$$

On the other hand, for any Lie group G, there is a smooth mapping

$$\exp: \mathfrak{g} \to G$$

characterized by the property that each x in $\mathfrak{g}$ gives rise to a homomorphism

$$\tau \mapsto \exp(\tau x)$$

from the Lie group of real numbers to G, the associated Lie algebra homomorphism being multiplication by x. These two exponential mappings are related by the identity

$$\mathrm{Ad}(\exp(x)) = e^{\mathrm{ad}(x)}.$$

See, for example, [7, p. 118].

Now if $|\det \mathrm{Ad}(g)|$ is identically equal to 1, it follows that

$$\det \mathrm{Ad}(\exp(x)) = \det e^{\mathrm{ad}(x)} = e^{\mathrm{trace}\, \mathrm{ad}(x)}$$

is identically 1, hence $\mathrm{trace}\, \mathrm{ad}(x) \equiv 0$. Conversely, if $\mathrm{trace}\, \mathrm{ad}(x) \equiv 0$ this argument shows that $\det \mathrm{Ad}(g) = 1$ for all g in the image of the exponential mapping. Using the inverse function theorem, this includes all g in some neighborhood of the identity. But G, being connected, is generated by any neighborhood of the identity; hence $\det \mathrm{Ad}(g)$ is identically equal to 1. ∎

A Lie algebra which satisfies this condition $\mathrm{trace}\, \mathrm{ad}(x) \equiv 0$ will be called a *unimodular Lie algebra*.

Now let $\mathfrak{g}$ be a completely arbitrary Lie algebra. Using the Jacobi identity

$$\mathrm{ad}[x, y] = \mathrm{ad}(x)\, \mathrm{ad}(y) - \mathrm{ad}(y)\, \mathrm{ad}(x)$$

we see that $\mathrm{ad}[x, y]$ has trace zero. *Therefore the linear mapping*

$$x \mapsto \mathrm{trace}\, \mathrm{ad}(x)$$

from $\mathfrak{g}$ *to the commutative Lie algebra* $\mathbf{R}$ *is actually a homomorphism of Lie algebras.* In particular its kernel

$$\mathfrak{u} = \{x \in \mathfrak{g} \mid \mathrm{trace}\, \mathrm{ad}(x) = 0\}$$

is an ideal, containing the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$. We will call $\mathfrak{u}$ the *unimodular kernel* of $\mathfrak{g}$. It is easy to check that $\mathfrak{u}$ itself is a unimodular Lie algebra. Here is an application:

LEMMA 6.4. *If the connected Lie group G has a left invariant metric with all Ricci curvatures $\geqslant 0$, then G is unimodular.*

Proof. Suppose on the contrary that G were not unimodular. Choosing a unit vector b orthogonal to the unimodular kernel, we would have trace $\mathrm{ad}(b) \neq 0$. Hence $\mathrm{ad}(b)$ could not be skew-adjoint, and it would follow by 2.3 that $r(b) < 0$. This contradiction completes the proof. ∎

The rest of this section will be concerned with proofs of statements from Section 4 concerning 3-dimensional Lie algebras.

Proof of Lemma 4.1. Let $\mathfrak{g}$ be a 3-dimensional Lie algebra with positive definite metric and with preferred orientation. Choosing an oriented orthonormal basis e_1, e_2, e_3, define the linear transformation $L: \mathfrak{g} \to \mathfrak{g}$ by $L(e_1) = [e_2, e_3]$, $L(e_2) = [e_3, e_1]$, $L(e_3) = [e_1, e_2]$. Then the identity $L(e_i \times e_j) = [e_i, e_j]$ is true for all basis elements, hence $L(x \times y) = [x, y]$ for all x and y. Setting

$$L(e_i) = \sum \alpha_{ij} e_j ,$$

inspection shows that

$$\begin{aligned} \text{trace ad}(e_1) &= -\alpha_{23} + \alpha_{32} \\ \text{trace ad}(e_2) &= -\alpha_{31} + \alpha_{13} \\ \text{trace ad}(e_3) &= -\alpha_{12} + \alpha_{21} . \end{aligned}$$

