The Geometry of Tensor Calculus, I

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INTRODUCTION

The goal of this first paper is to formalise the use of certain diagrams for a wide variety of situations in pure and applied mathematics. The main examples are the Feynman diagrams describing patterns of particle interactions in space–time. Other examples are given by circuit diagrams, networks, Petri nets, flow charts, and planar diagrams of knots or links.

Penrose [30, 31] was the first to use the graphical notation for calculating with tensors. It is now currently used by theoretical physicists as a private device for quickly verifying complicated tensor formulas. A striking aspect of the notation is that it is pictorial rather than sequential or alphabetical. This made it difficult to print, which partly explains why no rigorous theory was developed. We believe that a notation which is useful in private must be given a public value and that it should be provided with a firm theoretical foundation. Furthermore, printing techniques have improved drastically in recent years.

The non-linearity of the notation makes it better adapted to express parallel processing than the usual sequential notation. It is already a part of computer science at the hardware level (circuit diagrams) and the software level (flow charts). There is also some connection with the concept of pile [41] which gives a pictorial description of the elements of commutation monoids [5]. Other relevant areas where similar notations are developing are in proof theory [13] and rewriting systems [14].

The abstract concept of tensor category represents a reasonably general setting in which the notation is meaningful. It is not entirely new in

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category theory where it should not be confused with the usual diagrams. It was partly used by Eilenberg and Kelly [9], and further by Kelly [22] (implicitly in [24]), where it coincides with the diagrams used by Brauer [4] in his description of the Brauer algebra. Dubuc, in preparing his thesis [8], developed and "elevator calculus" for calculating in tensor categories, but this was not included in the written version. More recently, in a related context [25, 37, 38], Power [32] has used planar diagrams which are Poincaré dual to ours, while Aitchison [1] has indeed used string diagrams.

The recent developments in knot theory [15, 12, 39, 40] and quantum groups, [7, 29] have provided strong motivation for our work. Freyd and Yetter [42, 11, 43] have succeeded in giving a categorical description of knots and graphs embedded in 3-space. Their theory is a categorical formulation of the theory of Reidemeister moves [33]. Reshetikhin and Turaev [34] use representations of quantum groups to derive invariants of ribbon graphs. In a sense, our goal is opposite since we mainly want to show that diagrams can be used with profit in a wide variety of situations: in fact, in almost any context where a category equipped with a tensor product reveals itself.

The paper is made up of four chapters. In the first, we recall the basic algebraic structure, namely, *tensor category* (also called "monoidal category") which is simply a category with an associative (up to coherent isomorphism) *tensor product* operation. We introduce the concept of graph appropriate for both this paper and the next. We define the concept of *valuation* which labels the nodes of a graph with arrows from a tensor category $\mathscr{V}$ and labels the edges with objects of $\mathscr{V}$. A plane graph Γ equipped with a valuation v is called a *plane diagram* in $\mathscr{V}$. We then proceed to define an arrow $v(\Gamma)$ in $\mathscr{V}$, called the *value* of the diagram. The main result of this chapter is that the value is invariant under continuous deformation of plane diagrams. In Section 4, which can be skipped at a first reading, we check that free tensor categories can be described in terms of isotopy classes of plane diagrams.

In the second chapter we consider the case of *symmetric* tensor categories for which the tensor product is equipped with an extra structure of symmetry. In this situation we show that the value $v(\Gamma)$ of a diagram can be defined even when Γ is abstract (no planarity is needed). In the second section of this chapter we construct free symmetric tensor categories using isomorphism classes of abstract diagrams.

In Chapter 3 we consider the case of *braided* tensor categories [18]. In this situation the diagram Γ is embedded in 3-space. We prove that the value of a diagram is invariant under deformation. We then proceed to describe free braided tensor categories using isotopy classes of embedded diagrams in 3-space.

In Chapter 4 we introduce the concept of *balanced* tensor category. In this case the embedded graphs Γ are *framed*, or made of ribbons. Again we prove the invariance of the value of a ribbon diagram under continuous deformation and construct the free balanced tensor category from isotopy classes of ribbon diagrams.

Our second paper will deal with tensor categories in which the object have *duals*. In this situation more general diagrams can be used in which backtracking and looping can occur (see [19] for an example); the present paper deals with *progressive* diagrams only.

All the results proved here have a rather strong intuitive content which makes them look obvious. However, not unlike the Jordan Curve Theorem, many sketchy proofs turned out incorrect, and many easily formulated concepts turned out to be wrong. We had no choice but to perform a sharp analysis of all aspects, thereby increasing the length of the paper. We challenge the reader to devise shorter proofs.

CHAPTER 1. TENSOR CATEGORIES AND THEIR DIAGRAMS

1. *Tensor Categories*

Recall the concept of *tensor category* $\mathscr{V} = (\mathscr{V}, \otimes, a, l, r)$, also called "monoidal category" [26, 21, 10, 27, 34, 6]. This consists of a category $\mathscr{V}$, a functor $\otimes : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$ (called the *tensor product*), an object $I \in \mathscr{V}$ (called the *unit object*), and natural isomorphisms

$$a = a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$

$$l = l_A : I \otimes A \to A, \qquad r = r_A : A \otimes I \to A$$

(called the *constraints of associativity, left unit, right unit*, respectively) such that the following diagrams commute:

(a)   the pentagon for associativity
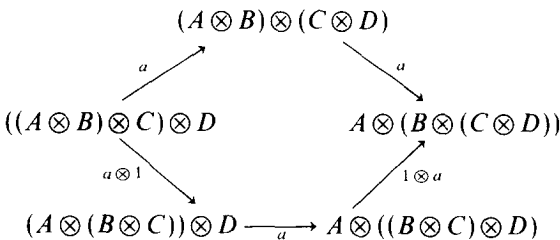
(b)   the triangle for the unit

$$(A \otimes I) \otimes B \xrightarrow{\ a\ } A \otimes (I \otimes B)$$

$$r \otimes 1 \searrow \qquad \swarrow 1 \otimes l$$

$$A \otimes B$$

Commutativity of the two triangles

$$(I \otimes A) \otimes B \xrightarrow{\ a\ } I \otimes (A \otimes B) \qquad (A \otimes B) \otimes I \xrightarrow{\ a\ } A \otimes (B \otimes I)$$

$$l \otimes 1 \searrow \qquad \swarrow l \qquad\qquad r \searrow \qquad \swarrow 1 \otimes r$$

$$A \otimes B \qquad\qquad\qquad A \otimes B$$

and the equality $r = l$: $I \otimes I \to I$ are (not so obvious) consequences of these axioms [20]. The coherence theorem of MacLane [26] states that *all* diagrams built up from a, l, r by tensoring, substituting, and composing, commute. It follows that all the objects obtained by computing the tensor product of a sequence $A_1 \otimes \cdots \otimes A_n$, by bracketing it differently, and by cancelling units are coherently identified with each other. More precisely, the different ways of computing the tensor product $A_1 \otimes \cdots \otimes A_n$ produce a *clique*; that is a non-empty family $(C_i | i \in I)$ of objects together with a family $(u_{ji}: C_i \to C_j | (i, j) \in I \times I)$ of maps such that $u_{ii} = 1$ and $u_{ki} = u_{kj} u_{ji}$ (so that $u_{ij} = u_{ji}^{-1}$). The cliques in $\mathscr{V}$ are the objects of a category clq $\mathscr{V}$ in which a map $f: (C_i | i \in I) \to (D_k | k \in K)$ is a family of maps $f_{ki}: C_i \to D_k$ such that

$$
\begin{array}{ccc}
C_i & \xrightarrow{\ f_{ki}\ } & D_k \\
{\scriptstyle u_{ji}}\downarrow & & \downarrow{\scriptstyle u_{mk}} \\
C_j & \xrightarrow{\ f_{mj}\ } & D_m
\end{array}
$$

commutes for every $(i, j) \in I^2$, $(k, m) \in K^2$. It is sometimes convenient to think of the n-fold tensor product as a functor

$$\mathscr{V}^n \to \text{clq } \mathscr{V}.$$

The functor $\mathscr{V} \to \text{clq } \mathscr{V}$, which associates to each $A \in \mathscr{V}$ the singleton clique $(A) \in \text{clq } \mathscr{V}$, is full and faithful. Since any clique is isomorphic to the singleton clique of any one of its members, this functor is an equivalence. This equivalence between $\mathscr{V}$ and clq $\mathscr{V}$ shows that the ambiguity which exists in computing the n-fold tensor product is not a real one.

Furthermore, any tensor category is equivalent to a *strict* one st($\mathscr{V}$)

[28]; that is, one in which each constraint is an identity arrow. The object of st($\mathcal{V}$) are words $w = A_1 A_2 \cdots A_m$ in objects of $\mathcal{V}$. An arrow $f: w \to w'$ is an arrow $f: [w] \to [w']$ in $\mathcal{V}$, where

$$[\varnothing] = I, \qquad [A] = A, \qquad \text{and} \qquad [A_1 \cdots A_{i+1}] = [A_1 \cdots A_i] \otimes A_{i+1}.$$

The tensor $\bar{\otimes}$ for st($\mathcal{V}$) is given by $v \bar{\otimes} w = vw$ and

$$
\begin{array}{ccc}
[v] \otimes [w] & \xrightarrow{f \otimes g} & [v'] \otimes [w'] \\
\Big\| & & \Big\| \\
[vw] & \xrightarrow[\overline{f \otimes g}]{} & [v'w'].
\end{array}
$$

In principle, most results obtained with the hypothesis that a tensor category is strict can be reformulated and proved without this condition. Since, in this paper, we wish to focus on aspects other than associativity of tensor product, we avoid putting brackets on n-fold tensor products when clarity is gained and rigor preserved.

EXAMPLE 1. Let k be a commutative ring. A *bialgebra* A over k is an associative algebra with unit equipped with a pair of algebra homomorphisms

$$\Delta: A \to A \otimes A, \qquad \varepsilon: A \to k,$$

(called the *diagonal* and the *co-unit*) such that the diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{\ \Delta\ } & A \otimes A \\
\Delta\Big\downarrow & & \Big\downarrow{\scriptstyle \Delta \otimes 1} \\
A \otimes A & \xrightarrow[1 \otimes \Delta]{} & A \otimes A \otimes A
\end{array}
\qquad
\begin{array}{ccc}
& A & \\
{\scriptstyle 1}\swarrow & \Big\downarrow{\scriptstyle \Delta} & \searrow{\scriptstyle 1} \\
A \xleftarrow[\varepsilon \otimes 1]{} & A \otimes A & \xrightarrow[1 \otimes \varepsilon]{} A
\end{array}
$$

commute. Let $\mathrm{Mod}(A)$ denote the category of left A-modules. For any $V, W \in \mathrm{Mod}(A)$, the tensor product $V \otimes W$ is an $(A \otimes A)$-module which becomes an A-module if we restrict the action of $A \otimes A$ along the diagonal $\Delta: A \to A \otimes A$. This defines a tensor product on $\mathrm{Mod}(A)$ yielding a tensor category for which the unit object is k equipped with the A-module structure given by $A \to k$.

In a tensor category there are two operations for constructing new arrows from old ones: composition $f \circ g$ and tensor product $f \otimes g$. Using ordinary algebraic notation, we are immediately faced with expressions like

$$(B \otimes c \otimes d) \circ (B \otimes B \otimes b \otimes C) \circ (a \otimes B \otimes C) = w_1$$
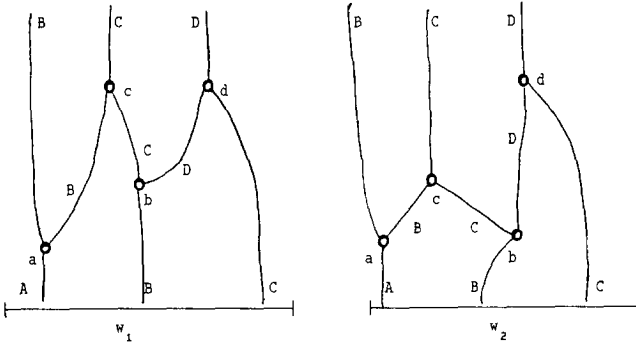
FIGURE 1.1

and

$$(B \otimes C \otimes d) \circ (B \otimes c \otimes D \otimes C) \circ (a \otimes b \otimes C) = w_2.$$

In this form it is sometimes unclear when two words like w_1, w_2 are equivalent; that is, when they can be proved to be equal from the tensor category axioms alone. The graphical notation we shall develop in this chapter will make it easier to detect such equalities, and hence will provide a convenient technique for computation in a tensor category.

To motivate the precise topological details of the graphical notation, we remark that the two words, w_1, w_2 given above will be respectively represented by the two diagrams in Fig. 1.1 In these diagrams the inner nodes represent the maps

$$a\colon A \to B \otimes B, \qquad b\colon B \to C \otimes D, \qquad c\colon B \otimes C \to C, \qquad d\colon D \otimes C \to D,$$

whereas the edges represent objects. It is obvious that the diagrams are deformations of one another; this will, after results below, enable us to deduce the equality $w_1 = w_2$.
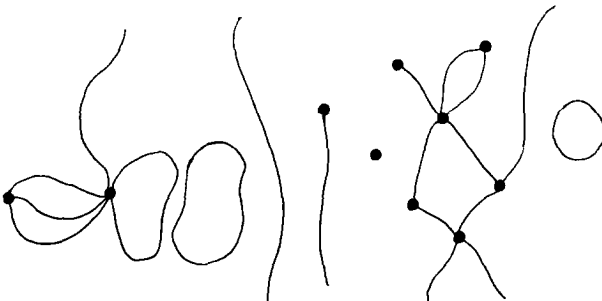

FIG. 1.2. A generalized graph.

2. Graphs

A *generalized* (*topological*) *graph* $G = (G, G_0)$ consists of a Hausdorff space G and a discrete closed subset $G_0 \subset G$ such that the complement $G - G_0$ is a 1-dimensional manifold without boundary. That is, $G - G_0$ is the topological sum of open intervals and circles. (See Fig. 1.2.)

An element of G_0 is called a *vertex* or *node*. A connected component of $G - G_0$ homeomorphic to an open interval is called an *open edge*; the other components are called circles. Each open edge e can be compactified to a *closed* edge $\hat{e}$ by adjoining two end-points. An edge e is called *pinned* when the inclusion $e \to G$ can be extended to a continuous map $\hat{e} \to G$ (called the *structure map*). When the inclusion $e \to G$ extends only to $\hat{e}$ minus one end-point, we call e *half-loose*. An edge is *loose* when it is neither pinned nor half-loose. (See Fig. 1.3.)

A *graph* is a generalised graph in which all the edges are pinned. It is called an *ordinary graph* when it has no circles.

A generalised graph G can be completed to a graph $G \cup \{\infty\}$ by attaching at ∞ all the loose ends. Conversely, given a pointed graph, the complement of the basepoint is a generalised graph whose loose edges are the loops at the basepoint.

In what follows we shall consider only *finite* generalised graphs: that is, G_0 and the set $\pi_0(G - G_0)$ of connected components of $G - G_0$ are finite. Obviously a finite generalised graph is a graph if and only if it is compact.

Write $\hat{G}$ for the compactification of the generalised graph adjoining one end-point to each half-pinned edge and two end-points to each loose edge. These extra points, along with G_0, are the nodes of a graph $\hat{G} = (\hat{G}, \hat{G}_0)$; the elements of $\hat{G}_0 - G_0$ are called the *outer nodes* of G. The graph $\hat{G}$, together with the set of outer nodes of G, is a "graph with boundary."

A *graph with boundary* $\Gamma = (\Gamma, \partial \Gamma)$ is a compact graph Γ together with a distinguished set ∂L of nodes of Γ such that each $x \in \partial \Gamma$ is of degree one. (The *degree* of a node is the number of connected components of $V - \{x\}$ where V is a sufficiently small connected neighbourhood of x.) Taking out
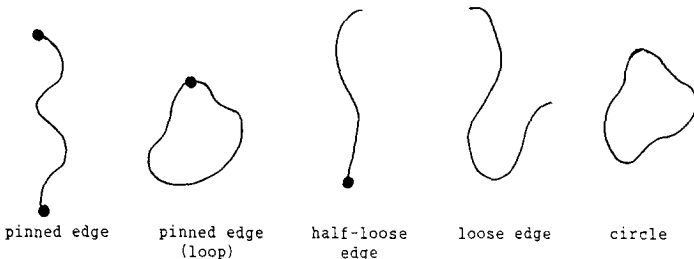

| pinned edge | pinned edge (loop) | half-loose edge | loose edge | circle |

FIGURE 1.3

the boundary $\partial \Gamma$ of Γ produces a generalised graph $G = \Gamma - \partial \Gamma$ whose compactification $\hat{G}$ is Γ and the set of outer nodes of G is $\partial \Gamma$. The elements of $\partial \Gamma$ are sometimes called the *outer nodes of* $(\Gamma, \partial \Gamma)$, and the nodes of $\Gamma - \partial \Gamma$ are called the *inner nodes of* $(\Gamma, \partial \Gamma)$.

An *isomorphism* $f: (\Gamma, \partial \Gamma) \to (\Omega, \partial \Omega)$ *of graphs with boundary* is a homeomorphism $f: \Gamma \to \Omega$ inducing bijections on the inner nodes and on the outer nodes.

An *oriented edge* of Γ is an edge e equipped with an orientation; or equivalently, with a linear order on $\partial \hat{e}$. The *source* $e(0)$ of an oriented edge e is the image of the first element of $\partial \hat{e}$ under the structure map $\hat{e} \to \Gamma$; the *target* $e(1)$ is the image of the last element; the *opposite edge* e^0 is obtained by taking the opposite orientation of e.

An *oriented graph* is a graph together with a choice of orientation for each of its edges and circles. For an oriented graph Γ, the *input* $\text{in}(x)$ of an inner node $x \in \Gamma_0$ is defined to be the set of oriented edges with target x; the *output* $\text{out}(x)$ of x is the set of those with source x.

A *polarised graph* is an oriented graph together with a choice of linear order on each $\text{in}(x)$ and $\text{out}(x)$.

A *progressive graph* is an oriented ordinary graph with no circuits. The *domain* dom Γ of a progressive graph Γ consists of the edges which have outer nodes as sources; the *codomain* cod Γ consists of the edges which have outer nodes as targets. In many situations we shall identify dom Γ and cod Γ with their corresponding sets of outer nodes.

A *parametrization* of an oriented edge e is the choice of an orientation-preserving homeomorphism $[0, 1] \overset{\sim}{\to} \hat{e}$. Composing it with the structure map $\hat{e} \to \Gamma$, we obtain a function $\gamma: [0, 1] \to \Gamma$. We call γ a *parametrised edge* with source $\gamma(0)$ and target $\gamma(1)$. The opposite parametrisation $\gamma^o(t) = \gamma(1 - t)$ is a parametrisation of the opposite edge. A *parametrisation* of a circle is of course a homeomorphism with the unit circle $\mathbb{S}^1 \subset \mathbb{C}$.

A *parametrised graph* is a graph together with a choice of parametrisation for each oriented edge and each oriented circle such that opposite edges and circles have the opposite parametrisations.

Except for Chapters 1 and 2, we shall assume that our graphs are smooth. All the results could be obtained with weaker assumptions, but would be the same in substance.

A graph is called *smooth* when each closed edge and each circle is equipped with a smooth (meaning C^∞) structure. Of course, parametrisations of smooth graphs will always be taken as smooth.

3. *Progressive Plane Diagrams*

The purpose of this section is to describe the diagrams appropriate for calculations in an arbitrary tensor category, to ascribe values to these diagrams, and to prove the value invariant under deformation.

