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Simplicial Homotopy Theory

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Contents

INTRODUCTION

What are (semi-)simplicial complexes, and what are they good for? A simplicial set is a combinatorial affair, i.e., a family of sets and maps between them, from which may be deduced the homotopy properties of a topological space.

The singular simplicial set $S(X)$ of a topological space X is defined as follows: For each integer $n \geqslant 0$,

$$S(X)_n = \{f : \delta[n] \to X\},$$

where f varies over all continuous maps of $\delta[n]$ (= the standard Euclidean simplex of dimension n) to X. For each integer $0 \leqslant i \leqslant n$, there are natural maps

$$d_i : S(X)_n \to S(X)_{n-1}, \qquad \text{the } i\text{-th face operator,}$$
$$s_i : S(X)_n \to S(X)_{n+1}, \qquad \text{the } i\text{-th degeneracy operator,}$$

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which satisfy the simplicial identities. $S(X)$ can be used to define some of the usual invariants (e.g., the homotopy and homology groups of X) and to prove some of the usual theorems (e.g., the Hurewicz theorem).

If X is the topological space of a polyhedron (sometimes called a simplicial complex), there is a simplicial set K determined by X as follows. Order the vertices of X, and for each integer $n \geqslant 0$, let

$$K_n = \{\langle v_0, ..., v_n\rangle\},$$

where the v_j are vertices (repetitions allowed) of a simplex of X with $v_0 \leqslant v_1 \leqslant \cdots \leqslant v_n$. Again there are face and degeneracy operators

$$d_i : K_n \to K_{n-1},$$
$$s_i : K_n \to K_{n+1},$$

obtained by deleting or repeating the i-th vertex, respectively.

We shall be concerned with describing the homotopy groups of a simplicial set (equivalently, of the underlying topological space) by its combinatorial structure, especially by bringing in some algebra. For example, Kan's construction assigns to each simplicial set K, a simplicial group GK (i.e., a family of groups, with face and degeneracy operators which are homomorphisms), which is the simplicial analogue of the loop space. Techniques of group theory applied to GK sometimes permit calculations of homotopy groups of K.

Broad Outline

Each section will have its own outline, together with some relevant references. Some additional comments: Sections 1–3 describe basic algebraic topology treated simplicially. Included are: The relation between simplicial sets and topological spaces; homotopy groups, homology groups, and the Hurewicz theorem; fibrations, minimal fibrations, and the Moore–Postnikov system. Also included is the $G(\cdot)$ construction, and the closely related $\overline{W}(\cdot)$ construction, which is the classifying complex for fibre bundles. Section 4 describes Milnor's $F(\cdot)$ construction, which is the "loops on the suspension." Some group-theoretic techniques are used to prove the Hilton–Milnor theorem, as well as theorems of James and Hopf. Section 5 is the standard relation between simplicial Abelian groups and Abelian chain complexes. Section 6 gives the simplicial fibre-bundle theory of Barratt, Guggenheim, and Moore; the main result is the classification of fibre bundles.

Sections 7–8 give the approach to the homotopy of a simplicial set by taking GK, filtering by its lower central series, and examining the quotients. In this way there arises a form of the Adams spectral sequence which works unstably as well. Section 9 describes a more general construction of Bousfield and Kan leading directly to an "unstable Adams spectral sequence." Section 10 is Bousfield's version (using GK) of Adams' cobar construction, which is useful in studying the differentials which occur in Sections 7–8. In Section 11 we give some applications to the J-homomorphism, the EHP-sequence, the Hopf-invariant, and Samelson and Whitehead products. Especially see the tables pertaining to the homotopy groups of spheres, unitary groups, etc. Section 12 is the simplicial extension theorem, which is used to establish the relation between simplicial homotopy and topological homotopy.

These notes are a slightly modified version of lectures given at the Matematisk Institut, Aarhus, Denmark, 1967–1968. I have been most strongly influenced by previous lecture notes of Barratt, Kan, and Moore. Covering simplicial theory in different ways are [L], [M], and [Q].

I have tried to show aspects of homotopy theory from the simplicial point of view. The topology underlying this is well represented in the literature in the papers of Adams, Barratt (unpublished), James, Mahowald, Toda, Whitehead, and others.

The proofs, where not indicated, are supposed to be straightforward.

1. Simplicial Sets

We begin by defining simplicial sets (older terms: semisimplicial complexes, c.s.s. complexes, s.s. complexes), or more generally, simplicial objects in a category $\mathscr{C}$, in case all the objects and maps are in $\mathscr{C}$.

The homotopy relation ($\simeq$) is defined for simplicial maps. Homotopy becomes an equivalence relation if the range is a Kan complex, i.e., a simplicial set satisfying the extension condition. More generally, homotopy in a category $\mathscr{C}$ is defined for simplicial objects in $\mathscr{C}$. It is an elementary but important principle that functors preserve homotopy.

We define minimal complexes, and show that any Kan complex has a minimal complex which is unique to within isomorphism, a useful property.

The singular and geometrical realization functors relate spaces to simplicial sets. To know that we are really getting anywhere with the latter, we need to know that these induce a one-one correspondence between homotopy types of CW spaces and Kan complexes, and also induce a one-one correspondence between homotopy classes of maps in the two categories. This can be deduced from the simplicial extension theorem, whose proof is deferred until Section 12.

The references for Section 1 are [E], [E, Z], [Mo1], [Mo2], [Mi1].

DEFINITION (1.1). A simplicial set K is a sequence of sets, $K = \{K_0, K_1, \ldots, K_n, \ldots\}$, together with functions

$$d_i : K_n \to K_{n-1},$$
$$s_i : K_n \to K_{n+1},$$

for each $0 \leqslant i \leqslant n$. These functions are required to satisfy the simplicial identities

$$d_i d_j = d_{j-1} d_i \qquad \text{for } i < j,$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j, \\ \text{identity} & \text{for } i = j, j+1, \\ s_j d_{i-1} & \text{for } i > j+1, \end{cases}$$

$$s_i s_j = s_{j+1} s_i \qquad \text{for } i \leqslant j.$$

Remark. Let $\mathcal{O}$ be the category of finite ordered sets and order-preserving maps. The above definition of a simplicial set is equivalent to asserting that K is a contravariant functor from $\mathcal{O}$ to the category of sets, where

$$K_n = K(\{0, 1, \ldots, n\}),$$
$$d_i = K(\text{the map which skips number } i),$$
$$s_i = K(\text{the map which repeats number } i).$$

One need not try to remember the simplicial identities—they may be recalled easily from the properties of the maps in $\mathcal{O}$.

A simplicial map $f : K \to L$ is a family of functions $f_n : K_n \to L_n$ commuting with the d_i and the s_i.

Elements $x \in K_n$ are called n-dimensional simplices; elements of K_0 are called vertices. A simplex x is called degenerate if $x = s_i y$ for some y, some i. Every simplex can be uniquely expressed in the form $x = s_{i_q} \cdots s_{i_1} y$, where y is nondegenerate and $i_q > \cdots > i_1 \geqslant 0$. For each integer $n \geqslant 0$, let $K^{(n)}$ be the n skeleton of K, that is, the smallest subsimplicial set of K which contains all the nondegenerate simplices of K of dimension $\leqslant n$. Arguments sometimes go by induction on the nondegenerate simplices, or by induction on the skeletons, in analogy with CW spaces.

If each of the K_n, and each of the d_i and s_i, are in a category $\mathscr{C}$, then K is called a simplicial object over $\mathscr{C}$; frequently the words "object over $\mathscr{C}$" are replaced by the generic name for objects in $\mathscr{C}$. Equivalently, by the remark above, a simplicial object over $\mathscr{C}$ is to be considered as a contravariant functor from $\mathscr{O}$ to $\mathscr{C}$. If $f : K \to L$ is a simplicial map with each f_n in $\mathscr{C}$, then f is called a simplicial map over $\mathscr{C}$.

TERMINOLOGY (1.2). What is usually called a simplicial complex is defined as follows. In Euclidean space R^N, any $q + 1$ points $p_0, \ldots, p_q$ in general position determine a (closed) simplex, say $\sigma[p_0, \ldots, p_q]$, by

$$\sigma[p_0, \ldots, p_q] = \{\Sigma a_i p_i\},$$

where $a_i \geqslant 0$, $\Sigma a_i = 1$. The corresponding open simplex $\sigma(p_0, \ldots, p_q)$ is the interior of $\sigma[p_0, \ldots, p_q]$ in the hyperplane. A simplicial complex X is to be a collection of open simplices such that (1) Every face of any simplex of X is again in X; (2) two distinct open simplices are disjoint.

