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Braided Tensor Categories

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INTRODUCTION

Categories enriched with tensor products, here called tensor categories, but also called monoidal categories, have been studied and used extensively in the literature [ML1, EK, ML2, SR, DM]. Large examples such as the categories of Abelian groups and of Banach spaces are important for studying mathematical structures. Small examples, as found in particular in algebraic topology, are important as mathematical structures in their own right.

Some tensor products behave like composition and so are not generally expected to be commutative. Yet categories with a "commutative" tensor product deserve special attention in the same way that commutative rings do in ring theory. Natural examples of commutativity are not strict in the sense of an equality $A \otimes B = B \otimes A$. Rather, natural isomorphisms $c_{A,B}: A \otimes B \to B \otimes A$ exist. In the case of categories of sets with structure $c_{A,B}$ is given on elements by a simple switch in order. It seemed reasonable,

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especially in view of Mac Lane's coherence theorem [ML1, ML2], to assume the symmetry condition

$$c_{B,A} \circ c_{A,B} = 1_{A \otimes B}. \tag{S}$$

Together with a condition (B1) of "bilinearity" expressing $c_{A,B \otimes C}$ in terms of $c_{A,B}$ and $c_{A,C}$, this is the notion of *symmetry* for a tensor category [EK]. The property (B2), expressing $c_{A \otimes B,C}$ in terms of $c_{A,C}$ and $c_{B,C}$, is then a consequence. A weaker notion of commutativity was contemplated in [KR], which kept (S) and replaced (B1).

During May 1985 we were led to analyze the dropping of (S) with the retention of (B1) and (B2). Such commutativities we call *braidings*. A tensor category equipped with a distinguished braiding is called *braided*.

Early justification for isolating the notion of braided tensor category came from homotopy theory. For one of us, this came via [JT] where the particular tensor categories, arising as algebraic homotopy 3-types of arc-connected, simply connected spaces, are what we call braided categorical groups: they are groupoids and each object has a quasi-inverse with respect to the tensor product.

For the other of us, initial motivation for braidings came from higher-order category theory. Bicategories with tensor product had been considered [Wlt] as a base for enriched category theory, and it was suggested that such a bicategory with one object might amount precisely to a symmetric tensor category (by a generalization of the argument [EH] commutativity of the higher homotopy groups). When we checked the details of this suggestion, it was a braiding rather than a symmetry which resulted.

The publication of this paper has been delayed, mainly by us. Although by now there are many publications concerning braided tensor categories, we feel there is still need for this fundamental paper on the subject to be widely available. In retrospect, the changes from our initial written version [JS1] are not so major, as we shall explain below.

Like the present paper, [JS1] included the definition, examples including the braid category as the free braided tensor category on a single generating object, the coherence theorem for braided tensor categories, the modified argument of [EH] put in terms of extra multiplications on tensor categories, a little bit of enriched category theory over a braided base, and a classification theorem for braided categorical groups (extending [SHX] which deals with the symmetric case).

Two talks in 1986 led to a widening interest in braided tensor categories. One was the talk of P. Freyd at the Category Theory Conference at Cambridge University; he described joint work with D. Yetter [FY1] in which they had discovered that their category of tangles was braided and autonomous (that is, each object has a dual; such tensor categories are also called "compact" and "rigid"), and was "free" in some sense. The other was V. Drinfel'd's International Congress talk on quantum groups where the "quasitriangular" bialgebras are examples of braided protensor categories. These talks provided classes of new examples of our structure: directly in the case of the former; by convolution [Da1] on the categories of representations in the latter case.

M. C. Shum began looking at coherence for autonomous braided tensor categories with a view to modifying the work of [KL] on the symmetric case. She discovered that the category of tangles was not the free autonomous braided tensor category in the most straightforward interpretation of this statement. The problem was resolved in [FY1] by considering *regular* isotopy classes of geometric tangles.

In mid-1987, we were led to deal with this problem in a different way: by looking at tangles of ribbons instead of tangles of strings. This led us to the notion of *balanced tensor categories* which, as well as a braiding, have a *twist* on each object; the free category as such is the category of braids on ribbons. Incorporating dual objects, we were led to define *tortile tensor categories.*

An interesting example of a braiding which we found at that time was on the tensor category of complex representations of the general linear groups $GL(n, F)$, where F is a fixed finite field. The proof of invertibility of this braiding turned out to be quite difficult and was not achieved until the next year.

Also in 1987, the first author's talk at the Category Theory Conference in Louvain-la-Neuve, entitled "Knot invariants revisited," discussed polynomial invariants from a categorical viewpoint. During the Conference, Iain Aitchison brought us a copy of a preprint (which became [T] from the Low-Dimensional Topology Conference at Sussex the previous week). Turaev's work involved a geometric tensor category but, instead of braidings, he used Yang–Baxter operators.

We decided it was necessary to make clear the connection between braidings and Yang–Baxter operators. Hence the 1988 version of the present paper dealt with this subject and included the example on representations of $GL(n, F)$. Some discussion of quantum groups was also included.

Since [FY1, FY2, JS2, JS4, SMC] have already appeared, we have decided to drop most of the references to quantum groups and tangles from this paper and to concentrate on a self-contained account of coherence for braided tensor categories, with full proofs which do not need reference to the literature on coherence theory. We found it necessary to begin by giving a version of Mac Lane's coherence theorem [ML1, ML2] for mere tensor categories. In fact, we hope our account will provide a key for entry into that literature.

Finally, we point out that the various sections are, to a large extent, independent.

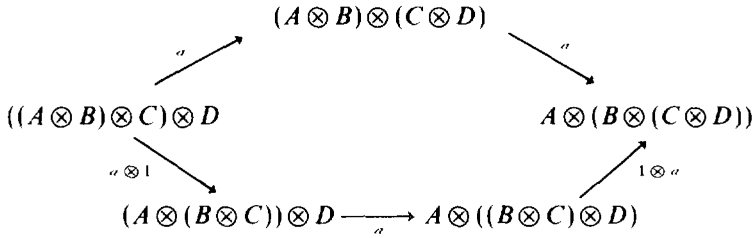
1. Tensor Categories

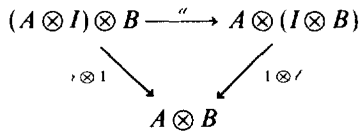
After defining the concepts, we provide a brief, yet thorough, account of the coherence theorems for tensor categories and for tensor functors. The main idea, which differs from that of Mac Lane [ML1, ML2], is based on the observation of Robert Gordon and John Power that Corollary 1.4 (below) follows from the Yoneda Lemma for bicategories [St1]. Some of the techniques we use are scattered through the literature [K3, K4, K5, BKP] and generalize to other categorical structures.

Recall [ML2] that a *tensor* (or "monoidal") *category* $\mathscr{V} = (\mathscr{V}, \otimes, I, a, \ell, r)$ consists of a category $\mathscr{V}$, a functor $\otimes : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$ (called the *tensor product*), an object $I \in \mathscr{V}$ (called the *unit object*) and natural isomorphisms

$$a = a_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C),$$
$$\ell = \ell_A : I \otimes A \xrightarrow{\sim} A, \qquad r = r_A : A \otimes I \xrightarrow{\sim} A$$

(called the *associativity*, *left unit*, *right unit constraints*, respectively) such that, for all objects A, B, C, $D \in \mathscr{V}$, the following two diagrams (called the *associativity pentagon* and the *triangle for unit*) commute:



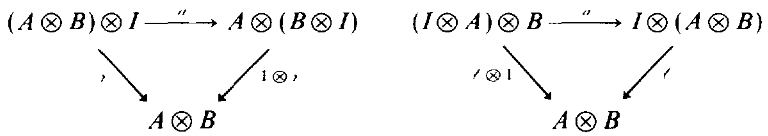


A tensor category is called *strict* when all the constraints $a_{A,B,C}$, ℓ_A, r_A are identity arrows.

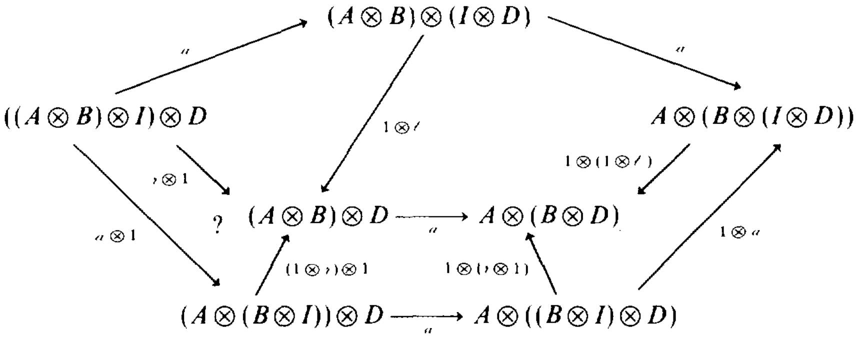
Proposition 1.1 [K2]. *In a tensor category, the equality*

$$\ell_I = r_I$$

holds, and the following two diagrams commute:

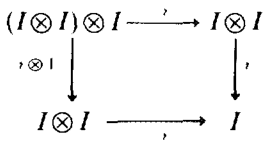


Proof. In the following diagram, the triangular region distinguished by the question mark (?) commutes, since all the other regions and the outside of the diagram commute and the arrows are all invertible. Now take $D = I$ and use the fact that r is natural and invertible to obtain commutativity of the first triangle of the proposition:



Commutativity of the second triangle is proved similarly.

The following square commutes by naturalness of r.



Since r is invertible this gives $r = r \otimes 1 : (I \otimes I) \otimes I \to I \otimes I$. But then, taking $A = B = I$ in the first triangle of the Proposition and the triangle for unit, and using the invertibility of a, we obtain $1 \otimes r = 1 \otimes l : I \otimes (I \otimes I) \to I \otimes I$. By naturalness and invertibility of l, we deduce that $r = l : I \otimes I \to I$.

Q.E.D.

Suppose $\mathscr{V}$, $\mathscr{W}$ are tensor categories. A *tensor functor* $F = (F, \phi_2, \phi_0)$: $\mathscr{V} \to \mathscr{W}$ consists of a functor $F: \mathscr{V} \to \mathscr{W}$, a family of natural isomorphisms

$$\phi_{2,A,B}: FA \otimes FB \xrightarrow{\sim} F(A \otimes B),$$

and an isomorphism $\phi_0 : I \xrightarrow{\sim} FI$ such that the following three diagrams commute:

$$\begin{array}{ccccc} & & FA \otimes (FB \otimes FC) & & \\ & \nearrow^{a} & & \searrow^{1 \otimes \phi_2} & \\ (FA \otimes FB) \otimes FC & & & & FA \otimes F(B \otimes C) \\ \downarrow{\scriptstyle \phi_2 \otimes 1} & & & & \downarrow{\scriptstyle \phi_2} \\ F(A \otimes B) \otimes FC & & & & F(A \otimes (B \otimes C)) \\ & \searrow_{\phi_2} & & \nearrow_{Fa} & \\ & & F((A \otimes B) \otimes C) & & \end{array}$$

$$\begin{array}{ccc} FA \otimes I & \xrightarrow{r} & FA \\ {\scriptstyle 1 \otimes \phi_0}\downarrow & & \uparrow{\scriptstyle Fr} \\ FA \otimes FI & \xrightarrow[\phi_2]{} & F(A \otimes I) \end{array} \qquad \begin{array}{ccc} I \otimes FA & \xrightarrow{l} & FA \\ {\scriptstyle \phi_0 \otimes 1}\downarrow & & \uparrow{\scriptstyle Fl} \\ FI \otimes FA & \xrightarrow[\phi_2]{} & F(I \otimes A) \end{array}$$

The tensor functor is called *strict* when each of the isomorphisms $\phi_{2,A,B}$, ϕ_0 is an identity. Suppose $G : \mathscr{V} \to \mathscr{W}$ is also a tensor functor. A *morphism* $\theta : F \to G$ *of tensor functors* is a natural transformation $\theta : F \to G$ such that the two following diagrams commute:

$$\begin{array}{ccc} FA \otimes FB & \xrightarrow{\phi_2} & F(A \otimes B) \\ {\scriptstyle \theta_A \otimes \theta_B}\downarrow & & \downarrow{\scriptstyle \theta_{A \otimes B}} \\ GA \otimes GB & \xrightarrow[\phi_2]{} & G(A \otimes B) \end{array} \qquad \begin{array}{ccc} & \overset{\phi_0}{\nearrow} & FI \\ I & & \downarrow{\scriptstyle \theta_I} \\ & \underset{\phi_0}{\searrow} & GI \end{array}$$

In order to express the coherence theorem for tensor categories, we need to consider the free tensor category $\mathscr{F}\mathscr{C}$ generated by a category $\mathscr{C}$. The objects of $\mathscr{F}\mathscr{C}$ are given inductively by the requirement that they include the objects of $\mathscr{C}$, a new object I, and an object $(M \otimes N)$ for any two objects $M, N \in \mathscr{F}\mathscr{C}$. The arrows of $\mathscr{F}\mathscr{C}$ are equivalence classes of arrows built up formally from the arrows of $\mathscr{C}$ and the basic constraints a, l, r by tensoring, substituting, inverting, and composing, where the equivalence relation is generated by the axioms for a tensor category. There is a functor $i : \mathscr{C} \to \mathscr{F}\mathscr{C}$ such that, for each functor $F : \mathscr{C} \to \mathscr{V}$ into a tensor category $\mathscr{V}$, there exists a unique strict tensor functor $F' : \mathscr{F}\mathscr{C} \to \mathscr{V}$ such that the following triangle commutes:

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{i} & \mathscr{F}\mathscr{C} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle F'} & \\ & \mathscr{V} & \end{array}$$

Let $\mathscr{F}_s\mathscr{C}$ denote the free *strict* tensor category on the category $\mathscr{C}$; so $\mathscr{F}_s\mathscr{C}$ is just the free monoid on $\mathscr{C}$. The objects are words $A_1A_2 \dots A_n$ in objects of $\mathscr{C}$ and the arrows $f_1f_2 \dots f_n: A_1A_2 \dots A_n \to B_1B_2 \dots B_n$ are words in arrows $f_i: A_i \to B_i$ of $\mathscr{C}$; so $\mathscr{F}_s\mathscr{C}$ is the disjoint union of the categories $\mathscr{C}^n$, $n > 0$. The tensor in $\mathscr{F}_s\mathscr{C}$ is concatenation. Then there is a unique strict tensor functor $\Gamma: \mathscr{F}\mathscr{C} \to \mathscr{F}_s\mathscr{C}$ such that $\Gamma \circ i$ is the inclusion of the generators. In fact, Γ is the universal strict tensor functor out of $\mathscr{F}\mathscr{C}$ which forces the constraint isomorphisms a, ℓ, r to be identities. We prove

THEOREM 1.2 (Coherence for Tensor Categories). *For all categories $\mathscr{C}$, the functor*

$$\Gamma: \mathscr{F}\mathscr{C} \to \mathscr{F}_s\mathscr{C}$$

is an equivalence.

We begin by constructing, from any tensor category $\mathscr{V}$, a strict one denoted by $\mathbf{e}(\mathscr{V})$. The construction is inspired by the Cayley Theorem in which a monoid M is embedded in the monoid of endofunctions $\mathrm{End}(M)$ by using the left regular representation. The image of M in $\mathrm{End}(M)$ can be characterized as the self maps $M \to M$ commuting with all right translations.

The objects (E, ρ) of $\mathbf{e}(\mathscr{V})$ consist of a functor $E: \mathscr{V} \to \mathscr{V}$ and a natural family of isomorphisms

$$\rho_{A,B}: (EA) \otimes B \xrightarrow{\sim} E(A \otimes B).$$

The natural transformation ρ might be thought of as a witness of the fact that E commutes with right translations $(-) \otimes B$. An arrow $\theta: (E, \rho) \to (E', \rho')$ is a natural transformation $\theta: E \to E'$ such that the following square commutes for all objects A, B of $\mathscr{V}$.

$$\begin{array}{ccc} (EA)\otimes B & \xrightarrow{\rho_{A,B}} & E(A\otimes B) \\ {\scriptstyle \theta_A \otimes 1_B}\downarrow & & \downarrow{\scriptstyle \theta_{A\otimes B}} \\ (E'A)\otimes B & \xrightarrow[\rho'_{A,B}]{} & E'(A\otimes B) \end{array}$$

Composition is the vertical composition of natural transformations. The tensor product for $\mathbf{e}(\mathscr{V})$ is given on objects (E, ρ), (F, σ) by the equation

$$(E, \rho) \otimes (F, \sigma) = (E \circ F, \tau),$$

where $\tau_{A,B}$ is the composite

$$(EFA) \otimes B \xrightarrow{\rho_{FA,B}} E(FA \otimes B) \xrightarrow{E\sigma_{A,B}} EF(A \otimes B);$$

the tensor product on arrows is horizontal composition of natural transformations. Thus $\mathbf{e}(\mathscr{V})$ becomes a strict tensor category with unit object $(1_{\mathscr{V}}, 1_{\otimes})$.

There is a functor $L\colon \mathscr{V} \to \mathbf{e}(\mathscr{V})$ given by

$$LX = (X \otimes -, a_{X,\ ,\ }), \qquad Lf = f \otimes -.$$

PROPOSITION 1.3. *$L\colon \mathscr{V} \to \mathbf{e}(\mathscr{V})$ is a fully faithful tensor functor.*

Proof. Suppose $\theta\colon LX \to LY$ in $\mathbf{e}(\mathscr{V})$. Let $f\colon X \to Y$ be the composite

$$X \xrightarrow{\imath_X} X \otimes I \xrightarrow{\theta_I} Y \otimes I \xrightarrow{\imath_Y^{-1}} Y.$$

It follows from commutativity of the diagram

$$\begin{array}{ccccccc}
X \otimes A & \xrightarrow{\imath_X^{-1} \otimes 1} & (X \otimes I) \otimes A & \xrightarrow{a_{X,I,A}} & X \otimes (I \otimes A) & \xrightarrow{1_X \otimes \ell_A} & X \otimes A \\
{\scriptstyle f \otimes 1_A}\downarrow & & \downarrow{\scriptstyle \theta_I \otimes 1_A} & & \downarrow{\scriptstyle \theta_{I \otimes A}} & & \downarrow{\scriptstyle \theta_A} \\
Y \otimes A & \xrightarrow[\imath_Y^{-1} \otimes 1]{} & (Y \otimes I) \otimes A & \xrightarrow[a_{Y,I,A}]{} & Y \otimes (I \otimes A) & \xrightarrow[1_Y \otimes \ell_A]{} & Y \otimes A
\end{array}$$

and the triangle for unit that $\theta_A = f \otimes 1_A = Lf$. If $Lf = Lf'$ then $f \otimes 1_I = f' \otimes 1_I$, so $f = f'$ by naturality of $\imath$. So L is fully faithful.

To see that L is a tensor functor, we note that we have

$$\ell^{-1}\colon (1_{\mathscr{V}}, 1_{\otimes}) \to (I \otimes -, a_{I,\ ,\ }),$$

which gives an arrow $\phi_0\colon I \to LI$ by Proposition 1.1, and we have

$$a_{X,Y,}^{-1}\colon (X \otimes (Y \otimes -), (X \otimes a_{Y,\ ,\ }) \circ (a_{X,Y \otimes\ ,\ }))$$
$$\to ((X \otimes Y) \otimes -, a_{X \otimes Y,\ ,\ }),$$

which gives an arrow $\phi_2\colon LX \otimes LY \to L(X \otimes Y)$ by the associativity pentagon. The hexagon diagram involved in showing that this gives a tensor functor reduces to a pentagon since $\mathbf{e}(\mathscr{V})$ is strictly associative; this pentagon is again that for associativity. The square diagrams reduce to triangles, of which one is the triangle for unit, and the other is from Proposition 1.1. Q.E.D.