DEFINITION 1.1. Let $a < b$ be real numbers. A *progressive plane graph* (*between the levels a and b*) is a graph Γ (with boundary) embedded in $\mathbb{R} \times [a, b]$ such that

(i) $\partial \Gamma = \Gamma \cap (\mathbb{R} \times \{a, b\})$, and

(ii) the second projection $\mathrm{pr}_2 : \mathbb{R} \times [a, b] \to [a, b]$ is injective on each connected component of $\Gamma - \Gamma_0$.

Each progressive plane graph Γ is both progressive and polarised in the sence of Section 2. Each edge e is given the orientation with $\mathrm{pr}_2\, e(0) < \mathrm{pr}_2\, e(1)$. Condition (ii) certainly excludes circles and circuits. Also, $\mathrm{in}(x)$ and $\mathrm{out}(x)$ can be linearly ordered as follows. Choose $u \in [a, b]$ smaller than but close enough to $\mathrm{pr}_2(x)$. Then each edge $e \in \mathrm{in}(x)$ intersects the line $\mathbb{R} \times \{u\}$ in one point which is different for different edges. This defines a bijection between $\mathrm{in}(x)$ and a subset of $\mathbb{R} \times \{u\}$ ($\cong \mathbb{R}$), and so induces a linear order on $\mathrm{in}(x)$. The order on $\mathrm{out}(x)$ is defined similarly by intersecting with $\mathbb{R} \times \{u\}$ for u larger than but close to $\mathrm{pr}_2(x)$.

Notice that dom Γ and cod Γ are naturally linearly ordered as subsets of $\mathbb{R} \times \{a\}$ and $\mathbb{R} \times \{b\}$, respectively.

A number $u \in [a, b]$ is called a *regular level* for Γ when the line $\mathbb{R} \times \{u\}$ contains no inner nodes. If $c < d$ are regular levels of Γ, we write $\Gamma[c, d]$ for the graph $\Gamma \cap (\mathbb{R} \times [c, d])$ whose set of inner nodes is $(\Gamma_0 - \partial \Gamma) \cap (\mathbb{R} \times [c, d])$ and whose set of outer nodes is $\Gamma \cap (\mathbb{R} \times \{c, d\})$. The graph $\Gamma[c, d]$ is a progressive plane graph between the levels c and d; it is called a *layer* of Γ.

Suppose Γ is the disjoint union of two subgraphs Γ^1 and Γ^2. We shall say that the pair (Γ^1, Γ^2) is a *tensor decomposition* of Γ, and write $\Gamma = \Gamma^1 \otimes \Gamma^2$, when there exists a number ξ such that

$$\Gamma^1 \subseteq (-\infty, \xi) \times [a, b] \qquad \text{and} \qquad \Gamma^2 \subseteq (\xi, \infty) \times [a, b].$$

This notion extends in the obvious way to n-fold tensor decompositions

$$\Gamma = \Gamma^1 \otimes \cdots \otimes \Gamma^n.$$

DEFINITION 1.2. Let $(\Gamma, 2\Gamma)$ denote a graph with boundary. A *deformation of progressive plane graphs* (between levels a and b) is a continuous function

$$h : \Gamma \times [0, 1] \to \mathbb{R} \times [a, b]$$

such that, for all $t \in [0, 1]$, the function

$$h(-, t) : \Gamma \to \mathbb{R} \times [a, b]$$

is an embedding whose image is a progressive plane graph $(\Gamma(t), 2\Gamma(t))$ between the levels a and b.

In view of the isomorphism $\Gamma \cong \Gamma(t)$, it is possible to define many struc-tures on Γ by transporting the structures defined on the $\Gamma(t)$ for some $t \in [0, 1]$. Some of these structures on Γ are independent of the choice of $t \in [0, 1]$. For example, our definition of deformation forces the inner and outer nodes to keep their natures independent of t.

For any inner node $x \in \Gamma_0$ we can define $\mathrm{in}(x)$ and $\mathrm{out}(x)$ as the ordered sets of edges corresponding to $\mathrm{in}(h(x, t))$ and $\mathrm{out}(h(x, t))$ via the isomor-phism $\Gamma \cong \Gamma(t)$. It is easy to see that these ordered sets do not depend on the choice of $t \in [0, 1]$.

DEFINITION 1.3. A *valuation* $v: \Gamma \to \mathscr{V}$ of a progressive plane graph Γ in a tensor category $\mathscr{V}$ is a pair of functions

$$v_0: \Gamma_1 \to \mathrm{obj}\ \mathscr{V}, \qquad v_1: \Gamma_0 - \partial\Gamma \to \mathrm{arr}\ \mathscr{V},$$

where Γ_1 is the set of edges of Γ and $\Gamma_0 - \partial\Gamma$ is the set of inner nodes, such that, for all inner nodes x of Γ,

$$v_1(x): v_0(\gamma_1) \otimes \cdots \otimes v_0(\gamma_m) \to v_0(\delta_1) \otimes \cdots \otimes v_0(\delta_n)$$

where $\gamma_1 < \cdots < \gamma_m$, $\delta_1 < \cdots < \delta_n$ are the ordered lists of elements of $\mathrm{in}(x)$, $\mathrm{out}(x)$, respectively. The pair (Γ, v) is called a (*progressive plane*) *diagram in* $\mathscr{V}$, and is denoted merely by Γ when the context is clear.

The *domain* and *codomain* of a diagram (Γ, v) are the families of objects

$$\mathrm{dom}(\Gamma, v) = (v_0(z) \mid z \in \mathrm{dom}\ \Gamma), \qquad \mathrm{cod}(\Gamma, v) = (v_0(z) \mid z \in \mathrm{cod}\ \Gamma)$$

indexed by the linearly ordered sets $\mathrm{dom}\ \Gamma$, $\mathrm{cod}\ \Gamma$.

If $c < d$ are regular levels for a diagram $\Gamma = (\Gamma, v)$ the valuation v "restricts" in an obvious way to a valuation on the layer $\Gamma[c, d]$ and we also denote this by v. Similarly, if $\Gamma = \Gamma^1 \otimes \Gamma^2$, the valuation v restricts to valuations on Γ^1 and Γ^2 again denoted by v.

If $h: \Gamma \times [0, 1] \to \mathbb{R} \times [a, b]$ is a deformation of progressive plane graphs then a valuation defined on one $\Gamma(t_0)$ for some $t_0 \in [0, 1]$ can be trans-ported along the isomorphisms $\Gamma(t_0) \cong \Gamma \cong \Gamma(t)$ to a valuation on $\Gamma(t)$ for all $t \in [0, 1]$. In this way h becomes a *deformation of diagrams*. Our inten-tion now is to assign a *value* $v(\Gamma) \in \mathrm{arr}\ \mathscr{V}$ to each progressive plane diagram (Γ, v) in $\mathscr{V}$ and prove it invariant under deformation of diagrams. To do this we must subdivide the diagram into simpler parts and "integrate" the result by composing and tensoring.

A diagram Γ is called *prime* when it is connected and has precisely one inner node x. In this case, we define the value of Γ by the equality

$$v(\Gamma) = v_1(x).$$

A diagram Γ is called *invertible* when it has no inner nodes. In such a diagram we have bijections

$$\text{dom } \Gamma \cong \pi_0(\Gamma) \cong \text{cod } \Gamma$$

between the domain, connected components, and codomain of the graph Γ such that the composite is order-preserving. Thus we obtain a linear order $e_1 < e_2 < \cdots < e_n$ on the set $\pi_0(\Gamma)$. In this case, the value $v(\Gamma)$ of Γ is defined to be the identity arrow of $v_0(e_1) \otimes \cdots \otimes v_0(e_n)$.

A diagram Γ is called *elementary* when it has a tensor decomposition $\Gamma = \Gamma^1 \otimes \cdots \otimes \Gamma^n$ with each Γ^i $(1 \leqslant i \leqslant n)$ either prime or invertible. In this case, we define

$$v(\Gamma) = v(\Gamma') \otimes \cdots \otimes v(\Gamma'').$$

That this is independent of the choice of tensor decomposition follows from the facts that

— prime diagrams are tensor indecomposable, and

— if $\Omega^1 \otimes \Omega^2$ is invertible then $\pi_0(\Omega^1 \otimes \Omega^2)$ is the ordered sum of $\pi_0(\Omega^1)$ and $\pi_0(\Omega^2)$.

Remark. The reader may feel that a more restrictive notion of "elementary diagram" should have been used, namely, those diagrams which admit a tensor decomposition into primes and *connected* invertibles. It is true that every diagram decomposes into layers of this kind. However, the restricted notion is not inherited by taking further layers (see Fig. 1.4).
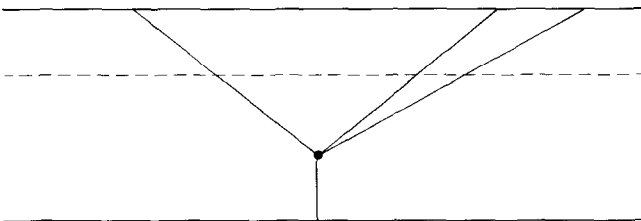

FIG. 1.4.   Top layer not elementary in restricted sense.

PROPOSITION 1.1. *If u is a regular level for an elementary diagram Γ between levels a and b then $\Gamma[a, u]$, $\Gamma[u, b]$ are elementary, and*

$$v(\Gamma) = v(\Gamma[u, b]) \circ v(\Gamma[a, u]).$$

Proof. We use induction on the number of factors needed to tensor decompose Γ into primes and invertibles. If $\Gamma = \Gamma^1 \otimes \Gamma^2$ is a proper decomposition, put $f_i = v(\Gamma^i[a, u])$, $g_i = v(\Gamma^i[u, b])$ for $i = 1, 2$, and we have

$$\begin{aligned}
v(\Gamma) &= v(\Gamma^1) \otimes v(\Gamma^2) && \text{by definition;} \\
&= (g_1 \circ f_1) \otimes (g_2 \circ f_2) && \text{by induction;} \\
&= (g_1 \otimes g_2) \circ (f_1 \otimes f_2) && \text{by functoriality of } \otimes, \\
&= v(\Gamma[u, b]) \circ v(\Gamma[a, u]) && \text{by definition.} \quad \blacksquare
\end{aligned}$$

For any progressive plane diagram Γ between levels a and b, we now define the value by

$$v(\Gamma) = v(\Gamma[u_{n-1}, u_n]) \circ \cdots \circ v(\Gamma[u_0, u_1]),$$

where $a = u_0 < u_1 < \cdots < u_n = b$ are regular levels for Γ such that each layer $\Gamma[u_{i-1}, u_i]$ is elementary for $1 \leqslant i \leqslant n$. The existence of such regular levels can be seen by choosing $u_1, ..., u_{n-1}$ to be numbers close enough to, and on both sides of, each critical ($=$ non-regular) level. The independence of the definition under different choices amounts to independence under a refinement, which follows from Proposition 1.1.

THEOREM 1.2. *If $h: \Gamma \times [0, 1] \to \mathbb{R} \times [a, b]$ is a deformation of progressive plane diagrams then*

$$v(\Gamma(0)) = v(\Gamma(1)).$$

Proof. Since $[0, 1]$ is connected, it suffices to show that $v(\Gamma(t))$ is a locally constant function of t. Take $0 \leqslant t_0 \leqslant 1$. We shall prove that $v(\Gamma(t)) = v(\Gamma(t_0))$ for t close enough to t_0. Choose regular levels $a = u_0 < u_1 < = \cdots < u_n = b$ for $\Gamma(t_0)$ such that each $\Gamma(t_0)[u_{i-1}, u_i]$ is elementary. Then, for t close enough to t_0, these levels are regular for $\Gamma(t)$, and each $\Gamma(t)[u_{i-1}, u_i]$ is elementary. It remains to be proven that, for t close enough to t_0, we have

$$v(\Gamma(t)[c, d]) = v(\Gamma(t_0)[c, d]),$$

where $c = u_{i-1}$, $d = u_i$. Choose $\xi_0 < \xi_1 < \cdots < \xi_r$ such that $\Gamma(t_0)[c, d] = \Gamma^1(t_0) \otimes \cdots \otimes \Gamma^r(t_0)$, where each $\Gamma^j(t_0) = \Gamma(t_0) \cap ((\xi_{j-1}, \xi_j) \times [c, d])$ is

either prime or invertible. For t close enough to t_0, we also have $\Gamma(t)[c, d] = \Gamma^1(t) \otimes \cdots \otimes \Gamma^r(t)$, where each $\Gamma^j(t) = \Gamma(t) \cap ((\xi_{j-1}, \xi_j) \times [c, d])$ is either prime or invertible depending on the nature of $\Gamma^j(t_0)$. It remains to be shown that $v(\Gamma^j(t) = v(\Gamma^j(t_0))$. If $\Gamma^j(t_0)$ is invertible then the deformation h gives an order isomorphism $\pi_0(\Gamma^j(t)) \cong \pi_0(\Gamma^j(t_0))$ compatible with v_0; so the result follows. If $\Gamma^j(t_0)$ is prime with inner node x_0 and $\Gamma^j(t)$ has inner node x, we have

$$v(\Gamma^j(t)) = v_1(x) = v_1(x_0) = v(\Gamma^j(t_0)). \quad \blacksquare$$

4. Free Tensor Categories

Let $\mathscr{V}$, $\mathscr{W}$ denote tensor categories. A *tensor functor* $[10, 35]$

$$F = (F, \phi_2, \phi_0) \colon \mathscr{V} \to \mathscr{W}$$

(also called "strong monoidal functor") consists of a functor $F \colon \mathscr{V} \to \mathscr{W}$, a natural isomorphism

$$\phi_2 = \phi_{2A, B} \colon FA \otimes FB \to F(A \otimes B),$$

and an isomorphism $\phi_0 \colon I \to FI$, such that the following diagrams commute:

$$
\begin{array}{ccc}
FA \otimes FB \otimes FC & \xrightarrow{\phi_2 \otimes 1} & F(A \otimes B) \otimes FC \\
\downarrow{\scriptstyle 1 \otimes \phi_2} & & \downarrow{\scriptstyle \phi_2} \\
FA \otimes F(B \otimes C) & \xrightarrow[\phi_2]{} & F(A \otimes B \otimes C)
\end{array}
$$

$$
\begin{array}{ccc}
 & FA & \\
{\scriptstyle F(\phi_0 \otimes 1)}\swarrow & & \searrow{\scriptstyle \phi_0 \otimes 1} \\
F(I \otimes A) \xrightarrow[\phi_2]{} & & FI \otimes FA
\end{array}
\qquad
\begin{array}{ccc}
 & FA & \\
{\scriptstyle F(1 \otimes \phi_0)}\swarrow & & \searrow{\scriptstyle 1 \otimes \phi_0} \\
F(A \otimes I) \xrightarrow[\phi_2]{} & & FA \otimes F
\end{array}
$$

Call F *strict* when ϕ_2, ϕ_0 are identities.

We can define natural isomorphisms

$$\phi_n \colon FA_1 \otimes \cdots \otimes FA_n \to F(A_1 \otimes \cdots \otimes A_n)$$

inductively as follows: ϕ_0 is given, ϕ_1 is the identity, ϕ_2 is given, and ϕ_{n+1} is the composite

$$FA_1 \otimes \cdots \otimes FA_{n+1} \xrightarrow{1 \otimes \phi_n} FA_1 \otimes F(A_2 \otimes \cdots \otimes A_{n+1})$$

$$\xrightarrow{\phi_2} F(A_1 \otimes \cdots \otimes A_{n+1}).$$

The following triangle then commutes.

$$FA_1 \otimes \cdots \otimes FA_m \otimes FB_1 \otimes \cdots \otimes FB_n \xrightarrow{\phi_{m+n}} F(A_1 \otimes \cdots \otimes A_m \otimes B_1 \otimes \cdots \otimes B_n)$$

with $\phi_m \otimes \phi_n$ down to $F(A_1 \otimes \cdots \otimes A_m) \otimes F(B_1 \otimes \cdots \otimes B_n)$ and ϕ_2 up to the top right.

Tensor functors $\mathcal{U} \to \mathcal{V}$, $\mathcal{V} \to \mathcal{W}$ can be composed in the obvious manner [10, 35]. Suppose $F, G : \mathcal{V} \to \mathcal{W}$ are tensor functors. A *tensor transformation* $\alpha : F \to G$ is a natural transformation α such that the following diagrams commute:

$$
\begin{array}{ccc}
FA \otimes FB & \xrightarrow{\phi_2} & F(A \otimes B) \\
{\scriptstyle \alpha_A \otimes \alpha_B} \downarrow & & \downarrow {\scriptstyle \alpha_{A \otimes B}} \\
GA \otimes GB_2 & \xrightarrow[\phi_2]{} & G(A \otimes B)
\end{array}
\qquad
\begin{array}{c}
FI \\
\phi_0 \nearrow \quad \downarrow {\scriptstyle \alpha_I} \\
I \\
\phi_0 \searrow \quad \downarrow \\
GI
\end{array}
$$

We write $\mathrm{Ten}(\mathcal{V}, \mathcal{W})$ for the category of tensor functors $\mathcal{V}$, $\mathcal{W}$ and tensor transformations between them.

The following concept has appeared with a variety of terminology and in varying generality [3, 28, 13].

DEFINITION 1.4. A *tensor scheme* $\mathcal{D}$ consists of two sets $\mathrm{obj}\,\mathcal{D}$ and $\mathrm{mor}\,\mathcal{D}$ together with a function which assigns to each element $d \in \mathrm{mor}\,\mathcal{D}$ a pair $(d(0), d(1))$ of words in the elements of $\mathrm{obj}\,\mathcal{D}$. Write

$$d : X_1 \cdots X_m \to Y_1 \cdots Y_n$$

for $d \in \mathrm{mor}\,\mathcal{D}$ with $d(0) = X_1 \cdots X_m$, $d(1) = Y_1 \cdots Y_n$.

Each tensor scheme $\mathcal{D}$ and tensor category $\mathcal{V}$ determine a category $[\mathcal{D}, \mathcal{V}]$ described as follows. An object K is a pair of functions

$$K : \mathrm{obj}\,\mathcal{D} \to \mathrm{obj}\,\mathcal{V}, \qquad K : \mathrm{mor}\,\mathcal{D} \to \mathrm{mor}\,\mathcal{V}$$

such that, for all $d : X_1 \cdots X_m \to Y_1 \cdots Y_n$, we have

$$Kd : KX_1 \otimes \cdots \otimes KX_m \to KY_1 \otimes \cdots \otimes KY_n.$$

A morphism $\kappa : K \to L$ is a family of arrows

$$\kappa_X : KX \to LX, \qquad X \in \mathrm{Obj}\,\mathcal{D},$$

in $\mathscr{V}$ such that, for all d (as before), the following square commutes:

$$
\begin{array}{ccc}
KX_1 \otimes \cdots \otimes KX_m & \xrightarrow{\ Kd\ } & KY_1 \otimes \cdots \otimes KY_n. \\
{\scriptstyle \kappa_{X_1} \otimes \cdots \otimes \kappa_{X_m}} \downarrow & & \downarrow {\scriptstyle \kappa_{X_1} \otimes \cdots \otimes \kappa_{X_m}} \\
LX_1 \otimes \cdots \otimes LX_m & \xrightarrow[\ Ld\ ]{} & LY_1 \otimes \cdots LY_n.
\end{array}
$$

There is a "composition" functor

$$
\mathrm{Ten}(\mathscr{V}, \mathscr{W}) \times [\mathscr{D}, \mathscr{V}] \to [\mathscr{D}, \mathscr{W}]
$$

$$
(F, K) \mapsto F \circ K, \qquad (\alpha, \kappa) \mapsto \alpha \circ \kappa,
$$

where $(F \circ K)X = FKX$, $(F \circ K)d$ is the composite

$$
FKX_1 \otimes \cdots \otimes FKX_m \xrightarrow{\ \phi_m\ } F(KX_1 \otimes \cdots \otimes KX_m)
$$

$$
\xrightarrow{\ FKd\ } F(KY_1 \otimes \cdots \otimes FY_n)
$$

$$
\xrightarrow{\ \phi_n^{-1}\ } FKY_1 \otimes \cdots \otimes FKY_n,
$$

and $(\alpha \circ \kappa)_X = \alpha_{LX} \circ F\kappa_X$.