As will be seen, such a simplicial complex X determines a simplicial set K by adding degenerate simplices to the collection, but not every simplicial set K is of this form. We shall call such special simplicial sets (arising from a simplicial complex) *polyhedral* (see Section 12).

Following the usual terminology, we make other uses of the word complex, not necessarily as simplicial complexes, and keep the terms simplicial complex or polyhedral simplicial set where appropriate. To anticipate: A simplicial set L which is a subsimplicial set of a simplicial set K (i.e., for each n, $L_n \subset K_n$, and L is closed under the d_i and s_i) is called a subcomplex of K. A simplicial set satisfying the extension condition (1.12) is called a Kan complex; a minimal complex (1.20) is a special sort of Kan complex, etc. Another, different use of the word complex occurs in Section 3 as chain complex.

EXAMPLES (1.3). Let X be a simplicial complex, and let the vertices of X be ordered. Let K be the simplicial set

$$K_n = \{\langle v_0, \ldots, v_n\rangle : \text{the } v_i \text{ are vertices of a simplex of } X \text{ with } v_0 \leqslant \cdots \leqslant v_n\},$$

$$d_i\langle v_0, \ldots, v_n\rangle = \langle v_0, \ldots, \hat{v}_i, \ldots, v_n\rangle,$$

$$s_i\langle v_0, \ldots, v_n\rangle = \langle v_0, \ldots, v_i, v_i, \ldots, v_n\rangle.$$

More examples (1.4). The standard n simplex $\Delta[n]$ is the simplicial set with vertices $0, 1, 2, \ldots, n$, where

$$(\Delta[n])_q = \{\langle v_0, \ldots, v_q\rangle : 0 \leqslant v_0 \leqslant \cdots < v_q \leqslant n\}.$$

Let $i_n = \langle 0, i, \ldots, n\rangle \in (\Delta[n])_n$. The "boundary of $\Delta[n]$" is $\dot{\Delta}[n] = \Delta[n]^{(n-1)} =$ the $n-1$ skeleton. The n sphere S^n is the quotient simplicial set $\Delta[n]/\dot{\Delta}[n]$. Thus S^n has two nondegenerate simplices, a vertex which we call $*$, and σ_n in dimension n, which is the image of i_n. In dimensions $n+q$, S^n has the iterated degeneracy of $*$, and simplices $s_{i_q} \cdots s_{i_1}\sigma_n$, where $n + q > i_q > \cdots > i_1 \geqslant 0$.

Proposition (1.5). *Let K be a simplicial set, and $x \in K_n$. Then there is a unique simplicial map*

$$f_x : \Delta[n] \to K$$

such that $f_x(i_n) = x$. Call f_x the representing map for x. If $x \in K_n$, with all $d_i x = *$, then f_x passes to the quotient

$$\bar{f}_x : S^n \to K.$$

with $\bar{f}_x(\sigma_n) = x$.

Prolongation (1.6). An important construction is the following, which is called by Dold the "prolongation of T." If T is a covariant functor from $\mathscr{C}$ to $\mathscr{D}$, and K is a simplicial object over $\mathscr{C}$, let TK be the simplicial object over $\mathscr{D}$ obtained by applying T to each K_n and to each d_i and s_i. Equivalently, regarding K as a functor from $\mathscr{O}$ to $\mathscr{C}$, TK is the composed functor $T \circ K$ from $\mathscr{O}$ to $\mathscr{D}$. If $f : K \to L$ is a simplicial map over $\mathscr{C}$, then $Tf : TK \to TL$ will be a simplicial map over $\mathscr{D}$.

Example. Let $Z(\cdot)$ be the free Abelian group functor. Then the simplicial Abelian group $Z(S^n)/Z(*)$ will be an Eilenberg–MacLane complex $K(Z, n)$.

Proposition (1.7). *Let A be a simplicial Abelian group and let $a \in A_n$. Then there is a unique simplicial homomorphism*

$$f_a : Z(\Delta[n]) \to A$$

such that $f_a(i_n) = a$. *If* $a \in A_n$, *with all* $d_i a = 0$, *then* f_a *passes to the quotient simplicial group*

$$\bar{f}_a : K(Z, n) \to A$$

with $\bar{f}_a(\sigma_n) = a$.

SOME CONSTRUCTIONS (1.8). Let K and L be simplicial sets. The *Cartesian product* $K \times L$ is to be the simplicial set

$$(K \times L)_n = K_n \times L_n,$$
$$d_i(x, y) = (d_i x, d_i y),$$
$$s_i(x, y) = (s_i x, s_i y).$$

A *base point* in K is a vertex $*$ and all of its degeneracies, also denoted by $*$. A map of simplicial sets with base points means $*$ goes to $*$. A simplicial set is called *reduced* if there is only one vertex.

The *wedge* $K \vee L$ is the subcomplex of $K \times L$ consisting of all $(x, *)$ and all $(*, y)$. The *edge* (or *smash*) is $K \wedge L = K \times L / K \vee L$.

The cone CK is defined by

$$(CK)_n = \{(x, q) : x \in K_{n-q}, 0 \leqslant k \leqslant n\} \quad \text{with} \quad (*, q) \quad \text{all identined to } *,$$

$$d_i(x, q) = \begin{cases} (x, q-1) & \text{for } 0 \leqslant i < q, \\ (d_{i-q}x, q) & \text{for } q \leqslant i \leqslant n, \end{cases}$$

$$s_i(x, q) = \begin{cases} (x, q+1) & \text{for } 0 \leqslant i < q, \\ (s_{i-q}x, q) & \text{for } q \leqslant i \leqslant n. \end{cases}$$

The *reduced suspension* $SK = CK/K$, the quotient simplicial set.

Let I be the simplicial set $\Delta[1]$. In I, let 0 stand for the vertex $\langle 0 \rangle$ and for any of its degeneracies $\langle 0, \ldots, 0 \rangle$; similarly for the vertex 1.

DEFINITION (1.9). If $f, g : K \to L$ are simplicial maps of simplicial sets, we call f *homotopic* to g if there is a simplicial map

$$F : K \times I \to L$$
$$\text{with} \quad F(x, 0) = f(x), \quad F(x, 1) = g(x),$$

in which case write $f \simeq g$. If $M \subset K$ is a subcomplex, write $f \simeq g$ (rel M) if F is constant on M. Note that a homotopy F can be regarded as a family of functions (*not* simplicial maps)

$$\{F_t\} : K_n \to L_n$$

for each integer $n \geqslant 0$ and each $t \in I_n$, where $F_t(x) = F(x, t)$. The F_t satisfy some conditions: For each $t \in I_n$,

$$d_i \circ F_t = F_{d_i t} \circ d_i ,$$

$$s_i \circ F_t = F_{s_i t} \circ s_i .$$

If all of the F_t are in a category $\mathscr{C}$, we say that F is a homotopy in $\mathscr{C}$.

PRINCIPLE (1.10). *Functors preserve homotopy: Let T be a functor from $\mathscr{C}$ to $\mathscr{D}$, and suppose*

$$\{F_t\} : f \simeq g : K \to L \text{ is a homotopy in } \mathscr{C}$$

Then

$$\{TF_t\} : Tf \simeq Tg : TK \to TL \text{ is a homotopy in } \mathscr{D}.$$

Note that even if $F : K \times I \to L$ is itself a map in some category, $TF : T(K \times I) \to TL$ is not, in general, another homotopy.

Let $\Lambda^k[n]$ be the subcomplex of $\Delta[n]$ generated by all $d_i(i_n)$ for $i \neq k$.

DEFINITION (1.12). We say that a simplicial set K satisfies the *extension condition* if every map $f : \Lambda^k[n] \to K$ has an extension $g : \Delta[n] \to K$.

Simplicial sets satisfying the extension condition are *Kan complexes*. As we see later, the singular simplicial set of a space is a Kan complex; a simplicial group is a Kan complex. But simplicial sets arising from geometrical simplicial complexes are *not* Kan complexes, and neither are $\Delta[n]$, $\dot{\Delta}[n]$, S^n, and other simple simplicial sets.