Thus $\mathfrak{g}$ is unimodular if and only if the matrix (α_{ij}) is symmetric, or in other words if and only if the linear transformation L is self adjoint. ∎

Proof of Theorem 4.3. We must compute curvature under the hypothesis that $\mathfrak{g}$ has an orthonormal basis with

$$[e_2, e_3] = \lambda_1 e_1 , \qquad [e_3, e_1] = \lambda_2 e_2 , \qquad [e_1, e_2] = \lambda_3 e_3 .$$

Using 5.3 or 5.4 we easily obtain the formula $\nabla_{e_i} v = \mu_i e_i \times v$ for any vector v, or briefly

$$\nabla_{e_i} = \mu_i e_i \times ,$$

where

$$\mu_1 = \tfrac{1}{2}(-\lambda_1 + \lambda_2 + \lambda_3), \quad \mu_2 = \tfrac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3), \quad \mu_3 = \tfrac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3).$$

For example $\nabla_{e_1} = \mu_1 e_1 \times$ is the skew-adjoint transformation defined by

$$e_1 \mapsto 0, \qquad e_2 \mapsto \mu_1 e_3, \qquad e_3 \mapsto -\mu_1 e_2 .$$

Using the Jacobi identity

$$e_1 \times (e_2 \times v) - e_2 \times (e_1 \times v) = (e_1 \times e_2) \times v$$

it follows that

$$R_{e_1 e_2} = \nabla_{\lambda_3 e_3} - \nabla_{e_1} \nabla_{e_2} + \nabla_{e_2} \nabla_{e_1}$$

is equal to the linear transformation $(\lambda_3 \mu_3 - \mu_1 \mu_2) e_3 \times$. Hence

$$R_{e_1 e_2}(e_1) = (\lambda_3 \mu_3 - \mu_1 \mu_2)\, e_2 ,$$

with similar formulas for the other $R_{e_i e_j}(e_i)$. Recalling the definition

$$\hat{r}(x) = \sum R_{e_i x}(e_i)$$

from Section 2, it follows that

$$\hat{r}(e_2) = (\lambda_3 \mu_3 - \mu_1 \mu_2)\, e_2 + (\lambda_1 \mu_1 - \mu_2 \mu_3)\, e_2 = 2\mu_1 \mu_3 e_2 ,$$

with similar formulas for the other $\hat{r}(e_i)$. Thus e_1, e_2, e_3 are eigenvectors of the Ricci transformation $\hat{r}$, with corresponding eigenvalues equal to $2\mu_2\mu_3$, $2\mu_1\mu_3$, and $2\mu_1\mu_2$, respectively. ▮

Proof of Lemma 4.10. We now consider a 3-dimensional Lie algebra $\mathfrak{g}$ which is *not* unimodular. Its unimodular kernel $\mathfrak{u}$, being 2-dimensional and unimodular, must clearly be commutative. Choose e_1 in $\mathfrak{g}$ so that trace $\mathrm{ad}(e_1) = 2$. Since $\mathfrak{u}$ is commutative, the linear transformation

$$L(u) = [e_1 , u]$$

from $\mathfrak{u}$ to itself, with trace 2, is independent of the particular choice of e_1.

If L maps each vector to a multiple of itself, then we are in the special case of Section 1.7 (and in fact L must be the identity map). *Otherwise, the determinant D of L provides a complete isomorphism invariant.* For choosing e_2 so that the vectors e_2 and $L(e_2) = e_3$ are linearly independent, the conditions trace$(L) = 2$, det$(L) = D$ imply that

$$L(e_2) = e_3$$
$$L(e_3) = -De_2 + 2e_3 .$$

Thus the bracket product operation (with respect to a carefully chosen basis) is uniquely determined. ∎

Suppose that we are given a positive definite metric on the non-unimodular Lie algebra $\mathfrak{g}$. Choose an orthonormal basis e_1, e_2, e_3 so that e_1 is orthogonal to $\mathfrak{u}$, and so that the two image vectors $[e_1, e_2]$ and $[e_1, e_3]$ are mutually orthogonal. The bracket product can then be expressed as

$$[e_1, e_2] = \alpha e_2 + \beta e_3$$
$$[e_1, e_3] = \gamma e_2 + \delta e_3$$

and $[e_2, e_3] = 0$; with $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$.

