ADVANCES IN MATHEMATICS 87, 93–147 (1991)

# The Gauss–Green Theorem

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In the $m$-dimensional Euclidean space, we establish the Gauss-Green theorem for any bounded set of bounded variation, and any bounded vector field continuous outside a set of $(m-1)$-dimensional Hausdorff measure zero and almost differentiable outside a set of $\sigma$-finite $(m-1)$-dimensional Hausdorff measure. 

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## 1. INTRODUCTION

The importance of the Gauss–Green theorem in mathematics and its applications is well recognized and requires no discussion. As the divergence of a noncontinuously differentiable vector field need not be Lebesgue integrable, it is clear that formulating the Gauss–Green theorem by means of the Lebesgue integral creates an artificial restriction. An early recognition of this fact led to the development of the Denjoy–Perron integral for which the unrestricted fundamental theorem of calculus (i.e., the one-dimensional Gauss–Green theorem) holds. In spite of many efforts, no substantial progress was made in the higher-dimensional case for nearly seventy years. Only in the eighties were several extensions of the multidimensional Lebesgue integral shown to integrate the divergence of *any* differentiable vector field (see [23, 22, 14, 16, 17, 15, 30, 26, 28, 27, 29, 31],



0001-8708/91 $7.50

and others). Unfortunately, the averaging processes employed have serious deficiencies varying only in degree of severity. Some lack certain basic properties expected of any integral, others either are coordinate bound or cannot be applied to vector fields with many singularities, and all place unnatural restrictions on the domains of integration.

In the present paper, elaborating on the ideas of Henstock [10, Sect. 5] and Besicovitch [1], we define a variational type averaging process on the family $BV$ of all bounded subsets of $\mathbf{R}^m$ of bounded variation. Since this averaging process is not additive in the usual sense, we use it only as an intermediate gadget to which the extension method of Mařík (see [19, 12, 18, 21, 31]) is applied. As a result we obtain a well-behaved coordinate free integral which integrates the divergence of any bounded vector field continuous outside a set of $(m-1)$-dimensional Hausdorff measure zero and almost differentiable outside a set of $\sigma$-finite $(m-1)$-dimensional Hausdorff measure. For such vector fields the Gauss–Green formula holds.

The approach described above reaches the limits of generality. The family $BV$ contains the most general bounded subsets of $\mathbf{R}^m$ for which the surface area and exterior normal can still be profitably defined. Moreover, $BV$ has a compactness property which has been utilized for solving variational problems of geometric measure theory (cf. [8]), as well as for finding weak solutions of conservation laws (cf. [35]). As this important property is usually lost when restrictions are placed on boundaries of sets, it is critical to have the integral defined on *all* members of $BV$. While we restricted our attention to bounded sets and bounded vector fields, the inclusion of arbitrary Caccioppoli sets and unbounded vector fields in our framework appears possible under appropriate growth conditions balancing the perimeters of sets against the magnitude of vector fields. No topological restrictions are placed on the exceptional sets for differentiability and continuity, and in terms of the $(m-1)$-dimensional Hausdorff measure, these sets are as large as one may hope for (cf. Remarks 5.20,2 and 10.10,2). We note that in the Lebesgue integral setting, our exceptional sets for differentiability were used in [34]. Since the integral is invariant with respect to lipeomorphisms (i.e., bilipschitzian maps) it can be applied to more general geometric objects than differentiable manifolds.

Two important topics, integration by parts and interpretation of integrable functions as currents, are not included. They wil be addressed in a separate paper.

Our paper is organized as follows. After some general preliminaries, we discuss sets of bounded variation in Section 3. Section 4 deals with continuous additive functions of sets of bounded variation. It contains a proof of the fundamental result (Proposition 4.6), akin to Cousin's lemma (see [22]), on which the further exposition is based. The variational integral is introduced in Section 5, and its Perron and Riemann type definitions are given in Sections 6 and 7. Convergence among sets of bounded variation, defined in Section 8, is applied in Section 9 where the variational integral is extended to the continous integral. The final extension of the continuous integral is made in Section 10.

## 2. Preliminaries

Throughout this paper, $m \geqslant 1$ is a fixed integer. The set of all real numbers is denoted by $\mathbf{R}$, and the $m$-fold Cartesian product of $\mathbf{R}$ is denoted by $\mathbf{R}^m$. For $x = (\xi_1, ..., \xi_m)$ and $y = (\eta_1, ..., \eta_m)$ in $\mathbf{R}^m$, we let $x \cdot y = \sum_{i=1}^m \xi_i \eta_i$, $\|x\| = \sqrt{x \cdot x}$, and $|x| = \max\{|\xi|, ..., |\xi_m|\}$. In $\mathbf{R}^m$, we use exclusively the metric induced by the norm $|x|$. If $E \subset \mathbf{R}^m$, then cl $E$, int $E$, bd $E$, and $d(E)$ denote, respectively, the closure, interior, boundary, and diameter of $E$. For $x \in \mathbf{R}^m$ and $\varepsilon > 0$, we set $U(x, \varepsilon) = \{y \in \mathbf{R}^m : |x - y| < \varepsilon\}$.

A *cube* is the product of $m$ bounded one-dimensinal invervals of equal positive length. A cube is called a *square* if $m = 2$. A *dyadic cube* is the product $\prod_{i=1}^m [k_i 2^{-n}, (k_i + 1) 2^{-n})$ where $k_1, ..., k_m$ and $n$ are integers with $n \geqslant 0$.

As is customary in geometric measure theory, a *measure* in a metric space $X$ means always an *outer measure* in $X$ (see [6, Sect. 1, p. 6]). If $k \geqslant 1$ is an integer, we denote by $\lambda_k$ the $k$-dimensional Lebesgue measure in $\mathbf{R}^k$. We write $\lambda$ instead of $\lambda_1$, and $|E|$ instead of $\lambda_m(E)$ for each $E \subset \mathbf{R}^m$. Unless specified otherwise, the words "measure" and "measurable," as well as the expressions "almost all" and "almost everywhere," refer to the measure $\lambda_m$.

Let $E \subset \mathbf{R}^m$ be measurable and let $x \in \mathbf{R}^m$. We say that $x$ is, respectively, a *density* or *dispersion* point of $E$ whenever

$$\liminf_{\varepsilon \to 0+} \frac{|E \cap U(x, \varepsilon)|}{(2\varepsilon)^m} = 1 \qquad \text{or} \qquad \limsup_{\varepsilon \to 0+} \frac{|E \cap U(x, \varepsilon)|}{(2\varepsilon)^m} = 0.$$

The set of all density points of $E$ is called the *essential interior* of $E$, denoted by $\mathrm{int}_e\, E$; the complement of the set of all dispersion points of $E$ is called the *essential closure* of $E$, denoted by $\mathrm{cl}_e\, E$. The *essential boundary* of $E$ is the set $\mathrm{bd}_e\, E = \mathrm{cl}_e\, E - \mathrm{int}_e\, E$. It is easy to verify that $\mathrm{int}\, E \subset \mathrm{int}_e\, E \subset \mathrm{cl}_e\, E \subset \mathrm{cl}\, E$, $\mathrm{bd}_e\, E \subset \mathrm{bd}\, E$, and $\mathbf{R}^m - \mathrm{cl}_e\, E = \mathrm{int}_e(\mathbf{R}^m - E)$. By [33, Chap. IV, Theorem (6.1), p. 117], the sets $E$, $\mathrm{int}_e\, E$, and $\mathrm{cl}_e\, E$ differ only by sets of measure zero; in particular, $|\mathrm{bd}_e\, E| = 0$. We call the set $E$ *dispersed* whenever $\mathrm{cl}_e\, E$ is a proper subset of cl $E$.

Example 2.1. Let $m = 1$, and let $C$ be the Cantor set obtained from the interval $[0, 1]$ by successively removing the open middle intervals of length $2^{-2k-2}$, $k = 0, 1, ...$. The family of $2^k$ intervals, each of length $2^{-2k-2}$, removed at the $k$th step is denoted by $\mathscr{C}_k$. If $\Delta_k$, $k \geqslant 1$, is any connected component of $[0, 1] - \bigcup_{i=1}^{k-1} (\bigcup \mathscr{C}_i)$, then $|\Delta_k| = 2^{-k-1} + 2^{-2k-1}$ and $|C \cap \Delta_k| = 2^{-k-1}$. Thus $\lim(|C \cap \Delta_k| / |\Delta_k|) = 1$. Now let $x \in C$, and for $k = 1, 2, \ldots$, let $\varepsilon_k$ be the least positive number such that the closure of $U_k = U(x, \varepsilon_k)$ contains the component $\Delta_k$ containing $x$. Then $|U_k| \leqslant 2\,|\Delta_k|$, and we see that $\lim \varepsilon_k = 0$ and

$$\liminf \frac{|C \cap U_k|}{2\varepsilon_k} \geqslant \frac{1}{2}.$$

Thus $C$, being closed, is nondispersed. Moreover, $\operatorname{int}_e C \neq \varnothing$ because $|C| > 0$. As $C$ is nowhere dense, the set $[0, 1] - C$ is dispersed.

Later we shall need the following simple lemma.

LEMMA 2.2. *Let $A \subset \mathbf{R}^m$ be measurable, let $x \in \operatorname{int}_e A$, and let $\{\varepsilon_n\}$ be a sequence of positive numbers with $\lim \varepsilon_n = 0$. If $\{A_n\}$ is a sequence of measurable sets such that $A_n \subset U(x, \varepsilon_n)$ and $|A_n| \geqslant \alpha\,|U(x, \varepsilon_n)|$ for $n = 1, 2, \ldots$ and a fixed $\alpha > 0$, then*

$$\lim \frac{|A \cap A_n|}{|A_n|} = 1.$$

*Proof.* Assume that $\liminf(|A \cap A_n| / |A_n|) < \beta < 1$, and find a $\gamma < 1$ such that $1 - \gamma < \alpha(1 - \beta)$. For $n = 1, 2, \ldots$, let $U_n = U(x, \varepsilon_n)$. Since $\lim(|A \cap U_n| / |U_n|) = 1$, there is an integer $p \geqslant 1$ such that $|A \cap A_p| \leqslant \beta\,|A_p|$ and $|A \cap U_p| \geqslant \gamma\,|U_p|$. A contradiction follows:

$$\begin{aligned}
|A \cap (U_p - A_p)| &\geqslant \gamma\,|U_p| - \beta\,|A_p| \\
&= |U_p| - |A_p| - (1 - \gamma)\,|U_p| + (1 - \beta)\,|A_p| \\
&\geqslant |A \cap (U_p - A_p)| + [\alpha(1 - \beta) - (1 - \gamma)]\,|U_p| \\
&> |A \cap (U_p - A_p)|.
\end{aligned}$$

COROLLARY 2.3. *Let $A \subset \mathbf{R}^m$ be measurable, let $x \in \operatorname{int}_e A$, and let $\{B_n\}$ be a sequence of cubes such that $x \in \operatorname{cl} B_n$, $n = 1, 2, \ldots$, and $\lim d(B_n) = 0$. Then $x \in \operatorname{cl}_e(A \cap B_n)$, $n = 1, 2, \ldots$, and $\lim(|A \cap B_n| / |B_n|) = 1$.*

*Proof.* Given an integer $n \geqslant 1$, there is an $\eta > 0$ such that for $U_k = U(x, \eta/k)$, $k = 1, 2, \ldots$, we have $|B_n \cap U_k| \geqslant 2^{-m}\,|U_k|$. Thus by Lemma 2.2,

$$\liminf_{k \to \infty} \frac{|(A \cap B_n) \cap U_k|}{|U_k|} = \liminf_{k \to \infty} \left( \frac{|A \cap (B_n \cap U_k)|}{|B_n \cap U_k|} \cdot \frac{|B_n \cap U_k|}{|U_k|} \right) \geqslant 2^{-m}$$

and we see that $x \in \mathrm{cl}_e(A \cap B_n)$. Now if $\varepsilon_n = 2d(B_n)$, then $B_n \subset U(x, \varepsilon_n)$ and $|B_n| = 4^{-m} |U(x, \varepsilon_n)|$. Hence the rest of the corollary follows directly from Lemma 2.2.

By $\mathscr{H}$ we denote the $(m-1)$-dimensional Hausdorff measure in $\mathbf{R}^m$ defined so that $\mathscr{H}(E)$ is the counting measure of $E$ if $m=1$, and $\mathscr{H}(E) = \lambda_{m-1}(E)$ whenever $m \geqslant 2$ and $E \subset \mathbf{R}^{m-1}$ (cf. [7, Sect. 2.10.2, p. 171]). A subset of $\mathbf{R}^m$ is called *thin* if its $\mathscr{H}$ measure is $\sigma$-finite. Thus the thin sets in this paper are appreciably *larger* than those considered in [28, 31]; in particular, they need not be compact. If $T$ is a thin set, then $|T| = 0$ by [6, Sect. 1.2, p. 7]. A set $E \subset \mathbf{R}^m$ is called, respectively, *solid* or *opaque* whenever $\mathrm{cl}\, E - \mathrm{cl}_e\, E$ or $\mathrm{int}_e\, E - \mathrm{int}\, E$ is thin. Clearly, a set is solid if and only if its complement is opaque. Each nondispersed set is solid but not vice versa (see Example 2.4 below).

EXAMPLE 2.4. Let $m = 2$, let $C$ and $\mathscr{C}_k$ be as in Example 2.1, and set

$$D = \bigcup_{k=0}^{\infty} \bigcup_{U \in \mathscr{C}_k} [U \times (0, 2^{-2k-2})].$$

Since $\mathscr{H}(\mathrm{bd}\, D) = 5/2$, we see that $D$ is solid. On the other hand, $C \times \{0\} \subset \mathrm{cl}\, D$ and it is easy to see that $(x, 0)$ is a dispersion point of $D$ for each $x \in \mathrm{int}_e\, C$. It follows that $D$ is a dispersed set.

The next lemma follows from [6, Theorem 5.1, p. 65].

LEMMA 2.5. *There is a constant $\kappa > 0$ depending only on $m$ and having the following property: if $E \subset \mathbf{R}^m$ and $\mathscr{H}(E) < a$, then for each $\delta > 0$ we can find a sequence $\{B_n\}$ of dyadic cubes of diameters less than $\delta$ such that*

$$E \subset \mathrm{int}\left(\bigcup_n B_n\right) \qquad \text{and} \qquad \sum_n [d(B_n)]^{m-1} < \kappa a.$$

Unless stated differently, a "function" always means a real-valued function. If $f$ is a function on a set $A$ and $B \subset A$, we denote by $f \restriction B$ the restriction of $f$ to $B$; when no confusion can arise we write $f$ instead of $f \restriction B$. The algebraic operations, order, and convergence among functions on the same set are defined pointwise.

## 3. BV SETS

A set of *bounded variation* (abbreviated as BV set) is a bounded measurable set $A \subset \mathbf{R}^m$ such that the distributional gradient of its characteristic function is a vector-valued measure in $\mathbf{R}^m$ whose variation $\sigma_A$, called the *surface measure* of $A$, is finite. By [7, Sect. 2.10.6, p. 173, and Theorem 4.5.11, p. 506], a bounded set $A \subset \mathbf{R}^m$ is a BV set if and only if $\mathscr{H}(\mathrm{bd}_e A) < +\infty$. For the rich theory of BV sets, which has evolved over the past thirty years, we refer to [5, 4, 20, 35, 7, 8, and 24, Chap. 6]. In this section we merely summarize the basics and establish a few facts for future use.

Let $A$ be a BV set. Then $\sigma_A(E) = \mathscr{H}(E \cap \mathrm{bd}_e A)$ for each $E \subset \mathbf{R}^m$. The number $\|A\| = \sigma_A(\mathbf{R}^m)$ is called the *perimeter* of $A$. An $x \in \mathbf{R}^m$ is called a *surface dispersion point* of $A$ whenever

$$\lim_{\varepsilon \to 0+} \frac{\sigma_A[U(x, \varepsilon)]}{(2\varepsilon)^{m-1}} = 0.$$

The *critical boundary* of $A$, denoted by $\mathrm{bd}_c A$, is the set of all $x \in \mathbf{R}^m$ which are not surface dispersion points of $A$. By [35, Sect. 2, Subsect. 4], $\mathrm{bd}_e A \subset \mathrm{bd}_c A \subset \mathrm{bd}\, A$ and $\mathscr{H}(\mathrm{bd}_c A - \mathrm{bd}_e A) = 0$. The sets $\mathrm{int}_c A = \mathrm{int}_e A - \mathrm{bd}_c A$ and $\mathrm{cl}_c A = (\mathrm{cl}_e A) \cup (\mathrm{bd}_c A)$ are called the *critical interior* and *critical closure* of $A$, respectively. Finally, there is a Borel vector field $n_A$ on $\mathbf{R}^m$, called the *Federer exterior normal* of $A$, such that

$$\sigma_A(B) = \int_B \|n_A\|\, d\mathscr{H} \qquad \text{and} \qquad \int_A \operatorname{div} v\, d\lambda_m = \int_{\mathrm{bd}\, A} v \cdot n_A\, d\mathscr{H}$$

for every $\mathscr{H}$-measurable set $B \subset \mathbf{R}^m$ and every vector field $v$ continuously differentiable in a neighborhood of $\mathrm{cl}\, A$ (see [7, Chap. 4]).

The family of all BV sets, denoted by $BV$, is a ring (cf. [9, Sect. 4, p. 19]) and

$$\max\{\|A \cup B\|, \|A \cap B\|, \|A - B\|\} \leqslant \|A\| + \|B\|$$

for all $A$, $B$ in $BV$. Since $\mathrm{cl}_e(A \cup B) = (\mathrm{cl}_e A) \cup (\mathrm{cl}_e B)$ for all $A, B \subset \mathbf{R}^m$, the families of all solid and all nondispersed sets are closed with respect to finite unions. Example 3.1 below shows, however, that neither family is a ring. There is a subring $\mathscr{A}$ of $BV$ consisting of all BV sets whose boundaries are thin, or alternately, which are simultaneously solid and opaque. If $E \subset \mathbf{R}^m$ is any set, we let $BV_E = \{A \in BV : A \subset E\}$.

EXAMPLE 3.1. Assume that $m = 2$. Let $P = P_{0,1} = [-1/2, 1/2]^2$, and denote by $Q_{0,1}$ the open square of diameter $3^{-1}$ concentric with $P_{0,1}$. Divide $P_{0,1} - Q_{0,1}$ into nonoverlapping closed squares $P_{1,i}$, $i = 1, ..., 8$, each of diameter $3^{-1}$, and denote by $Q_{1,i}$ the open square of diameter $3^{-2}$ concentric with $P_{1,i}$. Now divide $\bigcup_{i=1}^{8} (P_{1,i} - Q_{1,i})$ into nonoverlapping closed squares $P_{2,i}$, $i = 1, ..., 8^2$, each of diameter $3^{-2}$, and denote by $Q_{2,i}$ the open square of diameter $3^{-3}$ concentric with $P_{2,i}$. Proceeding inductively, at the

$n$th step we construct closed nonoverlapping squares $P_{n,i}$, $i=1, \ldots, 8^n$, each of diameter $3^{-n}$, and open squares $Q_{n,i}$ of diameter $3^{-n-1}$ concentric with $P_{n,i}$.

Let $R_{n,i}$ be an open square of diameter $12^{-n-1}$ concentric with $Q_{n,i}$, and let

$$R = \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{8^n} R_{n,i} \qquad \text{and} \qquad S = P - R.$$

An easy calculation shows that $|P_{n,i}| = 3^{-2n}$ and $|S \cap P_{n,i}| = 3^{-2n}(1 - 4^{-2n-1}/34)$. Thus $\lim_{n\to\infty} (|S \cap P_{n,i}|/|P_{n,i}|) = 1$. If $x$ belongs to

$$S \cap \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{8^n} Q_{n,i} = \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{8^n} (Q_{n,i} - R_{n,i}),$$

then clearly $x \in \mathrm{cl}_e\, S$. If $x \in S - \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{8^n} Q_{n,i}$, let $\varepsilon_n$ be the least positive number such that $U_n = U(x, \varepsilon_n)$ contains a $P_{n,i}$ containing $x$. Then $|U_n| \leqslant 4\, |P_{n,i}|$, and we see that $\lim \varepsilon_n = 0$ and

$$\liminf \frac{|S \cap U_n|}{(2\varepsilon_n)^2} \geqslant \frac{1}{4}.$$

Thus again $x \in \mathrm{cl}_e\, S$, and we conclude that $S$ is nondispersed.

Since $\sum_{n=0}^{\infty} \sum_{i=1}^{8^n} \|R_{n,i}\| < +\infty$, we see that $R$ is a BV set, and hence $\mathscr{H}(\mathrm{bd}_e\, R) < +\infty$. On the other hand, the set $C = P - \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{8^n} Q_{n,i}$ is a subset of bd $R$, and it follows from [6, Theorem 8.6, p. 122] that $C$ is not thin. As $R$ is open, it is not solid.

The next proposition, suggested in part by the author, is due to G. Congedo and I. Tamanini (see [3, 36]). It shows that any BV set can be approximated *from inside* by solid BV sets (cf. [8, Remark 1.27, p. 24]).

PROPOSITION 3.2. *For each BV set $A$ there are nondispersed BV sets $A_n \subset A$ such that $\|A_n\| \leqslant \|A\|$ and $|A - A_n| \leqslant \|A\|/n$, $n = 1, 2, \ldots$.*

*Proof.* Fix an integer $n \geqslant 1$. Using the usual compactness argument in $BV_A$ (cf. [8, Proof of Theorem 1.20, p. 18]), it is easy to find an $E \in BV_A$ such that

$$\|E\| - n\,|E| \leqslant \|B\| - n\,|B|$$

for each $B \in BV_A$. Since $B = A$ yields $\|E\| \leqslant \|A\|$ and $n\,|A - E| \leqslant \|A\|$, it suffices to show that $E$ is not dispersed.