PROPOSITION (1.13). *Let K be a simplicial set. Then K satisfies the extension condition $\Leftrightarrow$ for every collection $y_0, \ldots, \hat{y}_k, \ldots, y_n$ of simplices in K_{n-1}, with $d_i y_j = d_{j-1} y_i$ for $i < j$, $i \neq k$, $j \neq k$, and there is a simplex $y \in K_n$ with $d_i y = y_i$, $i \neq k$.*

This matching face property is sometimes taken as the definition of the extension condition.

LEMMA (1.14). *Let $A \subset B$ denote any of the following pairs:*

$$\Delta[n] \times \Lambda^k[m] \subset \Delta[n] \times \Delta[m],$$

$$(\Delta[n] \times \Lambda^k [m]) \cup (\dot{\Delta}[n] \times \Delta[m]) \subset \Delta[n] \times \Delta[m].$$

Then for any Kan complex K and simplicial map $f : A \to K$, f can be extended to a map $g : B \to K$.

Proof. For each such pair, A can be obtained from B by successively adjoining a simplex and one of its faces, all other faces already lying in a subcomplex. Iterated application of the extension condition gives the lemma.

Remark. Of course this lemma is a special case of the simplicial extension theorem and could better be deduced from it.

LEMMA (1.15). *Let K be a Kan complex, L any simplicial set. Then any map $f : L \times \Lambda^k[m] \to K$ can be extended to a map $g : L \times \Delta[m] \to K$.*

Proof. By skeletons of L; the step from $L^{(n-1)}$ to $L^{(n)}$ is shown by the preceding lemma applied to each nondegenerate simplex of L_n.

COROLLARY (1.16). *If K is a Kan complex, the homotopy relation ($\simeq$) is an equivalence relation on the set of maps from L to K.*

The "inclusion as i-th face" map $\epsilon_i : \Delta[n-1] \to \Delta[n]$ and "i-th projection" $\eta_i : \Delta[n] \to \Delta[n-1]$ are the simplicial maps defined on the vertices by

$$\epsilon_i(q) = \begin{cases} q, & q < i, \\ q+1, & q \geqslant i, \end{cases}$$

$$\eta_i(q) = \begin{cases} q, & q \leqslant i, \\ q-1, & q > i. \end{cases}$$

Thus if $x \in K_n$ is represented by $f_x : \Delta[n] \to K$, $d_i x \in K_{n-1}$ is represented by $f_x \circ \epsilon_i$ and $s_i x$ is represented by $f_x \circ \eta_i$.

DEFINITION (1.16). Let L and K be simplicial sets. Then the function complex K^L is defined by

$$(K^L)_n = \{\text{simplicial maps } f : L \times \Delta[n] \to K\}$$

where

$$d_i f = f \circ (1_L \times \epsilon_i),$$

$$s_i f = f \circ (1_L \times \eta_i)$$

Remark. Careful scrutiny would reveal that this is very analogous to the function space definition for spaces.

PROPOSITION (1.17). *If K is a Kan complex, so is K^L.*

Proof. Let $f_0, \ldots, \hat{f}_k, \ldots, f_n \in (K^L)_{n-1}$ have matching faces, i.e., each $f_i : L \times \Delta[n-1] \to K$, and if we consider each such $\Delta[n-1]$ as the i-th face of $\Delta[n]$, the f_i agree on $(\Delta[n])^{(n-2)}$ to give a simplicial map $f : L \times \Lambda^k[n] \to K$. By (1.15) f extends to $g : L \times \Delta[n] \to K$. Then $g \in (K^L)_n$ is the required simplex with $d_i g = f_i$, $i \neq k$.

As usual, a simplicial map $f : L \to K$ is called a homotopy equivalence if there is a simplicial map $g : K \to L$ with $f \circ g \simeq 1_K$ and $g \circ f \simeq 1_L$, in which case write $L \simeq K$. Also there are the usual definitions of retract, deformation retract, strong deformation retract, etc.

DEFINITION (1.18). Let K be a Kan complex. Then for $x, y \in K_n$, call $x \simeq y$ if the representing maps f_x and f_y are homotopic (rel $\dot{\Delta}[n]$).

PROPOSITION (1.19). *Let K be a Kan complex, $x, y \in K_n$. Then $x \simeq y \Leftrightarrow d_i x = d_i y$ for all i, and for some $0 \leqslant k \leqslant n$ there is $w \in K_{n+1}$ with $d_k w = x$, $d_{k+1} w = y$, and $d_i w = d_i s_k x = d_i s_k y$, $k \neq i \neq k+1$.*

DEFINITION (1.20). A Kan complex M is called *minimal* if $x \simeq y \Rightarrow x = y$. If K is a Kan complex, then a subsimplicial set $M \subset K$ is called a *minimal subcomplex* of K if M is minimal and if M is a strong deformation retract of K.

PROPOSITION (1.21). *If M is minimal $\Leftrightarrow$ whenever $v, w \in K_{n+1}$ with $d_i v = d_i w$, $i \neq k$, then $d_k v = d_k w$.*

LEMMA (1.22). *Let K be a Kan complex.*

(1) *Suppose $x \in K_n$, $v \in K_{n-1}$ with $d_k x \simeq v$. Then there exists $z \in K_n$ with $d_i z = d_i x$ for $i \neq k$ and $d_k z = v$.*

(2) *Let $F : \Delta[n] \times I \to K$ and let $x = F(i_n, 1)$. Suppose $x \simeq y$. Then there exists $G : \Delta[n] \times I \to K$ with $G(i_n, 1) = y$ and $F = G$ on the subcomplex $(\Delta[n] \times 0) \cup (\dot{\Delta}[n] \times I)$.*

THEOREM (1.23). *Any Kan complex K contains a minimal subcomplex M; any two such minimal subcomplexes of K are isomorphic.*

Proof. Let M_0 consist of a choice of one vertex in each homotopy class in K_0. Suppose M_j is defined for all $0 \leqslant j \leqslant n-1$. Let M_n

consist of a choice of one simplex from each homotopy class in K_n of simplices all of whose faces are in M_{n-1}, choosing always a degenerate one if possible.

To show that M is a Kan complex, let $y_0, \ldots, \hat{y}_k, \ldots, y_n \in M_{n-1}$ have matching faces. Then there exists $y \in K_n$, $d_i y = y_i$, $i \neq k$. As $d_k y$ has all of its faces in M_{n-1}, it is homotopic in K to (a unique) $w \in M_{n-1}$. Then let $z \in K_n$ with $d_i z = d_i y$, $i \neq k$, and suppose $d_k z = w$. Then $z \simeq x$ (homotopy in K) where $x \in M_n$ satisfies $d_i x = y_i$ for $i \neq k$.

To show that M is a deformation retract of K, we construct $F : K \times I \to K$. To define F on $K^{(0)} \times I$ choose for each vertex $v \in K_0$ a simplex $x \in K_1$ with $d_0 x = v$ and $d_1 x = w$, where $v \simeq w$, $w \in M$. Let $F(v, i_1) = x$. Suppose F is defined on $(K^{(n-1)} \times I) \cup (K \times 0)$. For each nondegenerate $y \in K_n$, F defines a map

$$G_y : (\dot{\Delta}[n] \times I) \cup (\Delta[n] \times 0) \to K$$

Extend G_y to $\Delta[n] \times I$. Then $G_y(i_n \times 1)$ has all of its faces in M_{n-1}, and so is homotopic to a simplex z in M_n. Now construct $G_y : \Delta[n] \times I \to K$ with $G_y'(i_n \times 1) = z$, $G_y' = G_y$ elsewhere. Putting these G_y' together gives F on $K^{(n)} \times I$, and thence F on $K \times I$.

LEMMA (1.24). *Let $f \simeq g : K \to M$ where M is minimal. Then if f is an isomorphism, so is g.*

Proof (sketch). The homotopy $F : K \times I \to M$ yields a family $\{F_t\} : K_n \to M_n$ for each $n \geqslant 0$, each $t \in I_n$, as in (1.9). Then F_0 is an isomorphism, and so is F_t for each $t \in I_n$. This latter can also be shown by induction on t, as t "crosses" I_n, using the minimality of M.

Remark. If it is only assumed that f is one-one, then so will g be one-one. But if f is only assumed to be onto, then g may not be onto.