DEFINITION 1.5. A tensor category F is said to be *free on the tensor scheme* $\mathscr{D}$ when there exists an object N of $[\mathscr{D}, \mathscr{F}]$ such that the functor

$$
- \circ N \colon \mathrm{Ten}(\mathscr{F}, \mathscr{V}) \to [\mathscr{D}, \mathscr{V}]
$$

is an equivalence of categories for all tensor categories $\mathscr{V}$.

If $\mathscr{F}'$ is also a free tensor category on $\mathscr{D}$ then there exists an equivalence of tensor categories $\mathscr{F} \rightsquigarrow \mathscr{F}'$. The existence of free tensor categories can be proved algebraically, but our purpose here is to provide a topological construction in terms of plane graphs.

A plane graph Γ will be called *boxed* when it is between levels -1 and $+1$, and is contained in $(-1, 1) \times [-1, 1]$. Write $\Gamma \colon m \to n$ when m, n are the cardinalities of dom Γ, cod Γ, respectively.

In defining operations on boxed graphs we shall use the functions $\gamma, \tau \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$
\gamma(x, t) = (x, \tfrac{1}{3}t), \qquad \tau(x, t) = (\tfrac{1}{2}x, t)
$$

and the points $e_1 = (1, 0)$, $e_2 = (0, 1) \in \mathbb{R}^2$. Notation such as $\gamma(S + e_2)$, for $S \subset \mathbb{R}^2$, denotes the set

$$
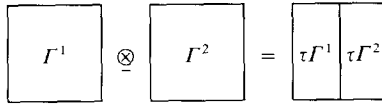\{(x, \tfrac{1}{3}(t + 1)) \in \mathbb{R}^2 \mid (x, t) \in S\}.
$$

FIGURE 1.5

The *tensor product* $\Gamma^1 \otimes \Gamma^2$ of two boxed plane graphs Γ^1, Γ^2 is the space $\tau((\Gamma^1 - e_1) \cup (\Gamma^2 + e_1))$ with $\tau((\Gamma_0^1 - e_i) \cup (\Gamma_0^2 + e_1))$ as the set of nodes. (Ignoring translations, we depict this as in Fig. 1.5).

Suppose $\Gamma: m \to n$, $\Omega: n \to p$ are boxed plane graphs. Let $a_1 < a_2 < \cdots < a_n$ be the elements of the codomain, and let I be the set of inner nodes of the graph $\gamma(\Gamma - 2e_2)$. Let $b_1 < b_2 < \cdots < b_n$ be the elements of the domain, and let J be the set of inner nodes of the graph $\gamma(\Omega + 2e_2)$. The *composite* $\Omega \circ \Gamma: m \to p$ is the plane graph consisting of the space

$$\Omega \circ \Gamma = \gamma((\Gamma - 2e_2) \cup [a_1, b_1] \cup \cdots \cup [a_n, b_n] \cup (\Omega + 2e_2))$$

with $I \cup J$ as the set of inner nodes, as in Fig. 1.6.

The concept of valuation in a tensor category $\mathcal{V}$ (Definition 1.3) needs neither composition nor tensor products of morphisms in $\mathcal{V}$ (these are needed in calculating the value). We therefore define a *valuation* $v: \Gamma \to \mathcal{D}$ of a progressive plane graph Γ in a tensor scheme $\mathcal{D}$ just as in $\mathcal{V}$ except that the tensor products in the domain and codomain of $v_1(x)$ must be replaced by words in the elements of obj $\mathcal{D}$. This gives the notion of a *progressive plane diagram* (Γ, v) *in a tensor scheme*. The *domain* [*codomain*] of such a diagram is defined to be the word $v_0(z_1) \cdots v_0(z_n)$, where $z_1 < \cdots < z_n$ are the elements of dom Γ [cod Γ].

The tensor product and composite of boxed progressive plane diagrams in a tensor scheme can now be defined in the obvious way. A tensor product $\Gamma^1 \otimes \Gamma^2$ has a tensor decomposition into subgraphs isomorphic to Γ^1, Γ^2; so valuations on Γ^1, Γ^2 transport to the subgraphs and these together give a valuation on Γ^1, Γ^2. If the codomain of (Γ, v) agrees with the domain of (Ω, w) then there is a unique valuation on $\Omega \circ \Gamma$ whose restriction to $(\Omega \circ \Gamma)[-1, -\frac{1}{3}]$ transports to v under the canonical
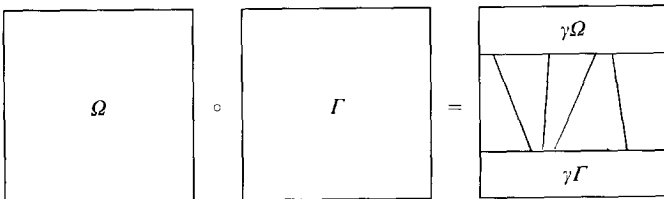

FIGURE 1.6

isomorphism with Γ, and whose restriction to $(\Omega \circ \Gamma)[\frac{1}{3}, 1]$ transports to w under the canonical isomorphism with Ω (note that the layer $(\Omega \circ \Gamma)[-\frac{1}{3}, \frac{1}{3}]$ has no inner nodes).

For each tensor scheme $\mathscr{D}$, there is a strict tensor category $\mathbb{F}(\mathscr{D})$ defined as follows. The objects are words in elements of obj $\mathscr{D}$. The morphisms are deformation classes of boxed progressive plane diagrams in $\mathscr{D}$. The domain, codomain and composition of morphisms are induced on deformation classes by the corresponding operations for the diagrams. Identity morphisms are deformation classes of diagrams with invertible graph. The tensor product on objects is given by juxtaposition of words, and on morphisms is induced by the tensor product of boxed diagrams. We shall now see, not only that it is a tensor category, but that $\mathbb{F}(\mathscr{D})$ is free.

THEOREM 1.2. $\mathbb{F}(\mathscr{D})$ *is the free tensor category on the tensor scheme* $\mathscr{D}$.

Proof. For boxed plane graphs, we have:

(a) If Γ^1 is a deformation of Γ^2 (written $\Gamma^1 \sim \Gamma^2$) then dom $\Gamma^1 =$ dom Γ^2 and cod $\Gamma^1 =$ cod Γ^2;

(b) If $\Gamma^1 \sim \Gamma^2$ and $\Omega^1 \sim \Omega^2$ then $\Gamma^1 \otimes \Omega^1 \sim \Gamma^2 \otimes \Omega^2$;

(c) If $\Gamma^1 \sim \Gamma^2$ and $\Omega^1 \sim \Omega^2$ then $\Omega^1 \circ \Gamma^1 \sim \Omega^2 \circ \Gamma^2$ (where it is understood that the composites are defined);

(d) $(\Gamma^1 \otimes \Gamma^2) \otimes \Gamma^3 \sim \Gamma^1 \otimes (\Gamma^2 \otimes \Gamma^3)$ (Fig 1.7);

(e) $(\Lambda \circ \Omega) \circ \Gamma \sim \Lambda \circ (\Omega \circ \Gamma)$ (Fig. 1.8);

(f) $(\Omega^1 \circ \Gamma^1) \otimes (\Omega^2 \circ \Gamma^2) = (\Omega^1 \otimes \Omega^1) \circ (\Gamma^1 \otimes \Gamma)^2$ (Fig. 1.9).

It follows from (a)–(f) that $\mathbb{F}(\mathscr{D})$ is a strict tensor category.

Define N to be the object of $[\mathscr{D}, \mathbb{F}(\mathscr{D})]$ described as follows. For $X \in$ obj $\mathscr{D}$, let $NX = X$ as a one-letter word. For $d: X_1 \cdots X_m \to Y_1 \cdots Y_n$ in mor D, take $Nd: X_1 \cdots X_m \to Y_1 \cdots Y_n$ to be the class of the diagram (Γ, v), where Γ is the union of the line segments

$$\left[\left(-1 + \frac{2h-1}{m}, -1 \right), (0, 0) \right], \qquad \left[(0, 0), \left(-1 + \frac{2k-1}{n}, 1 \right) \right]$$
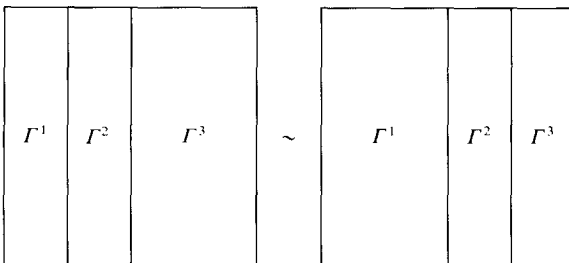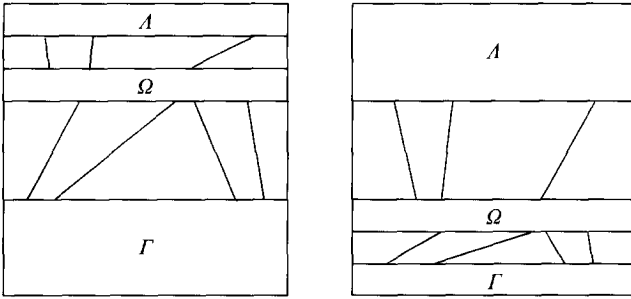

FIGURE 1.7

FIGURE 1.8

for $1 \leqslant h \leqslant m$, $1 \leqslant k \leqslant n$, in the plane $\mathbb{R}^2$ with the origin $(0, 0)$ as the only inner node, where v_0 takes the line segments to X_h, Y_k, respectively, and where $v_1(0, 0) = d$.

Without loss of generality we may suppose $\mathscr{V}$ to be a strict tensor category and prove that

$$- \circ N: \mathrm{Ten}(\mathbb{F}(\mathscr{D}), \mathscr{V}) \to [\mathscr{D}, \mathscr{V}]$$

is an equivalence of categories. For such a $\mathscr{V}$ we shall show that $- \circ N$ is surjective on objects and fully faithful. Take any object K of $[\mathscr{D}, \mathscr{V}]$. In fact we shall produce the unique strict tensor functor $T: \mathbb{F}(\mathscr{D}) \to \mathscr{V}$ with $T \circ N = K$. Since $\mathrm{obj}\,\mathbb{F}(\mathscr{D})$ is the free monoid on $\mathrm{obj}\,\mathscr{D}$, certainly T is uniquely determined on objects if it is to preserve tensor. Let (Γ, v) be a boxed progressive plane diagram in $\mathscr{D}$. Then (Γ, Kv) is a boxed progressive plane diagram in V. Define T to take the deformation class of (Γ, v) to the value $(Kv)(\Gamma)$ of (Γ, Kv). This is well defined by Theorem 1.2. It clearly preserves domain and codomain, and has $T \circ N = K$. That T preserves composition and tensor product, and is unique, follows from the definition of the value of a diagram which builds up from the values of diagrams with at most one inner node.
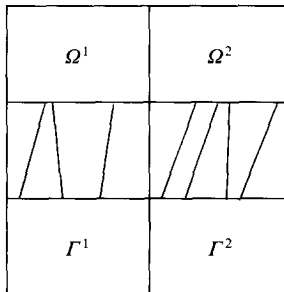

FIGURE 1.9

Suppose $F, G: \mathbb{F}(\mathscr{D}) \to \mathscr{V}$ are arbitrary tensor functors and suppose $\kappa: F \circ N \to G \circ N$ is a morphism of $[\mathscr{D}, \mathscr{V}]$. If we are to have $\alpha: F \to G$ with $\alpha \circ N = \kappa$, we are forced to define $\alpha_{X_1 \cdots X_m}: F(X_1 \cdots X_m) \to G(X_1 \cdots X_m)$ to be

$$\phi_m \circ (\kappa_{X_1} \otimes \cdots \otimes \kappa_{X_m}) \circ \phi_m^{-1}$$

Compatibility of α with ϕ_n is automatic. Naturality reduces to the case of morphisms represented by diagrams with at most one inner node, and for this we invoke the conditions satisfied by κ as a morphism of $[\mathscr{D}, \mathscr{V}]$. This proves $- \circ N$ is fully faithful. ∎

Remark. The above shows that $- \circ N$ gives an *isomorphism* of categories between the full subcategory of Ten($\mathbb{F}(\mathscr{D})$, $\mathscr{V}$) consisting of the *strict* tensor functors (that is, those with ϕ_0, ϕ_2 identities) and the category $[\mathscr{D}, \mathscr{V}]$. This is consistent with the "flexibility" of the structure of tensor category.

CHAPTER 2. SYMMETRIC TENSOR CATEGORIES AND PROGRESSIVE POLARISED DIAGRAMS

1. *The Value of a Progressive Polarised Diagram*

Recall [10, 27, 35] that a *symmetry* for a tensor category $\mathscr{V}$ is a natural family of isomorphisms

$$c_{A,B}: A \otimes B \to B \otimes A$$

such that the following two diagrams commute:

(S)

$$A \otimes B \xrightarrow{c_{A,B}} B \otimes A$$

with $1_{A \otimes B}$ and $c_{B,A}$ to $A \otimes B$

(B)

$$A \otimes B \otimes C \xrightarrow{c_{A \otimes B, C}} C \otimes A \otimes B$$

with $1 \otimes c_{B,C}$ and $c_{A,C} \otimes 1$ to $A \otimes C \otimes B$

(We continue to assume $\mathscr{V}$ is strict, for simplicity of exposition.) A *symmetric tensor category* is a tensor category with a distinguished symmetry.

EXAMPLE 2. A k-bialgebra A (see Example 1) is *commutative* when

$\varDelta = \sigma \circ \varDelta$, where $\sigma: A \otimes A \to A \otimes A$ is the switch map $\sigma(a \otimes b) = b \otimes a$. Then the switch map

$$c: V \otimes W \to W \otimes V, \qquad c(v \otimes w) = w \otimes v,$$

is a symmetry for $\mathrm{Mod}(A)$.

Suppose Γ is a progressive polarised graph (as defined in Chapter 1, Section 2) and suppose $\mathscr{V}$ is a symmetric tensor category. A *valuation* $v: \Gamma \to \mathscr{V}$ of Γ in $\mathscr{V}$ is defined precisely as in Definition 1.3 (all that is used is the order on each $\mathrm{in}(x)$ and $\mathrm{out}(x)$, and the tensor product in $\mathscr{V}$). The pair $\Gamma = (\Gamma, v)$ is called a (*progressive polarised*) *diagram* in the symmetric tensor category $\mathscr{V}$.

To define the *value* $v(\Gamma)$ of a diagram as an arrow in $\mathscr{V}$, we need the domain $\mathrm{dom}\,\Gamma$ and codomain $\mathrm{cod}\,\Gamma$ to be linearly ordered. These linear orders are not part of the definition of diagram because we need to consider "layers" which do not have natural linear orders on their domains and codomains. Our strategy is to define the value $\bar{v}(\Gamma)$ in an extension $\bar{\mathscr{V}}$ of the category $\mathscr{V}$; this will not require artificial choices of linear orders.

The category $\bar{\mathscr{V}}$ has object families $(A_s \,|\, s \in S)$ of objects of $\mathscr{V}$ indexed by finite sets S. Each such object gives rise to a clique (see Chapter 1, Section 1)

$$\left(\bigotimes_{k=1}^{m} A_{\phi(k)} \,\middle|\, \phi: [1, m] \stackrel{\sim}{\to} S \right)$$

in $\mathscr{V}$ indexed by bijections $\dot\phi: [1, m] \to S$, where $[1, m] = \{1, 2, ..., m\}$; the maps for the clique are

$$\langle \psi^{-1}\phi \rangle: \bigotimes_{k=1}^{m} A_{\phi(k)} \to \bigotimes_{k=1}^{m} A_{\psi(k)},$$

where $\langle \sigma \rangle$ for a permutation σ will now be described. If σ is the simple transposition interchanging i and $i+1$ then $\langle \sigma \rangle = 1 \otimes \cdots \otimes c \otimes \cdots \otimes 1$ ($(m-1)$ terms with c in the ith position). For a general permutation τ, decompose it as a product of simple transpositions σ, and obtain $\langle \tau \rangle$ as the composite of the corresponding $\langle \sigma \rangle$. That this is well-defined and yields a clique follows from MacLane's coherence theorem [26] for symmetry.

The arrows $f: (A_s \,|\, s \in S) \to (B_t \,|\, t \in T)$ in $\bar{\mathscr{V}}$ are precisely maps between the associated cliques. Another description of the arrows of $\bar{\mathscr{V}}$ will be useful. In general, for cliques $(C_i \,|\, i \in I)$, $(D_j \,|\, j \in J)$ in any category A, the set of clique maps from the first to the second is isomorphic to the quotient set

$$\sum_{(i,j) \in I \times J} A(C_i, D_j) / \sim,$$

where $(f: C_i \to D_j) \sim (g: C_k \to D_m)$ when $g \circ u_{ki} = u_{mj} \circ f$. Thus, an arrow

$$[\phi, f, \psi]: (A_s \,|\, s \in S) \to (B_t \,|\, t \in T)$$

in $\bar{\mathscr{V}}$ is an equivalence class of triples (ϕ, f, ψ) consisting of linear orderings $\phi: [1, m] \to S$, $\psi: [1, n] \to T$ of S, T and an arrow $f: \bigotimes_{h=1}^{m} A_{\phi(h)} \to \bigotimes_{k=1}^{n} B_{\psi(k)}$ in $\mathscr{V}$, where (ϕ, f, ψ) is equivalent to (ϕ', f', ψ') if and only if the following square commutes:

$$
\begin{array}{ccc}
\displaystyle\bigotimes_{h=1}^{m} A_{\phi(h)} & \xrightarrow{\ f\ } & \displaystyle\bigotimes_{k=1}^{n} B_{\psi(k)} \\[2mm]
\langle \phi'^{-1} \ \phi \rangle \Big\downarrow & & \Big\downarrow \langle \psi'^{-1} \ \psi \rangle \\[2mm]
\displaystyle\bigotimes_{h=1}^{m} A_{\phi'(h)} & \xrightarrow{\ f'\ } & \displaystyle\bigotimes_{k=1}^{n} B_{\psi'(k)}
\end{array}
$$

The functor $\mathscr{V} \to \bar{\mathscr{V}}$, obtained by regarding objects of $\mathscr{V}$ as a singleton families, is an equivalence of categories.

For any valuation $v: \Gamma \to \mathscr{V}$, we put $v(s) = v_0(s) \in \text{obj } \mathscr{V}$ for $s \in \text{dom } \Gamma \cup \text{cod } \Gamma$. The intended value $\bar{v}(\Gamma)$ of Γ will be an arrow

$$\bar{v}(\Gamma): (v(s) \,|\, s \in \text{dom } \Gamma) \to (v(t) \,|\, t \in \text{cod } \Gamma)$$

in $\bar{\mathscr{V}}$. Obviously, if dom Γ and cod Γ are linearly ordered, we obtain an arrow $v(\Gamma)$ in $\mathscr{V}$.