Remarks. If we further normalize by requiring that $\alpha \geqslant \delta$, $\beta \geqslant \gamma$, and $\alpha + \delta > 0$, then these structure constants α, β, γ, δ are uniquely determined. Note that the determinant invariant of 4.10 is now given by $D = 4(\alpha\delta - \beta\gamma)/(\alpha + \delta)^2$.

LEMMA 6.5. *This basis also diagonalizes the Ricci quadratic form, the principal Ricci curvatures being*

$$r(e_1) = -\alpha^2 - \delta^2 - \tfrac{1}{2}(\beta + \gamma)^2$$
$$r(e_2) = -\alpha(\alpha + \delta) + \tfrac{1}{2}(\gamma^2 - \beta^2)$$
$$r(e_3) = -\delta(\alpha + \delta) + \tfrac{1}{2}(\beta^2 - \gamma^2).$$

The proof, based on Section 5.5, is tedious but straightforward, and will be left to the reader. ∎

The *proof of Theorem* 4.11 will be given in outline only. To simplify the formulas in 6.5, let us make a scale change if necessary so that $\alpha + \delta = 2$. Then setting

$$\alpha = 1 + \xi, \qquad \beta = (1 + \xi)\eta,$$
$$\gamma = -(1 - \xi)\eta, \qquad \delta = 1 - \xi,$$

we may assume that $\xi \geqslant 0$, $\eta \geqslant 0$. The special case $\xi = \eta = 0$ (corresponding to Example 1.7) must be excluded. The expressions for principal curvatures now take the form

$$r(e_1) = -2(1 + \xi^2(1 + \eta^2)) \leqslant -2$$
$$r(e_2) = -2(1 + \xi(1 + \eta^2)) \leqslant -2$$
$$r(e_3) = -2(1 - \xi(1 + \eta^2)),$$

and the determinant becomes

$$D = (1 - \xi^2)(1 + \eta^2).$$

As examples, taking $\xi = 0$ we obtain metrics of constant negative curvature for every $D > 1$, and taking $\eta = 0$ we obtain metrics of negative sectional curvature whenever $0 < D < 1$. Further details will be left to the reader. ▌

7. Bi-invariant Metrics

Recall that a Riemannian metric on G is called *bi-invariant* if it is invariant under both left and right translation. We will first outline the classical theory.

Lemma 7.1. *A left invariant metric on G is also right invariant if and only if, for each group element g, the linear transformation*

$$\mathrm{Ad}(g): \mathfrak{g} \to \mathfrak{g}$$

is an isometry with respect to the induced metric on the Lie algebra $\mathfrak{g}$.

(Compare Section 6.1.) In other words the homomorphism $g \mapsto \mathrm{Ad}(g)$ must map G into the orthogonal group $O(n)$ consisting of all linear isometries of $\mathfrak{g}$.

Proof. Let $l_g: G \to G$ denote left translation by g and let r_g denote right translation. Thus $\mathrm{Ad}(g)$ is induced by the smooth mapping $l_g r_g^{-1}$ from G to itself. Since the metric μ is left invariant, we have $l_g{}^*\mu = \mu$. If μ is also right invariant, $r_g{}^*\mu = \mu$, then evidently

$$(l_g r_g^{-1})^* \mu = \mu$$

so that $\mathrm{Ad}(g)$ is an isometry, and conversely. ▌

LEMMA 7.2. *In the case of a connected group G, a left invariant metric is actually bi-invariant if and only if the linear transformation* $\mathrm{ad}(x)$ *is skew-adjoint for every x in the Lie algebra of G.*