Subtracting from $E$ a set of measure zero, we may assume that $|E \cap U(x, \varepsilon)| > 0$ for every $x \in \mathrm{cl}\, E$ and every $\varepsilon > 0$ (cf. [8, Proof of Proposition 3.1, p. 42]). Select an $x \in \mathrm{cl}\, E$, and for $t > 0$ let $U_t = U(x, t)$. The inequalities and equations which follow hold for $\lambda$-almost all $t > 0$; some hold for all $t > 0$, but this is irrelevant. The isoperimetric inequality (see [8, Theorem 1.29, p. 25]) provides a positive constant $\alpha$, depending only on $m$, such that

$$\alpha \leqslant \|E \cap U_t\| \cdot |E \cap U_t|^{(1/m)-1}.$$

The minimality of $E$ gives

$$\|E\| - n\,|E| \leqslant \|E - U_t\| - n\,|E - U_t|,$$

and hence using the equations

$$\|E \cap U_t\| = \sigma_E(U_t) + \sigma_{U_t}(E),$$
$$\|E - U_t\| = \|E\| - \sigma_E(U_t) + \sigma_{U_t}(E)$$

established in [4] (see also [24, Section 6.2.3, Lemma 4, p. 306]), we obtain

$$\alpha \leqslant 2\sigma_{U_t}(E)\,|E \cap U_t|^{(1/m)-1} + n\,|E \cap U_t|^{1/m}.$$

If $x = (\xi_1, ..., \xi_m)$, then applying Fubini's theorem to the sets

$$\{(\eta_1, ..., \eta_m) \in E \cap U_t : |\eta_j - \xi_j| \leqslant |\eta_i - \xi_i|, j = 1, ..., m\}$$

for $i = 1, ..., m$, it is easy to verify that

$$|E \cap U_t| = \int_0^t \sigma_{U_s}(E)\,d\lambda(s) \qquad \text{or equivalently} \qquad \sigma_{U_t}(E) = \frac{d}{dt}\,|E \cap U_t|.$$

Thus

$$\alpha \leqslant 2m \frac{d}{dt}\,(|E \cap U_t|^{1/m}) + n\,|E \cap U_t|^{1/m}.$$

Dividing the last inequality by an $\varepsilon > 0$ and integrating over the interval $(0, \varepsilon)$ yields

$$\alpha \leqslant m \left(\frac{|E \cap U_\varepsilon|}{(2\varepsilon)^m}\right)^{1/m} + n\,|E \cap U_\varepsilon|^{1/m},$$

from which we conclude that $x \in \mathrm{cl}_e\, E$.

*Note.* It is possible to show that the sequence $\{A_n\}$ constructed according to the previous proof is increasing, but we shall not need this.

*Remark* 3.3. It would be interesting to know whether each BV set $A$ contains a sequence $\{A_n\}$ of nondispersed BV subsets of $A$ such that $\lim \|A - A_n\| = 0$ (cf. Remark 10.10,1 and Note added in proof).

Following [28, Sect. 2], we define the *regularity* of a BV set $A$ as the number

$$r(A) = \begin{cases} \dfrac{|A|}{d(A)\, \|A\|} & \text{if } \; d(A)\, \|A\| > 0, \\ 0 & \text{otherwise,} \end{cases}$$

which controls simultaneously the shape and perimeter of $A$.

LEMMA 3.4. *Let $A \in BV$, let $x \in \operatorname{int}_c A$, and let $\{B_k\}$ be a sequence of cubes such that $x \in \operatorname{cl} B_k$, $k = 1, 2, \ldots$, and $\lim d(B_k) = 0$. Then $x \in \operatorname{cl}_e(A \cap B_k)$, $k = 1, 2, \ldots$, and*

$$\liminf r(A \cap B_k) \geqslant \frac{1}{2m}.$$

*Proof.* Set $\varepsilon_k = d(B_k)$ and $U_k = U(x, \varepsilon_k)$, and observe that

$$\|A \cap B_k\| \leqslant \sigma_A(U_k) + \|B_k\|, \qquad k = 1, 2, \ldots.$$

Since $x$ is a surface dispersion point of $A$,

$$\limsup \frac{\|A \cap B_k\|}{\varepsilon_k^{m-1}} \leqslant 2m.$$

As $x$ is also a density point of $A$, by Lemma 2.3, $x \in \operatorname{cl}_e(A \cap B_k)$, $k = 1, 2, \ldots$, and

$$\liminf r(A \cap B_k) = \liminf \left( \frac{|A \cap B_k|}{|B_k|} \cdot \frac{\varepsilon_k^{m-1}}{\|A \cap B_k\|} \right) \geqslant \frac{1}{2m}.$$

*Note.* The estimate of Lemma 3.4 cannot be improved because the regularity of any cube is $1/(2m)$.

## 4. CONTINUOUS ADDITIVE FUNCTIONS

A *division* of a BV set $A$ is a finite disjoint family of BV sets whose union is $A$.

DEFINITION 4.1. Let $A \in BV$ and let $F$ be a function on $BV_A$. We say that the function $F$ is

1. *superadditive* if $F(B) \geqslant \sum_{D \in \mathscr{D}} F(D)$ for each $B \in BV_A$ and each division $\mathscr{D}$ of $B$;

2. *lower continuous* if given $\varepsilon > 0$, there is a $\delta > 0$ such that $F(B) > -\varepsilon$ for each $B \in BV_A$ with $|B| < \delta$ and $\|B\| < 1/\varepsilon$;

3. *additive* or *continuous* if both $F$ and $-F$ are superadditive or lower continuous, respectively.

EXAMPLE 4.2. Let $A$ be a BV set.

1. If $f$ is a measurable function on $A$ with $\int_A |f| \, d\lambda_m < +\infty$ and $F(B) = \int_B f \, d\lambda_m$ for each $B \in BV_A$, then $F$ is a continuous additive function on $BV_A$ by the absolute continuity of the Lebesgue integral.

2. If $v$ is a continuous vector field on $\operatorname{cl} A$ and $F(B) = \int_{\operatorname{bd} B} v \cdot n_B \, d\mathscr{H}$ for each $B \in BV_A$, then $F$ is a continuous additive function on $BV_A$. To see this, choose an $\varepsilon > 0$, and find a vector field $w$ with polynomial coordinates so that $\|v(x) - w(x)\| < \varepsilon^2/2$ for all $x \in \operatorname{cl} A$. If $c = \sup_{x \in \operatorname{cl} A} |\operatorname{div} w(x)|$ and $B \in BV_A$, then

$$|F(B)| \leqslant \int_{\operatorname{bd}_e B} \|v - w\| \, d\mathscr{H} + \left| \int_{\operatorname{bd} B} w \cdot n_B \, d\mathscr{H} \right|$$
$$\leqslant \frac{\varepsilon^2}{2} \|B\| + \int_B |\operatorname{div} w| \, d\lambda_m \leqslant \frac{\varepsilon^2}{2} \|B\| + c\,|B|.$$

Thus $|F(B)| < \varepsilon$ whenever $\|B\| < 1/\varepsilon$ and $2c\,|B| < \varepsilon$.

Let $A \in BV$ and let $T$ be a thin set. A collection (possibly empty) $P = \{(K_1, x_1), ..., (K_p, x_p)\}$ where $K_1, ..., K_p$ are disjoint dyadic subcubes of $A$ and $x_i \in \operatorname{cl} K_i - T$, $i = 1, ..., p$, is called a *dyadic partition* in $A$ mod $T$. Given a positive function $\delta$ on $\operatorname{cl}_e A - T$, we say that the dyadic partition $P$ is *$\delta$-fine* whenever $d(K_i) < \delta(x_i)$, $i = 1, ..., p$.

The following existence result is sometimes referred to as Cousin's lemma. For its proof, which is a simple compactness argument, we refer to [25, Chap. IV, Theorem 3-1, p. 258].

LEMMA 4.3. *Let $L$ be a dyadic cube, and let $\delta$ be a positive function on $\operatorname{cl} L$. Then there is a $\delta$-fine dyadic partition $\{(L_1, x_1), ..., (L_p, x_p)\}$ in $L$ mod $\varnothing$ with $\bigcup_{j=1}^{p} L_j = L$.*

The next lemma is due to E. J. Howard (see [13]).

LEMMA 4.4. *Let $L$ be a dyadic cube, and let $F$ be a superadditive lower continuous function on $BV_L$. Given an $\varepsilon > 0$, a thin set $T$, and a positive function $\delta$ on* $\operatorname{cl} L - T$, *there is a $\delta$-fine dyadic partition $\{(L_1, x_1), ..., (L_q, x_q)\}$ in $L \bmod T$ such that $F(L - \bigcup_{j=1}^{q} L_j) > -\varepsilon$.*

*Proof.* According to [6, Theorem 1.6(a), p. 8], there is a sequence $\{T_i\}$ of sets such that $T = \bigcup_i T_i$ and $\mathscr{H}(T_i) < 2$ for all $i$. For $i = 1, 2, ...,$ set $\varepsilon_i = \min\{1/(4m\kappa), 2^{-i}\varepsilon\}$, where $\kappa$ is a positive constant from Lemma 2.5, and find an $\eta_i > 0$ so that $F(B) > -\varepsilon_i$ for each $B \in BV_L$ with $|B| < \eta_i$ and $\|B\| < 1/\varepsilon_i$.

Fix an integer $i \geqslant 1$, and use Lemma 2.5 to find a countable family $\mathscr{C}_i$ of dyadic cubes of diameters less than $\eta_i \varepsilon_i$ so that

$$T_i \subset \operatorname{int}\left(\bigcup \mathscr{C}_i\right) \qquad \text{and} \qquad \sum_{C \in \mathscr{C}_i} [d(C)]^{m-1} < 2\kappa.$$

If $\mathscr{E}$ is a finite disjoint subfamily of $\mathscr{C}_i$ and $E = \bigcup \mathscr{E}$, then

$$\|E\| \leqslant \sum_{C \in \mathscr{E}} \|C\| \leqslant 2m \sum_{C \in \mathscr{C}_i} [d(C)]^{m-1} < 4m\kappa \leqslant \frac{1}{\varepsilon_i},$$

$$|E| = \sum_{C \in \mathscr{E}} |C| \leqslant \eta_i \varepsilon_i \sum_{C \in \mathscr{E}} \|C\| < \eta_i,$$

and hence $F(E) > -\varepsilon_i$.

Let $\mathscr{C}$ be a disjoint subfamily of $\bigcup_i \mathscr{C}_i$ with $\bigcup \mathscr{C} = \bigcup_i (\bigcup \mathscr{C}_i)$, and extend $\delta$ to a positive function on $\operatorname{cl} L$ by setting

$$\delta(x) = \min\{d(C) : C \in \mathscr{C}, x \in \operatorname{cl} C\}$$

for each $x \in T \cap \operatorname{cl} L$. By Lemma 4.3, there is a $\delta$-fine dyadic partition $P = \{(L_1, x_1), ..., (L_p, x_p)\}$ in $L \bmod \varnothing$ with $\bigcup_{j=1}^{p} L_j = L$. Let $\mathscr{D}$ consist of all $C \in \mathscr{C}$ such that $C \subset L$ and $L_j \subset C$ for some $j = 1, ..., p$. Thus $\mathscr{D}$ is a finite family, and as $T \subset \operatorname{int}(\bigcup \mathscr{C})$, our definition of $\delta$ on $T \cap \operatorname{cl} L$ implies that $L_j \subset \bigcup \mathscr{D}$ whenever $x_j \in T$. If $L_j$ meets a $D \in \mathscr{D}$ and $L_j \not\subset D$, then $D \subset L_j$ because both $D$ and $L_j$ are dyadic cubes. This leads to a contradiction, since by the definition of $\mathscr{D}$, there is an $L_k \subset D$ for a $k \neq j$ and $L_j \cap L_k = \varnothing$. We conclude that for each $j = 1, ..., p$, either $L_j \subset \bigcup \mathscr{D}$ or $L_j \cap (\bigcup \mathscr{D}) = \varnothing$. Thus after a suitable reordering, $\bigcup \mathscr{D} = L - \bigcup_{j=1}^{q} L_j$ for an integer $q$ with $0 \leqslant q \leqslant p$, and we see that $\{(L_1, x_1), ..., (L_q, x_q)\}$ is a $\delta$-fine dyadic partition in $L \bmod T$.

Let $\mathscr{D}_1 = \mathscr{D} \cap \mathscr{C}_1$, and for $i = 1, 2, ...,$ set $\mathscr{D}_i = \mathscr{D} \cap \mathscr{C}_i - \bigcup_{j=1}^{i-1} \mathscr{D}_j$. Then $\mathscr{D}$ is the disjoint union of the $\mathscr{D}_i$'s and, as $\mathscr{D}$ is finite, there is an integer $s \geqslant 1$ such that $\mathscr{D} = \bigcup_{i=1}^{s} \mathscr{D}_i$. If $D_i = \bigcup \mathscr{D}_i$, then $F(D_i) > -\varepsilon_i$, and $\bigcup \mathscr{D}$ is the disjoint union of $D_1, ..., D_s$. Thus

$$F\left(L - \bigcup_{j=1}^{q} L_j\right) = F\left(\bigcup_{i=1}^{s} D_i\right) \geqslant \sum_{i=1}^{s} F(D_i) > -\sum_{i=1}^{s} \varepsilon_i > -\varepsilon.$$

LEMMA 4.5. *Let $L$ be a dyadic cube, and let $F$ be a superadditive lower continuous function on $BV_L$ with $F(L)<0$. If $T$ is a thin set, then there is an $x \in \mathrm{cl}\, L-T$ and a sequence $\{B_k\}$ of dyadic subcubes of $L$ such that $x \in \mathrm{cl}\, B_k$, $F(B_k)<0$, $k=1, 2, \ldots$, and* $\lim d(B_k)=0$.

*Proof.* If the lemma is false, we can find a positive function $\delta$ on $\mathrm{cl}\, L-T$ such that $F(B) \geqslant 0$ for each dyadic cube $B \subset L$ with $x \in \mathrm{cl}\, B$ and $d(B)<\delta(x)$. By Lemma 4.4, there is a $\delta$-fine dyadic partition $\{(L_1, x_1), \ldots, (L_q, x_q)\}$ in $L$ mod $T$ such that $F(L-\bigcup_{j=1}^{q} L_j)>F(L)$. This is a contradiction since

$$F(L) \geqslant \sum_{j=1}^{q} F(L_j)+F\left(L-\bigcup_{j=1}^{q} L_j\right) \geqslant F\left(L-\bigcup_{j=1}^{q} L_j\right).$$

PROPOSITION 4.6. *Let $A \in BV$, let $T$ be a thin set, and let $F$ be a superadditive lower continuous function on $BV_A$. If $F(A)<0$ and $\varepsilon<1/(2m)$, then there is an $x \in \mathrm{cl}_e\, A-T$, and a sequence $\{C_k\}$ in $BV_A$ such that $x \in \mathrm{cl}_e\, C_k$, $r(C_k)>\varepsilon$, $F(C_k)<0$, $k=1, 2, \ldots$, and* $\lim d(C_k)=0$.

*Proof.* It follows from Proposition 3.2 that $F(B)<0$ for some nondispersed set $B \in BV_A$, so we may assume that $A$ is solid (in fact, nondispersed, but we do not need this). Consequently, we may also assume that $\mathrm{cl}\, A-\mathrm{int}_c\, E \subset T$. Let $K=\prod_{i=1}^{m} [p_i, p_i+q)$ be such that $q \geqslant 3$ and $p_1, \ldots, p_m$ are integers, and $\mathrm{cl}\, A \subset \prod_{i=1}^{m} (p_i+1, \quad p_i+q-1)$. Setting $G(B)=F(A \cap B)$ for each $B \in BV_K$ extends $F$ to a superadditive lower continuous function $G$ on $BV_K$. Since $K$ is a finite union of dyadic cubes, it is easy to verify that Lemma 4.5 holds for $K$. Thus there are an $x \in \mathrm{cl}\, K-T$ and a sequence $\{B_k\}$ of dyadic subcubes of $K$ such that $x \in \mathrm{cl}\, B_k$, $G(B_k)<0$, $k=1, 2, \ldots$, and $\lim d(B_k)=0$; in particular, $x \in \mathrm{cl}\, A$. By the choice of $T$, we have $x \in \mathrm{int}_c\, A$, and it follows from Lemma 3.4 that the sequence $\{A \cap B_k\}$ has a subsequence $\{C_k\}$ with $r(C_k)>\varepsilon$ for all $k$. Observing that $F(C_k)=G(B_{n_k})$ for an integer $n_k \geqslant 1$ completes the argument.

## 5. THE VARIATIONAL INTEGRAL

Let $A \in BV$, and let $f$ and $F$ be functions defined on $\mathrm{cl}_e\, A$ and $BV_A$, respectively. Given $\varepsilon>0$ and a thin set $T$, an *$\varepsilon$-majorant* of the pair $(f, F)$ in $A$ mod $T$ is a nonnegative superadditive function $M$ on $BV_A$ satisfying the following conditions: $M(A)<\varepsilon$, and for each $x \in \mathrm{cl}_e\, A-T$ there is a $\delta>0$ such that

$$|\, f(x)\, |B|-F(B)| \leqslant M(B)$$

for every $B \in BV_A$ with $x \in \mathrm{cl}\, B$, $d(B)<\delta$, and $r(B)>\varepsilon$.

DEFINITION 5.1. Let $A \in BV$ and let $f$ be a function on $\mathrm{cl}_e A$. We say that $f$ is *variationally integrable* (abbreviated as v-integrable) in $A$ if there is a continuous additive function $F$ on $BV_A$ which satisfies the following condition: for each $\varepsilon > 0$ there is a thin set $T$ such that the pair $(f, F)$ has an $\varepsilon$-majorant in $A$ mod $T$.

Let $A$ be a BV set. The family of all v-integrable functions in $A$ is denoted by $\mathscr{I}_v(A)$. If $f \in \mathscr{I}_v(A)$, then each continuous additive function $F$ on $BV_A$ which satisfies the condition of Definition 5.1 is called *an indefinite v-integral* of $f$ in $A$.

*Remark* 5.2. A few comments on the definition of the variational interal are in order.

1. Since a countable union of thin sets is again thin, it is easy to see that the thin set $T$ of Definition 5.1 can be selected independently of $\varepsilon$.

2. The ambiguity concerning an indefinite v-integral is only temporary, as we show in Corollary 5.5 that each integrable function has only one indefinite v-integral. We also show that defining $f$ on $\mathrm{cl}_e A$ rather than on $A$ is inconsequential (cf. Remark 5.10).

3. It will be shown in Section 7 that the value of an indefinite integral is approximated by the usual Riemann sums (see Proposition 7.8). Thus the v-integral is a genuine *averaging process.*

4. The definition of an $\varepsilon$-majorant can be modified in two distinct ways (cf. Remark 6.4):

(a) The condition $x \in \mathrm{cl}\, B$ may be replaced by $x \in \mathrm{cl}_e B$. We shall see in Section 7 (Propositions 7.7 and 7.10) that this modification has no effect on the v-integral as long as $m \geqslant 2$. We do not know if the same is true for $m = 1$ (cf. Remark 6.10).

(b) We may require that $M$ is additive rather than superadditive. With the exception of Section 7, all results of this paper hold for the v-integral defined by means of additive $\varepsilon$-majorants, called the *va-integral.* Obviously, the v-integral extends the va-integral; however, it is unclear whether this extension is *proper* (cf. [11]).

5. Superficially, the v-integrability given by Definition 5.1 looks very similar to that presented in [31, Definition 3.1]. However, a closer examination reveals two significant differences:

(a) The thin sets are larger, and they are used in a less restrictive manner.

(b) There are no restrictions on boundaries of integration domains.

The inclusion of a rather complicated Proposition 4.6 is necessitated by (a), and noticeable complications arise from (b). In particular, (b) is responsible for a deficiency in the additivity of the $v$-integral defined here (see Propositions 5.7 and 9.5,3, and Example 5.21). Of course, the above differences also bring substantial gains already discussed in the introduction.

PROPOSITION 5.3. *Let $A \in BV$, $f \in \mathscr{I}_v(A)$, and let $F$ be an indefinite $v$-integral of $f$ in $A$. If $B \in BV_A$, then $f \upharpoonright \mathrm{cl}_e B$ belongs to $\mathscr{I}_v(B)$ and $F \upharpoonright BV_B$ is an indefinite $v$-integral of $f \upharpoonright \mathrm{cl}_e B$ in $B$.*

This proposition is an obvious consequence of Definition 5.1.

LEMMA 5.4. *Let $A \in BV$, and for $i=1, 2$ let $F_i$ be an indefinite $v$-integral of $f_i \in \mathscr{I}_v(A)$. If $f_1 \leqslant f_2$ then $F_1 \leqslant F_2$.*

*Proof.* It suffices to show that the assumption $F_2(A) < F_1(A)$ leads to a contradiction. Choose a positive $\varepsilon < 1/(2m)$ such that $F_2(A) + 2\varepsilon < F_1(A)$. For $i = 1, 2$ we can find thin sets $T_i$ and $\varepsilon$-majorants $M_i$ of the pairs $(f_i, F_i)$ in $A$ mod $T_i$. Then $T = T_1 \cup T_2$ is a thin set, and

$$F = F_2 - F_1 + M_1 + M_2$$

is a superadditive lower continuous function on $BV_A$ with $F(A) < 0$. By Proposition 4.6, there is an $x \in \mathrm{cl}_e A - T$, and a sequence $\{C_k\}$ in $BV_A$ such that $x \in \mathrm{cl}\, C_k$, $r(C_k) > \varepsilon$, $F(C_k) < 0$, $k = 1, 2, \ldots$, and $\lim d(C_k) = 0$. Thus

$$|f_i(x)\, |C_p| - F_i(C_p)| \leqslant M_i(C_p)$$

for $i = 1, 2$ and some integer $p \geqslant 1$. This implies that

$$F_1(C_p) - M_1(C_p) \leqslant f_1(x)\, |C_p| \leqslant f_2(x)\, |C_p| \leqslant F_2(C_p) + M_2(C_p)$$

and consequently $F(C_p) \geqslant 0$, contrary to our assumption.

COROLLARY 5.5. *If $A \in BV$ and $f \in \mathscr{I}_v(A)$, then all indefinite $v$-integrals of $f$ in $A$ are equal.*

In view of the previous corollary, if $A \in BV$ and $f \in \mathscr{I}_v(A)$, we can talk about *the indefinite $v$-integral* of $f$ in $A$, denoted by $I_v(f, \cdot)$; the number $I_v(f, A)$ is called the *$v$-integral* of $f$ over $A$. Observe that $I_v(f \upharpoonright \mathrm{cl}_e B, \cdot) = I_v(f, \cdot) \upharpoonright BV_B$ for each $B \in BV_A$.

PROPOSITION 5.6. *If $A \in BV$, then $\mathscr{I}_v(A)$ is a linear space and the map $f \mapsto I_v(f, A)$ is a nonnegative linear functional on $\mathscr{I}_v(A)$.*

*Proof.* If $f \in \mathscr{I}_v(A)$ is a nonnegative function, then $I_v(f, A) \geqslant 0$ by Lemma 5.4. The rest of the proposition follows directly from Definition 5.1.