As in Chapter 1, our definition of value will involve some choices. To show the independence of these choices we need some formal properties of iterated tensor products. For these we introduce a category $\text{sp}(I, \mathscr{V})$ for each finite set I.

Objects of $\text{sp}(I, \mathscr{V})$ have the form $(A_s \,|\, s \in S \xrightarrow{\phi} I)$, where $(A_s \,|\, s \in S)$ is a family of objects of $\mathscr{V}$ indexed by the finite set S, and $\phi: S \to I$ is a function equipped with a linear order on each fibre $\phi^{-1}(i)$. A map $(A_s \,|\, s \in S \xrightarrow{\phi} I) \to (B_t \,|\, t \in T \xrightarrow{\psi} I)$ in $\text{sp}(I, \mathscr{V})$ is a family

$$\left(u_i: \bigotimes_{\phi(s)=i} A_s \to \bigotimes_{\psi(t)=i} B_t \,\Big|\, i \in I \right)$$

of maps u_i in $\mathscr{V}$ indexed by I.

We now define a functor

$$\bar{\bigotimes}_{i \in I}: \text{sp}(I, \mathscr{V}) \to \bar{\mathscr{V}}$$

which is given on objects by

$$\overline{\bigotimes_{i \in I}} (A_s \,|\, s \in S \xrightarrow{\phi} I) = (A_s \,|\, s \in S).$$

Suppose $(u_i \,|\, i \in I): (A_s \,|\, s \in S \xrightarrow{\phi} I) \to (B_t \,|\, t \in T \xrightarrow{\psi} I)$ is a map in $\mathrm{sp}(I, \mathscr{V})$. Choose a linear order on I; this together with the linear orders on the fibres of ϕ, ψ, determines linear orders on S, T (ordinal sums of the fibres). The arrow

$$\overline{\bigotimes_{i \in I}} u_i : (A_s \,|\, s \in S) \to (B_t \,|\, t \in T),$$

in $\mathscr{V}$ is the map of cliques determined by these orders on S and T, and the map

$$\bigotimes_{i \in I} u_i : \bigotimes_{i \in I} \bigotimes_{\phi(s) = i} A_s \to \bigotimes_{i \in I} \bigotimes_{\psi(t) = i} B_t$$

in $\mathscr{V}$. Different choices of linear order on I lead to the same map of cliques.

Given a function $\kappa : I \to J$ equipped with a linear order on each fibre $\kappa^{-1}(j)$, there is a functor

$$\kappa_* : \mathrm{sp}(I, \mathscr{V}) \to \mathrm{sp}(J, \mathscr{V})$$

given on objects by

$$\kappa_*(A_s \,|\, s \in S \xrightarrow{\phi} I) = (A_s \,|\, s \in S \xrightarrow{\kappa\phi} J),$$

where the linear order on the fibres of $\kappa\phi$ is such that $s \leqslant s'$ in $(\kappa\phi)^{-1}(j)$ when either $\phi(s) < \phi(s')$ in $\kappa^{-1}(j)$, or $\phi(s) = \phi(s') = i$ and $s \leqslant s'$ in $\phi^{-1}(i)$. One maps, κ_* is given by $\kappa_*(u_i \,|\, i \in I) = (\bigotimes_{k(i) = j} u_i \,|\, j \in J)$.

The required associativity of iterated tensor product can be expressed as a commutative triangle of functors:

This is trivial on objects, while on maps it amounts to the equality

$$\overline{\bigotimes_{i \in I}} u_i = \overline{\bigotimes_{j \in J}} \bigotimes_{k(i) = j} u_i$$

which is proved by choosing a linear order on J, and then obtaining linear orders on S, T as before.

The value of a diagram will be defined by cutting into layers just as in the plane case. Since in this case we do not have the aid of the embedding in the plane to give us levels, here we need a different approach to layers.

Suppose Γ is a progressive graph with boundary. Let $\mathring{\Gamma}$ denote the set of inner nodes of Γ as a partially ordered set ($x \leqslant y$ when there is a directed path from x to y in Γ). Define a *level* a of Γ to be an initial segment of $\mathring{\Gamma}$ (that is, $x \leqslant y \in a$ implies $x \in a$). The smallest level is $\varnothing$ and the largest level is $\mathring{\Gamma}$.

An edge e is said to be *cut* by level a when $e(0) \in a \cup \operatorname{dom} \Gamma$ and $e(1) \in (\mathring{\Gamma} - a) \cup \operatorname{cod} \Gamma$. Let $\operatorname{cut}(a)$ denote the set of edges cut by a. Note that we have bijections

$$\operatorname{cut}(\varnothing) \simeq \operatorname{dom} \Gamma, \qquad \operatorname{cut}(\mathring{\Gamma}) \simeq \operatorname{cod} \Gamma.$$

An *interval* in Γ is a pair of levels $a \subseteq b$. Define the *layer* $\Gamma[a, b]$ (see Fig. 2.1) to be the generalised oriented graph whose nodes are the elements of $b - a$, whose pinned edges are those edges e of Γ with source and target in $b - a$, and whose loose and half-loose edges are the elements of $\operatorname{cut}(a) \cup \operatorname{cut}(b)$. We have bijections

$$\operatorname{cut}(a) \simeq \operatorname{dom} \Gamma[a, b], \qquad \operatorname{cut}(b) \simeq \operatorname{cod} \Gamma[a, b].$$

We now proceed to define the value $\bar{v}(\Gamma) \in \operatorname{arr} \mathcal{\bar{V}}$ of a progressive polarised diagram Γ.

Call Γ *elementary* when the inner nodes are incomparable; that is, when the order on $\mathring{\Gamma}$ is discrete. Put $I = \pi_0(\Gamma)$, the set of connected components of Γ. For each $i \in I$, define an arrow u_i of $\mathcal{\bar{V}}$ by

$$u_i = \begin{cases} v_1(x) & \text{when } i \text{ contains a single inner node } x, \\ 1_{v_0(i)} & \text{when } i \text{ is a single edge.} \end{cases}$$

The functions $\operatorname{dom} \Gamma \to I$, $\operatorname{cod} \Gamma \to I$ (taking the outer nodes to the com-
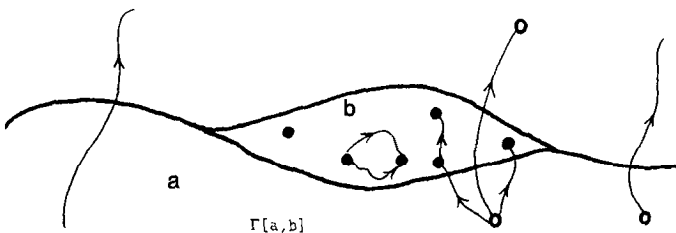

FIG. 2.1. $\Gamma[a, b]$.

ponents in which they lie) have fibre over i either a singleton or in natural bijection with $\text{in}(x)$, $\text{out}(x)$, respectively; so the fibres are linearly ordered. Thus we have a map

$$(u_i \mid i \in I): (v(s) \mid s \in \text{dom } \Gamma \to I) \to (v(t) \mid t \in \text{cod } \Gamma \to I)$$
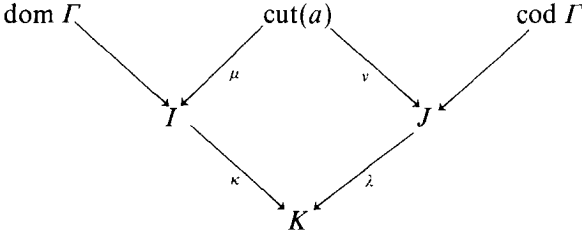
in $\text{sp}(I, \mathscr{V})$. Define

$$\bar{v}(\Gamma) = \bigotimes_{i \in I} u_i: (v(s) \mid s \in \text{dom } \Gamma) \to (v(t) \mid t \in \text{cod } \Gamma)$$

in $\bar{\mathscr{V}}$.

PROPOSITION 2.1. *If a is a level for an elementary diagram Γ then $\Gamma[\varnothing, a]$, $\Gamma[a, \mathring{\Gamma}]$ are elementary, and*

$$\bar{v}(\Gamma) = \bar{v}(\Gamma[a, \mathring{\Gamma}]) \circ \bar{v}(\Gamma[\varnothing, a]).$$

Proof. Let $I = \pi_0 \Gamma[\varnothing, a])$, $J = \pi_0 \Gamma[a, \mathring{\Gamma}]$, $K = \pi_0 \Gamma$ so that we have a diagram of functions

whose fibres are linearly ordered and the diamond commutes with $(\kappa\mu)^{-1}(k) = (\lambda v)^{-1}(k)$ as ordered sets. We have

$$\bar{v}(\Gamma[\varnothing, a]) = \bigotimes_{i \in I} f_i: (v(s) \mid s \in \text{dom } \Gamma) \to (v_0(e) \mid e \in \text{cut}(a))$$

$$\bar{v}(\Gamma[a, \mathring{\Gamma}]) = \bigotimes_{j \in J} g_i: (v_0(e) \mid e \in \text{cut}(a)) \to (v(t) \mid t \in \text{cod } \Gamma)$$

$$\bar{v}(\Gamma) = \bigotimes_{k \in K} h_k: (v(s) \mid s \in \text{dom } \Gamma) \to (v(t) \mid t \in \text{cod } \Gamma).$$

For each $k \in K$, either $\kappa^{-1}(k)$ is a singleton and g_j is an identity for all $j \in \lambda^{-1}(k)$, or $\lambda^{-1}(k)$ is a singleton and f_i is an identity for all $i \in \kappa^{-1}(k)$; so we see that

$$h_k = \bigotimes_{\lambda(j) = k} g_j \circ \bigotimes_{\kappa(i) = k} f_i.$$

Hence

$$\bar{v}(\Gamma) = \bar{\bigotimes_{k \in K}} \, h_k$$

$$= \left(\bar{\bigotimes_{k \in K}} \bigotimes_{\lambda(j) = k} g_j \right) \circ \left(\bar{\bigotimes_{k \in K}} \bigotimes_{\kappa(j) = k} f_i \right) \qquad \text{by functoriality of } \bar{\bigotimes_{k \in K}}$$

$$= \bar{\bigotimes_{j \in J}} \, g_i \circ \bar{\bigotimes_{i \in I}} \, f_i \qquad \text{by associativity of } \bar{\otimes}$$

$$= \bar{v}(\Gamma[a, \mathring{\Gamma}]) \circ \bar{v}(\Gamma[\varnothing, a]). \quad \blacksquare$$

For any progressive polarised diagram Γ, choose a maximal chain

$$\varnothing \subset a_1 \subset a_2 \subset \cdots \subset a_n = \mathring{\Gamma}$$

of levels of Γ. Then each layer $\Gamma[a_i, a_{i+1}]$ has precisely one inner node and so is elementary, Define

$$\bar{v}(\Gamma) = \bar{v}(\Gamma[a_{n-1}, a_n]) \circ \cdots \circ \bar{v}(\Gamma[\varnothing, a_1]).$$

We must see that this definition is independent of the choice of maximal chain. It is possible to move from any maximal chain to any other by a finite sequence of steps each of which involves replacement of a single a_i by another level. To see this, note that a maximal chain amounts to a linear order on $\mathring{\Gamma}$ which refines the partial order, and observe that we can pass from one such linear order to any given new one by interchanging consecutive pairs of elements to move them into the new order. To see that $\bar{v}(\Gamma)$ is well defined now comes to showing that, for levels $a \subset b \subset c$ and $a \subset b' \subset c$ where $b - a = \{w\} = c - b'$, $c - b = \{y\} = b' - a$, we have

$$\bar{v}(\Gamma[b, c]) \circ \bar{v}[\Gamma[a, b]) = \bar{v}(\Gamma[b', c]) \circ \bar{v}(\Gamma[a, b']).$$

Since x, y are incomparable, the diagram $\Gamma[a, c]$ is elementary and Proposition 2.1 applies to show that both sides of the last equation are equal to $\bar{v}(\Gamma[a, c])$.

An isomorphism $f: \Gamma \to \Omega$ of progressive polarised diagrams Γ, Ω is an isomorphism of graphs with boundary which preserves orientation and the orders on each input and output, and is compatible with the valuations.

THEOREM 2.2. If $f: \Gamma \to \Omega$ is an isomorphism of progressive polarised diagrams in a symmetric tensor category $\mathscr{V}$ then the square

$$
\begin{array}{ccc}
(v(s) \mid s \in \text{dom } \Gamma) & \xrightarrow{\;\sim\;} & (v(s') \mid s' \in \text{dom } \Omega) \\
{\scriptstyle \bar{v}(\Gamma)} \downarrow & & \downarrow {\scriptstyle \bar{v}(\Omega)} \\
(v(t) \mid t \in \text{cod } \Gamma) & \xrightarrow{\;\sim\;} & (v(t') \mid t' \in \text{cod } \Omega)
\end{array}
$$

commutes in $\mathscr{V}$ where the horizontal isomorphisms are determined by the bijections dom $\Gamma \simeq$ dom Ω, cod $\Gamma \simeq$ cod Ω *induced by* f.

A progressive graph Γ is called *anchored* when it is equipped with linear orders on dom Γ and cod Γ. In this case, $\bar{v}(\Gamma)$ determines a map

$$v(\Gamma): \bigotimes_{s \in \text{dom}\,\Gamma} v(s) \to \bigotimes_{t \in \text{cod}\,\Gamma} v(t)$$

in $\mathscr{V}$. An isomorphism $f: \Gamma \to \Omega$ of progressive graphs is *anchored* when the bijections dom $\Gamma \simeq$ dom Ω, cod $\Gamma \simeq$ cod Ω induced by f are order preserving.

COROLLARY 2.3. *If* $f: \Gamma \to \Omega$ *is an anchored isomorphism of progressive polarised diagrams then* $v(\Gamma) = v(\Omega)$.

2. Free Symmetric Tensor Categories

Suppose $\mathscr{V}$, $\mathscr{W}$ are symmetric tensor categories. A tensor functor $F: \mathscr{V} \to \mathscr{W}$ is called *symmetric* [10] when, for all $A, B \in \text{obj}\,\mathscr{V}$, the following square commutes:

$$
\begin{array}{ccc}
F(A) \otimes F(B) & \xrightarrow{\phi_2} & F(A \otimes B) \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle Fc} \\
F(B) \otimes FA & \xrightarrow[\phi_2]{} & F(B \otimes A)
\end{array}
$$

Write STen($\mathscr{V}$, $\mathscr{W}$) for the category of symmetric tensor functors $\mathscr{V} \to \mathscr{W}$ and tensor transformations between them.

DEFINITION 2.1. A symmetric tensor category F is said to be *free, on the tensor scheme* $\mathscr{D}$, when there exists an object N of $[\mathscr{D}, \mathscr{F}]$ such that the functor

$$- \circ N: \text{STen}(\mathscr{F}, \mathscr{V}) \to [\mathscr{D}, \mathscr{V}]$$

is an equivalence of categories for all symmetric tensor categories $\mathscr{V}$.

The notion of *valuation* $v: \Gamma \to \mathscr{D}$ of a progressive polarised graph Γ in a tensor scheme $\mathscr{D}$ is defined just as for a progressive plane graph (Chapter 1, Section 4). Call (Γ, v) a *progressive polarised diagram in* $\mathscr{D}$. When Γ is anchored, the *domain* [*codomain*] of (Γ, v) is defined to be the word $v_0(z_1) \cdots v_0(z_n)$ where $z_1 < \cdots < z_n$ are the elements of dom Γ[cod Γ].

The *tensor product* $\Gamma^1 \otimes \Gamma^2$ of two progressive polarised diagrams Γ^1, Γ^2 is the diagram whose graph is the disjoint union $\Gamma^1 + \Gamma^2$ (with inner

nodes those of Γ^1 and of Γ^2) and whose valuation restricts to Γ^1, Γ^2 to give their valuations.

Suppose Γ, Ω are anchored progressive polarised diagrams in $\mathscr{D}$ with $\text{cod}(\Gamma, v) = \text{dom}(\Omega, v)$ (as words in the elements of obj $\mathscr{D}$). This produces an order-preserving bijection cod $\Gamma \simeq \text{dom } \Omega$. The *composite* $\Omega \circ \Gamma$ is the diagram defined as follows. The graph is obtained from the disjoint union $\Omega + \Gamma$ by identifying the outer nodes which correspond under cod $\Gamma \simeq \text{dom } \Omega$. The inner nodes and edges are those of Γ and of Ω, except for the edges of Γ which have target an outer node and the edges of Ω which have source an outer node; these edges pair up via corresponding outer nodes, each pair contributing an edge to $\Omega \circ \Gamma$. As paired edges have equal values, we obtain a valuation on $\Omega \circ \Gamma$.

For each tensor scheme $\mathscr{D}$, there is a symmetric strict tensor category $\mathbb{F}_s(\mathscr{D})$ defined as follows. The objects are words in elements of obj $\mathscr{D}$. The morphisms are anchored isomorphism classes of anchored progressive polarised diagrams in $\mathscr{D}$. The domain, codomain, composition, and tensor product are induced on anchored isomorphism classes by the corresponding operations on diagrams. The symmetry $c_{v,w}: VW \to WV$, where V, W are the words $X_1 \cdots X_m$, $Y_1 \cdots Y_n$, respectively, is the anchored isomorphism class of the achored diagram (Γ, v) described as follows. The graph Γ is the union of $m + n$ disjoint closed intervals $[a_i, b_i]$, $[c_j, d_j]$ $(i = 1, ..., m; j = 1, ..., n)$ with their natural orientation and with no inner nodes. The anchoring of Γ is given by

$$\text{dom } \Gamma = \{a_1 < \cdots < a_m < c_1 < \cdots < c_n\}$$
$$\text{cod } \Gamma = \{d_1 < \cdots < d_n < b_1 < \cdots < b_m\}.$$

The valuation is given by

$$v_0(a_i, b_i) = X_i, \qquad v_0(c_j, d_j) = Y_j.$$

THEOREM 2.3. $\mathbb{F}_s(\mathscr{D})$ *is the free symmetric tensor category on the tensor scheme* $\mathscr{D}$.

Proof. The fact that $\mathbb{F}_s(\mathscr{D})$ is a symmetric strict category is straightforward. Each progressive plane diagram is an anchored progressive polarised diagram (yielding $\mathbb{F}(\mathscr{D}) \to \mathbb{F}_s(\mathscr{D})$), so we have an object N of $[\mathscr{D}, \mathbb{F}_s(\mathscr{D})]$ defined as in Theorem 1.3. As in the proof of Theorem 1.3, we take $\mathscr{V}$ to be strict tensor category, this time, with a symmetry. The proof then follows closely that of Theorem 1.3. If (Γ, v) is the diagram in the definition of the symmetry on $\mathbb{F}_s(\mathscr{D})$ then the value $(Kv)(\Gamma)$:

$TV \otimes TW \to TW \otimes TV$ can be seen, from the definition, to be $\langle \alpha \rangle$, where α is the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & m+n \\ m+1 & m+2 & \cdots & m+n & 1 & 2 & \cdots & n \end{pmatrix}.$$

It follows that T is symmetric. The remaining details are left to the reader. ∎

CHAPTER 3.   BRAIDED TENSOR CATEGORIES AND $3D$-DIAGRAMS

1. *Braided tensor categories*

Suppose $\mathscr{V}$ is a tensor category. We suppose for simplicity that $\mathscr{V}$ is strict.