COROLLARY (1.25). *Let $f : M \to M$ be a homotopy equivalence of minimal complexes. Then f is an isomorphism onto.*

COROLLARY (1.26). *Let $f : K \to L$ be a homotopy equivalence. Then any minimal complex M of K is isomorphic to any minimal complex N of L.*

Proof. The composite $M \to K \to L \to N$ is a homotopy equivalence, and hence is an isomorphism.

Euclidean Simplices (1.27). Let $\delta[n] \subset R^{n+1}$ be the topological space (as a subset of R^{n+1})

$$\delta[n] = \{(x_0, .., x_n) : \Sigma x_i = 1, x_i \geqslant 0\}.$$

Let $\epsilon_i : \delta[n-1] \to \delta[n]$ and $\eta_i : \delta[n+1] \to \delta[n]$ be defined by

$$\epsilon_i(x_0, ..., x_{n-1}) = (x_0, ..., 0, ..., x_{n-1}),$$
$$\eta_i(x_0, ..., x_{n+1}) = (x_0, ..., x_i + x_{i+1}, ..., x_{n+1}).$$

Singular Simplicial Set (1.28). For each topological space X, let $S(X)$ be the simplicial set

$$S(X)_n = \{\text{all continuous } f : \delta[n] \to X\},$$
$$d_i(f) = f \circ \epsilon_i,$$
$$s_i(f) = f \circ \eta_i.$$

Then S is a functor from $\mathscr{T}$, the category of topological spaces, to $\mathscr{S}$, the category of simplicial sets.

Geometrical Realization (1.29). Let K be a simplicial set, and let $\overline{RK}$ be the topological space (as the disjoint union)

$$\overline{RK} = \bigcup_{x \in K} (\delta[\dim x], x).$$

Denote an equivalence relation on $\overline{RK}$ by $(p, x) \sim (y, q)$ if either

(1) $d_i x = y$ and $\epsilon_i(q) = p$;

(2) $s_i x = y$ and $\eta_i(q) = p$.

Then let $RK = \overline{RK}/(\sim)$ with the identification topology. It is easy to see that R defines a functor from $\mathscr{S}$ to $\mathscr{T}$.

In fact, each geometrical realization is a CW space; "CW complex" is Whitehead's term, but "complex" has too many other meanings to be usable here. If X is a geometrical simplicial complex, and K is constructed from X as in (1.3), then RK is homeomorphic to the topological space of X.

If K and L are simplicial sets, then there is a natural map

$$m : R(K \times L) \to RK \times RL$$

which is continuous, one-one, and onto, but not a homeomorphism in general. However, m will be a homeomorphism if either

(i) Both K and L are countable;

(ii) One of the two spaces RK or RL is locally finite.

For details, see Ref. [Mil]. In particular, since $R(\Delta[1]) = \delta[1]$ (= the unit interval), which is a finite CW space,

$$R(K \times \Delta[1]) \cong RK \times \delta[1].$$

This geometrical realization is related to the following construction. For a simplicial set K, let $\bar{K}$ be the simplicial set which is the disjoint union

$$\bar{K} = \bigcup_{x \in K} (\Delta[\dim x], x)$$

and for $(u, x) \in \bar{K}_q$, i.e., for $u \in \Delta[\dim x]_q$, $d_i(u, x) = (d_i u, x)$ and $s_i(u, x) = (s_i u, x)$. Let an equivalence relation on $\bar{K}$ be defined by $(u, x) \sim (v, y)$ if either

(1) $d_i x = y$ and $\epsilon_i(v) = u$;

(2) $s_i x = y$ and $\eta_i(v) = u$.

Then there is a natural equivalence $K \approx \bar{K}(\sim)$.

This construction is useful, for example, in the following way. To subdivide K, first subdivide each $\Delta[n] \subset \bar{K}$, put on an equivalence, and pass to the quotient set. See (12.4).

Adjoint Maps (1.30). Suppose K is a simplicial set, X is a topological space, and $f : K \to SX$ is a simplicial map. Let $\tilde{f} : RK \to X$ be the function

$$\tilde{f}(p, x) = f(x)(p)$$

for each $x \in K_n$, $p \in \delta[n]$. It is easily verified that $\tilde{f}$ is well-defined and continuous.

On the other hand, suppose $g : RK \to X$ is a continuous map. Let $\tilde{g} : K \to SX$ be the function

$$\tilde{g}(x)(p) = g(p, x)$$

for $x \in K_n$, $p \in \delta[n]$. Then $\tilde{g}$ is easily seen to be a simplicial map.

PROPOSITION (1.31). *There is a one-one correspondence*

$$\{\textit{simplicial maps } K \to SX\} \leftrightarrows \{\textit{continuous maps } RK \to X\}$$

which is natural in K and X, and preserves homotopy.

DEFINITION (1.32). If L is a Kan complex and K any simplicial set, let $[K \to L]$ stand for the set of homotopy classes of maps of K to L. Also, for topological spaces X and Y, let $[X \to Y]$ stand for the set of homotopy classes of maps of X to Y.

PROPOSITION (1.33). (1) *The functor $R : \mathscr{S} \to \mathscr{T}$ preserves homotopy, and defines a function*

$$[K \to L] \to [RK \to RL].$$

(2) *The functor $S : \mathscr{T} \to \mathscr{S}$ takes topological spaces to Kan complexes, preserves homotopy, and defines a function*

$$[X \to Y] \to [SX \to SY].$$

Proof. (1) Suppose

$$F : K \times \Delta[1] \to L$$

is a simplicial homotopy. Then

$$RF : R(K \times \Delta[1]) \to RL$$

will be the continuous homotopy, since $R(K \times \Delta[1])$ is homeomorphic to $RK \times \delta[1]$ by (1.29).

(2) Suppose

$$H : Y \times \delta[1] \to X$$

is a continuous homotopy. Then the required simplicial homotopy will be the composite

$$SY \times \Delta[1] \to SY \times S(\delta[1]) \to S(Y \times \delta[1]) \xrightarrow{S(H)} SX.$$

EQUIVALENCE OF SIMPLICIAL AND TOPOLOGICAL HOMOTOPY (1.34). Proposition (1.33) is not good enough. To establish the connection between simplicial homotopy and topological homotopy, we need the following.

THEOREM (1.35). (1) *For K a simplicial set and L a Kan simplicial set, R induces a one-one correspondence*

$$[K \to L] \to [RK \to RL].$$

(2) *For Y a topological space, and X a CW space, S induces a one-one correspondence*

$$[X \to Y] \to [SX \to SY].$$

(3) *R and S induce a one-one correspondence between homotopy types of Kan complexes and homotopy types of CW spaces.*

THEOREM (1.36) (Simplicial extension theorem). *Let K be a simplicial set and L a Kan complex. Suppose there is a continuous map $p : RK \to RL$. Then there is a simplicial map $g : K \to L$ with $Rg \simeq p : RK \to RL$.*

Proof (In Section 12). It takes some subdivision and a semi-simplicial approximation theorem—a little complicated, but basically elementary. (1.35) then follows easily from (1.36).

2. HOMOTOPY GROUPS

For each Kan complex K, with base point $*$, the n-th homotopy set $\pi_n(K, *)$ is defined, becomes a group for $n \geqslant 1$, and becomes an Abelian group for $n \geqslant 2$. It is elementary to prove the homotopy addition theorem for Kan complexes.

Fibre maps of simplicial sets are defined by a fill-in property similar to the extension condition. Fibre maps have the usual properties, like the long exact sequence in homotopy, existence of induced fibre maps, etc.

Following the treatment of Moore [Mo2], we introduce the Moore–Postnikov system of a simplicial set. For a minimal complex K, the maps in the Postnikov system for K are minimal fibre maps. This can be used to show that a weak homotopy equivalence between minimal complexes is an isomorphism; thus, a weak homotopy equivalence between Kan complexes is a homotopy equivalence.

The reference for Section 2 is [Mo2].

DEFINITION (2.1). For each Kan complex K with base point $*$, the n-th homotopy set $\pi_n(K, *)$ is to be the set of $\simeq$ equivalence classes of simplices of K_n all of whose faces are at the base point $*$. Write $[x]$ for the equivalence class in $\pi_n(K, *)$ containing x.

For each $n \geqslant 1$, define an operation $+$ in $\pi_n(K, *)$ as follows. For $a, b \in \pi_n(K, *)$, put

$$a + b = [d_1 w],$$

where $w \in K_{n+1}$ satisfies $[d_0 w] = b$, $[d_2 w] = a$, and $d_i w = *$ for all $3 \leqslant i \leqslant n + 1$.