Proof. If g is sufficiently close to the identity then $g = \exp(x)$ for some uniquely defined x close to zero. As in Section 6.3 we use the identity

$$\mathrm{Ad}(g) = \mathrm{Ad}(\exp(x)) = e^{\mathrm{ad}(x)}.$$

Recall that $\mathrm{Ad}(g)$ is orthogonal if and only if

$$\mathrm{Ad}(g)^{-1} = \mathrm{Ad}(g)^*.$$

Since the left side equals $e^{-\mathrm{ad}(x)}$ while the right side equals $e^{\mathrm{ad}(x)^*}$, this condition is satisfied if and only if

$$-\mathrm{ad}(x) = \mathrm{ad}(x)^*$$

so that $\mathrm{ad}(x)$ is skew-adjoint. Since a connected Lie group is generated by any neighborhood of the identity, and since products of orthogonal transformations are orthogonal, the conclusion follows. ∎

DEFINITION. It will be convenient to say that a metric on $\mathfrak{g}$ is *bi-invariant* if every $\mathrm{ad}(x)$ is skew-adjoint. Note that a bi-invariant metric on $\mathfrak{g}$ induces a bi-invariant metric on any subalgebra of $\mathfrak{g}$.

Using such a bi-invariant metric, the last two terms in formula 5.3 cancel so that we obtain simply

$$\nabla_x = \tfrac{1}{2}\mathrm{ad}(x).$$

The curvature transformation R_{xy} then is equal to $\frac{1}{2}\mathrm{ad}([x, y]) - \frac{1}{4}\mathrm{ad}(x)\,\mathrm{ad}(y) + \frac{1}{4}\mathrm{ad}(y)\,\mathrm{ad}(x)$. Using the Jacobi identity $\mathrm{ad}[x, y] = \mathrm{ad}(x)\,\mathrm{ad}(y) - \mathrm{ad}(y)\,\mathrm{ad}(x)$ this reduces to

$$R_{xy} = \tfrac{1}{4}\mathrm{ad}[x, y].$$

Hence the biquadratic curvature function $\kappa(x, y) = \langle R_{xy}(x), y\rangle$ equals $\frac{1}{4}\langle[[x, y], x], y\rangle$. Using skew-adjointness once more, this can be written as

$$\kappa(x, y) = \tfrac{1}{4}\langle[x, y], [x, y]\rangle. \tag{7.3}$$

Thus $\kappa(x, y) \geqslant 0$, *with equality if and only if* $[x, y] = 0$. It follows that all Ricci curvatures satisfy $r(u) \geqslant 0$, with equality if and only if u belongs to the center of $\mathfrak{g}$.

We can give a completely explicit formula for the Ricci quadratic form as follows. Recall that the *Killing form* β is defined by $\beta(x, y) = \operatorname{trace}(\operatorname{ad}(x)\operatorname{ad}(y))$. Since $r(x)$ can be defined as the trace of the linear transformation

$$y \mapsto R_{xy}(x) = \tfrac{1}{4}[[x, y], x] = -\tfrac{1}{4}\operatorname{ad}(x)^2 y,$$

it follows that $r(x) = -\frac{1}{4}\beta(x, x)$. *Thus the quadratic form $r(x)$ is independent of the particular choice of bi-invariant metric.*

One important property of bi-invariant metrics is the following. Recall that a Lie algebra is *simple* if it contains no ideals other than 0 and itself.

LEMMA 7.4. *If the metric on $\mathfrak{g}$ is bi-invariant, then the orthogonal complement of any ideal is itself an ideal. Hence $\mathfrak{g}$ can be expressed as an orthogonal direct sum of simple ideals.*

Proof. If y is orthogonal to the ideal $\mathfrak{a}$, then we must prove that $[x, y]$ is orthogonal to $\mathfrak{a}$. But

$$\langle [x, y], a\rangle = -\langle y, [x, a]\rangle = 0$$

for any a in $\mathfrak{a}$. Thus $\mathfrak{g}$ splits as a direct sum $\mathfrak{a} \oplus \mathfrak{a}^{\perp}$ of ideals. The conclusion now follows by an easy induction. ∎

If $\mathfrak{g}$ equals the orthogonal direct sum $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k$ of simple ideals, then the simply connected Lie group $\tilde{G}$ can be expressed correspondingly as the cartesian product $A_1 \times \cdots \times A_k$ of normal subgroups. For each simply connected factor A_i there are two possibilities:

Case 1. *If $\mathfrak{a}_i$ is commutative, hence* 1*-dimensional, then* $A_i \cong \mathbf{R}$.