Recall [17, 18] that a braiding for $\mathscr{V}$ consists of a natural family of isomorphisms

$$c = c_{A,B} : A \otimes B \to B \otimes A$$

in $\mathscr{V}$ such that the following two diagrams commute:

(B)

$$\begin{array}{ccc}
A \otimes B \otimes C \xrightarrow{c_{A,B \otimes C}} B \otimes C \otimes A & \qquad & A \otimes B \otimes C \xrightarrow{c_{A \otimes B, C}} C \otimes A \otimes B \\
\scriptstyle{c_{A,B} \otimes 1} \searrow \quad \nearrow \scriptstyle{1 \otimes c_{A,C}} & & \scriptstyle{1 \otimes c_{B,C}} \searrow \quad \nearrow \scriptstyle{c_{A,C} \otimes 1} \\
B \otimes A \otimes C & & A \otimes C \otimes B
\end{array}$$

The pair $(\mathscr{V}, c)$ is called a *braided tensor category*. Recall [17, 18] that commutativity of the following *Yang–Baxter hexagon* is a consequence of (B) and naturality:

(YB)

$$\begin{array}{ccc}
& A \otimes C \otimes B \xrightarrow{c \otimes 1} C \otimes A \otimes B & \\
\scriptstyle{1 \otimes c} \nearrow & & \searrow \scriptstyle{1 \otimes c} \\
A \otimes B \otimes C & & C \otimes B \otimes A \\
\scriptstyle{c \otimes 1} \searrow & & \nearrow \scriptstyle{c \otimes 1} \\
& B \otimes A \otimes C \xrightarrow[1 \otimes c]{} B \otimes C \otimes A &
\end{array}$$

EXAMPLE 3. A *braiding* for a k-algebra A is an invertible element $\gamma \in A \otimes A$ such that

$$\gamma \cdot \Delta_{21} = \Delta \cdot \gamma$$

$$(\Delta \otimes 1_A)(\gamma) = (1 \otimes \gamma) \cdot (\gamma \otimes 1)_{132}$$

$$(1_A \otimes \Delta)(\gamma) = (\gamma \otimes 1) \cdot (1 \otimes \gamma)_{213},$$

where, for example, ξ_{213} is the image of $\xi \in A \otimes A \otimes A$ under $A \otimes A \otimes A \rightarrow A \otimes A \otimes$, $a_1 \otimes a_2 \otimes a_3 \mapsto a_2 \otimes a_1 \otimes a_3$. (Braided bialgebras are called "quasitriangular" by Drinfel'd [7].) Braidings $c: V \otimes W \rightarrow W \otimes V$ for Mod(A) are in bijection with braidings for A via the formula

$$c(v \otimes w) = \gamma(w \otimes v).$$

A tensor functor $F: \mathscr{V} \rightarrow \mathscr{W}$ (Chapter 2, Section 2) between braided tensor categories $\mathscr{V}$, $\mathscr{W}$ is called *braided* when the following diagram commutes:

$$
\begin{array}{ccc}
FA \otimes FB & \xrightarrow{\ \phi\ } & F(A \otimes B) \\
{\scriptstyle c}\big\downarrow & & \big\downarrow{\scriptstyle Fc} \\
FB \otimes FA & \xrightarrow[\ \phi\ ]{} & F(B \otimes A)
\end{array}
$$

Let us recall the notion of *fundamental groupoid* $\pi_1(X)$ of a space X. This is the groupoid whose objects are the points x of the space X and whose arrows $[\omega]: x \rightarrow y$ are homotopy classes of paths of paths $\omega: [0, 1] \rightarrow X$ with $\omega(0) = x$, $\omega(1) = y$, where the homotopies are constant at the endpoints. If Λ is any set and $\rho: \Lambda \rightarrow X$ is a function, we write $\pi_1(X, \rho)$ for the groupoid whose objects are elements of Λ and whose arrows $\lambda \rightarrow \mu$ are just the arrows $\rho(\lambda) \rightarrow \rho(\mu)$ in $\pi_1(X)$. The obvious functor

$$\pi_1(X, \rho) \rightarrow \pi_1(X)$$

is an equivalence of groupoids provided $\pi_0(\rho): \Lambda \rightarrow \pi_0(X)$ is surjective (where $\pi_0(X)$ is the set of path components of X).

Let C_n denote the space of subsets of the Euclidean plane $\mathbb{R}^2$ of cardinality precisely n. The *configuration space* C *of distinct points in* $\mathbb{R}^2$ is the disjoint union

$$C = \sum_{n \geqslant 0} C_n.$$

A path ω from S to T in C can be depicted by a diagram as in Fig. 3.1, where $\omega(t)$ is obtained from the intersection of the curves with the horizontal plane $\mathbb{R}^2 \times \{t\}$.
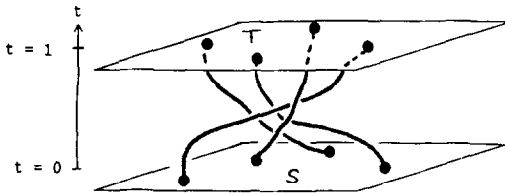
FIGURE 3.1

Define a function $\rho \colon \mathbb{N} \to C$ by

$$\rho(n) = \{1, 2, ..., n\},$$

where $1, 2, 3, ...$ denote integer points on the x-axis of $\mathbb{R}^2$.

The *braid category* $\mathbb{B}$ is defined to be the groupoid $\pi_1(C, \rho)$. For $m \neq n$, the homset $\mathbb{B}(m, n)$ is empty. The group $\mathbb{B}(n, n)$ of automorphisms of n in $\mathbb{B}$ is precisely the *braid group* $\mathbb{B}_n$ *on n strings* in the sense of Artin [2]. For $1 \leqslant i \leqslant n-1$, let $s_i \in \mathbb{B}_n$ be the homotopy class of the path depicted in Fig. 3.2. A presentation for the group $\mathbb{B}_n$ is given by the generators s_i for $1 \leqslant i \leqslant n-1$ and the relations

(A1) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $1 \leqslant i \leqslant n-2$

(A2) $s_i s_j = s_j s_i$ for $|i - j| > 1$.

A presentation for the *symmetric group* $\mathbb{S}_n$ on n symbols is obtained by imposing the extra relation

$$s_i s_i = 1.$$

It follows that we have a canonical surjective homomorphism

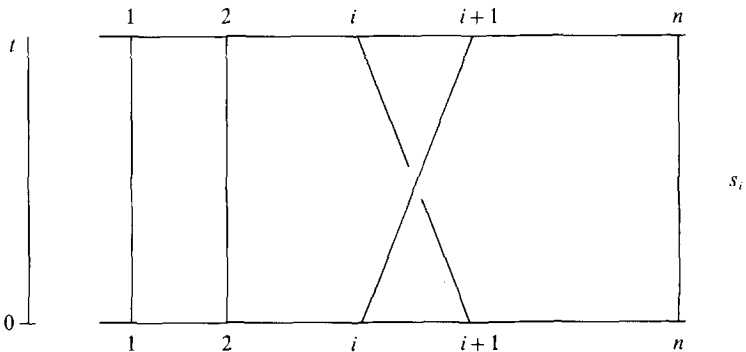$$\mathbb{B}_n \to \mathbb{S}_n, \qquad \alpha \mapsto \underline{\alpha}.$$

FIGURE 3.2

The braid category $\mathbb{B}$ becomes a strict tensor category by means of the operation of *addition of braids*

$$+: \mathbb{B}_m \times \mathbb{B}_n \to \mathbb{B}_{m+n}, \ (s_i, s_j) \mapsto s_i s_{m+j}$$

depicted by juxtaposition of diagrams. The unit object I is 0. A braiding for $\mathbb{B}$ is given by the arrows

$$c = c_{m,n}: m + n \to n + m$$

illustrated by Fig. 3.3.

THEOREM 3.1. $\mathbb{B}$ *is the free braided strict tensor category on one generating object.*

Proof.   The statement means that, for each object A of a strict tensor category $\mathcal{V}$, there is a unique braided strict tensor functor $F: \mathbb{B} \to \mathcal{V}$ with $F1 = A$. On objects F is defined by

$$Fn = A^{\otimes n} = A \otimes \cdots \otimes A \qquad (n \text{ terms}).$$

On arrows F is given by the monoid homomorphism

$$F: \mathbb{B}_n \to \mathcal{V}(A^{\otimes n}, A^{\otimes n})$$

determined by putting

$$Fs_i = 1_{F(i-1)} \otimes c_{A,A} \otimes 1_{F(n-i-1)}: A^{\otimes n} \to A^{\otimes n}.$$
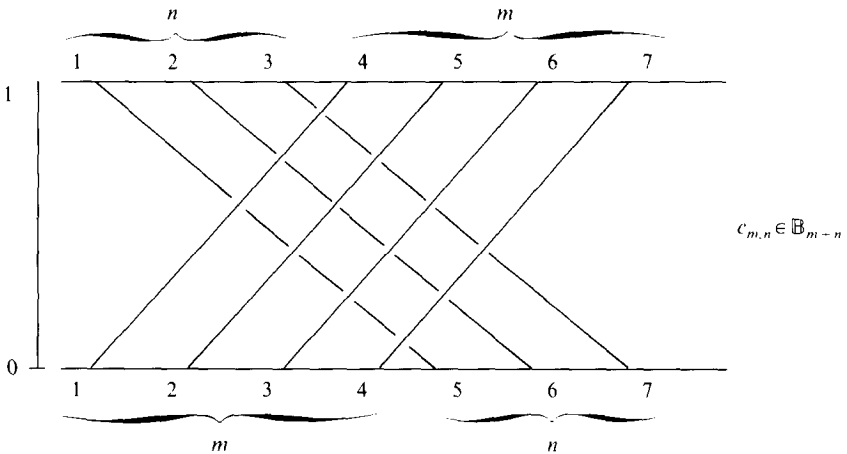

FIGURE 3.3

That this homomorphism is well defined follows from the Yang–Baxter hexagon (YB) and the functoriality of tensor which show that relations (A1), (A2) are respected. That F is braided follows inductively from the conditions (B) on a brading. ∎

We shall have need for a version of $\mathbb{B}$ which is "labelled" by the morphisms of some category $\mathscr{A}$.

First we describe the category $\mathbb{B}_n \int \mathscr{A}$ of braids on n labelled strings. The objects of $\mathbb{B}_n \int \mathscr{A}$ are n-tuples of objects of $\mathscr{A}$. The arrows

$$(\alpha; f_1, f_2, ..., f_n): (A_1, A_2, ..., A_n) \to (B_1, B_2, ..., B_n)$$

consist of $\alpha \in \mathbb{B}_n$ and $f_i: A_i \to B_{\chi(i)}$ in $\mathscr{A}$ for $1 \leqslant i \leqslant n$ (illustrated in Fig. 3.4). Composition of labelled braids is performed by composing the labels on each string of the composed braid. The operation of addition of braids extends obviously to labelled braids

$$\mathbb{B}_m \int \mathscr{A} \times \mathbb{B}_n \int \mathscr{A} \to \mathbb{B}_{m+n} \int \mathscr{A}.$$

Now we define the braided strict tensor category $\mathbb{B} \int \mathscr{A}$ of braids having their strings labelled by arrows of $\mathscr{A}$ to be the disjoint union

$$\mathbb{B} \int \mathscr{A} = \sum_{n \geqslant 0} \mathbb{B}_n \int \mathscr{A}$$

with addition as the tensor product, and with braiding given by $c_{m,n} \in \mathbb{B}$ labelled with identity arrows. Each arrow of $\mathscr{A}$ can be seen as a braid on one string $\mathscr{A} = \mathbb{B}_1 \int \mathscr{A}$, so we have an inclusion $\mathscr{A} \subsetneq \mathbb{B} \int \mathscr{A}$. Theorem 3.1 generalizes to the next result which can be proved similarly.

THEOREM 3.2. $\mathbb{B} \int \mathscr{A}$ is the free braided strict tensor category generated by $\mathscr{A}$.
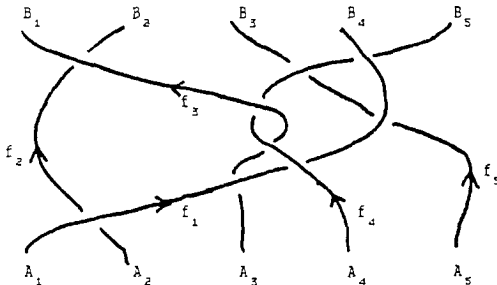

FIGURE 3.4

This result should be compared with the consequence [23], of MacLane's coherence result that

$$\mathbb{S}\int\mathscr{A} = \sum_{n \geqslant 0} \mathbb{S}_n \int \mathscr{A}$$

is the free symmetric tensor category generated by $\mathscr{A}$. The meaning of our result is that each functor $T: \mathscr{A} \to \mathscr{V}$ into a braided strict tensor category $\mathscr{V}$ has a unique braided strict tensor functor $F: \mathbb{B}\int\mathscr{A} \to \mathscr{V}$ extending it. For each functor $S: \mathscr{A} \to \mathscr{B}$, we thus have a braided strict tensor functor

$$\mathbb{B}\int S: \mathbb{B}\int\mathscr{A} \to \mathbb{B}\int\mathscr{B}$$

whose restriction to $\mathscr{A}$ is the composite

$$\mathscr{A} \xrightarrow{\ S\ } \mathscr{B} \hookrightarrow \mathbb{B}\int\mathscr{B}.$$

Extending the identity functor of any braided strict tensor category $\mathscr{V}$, we have a braided strict tensor functor

$$\otimes : \mathbb{B}\int\mathscr{V} \to \mathscr{V}$$

which assigns to each $(\alpha; f_1, ..., f_n): (A_1, ..., A_n) \to (B_1, ..., B_n)$ in $\mathbb{B}_n\int\mathscr{V}$ and arrow $\alpha[f_1, ..., f_n]: A_1 \otimes \cdots \otimes A_n \to B_1 \otimes \cdots \otimes B_n$ in $\mathscr{V}$. In particular, we have an operation of *substitution of braids*

$$\otimes : \mathbb{B}\int\mathbb{B} \to \mathbb{B}$$

as illustrated in Fig. 3.5, and also of $\mathscr{A}$-labelled braids

$$\otimes : \mathbb{B}\int\mathbb{B}\int\mathscr{A} \to \mathbb{B}\int\mathscr{A}.$$

The following commutative square expresses the *associativity* of $\otimes : \mathbb{B}\int\mathscr{V} \to \mathscr{V}$:

$$
\begin{array}{ccc}
\mathbb{B}\int\mathbb{B}\int\mathscr{V} & \xrightarrow{\ \otimes\ } & \mathbb{B}\int\mathscr{V} \\
{\scriptstyle \mathbb{B}\int\otimes}\Big\downarrow & & \Big\downarrow{\scriptstyle \otimes} \\
\mathbb{B}\int\mathscr{V} & \xrightarrow[\ \otimes\ ]{} & \mathscr{V}
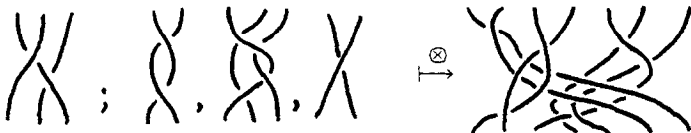\end{array}
$$

FIGURE 3.5

2. *Progressive Polarised 3D Diagrams*

This section concerns the diagrams appropriate for braided tensor categories. An explained at the end of Chapter 1, Section 2, we shall use smooth graphs henceforth.

Let $a < b$ be real numbers. We shall often make use of the *front projection*

$$fr: \mathbb{R}^2 \times [a, b] \to \mathbb{R} \times [a, b]$$

given by $fr(x, y, t) = (x, t)$.

DEFINITION 3.1.   *A progressive polarized* (*smooth*) *3D graph* (*between the levels a and b*) is a smooth embedded graph Γ (with boundary) in $\mathbb{R}^2 \times [a, b]$ such that

(i) $\partial \Gamma = (\mathbb{R}^2 \times \{a, b\}) \cap \Gamma$ and the edges meet $\mathbb{R}^2 \times \{a, b\}$ transversally,

(ii)   the second projection

$$pr_2: \mathbb{R}^2 \times [a, b] \to [a, b]$$

is a smooth embedding on each connected component of $\Gamma - \Gamma_0$,

(iii)   for any pair of distinct edges $\gamma_1(s)$, $\gamma_2(s)$ with $\gamma_1(0) = \gamma_2(0)$ $[\gamma_1(1) = \gamma_2(1)]$, the unit tangent vectors at $s = 0$ $[s = 1]$ of the projected curves $fr\,\gamma_1(s)$ and $fr\,\gamma_2(s)$ are distinct.

Each progressive polarised $3D$-graph is both progressive and polarised in the sense of Chapter 1, Section 2. Each edge e will be given the orientation with $pr_2\, e(0) < pr_2\, e(1)$. Condition (ii) excludes circles and circuits. The sets $in(x)$ and $out(x)$ are linearly ordered as follows. If $u \in [a, b]$ is chosen smaller than but close enough to $pr_2(x)$, each edge $\gamma \in in(x)$ meets the plane $\mathbb{R}^2 \times \{u\}$ in a single point which is different for different edges. So the front projection defines a bijection between $in(x)$ and a subset of $\mathbb{R} \times \{u\}$, and so induces a linear order on $in(x)$. Similarly for $out(x)$.

A number $u \in [a, b]$ is called a *regular level* for Γ when the plane $\mathbb{R}^2 \times \{u\}$ contains no inner nodes. If $u < v$ are regular levels of Γ, we write $\Gamma[u, v]$ for the graph $\Gamma \cap (\mathbb{R}^3 \times [u, v])$ whose set of inner nodes is $\Gamma_0 \cap (\mathbb{R}^2 \times (u, v))$ and whose set of outer nodes is $\Gamma \cap (\mathbb{R}^2 \times \{u, v\})$. The graph $\Gamma[u, v]$ is a progressive polarised $3D$-graph between the levels u and v; it is called a *layer* of Γ.

There is also a concept of *tensor decomposition* for a $3D$-graph Γ which we shall now describe. By a *standard rectangle* $R \subseteq \mathbb{R}^2$ we mean a product $(\alpha, \beta) \times (\gamma, \delta)$ of two open intervals. A tensor decomposition of Γ is a family $(\Gamma^R \mid R \in \mathscr{R})$ of subgraphs Γ^R of Γ indexed by a set $\mathscr{R}$ of pairwise disjoint

standard rectangles $R \subseteq \mathbb{R}^2$ such that $\Gamma^R \subseteq R \times \mathbb{R}$ for each $R \in \mathscr{R}$ and Γ is the union of the Γ^R. We denote this by

$$\Gamma = \bigotimes_{R \in \mathscr{R}} \Gamma^R.$$

DEFINITION 3.2. *A deformation of progressive polarised (smooth) 3D graphs (between levels a and b)* is a continuous function

$$h: \Gamma \times [0, 1] \to \mathbb{R}^2 \times [a, b]$$

such that, for all $t \in [0, 1]$, the function

$$h(-, t): \Gamma \to \mathbb{R}^2 \times [a, b]$$

is a smooth embedding of $(\Gamma, \partial\Gamma)$ whose image $(\Gamma(t), \partial\Gamma(t))$ is a progressive polarised 3D-graph between levels a and b (Definition 3.1), and

$$\frac{d}{ds} h(\gamma(s), t)$$

is a continuous function on $[0, 1] \times [0, 1]$ for each edge γ of Γ.

For each inner node x of Γ, we can define $\text{in}(x)$ and $\text{out}(x)$ as ordered sets just as in the plane case (Chapter 1, Section 3).