COROLLARY 1.4. *Every tensor category is tensor equivalent to a strict one.*

Proof. Take the full image of L. Q.E.D.

This Corollary 1.4 by itself is a coherence result. However, it is a weaker statement than the Coherence Theorem (Theorem 1.2) because it uses a tensor functor while the Coherence Theorem involves a *strict* tensor functor. Later we prove a coherence theorem for tensor functors.

We describe the construction for functors of the analogue of the equalizer for maps. Given functors $S, T\colon \mathscr{A} \to \mathscr{X}$, the category $\mathrm{Eq}(S, T)$ has objects (A, h) consisting of an object $A \in \mathscr{A}$ and an isomorphism $h\colon SA \xrightarrow{\sim} TA$ in $\mathscr{X}$, and has arrows $f\colon (A, h) \to (A', h')$, those arrows $f\colon A \to A'$ in $\mathscr{A}$ for which the following square commutes:

$$\begin{array}{ccc} SA & \xrightarrow{h} & TA \\ {\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Tf} \\ SA' & \xrightarrow[h']{} & TA' \end{array}$$

There is a functor $P\colon \mathrm{Eq}(S, T) \to \mathscr{A}$ given by $P(A, h) = A$ and $Pf = f$. There is a natural isomorphism $\sigma\colon SP \xrightarrow{\sim} TP$ whose component at (A, h) is h. If S, T are tensor functors between tensor categories $\mathscr{A}$, $\mathscr{X}$ then a straightforward calculation shows that $\mathrm{Eq}(S, T)$ supports a unique structure of tensor category such that P becomes a *strict* tensor functor and σ becomes an isomorphism of tensor functors.

We now prove the following "flexibility" result.

PROPOSITION 1.5. *Every tensor functor $T\colon \mathscr{F}\mathscr{C} \to \mathscr{V}$ is isomorphic to a strict tensor functor $S\colon \mathscr{F}\mathscr{C} \to \mathscr{V}$.*

Proof. By freeness of $\mathscr{F}\mathscr{C}$, there is a unique strict tensor functor $S\colon \mathscr{F}\mathscr{C} \to \mathscr{V}$ such that $Si = Ti\colon \mathscr{C} \to \mathscr{V}$. Also, there exists a unique functor $F\colon \mathscr{C} \to \mathrm{Eq}(S, T)$ such that $PF = i$ and σF is an identity. So we have a unique strict tensor functor $F'\colon \mathscr{F}\mathscr{C} \to \mathrm{Eq}(S, T)$ with $F'i = F$. By freeness of $\mathscr{F}\mathscr{C}$, the strictness of PF', and the equality $PF'i = 1_{\mathscr{F}\mathscr{C}}\, i$, we obtain the equality $PF' = 1_{\mathscr{F}\mathscr{C}}$. So we have the isomorphism $\sigma F'\colon S \to T$ of tensor functors. Q.E.D.

We need one final observation before giving the proof of Theorem 1.2. Each functor $F\colon \mathscr{C} \to \mathscr{V}$ into a tensor category extends to a functor $F'\colon \mathscr{F}_s\mathscr{C} \to \mathscr{V}$ (not *a priori* a tensor functor) whose restriction F'_n to $\mathscr{C}^n$ is defined inductively by

$$F'_0 = I, \qquad F'_1 = F, \qquad F'_{n+1} = (\mathscr{C}^{n+1} = \mathscr{C}^n \times \mathscr{C} \xrightarrow{F'_n \times F} \mathscr{V} \times \mathscr{V} \xrightarrow{\otimes} \mathscr{V}).$$

Proof of Theorem 1.2. First observe that $\Gamma\colon \mathscr{F}\mathscr{C} \to \mathscr{F}_s C$ is surjective on objects and full since it is a splitting for the extension $i'\colon \mathscr{F}_s\mathscr{C} \to \mathscr{F}\mathscr{C}$ of the functor $i\colon \mathscr{C} \to \mathscr{F}\mathscr{C}$. So it remains to prove that Γ is faithful. Proposition 1.3 gives the construction of a faithful tensor functor $T\colon \mathscr{F}\mathscr{C} \to \mathscr{W}$ with $\mathscr{W}$ strict. By Proposition 1.5, we have an isomorphism $S \xrightarrow{\sim} T$ with

$S: \mathcal{F}\mathcal{C} \to \mathcal{W}$ strict. By the universal property of $\mathcal{F}_s\mathcal{C}$ and $\mathcal{F}\mathcal{C}$, there is a unique strict tensor functor $R: \mathcal{F}_s\mathcal{C} \to \mathcal{W}$ such that $R\Gamma = S$. But S is faithful since it is isomorphic to T, and T is faithful. Then $R\Gamma = S$ implies that Γ is faithful. Q.E.D.

COROLLARY 1.6. *In the free tensor category generated by a set A of objects, every diagram commutes.*

Proof. We have an equivalence $\mathcal{F}A \simeq \mathcal{F}_s A$, and the category $\mathcal{F}_s A$ is discrete. Q.E.D.

To relate Corollary 1.6 to the coherence theorem of Mac Lane [ML2, p. 165], it is convenient to use the notion of clique. Recall from [JS2] that a *clique* in a category is a non-empty family $(C_k \mid k \in K)$ of objects and a family of arrows $(u_{kh}: C_k \to C_h \mid (k, h) \in K \times K)$ such that $u_{kh} \circ u_{jk} = u_{jh}$ and $u_{kk} = 1$ (so that u_{kh} has inverse u_{hk}). In other words, a clique is a functor whose domain is a category equivalent to the category $\mathbf{1}$ (with one object and an identity arrow).

Now consider a tensor category $\mathcal{V}$ with set of objects $K = \text{obj } \mathcal{V}$, and the unique strict tensor functor $t: \mathcal{F}(K) \to \mathcal{V}$ which extends the inclusion of the discrete category K in $\mathcal{V}$. By Corollary 1.6, the fibre $\Gamma^{-1}(w)$ of the functor $\Gamma: \mathcal{F}(K) \simeq \mathcal{F}_s(K)$ over an object $w = A_1 A_2 \ldots A_n$ is a category equivalent to the category $\mathbf{1}$. The image of $\Gamma^{-1}(w)$ under $t: \mathcal{F}(K) \to \mathcal{V}$ gives a clique in $\mathcal{V}$. For example, if $w = ABCDE$, the clique gives a canonical isomorphism

$$(A \otimes I) \otimes ((B \otimes C) \otimes (I \otimes (D \otimes E)))$$
$$\xrightarrow{\sim} (I \otimes (A \otimes B)) \otimes ((I \otimes I) \otimes ((C \otimes D) \otimes E))).$$

For each tensor category $\mathcal{V}$, there is a *strict* tensor category $\mathbf{st}(\mathcal{V})$. The objects of $\mathbf{st}(\mathcal{V})$ are words $w = A_1 A_2 \ldots A_m$ in objects of $\mathcal{V}$. An arrow $f: W \to W'$ is an arrow $f: [w] \to [w']$ in $\mathcal{V}$, where we define

$$[\varnothing] = I, \qquad [A] = A, \qquad \text{and} \qquad [A_1 A_2 \ldots A_{m+1}] = [A_1 A_2 \ldots A_m] f A_{m+1}.$$

Composition is just that of $\mathcal{V}$. In other words, the extension $1'_{\mathcal{V}}: \mathcal{F}_s(\mathcal{V}) \to \mathcal{V}$ of the identity $1_{\mathcal{V}}$ of $\mathcal{V}$ factors as a composite of a functor $\mathcal{F}_s(\mathcal{V}) \to \mathbf{st}(\mathcal{V})$ which is the identity on objects and a fully faithful functor $[\ ]: \mathbf{st}(\mathcal{V}) \to \mathcal{V}$. The tensor $\bar{\otimes}$ for $\mathbf{st}(\mathcal{V})$ is given by $v \bar{\otimes} w = vw$ and by commutativity of the square

$$\begin{array}{ccc}
[v] \otimes [w] & \xrightarrow{f \otimes g} & [v'] \otimes [w'] \\
\wr \downarrow & & \wr \downarrow \\
[vw] & \xrightarrow[f \bar{\otimes} g]{} & [v'w']
\end{array}$$

in which the vertical isomorphisms are those of the clique. Thus $[\ ]: \mathbf{st}(\mathcal{V}) \to \mathcal{V}$ becomes a tensor functor. The functors $s: \mathcal{V} \to \mathbf{st}(\mathcal{V})$, which takes each object $A \in \mathcal{V}$ to the word A of length 1 and each arrow to itself, is a splitting of the functor $[\ ]: \mathbf{st}(\mathcal{V}) \to \mathcal{V}$. Since the tensor function $[\ ]: \mathbf{st}(\mathcal{V}) \to \mathcal{V}$ provides a fully faithful retraction for s, it follows that $s: \mathcal{V} \to \mathbf{st}(\mathcal{V})$ is an equivalence of tensor categories. This gives another proof of Corollary 1.4, but this proof uses the coherence theorem.

We would like to show that each tensor functor $T: \mathcal{V} \to \mathcal{W}$ induces a strict tensor functor $\mathbf{st}(T): \mathbf{st}(\mathcal{V}) \to \mathbf{st}(\mathcal{W})$. The way to define the functor $\mathbf{st}(T)$ on objects is obvious, but to define it on arrows requires a coherence theorem for tensor functors which we now develop.

There is a tensor functor $H': \mathcal{F}_0 H \to \mathcal{F}_1 H$ generated freely by a given functor $H: \mathcal{C} \to \mathcal{D}$. That is, $\mathcal{F}_0 H$ is the free tensor category $\mathcal{F}\mathcal{C}$, and $\mathcal{F}_1 H$ is the tensor category formally generated by $\mathcal{D}$ and the structural isomorphisms ϕ_2, ϕ_0 of a tensor functor extending H to the tensor functor H' on $\mathcal{F}\mathcal{C}$. More precisely, there is a commutative square of functors

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{H} & \mathcal{D} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ \mathcal{F}\mathcal{C} & \xrightarrow[H']{} & \mathcal{F}_1 H \end{array}$$

such that, for all commutative squares

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{H} & \mathcal{D} \\ {\scriptstyle M}\downarrow & & \downarrow{\scriptstyle N} \\ \mathcal{V} & \xrightarrow[T]{} & \mathcal{W} \end{array}$$

with T a tensor functor, there exists a unique commutative square of tensor functors

$$\begin{array}{ccc} \mathcal{F}\mathcal{C} & \xrightarrow{H'} & \mathcal{F}_1 H \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle V} \\ \mathcal{V} & \xrightarrow[T]{} & \mathcal{W} \end{array}$$

with U, V strict and $Ui = M$, $Vj = N$. In particular, there is a commutative square

$$\begin{array}{ccc} \mathcal{F}\mathcal{C} & \xrightarrow{H'} & \mathcal{F}_1 H \\ {\scriptstyle \Gamma}\downarrow & & \downarrow{\scriptstyle \Delta} \\ \mathcal{F}_s\mathcal{C} & \xrightarrow[\mathcal{F}_s H]{} & \mathcal{F}_s\mathcal{D} \end{array}$$

where Δ is strict and $\Delta j: \mathcal{D} \to \mathcal{F}_s\mathcal{D}$ is the inclusion of the generators.

THEOREM 1.7 (Coherence for Tensor Functors). *For all functors* $H: \mathscr{C} \to \mathscr{D}$, *the functor* $\Delta: \mathscr{F}_1 H \to \mathscr{F}_s \mathscr{D}$ *is an equivalence.*

COROLLARY 1.8. *In the codomain category* $\mathscr{F}_1 \xi$ *of the free tensor functor generated by a function* $\xi: A \to M$, *every diagram commutes.*

Proof. The functor $\Delta: \mathscr{F}_1 \xi \to \mathscr{F}_s M$ is an equivalence, and $\mathscr{F}_s M$ is discrete. Q.E.D.

LEMMA 1.9. *Suppose the following squares of tensor functors commute for* M, N_i *strict and* $i = 1, 2$:

$$\begin{array}{ccc} \mathscr{F}\mathscr{C} & \xrightarrow{H'} & \mathscr{F}_1 H \\ {\scriptstyle M}\downarrow & & \downarrow{\scriptstyle N_i} \\ \mathscr{V} & \xrightarrow[T_i]{} & \mathscr{W} \end{array}$$

For each isomorphism $\tau: T_1 \to T_2$ *of tensor functors, there exists a unique isomorphism* $\nu: N_1 \to N_2$ *of tensor functors such that*

$$\nu H' = \tau M.$$

Proof. Let $\mathscr{W}^I$ be the tensor category whose objects are invertible arrows in $\mathscr{W}$ and whose morphisms are commutative squares. The isomorphism τ induces a tensor functor $T: \mathscr{V} \to \mathscr{W}^I$. Applying the universal property of H', we obtain a commutative square

$$\begin{array}{ccc} \mathscr{F}\mathscr{C} & \xrightarrow{H'} & \mathscr{F}_1 H \\ {\scriptstyle M}\downarrow & & \downarrow{\scriptstyle N} \\ \mathscr{V} & \xrightarrow[T]{} & \mathscr{W} \end{array}$$

from which the result follows. Q.E.D.

Proof of Theorem 1.7. The functor Δ is surjective on objects and full since it is a splitting for the extension $j': \mathscr{F}_s \mathscr{D} \to \mathscr{F}_1 H$ of j. It remains to prove Δ faithful.

By Proposition 1.5, the tensor functor H' is isomorphic to a strict tensor functor $S: \mathscr{F}\mathscr{C} \to \mathscr{F}_1 H$. The universal property of H' gives the commutative square

$$\begin{array}{ccc} \mathscr{F}\mathscr{C} & \xrightarrow{H'} & \mathscr{F}_1 H \\ \Vert & & \downarrow{\scriptstyle E} \\ \mathscr{F} C & \xrightarrow[S]{} & \mathscr{F}_1 H \end{array}$$

We also have the identity square

$$\begin{array}{ccc} \mathscr{F}\mathscr{C} & \xrightarrow{H'} & \mathscr{F}_1 H \\ \| & & \|\, 1 \\ \mathscr{F}\mathscr{C} & \xrightarrow[H']{} & \mathscr{F}_1 H \end{array}$$

and $S \cong H'$, so we can apply Lemma 1.9. This shows that $E \cong 1$, and so E is an equivalence. From the universal property of H', we have a commutative square

$$\begin{array}{ccc} \mathscr{F}\mathscr{C} & \xrightarrow{H'} & \mathscr{F}_1 H \\ \| & & \downarrow \Delta_1 \\ \mathscr{F}\mathscr{C} & \xrightarrow[\mathscr{F} H]{} & \mathscr{F}\mathscr{D} \end{array}$$

and also $\Gamma \Delta_1 = \Delta$. By Theorem 1.2, it suffices to prove that Δ_1 is faithful. Since S is strict, E factors through Δ_1. Since E is an equivalence, Δ_1 is faithful. Q.E.D.

We now return to the question of how to define $\mathbf{st}(T)\colon \mathbf{st}(\mathscr{V}) \to \mathbf{st}(\mathscr{W})$ for any tensor functor $T\colon \mathscr{V} \to \mathscr{W}$. For an object $w = A_1 \ldots A_m$ of $\mathbf{st}(\mathscr{V})$, define

$$\mathrm{st}(T)(w) = (TA_1) \ldots (TA_m).$$

For an arrow $f\colon w \to w'$, define

$$\mathbf{st}(T)(f)\colon [TA_1 \ldots TA_m] \to [TA'_1 \ldots TA'_n]$$

by requiring commutativity of the square

$$\begin{array}{ccc} [TA_1 \ldots TA_m] & \xrightarrow{\mathrm{st}(T)(f)} & [TA'_1 \ldots TA'_n] \\ \wr\,\big|\, 1 & & \wr\,\big|\, 1 \\ T[A_1 \ldots A_m] & \xrightarrow[T(f)]{} & T[A'_1 \ldots A'_n] \end{array}$$

where the vertical sides are those of the clique in $\mathscr{W}$ obtained using Corollary 1.8. This construction produces a strict tensor functor $\mathbf{st}(T)\colon \mathbf{st}(\mathscr{V}) \to \mathbf{st}(\mathscr{W})$. Composition of tensor functors is preserved by the construction, so we are able to replace diagrams of tensor categories and tensor functors by equivalent diagrams of strict tensor categories and strict tensor functors.

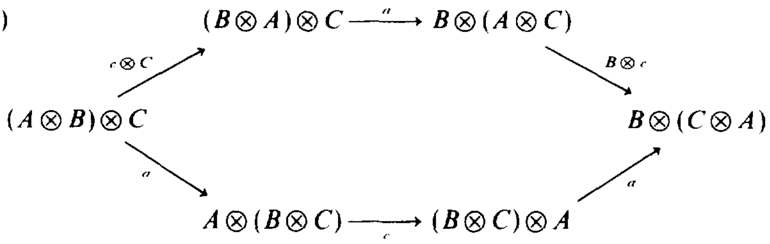
2. Braidings and Yang–Baxter Operators

Definition 2.1. A *braiding* for a tensor category $\mathscr{V}$ consists of a natural family of isomorphisms

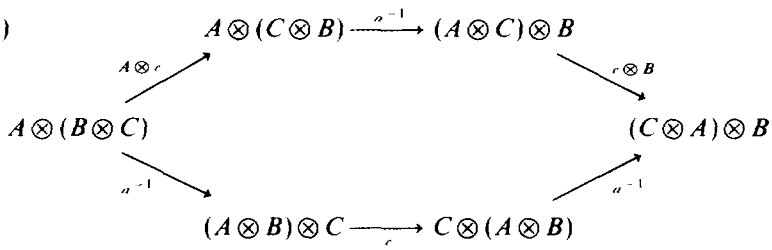
$$c = c_{A,B} \colon A \otimes B \xrightarrow{\sim} B \otimes A$$

in $\mathscr{V}$ such that the two diagrams (B1) and (B2) commute:

(B1)



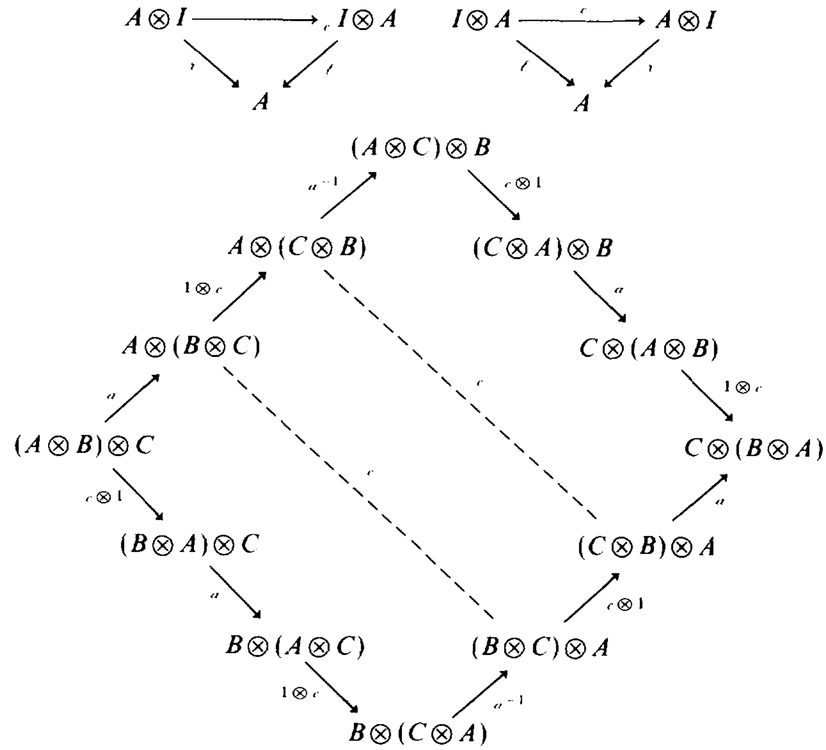
(B2)



If c is a braiding then so too is c' given by $c'_{A,B} = (c_{B,A})^{-1}$ since (B2) is just obtained from (B1) by replacing c with c'. A *symmetry* [EK] is a braiding for which $c = c'$.