PROPOSITION 5.7. *Let $\mathscr{D}$ be a division of a BV set $A$, and let $f$ be a function on $\mathrm{cl}_e A$ which is v-integrable in each $D \in \mathscr{D}$. If $\mathscr{D}$ consists of solid sets, then $f$ is v-integrable in $A$.*

*Proof.* For each $D \in \mathscr{D}$, let $f_D = f \restriction \mathrm{cl}_e D$ and $F_D = I_v(f_D, \cdot)$. If

$$F(B) = \sum_{D \in \mathscr{D}} F_D(B \cap D)$$

for every $B \in BV_A$, then $F$ is a continuous additive function on $BV_A$, and we show that $F = I_v(f, \cdot)$. Let $n$ be the number of elements in $\mathscr{D}$ and let $\varepsilon > 0$. Given $D \in \mathscr{D}$, there is a thin set $T_D$ such that the pair $(f_D, F_D)$ has an $(\varepsilon/n)$-majorant $M_D$ in $D \bmod T_D$. Setting

$$M(B) = \sum_{D \in \mathscr{D}} M_D(B \cap D)$$

for each $B \in BV_A$, we see that $M$ is a nonnegative superadditive function on $BV_A$ and $M(A) < \varepsilon$. If $\mathscr{D}$ consists of solid sets, then the set

$$T = \bigcup_{D \in \mathscr{D}} [T_D \cup (\mathrm{cl}\, D - \mathrm{int}_e D)]$$

is thin. To show that $M$ is an $\varepsilon$-majorant of the pair $(f, F)$ in $A \bmod T$, select an $x \in \mathrm{cl}_e A - T$. By the choice of $T$, we have $x \in \mathrm{int}_e D_x$ for some $D_x \in \mathscr{D}$, and $x \notin \mathrm{cl}\, D$ for every $D \in \mathscr{D}$ different from $D_x$. Thus we can find an $\eta > 0$ so that $D \cap U(x, \eta) = \emptyset$ whenever $D \in \mathscr{D}$ and $D \neq D_x$. Now there is a positive $\delta < \eta$ such that

$$M(B) = M_{D_x}(B) \geqslant |f_{D_x}(x)\,|B| - F_{D_x}(B)| = |f(x)\,|B| - F(B)|$$

for each $B \in BV_A$ with $x \in \mathrm{cl}\, B$, $r(B) > \varepsilon$, and $d(B) < \delta$. Indeed $\varepsilon \geqslant \varepsilon/n$, and $x \in \mathrm{cl}\, B$ together with $d(B) < \delta$ imply that $B \subset D_x$.

*Note.* The set $A$ from the previous proposition, having a solid division, is solid. Example 5.21 shows that assuming $A$ alone is solid is not sufficient for the validity of Proposition 5.7 (cf. Proposition 9.5,3).

For a measurable set $E \subset \mathbf{R}^m$, we denote by $\mathscr{L}_1(E)$ the family of all measurable functions $f$ on $E$ for which the Lebesgue integral $\int_E |f|\, d\lambda_m$ is finite.

PROPOSITION 5.8. *If $A \in BV$, then $\mathscr{L}_1(\mathrm{cl}_e A) \subset \mathscr{I}_v(A)$ and $I_v(f, A) = \int_A f\, d\lambda_m$ for each $f \in \mathscr{L}_1(\mathrm{cl}_e A)$.*

*Proof.* Let $f \in \mathscr{L}_1(\mathrm{cl}_e A)$ and $F(B) = \int_B f\, d\lambda_m$ for every $B \in BV_A$. By Example 4.2,1, $F$ is a continuous additive function on $BV_A$ and we show that $F = I_v(f, \cdot)$. Given $\varepsilon > 0$, we can find *extended* real-valued functions $g$ and $h$ on $\mathrm{cl}_e A$ which are, respectively, upper and lower semicontinuous, and such that

$$g \leqslant f \leqslant h \qquad \text{and} \qquad \int_A (h-g)\, d\lambda_m < \frac{\varepsilon}{2}$$

(see [32, Theorem 2.25, p. 56]). Setting

$$M(B) = \frac{\varepsilon\, |B|}{2(1+|A|)} + \int_B (h-g)\, d\lambda_m$$

for each $B \in BV_A$, we see that $M$ is a nonnegative additive function on $BV_A$, and $M(A) < \varepsilon$. To show that $M$ is an $\varepsilon$-majorant of the pair $(f, F)$ in $A$ mod $\varnothing$, select an $x \in \mathrm{cl}_e A$ and find a $\delta > 0$ so that

$$g(y) < f(x) + \frac{\varepsilon}{2(1+|A|)} \qquad \text{and} \qquad h(y) > f(x) - \frac{\varepsilon}{2(1+|A|)}$$

for all $y \in U(x, \delta) \cap \mathrm{cl}_e A$. Now let $B \in BV_A$, $x \in \mathrm{cl}\, B$, and $d(B) < \delta$. Then

$$\int_B g\, d\lambda_m - \frac{\varepsilon\, |B|}{2(1+|A|)} \leqslant f(x)\, |B| \leqslant \int_B h\, d\lambda_m + \frac{\varepsilon\, |B|}{2(1+|A|)},$$

$$\int_B g\, d\lambda_m \leqslant F(B) \leqslant \int_B h\, d\lambda_m,$$

and consequently

$$|f(x)\, |B| - F(B)| \leqslant M(B).$$

COROLLARY 5.9. *Let $A \in BV$ and let $f$ and $g$ be functions on $\mathrm{cl}_e A$ which are equal almost everywhere. Then $f \in \mathscr{I}_v(A)$ if and only if $g \in \mathscr{I}_v(A)$, in which case $I_v(f, A) = I_v(g, A)$.*

*Remark* 5.10. By means of Corollary 5.9, in the obvious way we can and will extend the definitions of v-integrability and the v-integral to functions defined almost everywhere in their integration domains. In particular, we shall always view a v-integrable function in a BV set $A$ as being defined on $A$, or almost everywhere in $A$, and only when needed, we extend it arbitrarily to $\mathrm{cl}_e A$ (cf. Remark 5.5,2).

We say that a function $F$ defined on $BV$ is *derivable* at $x \in \mathbf{R}^m$ if there exists a finite

$$\lim \frac{F(B_n)}{|B_n|}$$

for each sequence $\{B_n\}$ of closed cubes such that $x \in B_n$, $n = 1, 2, \ldots$, and $\lim d(B_n) = 0$. When all these limits exist they have the same value, denoted by $F'(x)$.

PROPOSITION 5.11. *Let $A \in BV$, $f \in \mathscr{I}_v(A)$, and let $F(B) = I_v(f, A \cap B)$ for each $B \in BV$. Then for almost all $x \in A$ the function $F$ is derivable at $x$ and $F'(x) = f(x)$.*

*Proof.* In view of Remark 5.2,1, there is a thin set $T$ such that given $\varepsilon > 0$, the pair $(f, F)$ has an $\varepsilon$-majorant in $A$ mod $T$. Let $E$ be the set of all $x \in \operatorname{int}_c A - T$ for which either $F$ is not derivable at $x$ or $F'(x) \neq f(x)$. If $x \in E$ and $\{B_n\}$ is a sequence of closed cubes such that $x \in B_n$, $n = 1, 2, \ldots$, and $\lim d(B_n) = 0$, then by Corollary 2.3, the limits

$$\lim \frac{F(B_n)}{|B_n|} \qquad \text{and} \qquad \lim \frac{F(A \cap B_n)}{|A \cap B_n|}$$

either both do not exist, or both exist and have the same value. Thus given $x \in E$, we can find an $\alpha(x) > 0$ such that for each $\delta > 0$ there is a closed cube $B$ with $x \in B$, $d(B) < \delta$, and

$$\left| \frac{F(A \cap B)}{|A \cap B|} - f(x) \right| \geqslant \alpha(x).$$

Fix an integer $n \geqslant 1$ and let $E_n = \{x \in E : \alpha(x) \geqslant 1/n\}$. Choose a positive $\varepsilon < 1/(2m)$ and find an $(\varepsilon/n)$-majorant $M$ of the pair $(f, F)$ in $A$ mod $T$. It follows from Lemma 3.4 that for each $x \in E_n$ there is a $\delta(x) > 0$ such that

$$|f(x)\, |A \cap B| - F(A \cap B)| \leqslant M(A \cap B)$$

for every closed cube $B$ with $x \in B$ and $d(B) < \delta(x)$. Now let $\mathscr{V}$ be the family of all closed cubes $B$ such that $d(B) < \delta(x_B)$ for some $x_B \in B \cap E_n$ and

$$|F(A \cap B) - f(x_B)\, |A \cap B|| \geqslant \frac{|A \cap B|}{n}.$$

It is easy to see that $\mathscr{V}$ covers $E_n$ in the sense of Vitali, and so by [33, Chap. IV, Theorem (3.1), p. 109], there is a disjoint sequence $\{B_k\}$ in $\mathscr{V}$ such that $|E_n - \bigcup_{k=1}^{\infty} B_k| = 0$. Since $|E_n - A| = 0$ and

$$\begin{aligned}
\sum_{k=1}^{p} |A \cap B_k| &\leqslant n \sum_{k=1}^{p} |F(A \cap B_k) - f(x_{B_k})\, |A \cap B_k|| \\
&\leqslant n \sum_{k=1}^{p} M(A \cap B_k) \leqslant nM\left(A \cap \bigcup_{k=1}^{p} B_k\right) \leqslant nM(A) < \varepsilon
\end{aligned}$$

for $p=1, 2, \ldots$, we have

$$|E_n| \leqslant \left| \bigcup_{k=1}^{\infty} (A \cap B_k) \right| = \sum_{k=1}^{\infty} |A \cap B_k| \leqslant \varepsilon,$$

and the arbitrariness of $\varepsilon$ implies that $|E_n|=0$. Observing that $E=\bigcup_{n=1}^{\infty} E_n$ and $|(A-\operatorname{int}_c A) \cup T|=0$ completes the proof.

*Note.* By improving on Lemma 3.4, we can obtain a stronger derivability result similar to that of [28, Proposition 4.2] (cf. the beginning of Section 6).

COROLLARY 5.12. *If $A \in BV$, then each $f \in \mathscr{I}_v(A)$ is measurable.*

The corollary follows from Proposition 5.11 by standard arguments (see [33, Chap. IV, Theorem (4.2), p. 112]).

LEMMA 5.13. *Let $E$ be a bounded measurable subset of $\mathbf{R}^m$, let $\mathscr{F}$ be a linear space of measurable functions on $E$ containing $\mathscr{L}_1(E)$, and let $J$ be a nonnegative linear functional on $\mathscr{F}$ such that $J(f)=\int_E f\, d\lambda_m$ for each $f \in \mathscr{L}_1(E)$. Then a function $f$ on $E$ belongs to $\mathscr{L}_1(E)$ whenever $f$ and $|f|$ belong to $\mathscr{F}$. Moreover, if $\{f_n\}$ is a sequence in $\mathscr{F}$ and* $\lim f_n=f$, *then $f \in \mathscr{F}$ and $J(f)=\lim J(f_n)$ whenever either of the following conditions holds:*

(a) *$f_n \leqslant f_{n+1}$, $n=1, 2, \ldots$, and* $\lim J(f_n) < +\infty$;

(b) *$g \leqslant f_n \leqslant h$ for some $g, h \in \mathscr{F}$ and $n=1, 2, \ldots$.*

*Proof.* If $f$ and $|f|$ belong to $\mathscr{F}$, then for $n=1, 2, \ldots$, the function $k_n=\min\{|f|, n\}$ belongs to $\mathscr{L}_1(E)$, and hence to $\mathscr{F}$. Since

$$\int_E |f|\, d\lambda_m = \lim \int_E k_n\, d\lambda_m = \lim J(k_n) \leqslant J(|f|) < +\infty,$$

we have $f \in \mathscr{L}_1(E)$. The rest of the lemma follows from the monotone and dominated convergence theorems applied to the sequences $\{f_n - f_1\}$ and $\{f_n - g\}$, respectively.

*Note.* The assumptions of Lemma 5.13 are unnecessarily restrictive (cf. [31, Proposition 3.9]) but we do not need a larger generality.

COROLLARY 5.14. *For a fuction $f$ defined on a $BV$ set $A$ the following statements are true.*

1. *$f$ belongs to $\mathscr{L}_1(A)$ if and only if both $f$ and $|f|$ belong to $\mathscr{I}_v(A)$.*
2. *$f=0$ almost everywhere if and only if $f \in \mathscr{I}_v(A)$ and $I_v(f, \cdot)=0$.*
3. *If $\{f_n\}$ is a sequence in $\mathscr{I}_v(A)$ and* $\lim f_n=f$, *then $f \in \mathscr{I}_v(A)$ and $I_v(f, A)=\lim I_v(f_n, A)$ whenever either of the following conditions holds:*

(a) $f_n \leqslant f_{n+1}$, $n = 1, 2, \ldots$, *and* $\lim I_v(f_n, A) < +\infty$;

(b) $g \leqslant f_n \leqslant h$ *for some* $g, h \in \mathscr{I}_v(A)$ *and* $n = 1, 2, \ldots$.

The next proposition is a useful necessary condition for v-integrability (cf. Examples 5.21 and 6.9).

PROPOSITION 5.15. *If $f$ is a v-integrable function in a BV set $A$, then there is a thin set $T$ with the following property: given $\varepsilon > 0$, we can find a positive function $\delta$ on* $\mathrm{cl}_e A - T$ *such that*

$$\sum_{n=1}^{\infty} |f(x_n)\,|B_n| - I_v(f, B_n)| < \varepsilon$$

*for each sequence $\{x_n\}$ in $\mathrm{cl}_e A - T$ and each sequence $\{B_n\}$ of disjoint sets from $BV_A$ with $d(B_n \cup \{x_n\}) < \delta(x_n)$ and $r(B_n \cup \{x_n\}) > \varepsilon$, $n = 1, 2, \ldots$.*

*Proof.* Let $F = I_v(f, \cdot)$. In view of Remark 5.2,1, there is a thin set $T$ such that given $\varepsilon > 0$, the pair $(f, F)$ has an $\varepsilon$-majorant $M$ in $A$ mod $T$. For each $x \in \mathrm{cl}_e A - T$ there is a $\delta(x) > 0$ such that

$$|f(x)\,|B| - F(B)| \leqslant M(B)$$

for every $B \in BV_A$ with $x \in \mathrm{cl}\, B$, $d(B) < \delta(x)$, and $r(B) > \varepsilon$. Let $\delta$ be the function $x \mapsto \delta(x)$, and choose sequences $\{x_n\}$ and $\{B_n\}$ as in the proposition. Since each $x_n$ is a cluster point of $A$, the set $A$ contains disjoint countable sets $C_n$ with $x_n \in \mathrm{cl}\, C_n$, $n = 1, 2, \ldots$. If $D_n = (B_n - \bigcup_{k=1}^{\infty} C_k) \cup C_n$, then the sets $D_n$ are disjoint, and $F(D_n) = F(B_n)$ by the additivity and continuity of $F$. By making the sets $C_n$ sufficiently small, we may assume that $d(D_n) < \delta(x_n)$ and $r(D_n) > \varepsilon$. Thus

$$\begin{aligned}\sum_{n=1}^{p} |f(x_n)\,|B_n| - F(B_n)| &= \sum_{n=1}^{p} |f(x_n)\,|D_n| - F(D_n)| \\ &\leqslant \sum_{n=1}^{p} M(D_n) \leqslant M\left(\bigcup_{n=1}^{p} D_n\right) \leqslant M(A)\end{aligned}$$

for each integer $p \geqslant 1$, and as $M(A) < \varepsilon$, the proposition follows.

If $f$ is a function defined on an open set $U \subset \mathbf{R}^m$, we define the differentiability of $f$ at $x \in U$ in the usual way (see [32, Definition 7.22, p. 150]). Thus differentiability implies continuity and the existence of partial derivatives, which *need not* be continuous. For $i = 1, \ldots, m$ the $i$th partial derivative of $f$ is denoted by $\partial_i f$, and if $v = (f_1, \ldots, f_m)$ is a differentiable vector field, we set $\mathrm{div}\, v = \sum_{i=1}^{m} \partial_i f_i$.

Now let $f$ be a function defined on an *arbitrary* set $A \subset \mathbf{R}^m$, and let $E$ be a *measurable* subset of $A$. We say that $f$ is *differentiable* on $E$ whenever $f$ can be extended to a function $g$ such that the domain of $g$ is a neighborhood of $E$ and $g$ is differentiable at each $x \in E$. Given such an extension $g$ and $x \in E$, we set $\partial_i f(x) = \partial_i g(x)$ for $i = 1, ..., m$, and show that, up to a set of measure zero, thus defined functions $\partial_i f$ on $E$ do not depend on the choice of $g$.

LEMMA 5.16. *Let $E$ be a measurable subset of an open set $U \subset \mathbf{R}^m$, and let $g$ and $h$ be functions on $U$ which have partial derivatives at each $x \in E$. If $g(x) = h(x)$ for all $x \in E$, then $\partial_i g(x) = \partial_i h(x)$ for $i = 1, ..., m$ and almost all $x \in E$.*

*Proof.* Suppose that $|\{x \in E : \partial_1 g(x) \neq \partial_1 h(x)\}| > 0$. By Fubini's theorem, there is a $\xi \in \mathbf{R}^{m-1}$ such that the set $S \subset \mathbf{R}$ of those $s$ for which $(s, \xi) \in E$ and $\partial_1 g(s, \xi) \neq \partial_1 h(s, \xi)$ has a positive measure $\lambda$. In particular, there is a $t \in S$ and a sequence $\{t_n\}$ in $S - \{t\}$ with $\lim t_n = t$. From this we obtain that $\partial_1 g(t, \xi) = \partial_1 h(t, \xi)$, a contradiction. The lemma follows by symmetry.

LEMMA 5.17. *Let $v$ be a bounded vector field on a set $U \subset \mathbf{R}^m$ which is differentiable at $x \in \operatorname{int} U$. Then given $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\left| \operatorname{div} v(x)\, |B| - \int_{\operatorname{bd} B} v \cdot n_B \, d\mathscr{H} \right| < \varepsilon\, |B|$$

*for each $B \in BV_U$ for which $x \in \operatorname{cl} B$, $d(B) < \delta$, $r(B) > \varepsilon$, and $v$ is $\mathscr{H}$-measurable on $\operatorname{bd}_e B$.*

*Proof.* For $y \in \mathbf{R}^m$, let $w(y) = Dv(x) \cdot (y - x)$ where $Dv(x)$ is the Jacobi matrix at $x$ of the map $v : U \to \mathbf{R}^m$. Then $\operatorname{div} w(y) = \operatorname{div} v(x)$ for each $y \in \mathbf{R}^m$, and there is a nonnegative function $h$ on $U$ such that $\lim_{y \to x} h(y) = 0$ and $\|v(y) - v(x) - w(y)\| \leqslant h(y)\, |y - x|$ for each $y \in U$. Given $\varepsilon > 0$, choose $\delta > 0$ so that $U(x, \delta) \subset U$ and $h(y) < \varepsilon^2$ whenever $y \in U(x, \delta)$. Now if $B \in BV_U$ is such that $x \in \operatorname{cl} B$, $d(B) < \delta$, $r(B) > \varepsilon$, and $v$ is $\mathscr{H}$-measurable on $\operatorname{bd}_e B$, then

$$\begin{aligned}
&\left| \operatorname{div} v(x)\, |B| - \int_{\operatorname{bd} B} v \cdot n_B \, d\mathscr{H} \right| \\
&\quad = \left| \int_B \operatorname{div} w(y)\, d\lambda_m(y) - \int_{\operatorname{bd} B} [v(y) - v(x)] \cdot n_B(y)\, d\mathscr{H}(y) \right| \\
&\quad = \left| \int_{\operatorname{bd} B} [w(y) - v(y) + v(x)] \cdot n_B(y)\, d\mathscr{H}(y) \right| \\
&\quad \leqslant \int_{\operatorname{bd}_e B} \|v(y) - v(x) - w(y)\|\, d\mathscr{H}(y) \\
&\quad \leqslant \int_{\operatorname{bd}_e B} h(y)\, |y - x|\, d\mathscr{H}(y) \leqslant \varepsilon^2 d(B)\, \|B\| < \varepsilon\, |B|.
\end{aligned}$$

LEMMA 5.18. *If $A \in BV$ and $C \subset A$ has measure zero, then given $\varepsilon > 0$, there is a nonnegative additive function $H$ having the following properties: $H(A) < \varepsilon$, and for each $x \in C$ and each positive integer $n$ we can find a $\delta > 0$ so that $H(B) \geqslant n\,|B|$ whenever $B \in BV_A$, $x \in \mathrm{cl}\,B$ and $d(B) < \delta$.*

*Proof.* Find a decreasing sequence $\{U_n\}$ of open sets containing $C$ so that $|U_n| < \varepsilon/2^n$, $n = 1, 2, \ldots$, and for $B \in BV_A$ set $H(B) = \sum_{n=1}^{\infty} |B \cap U_n|$. Clearly, $0 \leqslant H < \varepsilon$ and $H$ is additive. Given $x \in C$ and an integer $n \geqslant 1$, there is a $\delta > 0$ such that $U(x, \delta) \subset U_n$. Now if $B \in BV_A$, $x \in \mathrm{cl}\,B$, and $d(B) < \delta$, then $H(B) \geqslant n\,|B|$ and the lemma is proved.

A vector field $v$ on an open set $U \subset \mathbf{R}^m$ is called *almost differentiable* at $x \in U$ if

$$\limsup_{y \to x} \frac{|v(y) - v(x)|}{|y - x|} < +\infty.$$

Now let $v$ be a vector field defined on an arbitrary set $A \subset \mathbf{R}^m$, and let $E$ be a measurable subset of $A$. We say that $v$ is almost differentiable on $E$ whenever $v$ can be extended to a vector field $w$ such that the domain of $w$ is a neighborhood of $E$ and $w$ is almost differentiable at each $x \in E$. By the Stepanoff theorem (see [7, Theorem 3.1.9, p. 218]), $w$ is differentiable almost everywhere in $E$, and by Lemma 5.16, almost everywhere in $E$, $\mathrm{div}\,w$ is determined uniquely by $v$. Thus in $E$ we set $\mathrm{div}\,v = \mathrm{div}\,w$.