PROPOSITION (2.2). *Under this operation, $\pi_n(K, *)$ becomes a group for $n \geqslant 1$; $\pi_n(K, *)$ is an Abelian group for $n \geqslant 2$.*

As usual, if G is a simplicial group (with identity e as base point), then the two group structures in $\pi_n(G, e)$ agree, and are both Abelian even for $n = 1$.

Henceforth simplicial sets mean with basepoint, maps are to preserve $*$, and write $\pi_n(K)$ for $\pi_n(K, *)$. If $f : K \to L$ is a map of Kan complexes, let f_* be the function $f_* : \pi_n(K) \to \pi_n(L)$ defined by

$$f_*[x] = [f(x)].$$

PROPOSITION (2.3). *Let $f : K \to L$ be a map of Kan complexes. Then $f_* : \pi_n(K) \to \pi_n(L)$ is well-defined, and f_* is a homomorphism for $n \geqslant 1$. Furthermore, if $f \simeq g : K \to L$* (rel $*$), *then $f_* = g_*$.*

THEOREM (2.4) (Homotopy addition theorem). *Let K be a Kan complex, and $y_i \in K_n$ for $0 \leqslant i \leqslant n + 1$, with all $d_j y_i = *$. Then in $\pi_n(K)$,*

$$[y_0] - [y_1] + [y_2] - \cdots + (-1)^{n+1}[y_{n+1}] = 0$$

$\Leftrightarrow$ *there is $y \in K_{n+1}$ with $d_i y = y_i$ for $0 \leqslant i \leqslant n + 1$.*

Proof ($\Leftarrow$). For $n = 2$, the statement is the definition of $+$ in $\pi_2(K)$. For $n > 2$, first consider the special case where $y_i = *$ for $i \neq q$, $q + 1, q + 2$. If also $q = 0$, the statement is the definition of $+$ in $\pi_n(K)$. For $q > 0$, use the extension condition on K to move the relation one step to the left. That is, take $w \in K_{n+2}$ with $d_{q-1} w = s_{q+1} y_{q+2}$, $d_q w = y$,

$d_{q+1}w = s_q y_q$, $d_{q+3}w = s_{q-1}y_{q+2}$, and all other faces of w except $d_{q+2}w$ are to be $*$. The faces match, as seen from the following table:

	$\cdots$	d_{q-1}	d_q	d_{q+1}	d_{q+2}	$\cdots$
$d_{q-1}w = s_{q+1}y_{q+2}$	$\cdots$	$*$	$*$	y_{q+2}	y_{q+2}	$\cdots$
$d_q w = y$	$\cdots$	$*$	y_q	y_{q+1}	y_{q+2}	$\cdots$
$d_{q+1}w = s_q y_q$	$\cdots$	$*$	y_q	y_q	$*$	$\cdots$
$d_{q+2}w$	$\cdots$	—	—	—	—	$\cdots$
$d_{q+3}w = s_{q-1}y_{q+2}$	$\cdots$	y_{q+2}	y_{q+2}	$*$	$*$	$\cdots$

Then the only non-$*$ faces of $z = d_{q+2}w$ are $d_{q-1}z = y_{q+2}$, $d_q z = y_{q+1}$, $d_{q+1}z = y_q$. It now follows by repeating this move q times that the formula holds for such special case.

For the general case, make use of the following:

SUBLEMMA. *Let K be a Kan complex, and let $x \in K_{n+1}$ with $d_i x = x_i$ for $0 \leqslant i \leqslant n+1$. Let q be any integer $\leqslant n-1$, and let $w \in K_{n+1}$ have faces $d_i w = x_i$ for $i \neq q, q+1$, and $d_q w = *$, $d_{q+1}w = z$. Then in $\pi_q(K)$*

$$(-1)^q [x_q] + (-1)^{q+1} [x_{q+1}] + (-1)^{q+2} [z] = 0.$$

Proof. Use the extension condition on K to find $v \in K_{n+2}$ whose faces are $d_i v = s_{q+1}x_i$ for $i \leqslant q-1$, $d_q v = s_q x_q$, $d_{q+2}v = x$, $d_{q+3}v = w$, and $d_i v = s_{q+2}x_{i-1}$ for $i \geqslant q+4$. The faces match, as seen from the following table:

	$\cdots$	d_{q-1}	d_q	d_{q+1}	d_{q+2}	d_{q+3}	$\cdots$
$d_{q-1}v = s_{q+1}x_{q-1}$	$\cdots$	$*$	$*$	x_{q-1}	x_{q-1}	$*$	$\cdots$
$d_q v = s_q x_q$	$\cdots$	$*$	x_q	x_q	$*$	$*$	$\cdots$
$d_{q+1}v$	$\cdots$	—	—	—	—	—	$\cdots$
$d_{q+2}v = x$	$\cdots$	x_{q-1}	x_q	x_{q+1}	x_{q+2}	x_{q+3}	$\cdots$
$d_{q+3}v = w$	$\cdots$	x_{q-1}	$*$	z	x_{q+2}	x_{q+3}	$\cdots$
$d_{q+4}v = s_{q+2}x_{q+3}$	$\cdots$	$*$	$*$	$*$	x_{q+3}	x_{q+3}	$\cdots$

Then the only non-$*$ faces of $d_{q+1}v = u$ are $d_q u = x_q$, $d_{q+1}u = x_{q+1}$, $d_{q+2}u = z$, so the asserted relation follows by using the special case of the theorem.

To conclude the general case, let $y \in K_{n+1}$, with $d_i y = y_i$, for $0 \leqslant i \leqslant n+1$. For each $0 \leqslant q \leqslant n-1$, let $w_q \in K_{n+1}$ satisfy $d_i w_q = *$ for $i \leqslant q$, $d_i w_q = y_i$ for $i \geqslant q+2$, and suppose $d_{q+1} w = z_q$.
Then the sublemma applied to w_q and w_{q+1} shows that

$$(-1)^q [z_q] + (-1)^{q+1} [y_{q+1}] + (-1)^{q+2} [z_{q+1}] = 0.$$

Summing these gives the result.

Proof of ($\Rightarrow$). Suppose $y_0, y_1, \ldots, y_{n+1}$ are in K_n with all $d_j y_i = *$, and that

$$[y_0] - [y_1] + \cdots + (-1)^{n+1} [y_{n+1}] = 0.$$

By the extension condition applied to $\hat{y}_0, y_1, \ldots, y_{n+1}$, there is $x \in K_{n+1}$ with $d_i x = y_i$ for $i \neq 0$. By the previous steps,

$$[d_0 x] - [y_1] + \cdots + (-1)^{n+1} [y_{n+1}] = 0.$$

Then $d_0 x \simeq y_0$, so there is $w \in K_{n+1}$ with $d_0 w = d_0 x$, $d_1 w = y_0$, and $d_i w = *$ for $i \geqslant 2$. Another use of the extension condition shows that there is $z \in K_{n+2}$ with $d_0 z = w$, $d_1 z = x$, and $d_i z = s_{i-2} y_{i-1}$ for $3 \leqslant i \leqslant n+2$. Then $y = d_2 z$ satisfies $d_i y = y_i$ as required.

DEFINITION (2.5). A simplicial map $p : E \to B$ is called a *fibre map* if whenever $f : \Lambda^k[n] \to E$ and $g : \Delta[n] \to B$ with $p \circ f = g \mid \Lambda^k[n]$, then there is an extension of f to a map $f' : \Delta[n] \to E$, with $p \circ f' = g$.
The picture for this is the diagram

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "g"] \arrow[ur, dashed, "f'"] & B \end{tikzcd}$$

Call f' an extension of f which covers g. There is also a matching face definition of a fibre map, analogous to the one for the extension condition. Notice that the map $K \to *$ is a fibre map $\Leftrightarrow$ K satisfies the extension condition.

Remark. It is easy to show that if $f : X \to Y$ is a Serre fibre map of topological spaces, then $Sf : SX \to SY$ is a simplicial fibre map. The other way is also valid, as shown by Barratt (unpublished) and Quillen [Q2]: If $p : E \to B$ is a simplicial fibre map, then $Rp : RE \to RB$ is a Serre fibre map.