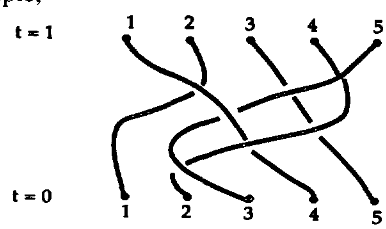
Definition 2.2. A *braided tensor category* is a pair $(\mathscr{V}, c)$ consisting of a tensor category $\mathscr{V}$ and a braiding c.

Proposition 2.1. *In a braided tensor category, the following diagrams commute*:

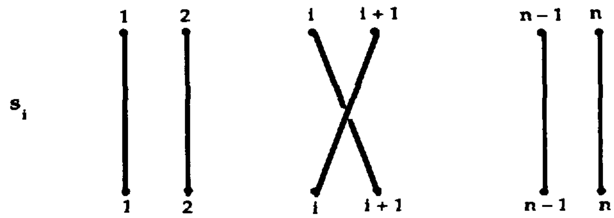


Proof. For the first triangle, take $B = C = I$ in diagram (B1), and use the coherence of a, $\imath$, ℓ and the invertibility of $c_{A,I}$. The second triangle is obtained similarly from (B2) with $A = B = I$. The big diagram is shown to be commutative by constructing the indicated dashed arrows: two of the regions commute by (B1) and the quadrilateral commutes by naturality of c. Q.E.D.

EXAMPLE 2.1 (Braids, and labelled braids on strings). Let P denote a Euclidean plane and let $C_n(P)$ be the space of subsets of P of cardinality n. The *Artin braid group* $\mathfrak{B}_n$ is the fundamental group of $C_n(P)$. Denoting some n distinct collinear points of P by $1, 2, \ldots, n$, we can describe a loop $\omega\colon [0, 1] \to C_n(P)$ at the point $\{1, 2, \ldots, n\}$ of $C_n(P)$ by its graph in $[0, 1] \times P$; for example,



where a horizontal cross section by P at level $t \in [0, 1]$ intersects the curves (called the *strings*) in the subset $\omega(t)$ of P. Let s_i be the braid depicted by



A presentation [A, B] for $\mathfrak{B}_n$ is given by the generators $s_1, ..., s_{n-1}$ and the relations

$$\text{(A1)} \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad \text{for} \quad 1 \leqslant i \leqslant n-2,$$

$$\text{(A2)} \quad s_i s_j = s_j s_i \qquad \text{for} \quad 1 \leqslant i < j-1 \leqslant n-2.$$

The *braid category* B is the disjoint union of the $\mathfrak{B}_n$. More explicitly, the objects of $\mathfrak{B}$ are the natural numbers 0, 1, 2, ..., the homsets are given by

$$\mathfrak{B}(m, n) = \begin{cases} \mathfrak{B}_n & \text{when} \quad m = n \\ \varnothing & \text{otherwise,} \end{cases}$$

and composition is the multiplication of the braid groups. The category $\mathfrak{B}$ is equipped with a strict tensor structure defined by *addition of braids*,

$$\oplus : \mathfrak{B}_m \times \mathfrak{B}_n \to \mathfrak{B}_{m+n},$$

which is algebraically described by the equation

$$s_i \oplus s_j = s_i s_{m+j} \; (= s_{m+j} s_i),$$

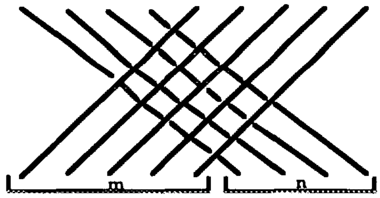
and is pictured as in the following example:

α β α⊕β

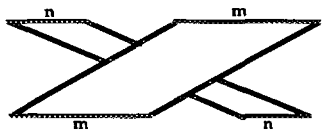
A braiding for $\mathfrak{B}$ is given by the elements

$$c = c_{m,n} : m + n \to n + m,$$

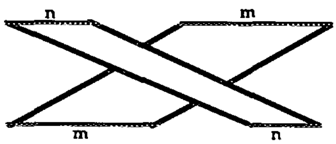
illustrated by the following diagram:



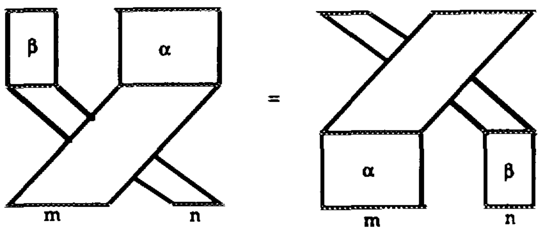
This can also be depicted as follows:



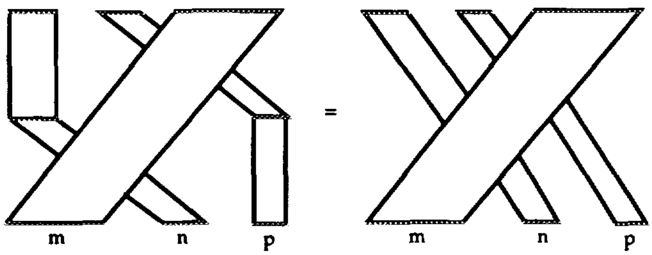
The picture for $c'_{m,n} = (c_{n,m})^{-1}$ is then as follows:



Naturalness of $c_{m,n}$ is proved pictorially by the equality



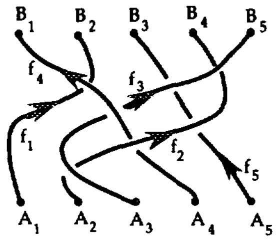
for all $\alpha \in \mathfrak{B}_m$, $\beta \in \mathfrak{B}_n$. Axiom (B2) is proved pictorially by the equality



More generally, for any category $\mathscr{A}$, there is a braided strict tensor category $\mathfrak{B} \int \mathscr{A}$ of braids having their strings labelled by arrows of $\mathscr{A}$. (The notation $\mathfrak{B} \int \mathscr{A}$ is intended to indicate that it is a wreath product in a generalized sense [K5].) The objects of $\mathfrak{B} \int \mathscr{A}$ are finite sequences of objects of $\mathscr{A}$. An arrow

$$(\alpha, f_1, ..., f_n): (A_1, ..., A_n) \to (B_1, ..., B_n)$$

consists of $\alpha \in \mathfrak{B}_n$ and $f_i \in \mathscr{A}(A_i, B_{\alpha(i)})$, where $i \mapsto \alpha(i)$ is the permutation defined by α. Such an arrow can be viewed as the braid α labelled by $f_1, ..., f_n$ as, for example:



Composition of labelled braids is performed by composing the label on each string of the composite braid. The operation of addition of braids extends in the obvious way to labelled braids $\mathfrak{B} \int \mathscr{A} \times \mathfrak{B} \int \mathscr{A} \to \mathfrak{B} \int \mathscr{A}$, yielding a tensor structure on $\mathfrak{B} \int \mathscr{A}$. There is an obvious braiding on $\mathfrak{B} \int \mathscr{A}$ obtained from the braiding on $\mathfrak{B}$ by labelling the strings with identity arrows. We have an inclusion functor

$$i: \mathscr{A} \to \mathfrak{B} \int \mathscr{A}$$

identifying $\mathscr{A}$ with the labelled braids with a single string.

EXAMPLE 2.2 (Super Vector Spaces). (This example is further explained in Section 3 on categorical groups; also see [MS, Lyu].) Consider the category $\mathbf{Z}_2\, \mathscr{V}ect_{\mathbb{C}}$ of $\mathbf{Z}_2$-graded complex vector spaces. The objects are pairs (A_0, A_1) of vectors spaces and the arrows are pairs (f_0, f_1) of linear maps. There is a familiar tensor product on this category given by

$$(A_0, A_1) \otimes (B_0, B_1) = (A_0 \otimes B_0 \oplus A_1 \otimes B_1, A_0 \otimes B_1 \oplus A_1 \otimes B_0).$$

However, apart from the familiar associativity constraint and symmetry, we also have the associativity constraint and braiding which, for homogeneous x, y, z, are given by

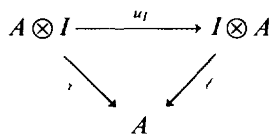
$$a((x\otimes y)\otimes z)=\begin{cases}(-1)\,x\otimes(y\otimes z) & \text{for } x, y, z \text{ all odd,}\\ x\otimes(y\otimes z) & \text{otherwise,}\end{cases}$$

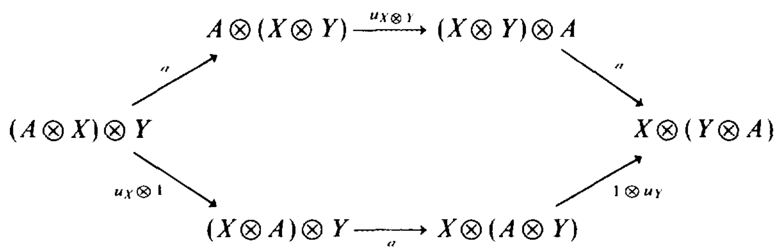
$$c(x\otimes y)=\begin{cases}\sqrt{-1}\;y\otimes x & \text{for } x, y \text{ both odd,}\\ y\otimes x & \text{otherwise.}\end{cases}$$

EXAMPLE 2.3 (The Centre of a Tensor Category). (See [JS3, Mj2].) The *centre* $\mathcal{Z}_{\mathcal{V}}$ of $\mathcal{V}$ is the category whose objects are pairs (A, u), where $A\in\mathcal{V}$ and

$$u\colon A\otimes - \xrightarrow{\sim} -\otimes A$$

is a natural isomorphism such that the following two conditions hold:





An arrow $f\colon (A, a)\to(B, b)$ in $\mathcal{Z}_{\mathcal{V}}$ is an arrow $f\colon A\to B$ such that, for all $X\in\mathcal{V}$,

$$b_X\circ(f\otimes 1)=(1\otimes f)\circ a_X.$$

Moreover, $\mathcal{Z}_{\mathcal{V}}$ becomes a braided tensor category with tensor product given by

$$(A, a)\otimes(B, b)=(A\otimes B, (a\otimes 1)\circ(1\otimes b))$$

and braiding given by

$$c_{(A,a),(B,b)}=a_B\colon (A, a)\otimes(B, b)\to(B, b)\otimes(A, a).$$

DEFINITION 2.3. Suppose $\mathcal{V}$, $\mathcal{W}$ are braided tensor categories. A tensor functor $F\colon\mathcal{V}\to\mathcal{W}$ is said to be *braided* when the following square commutes:

$$\begin{array}{ccc} FA \otimes FB & \xrightarrow{\phi_2} & F(A \otimes B) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle Fc} \\ FB \otimes FA & \xrightarrow[\phi_2]{} & F(B \otimes A) \end{array}$$

EXAMPLE 2.4. Suppose $F: \mathcal{V} \to \mathcal{W}$ is a tensor equivalence between tensor categories $\mathcal{V}$, $\mathcal{W}$. Each braiding c for $\mathcal{V}$ *transports along* F to a unique braiding on $\mathcal{W}$ such that F becomes braided. In particular, if $\mathcal{V}$ is braided, so is $\mathbf{st}(\mathcal{V})$.

EXAMPLE 2.5. For any tensor category $\mathcal{V}$, there is a tensor category $\mathcal{V}^{\text{rev}}$, called the *reverse* of $\mathcal{V}$, with the same underlying category, but with tensor product $\otimes'$ given by

$$A \otimes' B = B \otimes A.$$

If $\mathcal{V}$ is braided then the identity functor $1_{\mathcal{V}}$ of $\mathcal{V}$ together with $\phi_2 = c$ and $\phi_0 = 1_I$ forms a tensor isomorphism $\mathcal{V} \xrightarrow{\sim} \mathcal{V}^{\text{rev}}$. The proof requires commutativity of the square

$$\begin{array}{ccc} A \otimes B \otimes C & \xrightarrow{c \otimes 1} & B \otimes A \otimes C \\ {\scriptstyle 1 \otimes c}\downarrow & & \downarrow{\scriptstyle c} \\ A \otimes C \otimes B & \xrightarrow[c]{} & C \otimes B \otimes A \end{array}$$

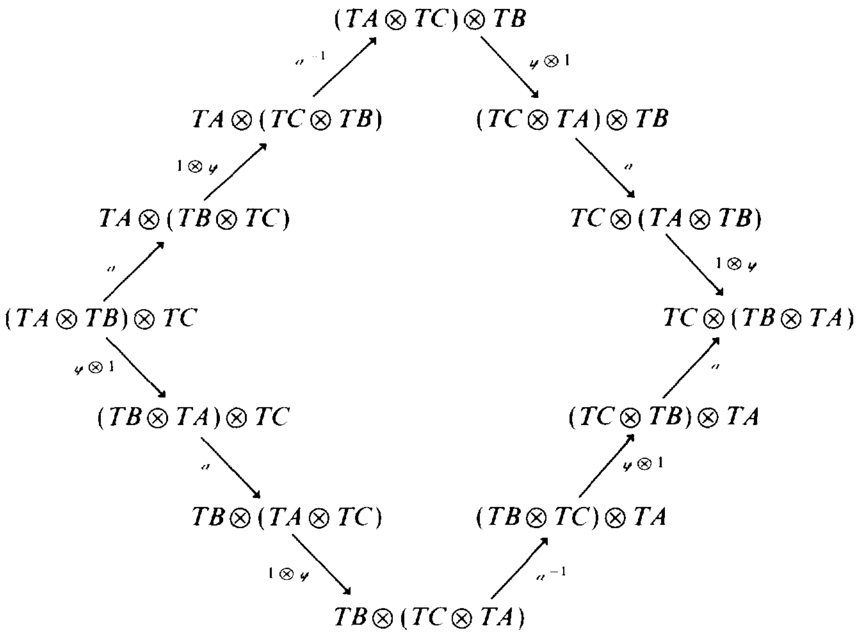
(which does commute since it becomes the third diagram of Proposition 2.1 when we replace the bottom side using (B1) and the right side using (B2)). Moreover, $\mathcal{V}^{\text{rev}}$ is braided via $c: A \otimes' B \to B \otimes' A$, and $\mathcal{V} \xrightarrow{\sim} \mathcal{V}^{\text{rev}}$ is a braided tensor isomorphism.

The concept of a Yang-Baxter operator on an object in a tensor category has been considered at various levels of generality in [Dr, Man, T]. We need to extend this somewhat. Let $T: \mathcal{A} \to \mathcal{V}$ be a functor from a category $\mathcal{A}$ to a tensor category $\mathcal{V}$.

DEFINITION 2.4. A *Yang–Baxter operator on* T is a natural family of isomorphisms

$$y = y_{A,B}: TA \otimes TB \xrightarrow{\sim} TB \otimes TA$$

such that the following diagram commutes:



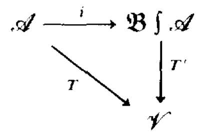
When $\mathscr{A}$ is the category **1** with a single object A (and a single arrow 1_A), a Yang–Baxter operator on T is the same as a Yang–Baxter operator on the object $X = T(A)$ in $\mathscr{V}$.

EXAMPLE 2.6. By Proposition 2.1, any functor $T: \mathscr{A} \to \mathscr{V}$ into a braided tensor category $\mathscr{V}$ comes equipped with a Yang–Baxter operator obtained from the braiding of $\mathscr{V}$:

$$y_{A,B} = c_{TA,TB}: TA \otimes TB \cong TB \otimes TA.$$

In particular, we obtain a Yang–Baxter operator on the inclusion functor $i: \mathscr{A} \to \mathfrak{B} \int \mathscr{A}$; we shall denote it by x.

PROPOSITION 2.2. (a) *For any strict tensor category $\mathscr{V}$ and any Yang–Baxter operator y on $T: \mathscr{A} \to \mathscr{V}$, there exists a unique strict tensor functor $T': \mathfrak{B} \int \mathscr{A} \to \mathscr{V}$ such that the following triangle commutes and $T'(x) = y$:*



(b) *$\mathfrak{B} \int \mathscr{A}$ is the free braided strict tensor category on $\mathscr{A}$.*

Proof. *Step* 1. (a) *holds for* $\mathscr{A} = \mathbf{1}$. Given a Yang-Baxter operator y on X in $\mathscr{V}$, define $T': \mathfrak{B} \to \mathscr{V}$ on objects by $T'n = X^{\otimes n}$ (the nth tensor power of X). For $0 \leqslant i < n$, define

$$y_i = X^{\otimes (i-1)} \otimes y \otimes X^{\otimes (n-i-1)} : X^{\otimes n} \to X^{\otimes n}.$$

These satisfy the relations given in Example 1.1 for the presentation of the Artin braid group $\mathfrak{B}_n$; (A1) holds since y is a Yang–Baxter operator, and (A2) holds by functoriality of $\otimes$. Thus we obtain a monoid homomorphism $T'_n: \mathfrak{B}_n \to \mathscr{V}(X^{\otimes n}, X^{\otimes n})$ taking s_i to y_i for all $0 \leqslant i < n$. This defines T' on arrows. Clearly this T' is the unique strict tensor functor with the properties claimed.