THEOREM 5.19. *Let $A \in BV$, and let $T$ be a thin set. Suppose that $v$ is a continuous vector field on $\mathrm{cl}\,A$ which is almost differentiable on $\mathrm{cl}_e\,A - T$. Then $\mathrm{div}\,v$ is v-integrable in $A$ and*

$$I_v(\mathrm{div}\,v, A) = \int_{\mathrm{bd}\,A} v \cdot n_A \, d\mathscr{H}.$$

*Proof.* By our assumption, $v$ is extendable to a vector field $w$ such that $w$ is defined on a set $U$ whose interior contains $\mathrm{cl}_e\,A - T$, and $w$ is almost differentiable at every $x \in \mathrm{cl}_e\,A - T$. Since $w \restriction \mathrm{cl}\,A = v$ is continuous on $\mathrm{cl}\,A$, the function $F$ on $BV_A$ defined by

$$F(B) = \int_{\mathrm{bd}\,B} v \cdot n_B \, d\mathscr{H} = \int_{\mathrm{bd}\,B} w \cdot n_B \, d\mathscr{H}$$

is additive and continuous according to Example 4.2,2. We show that $F = I_v(\mathrm{div}\,v, \cdot)$. By Stepanoff's theorem (see [7, Theorem 3.1.9, p. 218]), there is a set $C \subset \mathrm{cl}_e\,A - T$ such that $|C| = 0$ and $w$ is differentiable on $\mathrm{cl}_e\,A - (T \cup C)$. With no loss of generality, we extend $\mathrm{div}\,w$ to $\mathrm{cl}_e\,A$ by zero (see Remark 5.10). Given $\varepsilon > 0$, let $H$ be a function on $BV_A$ associated with $C$ and $\varepsilon/2$ according to Lemma 5.18, and set

$$M(B) = \frac{\varepsilon\,|B|}{2(1+|A|)} + H(B)$$

for each $B \in BV_A$. Clearly, $M$ is a nonnegative additive function on $BV_A$ and $M(A) < \varepsilon$.

Let $x \in C$. There is an integer $k \geqslant 1$ and a $\delta > 0$ such that

$$\|w(y) - w(x)\| \leqslant k\,|y-x| \qquad \text{and} \qquad H(B) \geqslant \frac{k}{\varepsilon}\,|B|$$

for each $y \in U$ with $|y - x| < \delta$ and each $B \in BV_A$ with $x \in \operatorname{cl} B$ and $d(B) < \delta$. If $B \in BV_A$, $x \in \operatorname{cl} B$, $d(B) < \delta$, and $r(B) > \varepsilon$, then

$$\begin{aligned} |\operatorname{div} w(x)\,|B| - F(B)| &= \left| \int_{\operatorname{bd} B} [w(y) - w(x)] \cdot n_B(y)\, d\mathscr{H}(y) \right| \\ &\leqslant k \int_{\operatorname{bd}_e B} |y - x|\, d\mathscr{H}(y) \\ &\leqslant k d(B)\,\|B\| < \frac{k}{\varepsilon}\,|B| \leqslant H(B) < M(B). \end{aligned}$$

Let $x \in \operatorname{cl}_e A - (T \cup C)$. By Lemma 5.17, we can find a $\delta > 0$ so that

$$|\operatorname{div} w(x)\,|B| - F(B)| < \frac{\varepsilon\,|B|}{2(1+|A|)} \leqslant M(B)$$

for each $B \in BV_U$ for which $x \in \operatorname{cl} B$, $d(B) < \delta$, and $r(B) > \varepsilon$.

It follows that $M$ is an $\varepsilon$-majorant of the pair $(\operatorname{div} w, F)$ in $A$ mod $T$. In view of Lemma 5.16 and Remark 5.10, we conclude that

$$I_v(\operatorname{div} v, \cdot) = I_v(\operatorname{div} w, \cdot) = F.$$

*Remark* 5.20. The following points concerning Theorem 5.19 are noteworthy.

1. Neither side of the Gauss–Green formula depends on values of $v$ outside $\operatorname{cl}(\operatorname{cl}_e A)$, and it can be readily verified that the continuity of $v$ on $\operatorname{cl} A$ can be relaxed to that on $\operatorname{cl}(\operatorname{cl}_e A)$.

2. In terms of the measure $\mathscr{H}$, the exceptional set $T \cap \operatorname{cl}_e A$ is as large as possible. This is easily seen by considering the Cantor fuction on the Cantor ternary set (see [9, Sect. 19, Problem (3), p. 83]).

The next example is a modification of an example due to Z. Buczolich (cf. [2]).

EXAMPLE 5.21. Let $m=2$, and let $K=(a, a+h)\times(b, b+h)$ be a square where $a, b$, and $h$ are real numbers with $h>0$. For $(\xi, \eta)\in \mathbf{R}^2$ and $n=0, 1, \ldots$, set

$$v_n^K(\xi, \eta)=\begin{cases}\left(\dfrac{\pi}{4h}\left[1-\cos\dfrac{2^n\pi}{h}(\xi-a)\right]\sin\dfrac{\pi}{h}(\eta-b), 0\right) & \text{if } (\xi, \eta)\in K,\\ (0,0) & \text{otherwise.}\end{cases}$$

If $K^j=(a+(j-1)h/2^n,\ a+jh/2^n)\times(b, b+h)$, $j=1, \ldots, 2^n$, then $I_{\mathrm{v}}(\operatorname{div} v_n^K, K^j)=(-1)^{j-1}$.

Now adhering to the notation of Example 3.1, we let $v_{n,i}=2^{-4n+1}v_n^{R_{n,i}}$ for $i=1, \ldots, 8^n$ and $n=0, 1, \ldots$, and set

$$v=\sum_{n=0}^{\infty}\sum_{i=1}^{8^n} v_{n,i}.$$

The vector field $v$ is clearly differentiable in $R$, and since

$$\sum_{i=1}^{8^n}|v_{n,i}|\leqslant 6\pi(3/4)^n \qquad n=0, 1, \ldots,$$

$v$ is continuous in $\mathbf{R}^2$. We let $f(x)=\operatorname{div} v(x)$ if $x\in R$, and $f(x)=0$ if $x\in\mathbf{R}^2-R$. By Theorem 5.19 where $T=\mathrm{bd_e}R$, we see that $f$ is v-integrable in $R$ and $I_{\mathrm{v}}(f, R)=0$. It is also clear that $f$ is v-integrable in $S$ and $I_{\mathrm{v}}(f, S)=0$. In contrast, we shall see that $f$ is *not* v-integrable in $P$.

Proceeding towards a contradiction, suppose that $f$ is v-integrable in $P$, and find a thin set $T$ and a positive function $\delta$ on $P-T$ such that the conclusion of Proposition 5.15 holds for $\varepsilon=3^{-4}$. Let

$$C=\bigcap_{n=1}^{\infty}\bigcup_{i=1}^{8^n} P_{n,i}.$$

For $n=1, 2, \ldots$, denote by $\alpha_n$ the set of all integers $i$ with $1\leqslant i\leqslant 8^n$ for which there is an $x_{n,i}\in C\cap P_{n,i}-T$ such that $d(P_{n,i})<\delta(x_{n,i})$. Set $\beta_n=\{1, \ldots, 8^n\}-\alpha_n$, and denote the cardinalities of $\alpha_n$ and $\beta_n$ by $a_n$ and $b_n$, respectively. Now consider the closed sets

$$D_k=\bigcap_{n=1}^{k}\bigcup_{i\in\beta_n} P_{n,i} \qquad \text{and} \qquad D=\bigcap_{k=1}^{\infty} D_k\subset C.$$

If $x\in D-T$, then $\delta(x)>0$ and there is an integer $n\geqslant 0$ and $i\in\beta_n$ with

$x \in P_{n,i}$ and $d(P_{n,i}) < \delta(x)$. However, this is impossible since $i \notin \alpha_n$. Thus $D \subset T$, and we obtain a contradiction by showing that $D$ is not a thin set.

Set $\gamma_1 = \beta_1$, and for $k = 2, 3, \ldots$ let $\gamma_k$ be the set of all $i \in \beta_k$ for which $P_{k,i} \subset D_{k-1}$. Clearly, $D_k = \bigcup_{i \in \gamma_k} P_{k,i}$. If $c_k$ is the cardinality of $\gamma_k$, then

$$c_k \geqslant b_k - \sum_{n=1}^{k-1} 8^{k-n} a_n = 8^k \left(1 - \sum_{n=1}^{k} 8^{-n} a_n\right);$$

for each $P_{n,i}$, $1 \leqslant n \leqslant k$, covers precisely $8^{k-n}$ squares $P_{k,j}$. To calculate $\mathscr{H}(D)$, we obtain a lower estimate for the numbers $c_k$ by estimating the sum $\sum_{n=1}^{\infty} 8^{-n} a_n$.

For each $i \in \alpha_n$ and $n = 1, 2, \ldots$, there is a closed square $L_{n,i} \subset Q_{n,i} - \mathrm{cl}\, R_{n,i}$ of diameter $3^{-n-2}$. The sets $B_{n,i} = L_{n,i} \cup \bigcup_{j=1}^{2^n - 1} R_{n,i}^{2j}$, where $i \in \alpha_n$ and $n = 1, 2, \ldots$, are disjoint and a simple calculation reveals that

$$d(B_{n,i} \cup \{x_{n,i}\}) < \delta(x_{n,i}) \qquad \text{and} \qquad r(B_{n,i} \cup \{x_{n,i}\}) > 3^{-4}.$$

Since $f = 0$ on $L_{n,i} \cup \{x_{n,i}\}$, our choice of $T$ and $\delta$ yields

$$3^{-4} > \sum_{n=1}^{\infty} \sum_{i \in \alpha_n} |f(x_{n,i})\,|B_{n,i}| - I_v(f, B_{n,i})|$$
$$= \sum_{n=1}^{\infty} \sum_{i \in \alpha_n} \sum_{j=1}^{2^n - 1} |I_v(f, R_{n,i}^{2j})| = \sum_{n=1}^{\infty} 8^{-n} a_n,$$

and consequently, $c_k \geqslant 8^k/2$.

Let $s = \log 8/\log 3$, and let $\mathscr{H}^s$ be the $s$-dimensional Hausdorff measure in $\mathbf{R}^2$. Using covers by triadic squares, we show that $\mathscr{H}^s(D) > 0$. Since $s > 1$, it follows from [6, Sect. 1.2, p. 7] that the set $D$ is not thin.

A *triadic square* is the product $[i3^{-k}, (i+1)3^{-k}) \times [j3^{-k}, (j+1)3^{-k})$ where $i$, $j$, and $k$ are integers with $k \geqslant 0$. Cover $D$ by a sequence $\{U_r\}$ of triadic squares. For $r = 1, 2, \ldots$, let $\mathscr{U}_r$ be the collection of all triadic squares $U$ with $d(U) = d(U_r)$ and $(\mathrm{cl}\, U) \cap (\mathrm{cl}\, U_r) \neq \varnothing$, and set $V_r = \bigcup \mathscr{U}_r$. As the interiors of the $V_r$'s cover the compact set $D$, there is an integer $p \geqslant 1$ such that

$$D \subset \bigcup_{r=1}^{p} \mathrm{int}\, V_r \subset \mathrm{int}\left(\bigcup_{r=1}^{p} V_r\right).$$

It follows that $D_k \subset \bigcup_{r=1}^{p} V_r$ for all sufficiently large indices $k$. If $\bigcup_{r=1}^{p} \mathscr{U}_r$ consists of triadic squares $W_1, \ldots, W_q$ and $d(W_t) = 3^{-d_t}$, we select a fixed integer $k > \max\{d_t : t = 1, \ldots, q\}$ so that

$$D_k \subset \bigcup_{r=1}^{p} V_r = \bigcup_{t=1}^{q} W_t.$$

Each $W_t$ contains at most $8^{k-d_t}$ squares $P_{k,i}$, and the number of the squares $P_{k,i}$ contained in $\bigcup_{t=1}^{q} W_t$ is not less than the number of those contained in $D_k$. Hence

$$\sum_{t=1}^{q} 8^{k-d_t} \geqslant c_k \geqslant \frac{8k}{2},$$

and so $\sum_{t=1}^{q} 8^{-d_t} \geqslant 1/2$. Since $3^s = 8$, we have

$$\sum_r [d(U_r)]^s \geqslant \frac{1}{9} \sum_{t=1}^{q} 8^{-d_t} \geqslant \frac{1}{18},$$

and our assertion follows from [6, Theorem 5.1, p. 65].

## 6. A Perron Definition of the Variational Integral

We say that a sequence $\{A_n\}$ of $BV$ sets *shrinks* to a point $x \in \mathbf{R}^m$ whenever $x \in \operatorname{cl} A_n$, $n = 1, 2, \ldots$, $\lim d(A_n) = 0$, and $\inf r(A_n) > 0$. If $A$ is a BV set, $M$ is a function on $BV_A$, and $x \in \operatorname{cl}_e A$, we let

$$M_*(x) = \inf \liminf_{n \to \infty} \frac{M(B_n)}{|B_n|}$$

where the infimum is taken over all sequences $\{B_n\}$ in $BV_A$ which shrink to $x$.

Let $A \in BV$ and let $f$ be a function on $\operatorname{cl}_e A$. We say that a lower continuous superadditive function $M$ on $BV_A$ is a *majorant* of $f$ in $A$ if there is a thin set $T$ such that $M_*(x) \geqslant f(x)$ for each $x \in \operatorname{cl}_e A - T$. The *extended* real number

$$U(f, A) = \inf M(A),$$

where the infimum is taken over all majorants of $f$ in $A$, is called the *upper integral* of $f$ over $A$. The extended real valued function $B \mapsto U(f \restriction \operatorname{cl}_e B, B)$ on $BV_A$ is called the *indefinite upper integral* of $f$ in $A$, denoted by $U(f, \cdot)$. Observe that $U(f \restriction \operatorname{cl}_e B, \cdot) = U(f, \cdot) \restriction BV_B$ for each $B \in BV_A$. We say that $f$ is *Perron integrable* (abbreviated as P-integrable) in $A$ if

$$-U(-f, A) = U(f, A) \neq \pm\infty.$$

The family of all P-integrable functions in $A$ is denoted by $\mathscr{P}(A)$.

*Note.* The reader should carefully distinguish between $\varepsilon$-majorants and majorants, remembering that the former are applied to a pair of functions, while the latter are applied to a single function. Also the upper integral $U(f, A)$ and the neighborhood $U(x, \varepsilon)$ have nothing in common. We trust that when taken in the context, these superficial similarities will cause no confusion.

LEMMA 6.1. *If $A \in BV$ and $f$ is a function defined on $\mathrm{cl}_e A$, then $-U(-f, A) \leqslant U(f, A)$.*

*Proof.* Suppose that $U(f, A) < -U(-f, A)$. Then in $A$ there are majorants $M$ and $N$ of $f$ and $-f$, respectively, such that $M(A) < -N(A)$. Let $T$ be the union of the thin sets associated with $M$ and $N$, and choose an $\varepsilon > 0$ so that $M(A) + \varepsilon |A| < -N(A)$. Since the function $F = M + N + \varepsilon\lambda_m$ is superadditive and lower continuous and $F(A) < 0$, it follows from Proposition 4.6 that there is an $x \in \mathrm{cl}_e A - T$ and a sequence $\{C_k\}$ in $BV_A$ shrinking to $x$ such that $F(C_k) < 0$, $k = 1, 2, \ldots$. Thus

$$0 \geqslant F_*(x) \geqslant M_*(x) + N_*(x) + \varepsilon \geqslant \varepsilon,$$

a contradiction.

LEMMA 6.2. *Let $A \in BV$ and $f \in \mathscr{P}(A)$. Then $f \restriction \mathrm{cl}_e B \in \mathscr{P}(B)$ for each $B \in BV_A$, and the indefinite upper integral $U(f, \cdot)$ is a continuous additive function on $BV_A$.*

*Proof.* Observe first that if $M$ is a majorant of $f$ in $A$, then for each $B \in BV_A$ the restriction $M \restriction BV_B$ is a majorant of $f \restriction \mathrm{cl}_e B$ in $B$. Given $\varepsilon > 0$, in $A$ there are majorants $M$ and $N$ of $f$ and $-f$, respectively, such that

$$M(A) < U(f, A) + \frac{\varepsilon}{2} \qquad \text{and} \qquad N(A) < U(-f, A) + \frac{\varepsilon}{2}.$$

In view of Lemma 6.1 and the observation above,

$$-N(B) \leqslant -U(-f, B) \leqslant U(f, B) \leqslant M(B)$$

and we see that $U(f, \cdot)$ and $U(-f, \cdot)$ are real-valued functions on $BV_A$. Since $M + N$ is a nonnegative superadditive function on $BV_A$ and

$$M(A) + N(A) < U(f, A) + U(-f, A) + \varepsilon = \varepsilon,$$

it is easy to see that $M + N < \varepsilon$.

If $\mathscr{D}$ is a division of $A$, then by Lemma 6.1,

$$\begin{aligned}-\varepsilon < -[M(A)+N(A)] &\leqslant -M(A) - \sum_{D\in\mathscr{D}} N(D)\\ &\leqslant -U(f,A) - \sum_{D\in\mathscr{D}} U(-f,D)\\ &\leqslant U(-f,A) + \sum_{D\in\mathscr{D}} U(f,D) \leqslant N(A) + \sum_{D\in\mathscr{D}} M(D)\\ &\leqslant N(A)+M(A) < \varepsilon.\end{aligned}$$

By our assumption, $U(-f,A) = -U(f,A)$ and thus

$$\left|\sum_{D\in\mathscr{D}} U(f,D) - U(f,A)\right| < \varepsilon.$$

From the arbitrariness of $\varepsilon$ we obtain the additivity of $U(f,\cdot)$; the additivity of $U(-f,\cdot)$ follows by symmetry.

This in conjunction with Lemma 6.1 implies that $-U(-f,\cdot) = U(f,\cdot)$, or alternatively that $f \restriction \mathrm{cl}_e B \in \mathscr{P}(B)$ for each $B \in BV_A$.

As $M - U(f,\cdot)$ is a nonnegative superadditive function on $BV_A$ and $M(A) - U(f,A) < \varepsilon/2$, we have $M - U(f,\cdot) < \varepsilon/2$, and similarly $N - U(-f,\cdot) < \varepsilon/2$. By the lower continuity of $M$ and $N$, we can find a $\delta > 0$ so that

$$-\varepsilon < M(B) - \frac{\varepsilon}{2} < U(f,B) = -U(-f,B) < \frac{\varepsilon}{2} - N(B) < \varepsilon$$

for each $B \in BV_A$ with $|B| < \delta$ and $\|B\| < 1/\varepsilon < 2/\varepsilon$. This establishes the continuity of $U(f,\cdot)$.

PROPOSITION 6.3. *If $A \in BV$, then $\mathscr{P}(A) = \mathscr{I}_v(A)$ and $U(f,\cdot) = I_v(f,\cdot)$ for each $f \in \mathscr{P}(A)$.*

*Proof.* Choose an $\varepsilon > 0$ and suppose first that $f \in \mathscr{P}(A)$. In $A$ there are majorants $M$ and $N$ of $f$ and $-f$, respectively, such that $M(A) + N(A) < \varepsilon/2$. Let $T$ be the union of the thin sets associated with $M$ and $N$, and let

$$\varphi(B) = \frac{\varepsilon\,|B|}{2(1+|A|)}$$

for each $B \in BV_A$. Clearly, $H = M + N + \varphi$ is a nonnegative superadditive function on $BV_A$ and $H(A) < \varepsilon$. Fix an $x \in \mathrm{cl}_e A - T$. Since

$$(M+\varphi)_*(x) > f(x) \qquad \text{and} \qquad (N+\varphi)_*(x) > -f(x),$$

we can find a $\delta > 0$ so that

$$M(B) + \varphi(B) \geqslant f(x)\,|B| \geqslant -N(B) - \varphi(B)$$

for each $B \in BV_A$ with $x \in \operatorname{cl} B$, $d(B) < \delta$, and $r(B) > \varepsilon$. As $-N \leqslant U(f, \cdot) \leqslant M$, given such a set $B$ we also have

$$|f(x)\,|B| - U(f, B)| \leqslant H(B).$$

Thus $H$ is an $\varepsilon$-majorant of the pair $(f, U(f, \cdot))$ in $A$ mod $T$, and it follows from Lemma 6.2 that $f \in \mathscr{I}_{\mathrm{v}}(A)$ and $I_{\mathrm{v}}(f, \cdot) = U(f, \cdot)$.

Conversely, suppose that $f \in \mathscr{I}_{\mathrm{v}}(A)$, and for $n = 1, 2, \ldots$ find a thin set $T_n$ so that the pair $(f, I_{\mathrm{v}}(f, \cdot))$ has an $(\varepsilon/2^n)$-majorant $H_n$ in $A$ mod $T_n$. Let

$$T = \bigcup_{n=1}^{\infty} T_n \qquad \text{and} \qquad H = \sum_{n=1}^{\infty} H_n.$$

Now fix an $x \in \operatorname{cl}_e A - T$ and choose a sequence $\{B_k\}$ in $BV_A$ shrinking to $x$. As $r(B_k) > \varepsilon/2^p$ for some integer $p \geqslant 1$ and $k = 1, 2, \ldots$, we have

$$\begin{aligned} I_{\mathrm{v}}(f, B_k) - H(B_k) \leqslant I_{\mathrm{v}}(f, B_k) - H_p(B_k) &\leqslant f(x)\,|B_k| \\ &\leqslant I_{\mathrm{v}}(f, B_k) + H_p(B_k) \leqslant I_{\mathrm{v}}(f, B_k) + H(B_k) \end{aligned}$$

for all sufficiently large $k$. Consequently

$$[I_{\mathrm{v}}(f, \cdot) + H]_*(x) \geqslant f(x) \qquad \text{and} \qquad [H - I_{\mathrm{v}}(f, \cdot)]_*(x) \geqslant -f(x)$$

from which we see that in $A$, the functions $I_{\mathrm{v}}(f, \cdot) + H$ and $H - I_{\mathrm{v}}(f, \cdot)$ are majorants of $f$ and $-f$, respectively. Thus by Lemma 6.1

$$\begin{aligned} I_{\mathrm{v}}(f, A) - \varepsilon \leqslant I_{\mathrm{v}}(f, A) - H(A) &\leqslant -U(-f, A) \leqslant U(f, A) \\ &\leqslant I_{\mathrm{v}}(f, A) + H(A) \leqslant I_{\mathrm{v}}(f, A) + \varepsilon, \end{aligned}$$

and it follows from the arbitrariness of $\varepsilon$ that $U(f, A) = -U(-f, A) \neq \pm\infty$.

*Remark* 6.4. The following comments are related to those made in Remark 5.2,4:

(a) The definition of a sequence $\{A_n\}$ shrinking to a point $x$ can be modified so that $x \in \operatorname{cl} A_n$ is replaced by $x \in \operatorname{cl}_e A_n$. The resulting P-integral coincides with the v-integral modified according to Remark 5.2,4(a).

(b) We may require that majorants are additive rather than superadditive. It is easy to see that the P-integral defined by means of additive majorants, called the *Pa*-integral, coincides with the va-integral defined in Remark 5.2,4(b).