THEOREM (2.6) (Covering homotopy theorem). *Let $p : E \to B$ be a fibre map and let K be any simplicial set. Suppose there is a map $f : K \to E$ and a homotopy $F : K \times I \to B$ with $F(x, 0) = pf(x)$ for all $x \in K$. Then there is a (covering homotopy) $G : K \times I \to E$ with $p \circ G = F$ and $G(x, 0) = f(x)$.*

Proof. This is not hard using some earlier prismatic techniques.

DEFINITION (2.7). A sequence of simplicial maps

$$F \xrightarrow{i} E \xrightarrow{p} B$$

is called a *fibration* if p is a fibre map onto, and i maps F one-one and onto $p^{-1}(*)$.

Let $F \to E \to B$ be a fibration, where B is a Kan complex. For each integer $n \geqslant 1$, define $\partial : \pi_n(B) \to \pi_{n-1}(F)$ by $\partial b = [d_0 w]$ for $b \in \pi_n(B)$, where $w \in E_n$ satisfies $d_i w = *$ for $i > 0$ and $[p(w)] = b$.

THEOREM (2.8). *∂ is well-defined, independent of the choice of w. For $n \geqslant 1$, ∂ is a homomorphism. The sequence*

$$\cdots \to \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \to \cdots \to \pi_0(E) \to \pi_0(B) \to *$$

is exact; that is, the counter-image of $*$ of each map is the image of the previous map.

The proof is straightforward. Call this the long exact sequence in homotopy (LES in π) of the fibration.

PATHS AND LOOPS (2.9). For each Kan complex K, with base point $*$, the *path complex* PK is to be the simplicial set

$$(PK)_n = \{x \in K_{n+1} : d_1 \cdots d_{n+1} x = *\},$$

and for each $0 \leqslant i \leqslant n$, d_i on $(PK)_n$ is to be the restriction of d_{i+1} on K_{n+1}, and similarly for each s_i. Let $p : PK \to K$ be the simplicial map which sends $x \in (PK)_n$ to $d_0 x \in K_n$. The *loop complex* ΩK is to be the subcomplex $p^{-1}(*) \subset PK$. That is,

$$(\Omega K)_n = \{x \in K_{n+1} : d_1 \cdots d_{n+1} x = * \text{ and } d_0 x = *\}.$$

Then

$$\Omega K \xrightarrow{i} PK \xrightarrow{p} K$$

is a fibration with contractible total complex. In this case, of course,

$$\partial : \pi_n(K) \to \pi_{n-1}(\Omega K)$$

is an isomorphism for each $n \geqslant 1$, but possibly not onto for $n = 1$.

A map (f, g) from one fibre map $p' : E' \to B'$ to another $p : E \to B$ is a commutative diagram:

$$\begin{array}{ccc} E' & \xrightarrow{f} & E \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{g} & B \end{array}$$

A homotopy $(\theta, \psi) : (f, g) \simeq (h, k)$ is a commutative diagram:

$$\begin{array}{ccc} E' \times I & \xrightarrow{\theta} & E \\ \downarrow{\scriptstyle p' \times 1} & & \downarrow{\scriptstyle p} \\ B' \times I & \xrightarrow{\psi} & B \end{array}$$

where $\theta : f \simeq h$ and $\psi : g \simeq k$. The term *strong homotopy* is used if ψ is constant, i.e., if $\psi(b', t) = \psi(b', s)$ for all $b' \in B_n{}'$, and all $t, s \in I_n$.

INDUCED FIBRE MAPS (2.10). Let $p : E \to B$ be a fibre map, and let $f : A \to B$ be a simplicial map. Then as usual there is an induced fibre map $p' : E' \to A$, where

$$(E')_n = \{(x, a) \in E_n \times A_n : p(x) = f(a)\}$$

with the evident face and degeneracy operators. There is a map $(\bar{f}, f)$ of fibre maps

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \end{array}$$

For a special case, an inclusion $i : A \to B$ of a subsimplicial set induces a fibre map $E' \to A$, where $E' = p^{-1}(A)$. If v is any vertex of B, the simplicial set $p^{-1}(v)$, called the fibre over v, is a Kan complex.

PROPOSITION (2.11). *Let $p : E \to B$ be a fibre map, and let $f \simeq g : A \to B$. Then the fibre maps induced by f and g have the same homotopy type, by strong homotopies.*

Proof. Let $F : A \times I \to B$ be the homotopy between f and g. Let $p^0 : E^0 \to A$, $f^1 : E^1 \to A$, and $\bar{p} : \bar{E} \to A \times I$ be the fibre maps induced by f, g, and F, respectively. Thus p^0 is the part of $\bar{p}$ over $A \times 0$. There is a homotopy $F^0 : E^0 \times I \to \bar{E}$ with a commutative diagram

$$\begin{tikzcd} E^0 \times I \arrow[r, "F^0"] \arrow[d, "p^0 \times 1"] & \bar{E} \arrow[d, "p"] \\ A \times I \arrow[r, "="] & A \times I \end{tikzcd}$$

so that the composite

$$E^0 \approx E^0 \times 1 \to E^0 \times I \xrightarrow{F^0} E$$

can be considered as a map $u : E^0 \to E^1$. Similarly, there is a map $F^1 : E^1 \times I \to \bar{E}$ which produces a map $v : E^1 \to E^0$. By considering the diagram

$$\begin{tikzcd} E^0 \times \Lambda^1[2] \arrow[r, "\theta"] \arrow[d] & \bar{E} \arrow[d, "p"] \\ E^0 \times \Delta[2] \arrow[r] \arrow[ur, "\psi"] & A \times I \end{tikzcd}$$

where θ on $E^0 \times (d_0 i_2)$ is F^0, and θ on $E^0 \times (d_2 i_2)$ is the composite $F^1 \circ (u \times 1)$, there results a map ψ, which when restricted to $E^0 \times (d_1 i_2)$ gives $v \cdot u \simeq 1_{E^0}$. Similarly $u \cdot v \simeq 1_{E^1}$.

MOORE–POSTNIKOV SYSTEMS (2.12). Let K be a simplicial set. For each integer $n \geqslant 0$ let an equivalence relation R_n on K_n be defined by xR_ny for $x, y \in K_q$ if each face of x of dimension $\leqslant n$ agrees with the corresponding face of y. Equivalently, $xR_ny \Leftrightarrow$ the representing maps $f_x, f_y : \Delta[q] \to K$ agree on $\Delta[q]^{(n)}$. Let P_nK be the simplicial set where

$$(P_nK)_q = K_q/(R_n)$$

with face and degeneracy operators induced from those in K. The natural sequence of maps

$$K \to \cdots \to P_nK \to P_{n-1}K \to \cdots \to P_0K \to *$$

is called the Moore–Postnikov system for K.

LEMMA (2.13). *If K is a Kan complex, then*

(1) *Each $K \to P_nK$ is a fibre map onto*;

(2) *Each P_nK is a Kan complex*;

(3) *For each $n \geqslant m$, $p : P_nK \to P_mK$ is a fibre map onto.*

Proof. (1) For dimensions $q \leqslant n$, $K_q = (P_nK)_q$. For dimensions $q > n$, use the extension condition on K.

(2) Since p is a fibre map onto by (1), and K is a Kan complex, so is P_nK.

(3) Similar to (1), using (2).

Let E_nK be the subcomplex of K consisting of all simplices x, each face of x of dimension $< n$ being at the base point $*$. Thus

$$E_nK \to K \to P_{n-1}K$$

is a fibration.

THEOREM (2.14). *Let K be a Kan complex. Then*

$$\begin{aligned}
&(1)\ \pi_q(P_nK) = \pi_q(P_mK) && \text{for } q \leqslant m, n;\\
&(2)\ \pi_q(P_mK) = 0 && \text{for } q > 0;\\
&(3)\ \pi_q(E_{m+1}(P_nK)) = \begin{cases} \pi_q(P_nK) & \text{for } q > m;\\ 0 & \text{for } q \leqslant m,\\ 0 & \text{for } q > n.\end{cases}
\end{aligned}$$

Proof. $E_{m+1}(P_nK) = *$ for $q \leqslant m$. This and the LES in homotopy shows (1). If $x \in (P_mK)_q$ with all $d_ix = *$, then $x = *$ in P_mK for $q > m$, proving (2). The LES in homotopy shows (3).