Step 2. (a) *holds for general* $\mathscr{A}$. We use a technique of Kelly [K3, pp. 74–75] to reduce it to the special case already proved. The wreath product construction $\mathscr{D} \int \mathscr{A}$ can be defined for any category $\mathscr{A}$ equipped with a functor $\mathscr{A} \to \mathfrak{S}$ into the disjoint union $\mathfrak{S}$ of the symmetric groups. This construction has a right adjoint, denoted by $\{\mathscr{A}, \mathscr{B}\}$, which is a category equipped with a functor $\{\mathscr{A}, \mathscr{B}\} \to \mathfrak{S}$. More precisely, there is a natural bijection between functors $\mathscr{D} \int \mathscr{A} \to \mathscr{B}$ and functors $\mathscr{D} \to \{\mathscr{A}, \mathscr{B}\}$ over $\mathfrak{S}$. The explicit description of $\{\mathscr{A}, \mathscr{B}\}$ is as follows. For any natural number $n \in \mathbb{N}$, the symmetric group $\mathfrak{S}_n$ acts on the category $[\mathscr{A}^n, \mathscr{B}]$ of functors from the cartesian product $\mathscr{A}^n$ of n copies of $\mathscr{A}$ to $\mathscr{B}$ via the formula

$$(\sigma F)(A_1, ..., A_n) = F(A_{\sigma(1)}, ..., A_{\sigma(n)}).$$

This action is used to construct the semi-direct product

$$\{\mathscr{A}, \mathscr{B}\}_n = [\mathscr{A}^n, \mathscr{B}] \times \mathfrak{S}_n.$$

The objects of $\{\mathscr{A}, \mathscr{B}\}_n$ are the functors $F: \mathscr{A}^n \to \mathscr{B}$. An arrow $(u, \sigma): F \to G$ has $u: \sigma F \to G$ a natural transformation. The composition is given by the formula

$$(v, \tau) \circ (u, \sigma) = (v\tau \circ u, \tau\sigma).$$

For $\mathscr{V}$ a strict tensor category, there is a strict tensor product on $\{\mathscr{A}, \mathscr{V}\}$ for which $\{\mathscr{A}, \mathscr{V}\} \to \mathfrak{S}$ is a strict tensor functor: for $F: \mathscr{A}^n \to \mathscr{V}$, $G: \mathscr{A}^m \to \mathscr{V}$, take $F \otimes G: \mathscr{A}^{n+m} \to \mathscr{V}$ to be the functor given by

$$(F \otimes G)(A_1, ..., A_n, B_1, ..., B_m) = F(A_1, ..., A_n) \otimes G(B_1, ..., B_m).$$

Returning to the proof, we take any Yang–Baxter operator y on $T: \mathscr{A} \to \mathscr{V}$. Then we obtain a Yang–Baxter operator $(y, s_1): T \otimes T \to$

$T \otimes T$ on the object T of $\{\mathscr{A}, \mathscr{V}\}$. By Step 1, there is a unique strict tensor functor $T^{\#}: \mathfrak{B} \to \{\mathscr{A}, \mathscr{V}\}$ with

$$T^{\#}(1) = T \qquad \text{and} \qquad T^{\#}(s_1) = (y, s_1).$$

Now $\mathfrak{S}$ is a strict tensor category and $s_1 \in \mathfrak{S}_2$ is a Yang–Baxter operator therein. So, by uniqueness, the composite of $T^{\#}$ with the canonical functor $\{\mathscr{A}, \mathscr{V}\} \to \mathfrak{S}$ is just the canonical surjection $\mathfrak{B} \to \mathfrak{S}$. By adjointness, $T^{\#}$ corresponds to a well-defined functor $T': \mathfrak{B} \int \mathscr{A} \to \mathscr{V}$. Checking that T' is the unique strict tensor functor as required is then purely mechanical.

Step 3. (b) *holds*. We have already seen that $\mathfrak{B} \int \mathscr{A}$ is a braided strict tensor category. We must see that it is the free such generated by $\mathscr{A}$. So let $\mathscr{V}$ be any braided strict tensor category and let $T: \mathscr{A} \to \mathscr{V}$ be any functor. By Example 1.6, we obtain a Yang–Baxter operator y on T given by $y_{A,B} = c_{TA,TB}$. By Step 2, there exists a unique strict tensor functor $T': \mathfrak{B} \int \mathscr{A} \to \mathscr{V}$ such that $T' \circ i = T$ and $T'(c_{iA,iB}) = c_{TA,TB}$. It remains to show that T' satisfies the further equations needed for it to be braided. For each object B of $\mathscr{A}$, consider the set

$$\{A = (A_1, ..., A_n) \in \mathfrak{B} \textstyle\int \mathscr{A} \mid T'(c_{A,iB}) = c_{T'A,TB}\}.$$

Using the axioms for a braiding, we see that this set contains the unit object of $\mathfrak{B} \int \mathscr{A}$ and is closed under tensor product; so it contains all objects of $\mathfrak{B} \int \mathscr{A}$. Now consider the set

$$\{B = (B_1, ..., B_m) \in \mathfrak{B} \textstyle\int \mathscr{A} \mid T'(c_{A,B}) = c_{T'A,T'B} \text{ for all } A \in \mathfrak{B} \textstyle\int \mathscr{A}\}.$$

Again, using the axioms for a braiding, we see that this set contains the unit object of $\mathfrak{B} \int \mathscr{A}$ and is closed under tensor product; so it contains all objects of $\mathfrak{B} \int \mathscr{A}$. So T' is braided. Q.E.D.

COROLLARY 2.3. *For every braided tensor functor $F: \mathfrak{B} \int \mathscr{A} \to \mathscr{V}$ into a braided strict tensor category $\mathscr{V}$, there exist a braided strict tensor functor $S: \mathfrak{B} \int \mathscr{A} \to \mathscr{V}$ and an isomorphism $\sigma: F \cong S$ of tensor functors whose restriction $\sigma \circ i$ to $\mathscr{A}$ is the identity.*

Proof. By Proposition 2.2 (b) the condition $F \circ i = S \circ i$ defines S. Now proceed as in Proposition 1.5, noting that $\mathrm{Eq}(F, S)$ is a strict tensor category (since $\mathscr{V}$ is) and it admits a unique braiding for which $P: \mathrm{Eq}(F, S) \to \mathfrak{B} \int \mathscr{A}$ becomes a braided strict tensor functor. Q.E.D.

We can also deduce the "bicategorical" universe property of $\mathfrak{B} \int \mathscr{A}$. We need a little notation. Suppose $\mathscr{A}, \mathscr{B}$ are categories. We write $[\mathscr{A}, \mathscr{B}]$ for the category whose objects are functors $T: \mathscr{A} \to \mathscr{B}$ and whose arrows are natural transformations $\phi: T \to S$ between such functors. If $\mathscr{V}$, $\mathscr{W}$ are tensor categories, we write $\mathscr{T}en(\mathscr{V}, \mathscr{W})$ for the category whose objects are tensor functors $F: \mathscr{V} \to \mathscr{W}$ and whose arrows are morphisms $\psi: F \to G$ of tensor functors. If $\mathscr{V}$, $\mathscr{W}$ are braided tensor categories, we write $\mathscr{B}\mathscr{T}en(\mathscr{V}, \mathscr{W})$ for the category full subcategory of $\mathscr{T}en(\mathscr{V}, \mathscr{W})$ consisting of the braided tensor functors.

COROLLARY 2.4. *Suppose $\mathscr{A}$ is a category and $\mathscr{V}$ is a braided tensor category. Then restriction along $i: \mathscr{A} \to \mathfrak{B}\int\mathscr{A}$ determines an equivalence of categories*

$$\mathscr{B}\mathscr{T}en(\mathfrak{B}\textstyle\int\mathscr{A}, \mathscr{V}) \simeq [\mathscr{A}, \mathscr{V}].$$

Proof. From the following commutative square in which the vertical functors are equivalences induced by $s: \mathscr{V} \to \mathrm{st}(\mathscr{V})$, we see that it suffices to prove the result for $\mathscr{V}$ a braided strict tensor category.

$$\begin{array}{ccc}
\mathscr{B}\mathscr{T}en(\mathfrak{B}\textstyle\int\mathscr{A}, \mathscr{V}) & \longrightarrow & [\mathscr{A}, \mathscr{V}] \\
\wr\big\downarrow & & \wr\big\downarrow \\
\mathscr{B}\mathscr{T}en(\mathfrak{B}\textstyle\int\mathscr{A}, \mathrm{st}(\mathscr{V})) & \longrightarrow & [\mathscr{A}, \mathrm{st}(\mathscr{V})]
\end{array}$$

But then each $T \in [\mathscr{A}, \mathscr{V}]$ has the form $T = F \circ i$ for a unique braided *strict* tensor functor $F \in \mathscr{B}\mathscr{T}en(\mathfrak{B}\int\mathscr{A}, \mathscr{V})$ by Proposition 2.2(b).

It remains to show that $\mathscr{B}\mathscr{T}en(\mathfrak{B}\int\mathscr{A}, \mathscr{V}) \to [\mathscr{A}, \mathscr{V}]$ is fully faithful. So take any two braided tensor functors $F, G: \mathfrak{B}\int\mathscr{A} \to \mathscr{V}$ and any natural transformation $\phi: F \circ i \to G \circ i$. Let $S, T: \mathfrak{B}\int\mathscr{A} \to \mathscr{V}$ be the braided strict tensor functors isomorphic to F, G, respectively, as assured by Corollary 2.3. By using the category $[\mathbf{2}, \mathscr{V}]$ of arrows in $\mathscr{V}$ (which pointwise enherits a braided strict tensor structure from $\mathscr{V}$), Proposition 2.2(b) yields that the natural transformation $\phi: S \circ i \to T \circ i$ has the form $\phi = \alpha \circ i$ for a unique morphism $\alpha: S \to T$ tensor functors. Using the isomorphisms $F \cong S$, $G \cong T$, we obtain a unique morphism $\psi: F \to G$ of tensor functors such that $\phi = \psi \circ i$. Q.E.D.

We now consider coherence for braidings. After the coherence discussion of Section 1, it is natural to introduce the free braided tensor category $\mathscr{F}\ell\mathscr{A}$ generated by the category $\mathscr{A}$. For each braided tensor category $\mathscr{V}$, restriction along the inclusion $\mathscr{A} \to \mathscr{F}\ell\mathscr{A}$ determines a bijection between braided strict tensor functors $\mathscr{F}\ell\mathscr{A} \to \mathscr{V}$ and functors $\mathscr{A} \to \mathscr{V}$. In particular, there is a unique braided strict tensor functor $\Gamma: \mathscr{F}\ell\mathscr{A} \to \mathfrak{B}\int\mathscr{A}$ whose restriction to $\mathscr{A}$ is $i: \mathscr{A} \to \mathfrak{B}\int\mathscr{A}$.

THEOREM 2.5 (Coherence for Braided Tensor Categories). *$\Gamma: \mathscr{F}\ell\mathscr{A} \to \mathfrak{B}\int\mathscr{A}$ is a braided tensor equivalence.*

Proof. We have a pushout diagram

$$\begin{array}{ccc} \mathscr{F}\mathscr{A} & \xrightarrow{\Gamma} & \mathscr{F}_s\mathscr{A} \\ \downarrow & & \downarrow \\ \mathscr{F}\ell\mathscr{A} & \xrightarrow[\Gamma]{} & \mathfrak{B}\int\mathscr{A} \end{array}$$

of categories and functors. For we know by Proposition 2.2(b) that $\mathfrak{B}\int\mathscr{A}$ is the free braided strict tensor category and so is obtained from $\mathscr{F}\ell\mathscr{A}$ by forcing all the associativity isomorphisms to be identities. Also, $\mathscr{F}_s\mathscr{A}$ is obtained in the same way from $\mathscr{F}\mathscr{A}$. Moreover, these associativity isomorphisms are in bijection via the functor on the left side of the square.

It is a general property of pushouts in $\mathscr{C}at$ that, if the top functor is an equivalence and the left functor is injective on objects, then the bottom functor is an equivalence. So the result follows from Theorem 1.2. Q.E.D.

There is a canonical braided strict tensor functor $\mathfrak{B}\int\mathscr{A} \to \mathfrak{B}$ induced by the unique functor $\mathscr{A} \to \mathbf{1}$. Composing this with Γ we obtain a braided strict tensor functor

$$\mathscr{F}\ell\mathscr{A} \to \mathfrak{B}\int\mathscr{A} \to \mathfrak{B}$$

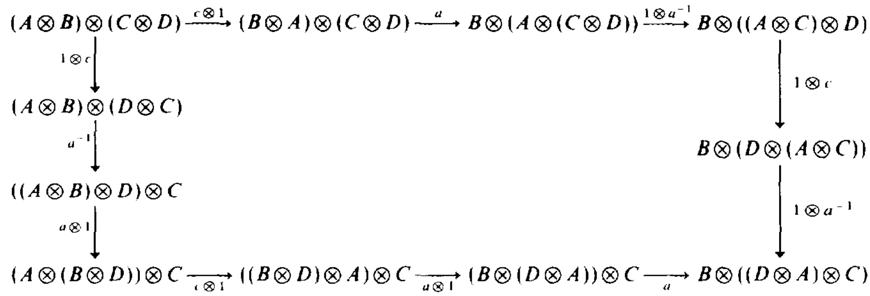
whose value at an arrow of $\mathscr{F}\ell\mathscr{A}$ is called the *underlying braid* of that arrow.

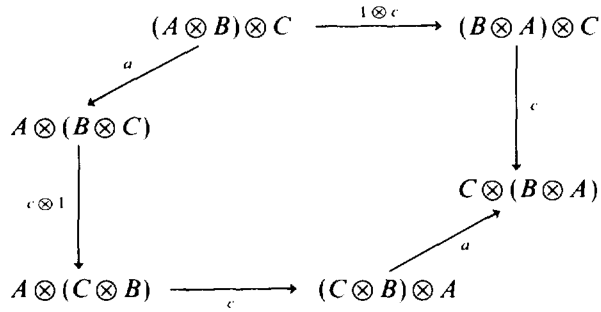
COROLLARY 2.6. *In the free braided tensor category generated by a set A of objects, a diagram commutes if and only if all legs with the same source and target have the same underlying braid.*

Proof. The arrows of $\mathfrak{B}\int A$ are braids labelled by elements of the set A, so $\mathfrak{B}\int A \to \mathfrak{B}$ is faithful. By Theorem 2.5, it follows that the composite $\mathscr{F}\ell A \to \mathfrak{B}\int A \to \mathfrak{B}$ is faithful. Q.E.D.

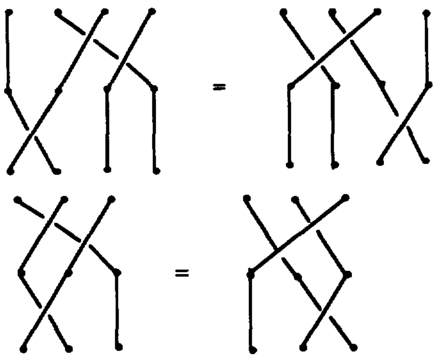
As an application of Corollary 2.6, we prove the commutativity in any braided tensor category of some particular diagrams. These results can also be proved directly (as for Proposition 2.1) by subdividing the diagrams into sub-regions, each of which commutes by an axiom for braided tensor categories.

PROPOSITION 2.7. *For all objects A, B, C, D of any braided tensor category $\mathscr{V}$, the following diagrams commute:*





Proof. It suffices to prove the commutativity of these diagrams in the free braided tensor category on four generating objects, and, for this, we need only check that the two legs of each diagram have the same underlying braid:



Q.E.D.

3. Braided Categorical Groups

In this section we study a special class of braided tensor categories and prove a classification theorem for that class.

DEFINITION 3.1. A *categorical group* $\mathscr{V}$ is a tensor category in which every arrow is invertible and, for each object A, there is an object A^* with an arrow $\varepsilon_A : A^* \otimes A \to I$. We say $\mathscr{V}$ is *braided* when it is equipped with a braiding. A categorical group is *strict* when it is strict as a tensor category and ε_A can be chosen to be an identity. Strict categorical groups are the group objects in the category of categories.

EXAMPLE 3.1. We can construct a braided strict categorical group $\mathscr{V}$ from any pair of abelian groups A, X and bilinear map $f: A \times A \to X$. The objects of $\mathscr{V}$ are the elements of A. The homsets are given by

$$\mathscr{V}(a, b) = \begin{cases} X & \text{for } a = b \\ \varnothing & \text{otherwise.} \end{cases}$$

Composition is addition in B. The tensor product is defined by

$$(x : a \to a) \otimes (y : b \to b) = (x + y : a + b \to a + b).$$

The braiding is

$$f(a, b) : a + b \to b + a.$$

In particular, for a commutative ring K, we can take $A = X = K$ under addition, while f is the multiplication operation $K \times K \to K$.

Remark 3.1. It is well known (see [BS] for the history) that strict categorical groups are the same as the "crossed modules" of Whitehead [W]. Recall that a *crossed module* $(N, E, \partial, *)$ consists of groups N, E, a group homomorphism $\partial : N \to E$, and an action of E on N subject to the following four axioms:

$$e * (uv) = (e * u)(e * v), \qquad (ef) * u = e * (f * u),$$

$$\partial(e * u) = e\partial(u)\, e^{-1}, \qquad \partial(u) * v = uvu^{-1}.$$

A strict categorical group $\mathscr{V}$ gives a crossed module as follows: E is the set of objects of $\mathscr{V}$, N is the set of arrows $u : A \to I$ into the unit object, ∂ is the source function $u \mapsto A$, and the action $*$ is conjugation,

$$B * (u : A \to I) = (B \otimes u \otimes B^* : B \otimes A \otimes B^* \to I).$$

The group structures on N and E are induced by the tensor product of $\mathscr{V}$.

Given any crossed module $(N, E, \partial, *)$, the corresponding strict categorical group $\mathscr{V}$ is described as follows. The objects are the elements $e \in E$. An arrow $u : e \to e'$ is an element $u \in N$ with $e = \partial(u)\, e'$. Composition is multiplication in N. The tensor product is given by

$$(u : e \to e') \otimes (v : f \to f') = (u(e' * v) : ef \to e'f').$$

Braidings for a strict categorical group are in bijection with "bracket operations" [C] for the corresponding crossed module. A *bracket operation* for a crossed module $(N, E, \partial, *)$ is a function $\{\ ,\ \}: E \times E \to N$ satisfying the five conditions

$$\{e, gf\} = \{e, f\}(f * \{e, g\}), \qquad \{ef, g\} = (e * \{f, g\})\{e, g\},$$
$$\partial\{e, f\} = efe^{-1}f^{-1}, \qquad \{\partial u, f\}(f * u) = u, \qquad \{e, \partial v\}\, v = e * v.$$

For a braided strict categorical group, the bracket operation on the corresponding crossed module is given by

$$\{A, B\} = (c_{A,B} \otimes A^* \otimes B^* : A \otimes B \otimes A^* \otimes B^* \to I).$$

Given a bracket operation on the crossed module $(N, E, \partial, *)$, the braiding on the corresponding strict categorical group is given by the equation

$$c_{e,f} = \{e, f\} : ef \to fe.$$

Suppose G, M are abelian groups. Then M can be regarded as a trivial G-module (via the action $x\mu = \mu$ for $x \in G$, $\mu \in M$) and the cohomology groups $H^n(G, M)$ can be considered. However, it is argued by Eilenberg and Mac Lane [E, ML0, EM] that these groups are inappropriate here since they take no account of the commutativity of G, and so should be replaced by groups $H^n_{\text{ab}}(G, M)$. We describe $H^3_{\text{ab}}(G, M)$.

An *abelian* 3-*cocycle* for G with coefficients in M is a pair (h, c), where $h: G^3 \to M$ is a "normalized 3-cocycle"

$$h(x, 0, y) = 0$$
$$h(x, y, z) + h(u, x + y, z) + h(u, x, y) = h(u, x, y + z) + h(u + x, y, z)$$

and $c: G^2 \to M$ is a function satisfying

$$h(y, z, x) + c(x, y + z) + h(x, y, z) = c(x, z) + h(y, x, z) + c(x, y)$$
$$-h(z, x, y) + c(x + y, z) - h(x, y, z) = c(x, z) - h(x, z, y) + c(y, z).$$

For any function $k: G^2 \to M$ satisfying

$$k(x, 0) = k(0, y) = 0,$$

the *coboundary* of k is the abelian 3-cocycle $\partial(k) = (h, c)$ defined by the equations

$$h(x, y, z) = k(y, z) - k(x + y, z) + k(x, y + z) - k(x, y)$$
$$c(x, y) = k(x, y) - k(y, x).$$

Then $H^3_{\text{ab}}(G, M)$ is the abelian group of abelian 3-cocycles modulo the coboundaries.