Let $E \subset \mathbf{R}^m$ be a measurable set. For a Lipschitzian map $\Phi: E \to \mathbf{R}^m$ (see [7, Sect. 2.2.7, p. 63]), we denote by $\det \Phi$ the determinant of the Jacobi matrix $D\Phi$ of $\Phi$. By the Kirszbraun and Rademacher theorems (see [7, Theorems 2.10.43 and 3.1.6, pp. 201 and 216]), the function $\det \Phi$ is defined almost everywhere in $E$, and by Lemma 5.16, it is determined uniquely up to a set of measure zero. A Lipschitzian map $\Phi: E \to \mathbf{R}^m$ is called a *lipeomorphism* if it is injective and the inverse map $\Phi^{-1}: \Phi(E) \to \mathbf{R}^m$ is also Lipschitzian. If $\Phi$ is a lipeomorphism, then $\det \Phi(x) \neq 0$ for almost all $x \in E$.

LEMMA 6.5. *Let $\Phi$ be a lipeomorphism of a measurable set $E \subset \mathbf{R}^m$. Then $\Phi$ extends uniquely to a lipeomorphism of* $\operatorname{cl} E$ *into $\mathbf{R}^m$, also denoted by $\Phi$, and $\Phi(\operatorname{cl}_e E) = \operatorname{cl}_e \Phi(E)$.*

*Proof.* There are positive constants $a$ and $b$ such that $a\,|x - x'| \leqslant |\Phi(x) - \Phi(x')| \leqslant b\,|x - x'|$ for all $x, x' \in E$. By the completeness of $\mathbf{R}^m$, the map $\Phi$ extends uniquely to $\operatorname{cl} E$. As the extended map, still denoted by $\Phi$, satisfies the above inequalities for all $x, x' \in \operatorname{cl} E$, it is a lipeomorphism. Let $x \in \operatorname{cl} E$ be a dispersion point of $E$, and let $y = \Phi(x)$. Since

$$\Phi(E) \cap U(y, \varepsilon) \subset \Phi(E \cap U(x, \varepsilon/a)),$$

$$|\Phi(E \cap U(x, \varepsilon/a))| \leqslant b^m\, |E \cap U(x, \varepsilon/a)|$$

for each $\varepsilon > 0$ (cf. [6, Lemma 1.8, p. 10]), we obtain

$$\limsup_{\varepsilon \to 0+} \frac{|\Phi(E) \cap U(y, \varepsilon)|}{\varepsilon^m} \leqslant \left(\frac{b}{a}\right)^m \limsup_{\varepsilon \to 0+} \frac{|E \cap U(x, \varepsilon/a)|}{(\varepsilon/a)^m} = 0.$$

Thus $y$ is a dispersion point of $\Phi(E)$, and the lemma follows by symmetry.

LEMMA 6.6. *Let $A \in BV$ and let $\Phi: A \to \mathbf{R}^m$ be a lipeomorphism with a Lipschitz constant $\alpha$. Then $B = \Phi(A)$ is a $BV$ set with $|B| \leqslant \alpha^m\, |A|$ and $\|B\| \leqslant \alpha^{m-1}\, \|A\|$.*

*Proof.* Since our argument relies on interpreting BV sets as *integral currents*, we shall employ the notation of [7, Chapter 4]. As $X = \mathbf{E}^m \llcorner A$ is an integral current, so is $\Phi_{\#}(X)$ (see [7, Sects. 4.5.1 and 4.1.14, pp. 474 and 370]). It follows from [7, Corollary 4.1.26, p. 383] that $\Phi_{\#}(X) = \mathbf{E}^m \llcorner h$, where $h$ is a function on $\mathbf{R}^m$ defined as follows:

$$h(y) = \begin{cases} \dfrac{\det \Phi(x)}{|\det \Phi(x)|} & \text{if } \; y = \Phi(x) \text{ and the fraction is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Phi$ is a lipeomorphism, $|h|$ is equal to the characteristic function of $B$ almost everywhere. Thus letting $Y = \mathbf{E}^m \llcorner B$, we have

$$|B| = \mathbf{M}(Y) = \mathbf{M}(\Phi_\#(X)) \leqslant \alpha^m \mathbf{M}(X) = \alpha^m |A|,$$

$$\begin{aligned}\|B\| &= \mathbf{M}(\partial Y) \leqslant \mathbf{M}(\partial \Phi_\#(X)) = \mathbf{M}(\Phi_\#(\partial X)) \\ &\leqslant \alpha^{m-1} \mathbf{M}(\partial X) = \alpha^{m-1} \|A\|.\end{aligned}$$

THEOREM 6.7. *Let $A \in BV$, let $\Phi \colon A \to \mathbf{R}^m$ be a lipeomorphism, and let $f \in \mathcal{I}_v(\Phi(A))$. Then $f \circ \Phi \cdot |\det \Phi|$ belongs to $\mathcal{I}_v(A)$ and*

$$I_v(f \circ \Phi \cdot |\det \Phi|, A) = I_v(f, \Phi(A)).$$

*Proof.* In accordance with Lemma 6.5, view $\Phi$ as a lipeomorphism of $\mathrm{cl}\, A$. By Lemma 6.6, there are positive constants $a$, $\alpha$, $b$, $\beta$, and $\gamma$ such that the following inequalities hold:

(1) $a\,|x - x'| \leqslant |\Phi(x) - \Phi(x')| \leqslant \alpha\,|x - x'|$ for each $x, x' \in \mathrm{cl}\, A$;

(2) $b\,|B| \leqslant |\Phi(B)| \leqslant \beta\,|B|$ for each $B \in BV_A$;

(3) $\|\Phi(B)\| \leqslant \gamma\,\|B\|$ for each $B \in BV_A$.

The isoperimetric inequality shows that $|B|/[d(B)]^m \geqslant c[r(B)]^m$ for each BV set $B$ and a positive constant $c$ depending only on $m$. Since $|\Phi(B)| = \int_B |\det \Phi|\, d\lambda_m$ for every measurable set $B \subset A$ (see [7, Theorem 3.2.3,(1), p. 243]), it follows from [33, Chap. IV, Theorem (6.3), p. 118] that for almost all $x \in A$,

$$\lim \frac{|\Phi(A_n)|}{|A_n|} = |\det \Phi(x)|$$

for every sequence $\{A_n\}$ in $BV_A$ shrinking to $x$. We denote by $C$ the set of all $x \in \mathrm{cl}_e\, A$ for which either $\det \Phi(x)$ is not defined, or there is a sequence $\{A_n\}$ in $BV_A$ shrinking to $x$ such that the above equation does not hold. Clearly, $|C| = 0$.

Choose an $\varepsilon > 0$, and find a majorant $M$ of $f$ in $\Phi(A)$ so that $M(\Phi(A)) < U(f, \Phi(A)) + \varepsilon$. Select a function $H$ on $BV_A$ associated with $C$ and $\varepsilon$ according to Lemma 5.18, and let

$$N(B) = M(\Phi(B)) + H(B)$$

for each $B \in BV_A$. In view of Lemma 6.6, $N$ is a well defined superadditive function on $BV_A$, and the lower continuity of $M$ together with (2) and (3) imply that it is lower continuous. There is a thin set $T$ such that $M_*(y) \geqslant f(y)$ for each $y \in \mathrm{cl}_e\, \Phi(A) - T$. By (1) and [6, Lemma 1.8, p. 10], the set $S = \Phi^{-1}(\Phi(\mathrm{cl}\, A) \cap T)$ is thin. Let $x \in \mathrm{cl}_e\, A - S$, $y = \Phi(x)$, let $\{A_n\}$ be a sequence in $BV_A$ shrinking to $x$, and for $n=1, 2, \ldots$, let $B_n=\Phi(A_n)$. According to Lemmas 6.5 and 6.6, $y \in \mathrm{cl}_e \Phi(A) - T$ and $\{B_n\}$ is a sequence in $BV_{\Phi(A)}$ shrinking to $y$. If $x \notin C$, then

$$\begin{aligned}\frac{|N(A_n)|}{|A_n|} &\geqslant \liminf \left(\frac{M(B_n)}{|B_n|} \cdot \frac{|\Phi(A_n)|}{|A_n|}\right) \geqslant M_*(y)\, |\det \Phi(x)| \\ &\geqslant f(y)\, |\det \Phi(x)|,\end{aligned}$$

and hence $N_*(x) \geqslant f(\Phi(x))\, |\det \Phi(x)|$. Now if $x \in C$, then it follows from Lemma 5.18 that $\lim(H(A_n)/|A_n|) = +\infty$. Since

$$\liminf \left(\frac{M(B_n)}{|B_n|} \cdot \frac{|\Phi(A_n)|}{|A_n|}\right) \geqslant b M_*(y)$$

(cf. (2)), we see that $N_*(x) = +\infty$, and hence again $N_*(x) \geqslant f(\Phi(x))\, |\det \Phi(x)|$. Consequently, $N$ is a majorant of $f \circ \Phi \cdot |\det \Phi|$ in $A$, and so

$$\begin{aligned}U(f \circ \Phi \cdot |\det \Phi|, A) &\leqslant N(A) \\ &= M(\Phi(A)) + H(A) < U(f, \Phi(A)) + 2\varepsilon.\end{aligned}$$

The arbitrariness of $\varepsilon$ implies that $U(f \circ \Phi \cdot |\det \Phi|, A) \leqslant U(f, \Phi(A))$. Applying this result to the function $-f$ and using Lemma 6.1 yields

$$\begin{aligned}-U(-f, \Phi(A)) &\leqslant -U(-f \circ \Phi \cdot |\det \Phi|, A) \\ &\leqslant U(f \circ \Phi \cdot |\det \Phi|, A) \leqslant U(f, \Phi(A)),\end{aligned}$$

and the theorem follows from Proposition 6.3.

Our next proposition compares the variational and Denjoy–Perron integrals in dimension one. For the definition and properties of the Denjoy–Perron integral (abbreviated as DP-integral) we refer to [33, Chap. VI, Sect. 6 and Chap. VIII, Sect. 5].

PROPOSITION 6.8. *Let $m=1$, and let $A=[a, b]$ where $a, b \in \mathbf{R}$ and $a<b$.*

1. *If $f \in \mathscr{I}_v(A)$, then $f$ is DP-integrable in $A$ and $I_v(f, A)$ is the value of the DP-integral of $f$ over $A$.*

2. *There is a function $f$ on $A$ which is DP-integrable but not v-integrable in $A$.*

*Proof.* We only prove the first statement. The function $f$ of the second statement is constructed in Example 6.9 below.

Let $f \in \mathscr{I}_v(A)$ and $\varepsilon > 0$. By Proposition 6.3, in $A$ there are majorants $M$ and $N$ of $f$ and $-f$, respectively, such that $M(A) + N(A) < \varepsilon/2$. Enumerate as $\{t_1, t_2, ...\}$ the union of the thin sets associated with $M$ and $N$. Let $[c, d]$ be a subinterval of $A=[a, b]$ with $c<d$. For $n=1, 2, ...$ set

$$\varphi_n([c, d])=\begin{cases} 1 & \text{if} \quad t_n \in (c, d), \\ 1/2 & \text{if} \quad t_n = c \quad \text{or} \quad t_n = d, \\ 0 & \text{if} \quad t_n \notin [c, d], \end{cases}$$

and let $\varphi = \sum_n 2^{-n-2}\varepsilon\varphi_n$. Moreover, set

$$M_\circ([c, d])=\begin{cases} M([a, d)) - M([a, c)) & \text{if} \quad d<b, \\ M([a, d]) - M([a, c)) & \text{if} \quad d=b, \end{cases}$$

and define $N_\circ$ analogously. Since $M$ and $N$ are lower continuous superadditive functions on $BV_A$, it is easy to verify that in $A$ the functions $M_\circ + \varphi$ and $-N_\circ - \varphi$ are, respectively, a majorant and a minorant of $f$ in the sense of [33, Chap. VI, Sect. 6, p. 201]. As

$$[M_\circ(A) + \varphi(A)] - [-N_\circ(A) - \varphi(A)] < \varepsilon,$$

it follows from [33, Chap. VIII, Sect. 3] that $f$ is DP-integrable in $A$, and we denote by $I$ the value of the DP-integral of $f$ over $A$. From the inequalities

$$-N(A) - \frac{\varepsilon}{2} \leqslant -N_\circ(A) - \varphi(A) \leqslant I \leqslant M_\circ(A) + \varphi(A) \leqslant M(A) + \frac{\varepsilon}{2},$$

$$-M(A) \leqslant -U(f, A) = U(-f, A) \leqslant N(A),$$

we obtain $|I - U(f, A)| \leqslant \varepsilon$, and by the arbitrariness of $\varepsilon$ and Proposition 6.3,

$$I = U(f, A) = I_v(f, A).$$

EXAMPLE 6.9. For $m=1$, we define first the DP-integrable function introduced in [28, Example 8.6].

If $K=(a, b)$ is a subinterval of $\mathbf{R}$ with $h=b-a>0$ and $n=1, 2, ...,$ let

$$K_{n_+} = (a + 2^{-2n}h, a + 2^{-2n+1}h),$$
$$K_{n_-} = (a + 2^{-2n+1}h, a + 2^{-2n+2}h),$$

and for $x \in \mathbf{R}$ set

$$f_K(x)=\begin{cases} 2^{2n}/n & \text{if} \quad x \in K_{n_+}, \\ -2^{2n-1}/n & \text{if} \quad x \in K_{n_-}, \\ 0 & \text{otherwise}. \end{cases}$$

Now let $C$ be any Cantor set in $A=[0,1]$ (e.g., take $C$ of Example 2.1), let $\mathscr{C}$ be the family of all connected components of $A-C$, and let $f=\sum_{K\in\mathscr{C}} f_K$. Since

$$\sup\left\{\left|\int_c^d f_K\,d\lambda\right| : [c,d]\subset K\right\}=|K|,$$

it is easy to see from [33, Chap. VIII, Theorem (5.1), p. 257] that $f$ is DP-integrable in $A$ and that the value of the DP-integral of $f$ over $A$ is zero. However, $f$ is not v-integrable in $A$.

Proceeding towards a contradiction, suppose that $f$ is v-integrable in $A$, and choose a thin set $T$ and a positive function $\delta$ on $A-T$. As $T$ is a countable set, $C-T$ is a $G_\delta$ set and hence it is completely metrizable. Using the Baire category theorem in $C-T$, there is an open interval $L$ with $L\cap(C-T)\neq\emptyset$, and an $\eta>0$ such that the set $E=\{x\in L\cap(C-T): \delta(x)\geqslant\eta\}$ is dense in $L\cap(C-T)$. Since $C$ is perfect and $T$ is countable, $E$ is dense in $L\cap C$. Select a $K=(a,b)$ in $\mathscr{C}$ with $a\in L\cap C$, and construct a sequence $\{x_n\}$ in $E$ so that $a-2^{-2n}|K|<x_n<x_{n+1}<a$, $n=1,2,\ldots$. Then $d(K_{n_+}\cup\{x_n\})\leqslant 3\cdot 2^{-2n}|K|$, $r(K_{n_+}\cup\{x_n\})\geqslant 1/6$, and

$$\sum_{n=1}^{\infty}|f(x_n)\,|K_{n_+}|-I_v(f,K_{n_+})|=\sum_{n=1}^{\infty} I_v(f,K_{n_+})=\sum_{n=1}^{\infty}\frac{|K|}{n}=+\infty.$$

Since $d(K_{n_+}\cup\{x_n\})<\eta\leqslant\delta(x_n)$ for all sufficiently large $n$, this contradicts Proposition 5.15.

*Remark* 6.10. Assuming that the function $f$ of Example 6.9 is integrable with respect to the v-integral modified according to Remark 5.2,4(a), we still obtain a contradiction. Indeed, if suffices to select $y_n$ so that $x_n<y_n<x_{n+1}$ and

$$\sum_{n=1}^{\infty}(|f(x_n)|\cdot|[x_n,y_n]|+|I_v(f,[x_n,y_n])|)<+\infty,$$

and observe that $x_n\in \mathrm{cl}_e(K_{n_+}\cup[x_n,y_n])$ and

$$\sum_{n=1}^{\infty}|f(x_n)\,|K_{n_+}\cup[x_n,y_n]|-I_v(f,K_{n_+}\cup[x_n,y_n])|=+\infty.$$

## 7. A. Riemann Definition of the Variational Integral

We begin by generalizing the concept of dyadic partition introduced in Section 4.

DEFINITION 7.1. Let $A \in BV$, let $T$ be a thin set, and let $\varepsilon > 0$. Furthermore, let $\delta$ be a positive function on $\mathrm{cl}_e A - T$, and let $H$ be a lower continuous superadditive function on $BV_A$. A *partition* in $A \bmod T$ is a collection (possibly empty) $P = \{(A_1, x_1), ..., (A_p, x_p)\}$, where $A_1, ..., A_p$ are disjoint BV subsets of $A$ and $x_i \in (\mathrm{cl}\, A_i) \cap (\mathrm{cl}_e A - T)$, $i = 1, ..., p$. We let $\bigcup P = \bigcup_{i=1}^{p} A_i$ and say that the partition $P$ is

1. *tight* if $x \in \mathrm{cl}_e A_i$, $i = 1, ..., p$;
2. an *$\varepsilon$-partition* if $r(A_i) > \varepsilon$, $i = 1, ..., p$;
3. *$\delta$-fine* if $d(A_i) < \delta(x_i)$, $i = 1, ..., p$;
4. *$H$-approximating* if $H(A - \bigcup P) > -1$.

The family of all $\delta$-fine $H$-approximating $\varepsilon$-partitions in $A \bmod T$ is denoted by $\Pi(A, T; \varepsilon, \delta, H)$.

The following lemma generalizes Lemma 4.4.

LEMMA 7.2. *Let $A \in BV$, let $T$ be a thin set, and let $\delta$ be a positive function on $\mathrm{cl}_e A - T$. If $0 < \varepsilon < 1/(2m)$ and $H$ is a superadditive lower continuous function on $BV_A$, then for each $\eta > 0$ there is a $\delta$-fine tight $\varepsilon$-partition $P$ in $A \bmod T$ such that $H(A - \bigcup P) > -\eta$. In particular, $\Pi(A, T; \varepsilon, \delta, H) \neq \emptyset$.*

*Proof.* Say that $B \in BV_A$ is *vile* if there is an $\eta > 0$ such that for each $\delta$-fine tight $\varepsilon$-partition $P$ in $B \bmod T$ we have $H(B - \bigcup P) \leqslant -\eta$, and observe that there is a largest $\eta$ with this property, denoted by $\eta_B$. For $B \in BV_A$ set

$$F(B) = \begin{cases} -\eta_B & \text{if } B \text{ is vile,} \\ 0 & \text{otherwise.} \end{cases}$$

Since $H(B) \leqslant F(B)$ for each vile set $B \in BV_A$, the lower continuity of $H$ implies that of $F$.

Let $B$ and $C$ be disjoint $BV$ subsets of $A$ and suppose that $F(B \cup C) < F(B) + F(C)$. By the definition of $F$, there are $\delta$-fine tight $\varepsilon$-partitions $P_B$ in $B \bmod T$ and $P_C$ in $C \bmod T$ such that

$$\begin{aligned} -\eta_{B \cup C} = F(B \cup C) &< H\left(B - \bigcup P_B\right) + H\left(C - \bigcup P_C\right) \\ &\leqslant H\left[B \cup C - \bigcup (P_B \cup P_C)\right]. \end{aligned}$$

As $P_B \cup P_C$ is a $\delta$-fine tight $\varepsilon$-partition in $B \cup C \bmod T$, this is a contradiction. It follows that $F$ is superadditive.

If $F(A)<0$, then by Proposition 4.6 there are an $x \in \mathrm{cl}_e A - T$ and a sequence $\{C_k\}$ in $BV_A$ such that $x \in \mathrm{cl}_e C_k$, $r(C_k)>\varepsilon$, $F(C_k)<0$, $k=1, 2, \ldots$, and $\lim d(C_k)=0$. Observe that $\{(C_n, x)\}$ is a $\delta$-fine tight $\varepsilon$-partition in $C_n$ mod $T$ whenever $d(C_n)<\delta(x)$. By the lower continuity of $H$ we have $0 \leqslant H(\varnothing)=H(C_n - C_n)$, so the sets $C_n$ are not vile for all sufficiently large $n$. This contradiction shows that $F(A)=0$, and the lemma is proved.

If $A \in BV$ and $f$ is a function on $\mathrm{cl}_e A$, then for each partition $P=\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in $A$ mod $\varnothing$, let

$$\sigma(f, P)=\sum_{i=1}^{p} f(x_i)\,|A_i|.$$

DEFINITION 7.3. Let $A \in BV$ and let $f$ be a function on $\mathrm{cl}_e A$. We say that $f$ is *Riemann integrable* (abbreviated as R-integrable) in $A$ if there is a real number $R$ which satisfies the following condition: given $\varepsilon>0$ there is a thin set $T$, a positive function $\delta$ on $\mathrm{cl}_e A - T$, and a lower continuous superadditive function $H$ on $BV_A$ such that

$$|\sigma(f, P)-R|<\varepsilon$$

for each $P \in \Pi(A, T; \varepsilon, \delta, H)$.

Let $A$ be a BV set. The family of all R-integrable functions in $A$ is denoted by $\mathscr{R}(A)$. Suppose that $f \in \mathscr{R}(A)$ and that the numbers $R_1$ and $R_2$ satisfy the condition of Definition 7.3. Choose a positive $\varepsilon<1/(2m)$, and if $T_i$, $\delta_i$, and $H_i$, $i=1, 2$, are associated with $R_i$ and $\varepsilon$ according to Definition 7.3, set $T=T_1 \cup T_2$, $\delta=\min\{\delta_1, \delta_2\}$ and $H=\min\{H_1, H_2\}$. As $H$ is lower continuous and superadditive, there is a $P \in \Pi(A, T; \varepsilon, \delta, H)$ by Lemma 7.2. Since

$$|R_1-R_2| \leqslant |R_1-\sigma(f, P)|+|\sigma(f, P)-R_2|<2\varepsilon,$$

the arbitrariness of $\varepsilon$ implies that $R_1=R_2$. Thus the number $R$ from Definition 7.3 is uniquely determined by $f$ and we denote it by $R(f, A)$.