Let $F_nK = E_n(P_nK)$, so that there are fibrations

$$F_nK \to P_nK \to P_{n-1}K.$$

LEMMA (2.15). $\pi_q(F_nK) = 0$ *for* $n \neq q$, *and* $\pi_n(F_nK) = \pi_n(K)$. *Thus F_nK is an Eilenberg–MacLane complex $K(\pi_n(K), n)$.*

MINIMAL FIBRE MAPS (2.16). Let $p : E \to B$ be a fibre map, and suppose we have a commutative diagram

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\
\Delta[n] \arrow[r, "g"'] \arrow[ur, dashed, "f'"] & B
\end{tikzcd}$$

DEFINITION (2.17). The fibre map $p : E \to B$ is called *minimal* if whenever there is such a diagram, the extension f' is uniquely determined on the missing face. That is, if f', f'' are two such extensions, $f'(d_k i_n) = f''(d_k i_n)$.

PROPOSITION (2.18). (1) *If $p : E \to B$ is a fibre map and E is a minimal complex, then p is minimal; if also p is onto, then B is a minimal complex.*

(2) *If p is a minimal fibre map, then each fibre is a minimal complex.*

This is straightforward from the definitions. Note that even if $p : E \to B$ is a minimal fibre map, and B is a minimal complex, then E might not be a minimal complex.

WEAK HOMOTOPY EQUIVALENCE (2.19). We say that a simplicial map $f : K \to L$ of connected simplicial sets is a weak homotopy equivalence if $f_* : \pi_q(K) \to \pi_q(L)$ is an isomorphism for all q.

THEOREM (2.20). *If $f : K \to L$ is a map of connected Kan complexes, then f is a weak homotopy equivalence $\Leftrightarrow$ f is a homotopy equivalence. A weak homotopy equivalence of two minimal complexes is an isomorphism.*

Proof. It is sufficient to prove the latter statement, so assume that $f : K \to L$ is a weak homotopy equivalence, where K and L are minimal complexes. Consider the Moore–Postnikov systems

$$\begin{array}{ccccccc} F & \to & P_nK & \xrightarrow{P_nf} & P_nL & \leftarrow & F' \\ & & \downarrow & & \downarrow & & \\ & & P_{n-1}K & \xrightarrow{P_{n-1}f} & P_{n-1}L & & \end{array}$$

By (2.13) and (2.18), each of the complexes in the diagram is a minimal complex, and each of the vertical maps is a minimal fibre map onto. As the map f induces $f_* : \pi_n(K) \approx \pi_n(L)$, the fibres $F = K(\pi_nK, n)$ and $F' = K(\pi_nL, n)$ must be isomorphic. Anticipating Section 6, we find that $P_nK \approx F \times {}_tP_{n-1}K$ and $P_nL \approx F' \times {}_tP_{n-1}L$, the twisted Cartesian products. It follows by induction that P_nf is an isomorphism for all n.

3. Group Complexes

There are two basic properties of simplicial groups, both due to Moore. First, each simplicial group satisfies the extension condition. Second, its homotopy groups $\pi_*(G)$ can be obtained as the *homology*, i.e., ker ∂/image ∂ of a certain chain complex (NG, ∂).

For a simplicial Abelian group A, there are two other chain complexes, the "total complex" with $\partial = \Sigma(-1)^i d_i$, and the "normalized chain complex," also with $\partial = \Sigma(-1)^i d_i$. All three chain complexes obtained from A have isomorphic homology groups, which by the preceding statements are isomorphic to the homotopy groups of A.

For each (reduced) simplicial set Kan [K1] constructs a simplicial group GK which has the properties for a "loop complex" for K; the proof of this is facilitated by introducing twisting functions and principal fibrations.

There is another construction, adjoint to the $G(\cdot)$ construction, which assigns to each simplicial group G a simplicial complex $\overline{W}G$, which is a "classifying complex" for G.

We write e for the identity of a multiplicative group, 0 for the identity of an additive group. The identity of a simplicial group will be taken as the base point. Recall that a simplicial set K is called a simplicial group if each K_n is a group and all d_i, s_i are homomorphisms.

The reference for Section 3 is [K1].

Lemma (3.1). *If G is a simplicial group, then G satisfies the extension conditions.*

Proof. Let $y_0, \ldots, \hat{y}_k, \ldots, y_n \in G_{n-1}$ have matching faces. Then put

$$\begin{aligned}
w_0 &= s_0 y_0, \\
w_i &= w_{i-1}(s_i d_i w_{i-1})^{-1} s_i y_i && \text{for} \quad 0 < i < k, \\
w_n &= w_{k-1}(s_{n-1} d_n w_{k-1})^{-1} s_{n-1} y_n, \\
w_i &= w_{i+1}(s_{i-1} d_i w_{i+1})^{-1} s_i y_i && \text{for} \quad k < i < n.
\end{aligned}$$

Then $w_{k+1} \in G_n$ is the desired simplex satisfying $d_i w_{k+1} = y_i$ for $i \neq k$.

Lemma (3.2). *If $f : G \to H$ is a homomorphism onto of simplicial groups, then f is a fibration.*

Proof. Suppose $y_0, \ldots, \hat{y}_k, \ldots, y_n \in G_{n-1}$ have matching faces, and $x \in H_n$ with $d_i x = f(y_i)$ for $i \neq k$. Take $v \in p^{-1}(x)$, and $z_i = v^{-1} y_i$.

Then the faces of z_i match, and the construction of (3.1) applied to the z_i yields $w \in G_n$ with $f(w) = e$ (= identity in H_n) and $d_i w = z_i$ for $i \neq k$. Then the simplex $v \cdot w \in G_n$ has the desired properties.

DEFINITION (3.3). A *chain complex* (C, ∂) is a sequence of groups and homomorphisms

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

with image ∂_{n+1} a normal subgroup of $\ker \partial_n$. For each integer n, the homology group $H_n(C, \partial)$ is defined to be the quotient group $\ker \partial_n / \text{image } \partial_{n+1}$.

EXAMPLE (3.4). If A is a simplicial Abelian group, then (A, ∂) where

$$\partial_n = \sum_{i=0}^{i=n} (-1)^i d_i ,$$

becomes a chain complex (of Abelian groups). It is an easy consequence of the simplicial identities that $\partial_n \circ \partial_{n+1} = 0$.

DEFINITION (3.5). For each simplicial set K, let $Z(K)$ be the simplicial free Abelian group generated by K, and let $C_*(K)$ be the chain complex $(Z(K), \partial)$. The *homology groups* $H_n(K)$ for each $n \geqslant 0$ are defined by

$$H_n(K) = H_n(C_*(K), \partial)$$

DEFINITION (3.6). For each simplicial group g, let (NG, ∂) be the chain complex

$$(NG)_n = \bigcap_{i \neq 0} \ker d_i ,$$

$$\partial_n = d_0 \text{ (restricted to } (NG)_n).$$

THEOREM (3.7). *For a simplicial group G, $\pi_n(G) \approx H_n(NG, \partial)$.*

Proof. Let $\omega : \ker \partial n \to \pi_n(G)$ be the function $\omega(x) = [x]$. Then it can easily be checked that ω is a homomorphism. Also, $x \in \ker \omega \Leftrightarrow z \in \text{image } \partial_{n+1}$, so induces an isomorphism.

COROLLARY (3.8). *A simplicial group considered as a simplicial set satisfying the extension condition will be a minimal complex $\Leftrightarrow$ the chain complex (NG, ∂) is minimal, i.e., each ∂_n is the null homomorphism.*

For any simplicial Abelian group A, let DA be the subgroup generated by all degenerate simplices. Then it is easily checked that

$$\partial_n(DA)_n \subset (DA)_{n-1}$$

and $(A/DA, \partial)$ also becomes a chain complex. The following is a standard exercise [Mac].

STANDARD EXERCISE (3.9). For a simplicial Abelian group A, the natural maps

$$H_n(NA, \partial) \leftarrow H_n(A, \partial) \rightarrow H_n(A/DA, \partial)$$

are isomorphisms.

Remark (3.10). Combined with (3.8), we have $H_n(K) \approx \pi_n(Z(K))$; that is, the *homology* groups of K are isomorphic to the *homotopy* groups of $Z(K)$, both being isomorphic to the homology groups of the chain complex $(C_*(K), \partial)$.

If $f, g : K \rightarrow L$, with $f \simeq g$, then also $Z(f) \simeq Z(g)$ as maps $Z(K) \rightarrow Z(L)$. In this case, $f_* = g_* : H_*(K) \rightarrow H_*(L)$. In particular, if K and L have the same homotopy type, then $H_*(K) \approx H_*(L)$.