A *valuation* $\Gamma \to \mathscr{V}$ of a progressive polarised 3D-graph Γ in a braided tensor category $\mathscr{V}$ is defined precisely as in the plane case (Definition 1.3). The pair (Γ, v) is called a *progressive polarised 3D diagram*. The *domain* and *codomain* of a diagram (Γ, v) are families

$$\text{dom}(\Gamma, v) = (v_0(z) \mid z \in \text{dom } \Gamma), \qquad \text{cod}(\Gamma, v) = (v_0(z) \mid z \in \text{cod } \Gamma)$$

of objects of $\mathscr{V}$ indexed by finite subsets $\text{dom } \Gamma$, $\text{cod } \Gamma$ of the planes $\mathbb{R}^2 \times \{a\}$, $\mathbb{R}^2 \times \{b\}$ (here an edge in $\text{dom } \Gamma$ [$\text{cod } \Gamma$] is identified with its source [target] in $\partial\Gamma$).

Valuations restrict to layers and to components of tensor decompositions just as in the plane case. Also *deformations of diagrams* are defined in the obvious way.

Just as in the symmetric case, to define the value $v(\Gamma)$ of a diagram, we need the domain and codomain of Γ to be linearly ordered. Because we first need to define the value of layers which do not have such linear orders, we again define the value $\bar{v}(\Gamma)$ in an extension $\bar{\mathscr{V}}$ of the braided tensor category $\mathscr{V}$. We are able to define $\bar{\mathscr{V}}$ and $\bar{v}(\Gamma)$ for prime and invertible diagrams Γ without further preparation. In order to define $\bar{v}(\Gamma)$ for all Γ and to prove it deformation invariant, we shall develop some topological constructions.

For finite subsets S, T of the plane $\mathbb{R}^2$, each arrow $\alpha: S \to T$ in the

fundamental groupoid $\pi_1(C)$ determined a bijection $\underline{\alpha}: S \to T$. If $A_1, ..., A_n$ are objects of $\mathscr{V}$ and $\alpha: \rho(n) \to \rho(n)$ is a braid we write

$$\langle \alpha \rangle: A_{\underline{\alpha}(1)} \otimes \cdots \otimes A_{\underline{\alpha}(n)} \to A_1 \otimes \cdots \otimes A_n$$

for the value of $\otimes: \mathbb{B} \int \mathscr{V} \to \mathscr{V}$ at the braid α labelled by identity arrows (this uses the braiding on $\mathscr{V}$).

The objects of the category $\bar{\mathscr{V}}$ are families $(A_s \,|\, s \in S)$ of objects A_s of $\mathscr{V}$ indexed by finite subsets $S \in C$ of $\mathbb{R}^2$. To each such object there corresponds a clique

$$\left(\bigotimes_{k=1}^{n} A_{\underline{\alpha}(k)} \,\middle|\, \alpha: \rho(n) \to S \text{ in } \pi_1(C) \right)$$

in $\mathscr{V}$ where the required maps $u_{\alpha,\beta}$ for the clique are given by $u_{\alpha,\beta} = \langle \beta^{-1}\alpha \rangle$. An arrow in $\bar{\mathscr{V}}$ is a map of the corresponding cliques. Regarding each object of $\mathscr{V}$ as a family indexed by $\rho(1) \in C$, we obtain an inclusion $\mathscr{V} \to \bar{\mathscr{V}}$ which is an equivalence of categories.

A diagram Γ is called *invertible* when it has no inner nodes. Such a Γ determines an arrow $\gamma: \operatorname{dom} \Gamma \to \operatorname{cod} \Gamma$ in $\pi_1(C)$ for which the following triangle of functions commutes:

$$\operatorname{dom} \Gamma \xrightarrow{\;\gamma\;} \operatorname{cod} \Gamma$$

$$v_0 \searrow \qquad \swarrow v_0$$

$$\operatorname{Obj} \mathscr{V}$$

The identity arrows $v_0(\underline{\alpha}(i)) \to v_0(\gamma(\underline{\alpha}(i)))$, where $\alpha: \rho(n) \to \operatorname{dom} \Gamma$ and $i \in \rho(n)$, determine an arrow

$$\bar{v}(\Gamma): (v_0(s) \,|\, s \in \operatorname{dom} \Gamma) \to (v_0(t) \,|\, t \in \operatorname{cod} \Gamma)$$

in $\bar{\mathscr{V}}$. This defines the value of an invertible diagram.

A diagram Γ is called *frontal* when the restriction of the front projection fr: $\mathbb{R}^2 \times [a, b] \to \mathbb{R} \times [a, b]$ to Γ is injective. In this case, the front projection fr(Γ) of Γ is a progressive polarised plane diagram and we can define the value of Γ by

$$v(\Gamma) = v(\operatorname{fr}(\Gamma))$$

(see Chapter 1, Section 3) which is an arrow of $\mathscr{V}$, and hence can be regarded as an arrow $\bar{v}(\Gamma)$ of $\bar{\mathscr{V}}$.

A diagram Γ is called *prime* when it is frontal, connected, and has precisely one inner node. In fact, we shall not need the above value of a

general frontal diagram, only of prime ones, in which case $v(\Gamma)$ is just $v_1(x)$, where x is the inner node of Γ.

A subset of $\mathbb{R}^2$ is called *frontal* when the first projection $\mathrm{pr}_1 \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is injective on it. Frontal subsets are naturally linearly ordered since pr_1 establishes a bijection with a subset of $\mathbb{R}$. If Γ is frontal then so are dom Γ and cod Γ.

A diagram Γ is called *elementary* when it has a tensor decomposition

$$\Gamma = \bigotimes_{R \in \mathcal{R}} \Gamma^R$$

in which each component Γ^R is either invertible or prime. We would like to define

$$\bar{v}(\Gamma) = \bar{\bigotimes}_{R \in \mathcal{R}} \bar{v}(\Gamma^R),$$

and so must develop the precise meaning an properties of $\bar{\bigotimes}_{R \in \mathcal{R}}$. This leads us to the following topological considerations.

Let $C[n]$ denote the space of injective functions $[1, n] \to \mathbb{R}^2$ as a subspace of $(\mathbb{R}^2)^n$. The symmetric group $\mathbb{S}_n$ acts on $C[n]$ by restriction of the action on the n-fold power of $\mathbb{R}^2$. Then

$$C_n = C[n]/\mathbb{S}_n.$$

Let $U[n]$ denote the universal covering space of C_n with basepoint $\rho(n) = \{1, 2, ..., n\} \in C_n$. The points of $U[n]$ are arrows $\alpha \colon \rho(n) \to S$ in the groupoid $\pi_1(C)$. Also $\mathbb{B}_n = \pi_1(C_n, \rho(n))$, which can be regarded as the group of deck transformations, acts on $U[n]$ and we have

$$C_n = U[n]/\mathbb{B}_n.$$

Furthermore, $U[n]$ is the universal covering space of $C[n]$, and the group of deck transformations here is the *pure braid group* $\mathbb{P}_n$ (which is the kernel of $\mathbb{B}_n \to \mathbb{S}_n$); so

$$C[n] = U[n]/\mathbb{P}_n.$$

For any space X, we put

$$C(X) = \sum_{n \geqslant 0} C[n] \times X^n/\mathbb{S}_n$$

which is *the space of finite sets of points in the plane labelled by X*. Note that we also have

$$C(X) = \sum_{n \geqslant 0} U[n] \times X^n/\mathbb{B}_n.$$

Since

$$U[n] \times X^n/\mathbb{B}_n = (U[n] \times X^n/\mathbb{P}_n)/(\mathbb{B}_n/\mathbb{P}_n)$$
$$= (U[n]/\mathbb{P}_n \times X^n)/\mathbb{S}_n$$

(because the action of $\mathbb{P}_n$ on X^n is trivial).

Using these constructions, we now enrich $\mathscr{V}$ which the structure of *topological category*; that is, we equip the sets $\mathscr{V}_0$ of objects and $\mathscr{V}_1$ of arrows of $\mathscr{V}$ with topologies such that the source and target functions $d_0, d_1 : \mathscr{V}_0 \to \mathscr{V}_1$, the function $\mathscr{V}_0 \to \mathscr{V}_1$ taking each object to its identity, and the composition function, are all continuous.

Define the set $H[m, n]$ to be the pullback

$$
\begin{array}{ccc}
H[m, n] & \longrightarrow & \mathscr{V}_1 \\
\downarrow & & \downarrow {\scriptstyle (d_0, d_1)} \\
\mathscr{V}_0^m \times \mathscr{V}_0^n & \xrightarrow{\otimes \times \otimes} & \mathscr{V}_0 \times \mathscr{V}_0
\end{array}
$$

where $d_0, d_1 : \mathscr{V}_0 \to \mathscr{V}_1$ are the source and target functions for the category $\mathscr{V}$ and

$$\otimes (A_1, ..., A_m) = A_1 \otimes \cdots \otimes A_m.$$

Elements of $H[m, n]$ will be denoted by expressions

$$f: \bigotimes_{i=1}^{m} A_i \to \bigotimes_{j=1}^{n} B_j.$$

The group $\mathbb{B}_n \times \mathbb{B}_n$ acts on $H[m, n]$ by means of the formula

$$\left(\bigotimes_{i=1}^{m} A_i \xrightarrow{\,f\,} \bigotimes_{j=1}^{n} B_j \right) (\alpha, \beta) = \left(\bigotimes_{i=1}^{m} A_{\alpha(i)} \xrightarrow{\langle \beta^{-1} \rangle \cdot f \cdot \langle \alpha \rangle} \bigotimes_{j=1}^{n} B_{\beta(j)} \right).$$

The space $\mathscr{V}_0$, $\mathscr{V}_1$ are defined by the equations

$$\mathscr{V}_0 = C(\mathscr{V}_0) = \sum_n U[n] \times \mathscr{V}_0^n/\mathbb{B}_n$$

and

$$\mathscr{V}_1 = \sum_{m,n} U[m] \times U[n] \times H[m, n]/\mathbb{B}_m \times \mathbb{B}_n.$$

Source and target maps $d_0, d_1 : \mathscr{V}_1 \to \mathscr{V}_0^n$ are induced by the equivariant projections

$$\mathscr{V}_0^m \leftarrow H[m, n] \to \mathscr{V}_0^n,$$

and provide the covering projection for $\bar{\mathscr{V}}_1$ as a covering space of $\bar{\mathscr{V}}_0 \times \bar{\mathscr{V}}_0$. (This implies that the homs of $\bar{\mathscr{V}}$ are discrete spaces since covering projections have discrete fibres,) the composition map

$$\bar{\mathscr{V}}_1 \times_{\mathscr{V}_0} \bar{\mathscr{V}}_1 \to \bar{\mathscr{V}}_1$$

is induced by the $\mathbb{B}_m \times \mathbb{B}_n \times \mathbb{B}_p$-equivariant function

$$H[m, n] \times_{H_0^n} H[n, p] \to H[m, p]$$

defined using composition in $\mathscr{V}$.

Tensor decompositions involve families of pairwise disjoint rectangles in the plane. To deal with this we introduce the space D of configurations of "little rectangles" in $\mathbb{R}^2$ which has the same homotopy type as the space C of configurations of points in $\mathbb{R}^2$.

More explicitly, let $D[n]$ denote the space of ordered n-tuples $(R_1, ..., R_n)$ of nonempty pairwise-disjoint (open) standard rectangles $R_i \subset \mathbb{R}^2$. Put

$$D = D[n]/\mathbb{S}_n \quad \text{and} \quad D = \sum_{n \geqslant 0} D_n.$$

The map $D[n] \to C[n]$ taking each n-tuple of rectangles to the n-tuple of centres is a homotopy equivalence which induces a homotopy equivalence $D \to C$. For any space X, define

$$D(X) = \sum_{n \geqslant 0} D[n] \times X^n/\mathbb{S}_n,$$

the space of configurations of "little rectangles" labelled by points of X.

In particular, we have the space $D(C)$ of little rectangles labelled by finite sets of points in $\mathbb{R}^2$. Let $[C] \subset D(C)$ be the subspace where the labelling sets of points are contained in the labelled rectangle; that is

$$[C] = \left\{ (S, \mathscr{R}) \mid S \in C, \mathscr{R} \in \mathscr{D}, S \subset \bigcup_{R \in \mathscr{R}} R \right\}.$$

Then we have the isomorphisms of groupoids

$$\pi_1[C] \simeq \mathbb{B} \int \mathbb{B}.$$

Note that each covering projection $X \to Y$ induces covering projections $C(X) \to C(Y)$ and $D(X) \to D(Y)$.

Each covering space X of C determines a covering space $[X]$ of $[C]$ via the pullback

$$
\begin{array}{ccc}
[X] & \longrightarrow & D(X) \\
\downarrow & \text{p.b.} & \downarrow \\
[C] & \longrightarrow & D(C)
\end{array}
$$

The fibre over $p = (S, \mathscr{R}) \in [C]$ is given by

$$[X]_p = \prod_{R \in \mathscr{R}} X_{S \cap R},$$

where X_A is denoting the fiber of X over $A \in C$.

We also have need of the space

$$[\![C]\!] = \{(S, \mathscr{R}_1, \mathscr{R}_2) \mid (S, \mathscr{R}_1) \in [C] \text{ and } \mathscr{R}_1 \text{ is a refinement of } \mathscr{R}_2\}.$$

($\mathscr{R}_1$ is a *refinement* of $\mathscr{R}_2$ when each rectangle in $\mathscr{R}_1$ is contained in some rectangle in $\mathscr{R}_2$.) We have a diagram of projection maps

$$[\![C]\!] \underset{p_2}{\overset{p_1}{\rightrightarrows}} [C] \xrightarrow{\ p\ } C$$

given by $p(S, \mathscr{R}) = S$, $p_1(S, \mathscr{R}_1, \mathscr{R}_2) = (S, \mathscr{R}_1)$, and $p_2(S, \mathscr{R}_1, \mathscr{R}_2) = (S, \mathscr{R}_2)$.

Each covering space X of C yields a covering space $[\![X]\!]$ of $[\![C]\!]$ whose fibre over $q = (S, \mathscr{R}_1, \mathscr{R}_2)$ is given by

$$[\![X]\!]_q = \prod_{R \in \mathscr{R}_1} X_{S \cap R}.$$

Thus we have a diagram of spaces and maps

$$
\begin{array}{ccccccc}
[\![X]\!] & \xrightarrow{\ \hat{p}_1\ } & [X] & \longrightarrow & D(X) & & X \\
\downarrow & \text{p.b.} & \downarrow & \text{p.b.} & \downarrow & & \downarrow \\
[\![C]\!] & \xrightarrow{\ p_1\ } & [C] & \longrightarrow & D(C) & & C
\end{array}
$$

Suppose $\otimes : [X] \to X$ is any map over $[C] \to C$. For each $(S, \mathscr{R}) \in [C]$ and each $(z_R \mid R \in \mathscr{R}) \in [X]$, where $z_R \in X_{R \cap S}$ for $R \in \mathscr{R}$, we obtain an element

$$\bigotimes_{R \in \mathscr{R}} z_R \in X_S.$$

We say that $\otimes : [X] \to X$ is *associative* when it satisfies the axioms

(i) $\displaystyle\bigotimes_{R \in \{R_1\}} z_R = z_{R_1}$ and

(ii) $\displaystyle\bigotimes_{R' \in \mathscr{R}_2} \left(\bigotimes_{\substack{R \subseteq R' \\ R \in \mathscr{R}_1}} z_R \right) = \bigotimes_{R \in \mathscr{R}_1} z_R.$

Axiom (ii) amounts to commutativity of the square

$$
\begin{array}{ccc}
[\![X]\!] & \xrightarrow{\ \hat{p}_1\ } & [X] \\
{\scriptstyle[\otimes]}\big\downarrow & & \big\downarrow{\scriptstyle\otimes} \\
[X] & \xrightarrow[\ \otimes\ ]{} & X
\end{array}
$$

where, for $q = (S, \mathscr{R}_1, \mathscr{R}_2) \in [\![C]\!]$ and $(z_R \mid R \in \mathscr{R}_1) \in [\![X]\!]_q$, the left side of the square is given by

$$[\otimes](z_R \mid R \in \mathscr{R}_1) = \left(\bigotimes_{R \subseteq R'} z_R \mid R' \in \mathscr{R}_2 \right).$$

In this case we call $\otimes$ a *multiplicative structure* on the covering space X of C.

A covering space X of C corresponds to a functor $F : \mathbb{B} \to \mathrm{Set}$ where $F[n] = X_{\rho(n)}$ and the action of $\mathbb{B}_n$ is from path lifting. Conversely, X can be entirely reconstructed from F via the formula

$$X = \sum_{n \geq 0} U[n] \times F[n] / \mathbb{B}_n.$$

A *multiplicative structure* on a functor $F : \mathbb{B} \to \mathrm{Set}$ is a natural family of functions

$$F[m] \times F[n] \to F[m+n], \qquad (u, v) \mapsto u \otimes v,$$

which is associative with a unit element $i \in F[0]$ and satisfies

$$F[c_{m,n}](u \otimes v) = v \otimes u.$$

(That is, F is a "braided monoidal functor" or "braided lax tensor functor".)

PROPOSITION 3.3.   *The multiplicative structures on X are in bijection with those on F.*

Proof. Let el(F) denote the category of elements of F. Using the multi-plicative structure on F, we can define a unique structure of braided tensor category on el(F) such that the functor el(F) → $\mathbb{B}$ is a braided strict tensor functor. By Theorem 3.2 we obtain a braided strict tensor functor $\otimes : \mathbb{B} \int \text{el}(F) \to \text{el}(F)$ whose restriction to el(F) is the identity functor. Furthermore, we have the associativity property

$$
\begin{array}{ccc}
\mathbb{B} \int \mathbb{B} \int \text{el}(F) & \xrightarrow{\otimes} & \mathbb{B} \int \text{el}(F) \\
{\scriptstyle \mathbb{B} \int \otimes} \downarrow & & \downarrow {\scriptstyle \otimes} \\
\mathbb{B} \int \text{el}(F) & \xrightarrow{\otimes} & \text{el}(F).
\end{array}
$$

Also $F: \mathbb{B} \to \text{Set}$ extends to a unique braided strict tensor functor $\hat{F}: \mathbb{B} \int \mathbb{B} \to \text{Set}$, which, in turn, extends to a unique braided strict tensor functor $\hat{\hat{F}}: \mathbb{B} \int \mathbb{B} \int \mathbb{B} \to \text{Set}$. Note that

$$
\text{el}(\hat{F}) = \mathbb{B} \int \text{el}(F) \qquad \text{and} \qquad \text{el}\,\hat{\hat{F}} = \mathbb{B} \int \mathbb{B} \int \text{el}(F).
$$

Thus we have a commutative diagram of functors

$$
\begin{array}{ccccc}
\text{el}(\hat{\hat{F}}) & \rightrightarrows & \text{el}(\hat{F}) & \xrightarrow{\otimes} & \text{el}(F) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{B} \int \mathbb{B} \int \mathbb{B} & \rightrightarrows & \mathbb{B} \int \mathbb{B} & \longrightarrow & \mathbb{B}
\end{array}
$$

(the top and bottom parts of the forks are producing two commutative squares). Given the equivalences

$$
\pi_1(C) \simeq \mathbb{B}, \qquad \pi_1[C] \simeq \mathbb{B} \int \mathbb{B}, \qquad \text{and} \qquad \pi_1[\![C]\!] = \mathbb{B} \int \mathbb{B} \int \mathbb{B},
$$

it is routine to check that the diagram transforms into a commutative diagram

$$
\begin{array}{ccccc}
[\![X]\!] & \overset{\hat{p}_1}{\underset{[\otimes]}{\rightrightarrows}} & [X] & \xrightarrow{\otimes} & X \\
\downarrow & & \downarrow & & \downarrow \\
[\![C]\!] & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & [C] & \longrightarrow & C.
\end{array} \qquad \blacksquare
$$

The reader will have no difficulty in modifying the above constructions and Proposition 3.3 to the case where X is a covering space of $C \times C$ with associated covering space $[X]$ of

$$
[C \times C] = \left\{ (S, T, R) \mid S, T \in C, \mathscr{R} \in D, S, T \subset \bigcup_{R \in \mathscr{R}} R \right\},
$$

and F is the corresponding functor $\mathbb{B} \times \mathbb{B} \to \text{Set}$.