LEMMA 7.4. *Let $A \in BV$ and let $f$ be a function on $\mathrm{cl}_e A$. Then $f \in \mathscr{R}(A)$ whenever for each $\varepsilon>0$ there is a thin set $T$, a positive function $\delta$ on $\mathrm{cl}_e A - T$, and a lower continuous superadditive function $H$ on $BV_A$ such that*

$$|\sigma(f, P)-\sigma(f, Q)|<\varepsilon$$

*for each $P$ and $Q$ in $\Pi(A, T; \varepsilon, \delta, H)$.*

*Proof.* For $\varepsilon_n = 1/(3mn)$, $n = 1, 2, \ldots$, find $T_n$, $\delta_n$, and $H_n$ so that the condition of the lemma is satisfied. We may assume that $T_n \subset T_{n+1}$, $\delta_{n+1} \leqslant \delta_n$, and $H_{n+1} \leqslant H_n$. By Lemma 7.2 there is a $P_n \in \Pi(A, T_n; \varepsilon_n, \delta_n, H_n)$, and by our assumptions the sequence $\{\sigma(f, P_n)\}$ is Cauchy; for $|\sigma(f, P_r) - \sigma(f, P_s)| < \varepsilon_r$ whenever $r \leqslant s$. Let $R = \lim \sigma(f, P_n)$, choose $\varepsilon > 0$, and find an integer $s \geqslant 1$ with $\varepsilon_s < \varepsilon/2$ and $|\sigma(f, P_s) - R| < \varepsilon/2$. Now if $P \in \Pi(A, T_s; \varepsilon, \delta_s, H_s)$ then

$$|\sigma(f, P) - R| \leqslant |\sigma(f, P) - \sigma(f, P_s)| + |\sigma(f, P_s) - R| < \varepsilon_s + \frac{\varepsilon}{2} < \varepsilon,$$

and the lemma is proved.

PROPOSITION 7.5. *Let $A \in BV$ and $f \in \mathscr{R}(A)$. Then $f \restriction \mathrm{cl}_e B$ belongs to $\mathscr{R}(B)$ for each $B \in BV_A$, and the function $R(f, \cdot)$ on $BV_A$ defined by $R(f, B) = R(f \restriction \mathrm{cl}_e B, B)$ is additive and continuous.*

*Proof.* Choose a positive $\varepsilon < 1/(2m)$, and find a thin set $T$, a positive function $\delta$ on $\mathrm{cl}_e A - T$, and a lower continuous superadditive function $H$ on $BV_A$ such that $|\sigma(f, P) - R(f, A)| < \varepsilon/2$ for each $P \in \Pi(A, T; \varepsilon, \delta, H)$.

For a set $B \in BV_A$ select $Q_i \in \Pi(B, T; \varepsilon, \delta, 2H)$, $i = 1, 2$, and use Lemma 7.2 to find a $Q \in \Pi(A - B, T; \varepsilon, \delta, 2H)$. Since

$$H\Big(A - \bigcup (Q_i \cup Q)\Big) \geqslant H\Big(B - \bigcup Q_i\Big) + H\Big((A - B) - \bigcup Q\Big) > -1,$$

the partition $P_i = Q_i \cup Q$ belongs to $\Pi(A, T; \varepsilon, \delta, H)$, and clearly $\sigma(f, P_i) = \sigma(f, Q_i) + \sigma(f, Q)$. Thus

$$\begin{aligned} |\sigma(f, Q_1) - \sigma(f, Q_2)| &= |\sigma(f, P_1) - \sigma(f, P_2)| \\ &\leqslant |\sigma(f, P_1) - R(f, A)| \\ &\quad + |R(f, A) - \sigma(f, P_2)| < \varepsilon \end{aligned}$$

and $f \restriction \mathrm{cl}_e B$ belongs to $\mathscr{R}(B)$ by Lemma 7.4.

Let $\mathscr{D}$ be a division of $A$ consisting of $n$ sets. For each $D \in \mathscr{D}$ there is a thin set $T_D$, a positive function $\delta_D$ on $\mathrm{cl}_e D - T_D$, and a lower continuous superadditive function $H_D$ on $BV_D$ such that $|\sigma(f, P_D) - R(f, D)| < \varepsilon/(2n)$ for each $P_D \in \Pi(D, T_D; \varepsilon, \delta_D, H_D)$. We may assume that $T \subset T_D$, $\delta_D \leqslant \delta \restriction (\mathrm{cl}_e D - T_D)$, and $H_D \leqslant (nH) \restriction BV_D$. Now if $P_D \in \Pi(D, T_D; \varepsilon, \delta_D, H_D)$, then $P = \bigcup_{D \in \mathscr{D}} P_D$ is in $\Pi(A, T; \varepsilon, \delta, H)$ since

$$H\Big(A - \bigcup P\Big) \geqslant \sum_{D \in \mathscr{D}} H\Big(D - \bigcup P_D\Big) \geqslant \frac{1}{n} \sum_{D \in \mathscr{D}} H_D\Big(D - \bigcup P_D\Big) > -1.$$

Therefore,

$$\left|R(f, A)-\sum_{D\in\mathscr{D}} R(f, D)\right| \leqslant |R(f, A)-\sigma(f, P)|$$

$$+\left|\sum_{D\in\mathscr{D}} \sigma(f, P_D)-\sum_{D\in\mathscr{D}} R(f, D)\right|$$

$$<\frac{\varepsilon}{2}+\sum_{D\in\mathscr{D}}\frac{\varepsilon}{2n}=\varepsilon$$

and the additivity of $R(f, \cdot)$ follows from the arbitrariness of $\varepsilon$.

As $H$ is lower continuous, there is an $\eta>0$ such that $H(B)>-1/2$ for each $B\in BV_A$ with $|B|<\eta$ and $\|B\|<1/\varepsilon$. Choose such a set $B\in BV_A$ and let $C=A-B$. Since $f\in\mathscr{R}(C)$, there are a thin set $T_C$, a positive function $\delta_C$ on $C-T_C$, and a lower continuous superadditive function $H_C$ on $BV_C$ such that $|\sigma(f, Q)-R(f, C)|<\varepsilon/2$ for each $Q\in\Pi(C, T_C; \varepsilon, \delta_C, H_C)$. We may assume that $T\subset T_C$, $\delta_C\leqslant\delta\restriction(\mathrm{cl}_e\, C-T_C)$, and $H_C\leqslant 2(H\restriction BV_C)$. By Lemma 7.2 there is a $Q\in\Pi(C, T_C; \varepsilon, \delta_C, H_C)$ and, as

$$H\left(A-\bigcup Q\right)\geqslant H\left(C-\bigcup Q\right)+H(B)>\tfrac{1}{2}H_C\left(C-\bigcup Q\right)-\tfrac{1}{2}>-1,$$

the partition $Q$ belongs also to $\Pi(A, T; \varepsilon, \delta, H)$. Thus

$$|R(f, B)|=|R(f, A)-R(f, C)|\leqslant|R(f, A)-\sigma(f, Q)|$$

$$+|\sigma(f, Q)-R(f, C)|<\varepsilon,$$

and the continuity of $R(f, \cdot)$ is established.

LEMMA 7.6. *Let $A\in BV$ and $f\in\mathscr{R}(A)$. For every $\varepsilon>0$ there is a thin set $T$ and a positive function $\delta$ on* $\mathrm{cl}_e\, A-T$ *such that*

$$\sum_{i=1}^{p}|f(x_i)\,|A_i|-R(f, A_i)|<\varepsilon$$

*for each $\delta$-fine $\varepsilon$-partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ in $A$* mod *$T$.*

*Proof.* Choose a positive $\varepsilon<1/(2m)$ and find a thin set $T$, a positive function $\delta$ on $\mathrm{cl}_e\, A-T$, and a lower continuous superadditive function $H$ on $BV_A$ so that $|\sigma(f, P)-R(f, A)|<\varepsilon/3$ for each $P\in\Pi(A, T; \varepsilon, \delta, H)$. By Lemma 7.2, each $\delta$-fine $\varepsilon$-partition $P$ in $A$ mod $T$ can be extended to a partition in $\Pi(A, T; \varepsilon, \delta, 2H)$, so it suffices to consider a partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ in $\Pi(A, T; \varepsilon, \delta, 2H)$. Lemma 7.2 and Proposition 7.5 imply that for $i=1, ..., p$, there is a $P_i\in\Pi(A_i, T; \varepsilon, \delta, 2pH)$ such that

$|\sigma(f, P_i) - R(f, A_i)| < \varepsilon/(3p)$. We may assume that $f(x_i)\,|A_i| \geqslant R(f, A_i)$ when $i = 1, ..., k$, and $f(x_i)\,|A_i| < R(f, A_i)$ when $i = k+1, ..., p$, where $0 \leqslant k \leqslant p$. Now

$$P_+ = \{(A_1, x_1), ..., (A_k, x_k)\} \cup \bigcup_{i=k+1}^{p} P_i,$$

$$P_- = \{(A_{k+1}, x_{k+1}), ..., (A_p, x_p)\} \cup \bigcup_{i=1}^{k} P_i$$

belong to $\Pi(A, T; \varepsilon, \delta, H)$ since

$$\begin{aligned} H\left(A - \bigcup P_+\right) &= H\left[\left(A - \bigcup_{i=1}^{p} A_i\right) \cup \bigcup_{i=k+1}^{p} \left(A_i - \bigcup P_i\right)\right] \\ &\geqslant H\left(A - \bigcup_{i=1}^{p} A_i\right) + \sum_{i=k+1}^{p} H\left(A_i - \bigcup P_i\right) \\ &> -\frac{1}{2} - (p-k)\frac{1}{2p} > -1, \end{aligned}$$

and similarly $H(A - \bigcup P_-) > -1$. Hence

$$\begin{aligned} \frac{\varepsilon}{3} > \sigma(f, P_+) - R(f, A) &= \sum_{i=1}^{k} |f(x_i)\,|A_i| - R(f, A_i)| \\ &\quad + \sum_{i=k+1}^{p} [\sigma(f, P_i) - R(f, A_i)] \\ &\geqslant \sum_{i=1}^{k} |f(x_i)\,|A_i| - R(f, A_i)| - \frac{\varepsilon(p-k)}{3p} \end{aligned}$$

and analogously

$$\frac{\varepsilon}{3} > \sum_{i=k+1}^{p} |f(x_i)\,|A_i| - R(f, A_i)| - \frac{\varepsilon k}{3p}.$$

Adding these inequalities yields

$$\sum_{i=1}^{p} |f(x_i)\,|A_i| - R(f, A_i)| < \varepsilon$$

and the proof is completed.

PROPOSITION 7.7. *Let $A \in BV$ and let $f$ be a function on* $\mathrm{cl_e}\, A$. *Then $f \in \mathscr{R}(A)$ if and only if there is a continuous additive function $F$ on $BV_A$ which satisfies the following condition: given $\varepsilon > 0$ there is a thin set $T$ and a positive function $\delta$ on* $\mathrm{cl_e}\, A - T$ *such that*

$$\sum_{i=1}^{p} |f(x_i)\, |A_i| - F(A_i)| < \varepsilon$$

*for each $\delta$-fine $\varepsilon$-partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ in $A$* mod $T$. *In particular, $R(f, A) = F(A)$.*

*Proof.* If $f \in \mathscr{R}(A)$ then, in view of Proposition 7.5 and Lemma 7.6, it suffices to let $F = R(f, \cdot)$. Conversely, let $F$ be a continuous additive function on $BV_A$ which satisfies the condition of the proposition, and let $\varepsilon > 0$. There is a thin set $T$ and a positive function $\delta$ on $\mathrm{cl_e}\, A - T$ such that

$$\sum_{i=1}^{p} |f(x_i)\, |A_i| - F(A_i)| < \frac{\varepsilon}{2}$$

for each $\delta$-fine $\varepsilon$-partition $P = \{(A_1, x_1), ..., (A_p, x_p)\}$ in $A$ mod $T$. Now $H = -2\,|F|/\varepsilon$ is a continuous superadditive function on $BV_A$, and if $P \in \Pi(A, T; \varepsilon, \delta, H)$ then

$$|\sigma(f, P) - F(A)| \leqslant \sum_{i=1}^{p} |f(x_i)\, |A_i| - F(A_i)| + \left|F\left(A - \bigcup P\right)\right| < \varepsilon.$$

It follows that $f \in \mathscr{R}(A)$ and $R(f, A) = F(A)$.

*Note.* The previous proposition relates the R-integral to that defined in [28, Definition 3.1]. The differences between these integrals are analogous to those mentioned in Remark 5.2,5.

PROPOSITION 7.8. *If $A \in BV$, then $\mathscr{R}(A) = \mathscr{I}_{\mathrm{v}}(A)$ and $R(f, \cdot) = I_{\mathrm{v}}(f, \cdot)$ for each $f \in \mathscr{R}(A)$.*

*Proof.* Choose an $\varepsilon > 0$, and suppose first that $f \in \mathscr{I}_{\mathrm{v}}(A)$. If $F = I_{\mathrm{v}}(f, \cdot)$, there is a thin set $T$ such that the pair $(f, F)$ has an $\varepsilon$-majorant $M$ in $A$ mod $T$. Consequently, we can find a positive function $\delta$ on $\mathrm{cl_e}\, A - T$ such that

$$|f(x)\, |B| - F(B)| \leqslant M(B)$$

for each $x \in \mathrm{cl_e}\, A - T$ and each $B \in BV_A$ with $x \in \mathrm{cl}\, B$, $d(B) < \delta(x)$, and $r(B) > \varepsilon$. Now if $\{(A_1, x_1), ..., (A_p, x_p)\}$ is a $\delta$-fine $\varepsilon$-partition in $A$ mod $T$, then

$$\sum_{i=1}^{p} |f(x_i)\, |A_i| - F(A_i)| \leqslant \sum_{i=1}^{p} M(A_i) \leqslant M(A) < \varepsilon,$$

and it follows from Proposition 7.7 that $f \in \mathscr{R}(A)$.

Conversely, let $f \in \mathscr{R}(A)$ and let $F = R(f, \cdot)$. By Lemma 7.6 there are a thin set $T$ and a positive function $\delta$ on $\mathrm{cl_e}\, A - T$ such that

$$\sum_{i=1}^{p} |f(x_i)\,|A_i| - F(A_i)| < \frac{\varepsilon}{2}$$

for each $\delta$-fine $\varepsilon$-partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ in $A \bmod T$. For $B \in BV_A$ let

$$M(B) = \sup \sum_{i=1}^{p} |f(x_i)\,|B_i| - F(B_i)|,$$

where the supremum is taken over all $\delta$-fine $\varepsilon$-partitions $\{(B_1, x_1), ..., (B_p, x_p)\}$ in $B \bmod T$. Clearly $0 \leqslant M \leqslant \varepsilon/2 < \varepsilon$, and

$$|f(x)\,|B| - F(B)| \leqslant M(B)$$

for each $x \in \mathrm{cl_e}\, A - T$ and $B \in BV_A$ with $x \in \mathrm{cl}\, B$, $d(B) < \delta(x)$ and $r(B) > \varepsilon$; for $\{(B, x)\}$ is a $\delta$-fine $\varepsilon$-partition in $B \bmod T$. Thus if $M$ is superadditive, it is an $\varepsilon$-majorant of the pair $(f, F)$, and we conclude that $f \in \mathscr{I}_v(A)$ and $F = I_v(f, \cdot)$.

To establish the superadditivity of $M$, let $B$ and $C$ be disjoint BV subsets of $A$, and suppose that $M(B \cup C) < M(B) + M(C)$. Then there are $\delta$-fine $\varepsilon$-partitions $P_B = \{(B_1, x_1), ..., (B_r, x_r)\}$ and $P_C = \{(C_1, y_1), ..., (C_s, y_s)\}$ in $B \bmod T$ and $C \bmod T$, respectively, such that

$$M(B \cup C) < \sum_{i=1}^{r} |f(x_i)\,|B_i| - F(B_i)| + \sum_{j=1}^{s} |f(y_j)\,|C_j| - F(C_j)|.$$

This is a contradiction since $P_B \cup P_C$ is a $\delta$-fine $\varepsilon$-partition in $B \cup C \bmod T$.

By allowing only tight partitions in Definition 7.3, we produce an R-integral which coincides with the v-integral modified according to Remark 5.2,4(a). We shall prove next (cf. Proposition 7.10 in conjunction with Proposition 7.7) that if $m \geqslant 2$, this modification leads to no new integral.

LEMMA 7.9. *Let $A \in BV$, $x \in \mathrm{cl_e}\, A$, $\eta > 0$, and let $p \geqslant 1$ be an integer. If $m \geqslant 2$, then there are disjoint sets $C_1, ..., C_p$ in $BV_A$ such that for $i = 1, ..., p$ we have $x \in \mathrm{cl_e}\, C_i$ and*

$$\max\{d(C_i), |C_i|, \|C_i\|\} < \eta.$$

*Proof.* Let

$$\limsup_{\varepsilon \to 0+} \frac{|A \cap U(x, \varepsilon)|}{|U(x, \varepsilon)|} = \alpha > 0.$$

Using linear submanifolds passing through $x$, divide $\mathbf{R}^m$ into disjoint segments $S_1, ..., S_k$ so that

$$0 < \lim_{\varepsilon \to 0+} \frac{|S_i \cap U(x, \varepsilon)|}{|U(x, \varepsilon)|} < \frac{\alpha}{p}$$

for $i = 1, ..., k$. Since

$$\alpha = \limsup_{\varepsilon \to 0+} \sum_{i=1}^{k} \frac{|A \cap S_i \cap U(x, \varepsilon)|}{|U(x, \varepsilon)|} \leqslant \sum_{i=1}^{k} \limsup_{\varepsilon \to 0+} \frac{|A \cap S_i \cap U(x, \varepsilon)|}{|U(x, \varepsilon)|},$$

there are at least $p$ segments, say $S_1, ..., S_p$, such that $x \in \mathrm{cl}_e(A \cap S_i \cap U(x, \varepsilon))$ for each $\varepsilon > 0$ and $i = 1, ..., p$. Now it suffices to let $C_i = A \cap S_i \cap U(x, \varepsilon)$ for a sufficiently small $\varepsilon > 0$ and $i = 1, ..., p$.

PROPOSITION 7.10. *Let $A \in BV$, let $T$ be a thin set, and let $\delta$ be a positive function on $\mathrm{cl}_e A - T$. Furthermore, let $\varepsilon > 0$, let $f$ be a function on $\mathrm{cl}_e A$, and let $F$ be a continuous additive function on $BV_A$. If $m \geqslant 2$ and the inequality*

$$\sum_{i=1}^{p} |f(x_i)\, |A_i| - F(A_i)| \leqslant \varepsilon$$

*holds for each $\delta$-fine tight $\varepsilon$-partition $P = \{(A_1, x_1), ..., (A_p, x_p)\}$ in $A$ mod $T$, then the same inequality holds for any $\delta$-fine $\varepsilon$-partition $P$ in $A$ mod $T$.*

*Proof.* Assume that there is a $\delta$-fine $\varepsilon$-partition $\{(B_1, x_1), ..., (B_p, x_p)\}$ in $A$ mod $T$ such that

$$\sum_{i=1}^{p} |f(x_i)\, |B_i| - F(B_i)| > \varepsilon.$$

In view of Lemma 7.9 and the continuity of $F$, we can find disjoint sets $C_1, ..., C_p$ in $BV_A$ such that if $A_i = (B_i - \bigcup_{j=1}^{p} C_j) \cup C_i$, then $\{(A_1, x_1), ..., (A_p, x_p)\}$ is a $\delta$-fine tight $\varepsilon$-partition in $A$ mod $T$, and

$$\sum_{i=1}^{p} (|f(x_i)| \cdot |C_i| + |F(C_i)|) + \sum_{i,j=1}^{p} (|f(x_i)| \cdot |B_i \cap C_j| + |F(B_i \cap C_j)|)$$
$$< \sum_{i=1}^{p} |f(x_i)\, |B_i| - F(B_i)| - \varepsilon.$$

The last inequality together with the equalities

$$|A_i| = |B_i| + |C_i| - \sum_{j=1}^{p} |B_i \cap C_j|,$$

$$F(A_i) = F(B_i) + F(C_i) - \sum_{j=1}^{p} F(B_i \cap C_j),$$

where $i = 1, ..., p$, imply that

$$\sum_{i=1}^{p} |f(x_i)\,|A_i| - F(A_i)| > \varepsilon,$$

and the proposition follows.

## 8. Convergence of BV Sets

In this section we discuss a convergence in $BV$ introduced by J. Mařík and developed in [19, 12, 18, 21]. The results will be applied in Section 7 to extending the variational integral (cf. [31, Sect. 4]).

We say that a sequence $\{A_n\}$ of BV sets *converges* to a BV set $A$, and write $\{A_n\} \to A$, whenever $A_n \subset A$, $n = 1, 2, ..., \sup \|A_n\| < +\infty$, and $\lim |A - A_n| = 0$ (cf. [18, Sect. 1]). If $\{A_n\}$ and $\{B_n\}$ are sequences of BV sets converging, respectively, to BV sets $A$ and $B$, then

$$\{A_n \cup B_n\} \to A \cup B \qquad \text{and} \qquad \{A_n \cap B_n\} \to A \cap B.$$

A family $\mathscr{C} \subset BV$ is called *closed* whenever for each sequence $\{C_n\}$ in $\mathscr{C}$ converging to a BV set $C$ we have $C \in \mathscr{C}$. The *closure* of a family $\mathscr{E} \subset BV$, denoted by $\mathrm{Cl}\,\mathscr{E}$, is the intersection of all closed subfamilies of $BV$ containing $\mathscr{E}$; clearly $\mathrm{Cl}\,\mathscr{E}$ is closed.

*Note.* In the definition of $\{A_n\} \to A$ one may assume that $A$ is any bounded subset of $\mathbf{R}^m$ and *deduce* from [8, Theorem 1.19, p. 17] that $A$ is, in fact, a BV set.

*Remark* 8.1. It will be convenient to describe the closure operation in $BV$ transfinitely.

1. Given an $\mathscr{F} \subset BV$, let $\mathrm{Cl}_1(\mathscr{F})$ be the collection of all $B \in BV$ for which there is a sequence $\{B_n\}$ in $\mathscr{F}$ with $\{B_n\} \to B$.

2. Given an $\mathscr{E} \subset BV$, set $\mathrm{Cl}_0\,\mathscr{E} = \mathscr{E}$, and assuming that $\mathrm{Cl}_\alpha\,\mathscr{E}$ has been defined for all ordinals $\alpha < \beta$ where $1 \leqslant \beta \leqslant \omega_1$, let $\mathrm{Cl}_\beta\,\mathscr{E} = \bigcup_{\alpha<\beta} \mathrm{Cl}_\alpha\,\mathscr{E}$ if $\beta$ is a limit ordinal, and $\mathrm{Cl}_\beta\,\mathscr{E} = \mathrm{Cl}_1(\mathrm{Cl}_\alpha\,\mathscr{E})$ if $\beta = \alpha + 1$.

Now for each $\mathscr{E} \subset BV$ it is easy to verify that $\mathrm{Cl}_\alpha \mathscr{E} \subset \mathrm{Cl}_\beta \mathscr{E}$ whenever $0 \leqslant \alpha \leqslant \beta \leqslant \omega$, and that $\mathrm{Cl}\, \mathscr{E} = \mathrm{Cl}_{\omega_1} \mathscr{E}$ (cf. [31, Proposition 4.3]).