Case 2. If $\mathfrak{a}_i$ is non-commutative, then the center of $\mathfrak{a}_i$ must be trivial, hence A_i has strictly positive Ricci curvature. Applying Myers' theorem as in Section 2.2, it follows that A_i is compact.

LEMMA 7.5. *The connected Lie group G admits a bi-invariant metric if and only if it is isomorphic to the cartesian product of a compact group and an additive vector group.*

Proof. If G admits a bi-invariant metric, then the argument above shows that the universal covering $\tilde{G}$ splits as the cartesian product of a compact group H and a vector group $\mathbf{R}^m$. The group G can be identified with the quotient $\tilde{G}/\Pi$ where Π is a discrete normal subgroup of $\tilde{G}$. Projecting Π into $\mathbf{R}^m$, let V be the vector space spanned by its image and let $V^\perp$ be the orthogonal complement. Then G is the cartesian product of the compact group $(H \times V)/\Pi$ and the vector group $V^\perp$.

The converse is straightforward. Any commutative group certainly admits a bi-invariant metric, and any compact group can be given a bi-invariant metric by starting with an arbitrary metric μ on the Lie algebra $\mathfrak{g}$ and then averaging $\mathrm{Ad}(g)^*\mu$ as g varies over G. See, for example, [3, p. 176]. ∎

In the case of a simple group this metric is essentially unique.

LEMMA 7.6. *If the Lie algebra* $\mathfrak{g}$ *of a compact Lie group is simple, then the bi-invariant metric is unique up to multiplication by a positive constant. Such a metric necessarily has constant Ricci curvature.*

Proof. Let $\langle x, y \rangle$ be one bi-invariant metric on $\mathfrak{g}$. Then any other metric on $\mathfrak{g}$ can be expressed as $\langle Sx, y \rangle$ where S is some self adjoint operator. If this new metric is also bi-invariant, then expressing the fact that $\mathrm{ad}(u)$ is skew-adjoint with respect to both metrics we see that $\mathrm{ad}(u)$ commutes with S, and hence maps each eigenspace of S into itself. This implies that each eigenspace is an ideal. Since $\mathfrak{g}$ is simple, it follows that S has only one eigenspace, say $Sx = \lambda x$ for all x. Thus the bi-invariant metric is essentially unique.

Choosing such a bi-invariant metric, consider the associated Ricci quadratic form $r(x)$. Setting $r(x) = \langle \hat{r}(x), x \rangle$ where $\hat{r}$ is self adjoint, the inner product $\langle \hat{r}(x), y \rangle$ can be considered as a Riemannian metric on G, since r is positive definite. Evidently this metric is invariant under both left and right translations. So the argument above proves that $\hat{r}(x) = \lambda x$ and hence

$$r(u) = \langle \hat{r}(u), u \rangle = \lambda$$

for every unit vector u, where $\lambda > 0$ is constant. ∎

If we make a scale change, that is, multiply the metric $\langle x, y \rangle$ by a positive constant, it is easy to check that the connection ∇, the Riemann tensor R, and the Ricci form $r(x)$ remain unchanged. Therefore, we may choose the metric so that

$$\langle x, x \rangle \equiv r(x),$$

or so that Ricci curvature is identically $+1$.

COROLLARY 7.7. *Any Lie group whose universal covering is compact admits a bi-invariant metric of constant Ricci curvature* $+1$.