For any normalized 3-cocycle $h: G^3 \to M$, there is a tensor category $\mathscr{V} = \mathscr{T}(G, M, h)$ defined as follows: the objects are elements of G; the homsets are given by

$$\mathscr{V}(x, y) = \begin{cases} M & \text{for } x = y \\ \varnothing & \text{for } x \neq y; \end{cases}$$

composition is addition in M; tensor product is given by

$$(\mu : x \to x) \otimes (\nu : y \to y) = (\mu + \nu : x + y \to x + y);$$

the associativity isomorphism is

$$h(x, y, z) : (x + y) + z \to x + (y + z);$$

and the 0 of G is the (strict) unit object.

A remarkable observation connecting braidings with cohomology is the following.

Proposition 3.1. *To say that c is a braiding for the tensor category $\mathscr{T}(G, M, h)$ is precisely to say that (h, c) is an abelian 3-cocycle for G with coefficients in M. Also, an equation*

$$(h, c) - (h', c') = \partial(k)$$

then determines a braided tensor isomorphism $\mathscr{T}(G, M, h) \cong \mathscr{T}(G, M, h')$.

Proof. The properties of c in the definition of abelian 3-cocycle amount precisely to the axioms (B1), (B2) for a braiding for the tensor category $\mathscr{T}(G, M, h)$. For the second sentence of the Proposition, the identity functor enriches to a tensor isomorphism via $\phi_2 = k(x, y) : x + y \to x + y$ using the first equation defining a coboundary, and the second equation yields that this isomorphism is braided. Q.E.D.

Remark 3.2. Suppose A is an abelian group and R is a commutative ring. Suppose (h, c) is an abelian 3-cocycle on A with coefficients in the multiplicative group $R^\times$ of units. Let $A\,\mathscr{M}od(R)$ denote the category of A-graded R-modules: the objects are families

$$M = (M_a \mid a \in A)$$

of R-modules and the arrows are families of R-module homomorphisms. There is a tensor on this category given by the "convolution formula"

$$(M \otimes N)_a = \sum_{b + c = a} M_b \otimes N_c.$$

The following equations define an associativity constraint a and a braiding c:

$$a((x\otimes y)\otimes z)=h(a,b,c)\,x\otimes(y\otimes z) \qquad \text{and}$$
$$c(x\otimes y)=c(a,b)\,y\otimes x \qquad \text{where}\quad x\in M_a,\ y\in N_b,\ z\in P_c.$$

In this way, $A\,\mathcal{M}od(R)$ becomes a braided tensor category.

A function $q: G\to M$ between abelian groups G, M is called *quadratic* when $q(x+y)-q(x)-q(y)$ is a bilinear function of x, y. This amounts to the conditions

$$q(-x)=q(x)$$
$$q(x+y+z)+q(x)+q(y)+q(z)=q(y+z)+q(z+x)+q(x+y).$$

The *trace* of an abelian 3-cocycle (h, c) is the function $q: G\to M$ given by

$$q(x)=c(x,x).$$

A calculation shows that traces are quadratic functions.

THEOREM 3.2 (Eilenberg and Mac Lane [E, ML0, EM]). *Trace determines an isomorphism between the group $H^3_{\mathrm{ab}}(G, M)$ and the group of quadratic functions from G to M.*

In the case where G is a cyclic group $\mathbf{Z}/n$ of order n, we have an isomorphism

$$H^3_{\mathrm{ab}}(\mathbf{Z}/n, M)\cong \mathrm{Hom}(\mathbf{Z}/(n^2, 2n), M).$$

Explicitly, given an element $v\in M$ such that $n^2v=2nv=0$, we can define a 3-cocycle (h, c) as

$$h(x,y,z)=\begin{cases}0 & \text{for}\quad y+z<n\\ xnv & \text{for}\quad y+z\geqslant n,\end{cases}$$
$$c(x\ y)=xyv,$$

where $x, y, z\in\{0, 1, ..., n-1\}$.

Combined with the above Remark 3.2, this gives the setting for Example 2.2.

We now state and prove a precise classification theorem for braided categorical groups. (The case of symmetry was dealt with by [SHX].) In this classification, two braided categorical groups which are connected by a braided tensor equivalence are considered the same. The problem arises of giving a *complete invariant* of this relation. (See [K1] for a categorical discussion of complete invariants.)

For any braided categorical group $\mathcal{V}$, consider the invariant made up of

(i) the group $\pi_0(\mathcal{V})$ of isomorphism classes of the objects in $\mathcal{V}$, where multiplication is induced by the tensor product,

(ii) the group $\pi_1(\mathcal{V}) = \mathcal{V}(I, I)$ of automorphisms of the unit object I, where the operation is composition,

(iii) the map $q: \pi_0(\mathcal{V}) \to \pi_1(\mathcal{V})$ assigning to each object its *signature*

$$q[A] = \operatorname{sgn}(A) = c_{A,A} \otimes A^* \otimes A^* = \{A, A\}.$$

The group $\pi_1(\mathcal{V})$ is abelian since the multiplication

$$\pi_1(\mathcal{V}) \times \pi_1(\mathcal{V}) \to \pi_1(\mathcal{V}), \qquad (u, v) \mapsto u \otimes v$$

is a group homomorphism [EH]. The group $\pi_0(\mathcal{V})$ is abelian because of the braiding. We see below that q is a quadratic map.

Let $\mathcal{Q}uad$ be the category whose objects (G, M, q) are quadratic maps $q: G \to M$ between abelian groups G, M, and whose arrows (f, p): $(G, M, q) \to (G', M', q')$ consist of group homomorphisms f, p such that we have a commutative square

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ M & \xrightarrow[p]{} & M' \end{array}$$

Let $\mathcal{BCG}$ denote the category whose objects are braided categorical groups and whose arrows are braided tensor functors. We have a functor

$$T: \mathcal{BCG} \to \mathcal{Q}uad, \qquad \mathcal{V} \mapsto (\pi_0(\mathcal{V}), \pi_1(\mathcal{V}), q).$$

THEOREM 3.3 (Classification of Braided Categorical Groups). *The functor $T: \mathcal{BCG} \to \mathcal{Q}uad$ has the following properties:*

(1) *for each object Q of $\mathcal{Q}uad$, there exists an object $\mathcal{V}$ of $\mathcal{BCG}$ with an isomorphism $T(\mathcal{V}) \cong Q$;*

(2) *for any isomorphism $\rho: T(\mathcal{V}) \xrightarrow{\sim} T(\mathcal{V}')$, there is an equivalence $F: \mathcal{V} \simeq \mathcal{V}'$ such that $T(F) = \rho$; and*

(3) *$T(F)$ is an isomorphism if and only if E is an equivalence.*

Proof. Recall that a category is *skeletal* when any two isomorphic objects are equal. Every category is equivalent to a skeletal one. A skeletal categorical group $\mathscr{V}$ is necessarily of the form $\mathscr{T}(G, M, h)$, where

$$G = \text{obj}\,\mathscr{V} = \pi_0(\mathscr{V}), \qquad M = \pi_1(\mathscr{V}) = \mathscr{V}(I, I),$$
$$h(x, y, z) = a_{x,y,z} \otimes z^* \otimes y^* \otimes x^*,$$

where $a_{x,y,z}\colon x \otimes y \otimes z \to x \otimes y \otimes z$ is the associativity isomorphism. If $\mathscr{V}$ is braided then we get an abelian 3-cocycle (h, c) with $c(x, y) = c_{x,y} \otimes x^* \otimes y^*$. This, together with Theorem 2, makes Property (1) obvious. Property (3) is a consequence of the observation that an arrow $F\colon \mathscr{V} \to \mathscr{V}'$ is an equivalence if and only if the maps $\pi_0(F)$, $\pi_1(F)$ are invertible. To prove Property (2) note that it reduces to the case where $\mathscr{V}$, $\mathscr{V}'$ are skeletal by using the fact that every object of $\mathscr{BCG}$ is equivalent to a skeletal one. Now observe that an arrow between skeletal braided categorical groups amounts to a triple

$$(f, p, k)\colon (G, M, h, c) \to (G', M', h', c')$$

consisting of group homomorphisms $f\colon G \to G'$, $p\colon M \to M'$, and $\partial(k) = (h'', c'')$ where

$$h''(x, y, z) = h'(gx, gy, gz) - ph(x, y, z), \qquad c''(x, y) = c'(gx, gy) - pc(x, y).$$

Given any arrow $(f, p)\colon (G, M, q) \to (G', M', q')$ in $\mathscr{Q}uad$, where q, q' are the traces of (h, c), (h', c'), we see that

$$(ph, pc), (h'g^3, c'g^2)$$

are abelian cocycles with the *same* trace. By Theorem 2, these cocycles differ by a coboundary $\partial(k)$; so, in fact, T is full. Q.E.D.

Remark 3.3. We reiterate that we actually proved T to be *full.* However, T is not faithful. In fact, for any objects $\mathscr{V}$, $\mathscr{V}'$ in $\mathscr{BCG}$ and any arrow $\rho\colon T(\mathscr{V}) \to T(\mathscr{V}')$ in $\mathscr{Q}uad$, isomorphism classes of arrows $F\colon \mathscr{V} \to \mathscr{V}$ in $\mathscr{BCG}$ with $T(F) = \rho$ are in bijection with elements of the group

$$\text{Ext}(\pi_0(\mathscr{V}), \pi_1(\mathscr{V}'))$$

(which is the same situation as for the complete invariants found in [St0]).

4. A Braiding for Representations of the Finite General Linear Groups

The main purpose of this Section is to describe a non-trivial braiding for the classical [Gn, Z] tensor product of finite dimensional complex representations of the general linear groups $GL_n(\mathbf{F}_q)$ for a fixed field $\mathbf{F}_q$ of cardinality q. Before considering this, we briefly review categories of representations of the symmetric groups $\mathfrak{S}_n$.

It is well known that to study representations of the symmetric groups, we should look at all the symmetric groups taken together. So we consider the category $\mathfrak{P}$ of finite sets and bijective functions; this category is equivalent to the disjoint union $\mathfrak{S}$ of the symmetric groups $\mathfrak{S}_n$, $n \in \mathbf{N}$. A *linear species* [J] is a functor

$$M: \mathfrak{P} \to \mathscr{V}ect_f,$$

where $\mathscr{V}ect_f$ is the category of finite dimensional complex vector spaces and linear functions. We write $\mathscr{R}\mathfrak{P}$ for the category $[\mathfrak{P}, \mathscr{V}ect_f]$ of linear species with the natural transformations as maps. A linear species determines (and is determined up to isomorphism by) a sequence of representations

$$\mathfrak{S}_n \times M[n] \to M[n]$$

of $\mathfrak{S}_n$, $n \in \mathbb{N}$, where $M[n]$ is the value of the functor M at the set $\{1, 2, ..., n\}$. The tensor product of linear species M, N is given by

$$(M \otimes N)(S) = \bigoplus_{T \subseteq S} M(T) \otimes N(S \backslash T),$$

where the direct sum runs over all subsets T of $S \in \mathfrak{P}$ and $S \backslash T$ denotes the complement of T in S. A bijection $\xi: S \xrightarrow{\sim} S'$ induces bijections

$$M(T) \xrightarrow{\sim} M(T'), \qquad N(S \backslash T) \xrightarrow{\sim} N(S' \backslash T')$$

for $T \subseteq S$ and $T' = \xi(T)$, and hence a linear isomorphism

$$(M \otimes N)(\xi): (M \otimes N)(S) \xrightarrow{\sim} (M \otimes N)(S').$$

Associativity constraints come from the identity

$$(M \otimes N \otimes P)(S) = \bigoplus_{U \subseteq T \subseteq S} M(U) \otimes N(T \backslash U) \otimes P(S \backslash T).$$

Furthermore, we have a natural isomorphism $c_{M,N}: M \otimes N \xrightarrow{\sim} N \otimes M$ whose component at $S \in \mathfrak{P}$ is the direct sum over all $T \subseteq S$ of the switch maps

$$M(T) \otimes N(S \backslash T) \xrightarrow{\sim} N(S \backslash T) \otimes M(T),$$

where the codomain is the direct summand of $(N \otimes M)(S)$ for the subset $S \setminus T$ of S. It is straightforward to verify that we have described a symmetric tensor category $\mathscr{R}\mathfrak{P}$ which has been studied [J] in connection with combinatorics.

Now we consider representations of finite general linear groups. Let $\mathfrak{G}(q)$ be the category whose objects are finite vector spaces over the field $\mathbf{F}_q$ and whose arrows are linear isomorphisms. The monoid of endomorphisms of $E \in \mathfrak{G}(q)$ is the finite general linear group $GL(E)$. A (finite dimensional complex) *representation* of $\mathfrak{G}(q)$ is a functor

$$M : \mathfrak{G}(q) \to \mathscr{V}ect_{\mathrm{f}}.$$

We write $\mathscr{R}\mathfrak{G}(q)$ for the category $[\mathfrak{G}(q), \mathscr{V}ect_{\mathrm{f}}]$ of representations of $\mathfrak{G}(q)$ with natural transformations as maps. A representation M determines (and is determined up to isomorphism by) a sequence of linear representations

$$GL(n, \mathbf{F}_q) \times M[n] \to M[n]$$

of $GL(n, \mathbf{F}_q)$, $n \geqslant 0$, where $M[n] = M(\mathbf{F}_q^n)$.

The classical tensor product of representations has the following simple expression in our setting: for representations M, N of $\mathfrak{G}(q)$, we put

$$(M \otimes N)(E) = \bigoplus_{A \leqslant E} M(A) \otimes N(E/A),$$

where the direct sum runs over the set of all subspaces A of E and E/A is the quotient space. Each linear isomorphism $E \cong E'$ induces isomorphisms $A \cong A'$, $E/A \cong E'/A'$, for each subspace A of E, and hence a linear isomorphism $(M \otimes N)(E) \cong (M \otimes N)(E')$. The associativity constraints come from the obvious identity

$$(M \otimes N \otimes P)(E) = \bigoplus_{A \leqslant B \leqslant E} M(A) \otimes N(B/A) \otimes P(E/B).$$

In this way we obtain a tensor category $\mathscr{R}\mathfrak{G}(q)$. The braiding for $\mathscr{R}\mathfrak{G}(q)$ is less classical.

THEOREM 4.1. *$\mathscr{R}\mathfrak{G}(q)$ is a braided tensor category.*

Proof. To describe the braiding $c: M \otimes N \to N \otimes M$, we have to define an isomorphism

$$\theta = c_{M,N}(E) : (M \otimes N) \xrightarrow{\sim} (N \otimes M)(E)$$

for each $E \in \mathfrak{G}(q)$. For each pair A, B of complementary subspaces of E, let

$$r_{A,B} : A \to E/B, \qquad s_{A,B} : E/A \to B$$

be the canonical isomorphisms. When A, B are not complementary, put $r_{A,B} = s_{A,B} = 0$. The composites

$$M(A) \otimes N(E/A) \xrightarrow{M(r_{A,B}) \otimes N(s_{A,B})} M(E/B) \otimes N(B) \xrightarrow{\text{switch}} N(B) \otimes M(E/B)$$

are the entries $\theta_{A,B}$ of the matrix $(\theta_{A,B})$ defining the map

$$\theta \colon \bigoplus_{A \leqslant E} M(A) \otimes N(E/A) \to \bigoplus_{B \leqslant E} N(B) \otimes M(E/B).$$

Naturality is clear. The invertibility of the map θ is non-trivial and will be proved in detail in [JS5]; however, see Theorem 4.2 below. It then remains to check the commutativity of (B1) and (B2). To do this, we use the 3-fold tensor product to transform (B1) into the triangle below.

$$\begin{array}{ccc} M \otimes N \otimes P & \longrightarrow & N \otimes P \otimes M \\ & \searrow \quad \nearrow & \\ & N \otimes M \otimes P & \end{array}$$

The symmetry of the tensor product of $\mathcal{V}ect_f$ allows us to translate the value of the above triangle at E to the triangle

$$\begin{array}{ccc} \bigoplus_{A \leqslant B \leqslant E} M(A) \otimes N(B/A) \otimes P(E/B) & \xrightarrow{\gamma} & \bigoplus_{C \leqslant D \leqslant E} M(E/D) \otimes N(C) \otimes P(D/C) \\ & {}_{\alpha}\searrow \quad \nearrow_{\beta} & \\ & \bigoplus_{C' \leqslant B' \leqslant E} M(B'/C') \otimes N(C') \otimes P(E/B') & \end{array}$$

where

(i) the matrix γ has component $\gamma_{A,B,C,D}$ zero unless A, D are complementary subspaces of E and the canonical isomorphism $E/A \cong D$ sends B/A to C, in which case $\gamma_{A,B,C,D}$ is induced by the isomorphisms

$$A \cong E/D, \qquad B/A \cong C, \qquad E/B \cong D/C;$$

(ii) the matrix α has component $\alpha_{A,B,C',B'}$ zero unless $B = B'$ and A, C' are complementary subspaces of B, in which case $\alpha_{A,B,C',B'}$ is induced by the isomorphisms

$$A \cong B/C', \qquad B/A \cong C', \qquad E/B \cong E/B';$$

(iii) the matrix β has component $\beta_{C',B',C,D}$ zero unless $C = C'$ and B'/C, D/C are complementary subspaces of E/C, in which case $\beta_{C',B',C,D}$ is induced by the isomorphisms

$$B'/C \cong E/D, \qquad C' \cong C, \qquad E/B' \cong D/C.$$

The desired result $\gamma = \beta \circ \alpha$ follows from the equation

$$\gamma_{A,B,C,D} = \beta_{C,B,C,D} \circ \alpha_{A,B,C,B}$$

which holds when either side is non-zero. Q.E.D.

The relationship between Hecke algebras and representations of the finite general linear groups is now explained without proof. Let Q denote the set of monic irreducible polynomials with nonzero constant term over the field $\mathbf{F}_q$. Let d_u denote the degree of $u \in Q$. Also, let $\varepsilon_u = 1$ if $u(0)$ is a square in $\mathbf{F}_q$ and $\varepsilon_u = -1$ otherwise. The *Hecke category based on* Q is the complex-additive strict tensor category $\mathfrak{H}(Q)$ presented as follows. The objects are the elements of Q. The arrows are generated by arrows

$$c_{u,v}\colon u \otimes v \to v \otimes u$$

subject to the relations

$$c_{u,v} \circ c_{v,u} = q^{d_u d_v} \qquad \text{for distinct } u, v, \qquad \text{and}$$
$$(c_{u,u} - \varepsilon_u q^{(1/2) d_u (d_u + 1)}) \circ (c_{u,u} + \varepsilon_u q^{(1/2) d_u (d_u - 1)}) = 0.$$

Gelfand and Graev [H, Appendix 3] have shown how to assign a cuspidal representation U_u of $GL(d_u, \mathbf{F}_q)$ to each $u \in Q$, and the calculations of [H; Chap. 1, Sect. 5] can be used to show that the assignment of the braiding $U_u \otimes U_v \to U_v \otimes U_u$ in $\mathcal{R}\mathfrak{G}(q)$ to $c_{u,v}\colon u \otimes v \to v \otimes u$ is compatible with source, target, tensor, and the above relations. So we obtain a complex-additive tensor functor

$$\mathfrak{H}(Q) \to \mathcal{R}\mathfrak{G}(q).$$

A result of Harish-Chandra [H; Chap. 1, Sect. 4] implies the next structure theorem for $\mathcal{R}\mathfrak{G}(q)$ from which follow the invertibility of the braiding of $\mathcal{R}\mathfrak{G}(q)$ and the braiding axioms for $\mathfrak{H}(Q)$.