If $\mathscr{E}$ and $\mathscr{E}'$ are subfamilies of $BV$, we let

$$\mathscr{E} \vee \mathscr{E}' = \{E \cup E' : E \in \mathscr{E}, E' \in \mathscr{E}'\},$$

$$\mathscr{E} \wedge \mathscr{E}' = \{E \cap E' : E \in \mathscr{E}, E' \in \mathscr{E}'\}.$$

LEMMA 8.2. *If $\mathscr{E}$ and $\mathscr{E}'$ are subfamilies of $BV$, then*

$$\mathrm{Cl}\, \mathscr{E} \vee \mathrm{Cl}\, \mathscr{E}' \subset \mathrm{Cl}(\mathscr{E} \vee \mathscr{E}') \qquad \textit{and} \qquad \mathrm{Cl}\, \mathscr{E} \wedge \mathrm{Cl}\, \mathscr{E}' \subset \mathrm{Cl}(\mathscr{E} \wedge \mathscr{E}').$$

*Proof.* It is easy to check that $\mathrm{Cl}_1 \mathscr{E} \vee \mathrm{Cl}_1 \mathscr{E}' \subset \mathrm{Cl}_1(\mathscr{E} \vee \mathscr{E}')$. From this it follows inductively that $\mathrm{Cl}_\alpha \mathscr{E} \vee \mathrm{Cl}_\alpha \mathscr{E}' \subset \mathrm{Cl}_\alpha(\mathscr{E} \vee \mathscr{E}')$ for each $\alpha \leqslant \omega_1$. The other inclusion is proved similarly.

Let $A$ be a BV set. A *kernel* of $A$ is any family $\mathscr{K} \subset BV_A$ with $A \in \mathrm{Cl}\, \mathscr{K}$. A kernel $\mathscr{K}$ of $A$ is called *nondispersed, solid,* or *opaque* if each $K \in \mathscr{K}$ is nondispersed, solid, or opaque, respectively.

LEMMA 8.3. *Let $\mathscr{K}$ and $\mathscr{K}'$ be kernels of $BV$ sets $A$ and $A'$, respectively. Then the following are true:*

1. *$\mathscr{K} \vee \mathscr{K}'$ and $\mathscr{K} \wedge \mathscr{K}'$ are kernels of $A \cup A'$ and $A \cap A'$, respectively.*
2. $|A - \bigcup \mathscr{K}| = 0$.
3. *$\mathscr{K}$ contains a countable kernel of $A$.*
4. *$\mathscr{K}$ is refined by a nondispersed kernel of $A$.*

*Proof.* 1. This follows directly from Lemma 8.2.

2. Let $B = A - \bigcup \mathscr{K}$ have a positive measure. Then the family $\mathscr{C} = \{C \in BV : |B \cap C| = 0\}$ is closed, $\mathscr{K} \subset \mathscr{C}$, and $A \notin \mathscr{C}$. This is a contradiction because $\mathrm{Cl}\, \mathscr{K} \subset \mathscr{C}$.

3. Let $\{\mathscr{K}_\gamma : \gamma \in \Gamma\}$ be the collection of all countable subfamilies of $\mathscr{K}$. As each sequence $\{K_n\}$ in $\mathscr{C} = \bigcup_{\gamma \in \Gamma} \mathrm{Cl}\, \mathscr{K}_\gamma$ is, in fact, a sequence in $\mathrm{Cl}\, \mathscr{K}_\gamma$ for some $\gamma \in \Gamma$, we see that $\mathscr{C}$ is a closed family containing $\mathscr{K}$. Thus $\mathrm{Cl}\, \mathscr{K} \subset \mathscr{C}$, and any $\mathscr{K}_\gamma$ with $A \in \mathrm{Cl}\, \mathscr{K}_\gamma$ is the desired countable kernel of $A$.

4. By Proposition 3.2, for each $K \in \mathscr{K}$ there is a sequence $\{K_n\}$ of nondispersed BV subsets of $K$ such that $\{K_n\} \to K$. Thus $\{K_n : K \in \mathscr{K}, n = 1, 2, ...\}$ is the desired refinement of $\mathscr{K}$.

LEMMA 8.4. *Let $\mathscr{K}$ be a kernel of a BV set $A$, and for each $K \in \mathscr{K}$ let $\mathscr{C}_K$ be a kernel of $K$. Then $\mathscr{C} = \bigcup_{K \in \mathscr{K}} \mathscr{C}_K$ is a kernel of $A$.*

*Proof.* Indeed $\mathscr{K} \subset \bigcup_{K \in \mathscr{K}} \mathrm{Cl}\, \mathscr{C}_K \subset \mathrm{Cl}\, \mathscr{C}$, and hence $\mathrm{Cl}\, \mathscr{K} \subset \mathrm{Cl}\, \mathscr{C}$.

Lemma 8.5. *Let $A \in BV$ and let $F$ be an additive function on $BV_A$. Then $F$ is lower continuous if and only if* $\limsup F(B_n) \leqslant F(B)$ *for each sequence $\{B_n\}$ in $BV_A$ converging to $B \in BV_A$. In particular, $F$ is continuous if and only if* $\lim F(B_n) = F(B)$ *for each sequence $\{B_n\}$ in $BV_A$ converging to $B \in BV_A$.*

*Proof.* Let $F$ be lower continuous, and let $\{B_n\}$ be a sequence in $BV_A$ converging to $B \in BV_A$. Choose an $\varepsilon > 0$ with $\|B\| + \sup \|B_n\| < 1/\varepsilon$, and find $\delta > 0$ such that $F(C) > -\varepsilon$ for each $C \in BV_A$ with $|C| < \delta$ and $\|C\| < 1/\varepsilon$. As $F$ is additive and

$$\|B - B_n\| \leqslant \|B\| + \|B_n\| < \frac{1}{\varepsilon}, \qquad n = 1, 2, ...,$$

we have

$$F(B_n) = F(B) - F(B - B_n) < F(B) + \varepsilon$$

whenever $|B - B_n| < \delta$. Since the last condition is satisfied for all sufficiently large $n$, we see that $\limsup F(B_n) \leqslant F(B)$. If $F$ is continuous, then also $\liminf F(B_n) \geqslant F(B)$ and consequently $\lim F(B_n) = F(B)$.

Conversely, if $F$ is not lower continuous, then there is an $\varepsilon > 0$ and a sequence $\{B_n\}$ in $BV_A$ with $|B_n| < 1/n$, $\|B_n\| < 1/\varepsilon$, and

$$-\varepsilon > F(B_n) = F(A) - F(A - B_n), \qquad n = 1, 2, ....$$

Thus $\{A - B_n\} \to A$, and yet $\limsup F(A - B_n) > F(A)$.

Corollary 8.6. *Let $A \in BV$ and let $F$ be an additive lower continuous function on $BV_A$. If $\mathscr{K}$ is a kernel of $A$ and $F(K) \geqslant 0$ for each $K \in \mathscr{K}$, then $F(A) \geqslant 0$.*

*Proof.* The family $\{B \in BV_A : F(B) \geqslant 0\}$ contains $\mathscr{K}$ and is closed by Lemma 8.5.

Proposition 8.7. *If $\mathscr{K}$ is a kernel of a $BV$ set $A$, then the set* $\mathrm{cl}_e A - \bigcup_{K \in \mathscr{K}} \mathrm{cl}_e K$ *is thin.*

The proof of this proposition, outlined by P. Mattila, requires two lemmas.

Lemma 8.8. *If $B$ is a $BV$ set, then* $\mathrm{int}_e B$ *can be covered by a countable family $\{V_k\}$ of open cubes such that*

$$\sum_k |V_k| \leqslant \alpha\, |B| \qquad \textit{and} \qquad \sum_k \|V_k\| \leqslant \beta\, \|B\|,$$

*where $\alpha$ and $\beta$ are positive constants depending only on $m$.*

*Proof.* We proceed as in [7, Proof of Corollary 4.5.8, p. 477]. If $x \in \operatorname{int}_e B$, then

$$\varepsilon \mapsto \frac{|B \cap U(x, \varepsilon)|}{|U(x, \varepsilon)|}$$

is a continuous function of $\varepsilon > 0$ which approaches 1 and 0 as $\varepsilon$ tends to 0 and $+\infty$, respectively. Thus for each $x \in \operatorname{int}_e B$ there is a $U_x = U(x, \varepsilon_x)$ such that

$$\frac{|U_x|}{2} = |U_x \cap B| = |U_x - B|.$$

It follows from [8, Corollary 1.29, p. 25] that

$$\|U_x\| = \frac{1}{2m} |U_x|^{(m-1)/m} \leqslant \gamma \sigma_B(U_x),$$

where $\gamma$ is a positive constant depending only on $m$. By [6, Lemma 1.9, p. 10], there is a countable set $C \subset \operatorname{int}_e B$ such that $\{U_x : x \in C\}$ is a disjoint family and $\operatorname{int}_e B \subset \bigcup_{x \in C} V_x$ where $V_x = U(x, 5\varepsilon_x)$. Hence

$$\sum_{x \in C} |V_x| = 2 \cdot 5^m \sum_{x \in C} |B \cap U_x| = 2 \cdot 5^m \left| B \cap \left( \bigcup_{x \in C} U_x \right) \right| \leqslant 2 \cdot 5^m |B|,$$

$$\sum_{x \in C} \|V_x\| \leqslant 5^{m-1} \gamma \sum_{x \in C} \sigma_B(U_x) = 5^{m-1} \gamma \sigma_B \left( \bigcup_{x \in C} U_x \right) \leqslant 5^{m-1} \gamma \|B\|,$$

and the lemma is established.

LEMMA 8.9. *Let $\{B_n\}$ be a sequence of BV sets. If* $\lim |B_n| = 0$, *then*

$$\mathscr{H}\left( \bigcap_{n=1}^{\infty} \operatorname{int}_e B_n \right) \leqslant \gamma \liminf \|B_n\|$$

*where $\gamma$ is a positive constant depending only on $m$.*

*Proof.* Assume that $a = \liminf \|B_n\|$ is finite and choose an $\varepsilon > 0$. If $\alpha$ and $\beta$ are the constants from Lemma 8.8, let $\gamma = \beta/(2m)$ and find an integer $p \geqslant 1$ such that $|B_p| < \varepsilon^m/\alpha$ and $\|B_p\| < a + \varepsilon/\gamma$. Let $\{V_k\}$ be a countable family of open cubes associated with $B_p$ according to Lemma 8.8. Then $\{V_k\}$ covers $\bigcap_{n=1}^{\infty} \operatorname{int}_e B_n$,

$$d(V_k) = |V_k|^{1/m} \leqslant (\alpha |B_p|)^{1/m} < \varepsilon, \quad k = 1, 2, \ldots,$$

$$\sum_k [d(V_k)]^{m-1} \leqslant \gamma \|B_p\| < \gamma a + \varepsilon,$$

and the lemma follows.

COROLLARY 8.10. *If $\{A_n\}$ is a sequence of BV sets converging to a BV set $A$, then*

$$\mathscr{H}\left(\mathrm{cl}_e A - \bigcup_{n=1}^{\infty} \mathrm{cl}_e A_n\right) < +\infty.$$

*Proof.* Since

$$\mathrm{cl}_e A - \bigcup_{n=1}^{\infty} \mathrm{cl}_e A_n \subset (\mathrm{bd}_e A) \cup \left(\mathrm{int}_e A - \bigcup_{n=1}^{\infty} \mathrm{cl}_e A_n\right)$$
$$= (\mathrm{bd}_e A) \cup \bigcap_{n=1}^{\infty} \mathrm{int}_e(A - A_n),$$

the corollary follows from Lemma 8.9.

*Proof of Proposition* 8.7. Suppose that $B = \mathrm{cl}_e A - \bigcup_{K \in \mathscr{K}} \mathrm{cl}_e K$ is not a thin set, and let $\mathscr{C}$ consist of all BV sets which meet $B$ in a thin set. Then $\mathscr{K} \subset \mathscr{C}$, $A \notin \mathscr{C}$, and by Corollary 8.10, $\mathscr{C}$ is closed. This is a contradiction as $\mathrm{Cl}\,\mathscr{K} \subset \mathscr{C}$.

## 9. THE CONTINUOUS INTEGRAL

Let $A \in BV$, and let $f$ and $F$ be functions defined on $A$ and $BV_A$, respectively. We denote by $\mathscr{I}_v(f, F; A)$ the family of all $B \in BV_A$ such that $f \restriction B \in \mathscr{I}_v(B)$ and $I_v(f \restriction B, \cdot) = F \restriction BV_B$.

DEFINITION 9.1. We say that a function $f$ on a $BV$ set $A$ is *continuously integrable* (abbreviated as c-integrable) in $A$ if there is a continuous additive function $F$ on $BV_A$ such that $\mathscr{I}_v(f, F; A)$ is a kernel of $A$.

Let $A$ be a BV set. The family of all c-integrable functions in $A$ is denoted by $\mathscr{I}_c(A)$. If $f \in \mathscr{I}_c(A)$, then each continuous function $F$ on $BV_A$ for which $\mathscr{I}_v(f, F; A)$ is a kernel of $A$ is called *an indefinite c-integral* of $f$ in $A$.

PROPOSITION 9.2. *Let $A \in BV$, $f \in \mathscr{I}_c(A)$, and let $F$ be an indefinite c-integral of $f$ in $A$. If $B \in BV_A$, then $f \restriction B$ belongs to $\mathscr{I}_c(B)$ and $F \restriction BV_B$ is an indefinite c-integral of $f \restriction B$ in $B$.*

*Proof.* By Proposition 5.3, $\mathscr{I}_v(f, F; B) = \{B\} \wedge \mathscr{I}_v(f, F; A)$ and so it suffices to apply Lemma 8.3,1.

LEMMA 9.3. *Let $A \in BV$, and for $i = 1, 2$ let $F_i$ be an indefinite c-integral of $f_i \in \mathscr{I}_c(A)$. If $f_1 \leqslant f_2$ then $F_1 \leqslant F_2$.*

*Proof.* By Lemma 8.3,1 the family $\mathscr{K} = \mathscr{I}_v(f_1, F_1; A) \wedge \mathscr{I}_v(f_2, F_2; A)$ is a kernel of $A$. If $F = F_2 - F_1$, then Lemma 5.4 implies that $F(K) \geqslant 0$ for each $K \in \mathscr{K}$. According to Corollary 8.6, $F(A) \geqslant 0$ and the lemma follows from Proposition 9.2.

COROLLARY 9.4. *If $A \in BV$ and $f \in \mathscr{I}_c(A)$, then all indefinite c-integrals of $f$ in $A$ are equal.*

In view of the previous corollary, if $A \in BV$ and $f \in \mathscr{I}_c(A)$, we can talk about *the indefinite c-integral* of $f$ in $A$, denoted by $I_c(f, \cdot)$; the number $I_c(f, A)$ is called the *c-integral* of $f$ over $A$. For each $B \in BV_A$, we have $I_c(f \restriction B, \cdot) = I_c(f, \cdot) \restriction BV_B$.

The next proposition summarizes the properties of the c-integral.

PROPOSITION 9.5. *For a function $f$ defined on a $BV$ set $A$ the following statements are true.*

1. *If $f \in \mathscr{I}_v(A)$, then $f \in \mathscr{I}_c(A)$ and $I_c(f, A) = I_v(f, A)$.*

2. *$\mathscr{I}_c(A)$ is a linear space and the map $f \mapsto I_c(f, A)$ is a nonnegative linear functional on $\mathscr{I}_c(A)$.*

3. *If $\mathscr{D}$ is a division of $A$, then $f$ is c-integrable in $A$ if and only if it is c-integrable in each $D \in \mathscr{D}$.*

4. *If $f \in \mathscr{I}_c(A)$ then $f$ is measurable.*

5. *$f$ belongs to $\mathscr{L}_1(A)$ if and only if both $f$ and $|f|$ belong to $\mathscr{I}_c(A)$.*

6. *$f = 0$ almost everywhere if and only if $f \in \mathscr{I}_c(A)$ and $I_c(f, \cdot) = 0$.*

7. *If $\{f_n\}$ is a sequence in $\mathscr{I}_c(A)$ and $\lim f_n = f$, then $f \in \mathscr{I}_c(A)$ and $I_c(f, A) = \lim I_c(f_n, A)$ whenever either of the following conditions holds:*

   (a) *$f_n \leqslant f_{n+1}$, $n = 1, 2, \ldots$, and $\lim I_c(f_n, A) < +\infty$;*

   (b) *$g \leqslant f_n \leqslant h$ for some $g, h \in \mathscr{I}_c(A)$ and $n = 1, 2, \ldots$.*

*Proof.* 1. This is obvious since $\{A\}$ is a kernel of $A$.

2. Here it suffices to use Propositions 5.3 and 5.6 in conjunction with Lemma 8.3,1.

3. As the converse follows from Proposition 9.2, suppose that $f \restriction D$ belongs to $\mathscr{I}_c(D)$ for each $D \in \mathscr{D}$, and let $F_D = I_c(f \restriction D, \cdot)$. If

$$F(B) = \sum_{D \in \mathscr{D}} F_D(B \cap D)$$

for every $B \in BV_A$, then $F$ is a continuous additive function on $BV_A$ and we show that $F = I_c(f, \cdot)$. By Lemma 8.3,4 and Proposition 5.3, each $D \in \mathscr{D}$ has a solid kernel $\mathscr{K}_D \subset \mathscr{I}_v(f, F; A)$; in fact, we can assume that $\mathscr{K}_D$ is a non-dispersed kernel but we do not need this. According to Lemma 8.3,1, the family $\mathscr{K} = \bigvee_{D \in \mathscr{D}} \mathscr{K}_D$ is a kernel of $A$, and $\mathscr{K} \subset \mathscr{I}_v(f, F; A)$ by Proposition 5.7.

4. This is a consequence of Corollary 5.12 and Lemma 8.3,2.

The remaining statements follow easily from Lemma 5.13.

*Remark* 9.6. By Proposition 9.5,1 we have $\mathscr{I}_v(A) \subset \mathscr{I}_c(A)$ for each BV set $A$. It follows from Proposition 9.5,3 and Example 5.21 that the previous inclusion is generally *proper* (cf. Remark 9.11 and Corollary 9.12).

Let $A \in BV$, and let $f$ and $F$ be functions defined on $A$ and $BV_A$, respectively. We denote by $\mathscr{I}_c(f, F; A)$ the family of all $B \in BV_A$ such that $f \restriction B \in \mathscr{I}_c(B)$ and $I_c(f \restriction B, \cdot) = F \restriction BV_B$. It is a direct consequence of Lemma 8.4 that Definition 9.1 produces no new integral when $\mathscr{I}_v(f, F; A)$ is replaced by $\mathscr{I}_c(f, F; A)$. We state this as a proposition.

PROPOSITION 9.7. *Let $f$ be a function on a BV set $A$. If there is a continuous additive function $F$ on $BV_A$ such that $\mathscr{I}_c(f, F; A)$ is a kernel of $A$, then $A \in \mathscr{I}_c(f, F; A)$.*

PROPOSITION 9.8. *A function $f$ on a BV set $A$ is c-integrable in $A$ whenever the following conditions hold:*

1. *there is a sequence $\{A_n\}$ in $BV_A$ converging to $A$ and such that $f$ is c-integrable in $A_n$, $n = 1, 2, \ldots$;*

2. *if $\{B_n\}$ is a sequence in $BV_A$ converging to $A$ and such that $f$ is c-integrable in $B_n$, $n = 1, 2, \ldots$, then a finite* $\lim I_c(f, B_n)$ *exists.*

*Proof.* Let $\mathscr{I}_c$ be the family of all BV subsets of $A$ on which $f$ is c-integrable, and set $F(B) = I_c(f, B)$ for each $B \in \mathscr{I}_c$. Note that $\mathscr{I}_c$ is an ideal in the ring $BV_A$. If $\{B_n\}$ and $\{C_n\}$ are two sequences in $\mathscr{I}_c$ converging to $A$, then so is the sequence $\{E_n\}$ where $E_{2n-1} = B_n$, $E_{2n} = C_n$, $n = 1, 2, \ldots$. It follows from condition 2 that $\lim F(B_n) = \lim F(C_n)$, and we denote this common value by $a$.

Let $B \in BV_A$, $C = A - B$, and let $\{E_n\}$ be a sequence in $\mathscr{I}_c$ converging to $A$. If $\limsup F(B \cap E_n) = +\infty$, then for $k = 1, 2, \ldots$, there is an integer $n_k \geqslant 1$ such that $F(B \cap E_{n_k}) \geqslant k - F(C \cap E_k)$. Thus letting $D_k = (B \cap E_{n_k}) \cup (C \cap E_k)$, we have $\{D_k\} \to A$ and $\lim F(D_k) = +\infty$, a contradiction. By symmetry, we conclude that the sequences $\{F(B \cap E_n)\}$ and $\{F(C \cap E_n)\}$ are bounded.

Now choose subsequences $\{B_{n\pm}\}$ and $\{C_{n\pm}\}$ of $\{B \cap E_n\}$ and $\{C \cap E_n\}$, respectively, so that $\lim F(B_{n\pm}) = b_\pm$ and $\lim F(C_{n\pm}) = c_\pm$ exist. With no loss of generality, we may assume that $b_- \leqslant b_+$ and $c_- \leqslant c_+$. Since $\{B_{n\pm} \cup D_{n\pm}\} \to A$, we have $a = b_- + c_- \leqslant b_+ + c_+ = a$ and consequently,

$b_- = b_+$ and $c_- = c_+$. In particular, a finite $\lim F(B \cap E_n) = b$ exists, and an argument analogous to the first part of the proof shows that $b$ does not depend on the choice of a sequence $\{E_n\}$ in $\mathscr{I}_c$ converging to $A$. In view of this, we can extend $F$ to $BV_A$ by letting $F(B) = \lim F(B \cap A_n)$, where $\{A_n\}$ is the sequence from condition 1.

The function $F$ is clearly additive. To show that $F$ is also continuous, choose a sequence $\{B_n\}$ in $BV_A$ which converges to a set $B \in BV_A$, and let $C = A - B$. For $k = 1, 2, \ldots$, there is an integer $n_k \geqslant 1$ such that $|F(B_k) - F(B_k \cap A_{n_k})| < 1/k$. The sets $E_k = (B_k \cap A_{n_k}) \cup (C \cap A_{n_k})$ belong to $\mathscr{I}_c$ and $\{E_k\} \to A$. By the previous part of the proof and our choice of $n_k$, we have

$$F(B) = \lim F(B \cap E_k) = \lim F(B_k \cap A_{n_k}) = \lim F(B_k).$$

This establishes the continuity of $F$.