In particular, we wish to apply Proposition 3.3 to the covering space $\bar{\mathscr{V}}_1$ of $C \times C$ with covering projection the composite

$$\bar{\mathscr{V}}_1 \xrightarrow{(d_0, d_1)} C(\mathscr{V}_0^{\cdot}) \times C(\mathscr{V}_0^{\cdot}) \to C \times C,$$

(where $C(\mathscr{V}_0^{\cdot}) \to C$ is induced by the projection $\mathscr{V}_0 \to 1$). The corresponding functor

$$H: \mathbb{B} \times \mathbb{B} \to \text{Set}, \qquad (m, n) \mapsto H[m, n],$$

has a canonical multiplicative structure

$$H[m, n] \times H[m', n'] \to H[m' + m', n + n']$$

$$(A \xrightarrow{f} B, C \xrightarrow{g} D) \mapsto (A \otimes C \xrightarrow{f \otimes g} B \otimes D).$$

Thus we obtain a corresponding multiplicative structure

$$\bar{\otimes} : [\bar{\mathscr{V}}_1] \to \bar{\mathscr{V}}_1$$

on $\bar{\mathscr{V}}_1$. The advantage of this construction is that we now have the continuity and associativity of $\bar{\otimes}$. However, an explicit description will be given below.

First note that arrows $(A_s | s \in S) \to (B_t | t \in T)$ in $\bar{\mathscr{V}}$ are determined uniquely by triples (α, f, β), where $\alpha: M \to S$, $\beta: N \to T$ are arrows in $\pi_1(C)$ with M, N frontal, and

$$f: \bigotimes_{i \in M} A_{\alpha(i)} \to \bigotimes_{j \in N} B_{\beta(j)}$$

is an arrow of $\mathscr{V}$. This is because each frontal $M \subset \mathbb{R}^2$ has a canonical path $M \to \rho(m)$ obtained by projecting M onto the x-axis and sliding the points along the x-axis.

Elements ϕ of $[\bar{\mathscr{V}}_1]$ are families $[\phi_R]_{R \in \mathscr{R}}$ of arrows

$$\phi_R : (A_s | s \in S \cap R) \to (B_t | t \in T \cap R)$$

in $\bar{\mathscr{V}}$ indexed by $R \in D$, where $S = \bigcup_{R \in \mathscr{R}} S \cap R$, $T = \bigcup_{R \in \mathscr{R}} T \cap R \in C$. The arrow

$$\bar{\otimes} \phi = \bar{\otimes}_{R \in \mathscr{R}} \phi_R : (A_s | s \in S) \to (B_t | t \in T)$$

can be described as follows. Choose any arrow $(\alpha, \beta, \gamma): (M, N, \mathscr{R}_0) \to (S, T, \mathscr{R})$ in $\pi_1([C \times C])$, where M, N are subsets of the x-axis and $\mathscr{R}_0$ is a set of pairwise disjoint rectangles with centres on the x-axis (so that $\mathscr{R}_0$

is naturally linearly ordered). The arrows $\alpha: M \to S$, $\beta: N \to T$ in $\pi_1(C)$ restrict to arrows $\alpha_R: M_R \to S \cap R$, $\beta_R: N_R \to T \cap R$, with $M_R \subseteq M$, $N_R \subseteq N$. So ϕ_R is represented by a triple (α_R, f_R, β_R) where

$$f_R: \bigotimes_{i \in M_R} A_{\alpha_R(i)} \to \bigotimes_{j \in N_R} B_{\beta_R(j)}$$

is in $\mathscr{V}$. Then $\bar{\otimes} \phi$ is represented by the triple

$$\left(\alpha, \bigotimes_{\rho \in \mathscr{R}_0} f_{\gamma(\rho)}, \beta \right).$$

From the above description it is easy to deduce the following *functoriality* of $\bar{\otimes}$:

$$\bar{\bigotimes_{R \in \mathscr{R}}} \psi_R \circ \bar{\bigotimes_{R \in \mathscr{R}}} \phi_R = \bar{\bigotimes_{R \in \mathscr{R}}} \psi_R \circ \phi_R.$$

Let us return to the definition of $\bar{v}(\Gamma)$. For elementary $\Gamma = \bigotimes_{R \in \mathscr{R}} \Gamma^R$, we have an element

$$(\bar{v}(\Gamma^R) \mid R \in \mathscr{R}) \in [\bar{\mathscr{V}_1}]$$

and we define

$$\bar{v}(\Gamma) = \bar{\bigotimes_{R \in \mathscr{R}}} \bar{v}(\Gamma^R).$$

That this definition is independent of the tensor decomposition follows from the associativity of $\bar{\otimes}$. Furthermore, using functoriality, we can prove the analogue of Propositions 1.1 and 2.1.

PROPOSITION 3.4. *If u is a regular level for an elementary diagram Γ between levels a and b then $\Gamma[a, u]$, $\Gamma[u, b]$ are elementary, and*

$$\bar{v}(\Gamma) = \bar{v}[\Gamma[u, b]) \circ \bar{v}(\Gamma[a, u]).$$

For any progressive polarized 3D diagram Γ between levels a and b, we now define

$$\bar{v}(\Gamma) = \bar{v}(\Gamma[u_{n-1}, u_n]) \circ \cdots \circ \bar{v}(\Gamma[u_0, u_1])$$

where $a = u_0 < u_1 < \cdots < u_n = b$ are regular levels for Γ such that each layer $\Gamma[u_{i-1}, u_i]$ is elementary for $1 \leqslant i \leqslant n$. (The existence and independence of the regular levels is just as for the plane case in Chapter 1.)

Before presenting the main result of this chapter, we shall point out that arrows $\gamma: S \to T$ in the groupoid $\pi_1(C)$ induce canonical isomorphisms in

$\mathscr{V}$. Suppose $(A_s | s \in S)$, $(B_t | t \in T)$ are objects of $\mathscr{V}$ such that $B_{\gamma(s)} = A_s$. There is an isomorphism

$$\langle \gamma \rangle: (A_s | s \in S) \overset{\sim}{\to} (B_t | t \in T)$$

in $\mathscr{V}$ determined by the triple $(\alpha, f, \gamma \circ \alpha)$, where $\alpha: M \to S$ is any arrow of $\pi_1(C)$ with M frontal and f is the identity arrow of $\bigotimes_{i \in M} A_{\alpha(i)}$.

THEOREM 3.5. *For any deformation*

$$h: \Gamma \times [0, 1] \to \mathbb{R}^2 \times [a, b]$$

of progressive polarised 3D diagrams in a braided tensor category $\mathscr{V}$, *the square*

$$(v(s) | s \in \operatorname{dom} \Gamma(0)) \overset{\sim}{\longrightarrow} (v(s) | s \in \operatorname{dom} \Gamma(1))$$

$$\left\downarrow{\scriptstyle \bar{v}(\Gamma(0))} \qquad\qquad\qquad\qquad \right\downarrow{\scriptstyle \bar{v}(\Gamma(1))}$$

$$(v(t) | t \in \operatorname{cod} \Gamma(0)) \overset{\sim}{\longrightarrow} (v(t) | t \in \operatorname{cod} \Gamma(1))$$

in $\mathscr{V}$ *commutes, where the horizontal isomorphisms are induced by the paths* $\tau \mapsto \operatorname{dom} \Gamma(\tau)$, $\tau \mapsto \operatorname{cod} \Gamma(\tau)$ *in* C.

Proof. By compactness and connectedness of $[0, 1]$, it suffices to show that, for all $\tau_0 \in [0, 1]$, the square

$$(v(s) | s \in \operatorname{dom} \Gamma(\tau_0)) \overset{\sim}{\longrightarrow} (v(s) | s \in \operatorname{dom} \Gamma(\tau))$$

$$\left\downarrow{\scriptstyle \bar{v}(\Gamma(\tau_0))} \qquad\qquad\qquad\qquad \right\downarrow{\scriptstyle \bar{v}(\Gamma(\tau))}$$

$$(v(t) | t \in \operatorname{cod} \Gamma(\tau_0)) \overset{\sim}{\longrightarrow} (v(t) | t \in \operatorname{cod} \Gamma(\tau))$$

commutes whenever τ is close enough to τ_0. As in the proof of Theorem 1.2, by choosing appropriate regular levels for $\Gamma(\tau_0)$ and restricting attention to the layers, we can assume $\Gamma(\tau_0)$ is elementary. So $\Gamma(\tau_0) = \bigotimes_{R \in \mathscr{R}} \Gamma(\tau_0)^R$ where each $\Gamma(\tau_0)^R$ is either prime or invertible. For τ near τ_0 it follows that $\Gamma(\tau) = \bigotimes_{R \in \mathscr{R}} \Gamma(\tau)^R$, where $\Gamma(\tau)^R$ is of the same nature as $\Gamma(\tau_0)^R$. The deformation h provides a path

$$(\bar{v}(\Gamma(\tau_0)^R) | R \in \mathscr{R}) \to (\bar{v}(\Gamma(\tau)^R) | R \in \mathscr{R})$$

in $[\mathscr{V}_1]$. By continuity of $\bigotimes: [\mathscr{V}_1] \to \mathscr{V}_1$, it follows that, for τ near τ_0, we have $\bar{v}(\Gamma(\tau))$ near $\bar{v}(\Gamma(\tau_0))$. So the two legs of the square are close in $\mathscr{V}_1$. But $(d_0, d_1): \mathscr{V}_1 \to \bar{V}_0 \times \mathscr{V}_0$ is a covering projection and so has discrete fibres. Since the two legs of the square have the same source and target in $\mathscr{V}_0$, their closeness implies their equality. ∎

A diagram Γ is called *anchored* when dom Γ and cod Γ are frontal. In this case, $\bar{v}(\Gamma)$ determines a map

$$v(\Gamma): \bigotimes_{s \in \text{dom}\, \Gamma} v(s) \to \bigotimes_{t \in \text{cod}\, \Gamma} v(t)$$

in $\mathscr{V}$. A deformation of diagrams is *anchored* when $\Gamma(t)$ is anchored for all $t \in [0, 1]$.

COROLLARY 3.6. *If $h: \Gamma \times [0, 1] \to \mathbb{R}^2 \times [a, b]$ is an anchored deformation or progressive polarised 3D diagrams in a braided tensor category $\mathscr{V}$ then*

$$v(\Gamma(0)) = v(\Gamma(1)).$$

3. *Free Braided Tensor Categories*

Suppose $\mathscr{V}$, $\mathscr{W}$ are braided tensor categories. A tensor functor $F: \mathscr{V} \to \mathscr{W}$ is called *braided* [17, 18] when it respects the braiding (in the same way that a symmetric tensor functor respects the symmetry; see Chapter 2, Section 2). Write BTen($\mathscr{V}, \mathscr{W}$) for the category of braided tensor functors $\mathscr{V} \to \mathscr{W}$ and tensor transformations between them. (In fact, BTen($\mathscr{V}, \mathscr{W}$) = STen($\mathscr{V}, \mathscr{W}$) when $\mathscr{V}$, $\mathscr{W}$ are both symmetric.)

DEFINITION 3.3. A braided tensor category $\mathscr{F}$ is said to be *free, on the tensor scheme $\mathscr{D}$*, when there exists an object N of $[\mathscr{D}, \mathscr{F}]$ such that the functor

$$- \circ N: \text{BTen}(\mathscr{F}, \mathscr{V}) \to [\mathscr{D}, \mathscr{V}]$$

is an equivalence of categories for all braided tensor categories $\mathscr{V}$.

A 3D graph Γ will be called *boxed* when it is between levels -1 and $+1$, and it is contained in the cube $(-1, 1)^2 \times [-1, 1]$. Write $\Gamma: m \to n$ when Γ is anchored and m, n are the cardinalities of dom Γ, cod Γ.

We shall use the functions $\gamma, \tau: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\gamma(x, y, t) = (x, y, \tfrac{1}{3}t), \qquad \tau(x, y, t) = (\tfrac{1}{2}x, y, t)$$

and the points $e_1 = (1, 0, 0)$, $e_3 = (0, 0, 1) \in \mathbb{R}^3$.

The *tensor product* $\Gamma^1 \otimes \Gamma^2$ of two boxed 3D graphs Γ^1, Γ^2 is defined to be $\tau((\Gamma^1 - e_1) \cup (\Gamma^2 + e_1))$ (as in the plane case of Chapter 1, Section 4).

Suppose $\Gamma: m \to n$, $\Omega: n \to p$ are boxed anchored 2D graphs. The *composite* $\Omega \circ \Gamma: m \to p$ is defined just as in the plane case (Chapter 1, Section 4) with e_2 replaced by e_3; however, the meetings across $t = \tfrac{1}{3}$ and

$t = \frac{2}{3}$ of the edges and line segments must be systematically made *smooth*. The order on the domains and codomains is given here by front projection.

The notion of a *progressive polarised* 3D *diagram* (Γ, v) *in a tensor scheme* should now be clear. The *domain* and *codomain* are defined as words just as before, provided (Γ, v) is anchored. Tensor product and composition are extended from graphs to diagrams just as before.

For each tensor category $\mathscr{D}$, there is a strict tensor category $\mathbb{F}_B(\mathscr{D})$ defined in the same way as $\mathbb{F}(\mathscr{D})$ except that "boxed progressive plane diagrams" are replaced by "boxed anchored progressive polarised 3D diagrams", and "deformation classes" are replaced by "anchored deformation classes".

We now describe a braiding on $\mathbb{F}_B(\mathscr{D})$. For this we use the canonical functor

$$\mathbb{B} \int \text{obj } \mathscr{D} \to \mathbb{F}_B(\mathscr{D})$$

which is an isomorphism onto the subcategory of $\mathbb{F}_B(\mathscr{D})$ consisting of all the objects and arrows which are anchored deformation classes of invertible 3D diagrams. The braiding for $\mathbb{B} \int \text{obj } \mathscr{D}$ (Section 1) carries over via the functor to a candidate for a braiding for $\mathbb{F}_B(\mathscr{D})$. All that remains to check is the naturality with respect to all the arrows of $\mathbb{F}_B(\mathscr{D})$. Every arrow is a composite of tensor products of classes of invertible diagrams and prime diagrams. So all that remains is the prime case, and this is indicated in Fig. 3.6.

The pattern of proof established for Theorems 1.3 and 2.3 can be followed to yield

THEOREM 3.7. $\mathbb{F}_B(\mathscr{D})$ *is the free braided tensor category on the tensor scheme* $\mathscr{D}$.
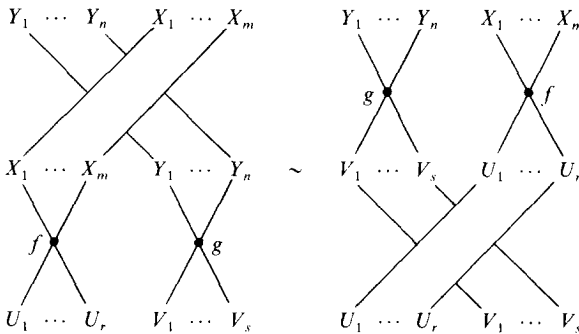

FIGURE 3.6

CHAPTER 4. BALANCED TENSOR CATEGORIES AND RIBBONS

1. *Balanced Tensor Categories*

Suppose $\mathscr{V}$ is a braided tensor category.

Recall [16, 36] that a *twist* for $\mathscr{V}$ consists of a natural family of automorphisms

$$\theta = \theta_A : A \to A$$

in $\mathscr{V}$ such that $\theta_I = 1_I$ and the following square commutes:

$$
\begin{array}{ccc}
A \otimes B & \xrightarrow{c_{A,B}} & B \otimes A \\
{\scriptstyle \theta_{A \otimes B}} \downarrow & & \downarrow {\scriptstyle \theta_B \otimes \theta_A} \\
A \otimes B & \xleftarrow{\ c_{B,A}\ } & B \otimes A
\end{array}
$$

A tensor category together with a distinguished braiding and a distinguished twist is called *balanced.*

EXAMPLE 4. A *twist* for the braided bialgebra (A, γ) is a central element $\tau \in A$ such that the following two equations hold:

$$\varepsilon(\tau) = 1 \in K \qquad \text{and} \qquad \Delta(\tau) = \gamma \cdot (\tau \otimes \tau) \cdot \gamma_{21} \in A \otimes A.$$

Twists for (A, γ) are in bijection with twists for the braided tensor category $\mathrm{Mod}(A)$.

A tensor functor $F : \mathscr{V} \to \mathscr{W}$ between balanced tensor categories $\mathscr{V}$, $\mathscr{W}$ is called *balanced* when it is braided (Chapter 3, Section 1) and

$$F\theta_A = \theta_{FA}.$$

Let $\tilde{C}$ denote the space $C(\mathbb{S}^1)$ (see Chapter 3, Section 2) of finite sets of distinct points in the plane $\mathbb{R}^2$ labelled by points of the unit circle $\mathbb{S}^1$. Points S of $\tilde{C}$ are *framed subsets* of $\mathbb{R}^2$ depicted as in Fig. 4.1 (where the
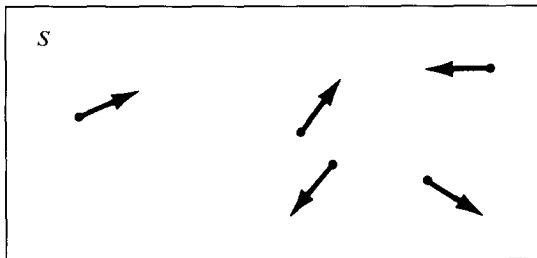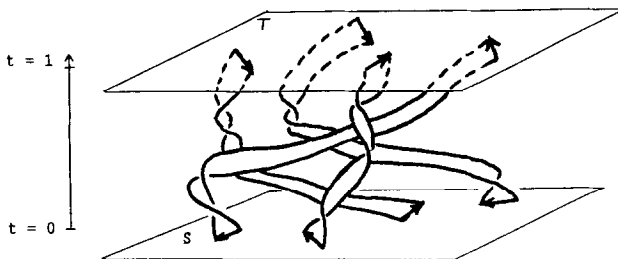

FIGURE 4.1

FIGURE 4.2

magnitude of the directed line segment is irrelevant, and made small enough so that it does not meet any of the others). A path ω from S to T in $\tilde{C}$ can be depicted as in Fig. 4.2.

Let $\tilde{\rho}: \mathbb{N} \to \tilde{C}$ be the function for which $\tilde{\rho}(n)$ is the set of points $1, 2, ..., n$ on the x-axis all labeled by $1 \in \mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ (Fig. 4.3).