THEOREM 4.2. *The functor $\mathfrak{H}(Q) \to \mathcal{R}\mathfrak{G}(q)$ is fully faithful. Moreover, every object of $\mathcal{R}\mathfrak{G}(q)$ is in the closure of the image of this functor under direct sums, retracts, and tensor products.*

5. Abstract Categorical Aspects of Braidings

We now present some results which are mainly of a categorical nature. We give a theoretical explanation of braidings by relating them to extra tensor products on a tensor category. (Indeed, this was one of our starting points for the notion of braiding following the suggestion of Walters [Wlt] that one-object bicategory with suitable tensor product should probably be a symmetric tensor category.) Symmetries are also "explainable." We also point out that categories enriched over braided tensor categories admit opposites and duals; classically [EK] this was just done for symmetric bases. Finally, we show that the concept of braiding can be abstracted to tensor objects in braided tensor 2-categories, and we give some applications.

We begin by observing that some of the commutative diagrams proved for a braiding can be taken as replacements for the defining axioms (B1) and (B2).

Proposition 5.1. *In a tensor category $\mathscr{V}$, a natural family of isomorphisms*

$$c_{A,B}\colon A\otimes B\to B\otimes A, \qquad A, B\in\mathscr{V}$$

is a braiding if and only if the two triangles of Proposition 2.1 *and the first diagram of Proposition* 2.7 *commute.*

Proof. Propositions 2.1 and 2.7 prove "only if." For the converse, take $C=I$ in the first diagram of Proposition 2.7 and use the triangles to obtain (B1). For (B2), take $D=I$. Q.E.D.

Let $\mathbb{1}$ denote the category with only one object, 0, and with only one arrow, an identity. Clearly there is a unique tensor category structure on $\mathbb{1}$. This is clearly the terminal object in the 2-category $\mathscr{T}en$ of tensor categories. For each tensor category $\mathscr{V}$, there is a tensor functor $I\colon\mathbb{1}\to\mathscr{V}$ which is unique up to isomorphism of tensor functors. The product $\mathscr{V}\times\mathscr{W}$ of two tensor categories $\mathscr{V}$, $\mathscr{W}$ is just their product as categories together with the tensor product given by the composite

$$\mathscr{V}\times\mathscr{W}\times\mathscr{V}\times\mathscr{W}\xrightarrow{1\times\text{switch}\times 1}\mathscr{V}\times\mathscr{V}\times\mathscr{W}\times\mathscr{W}\xrightarrow{\otimes\times\otimes}\mathscr{V}\times\mathscr{W}$$

and the constraints such that the projections are strict tensor functors. This is clearly the product in the 2-category $\mathscr{T}en$.

Let us define a *multiplication* on a tensor category $\mathscr{V}$ to be a tensor functor $\Phi\colon\mathscr{V}\times\mathscr{V}\to\mathscr{V}$ together with isomorphisms of tensor functors ρ, λ as illustrated below:

$$\begin{array}{ccccc} \mathscr{V} & \xrightarrow{(1_{\mathscr{V}},\, I)} & \mathscr{V} \times \mathscr{V} & \xleftarrow{(I,\, 1_{\mathscr{V}})} & \mathscr{V} \\ & {\scriptstyle 1_{\mathscr{V}}} \searrow \quad \overset{\Leftarrow}{\rho} & \big\downarrow {\scriptstyle \Phi} & \overset{\Rightarrow}{\lambda} \quad \swarrow {\scriptstyle 1_{\mathscr{V}}} & \\ & & \mathscr{V} & & \end{array}$$

The tensor structure on the functor $\Phi : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$ involves isomorphisms

$$\phi_0 : I \xrightarrow{\sim} \Phi(I, I),$$

$$\phi_{2; A, A', B, B'} : \Phi(A, A') \otimes \Phi(B, B') \xrightarrow{\sim} \Phi(A \otimes B, A' \otimes B')$$

satisfying various coherence conditions; see Section 1.

PROPOSITION 5.2. *In a tensor category $\mathscr{V}$, a family of arrows*

$$c_{A,B} : A \otimes B \to B \otimes A, \qquad A, B \in \mathscr{V}$$

is a braiding for $\mathscr{V}$ if and only if a multiplication Φ on $\mathscr{V}$ is defined by the assignements $\Phi = \otimes : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$, $\phi_0 = r_I^{-1} : I \xrightarrow{\sim} I \otimes I$, $\rho = r$, $\lambda = l$ and ϕ_2 is such that the square below commutes:

$$\begin{array}{ccc} (A \otimes A') \otimes (B \otimes B') & \xrightarrow{\phi_2} & (A \otimes B) \otimes (A' \otimes B') \\ \wr\big\downarrow & & \wr\big\downarrow \\ (A \otimes (A' \otimes B)) \otimes B' & \xrightarrow[(1 \otimes c) \otimes 1]{} & (A \otimes (B \otimes A')) \otimes B' \end{array}$$

Proof. By Corollary 1.4, it is possible to assume that $\mathscr{V}$ is a strict tensor category. It is clear then that Φ be a tensor functor is equivalent to the diagrams of Proposition 5.1 characterising a braiding, except that the diagrams are tensored on one side or other by extra objects. These extra objects cause no problem since tensoring preserves commutativity, and, in the other direction, the extra objects can be taken to be I. That ρ, λ are morphisms of tensor functors amounts to the two triangles of Proposition 2.1. Q.E.D.

The next result generalizes the fact that, if a binary operation on a monoid is a monoid homomorphism and has the unit of the monoid as a unit, then the operation is forced to be the monoid multiplication and the monoid is commutative [EH].

PROPOSITION 5.3. *For any multiplication Φ on a monoidal category $\mathscr{V}$, a braiding c for $\mathscr{V}$ is defined by the following commutative diagram:*

$$\begin{array}{ccc}
A\otimes B & \xrightarrow{c_{A,B}} & B\otimes A \\
\downarrow{\scriptstyle \lambda\otimes\rho} & & \downarrow{\scriptstyle \lambda^{-1}\otimes\rho^{-1}} \\
\Phi(I,A)\otimes\Phi(B,I) & & \Phi(B,I)\otimes\Phi(I,A) \\
\downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_2} \\
\Phi(I\otimes B, A\otimes I) & & \Phi(B\otimes I, I\otimes A) \\
& {\scriptstyle \Phi(\ell, r)}\searrow \quad \swarrow{\scriptstyle \Phi(r,\ell)} & \\
& \Phi(B,A) &
\end{array}$$

The multiplication obtained from this braiding c via Proposition 5.2 *is isomorphic* (*in the obvious sense*) *to Φ. If c' is any braiding for $\mathscr{V}$ and Φ is obtained from c' via Proposition* 5.2 *then $c = c'$.*

Proof. By Corollary 1.4, it is possible to assume $\mathscr{V}$ is a strict tensor category. The conditions that ρ, λ be morphisms of tensor functors include the condition that ρ_I, λ_I both be inverses for $\phi_0: I \xrightarrow{\sim} \Phi(I, I)$; so $\rho_I = \lambda_I$. This allows us to replace Φ by an isomorphic tensor functor which is a multiplication on $\mathscr{V}$ with ρ and λ both identity morphisms. Having done all this, let us now put

$$\phi(A, A', B, B') = \phi_{2:A,A',B,B'}$$

so that the conditions for a multiplication become that $\phi(A, I, B, I)$, $\phi(I, A, I, B)$, $\phi(A, A', I, I)$, $\phi(I, I, B, B')$ all be identity arrows,

$$\begin{aligned} &\phi(A\otimes B, A'\otimes B', C, C')\circ(\phi(A, A', B, B')\otimes 1_{\Phi(C,C')}) \\ &\quad = \phi(A, A', B\otimes C, B'\otimes C')\circ(1_{\Phi(A,A')}\otimes\phi(B, B', C, C')). \end{aligned}$$

These give the three equations

$$\phi(I, A\otimes B, C, I) = \phi(I, A, C, B)\circ(1_A\otimes\phi(I, B, C, I))$$

$$\phi(C, A, I, B)\circ(\phi(C, I, I, A)\otimes 1_B) = \phi(C, I, I, A\otimes B)$$

$$\phi(C, A, I, B)\circ(\phi(I, A, C, I)\otimes 1_B) = \phi(I, A, C, B)\circ(1_A\otimes\phi(C, I, I, B)).$$

Moreover, $c_{A,B} = \phi(B, I, I, A)^{-1}\circ\phi(I, A, B, I)$; so the three equations above easily give

$$c_{A\otimes B, C} = (c_{A,C}\otimes 1_B)\circ(1_A\otimes c_{B,C}),$$

which is axiom (B2) for a braiding; axiom (B1) is similar.

For the second sentence of the proposition, we have the tensor isomorphism

$$\phi(A, I, I, B)\colon A \otimes B \xrightarrow{\sim} \Phi(A, B).$$

For the third sentence, we have $\phi(A, A', B, B') = 1_A \otimes c'_{A', B} \otimes 1_{B'}$; so

$$\begin{aligned} c_{A,B} &= \beta(B, I, I, A)^{-1} \circ \phi(I, A, B, I) \\ &= (1_B \otimes c'_{I,I} \otimes 1_A)^{-1} \circ (1_I \otimes c'_{A,B} \otimes 1_I) \\ &= c'_{A,B}. \end{aligned}$$

Q.E.D.

PROPOSITION 5.4. *Suppose the braiding c corresponds to the multiplication Φ as in Proposition 5.3. Then c is a symmetry if and only if Φ is a braised tensor functor.*

Proof. As usual, we can assume $\mathcal{V}$ strict and, by Proposition 5.3, that Φ is defined in terms of c as in Proposition 5.2. Then to say Φ is braided is to say that the following square commutes:

$$\begin{array}{ccc} A \otimes A' \otimes B \otimes B' & \xrightarrow{c_{A,A'} \otimes c_{B,B'}} & A' \otimes A \otimes B' \otimes B \\ {\scriptstyle 1 \otimes c_{A',B} \otimes 1} \downarrow & & \downarrow {\scriptstyle 1 \otimes c_{A,B'} \otimes 1} \\ A \otimes B \otimes A' \otimes B' & \xrightarrow[c_{A \otimes B, A' \otimes B'}]{} & A' \otimes B' \otimes A \otimes B \end{array}$$

Take $A = B' = I$ in this square to see that c must be a symmetry. That the square does commute when c is a symmetry follows from Mac Lane's coherence theorem [ML1]. Q.E.D.

Remark 5.1. A more categorical formulation of Propositions 5.2, 5.3, and 5.4 is possible. We have mentioned that the 2-category $\mathcal{T}en$ of tensor categories has products. Let $\mathcal{BT}en$ denote the 2-category of braided tensor categories, braided tensor functors, and morphisms of tensor functors. Propositions 5.2 and 5.3 yield an equivalence of 2-categories

$$\mathcal{BT}en \simeq \mathrm{Mult}(\mathcal{T}en),$$

where $\mathrm{Mult}(\mathcal{T}en)$ denotes the 2-category of tensor categories with multiplication. Notice that the 2-category $\mathcal{BT}en$ has products formed as in $\mathcal{T}en$ with the obvious braidings. Let $\mathcal{ST}en$ denote the full sub-2-category of $\mathcal{BT}en$ consisting of the symmetric tensor categories. Proposition 5.4 gives equivalences of 2-categories

$$\mathcal{ST}en \simeq \mathrm{Mult}(\mathcal{BT}en) \simeq \mathrm{Mult}(\mathrm{Mult}(\mathcal{T}en)).$$

We now consider monoids in a tensor category $\mathcal{V}$. A *monoid* A in $\mathcal{V}$ consists of an object A and arrows $\mu: A \otimes A \to A$, $\eta: I \to A$ (called *multiplication* and *unit*) in $\mathcal{V}$ satisfying the usual associativity and unity conditions [ML2]. A *homomorphism of monoids* is an arrow $f: A \to B$ which commutes with the multiplication and unit. This gives a category $\mathrm{Mon}(\mathcal{V})$ of monoids in $\mathcal{V}$. (For example, if $\mathcal{V} = \mathcal{M}od_k$ is the category of modules over the commutative ring k, then $\mathrm{Mon}(\mathcal{M}od_k) = \mathcal{A}lg_k$ is the category of k-algebras.)

If $\mathcal{V}$ is braided, we can define a monoid structure on the tensor product $A \otimes B$ of two monoids A, B in $\mathcal{V}$ by means of the multiplication and unit

$$A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes c \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{\mu \otimes \mu} A \otimes B,$$

$$I \xrightarrow{\eta \otimes \eta} A \otimes B,$$

where we write as if $\mathcal{V}$ were strict. In this way, $\mathrm{Mon}(\mathcal{V})$ becomes a tensor category for which the forgetful functor $\mathrm{Mon}(\mathcal{V}) \to \mathcal{V}$ is a strict tensor functor. Also, for each monoid A, we obtain an *opposite* monoid A^{op} by replacing the multiplication μ by its composite with $c_{A,A}$. A monoid A is called *commutative* when it is equal to its opposite; that is, when the following triangle commutes:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{c_{A,A}} & A \otimes A \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & A & \end{array}$$

Remark 5.2. Category theorists will recognize that the above constructions for monoids carry over to enriched categories. Categories with homs enriched in a tensor (= monoidal) category $\mathcal{V}$ have been defined in [EK, pp. 495–496]; they are more briefly called *$\mathcal{V}$-categories*. (Monoids in $\mathcal{V}$ are one-object $\mathcal{V}$-categories.) For symmetric $\mathcal{V}$, Eilenberg and Kelly defined the notions of *opposite* and *tensor product* for $\mathcal{V}$-categories. This carries over using exactly the same definitions with $\mathcal{V}$ merely braided. There are just two warnings that need to be made.

(i) Opposite is not generally an involution. (Yet it is still possible to use the principle of *duality* in $\mathcal{V}$-category theory, since the assignment $\mathcal{A} \mapsto \mathcal{A}^{\mathrm{op}}$ is invertible.)

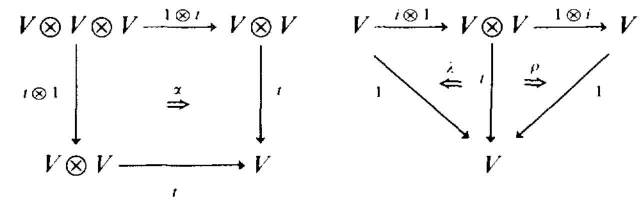
(ii) The tensor product on the (2-)category $\mathcal{V}$-*Cat* of $\mathcal{V}$-categories and $\mathcal{V}$-functors (and $\mathcal{V}$-natural transformations) yields a tensor (2-)

category. However, $\mathscr{V}$-$\mathscr{C}at$ is generally *not* braided in the way one might expect (unless $\mathscr{V}$ is symmetric).

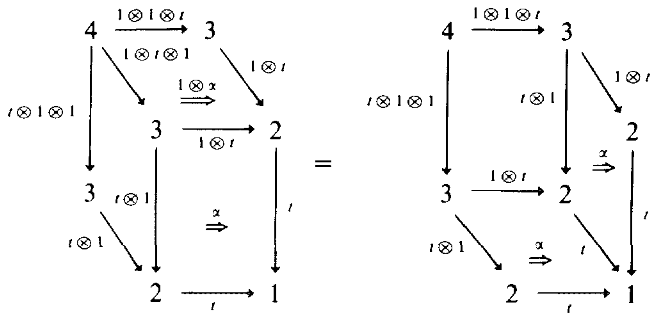
EXAMPLE 5.1. For any group G, Freyd and Yetter [FY1, FY2] have constructed a braided tensor category $\mathscr{C}r G$-$\mathscr{S}et$ of "crossed G-sets". A *G-set* is a set X together with a left action $G \times X \to X$, $(g, x) \mapsto gx$, so that $(gh)x = g(hx)$ and $1x = x$. A *crossed G-set* is a G-set X together with a function $|\ |: X \to G$, $x \mapsto |x|$, satisfying the condition $|gx| = g|x|\,g^{-1}$. A *morphism* $f: X \to Y$ of crossed *G-sets* is just a function satisfying $f(gx) = gf(x)$ and $|f(x)| = |x|$. The *tensor product* of crossed G-sets X, Y is their cartesian product $X \times Y$ as G-sets with $|(x, y)| = |x|\,|y|$. Thus we have a tensor category $\mathscr{C}r G$-$\mathscr{S}et$. It becomes braided by $c: X \times Y \to Y \times X$ given by $c(x, y) = (|x|\, y, x)$. (The usual switch map is not a morphism of crossed G-sets!) What we point out here is that each crossed module $(N, G, \partial, *)$ (see Remark 3.1) yields a *commutative* monoid N in the braided tensor category yields a tensor category $\mathscr{C}r G$-$\mathscr{S}et$. (The functions $*$, ∂ give N the structure of a crossed G-set, while the group multiplication of N makes it a monoid in $\mathscr{C}r G$-$\mathscr{S}et$; one of the crossed module axioms asserts commutativity.)

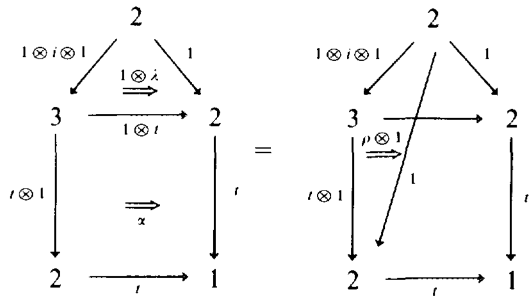
We now consider tensor objects in braided tensor 2-categories. A *tensor structure on a 2-category* $\mathscr{K}$ is just a tensor structure on the underlying category of $\mathscr{K}$, with the extra data needed for the tensor $\otimes : \mathscr{K} \times \mathscr{K} \to \mathscr{K}$ to be a 2-functor, and satisfying the condition that the constraints a, l, r should all be 2-natural. (The reader may need to refer to [KS] for the meaning of the 2-categorical pasting diagrams drawn below.)

DEFINITION 5.1. A *tensor object of* $\mathscr{K}$ consists of an object V of $\mathscr{K}$, arrows $t: V \otimes V \to V$, $i: I \to V$ (called the *tensor* and the *unit* for V), and invertible 2-cells



(called the *constraints of associativity, left unit, right unit*) such that the following two equations hold (where, for example, we have written 3 for $V \otimes V \otimes V$):





Call V *strict* when α, λ, ρ are identity 2-cells, so that V precisely amounts to a *monoid* in $\mathscr{K}$.

EXAMPLE 5.2. The 2-category $\mathscr{C}at$ of categories, functors, and natural transformations, with cartesian product as tensor product, is a tensor 2-category. A tensor object in $\mathscr{C}at$ is precisely a tensor category.

EXAMPLE 5.3. Let $\mathscr{P}rof$ denote the 2-category whose objects are categories and for which each homcategory $\mathscr{P}rof(\mathscr{A}, \mathscr{B})$ is the category of right-adjoint functors $[\mathscr{B}, \mathscr{S}et] \to [\mathscr{A}, \mathscr{S}et]$, where $[\mathscr{A}, \mathscr{B}]$ denotes the category of functors and natural transformations from $\mathscr{A}$ to $\mathscr{B}$. A right-adjoint functor $[\mathscr{B}, \mathscr{S}et] \to [\mathscr{A}, \mathscr{S}et]$ is determined up to isomorphism by the restriction of its left adjoints to representables, and so by a functor $\mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathscr{S}et$. The cartesian product of categories makes $\mathscr{P}rof$ a tensor 2-category. A tensor object in $\mathscr{P}rof$ is a *protensor category* (called "premonoidal" in [Da1]); the tensor is determined up to isomorphism by a functor $P\colon \mathscr{A}^{\mathrm{op}} \times \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathscr{S}et$.