Finally, $\mathscr{I}_v(f, F; A_n)$ is a kernel of $A_n$, and hence the c-integrability of $f$ in $A$ follows from Lemma 8.4.

*Remark* 9.9. It follows from Proposition 9.8 and Remark 8.1 that the c-integral is produced by forming "*improper*" *v-integrals* with respect to the convergence in $BV$ defined in Section 7, and iterating this process transfinitely. By Propositions 9.8, no further extension of the c-integral is possible in this manner. However, we shall see in Section 10 that the c-integral is still extendable by means of a *stronger* convergence in $BV$.

PROPOSITION 9.10. *Let $f$ be a c-integrable function in a $BV$ set $A$, and let $F = I_c(f, \cdot)$. If the family $\mathscr{I}_v(f, F; A)$ contains an opaque kernel of $A$, then $f$ is v-integrable in $A$ and $I_v(f, \cdot) = F$.*

*Proof.* By Lemma 8.3, $\mathscr{I}_v(f, F; A)$ contains a countable collection $\{K_1, K_2, \ldots\}$ of opaque sets which is a kernel of $A$. In view of Remark 5.2,1, for $n = 1, 2, \ldots$ we can find a thin set $T_n$ such that the pair $(f, F)$ has an $\varepsilon$-majorant in $K_n \bmod T_n$ for every $\varepsilon > 0$. Since each $K_n$ is opaque, Proposition 8.7 implies that the set

$$T = \left(\mathrm{cl}_e A - \bigcup_{n=1}^{\infty} \mathrm{int}\, K_n\right) \cup \bigcup_{n=1}^{\infty} T_n$$

is thin. Given $\varepsilon > 0$ and $n = 1, 2, \ldots$, the pair $(f, F)$ has an $(\varepsilon/2^n)$-majorant $M_n$ in $K_n \bmod T$. If

$$M(B) = \sum_{n=1}^{\infty} M_n(B \cap K_n)$$

for each $B \in BV_A$, then $M$ is a nonnegative additive function on $BV_A$ with $M(A) < \varepsilon$. Select an $x \in \mathrm{cl}_e A - T$ and find an integer $p \geqslant 1$ so that $x \in \mathrm{int}\, K_p$. There is a $\delta > 0$ such that

$$B \subset K_p \qquad \text{and} \qquad |f(x)\,|B| - F(B)| \leqslant M_p(B) \leqslant M(B)$$

for each $B \in BV_A$ with $x \in \mathrm{cl}\, B$, $d(B) < \delta$, and $r(B) > \varepsilon > \varepsilon/2^p$. It follows that $M$ is an $\varepsilon$-majorant of the pair $(f, F)$ in $A$ mod $T$, and the proposition is proved.

*Remark* 9.11. At the beginning of Section 3 we mentioned that $BV$ contains a subring $\mathscr{A}$ consisting of all BV sets whose boundary is thin. If we had used $\mathscr{A}$ rather than $BV$ to define the variational and continuous integrals, then both integrals would coincide. Indeed, since all sets from $\mathscr{A}$ are opaque, this follows from Proposition 9.10.

COROLLARY 9.12. *If $m = 1$, the variational and continuous integrals coincide.*

*Proof.* If $m = 1$, then up to a set of measure zero, each BV set is a finite union of nonempty open intervals whose closures are disjoint (see [35, Sect. 6] or [20, Theorem 33]). In particular, up to a set of measure zero, all BV sets are opaque, and the corollary follows from Proposition 9.10.

THEOREM 9.13. *Let $T$ be a thin set and let $\mathscr{K}$ be a kernel of a $BV$ set $A$. Suppose that $v$ is a continuous vector field on* $\mathrm{cl}(\mathrm{cl}_e A)$ *such that $v \restriction K$ is almost differentiable on* $\mathrm{cl}_e K - T$ *for each $K \in \mathscr{K}$. Then* $\mathrm{div}\, v$ *is c-integrable in $A$ and*

$$I_c(\mathrm{div}\, v, A) = \int_{\mathrm{bd}\, A} v \cdot n_A \, d\mathscr{H}.$$

The theorem is a direct consequence of Theorem 5.19 and Remark 5.20,1. Example 5.21 shows that the assumptions of Theorem 9.13 are weaker than those of Theorem 5.19.

THEOREM 9.14. *Let $A \in BV$, let $\Phi: A \to \mathbf{R}^m$ be a lipeomorphism, and let $f \in \mathscr{I}_c(\Phi(A))$. Then $f \circ \Phi \cdot |\det \Phi|$ belongs to $\mathscr{I}_c(A)$ and*

$$I_c(f \circ \Phi \cdot |\det \Phi|, A) = I_c(f, \Phi(A)).$$

*Proof.* It follows from Lemma 6.6 that $\{\Phi^{-1}(B_n)\} \to \Phi^{-1}(B)$ whenever $\{B_n\}$ is a sequence in $BV_{\Phi(A)}$ converging to $B \in BV_{\Phi(A)}$. Thus if $\mathscr{K}$ is a kernel of $\Phi(A)$, a simple transfinite induction shows that $\{\Phi^{-1}(K): K \in \mathscr{K}\}$ is a kernel of $A$ (cf. Remark 8.1). Now it suffices to apply Theorem 6.7.

## 10. The Integral

We say that a sequence $\{A_n\}$ of BV sets *converges strongly* to a BV set $A$, and write $\{A_n\} \xrightarrow{s} A$, whenever $A_n \subset A$, $n = 1, 2, \ldots$, and $\lim \|A - A_n\| = 0$ (cf. [21, Section 1] and [31, Section 5]). By the isoperimetric inequality (see [8, Theorem 1.29, p. 25]), a sequence $\{A_n\}$ in $BV$ which converges strongly to a BV set $A$ also converges to $A$. If $\{A_n\}$ and $\{B_n\}$ are sequences in $BV$ converging strongly to BV sets $A$ and $B$, respectively, then it follows from [21, Section 13] that

$$\{A_n \cup B_n\} \xrightarrow{s} A \cup B \qquad \text{and} \qquad \{A_n \cap B_n\} \xrightarrow{s} A \cap B.$$

A detailed proof of this fact can be found in [31, Lemma 5.6].

Definition 10.1. Let $A \in BV$ and let $F$ be a function defined on $BV_A$. We say that $F$ is *bounded* if given $\varepsilon > 0$, there is a $\delta > 0$ such that $|F(B)| < \varepsilon$ for each $B \in BV_A$ with $\|B\| < \delta$.

Example 10.2. Let $A$ be a BV set.

1. If $F$ is a continuous function on $BV_A$, then $F$ is bounded.
2. If $v$ is a bounded $\mathscr{H}$-measurable vector field on $\mathrm{cl}_e A$, and $F(B) = \int_{\mathrm{bd}\,B} v \cdot n_B \, d\mathscr{H}$ for each $B \in BV_A$, then $F$ is additive and bounded.

Lemma 10.3. *Let $A \in BV$ and let $F$ be an additive function on $BV_A$. Then $F$ is bounded if and only if* $\lim F(B_n) = F(B)$ *for each sequence $\{B_n\}$ in $BV_A$ converging strongly to $B \in BV_A$.*

For the proof, which is analogous to that of Lemma 8.5, we refer to [31, Lemma 5.3].

The next definition uses the family $\mathscr{I}_c(f, F; A)$ which has been defined in the paragraph preceding Proposition 9.7.

Definition 10.4. We say that a function $f$ on a BV set $A$ is *integrable* in $A$ if there is a bounded additive function $F$ on $BV_A$ and a sequence $\{A_n\}$ in $\mathscr{I}_c(f, F; A)$ converging strongly to $A$.

The family of all integrable functions in a BV set $A$ is denoted by $\mathscr{I}(A)$. Quite similarly to Section 7, we can show that the function $F$ from Definition 10.4 is determined uniquely by $f$. We call it the *indefinite integral* of $f$ in $A$, denoted by $I(f, \cdot)$; the number $I(f, A)$ is called the *integral* of $f$ over $A$. We have $I(f \restriction B, \cdot) = I(f, \cdot) \restriction BV_B$ for each $B \in BV_A$. It is easy to verify that statements analogous to 9.2–9.5 hold for the integral. Moreover, the next example shows that the inclusion $\mathscr{I}_c(A) \subset \mathscr{I}(A)$ is generally proper (cf. Proposition 10.8).

EXAMPLE 10.5. Let $m=2$. If $K$ is a closed square of diameter $h$ and center $z$, the affine map $\Phi_K(x)=(x-z)/h$ maps $K$ onto $[-1/2, 1/2]^2$. For $n=1, 2, \ldots$ and $i=1, \ldots, 2^n$, let $K(n, i)=[2^{-n}, 2^{-n+1}]\times[(i-1)2^{-n}, i2^{-n}]$, and let $k_n$ be the unique integer for which $2^n/n \leqslant k_n < (2^n/n)+1$. It is easy to check that the sets

$$A_j = \bigcup_{n=1}^{j} \bigcup_{i=1}^{k_n} K(n, i), \quad j=1, 2, \ldots, \qquad \text{and} \qquad A=\bigcup_{j=1}^{\infty} A_j$$

belong to $BV$, and that $\{A_j\} \xrightarrow{s} A$. If $v$ is the vector field from Example 5.21 and $f=\operatorname{div} v$, set

$$w=\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} v \circ \Phi_{K(n,i)} \qquad \text{and} \qquad g=\operatorname{div} w=\sum_{n=1}^{\infty} 2^n \sum_{i=1}^{k_n} f \circ \Phi_{K(n,i)}.$$

As $w$ is a bounded vector field in $\mathbf{R}^2$ with a single discontinuity at the origin, we can define a bounded additive function $G$ on $BV_A$ by letting $G(B)=\int_{\mathrm{bd}\,B} w \cdot n_B \, d\mathscr{H}$ for every $B \in BV_A$ (cf. Example 10.2,2). It follows that $g \in \mathscr{I}(A)$, since by Example 5.21 and Proposition 9.5,3, each $A_j$, $j=1, 2, \ldots$, belongs to $\mathscr{I}_c(g, G; A)$.

Now suppose that $g \in \mathscr{I}_c(A)$, and for $s=1, 2, \ldots$, let

$$B_s=\bigcup_{n=s}^{2s} \bigcup_{i=1}^{k_n} \Phi_{K(n,i)}^{-1}(R_{0,1}),$$

where $R_{0,1}$ is the square defined in Example 3.1. By Theorem 9.14,

$$\begin{aligned} I_c(g, B_s) &= \sum_{n=s}^{2s} \sum_{i=1}^{k_n} I_c(g, \Phi_{K(n,i)}^{-1}(R_{0,1})) = \sum_{n=s}^{2s} \sum_{i=1}^{k_n} 2^{-n} I_c(f, R_{0,1}) \\ &= \sum_{n=s}^{2s} k_n (2^{-n} \cdot 2) \geqslant 2 \sum_{n=s}^{2s} \frac{1}{n} \geqslant 2 \ln 2. \end{aligned}$$

Since

$$\|B_s\| \leqslant \sum_{n=s}^{2s} k_n \left(2^{-n} \frac{4}{12}\right) \leqslant \frac{1}{3} \sum_{n=s}^{2s} \left(\frac{1}{n}+2^{-n}\right) \leqslant \frac{1}{3}(2+\ln 2),$$

$$|B_s| = \sum_{n=s}^{2s} k_n \left(2^{-n} \frac{4}{12}\right)^2 \leqslant \frac{1}{9} \sum_{n=s}^{\infty} \left(\frac{1}{n}+2^{-n}\right) 2^{-n} \leqslant \frac{4}{9} 2^{-s},$$

it is obvious that $I_c(g, \cdot)$ is not continuous on $BV_A$, a contradiction.

PROPOSITION 10.6. *A function $f$ on a BV set $A$ is integrable in $A$ whenever the following conditions hold:*

1. *there is a sequence* $\{A_n\}$ *in* $BV_A$ *converging strongly to* $A$ *and such that* $f$ *is integrable in* $A_n$, $n = 1, 2, \ldots$;

2. *if* $\{B_n\}$ *is a sequence in* $BV_A$ *converging strongly to* $A$ *and such that* $f$ *is integrable in* $B_n$, $n = 1, 2, \ldots$, *then a finite* $\lim I(f, B_n)$ *exists.*

*Proof.* Proceeding as in the proof of Proposition 9.8 and using Lemma 10.3, we can define a bounded additive function $F$ on $BV_A$ by setting $F(B) = \lim I(f, B \cap A_n)$ for each $B \in BV_A$; here $\{A_n\}$ is the sequence from condition 1. Now for each $A_n$ there is a sequence $\{B_{k,n}\}_k$ in $\mathscr{I}_c(f, F; A_n)$ converging strongly to $A_n$. Given $n = 1, 2, \ldots$, find an integer $k_n$ so that $\|A_n - B_{k_n,n}\| \leqslant 1/n$. It is easy to see that $\{B_{k_n,n}\}$ converges strongly to $A$, and the proposition follows from Proposition 9.2.

*Remark* 10.7. It follows from Proposition 10.6 that the integral is an "*improper*" *c-integral*, and that there are *no "improper" integrals.*

PROPOSITION 10.8. *If* $m = 1$, *the integral coincides with the continuous and variational integrals.*

Indeed when $m = 1$, a sequence $\{A_n\}$ of BV sets converges strongly to a BV set $A$ if and only if $|A - A_n| = 0$ for all sufficiently large $n$, and the proposition follows from Corollary 9.12.

We say that a set $S \subset \mathbf{R}^m$ is *slight* whenever $\mathscr{H}(S) = 0$. Note that, in contrast with [28] and [31], no topological condition is imposed on slight sets.

THEOREM 10.9. *Let* $A$ *be a BV set, and let* $S$ *and* $T$ *be slight and thin sets, respectively. Suppose that* $v$ *is a bounded vector field on* $\mathrm{cl}\, A$ *which is continuous on* $\mathrm{cl}(\mathrm{cl}_e A) - S$ *and almost differentiable on* $\mathrm{cl}_e A - T$. *Then* $\mathrm{div}\, v$ *is integrable in* $A$ *and*

$$I(\mathrm{div}\, v, A) = \int_{\mathrm{bd}\, A} v \cdot n_A \, d\mathscr{H}.$$

*Proof.* Clearly, the vector field $v$ is $\mathscr{H}$-measurable. Hence setting $F(B) = \int_{\mathrm{bd}\, B} v \cdot n_B \, d\mathscr{H}$ for each $B \in BV_A$, we define a bounded additive function $F$ on $BV_A$ (cf. Example 10.2,2). It follows from Lemma 2.5 that there are sets $C_n \in BV$, $n = 1, 2, \ldots$, such that $S \subset \mathrm{int}\, C_n$ and $\|C_n\| \leqslant 1/n$. If $A_n = A - C_n$, then $S \cap \mathrm{cl}\, A_n = \varnothing$, and $\{A_n\} \xrightarrow{\mathrm{s}} A$. Thus according to Theorem 5.19 and Remark 5.20,1, $\{A_n\}$ is a sequence in $\mathscr{I}_c(\mathrm{div}\, v, F; A)$, and the proof is completed.

*Remark* 10.10. We make two comments about Theorem 10.9.

1. If there is a sequence $\{A_n\}$ of nondispersed BV subsets of $A$ such that $A_n \xrightarrow{\mathrm{s}} A$, then it suffices to assume that $v$ is continuous on $\mathrm{cl}_e A - S$ (cf. Remark 3.3 and Note added in proof).

2. By considering the characteristic function of the interval $[0, +\infty)$ in $\mathbf{R}$, it is easy to see that in terms of the measure $\mathscr{H}$, the exceptional set $S \cap \mathrm{cl}(\mathrm{cl}_e A)$ is as large as possible (cf. Remark 5.20,2).

THEOREM 10.11. *Let $A \in BV$, let $\Phi: A \to \mathbf{R}^m$ be a lipeomorphism, and let $f \in \mathscr{I}(\Phi(A))$. Then $f \circ \Phi \cdot |\det \Phi|$ belongs to $\mathscr{I}(A)$ and*

$$I(f \circ \Phi \cdot |\det \Phi|, A) = I(f, \Phi(A)).$$

*Proof.* By Lemma 6.6, $\{\Phi^{-1}(A_n)\} \xrightarrow{s} A$ whenever $\{A_n\}$ is a sequence in $BV_{\Phi(A)}$ converging strongly to $\Phi(A)$. The theorem follows from Theorem 9.14.

## ACKNOWLEDGMENTS

It is a pleasure to acknowledge the help received from Z. Buczolich, who constructed the essential part of Example 5.21, M. Miranda, who provided the proof of Proposition 3.2, and P. Mattila, who suggested how to establish Proposition 8.7. The author is also obliged to W. H. Fleming, G. Martin, F. Morgan, and L. Simon for valuable discussions concerning sets of bounded variation.

*Note added in proof.* After this writing was completed, the existence of a sequence of BV sets mentioned in Remarks 3.3 and 10.10,1 has been established by I. Tamanini and C. Giacomelli (see [37]).

## REFERENCES

1. A. S. BESICOVITCH, On sufficient conditions for a function to be analytic, and behaviour of analytic functions in the neighbourhood of non-isolated singular points, *Proc. London Math. Soc.* **32** (1931), 1–9.
2. Z. BUCZOLICH, Functions with all singular sets of Hausdorff dimension bigger than one, *Real Anal. Exchange* **15**(1) (1989–1990), 299–306.
3. G. CONGEDO AND I. TAMANINI, Note sulla regolarità dei minimi di funzionali del tipo dell'area, *Rend. Accad. Naz. XL Mem. Mat.* **106**(XII, 17) (1988), 239–257.
4. E. DEGIORGI, Nuovi teoremy relativi alle misure $(r-1)$-dimensionali in uno spazio ad $r$-dimensioni, *Ricerche Mat.* **4** (1955), 95–113.
5. E. DEGIORGI, Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad $r$-dimensioni, *Ann. Mat. Pura Appl.* **36** (1954), 191–213.
6. K. J. FALCONER, "The Geometry of Fractal Sets," Cambridge Univ. Press, Cambridge, U.K., 1985.
7. H. FEDERER, "Geometric Measure Theory," Springer-Verlag, New York, 1969.
8. E. GIUSTI, "Minimal Surfaces and Functions of Bounded Variation," Birkhäuser, Basel, 1984.
9. P. R. HALMOS, "Measure Theory," Van Nostrand, New York, 1950.
10. R. HENSTOCK, "Lectures on the Theory of Integration," World Scientific, Singapore, 1988.
11. R. HENSTOCK, Majorants in variational integration, *Canad. J. Math.* **18** (1966), 49–74.

12. J. HOLEC AND J. MAŘÍK, Continuous additive mappings, *Czechoslovak Math. J.* **14** (1965), 237–243.
13. E. J. HOWARD, Analyticity of almost everywhere differentiable functions, *Proc. Amer. Math. Soc.* **110** (1990), 745–753.
14. J. JARNÍK, J. KURZWEIL, AND Š. SCHWABIK, On Mawhin's approach to multiple nonabosolutely convergent integrals, *Časopis Pěst. Mat.* **108** (1983), 356–380.
15. J. JARNÍK AND J. KURZWEIL, A new and more powerful concept of the $PU$-integral, *Czechoslovak Math. J.* **38** (1988), 8–48.
16. J. JARNÍK AND J. KURZWEIL, A nonabsolutely convergent integral which admits $C^1$-transformations, *Časopis Pěst. Mat.* **109** (1984), 157–167.
17. J. JARNÍK AND J. KURZWEIL, A nonabsolutely convergent integral which admits transformation and can be used for integration on manifolds, *Czechoslovak Math. J.* **35** (1986), 116–139.
18. K. KARTÁK AND J. MAŘÍK, A non-absolutely convergent integral in $E_m$ and the theorem of Gauss, *Czechoslovak Math. J.* **15** (1965). 253–260.
19. J. MAŘÍK, Extensions of additive mappings, *Czechoslovak Math. J* **15** (1965), 244–252.
20. J. MAŘÍK, The surface integral, *Czechoslovak Math. J.* **6** (1956), 522–558.
21. J. MAŘÍK AND J. MATYSKA, On a generalization of the Lebesgue integral in $E_m$, *Czechoslovak Mat. J.* **15** (1965), 261–269.
22. J. MAWHIN, Generalized multiple Perron integrals and the Green–Goursat theorem for differentiable vector fields, *Czechoslovak Math. J.* **31** (1981), 614–632.
23. J. MAWHIN, Generalized Riemann integrals and the divergence theorem for differentiable vector fields, *in* "E. B. Christoffel," pp. 704–714, Birkhäuser, Basel, 1981.
24. V. G. MAZ'JA, "Sobolev Spaces," Springer-Verlag, New York, 1985.
25. E. J. MCSHANE, "Unified Integration," Academic Press, New York, 1983.
26. W. F. PFEFFER, The divergence theorem, *Trans. Amer. Math Soc.* **295** (1986), 665–685.
27. W. F. PFEFFER, Divergence theorem for vector fields with singularities, *in* "New Integrals" (P. S. Bullen, P. Y. Lee, J. L. Mawhin, P. Muldowney, and W. F. Pfeffer, Eds.), pp. 150–166, Lecture Notes in Mathematics, vol. 1419, Springer-Verlag, New York, 1990.
28. W. F. PFEFFER, The multidimensional fundamental theorem of calculus, *J. Austral. Math. Soc.* **43** (1987), 143–170.
29. W. F. PFEFFER, Stokes theorem for forms with singularities, *C.R. Acad. Sci. Paris Sér. I* **306** (1988), 589–592.
30. W. F. PFEFFER, Une intégrale riemannienne et le théorème de divergence, *C.R. Acad. Sci. Paris Sér. I* **299** (1984), 299–301.
31. W. F. PFEFFER AND WEI-CHI YANG, A multidimensional variational integral and its extensions, *Real Anal. Exchange* **15(1)** (1989–1990), 111–169.
32. W. RUDIN, "Real and Complex Analysis," McGraw–Hill, New York, 1987.
33. S. SAKS, "Theory of the Integral," Dover, New York, 1964.
34. V. L. SHAPIRO, The divergence theorem for discontinuous vector fields, *Ann. of Math.* **68** (1958), 604–624.
35. A. I. VOLPERT, The spaces $BV$ and quasilinear equations, *Math. USSR-Sb.* **2** (1967), 225–267.
36. I. TAMANINI AND C. GIACOMELLI, Approximation of Caccioppoli sets, with applications to problems in image segmentation, *Ann. Univ. Ferrara Sez. (N.S.)* **35** (1989), 187–213.
37. I. TAMANINI AND C. GIACOMELLI, Un tipo di approssimazione "dal'interno" degli insiemi di perimetro finito, *Rend. Mat. Accad. Lincei (9)* **1** (1990), 181–187.