The category $\tilde{\mathbb{B}}$ *of braids on ribbons* is defined to be the groupoid $\pi_1(\tilde{C}, \tilde{\rho})$. Let $s_i \in \tilde{\mathbb{B}}_n = \tilde{\mathbb{B}}(n, n)$ be the homotopy class of the path depicted in Fig. 4.4 for $1 \leqslant i \leqslant n - 1$, and in Fig. 4.5 for $i = n$. A presentation for the group $\tilde{\mathbb{B}}_n$ is given by the generators s_i for $1 \leqslant i \leqslant n$ and the relations (A1), (A2) of Chapter 3, Section 1 together with the relation

$$s_{n-1} s_n s_{n-1} s_n = s_n s_{n-1} s_n s_{n-1}.$$

The group $\tilde{\mathbb{B}}_n$ is the wreath product

$$\tilde{\mathbb{B}}_n = \mathbb{Z}^n \rtimes \mathbb{B}_n$$

obtained from the canonical homomorphisms $\mathbb{B}_n \to \mathbb{S}_n$, since a braid on ribbons is completely described by a braid on strings labelled by integers $n \in \mathbb{Z}$ indicating how many complete turns are performed on each ribbon of the braid. This leads us to the identification

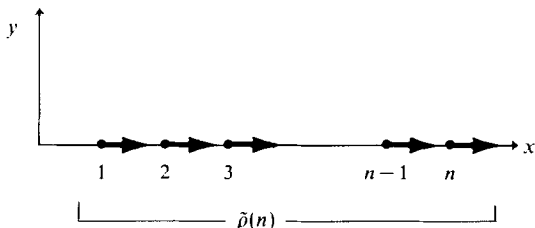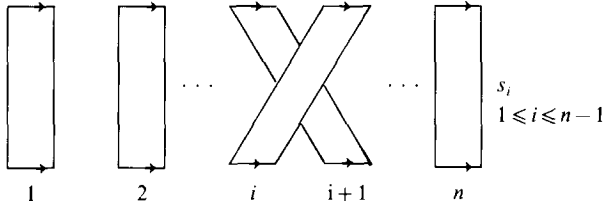$$\tilde{\mathbb{B}} = \mathbb{B} \wr \mathbb{Z},$$

FIGURE 4.3

FIGURE 4.4

where $\mathbb{Z}$ is the additive group of integers (as a one-object category). Hence $\tilde{\mathbb{B}}$ is equipped with a braided strict tensor category structure: it is the free such structure generated by a single object with a single automorphism (Theorem 3.2).

The tensor category $\tilde{\mathbb{B}}$ is balanced. The twist $\theta_n : n \to n$ can be viewed by taking the diagram for the identity arrow $n \to n$ in $\tilde{\mathbb{B}}$ consisting of n straight vertical ribbons, fixing the horizontal ends of the ribbons to two horizontal rods, and rotating the top rod through an angle of 2π in the right-hand screw direction with thumb pointing up (Fig. 4.6). In other words, $\theta_1 = s_1 \in \tilde{\mathbb{B}}_1$ and

$$\theta_{n+1} = c_{1,n}(\theta_1 \otimes \theta_n) c_{n,1}.$$

THEOREM 4.1. $\tilde{\mathbb{B}}$ *is the free balanced strict tensor category on one generating object.*

Proof. We must modify the proof of Theorem 3.1 for $\tilde{\mathbb{B}}$ in place of $\mathbb{B}$, and for $\mathscr{V}$ balanced. To define a balanced $F : \mathbb{B} \to \mathscr{V}$ we require the extra equation

$$Fs_n = 1_{F(n-1)} \otimes \theta_A. \quad \blacksquare$$

For each category $\mathscr{A}$, there is the category $\tilde{\mathbb{B}} \int \mathscr{A}$ of *braids on ribbons labelled by arrows of* $\mathscr{A}$. In fact

$$\tilde{\mathbb{B}} \int \mathscr{A} = \mathbb{B} \int \mathbb{Z} \int \mathscr{A} = \mathbb{B} \int (\mathbb{Z} \times \mathscr{A}).$$
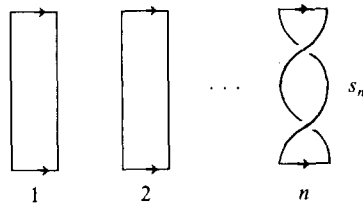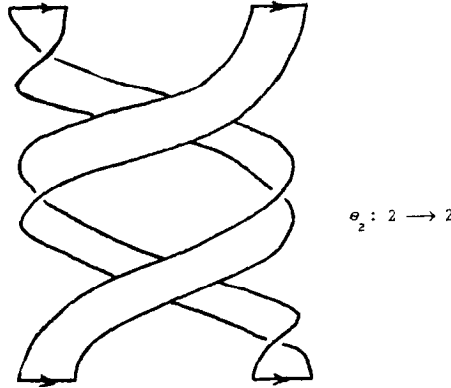

FIGURE 4.5

$e_2 : 2 \longrightarrow 2$

FIGURE 4.6

THEOREM 4.2. $\tilde{\mathbb{B}} \int \mathscr{A}$ is the free balanced strict tensor category generated by $\mathscr{A}$.

2. Progressive Polarised Framed 3D Diagrams

This section concerns the diagrams for balanced tensor categories.
For each point λ of the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$, we shall use the λ-projection

$$e_\lambda : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \xi \mapsto \langle \lambda, \xi \rangle \lambda;$$

its image is the line joining λ and the origin. The function

$$e_\lambda \times 1 : \mathbb{R}^2 \times [a, b] \to \mathbb{R}^2 \times [a, b], (\xi, t) \mapsto (e_\lambda(\xi), t)$$

will also be the λ-projection.
A *framing* for a graph Γ is a continuous function $fm : \Gamma \to \mathbb{S}^1$. A graph equipped with a framing is called a *framed graph*.

DEFINITION 4.1.   A *progressive polarised (smooth) framed 3D graph* (*between the levels a and b*) is a smooth embedded graph Γ (with boundary) in $\mathbb{R}^2 \times [a, b]$, equipped with a framing, such that

   (i) $\partial \Gamma = \Gamma \cap (\mathbb{R}^2 \times \{a, b\})$ and loose edges meet $\mathbb{R}^2 \times \{a, b\}$ transversally,

   (ii)   the second projection

$$pr_2 : \mathbb{R}^2 \times [a, b] \to [a, b]$$

is a smooth embedding on each connected component of $\Gamma - \Gamma_0$,

   (iii)   for any pair of distinct edges $\gamma_1(s)$, $\gamma_2(s)$ with $\gamma_1(0) = \gamma_2(0) = x$

$[\gamma_1(1) = \gamma_2(1) = x]$ and $fm(x) = \lambda$, the unit tangent vectors at $s = 0$ $[s = 1]$ of the λ-projected curves $(e_\lambda \times 1)\gamma_1(s)$ and $(e_\lambda \times 1)\gamma_2(s)$ are distinct.

The above Definition 4.1 differs from Definition 3.1 only in the presence of the framing and in condition (iii). Each $3D$ graph can be given the framing which is constant at $(1, 0) \in \mathbb{S}^1$; in that case the two conditions (iii) coincide.

The comments which follow Definition 3.1 can be made here with minor obvious changes.

A deformation of progressive polarised framed 3D graphs

$$h: \Gamma \times [0, 1] \to \mathbb{R}^2 \times [a, b]$$

is defined as in Definition 3.2 except that Γ should now be framed and each $\Gamma(t)$, with the induced framing, should be a progressive polarised *framed 3D* graph.

Valuations are defined just as in Chapter 3, Section 2. This gives the notion of *progressive polarised framed 3D diagram*. The domain and codomain of such a diagram are families of objects $\mathscr{V}$ indexed by framed subsets of the plane.

Henceforth, $\mathscr{V}$ will denote a balanced tensor category. We write $\underline{\alpha}: S \to T$ for the bijection determined by an arrow $\alpha: S \to T$ in the fundamental groupoid $\pi_1(\tilde{C})$. We write

$$\langle \alpha \rangle: A_{\underline{\alpha}(1)} \otimes \cdots \otimes A_{\underline{\alpha}(n)} \to A_i \otimes \cdots \otimes A_n$$

for the value of $\otimes: \tilde{\mathbb{B}} \int \mathscr{V} \to \mathscr{V}$ at $\alpha \in \tilde{\mathbb{B}}_n$ labelled by identity arrows of the objects A_i of $\mathscr{V}$.

There is a framed version of the category $\tilde{\mathscr{V}}$ of Chapter 3 which we denote by $\tilde{\mathscr{V}}$. Objects of $\tilde{\mathscr{V}}$ are families $(A_s | s \in S)$ of objects A_s of $\mathscr{V}$ indexed by framed subsets $S \in \tilde{C}$ of $\mathbb{R}^2$. Each such object has an associated clique

$$\left(\bigotimes_{k=1}^{n} A_{\underline{\alpha}(k)} \,\middle|\, \alpha: \tilde{\rho}(n) \to S \text{ in } \pi_1(\tilde{C}) \right)$$

in $\tilde{\mathscr{V}}$. An arrow in $\tilde{\mathscr{V}}$ is a map of the associated cliques. The inclusion $\mathscr{V} \to \tilde{\mathscr{V}}$ is an equivalence of categories.

The topological considerations of Chapter 3, Section 2 will now be modified to account for the framings.

Let $\tilde{C}[n]$ denote the space of partial maps from $\mathbb{R}^2$ to the unit circle $\mathbb{S}^1$ with domain of cardinality n. Putting $C[n] \times (S^1)^n = \tilde{C}[n]$, we have

$$\tilde{C}_n = \tilde{C}[n]/\mathbb{S}_n.$$

Let $\tilde{U}[n]$ denote the universal covering space of $\tilde{C}_n$ at the basepoint $\tilde{\rho}(n)$. We have

$$\tilde{C}_n = \tilde{U}[n]/\tilde{\mathbb{B}}_n \qquad \text{and} \qquad \tilde{C}[n] = \tilde{U}[n]/\tilde{\mathbb{P}}_n,$$

where $\tilde{\mathbb{P}}_n$ is the kernel of $\tilde{\mathbb{B}}_n \to \mathbb{S}_n$. For any space X, put

$$\tilde{C}(X) = \sum_{n \geqslant 0} \tilde{C}[n] \times X^n/\mathbb{S}_n = \sum_{n \geqslant 0} \tilde{U}[n] \times X^n/\tilde{\mathbb{B}}_n.$$

In fact, $\tilde{C} = C(\mathbb{S}^1)$ and $\tilde{C}(X) = C(\mathbb{S}^1 \times X)$ (see Section 3).

We write $\tilde{H}[m, n]$ for $H[m, n]$ acted on by $\tilde{\mathbb{B}}_m \times \tilde{\mathbb{B}}_n$ (now that $\mathscr{V}$ is balanced). We obtain spaces

$$\tilde{\mathscr{V}}_0 = \tilde{C}(\mathscr{V}_0) = \sum_n \tilde{U}[n] \times \mathscr{V}_0^n/\tilde{\mathbb{B}}_n$$

and

$$\tilde{\mathscr{V}}_1 = \sum_{m,n} \tilde{U}[m] \times \tilde{U}[n] \times \tilde{H}[m, n]/\tilde{\mathbb{B}}_m \times \tilde{\mathbb{B}}_n$$

which constitute a topological category $\tilde{\mathscr{V}}$ with

$$(d_0, d_1): \tilde{\mathscr{V}}_1 \to \tilde{\mathscr{V}}_0 \times \tilde{\mathscr{V}}_0$$

a covering projection.

The discussion of multiplicative structures (Proposition 3.3) will now be modified for the purposes of this Chapter.

Let $r_t: \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear map represented by the rotation matrix

$$\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix},$$

where $\phi = 2\pi t$. Each $S \in \tilde{C}_n$ can be regarded as a subset $S \cap \mathbb{R}^2 \times \mathbb{S}^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$. The image $(r_t \times r_t)(S)$ of S under the map

$$r_t \times r_t: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$$

is again in $\tilde{C}_n$. This gives a path

$$\gamma_n: [0, 1] \to \tilde{C}_n, \qquad t \mapsto (r_t \times r_t)(\tilde{\rho}(n))$$

in $\tilde{C}_n$ which is, in fact, a loop at $\tilde{\rho}(n)$. The homotopy class of γ_n is $\theta_n \in \pi_1(\tilde{C}, \tilde{\rho}(n)) = \tilde{\mathbb{B}}_n$.

A covering space X of $\tilde{C}$ is called *special* when each path γ_n in $\tilde{C}$ lifts to a closed path (a loop) in X. *Multiplicative structures* on such an X are defined, with the obvious changes, as in Chapter 3 and as indicated by the following diagram:

$$
\begin{array}{ccccc}
[\![X]\!] & \underset{[\otimes]}{\overset{\hat{p}_1}{\rightrightarrows}} & [X] & \overset{\otimes}{\longrightarrow} & X \\
\downarrow & & \downarrow & & \downarrow \\
[\![\tilde{C}]\!] & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & [\tilde{C}] & \longrightarrow & \tilde{C}.
\end{array}
$$

We point out the equivalences of groupoids

$$\pi_1(\tilde{C}) \simeq \mathbb{B}, \qquad \pi_1([\tilde{C}]) \simeq \mathbb{B} \wr \tilde{\mathbb{B}}, \qquad \pi_1([\![\tilde{C}]\!]) \simeq \mathbb{B} \wr \mathbb{B} \wr \tilde{\mathbb{B}}.$$

A *multiplicative structure* on a functor $F: \tilde{\mathbb{B}} \to \text{Set}$ is defined just as for a functor $\mathbb{B} \to \text{Set}$ (Chapter 3, Section 2) except that we now add the condition

$$F[\theta_n] = 1_{F[n]} : F[n] \to F[n].$$

PROPOSITION 4.3. *The multiplicative structures on a special covering space X of $\tilde{C}$ are in bijection with those on the corresponding functor $F: \tilde{\mathbb{B}} \to \text{Set}$.*

The functor

$$\tilde{H}: \tilde{\mathbb{B}} \times \tilde{\mathbb{B}} \to \text{Set}, \qquad (m, n) \mapsto \tilde{H}[m, n]$$

is multiplicative since $\tilde{H}[\theta_m, \theta_n]$ is the identity map. By Proposition 4.3, we obtain a corresponding multiplicative structure

$$\tilde{\otimes}: [\tilde{\mathscr{V}}_1] \to \tilde{\mathscr{V}}_1$$

on $\tilde{\mathscr{V}}_1$.

A framed subset M of $\mathbb{R}^2$ is called *frontal* when it is frontal as a mere subset and the framing is constant at $(1, 0) \in \mathbb{S}^1$. Arrows in $\tilde{\mathscr{V}}$ can be represented by triples (α, f, β) where $\alpha: M \to S$, $\beta: N \to T$ are arrows in $\pi_1(\tilde{C})$ with M, N frontal and f is an arrow in $\mathscr{V}$ with appropriate source and target (compare Chapter 3, Section 2). With this, an explicit description of $\tilde{\otimes}: [\tilde{\mathscr{V}}_1] \to \tilde{\mathscr{V}}_1$ can be given and the *functoriality* proved.

The *value* $\tilde{v}(\Gamma) \in \tilde{\mathscr{V}}_1$ of a progressive polarised framed 3D diagram Γ can now be constructed as in the unframed case (Chapter 3).

Call Γ *invertible* when it has no inner nodes. Such a Γ determines

an arrow $\gamma: \text{dom } \Gamma \to \text{cod } \Gamma$ in $\pi_1(\tilde{C})$ with $v_0 \circ \gamma = v_0$. Using this (as in Chapter 3, Section 2), we obtain the *value* of Γ as an arrow

$$\tilde{v}(\Gamma): (v_0(s) \mid s \in \text{dom } \Gamma) \to (v_0(t) \mid t \in \text{cod } \Gamma)$$

in $\tilde{\mathscr{V}}$.

Call Γ *prime* when it is connected, has precisely one inner node e, the λ-projection $e_\lambda \times 1$ is injective on Γ where $\lambda = \text{fm}(e)$ and moreover $\langle \lambda, \mu \rangle > 0$ for all values $\mu \in \mathbb{S}'$ of the framing of Γ. In the particular case where $\lambda = (1, 0)$ the diagram Γ is in a frontal position and its *value* $\tilde{v}(\Gamma) \in \tilde{V}$, can be defined as in Chapter 3, Section 2. In the general case, let r be a rotation of the plane such that $r\lambda = (1, 0)$. The *value* $\tilde{v}(\Gamma)$ can be defined by the formula

$$\tilde{v}(\Gamma) = r^{-1}\tilde{v}(r\Gamma).$$

Note that this formula is meaningful since the rotation group of the plane is acting on any special covering of $\tilde{C}$ or $\tilde{C} \times \tilde{C}$.

Call Γ *elementary* when it has a tensor decomposition

$$\Gamma = \bigotimes_{R \in \mathscr{R}} \Gamma^R$$

in which each Γ^R is either invertible or prime. Define

$$\tilde{v}(\Gamma) = \overset{\sim}{\bigotimes_{R \in \mathscr{R}}} \tilde{v}(\Gamma^R).$$

This is independent of the choices of $\mathscr{R}$ by associativity of $\tilde{\bigotimes}$. Proposition 3.4 holds with $\bar{v}$ replaced by $\tilde{v}$.

For an arbitrary Γ, define

$$\tilde{v}(\Gamma) = \tilde{v}(\Gamma[u_{n-1}, u_n]) \circ \cdots \circ \tilde{v}(\Gamma[u_0, u_1])$$

where $a = u_0 < u_1 < \cdots < u_n = b$ are regular levels for Γ such that each layer $\Gamma[u_{i-1}, u_i]$ is elementary for $1 \leqslant i \leqslant n$. The existence of such regular levels uses the continuity of the framing at the inner nodes.

THEOREM 4.4. *Theorem 3.5 holds with Γ framed, $\mathscr{V}$ balanced, and $\bar{v}$, $\mathscr{V}$, C replaced by $\tilde{v}$, $\tilde{\mathscr{V}}$, $\tilde{C}$, respectively.*

We also have the obvious modification of Corollary 3.6.

3. *Free Balanced Tensor Categories*

Suppose $\mathscr{V}$, $\mathscr{W}$ are balanced tensor categories. A tensor functor $F: \mathscr{V} \to \mathscr{W}$ is called *balanced* when it is braided and has $F(\theta_A) = \theta_{FA}$ for all

objects A of $\mathscr{V}$. Write $\tilde{\text{B}}\text{Ten}(\mathscr{V}, \mathscr{W})$ for the category of balanced tensor functors $\mathscr{V} \to \mathscr{W}$ and tensor transformations between them.

DEFINITION 4.2. A balanced tensor category $\mathscr{F}$ is *free, on the tensor scheme* $\mathscr{D}$, when there exists an object N of $[\mathscr{D}, \mathscr{F}]$ such that the functor

$$\circ N: \tilde{\text{B}}\text{Ten}(\mathscr{F}, \mathscr{V}) \to [\mathscr{D}, \mathscr{V}]$$

is an equivalence of categories for all balanced tensor categories $\mathscr{V}$.

We now proceed as in Chapter 3 Section 3 introducing framing in the obvious way. This leads to the strict tensor category $\mathbb{F}_{\tilde{B}}(\mathscr{D})$ whose arrows are deformation classes of boxed anchored progressive polarised framed $3D$ diagrams.

There is a canonical functor

$$\tilde{\text{B}}\text{-}\int \text{obj}\, \mathscr{D} \to \mathbb{F}_{\tilde{B}}(\mathscr{D})$$

onto the classes of invertible diagrams. Using this, we carry the twist of $\tilde{\text{B}}\int \text{obj}\, \mathscr{D}$ to obtain a twist for $\mathbb{F}_{\tilde{B}}(\mathscr{D})$. (The only point to check is the naturality with respect to classes of prime diagrams, and this is clear; we omit the picture.)

THEOREM 4.5. $\mathbb{F}_{\tilde{B}}(\mathscr{D})$ *is the free balanced category on the tensor scheme* $\mathscr{D}$.

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