EXAMPLE 5.4. Proposition 5.3 tells us that a tensor object in the cartesian tensor 2-category $\mathscr{T}en$ amounts to a braided tensor category.

Suppose $\mathscr{K}$, $\mathscr{L}$ are tensor 2-categories. A *tensor 2-functor* $\Phi\colon \mathscr{K} \to \mathscr{L}$ is just a tensor functor between the underlying tensor categories $\mathscr{K}\mathscr{L}$

enriched with the structure of a 2-functor and such that the constraints ϕ_0, ϕ_2 are 2-natural. We also need the notion of *weak tensor 2-functor* which is defined in the same way without the restriction that ϕ_0, ϕ_2 should be invertible.

PROPOSITION 5.5. *Weak tensor 2-functors take tensor objects to tensor objects.*

Proof. If V is a tensor object in $\mathscr{K}$ then ΦV becomes a tensor object in $\mathscr{L}$ with tensor equal to the composite

$$\Phi V \otimes \Phi V \xrightarrow{\phi_2} \Phi(V \otimes V) \xrightarrow{\Phi t} \Phi V.$$

Since Φ is a 2-functor, it preserves invertibility of 2-cells and any equations between pasted diagrams. Q.E.D.

EXAMPLE 5.5. The representable 2-functor $\mathscr{K}(I, -): \mathscr{K} \to \mathscr{C}at$ is a weak tensor 2-functor; so each tensor object V in $\mathscr{K}$ gives a tensor category $\mathscr{K}(I, V)$. Applying this to Example 3, we see that each protensor category $\mathscr{A}$ gives rise to a tensor category $\mathscr{P}rof(\mathbb{1}, \mathscr{A}) \simeq [\mathscr{A}, \mathscr{S}et]$ which is precisely $[\mathscr{A}, \mathscr{S}et]$ with the *convolution tensor product* of [Da1]. In fact, $[\mathscr{A}, \mathscr{S}et]$ is a biclosed tensor category [Da1].

There is an inclusion $\mathscr{C}at \to \mathscr{P}rof$ taking each functor $T: \mathscr{A} \to \mathscr{B}$ to restriction $[\mathscr{B}, \mathscr{S}et] \to [\mathscr{A}, \mathscr{S}et]$ along T. This inclusion is a tensor 2-functor, so every tensor category can be regarded as a protensor category. In this way, every example $\mathscr{V}$ of a tensor category gives an example of a biclosed tensor category $[\mathscr{V}, \mathscr{S}et]$.

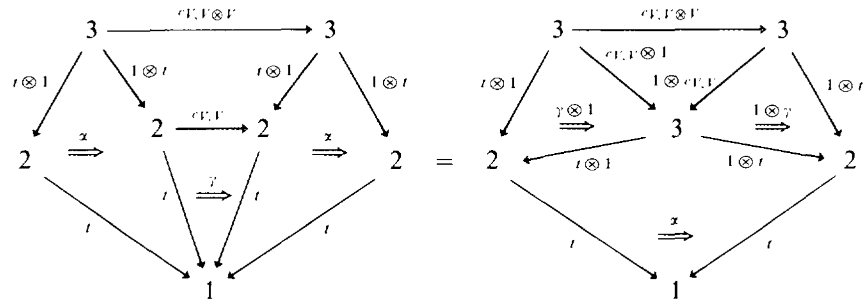
Now suppose the tensor 2-category $\mathscr{K}$ is *braided*; this just means that the underlying tensor category is braided and that the braiding is 2-natural.

DEFINITION 5.2. A *braiding* for a tensor object V in $\mathscr{K}$ is an invertible 2-cell γ as in the triangle

$$\begin{array}{ccc} V \otimes V & \xrightarrow{c_{V,V}} & V \otimes V \\ & {\scriptstyle t} \searrow \quad \overset{\gamma}{\Longrightarrow} \quad \swarrow {\scriptstyle t} & \\ & V & \end{array}$$

such that the equation

(B1)

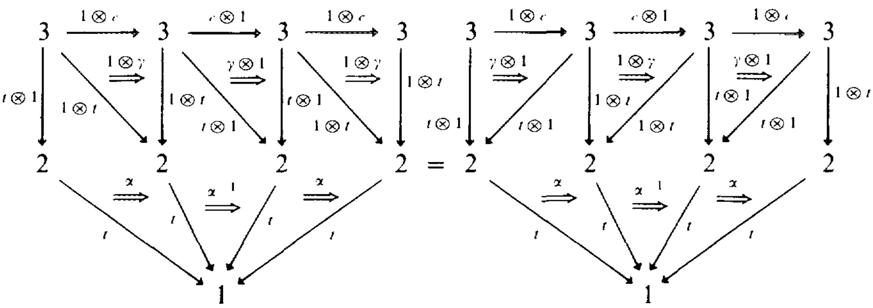


holds and so does the equation (B2) obtained from (B1) by replacing the braiding $c_{A,B}$ for $\mathscr{K}$ by $(c_{B,A})^{-1}$ and γ by $\gamma^{-1} \circ (c_{V,V})^{-1}$. A *braided tensor object* is a tensor object with a distinguished braiding.

PROPOSITION 5.6. (i) *Braided weak tensor 2-functors take braided tensor objects to braided tensor objects.*

(ii) *A braiding for a tensor object satisfies the following equality*:

(YB)



Note that the 2-naturalness of the braiding for $\mathscr{K}$ implies that the weak tensor 2-functor $\mathscr{K}(I, -): \mathscr{K} \to \mathscr{C}at$ is braided.

EXAMPLE 5.6. Both $\mathscr{C}at$ and $\mathscr{P}rof$ are symmetric, and braided tensor objects therein are respectively braided tensor categories and braided protensor categories. It follows that each braided protensor category $\mathscr{A}$ gives a convolution braided tensor category $[\mathscr{A}, \mathscr{S}et]$.

EXAMPLE 5.7. For any tensor category $\mathscr{V}$, the category Mon($\mathscr{V}$) of monoids in $\mathscr{V}$ (as described earlier in this section) becomes a 2-category on defining a 2-cell $\alpha : f \Rightarrow g$ between monoid homomorphisms $f, g : A \to B$ to be an arrow $\alpha : I \to B$ in $\mathcal{V}$ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & & \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \alpha \otimes 1} & & \\ B & \xrightarrow[1 \otimes \alpha]{} & B \otimes B & \xrightarrow[\mu]{} & B \end{array}$$

Suppose $\mathcal{V}$ is symmetric. Then, using the tensor product described earlier, $\mathrm{Mon}(\mathcal{V})$ becomes a symmetric tensor 2-category. Hence also $\mathrm{Mon}(\mathcal{V})^{\mathrm{op}}$ is a symmetric tensor 2-category. A *bialgebra in* $\mathcal{V}$ is a strict tensor object in $\mathrm{Mon}(\mathcal{V})^{\mathrm{op}}$. Thus we obtain the notion of *braiding on a bialgebra* A as used in the theory of quantum groups (see [JS4]). (A tensor object in $\mathrm{Mon}(\mathcal{V})^{\mathrm{op}}$ is a *quasi-bialgebra* as used by [Dr].) Moreover,

$$\mathrm{Mon}(\mathcal{V})^{\mathrm{op}} \to \mathcal{C}at, \qquad A \mapsto \mathrm{Mod}(A)$$

is a symmetric weak tensor 2-functor, so each braiding on A gives a braiding on the tensor category $\mathrm{Mod}(A)$ of A-modules [Mj1].

6. Balanced Tensor Categories

Many examples of tensor categories possess structure which is part-way between braiding and symmetry.

Definition 6.1. Suppose $\mathcal{V}$ is a braided tensor category. A (*full*) *twist* for $\mathcal{V}$ is a natural family of isomorphisms

$$\theta = \theta_A : A \xrightarrow{\sim} A$$

such that $\theta_I = 1_I$ and the following diagram (T) commutes:

(T)
$$\begin{array}{ccc} A \otimes B & \xrightarrow{c_{A,B}} & B \otimes A \\ {\scriptstyle \theta_{A \otimes B}}\downarrow & & \downarrow{\scriptstyle \theta_B \otimes \theta_A} \\ A \otimes B & \xleftarrow{c_{B,A}} & B \otimes A \end{array}$$

A tensor category equipped with a braiding and a twist is called *balanced*. A tensor functor $F : \mathcal{V} \to \mathcal{W}$ is called *balanced* when it is braided and preserves the twist (that is, $F\theta_A = \theta_{FA}$). We write $\boldsymbol{B}\mathcal{T}en(\mathcal{V}, \mathcal{W})$ for the category of balanced tensor functors and morphisms of tensor functors.

Remark 6.1. In example 2.5 we saw that the braiding $c_{A,B}$ determines a tensor isomorphism $\mathcal{V} \xrightarrow{\sim} \mathcal{V}^{\mathrm{rev}}$. The braiding $(c_{B,A})^{-1}$ determines another tensor isomorphism $\mathcal{V}^{\cdot} \xrightarrow{\sim} \mathcal{V}^{\cdot\,\mathrm{rev}}$. A twist is precisely a morphism between these two tensor isomorphisms.

Remark 6.2. In checking examples it is useful to observe that the braiding axiom (B2) is redundant in the list of axioms for a balanced tensor category.

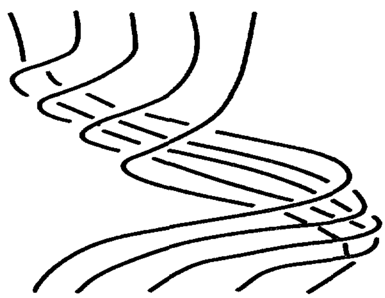
EXAMPLE 6.1. The identity arrows determine a twist for a braided tensor category if and only if the braiding is a symmetry. So symmetric tensor categories are balanced.

EXAMPLE 6.2. In Example 3.1 of a braided strict categorical group, constructed from a bilinear map $f\colon A \times A \to X$, there is a twist given by

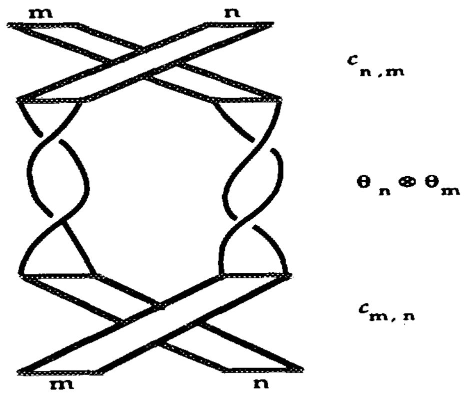
$$\theta_a = f(a, a)\colon a \to a.$$

EXAMPLE 6.3. The braid category $\mathfrak{B}$ (Example 2.1) is canonically balanced. The twist $\theta_n\colon n \to n$ can be viewed by taking the diagram for the identity braid on n vertical strings with the ends of the strings on two parallel horizontal rigid rods and rotating the top rod through 360° according to the right-hand screw law with thumb pointing upward (see below). In particular,

$$\theta_1 = 1_1\colon 1 \to 1 \qquad \text{and} \qquad \theta_2 = (c_{1,1})^2 = (s_1)^2\colon 2 \to 2.$$



The twist condition (T) can be verified by physically taking two adjacent ribbons representing the identity braids of m and n, performing the operations $c_{m,n}$, $\theta_n \otimes \theta_m$, $c_{n,m}$ in order, and observing that the result is a complete twist of the two ribbons taken together as one:

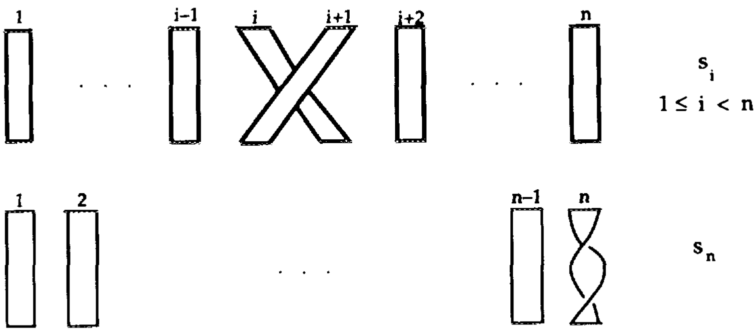


More generally, the labelled braid categories $\mathfrak{B} \int \mathcal{A}$ (Example 2.1) are all canonically balanced. Just label the strings of the twist for $\mathfrak{B}$ with identity arrows.

EXAMPLE 6.4. There is a tensor category **B** defined similarly to $\mathfrak{B}$ except that the arrows are braids on ribbons (instead of on strings) and it is permissible to twist the ribbons through full turns. The objects of **B** are the natural numbers, while the only non-empty homsets are the endomorphism monoids. The monoid $\mathbf{B}(n, n)$ is the group generated by $s_1, ..., s_{n-1}, s_n$ subject to the braid relations $(A1)$, (A2) (Example 2.1) for $s_1, ..., s_{n-1}$ and the further relation

$$s_{n-1} s_n s_{n-1} s_n = s_n s_{n-1} s_n s_{n-1}.$$

The pictorial representation of the generators is



A braid on ribbons is completely described by a braid on strings labelled by integers $n \in \mathbf{Z}$ indicating how many twists are performed on each ribbon of the braid and in which direction. Therefore, we can identify $\mathbf{B}$ with the braided tensor category $\mathfrak{B} \int \mathbb{Z}$, where $\mathbb{Z}$ denotes the additive monoid of integers regarded as a one-object category.

Now each ribbon has two edges which act as strings, so there is a faithful braided tensor functor $\mathbf{B} \to \mathfrak{B}$ taking n to $2n$, and taking the full twist $s_1: 1 \to 1$ in $\mathbf{B}$ on one ribbon to $(s_1)^2: 2 \to 2$ in $\mathfrak{B}$. Restriction of the canonical twist on $\mathfrak{B}$ along this functor makes $\mathbf{B}$ into a balanced tensor category. (Warning: this is not the same as the canonical twist on $\mathfrak{B} \int \mathbb{Z}$ as in Example 6.3.)

There is also a balanced category $\mathbf{B} \int \mathscr{A}$ of braids on labelled ribbons which can be identified with $\mathfrak{B} \int \mathbb{Z} \times \mathscr{A}$ as a braided tensor category, but with twist obtained by restriction along the functor $\mathbf{B} \int \mathscr{A} \to \mathfrak{B} \int \mathscr{A}$ induced by $\mathbf{B} \to \mathfrak{B}$.

Definition 6.2. Suppose $T: \mathscr{A} \to \mathscr{V}$ is a functor from a category $\mathscr{A}$ to a tensor category $\mathscr{V}$. A Yang–Baxter operator y on T (Definition 2.3) is said to be *balanced* when it is equipped with a natural family of isomorphisms

$$\theta = \theta_A : TA \xrightarrow{\sim} TA$$

such that the following two squares commute:

$$\begin{array}{ccccc} TA \otimes TB & \xrightarrow{y} & TB \otimes TA & \xrightarrow{y} & TA \otimes TB \\ \downarrow{\scriptstyle \theta \otimes 1} & & \downarrow{\scriptstyle 1 \otimes \theta} & & \downarrow{\scriptstyle \theta \otimes 1} \\ TA \otimes TB & \xrightarrow[y]{} & TB \otimes TA & \xrightarrow[y]{} & TA \otimes TB \end{array}$$

Example 6.5. Each functor $T: \mathscr{A} \to \mathscr{V}$ into a balanced tensor category admits a balanced Yang–Baxter operator $y_{A,B} = c_{TA,TB}$, $\theta = \theta_{TA}$. In particular, the inclusion $i: \mathscr{A} \to \mathbf{B} \int \mathscr{A}$ admits a canonical balanced Yang–Baxter operator c, ψ.

Remark 6.3. A functor $T: \mathscr{A} \to \mathscr{V}$ together with a natural automorphism θ of T precisely amounts to a functor $S: \mathbb{Z} \times \mathscr{A} \to \mathscr{V}$. A balanced Yang–Baxter operator (y, θ) on T precisely amounts to a Yang–Baxter operator y' on S. Equipped with these observations, the reader will find no difficulty in supplying proofs of Proposition 6.1 and Theorem 6.2. The same ideas are involved as for the corresponding results in Section 2.

Proposition 6.1. (a) *For any strict tensor category $\mathscr{V}$ and any balanced Yang–Baxter operator (y, θ) on $T: \mathscr{A} \to \mathscr{V}$, there exists a unique* *strict tensor functor* $T'\colon \mathbf{B}\int\mathcal{A} \to \mathcal{V}$ *such that* $T'(x) = y$, $T'(\psi) = \theta$, *and the following triangle commutes*:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ i\ } & \mathbf{B}\int\mathcal{A} \\ & {\scriptstyle T}\searrow & \downarrow{\scriptstyle T'} \\ & & \mathcal{V} \end{array}$$

(b) $\mathbf{B}\int\mathcal{A}$ *is the free balanced strict tensor category on* $\mathcal{A}$.

(c) *For every balanced tensor functor* $F\colon \mathbf{B}\int\mathcal{A} \to \mathcal{V}$ *into a balanced strict tensor category* $\mathcal{V}$, *there exists a balanced strict tensor functor* $S\colon \mathfrak{B}\int\mathcal{A} \to \mathcal{V}$ *and an isomorphism* $\sigma\colon F \cong S$ *of tensor functors whose restriction* $\sigma \circ i$ *to* $\mathcal{A}$ *is the identity.*

(d) *Suppose* $\mathcal{A}$ *is a category and* $\mathcal{V}$ *is a balanced tensor category. Then restriction along* $i\colon \mathcal{A} \to \mathbf{B}\int\mathcal{A}$ *determines an equivalence of categories*

$$\boldsymbol{B}\mathcal{T}\!\mathit{en}(\mathbf{B}\int\mathcal{A}, \mathcal{V}) \simeq [\mathcal{A}, \mathcal{V}].$$

To consider coherence for balanced tensor categories we introduce the free balanced tensor category $\mathcal{F}\boldsymbol{b}\mathcal{A}$ generated by the category $\mathcal{A}$. For each balanced tensor category $\mathcal{V}$, restriction along the inclusion $\mathcal{A} \to \mathcal{F}\boldsymbol{b}\mathcal{A}$ determines a bijection between balanced strict tensor functors $\mathcal{F}\boldsymbol{b}\mathcal{A} \to \mathcal{V}$ and functors $\mathcal{A} \to \mathcal{V}$. In particular, there is a unique braided strict tensor functor $\Gamma\colon \mathcal{F}\boldsymbol{b}\mathcal{A} \to \mathbf{B}\int\mathcal{A}$ whose restriction to $\mathcal{A}$ is $i\colon \mathcal{A} \to \mathbf{B}\int A$.

THEOREM 6.2 (Coherence for Balanced Tensor Categories). $\Gamma\colon \mathcal{F}\boldsymbol{b}\mathcal{A} \to \mathbf{B}\int\mathcal{A}$ *is a braided tensor equivalence.*

There is a canonical balanced strict tensor functor $\mathbf{B}\int\mathcal{A} \to \mathbf{B}$ induced by the unique functor $\mathcal{A} \to \mathbf{1}$. Composing this with Γ we obtain a balanced strict tensor functor

$$\mathcal{F}\boldsymbol{b}\mathcal{A} \to \mathbf{B}\int\mathcal{A} \to \mathbf{B}$$

whose value at an arrow of $\mathcal{F}\boldsymbol{b}\mathcal{A}$ is called the *underlying ribbon braid* of that arrow.