Proof. As noted above, $\tilde{G}$ splits as a cartesian product $A_1 \times \cdots \times A_k$ of simple groups. Each simple group has a unique bi-invariant metric with Ricci curvature $+1$, and the product metric then also has Ricci curvature identically equal to 1. ▌

In fact it is easy to check that there is precisely one such bi-invariant metric, namely $\langle x, y\rangle = -\frac{1}{4}\beta(x, y)$ where $\beta(x, y) = \operatorname{trace}(\operatorname{ad}(x)\operatorname{ad}(y))$ is the Killing form.

Next let us characterize flat left invariant metrics.

Proof of Theorem 1.5. Let G be a simply connected Lie group which admits a flat left invariant metric. If we ignore the group structure and think of G only as a complete Riemannian manifold it follows that G *is isometric to Euclidean space.* As an immediate consequence, note that *every compact subgroup of* G *is trivial.* (Compare Iwasawa's theorem quoted below.) For any compact subgroup, acting by left translation, would yield a compact group of isometries of Euclidean space. Using an averaging process, any such group of isometries would have a fixed point; but nontrivial left translations cannot have fixed points.

For any $\mathfrak{g}$ with metric, the correspondence $x \mapsto \nabla_x$ defines a linear mapping from $\mathfrak{g}$ to the Lie algebra $\mathfrak{o}(n)$ consisting of all skew-adjoint mappings from $\mathfrak{g}$ to itself. If the curvature tensor is identically zero

$$\nabla_{[x,y]} = \nabla_x\nabla_y - \nabla_y\nabla_x ,$$

then this correspondence is a homomorphism of Lie algebras. Hence its kernel $\mathfrak{u}$ is an ideal. Using the identity

$$[u, v] = \nabla_u v - \nabla_v u$$

it follows that $\mathfrak{u}$ is commutative.

Let $\mathfrak{b}$ be the orthogonal complement of $\mathfrak{u}$. For each $b \in \mathfrak{b}$ the identity

$$[b, u] = \nabla_b u - \nabla_u b = \nabla_b u$$

shows that the skew-adjoint transformation ∇_b maps the ideal $\mathfrak{u}$ into itself. Hence it maps the orthogonal complement $\mathfrak{b}$ into itself. *It follows that* $\mathfrak{b}$ *is a sub Lie algebra of* $\mathfrak{g}$.

Clearly $\mathfrak{b}$ maps isomorphically to a sub Lie algebra of $\mathfrak{o}(n)$. Since $\mathfrak{o}(n)$ is the Lie algebra of a compact group $O(n)$, it possesses a bi-invariant metric. Therefore $\mathfrak{b}$ possesses a bi-invariant metric, that is, one satisfying the conditions of 7.2. (Caution: This bi-invariant metric on $\mathfrak{b}$ may have nothing to do with the originally given metric.) Therefore, by 7.4, $\mathfrak{b}$ splits as a direct sum $\mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_k$ of simple ideals. If one of these simple ideals $\mathfrak{b}_i$ were non-commutative, then the corresponding simple Lie group B_i would be compact. Hence the inclusion $\mathfrak{b}_i \subset \mathfrak{b} \subset \mathfrak{g}$ would induce a non-trivial homomorphism $B_i \to G$. Hence G would contain a non-trivial compact subgroup; which is impossible. We conclude that each $\mathfrak{b}_i$ is commutative. *Therefore the Lie algebra* $\mathfrak{b}$ *is commutative.*

For each b in $\mathfrak{b}$, note that $\mathrm{ad}(b)$ is skew-adjoint. For $\mathrm{ad}(b)$ restricted to $\mathfrak{b}$ is trivial, while $\mathrm{ad}(b)$ restricted to $\mathfrak{u}$ coincides with the skew-adjoint transformation ∇_b.

Thus $\mathfrak{g}$ splits as an orthogonal direct sum $\mathfrak{u} \oplus \mathfrak{b}$ where $\mathfrak{u}$ is a commutative ideal, $\mathfrak{b}$ is a commutative sub algebra, and each $\mathrm{ad}(b)$ is skew-adjoint. Conversely, if these conditions are satisfied, then using 5.3 we see that

$$\nabla_u = 0, \qquad \nabla_b = \mathrm{ad}(b),$$

and it follows easily that the curvature tensor is identically zero. ∎

As a final application, we will construct metrics of positive scalar curvature. First recall the following basic result.

THEOREM OF IWASAWA. *If G is a connected Lie group, then:*

(a) *Every compact subgroup is contained in a maximal compact subgroup H, which is necessarily a connected Lie group.*

(b) *This maximal compact subgroup is unique up to conjugation.*

(c) *As a topological space, G is homeomorphic with the product of H and some Euclidean space $\mathbf{R}^m$.*

This statement was proved in [9], although substantial parts of it had been obtained earlier by Cartan and by Malcev. Here is an immediate consequence.

COROLLARY. *The universal covering of G is homeomorphic to Euclidean space if and only if every compact subgroup of G is commutative.*

The proof is easily supplied. ∎

Proof of Theorem 3.4. Let G be a connected Lie group, and suppose that G contains a compact non-commutative subgroup H. We must construct a metric of strictly positive scalar curvature. By Iwasawa's theorem, we may assume that H is connected. Since H is compact we can construct a positive definite metric on $\mathfrak{g}$ which is invariant under the linear transformation

$$\mathrm{Ad}(h)\colon \mathfrak{g} \to \mathfrak{g}$$

for every h in H. Using this metric, let $e_1, \ldots, e_m$ be an orthonormal basis for the Lie algebra of H, and extend to an orthonormal basis $e_1, \ldots, e_n$ for $\mathfrak{g}$. Inspecting the proof of 7.2, we see that the linear transformations

$$\mathrm{ad}(e_1), \ldots, \mathrm{ad}(e_m)$$

must be skew-adjoint. (However, the remaining transformations $\mathrm{ad}(e_{m+1}), \ldots, \mathrm{ad}(e_n)$ will not usually be skew-adjoint.)

Fixing any $\epsilon > 0$, consider a new basis $e_1', \ldots, e_n'$ defined by

$$e_1' = e_1, \ldots, e_m' = e_m, \qquad e_{m+1}' = \epsilon e_{m+1}, \ldots, e_n' = \epsilon e_n .$$

Choose a new metric so that this basis $e_1', \ldots, e_n'$ is orthonormal. The symbol $\mathfrak{g}_\epsilon$ will denote the Lie algebra $\mathfrak{g}$ provided with this new metric, and provided with this specified orthonormal basis. As in the proof of 2.5, we note that the structure constants $\langle [e_i', e_j'], e_k' \rangle$ associated with $\mathfrak{g}_\epsilon$ tend to well defined limits as $\epsilon \to 0$. Hence there is a well defined limit Lie algebra $\mathfrak{g}_0$, provided with a specified metric and orthonormal basis. Evidently $\mathfrak{g}_0$ splits as an orthogonal direct sum $\mathfrak{h} \oplus \mathfrak{u}$ where $\mathfrak{h}$ is the subalgebra spanned by $e_1', \ldots, e_m'$ and $\mathfrak{u}$ is the commutative ideal spanned by $e_{m+1}', \ldots, e_n'$. Note also that $\mathrm{ad}(b)$ is skew-adjoint for each $b \in \mathfrak{h}$. Applying 5.3 we see that $\nabla_u = 0$ for every u in $\mathfrak{u}$, hence $R_{xu} = 0$ and $\kappa(x, u) \neq 0$ for all x. In particular the Ricci curvature $r(u)$ is zero for $u \in \mathfrak{u}$. On the other hand for b in $\mathfrak{h}$ we have $r(b) \geqslant 0$ by 2.1, where equality does not always hold since $\mathfrak{h}$ is noncommutative. *Therefore the scalar curvature* $\rho = r(e_1') + \cdots + r(e_n')$ *of the limit algebra* $\mathfrak{g}_0$ *is strictly positive.* It follows by continuity that $\rho(\mathfrak{g}_\epsilon) > 0$ whenever ϵ is sufficiently small. ∎

Remark 7.8. This limit algebra $\mathfrak{g}_0$ provides an interesting example of a metric Lie algebra with all sectional curvatures $K \geqslant 0$.

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