COROLLARY 6.3. *In the free balanced tensor category generated by a set* A *of objects, a diagram commutes if and only if all legs with the same source and target have the same underlying ribbon braid.*

7. AUTONOMY

This section deals with dual objects in tensor categories and their interrelations with braidings, Yang–Baxter operators, and twists.

Suppose $\mathscr{V}$ is any tensor category. A *pairing* between two objects A, B of $\mathscr{V}$ is a map

$$\varepsilon\colon A \otimes B \to I.$$

For any objects X, Y and map $f\colon X \to B \otimes Y$, let $\varepsilon^{\#}(f)$ denote the composite map

$$A \otimes X \xrightarrow{1 \otimes f} A \otimes B \otimes Y \xrightarrow{\varepsilon \otimes 1} Y.$$

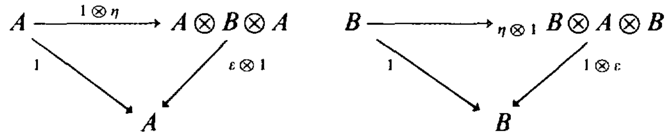
The pairing is *exact* when the function

$$\varepsilon^{\#}\colon \mathscr{V}(X, B \otimes Y) \to \mathscr{V}(A \otimes X, Y)$$

is bijective for all $X, Y \in \mathscr{V}$. This implies we have an adjunction of functors $A \otimes - \dashv B \otimes -$. It is easy to see that the pairing ε is exact if and only if there exists a map

$$\eta\colon I \to B \otimes A$$

such that the following two *adjunction triangles* commute (where we write as if $\mathscr{V}$ were strict):



It follows that we also have an adjunction of functors $- \otimes B \dashv - \otimes A$.

Definition 7.1. When the above adjunction triangles commute, we say that the pair (η, ε) is an *adjunction* between A and B, and that A [respectively, B] is *left adjoint* or *left dual* to B [respectively, *right adjoint* or *right dual* to A]; we write

$$(\eta, \varepsilon)\colon A \dashv B.$$

We call η the *unit* and ε the *counit* of the adjunction. If $(\eta, \varepsilon)\colon A \dashv B$ and $(\eta', \varepsilon')\colon A' \dashv B'$ then there is a bijection between maps $f\colon A \to A'$ and maps $g\colon B' \to B$ determined by any one of the four equations

$$f = (A' \otimes \eta) \circ (A' \otimes g \otimes A) \circ (\varepsilon' \otimes A),$$

$$\varepsilon' \circ (f \otimes B') = \varepsilon \circ (A \otimes g),$$

$$(B \otimes f) \circ \eta = (g \otimes A') \circ \eta',$$

$$g = (\eta' \otimes B) \circ (B' \otimes f \otimes B) \circ (B' \otimes \varepsilon).$$

We write $f \dashv g$ to indicate that f, g correspond under this bijection.

DEFINITION 7.2. A tensor category is *left* (*right*) *autonomous* when every object has a left (right) dual. It is *autonomous* when it is both left and right autonomous.

When $\mathscr{V}$ is left autonomous, we choose an adjunction

$$(\eta_A, \varepsilon_A): A^* \dashv A$$

from which we obtain a functor

$$(\ )^*: \mathscr{V}^{op} \to \mathscr{V}$$

determined on arrows $f: A \to B$ by the condition

$$f^* \dashv f.$$

The functor $(\ )^*$ is always fully faithful; it is an equivalence of categories if and only if $\mathscr{V}$ is autonomous.

Tensor functors preserve duals. More precisely, suppose that $F: \mathscr{V} \to \mathscr{W}$ is a tensor functor and $\varepsilon: A \otimes B \to I$ is a pairing in $\mathscr{V}$. Then a pairing $\varepsilon^F: FA \otimes FB \to I$ in $\mathscr{W}$ is given by the composite

$$FA \otimes FB \xrightarrow{\phi_2} F(A \otimes B) \xrightarrow{F} FI \xrightarrow{\phi_0} I.$$

If ε is exact then so is ε^F. So $A \dashv B$ implies $FA \dashv FB$. It is also easy to see that $f \dashv g$ implies $Ff \dashv Fg$.

Now we consider morphisms $\alpha: F \to G$ of tensor functors $F, G: \mathscr{V} \to \mathscr{W}$ in relation to duals.

PROPOSITION 7.1. *If $\mathscr{V}$ is left autonomous then $\mathscr{T}en(\mathscr{V}, \mathscr{W})$ is a groupoid. More explicitly, if $A \dashv B$ in $\mathscr{V}$ then $\alpha_B: FB \to GB$ is invertible with $\alpha_A \dashv \alpha_B^{-1}$.*

Proof. By the coherence results of Section 1 we may write as if F, G, $\mathscr{V}$, $\mathscr{W}$ were all strict. Let $g: GB \to FB$ be defined by $\alpha_A \dashv g$. So g is the composite

$$GB \xrightarrow{F\eta \otimes GB} FB \otimes FA \otimes GB \xrightarrow{FB \otimes \alpha_A \otimes GB} FB \otimes GA \otimes GB$$
$$\xrightarrow{FB \otimes G\varepsilon} FB.$$

Since α is a morphism of tensor functors, we have $\alpha_{B \otimes A} = (\alpha_B \otimes GA) \circ (FB \otimes \alpha_A)$ and $\alpha_{B \otimes A} \circ F\eta = G\eta$. Hence

$$\begin{aligned}\alpha_B \circ g &= (GB \otimes G\varepsilon) \circ (\alpha_B \otimes GA \otimes GB) \\ &\quad \circ (FB \otimes \alpha_A \otimes GB) \circ (F\eta \otimes GB) \\ &= (GB \otimes G\varepsilon) \circ (G\eta \otimes GB) = 1_{GB}.\end{aligned}$$

Similarly, $g \circ \alpha_B = 1_{FB}$; so $g = \alpha_B^{-1}$ as required. Q.E.D.

The next results deals with braidings and duals.

PROPOSITION 7.2. *Each left autonomous braided tensor category is autonomous.*

Proof. Example 2.5 shows that a braiding on $\mathscr{V}$ enriches the identity functor to a tensor functor $\mathscr{V} \to \mathscr{V}^{\text{rev}}$. Hence, an adjunction $(\eta, \varepsilon): A \dashv B$ in $\mathscr{V}$ yields an adjunction $A \dashv B$ in $\mathscr{V}^{\text{rev}}$; but this is just the adjunction $(\eta', \varepsilon'): B \dashv A$ in $\mathscr{V}$, where

$$\varepsilon' = (B \otimes A \xrightarrow{c_{B,A}} A \otimes B \xrightarrow{\varepsilon} I)$$

and

$$\eta' = (I \xrightarrow{\eta} B \otimes A \xrightarrow{c_{A,B}^{-1}} A \otimes B). \qquad \text{Q.E.D.}$$

Autonomous symmetric tensor categories were called "compact closed categories" and studied in depth in [KL]. Autonomous braided tensor categories are studied in [FY1, FY2].

Before looking at Yang–Baxter operators in the context of duals, we need to recall the idea of "mates under adjunction" [KS]. Suppose that we have two adjunctions

$$(\eta, \varepsilon): A^* \dashv A, \qquad (\zeta, \kappa): B^* \dashv B$$

in an arbitrary tensor category $\mathscr{V}$. The *mate* of a map $f: X \otimes A \to B \otimes Y$ under these adjunctions is the map

$$f^{(a)}: B^* \otimes X \to Y \otimes A^*$$

obtained as the composite

$$B^* \otimes X \xrightarrow{B^* \otimes X \otimes \eta} B^* \otimes X \otimes A \otimes A^* \xrightarrow{B^* \otimes f \otimes A^*} B^* \otimes B \otimes Y \otimes A^*$$
$$\xrightarrow{\kappa \otimes Y \otimes A^*} Y \otimes A^*.$$

The function

$$(\ )^{(a)}: \mathscr{V}(X \otimes A, B \otimes Y) \to \mathscr{V}(B^* \otimes X, Y \otimes A^*)$$

is the natural bijection obtained by composing the hom bijections for the two adjunctions of functors

$$- \otimes A \dashv - \otimes A^*, \qquad B^* \otimes - \dashv B \otimes -.$$

It is easy to see that $f^{(a}$ can be characterized as the only map $B^* \otimes X \to Y \otimes A^*$ rendering commutative either of the following two squares:

$$\begin{array}{ccc} X & \xrightarrow{\zeta \otimes 1} & B \otimes B^* \otimes A \\ {\scriptstyle 1 \otimes \eta} \downarrow & & \downarrow {\scriptstyle 1 \otimes f^{(a}} \\ X \otimes A \otimes A^* & \xrightarrow[f \otimes 1]{} & B \otimes Y \otimes A^* \end{array} \qquad \begin{array}{ccc} B^* \otimes X \otimes A & \xrightarrow{f^{(a} \otimes 1} & Y \otimes A^* \otimes A \\ {\scriptstyle 1 \otimes f} \downarrow & & \downarrow {\scriptstyle 1 \otimes \varepsilon} \\ B^* \otimes B \otimes Y & \xrightarrow[\kappa \otimes 1]{} & Y \end{array}$$

We have the special cases $f^{(a} = \varepsilon^*(f)$ when $B = B^* = I$, and $f^{(a} = f^*$ when $X = Y = I$. The following easyexercise is quite useful in dealing with mates.

PROPOSITION 7.3. *Suppose* $A^* \dashv A$, $B^* \dashv B$, $C^* \dashv C$ *and*

$$f: X \otimes A \to B \otimes Y, \qquad g: X' \otimes B \to C \otimes Y'.$$

Then the mate of the composite

$$X' \otimes X \otimes A \xrightarrow{X' \otimes f} X' \otimes B \otimes Y \xrightarrow{g \otimes Y} C \otimes Y' \otimes Y$$

is the composite

$$C^* \otimes X' \otimes X \xrightarrow{g^{(a} \otimes X} Y' \otimes B^* \otimes X \xrightarrow{Y' \otimes f^{(a}} Y' \otimes Y \otimes A^*.$$

A full proof of the following result appears in [JS4, Sect. 10, Proposition 8].

PROPOSITION 7.4. *In a braided tensor category, if* $B^* \dashv B$ *then the mate of* $c_{A,B}: A \otimes B \to B \otimes A$ *is* $(c_{A,B^*})^{-1}: B^* \otimes A \to A \otimes B^*$.

Remark 7.1. The calculation proving Proposition 7.4 also shows that the condition of invertibility, for a braiding in an autonomous tensor category, is redundant. This was observed by Yetter [Y].

DEFINITION 7.3. *Let* $T: \mathscr{A} \to \mathscr{V}$ be a functor from a category $\mathscr{A}$ to a tensor category $\mathscr{V}$. A Yang–Baxter operator y on T (Definition 2.4) is called *dualizable* when, for each $A \in \mathscr{A}$, the object TA has a left dual, and, for all $A, B \in \mathscr{A}$, the mates

$$y_{A,B}^{(a}, (y_{B,A}^{-1})^{(a} : (TB)^* \otimes TA \to TA \otimes (TB)^*$$

of the two maps $y_{A,B}, y_{B,A}^{-1}: TA \otimes TB \to TB \otimes TA$ are invertible.

Any braiding on an autonomous tensor category is a dualizable Yang–Baxter operator (Proposition 7.4).

A dualizable Yang–Baxter operator y on a functor $T: \mathscr{A} \to \mathscr{V}$ can be extended by duality to a Yang–Baxter operator y' on a functor $T': \mathscr{A}' \to \mathscr{V}$, where $\mathscr{A}'$ is the disjoint union $\mathscr{A} + \mathscr{A}^{\mathrm{op}}$ of the category $\mathscr{A}$ and its opposite $\mathscr{A}^{\mathrm{op}}$. To avoid ambiguities, we write A° and f° for the object and arrow in $\mathscr{A}^{\mathrm{op}}$ corresponding to A and f in $\mathscr{A}$. The extensions of T and y are given by the following equations:

$$T'(A) = T(A), \qquad T'(A^\circ) = T(A)^*,$$
$$T'(f) = T(f), \qquad T'(f^\circ) = T(f)^*,$$
$$y'_{A,B} = y_{A,B}, \qquad y'_{A,B^\circ} = (y^{\mathrm{op}}_{A,B})^{-1},$$
$$y'_{A^\circ,B} = (y^{-1}_{A,B})^{\mathrm{op}}, \qquad y'_{A^\circ,B^\circ} = (y_{A,B})^*.$$

The next result is applied in [JS4, Sect. 10].

Proposition 7.5. *The extension y' of a dualizable Yang–Baxter operator y is a Yang–Baxter operator.*

Proof. As usual, we write as if $\mathscr{V}$ were strict. For all $X, Y, Z \in \mathscr{A}'$, we need to prove commutativity of the following hexagon:

$$\begin{array}{ccccc} & TX\otimes TZ\otimes TY & \xrightarrow{y'\otimes 1} & TZ\otimes TX\otimes TY & \\ {}^{1\otimes y'}\nearrow & & & & \searrow{}^{1\otimes y'} \\ TX\otimes TY\otimes TZ & & & & TZ\otimes TY\otimes TX \\ {}_{y'\otimes 1}\searrow & & & & \nearrow{}_{y'\otimes 1} \\ & TY\otimes TX\otimes TZ & \xrightarrow[1\otimes y']{} & TY\otimes TZ\otimes TX & \end{array}$$

The eight cases arising from whether X, Y, Z are in $\mathscr{A}$ or $\mathscr{A}^{\mathrm{op}}$ can be reduced, by duality of various kinds, to the following three cases, where A, B, C are all in $\mathscr{A}$.

(i) $X = A$, $Y = B$, $Z = C$. Here the hexagon is already an instance of the Yang–Baxter condition for y itself.

(ii) $X = A^\circ$, $Y = B$, $Z = C$. Here the hexagon becomes the following:

$$\begin{array}{ccccc} & TA^*\otimes TC\otimes TB & \xrightarrow{(y^{-1}_{AC})^{\mathrm{op}}\otimes 1} & TC\otimes TA^*\otimes TB & \\ {}^{1\otimes y_{BC}}\nearrow & & & & \searrow{}^{1\otimes (y^{-1}_{AB})^{\mathrm{op}}} \\ TA^*\otimes TB\otimes TC & & & & TC\otimes TB\otimes TA^* \\ {}_{(y^{-1}_{AB})^{\mathrm{op}}\otimes 1}\searrow & & & & \nearrow{}_{y_{BC}\otimes 1} \\ & TB\otimes TA^*\otimes TC & \xrightarrow[1\otimes (y^{-1}_{AC})^{\mathrm{op}}]{} & TB\otimes TC\otimes TA^* & \end{array}$$

By mate naturality and Proposition 7.3, the upper leg of this hexagon is the mate of the composite

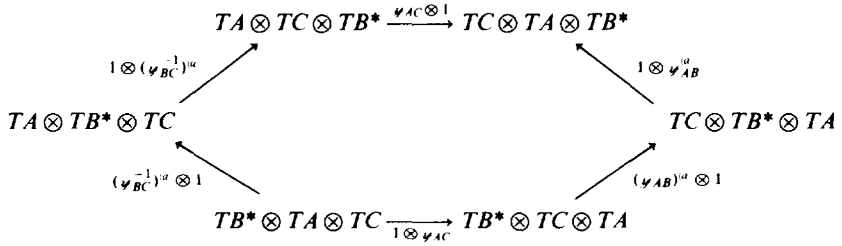
$$\bullet \xrightarrow{y_{BC} \otimes 1} \bullet \xrightarrow{1 \otimes y_{AB}^{-1}} \bullet \xrightarrow{y_{AC}^{-1} \otimes 1} \bullet$$

while the lower leg is the mate of the composite

$$\bullet \xrightarrow{1 \otimes y_{AC}^{-1}} \bullet \xrightarrow{y_{AB}^{-1} \otimes 1} \bullet \xrightarrow{1 \otimes y_{BC}} \bullet .$$

These two composites are equal by the Yang–Baxter condition for y itself.

(iii) $X = A, = B^{\circ}$, $Z = C$. After replacement of two of the arrows by their inverses, the hexagon here becomes the following:



The leftish leg of the last hexagon is the mate of the composite

$$\bullet \xrightarrow{1 \otimes y_{BC}^{-1}} \bullet \xrightarrow{y_{AB} \otimes 1} \bullet \xrightarrow{1 \otimes y_{AC}} \bullet .$$

The rightish leg is the mate of the composite

$$\bullet \xrightarrow{y_{AC} \otimes 1} \bullet \xrightarrow{1 \otimes y_{AB}} \bullet \xrightarrow{y_{BC}^{-1} \otimes 1} \bullet .$$

Again, these two composites are equal by the Yang–Baxter condition for y itself. Q.E.D.

DEFINITION 7.3. A tensor category is called *tortile* when it is autonomous and balanced, and, for all objects A,

$$\theta_{A^*} = \theta_A^* : A^* \to A^*.$$

Coherence for tortile tensor categories is dealt with by Shum [SMC]. The free structures are categories of tangles on ribbons. Compare [RT].

DEFINITION 7.5. A Yang–Baxter operator y on $T: \mathscr{A} \to \mathscr{V}$ is called *tortile* when it is dualizable and balanced and the following diagram commutes for all objects $A \in \mathscr{A}$:

$$\begin{array}{ccccc} TA & \xrightarrow{\theta_A} & TA & \xrightarrow{\theta_A} & TA \\ \downarrow{\scriptstyle 1 \otimes \eta_{TA}} & & & & \uparrow{\scriptstyle \varepsilon_{TA} \otimes 1} \\ TA \otimes TA \otimes TA^* & \xrightarrow[\psi_{A,A} \otimes 1]{} & TA \otimes TA \otimes TA^* & \xrightarrow[1 \otimes \psi^{-1}_{A,A}]{} & TA \otimes TA^* \otimes TA \end{array}$$

Proofs of the following result can be found in [JS3, Proposition 1, JS4, Sect. 10, Proposition 7].

PROPOSITION 7.6. *For a tortile tensor category* $\mathscr{V}$, *the pair* (c, θ) *is a tortile Yang–Baxter operator on the identity functor of* $\mathscr{V}$.

For any category $\mathscr{A}$, let $i: \mathscr{A} \to \mathbf{Ft}\mathscr{A}$ denote the universal functor out of $\mathscr{A}$ into a tensor category and equipped with a tortile Yang–Baxter operator (x, ψ). Write $\mathscr{C}_{\text{iso}}$ for the subcategory of the category $\mathscr{C}$ with the same objects but only the invertible arrows.

THEOREM 7.7. *The tortile Yang–Baxter operator* (x, ψ) *on* $i: \mathscr{A} \to \mathbf{Ft}\mathscr{A}$ *comes from a tortile tensor category structure on* $\mathbf{Ft}\mathscr{A}$, *and restriction along* i *determines an equivalence of groupoids*

$$\mathbf{B}\mathscr{T}\!en(\mathbf{Ft}\mathscr{A}, \mathscr{V}) \simeq [\mathscr{A}, \mathscr{V}]_{\text{iso}}$$

for all tortile tensor categories $\mathscr{V}$.

Proof. The case $\mathscr{A} = \mathbb{1}$ is proved in [JS3]. The general case follows by techniques used in the proof of Proposition 2.2. Q.E.D.

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