HUREWICZ HOMOMORPHISM (3.11). For any Kan complex K, the natural inclusion $K \rightarrow Z(K)$ induces homomorphisms for $n \geqslant 1$,

$$h_n : \pi_n(K) \rightarrow \pi_n(Z(K)) \approx H_n(K).$$

THEOREM (3.12) (Hurewicz). *If K is a connected Kan complex with $\pi_i(K) = 0$ for $i < n$ $(n \geqslant 2)$, then $h_n : \pi_n(K) \rightarrow H_n(K)$ is an isomorphism.*

Proof. It is sufficient to prove the theorem for a minimal complex M. In this case, $\pi_n(M) = M_n$, and consider

$$h_n : M_n \rightarrow H_n(M).$$

(1) h_n is onto: The elements of M_n generate $H_n(M)$ and if x, y are in M_n, then the formal sum $h_n(x) + h_n(y)$ in $H_n(M)$ is homologous to $h_n(z)$, where z in M_n is constructed using the extension condition in M.

(2) h_n is one-one: For if $h_n(x) = 0$, there must be $c = \Sigma b_j c_j \in C_{n+1}(M)$, with $\partial c = x$ in $C_n(M)$. By the homotopy addition theorem, $\partial c_j = 0$ in $\pi_n(M)$; hence $\partial \Sigma b_j c_j$, and therefore also x must be 0 in $\pi_n(M)$.

Remark. For $n = 1$, this method shows that if $h_1(x) = 0$, then x is a commutator. Thus the result for $n = 1$ becomes $\pi_1(K)/[\pi_1(K), \pi_1(K)] \approx H_1(K)$.

COROLLARY (3.13). *If K is a connected Kan complex with $\pi_1(K) = 0$ and $H_i(K) = 0$ for all $i \geqslant 2$, then K is contractible.*

DEFINITION (3.14). A *principle bundle* is a triple (G, t, K), where G is a simplicial group, K is a reduced simplicial set, and $t : K \to G$ is a function (called twisting function) decreasing dimension by one, satisfying

$$\begin{aligned} d_i t(x) &= t(d_{i+1}x) && \text{for } i > 0, \\ d_0 t(x) &= t(d_1 x) \cdot t(d_0 x)^{-1}, \\ s_i t(x) &= t(s_{i+1}x) && \text{for } i \geqslant 0, \\ * &= t(s_0 x). \end{aligned}$$

The bundle complex $G \times {}_tK$ is to be the simplicial set $(G \times {}_tK)_n = G_n \times K_n$ with face and degeneracy operators

$$\begin{aligned} d_0(a, x) &= (d_0 a \cdot t(x), d_0(x)), \\ d_i(a, x) &= (d_i a, d_i x) && \text{for } i > 0, \\ s_i(a, x) &= (s_i a, s_i x) && \text{for } i \geqslant 0. \end{aligned} \tag{3.14a}$$

It is then straightforward to verify that $G \times {}_tK$ becomes a simplicial set and that

$$G \to G \times {}_tK \to K$$

is a fibration. On the other hand, if we try to make a fibration $G \to G \times {}_tK \to K$, with face and degeneracy operators as in (3.14a), the function t which occurs will have to satisfy (3.14).

If $G \times {}_tK$ is contractible, then we call G a *loop group* for K, and K a *classifying complex* for G. In this case, $\pi_n(K) \approx \pi_{n-1}(G)$ for $n \geqslant 1$.

DEFINITION (3.15). For a reduced simplicial set K, let GK be the simplicial group defined by

(1) $(GK)_n$ is the group which has one generator $\bar{x}$ for every $x \in K_{n+1}$ and one relation $\overline{s_0 x} = e$ for every $x \in K_n$;

(2) The face and degeneracy operators are given by

$$d_0\bar{x} = (\overline{d_1 x}) \cdot (\overline{d_0 x})^{-1},$$

$$d_i\bar{x} = \overline{d_{i+1}x} \qquad \text{for} \quad i > 0,$$

$$s_i\bar{x} = \overline{s_{i+1}x}.$$

Thus $G(\cdot)$ is a functor from reduced simplicial sets to simplicial groups. In Ref. [K1], $G(\cdot)$ is defined for arbitrary simplicial sets with base point by means of a maximal tree.

Observe that $t : K \to GK$ by $t(x) = \bar{x}$ is a twisting function, which defines a simplicial set $EK = GK \times_t K$.

THEOREM (3.16). *For a (reduced) Kan complex K, GK is a loop group for K; that is, EK is contractible.*

Proof. The proof which closely follows Ref. [K1], consists in showing (1) EK is connected; (2) $\pi_1(EK) = 0$; (3) $H_n(K) = 0$ for each $n \geqslant 1$. Then (3.13) shows that EK is contractible.

We need a slightly different description of GK and EK. A "closed n-loop of length $2k$" is to be a sequence $(x_1, \ldots, x_{2k})$, $x_j \in K_{n+1}$, for which $d_0 x_{2i-1} = d_0 x_{2i}$ for each $0 < i \leqslant k$. An "n-path of length $2k + 1$" is to be a sequence $(x_1, \ldots, x_{2k+1})$, $x_j \in K_{n+1}$, also satisfying $d_0 x_{2i-1} = d_0 x_{2i}$ for each $0 < i \leqslant k$. (*Notice*: This is for reduced K, i.e., with only one vertex.) Among the closed n-loops of K, introduce an equivalence relation by

$$(x_1, \ldots, x_{2k}) \sim (x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2k})$$

whenever $x_j = x_{j+1}$. Let $\langle x_1, \ldots, x_{2k} \rangle$ be the class of $(x_1, \ldots, x_{2k})$ under the equivalence relation generated by $\sim$. For a closed n-loop $(x_1, \ldots, x_{2k})$, and for $0 \leqslant i \leqslant n$, let

$$d_i(x_1, \ldots, x_{2k}) = d_{i+1}x_1, \ldots, d_{i+1}x_{2k}),$$

$$s_i(x_1, \ldots, x_{2k}) = (s_{i+1}x_1, \ldots, d_{i+1}x_{2k}).$$

The d_i and the s_i respect the equivalence relation and $G'K$ becomes a simplicial set where $(G'K)_n$ is the set of equivalence classes of closed n-loops. It can easily be verified that the function $f : G'K \to GK$, defined by

$$f\langle x_1, \ldots, x_{2k} \rangle = \bar{x}_1(\bar{x}_2)^{-1}\, \bar{x}_3(\bar{x}_4)^{-1} \cdots \bar{x}_{2k-1}(\bar{x}_{2k})^{-1}$$

is a simplicial isomorphism, and thus by transport of structure, $G'K$ becomes a simplicial group.

Similarly define $E'K$ by equivalence classes of paths, and $f : E'K \to GK \times {}_tK$ is an isomorphism, where

$$f\langle x_1, \dots, x_{2k+1}\rangle = (f\langle x_1, \dots, x_{2k}\rangle, d_0x_{2k+1}).$$

We shall need to use the following, the proof of which is a standard exercise.

PROPOSITION (3.17). *Let X be a connected Kan complex, and $T \subset X$ a maximal tree; that is, T is a connected subcomplex of X containing all the vertices of X, but no reduced n-loops. Then $\pi_1(X) \approx F/R$, where F is the free group generated by the set X_1, and R is the normal subgroup of F generated by all $t \in T_1$, and all $(d_0w)(d_1w)^{-1}(d_2w)$ for $w \in X_2$ nondegenerate.*

We can now complete the proof of (3.16).

(1) EK is connected, since $\pi_1(EK) \to \pi_0(GK)$ is onto.

(2) We calculate $\pi_1(EK)$ by (3.17). Let T be the subcomplex of EK consisting of

$$T_0 = (EK)_0 = \{(g, *) : g \in (GK)_0\},$$
$$T_1 = \{(s_0g, x) : g \in (GK)_0, x \in K_1\},$$
$$T_n = \left\{\bigcup s_iT_{n-1}\right\} \qquad \text{for all} \quad n \geqslant 2.$$

The T is a maximal tree in EK and hence $\pi_1(EK)$ is isomorphic to F/R as in (3.17), where F is the free group generated by $(EK)_1$, and R is the normal subgroup containing all $(s_0\,g, x)$ and all $(d_0u)(d_1u)^{-1}(d_2u)$, where $u \in (EK)_2$ is nondegenerate. First take $u = (s_1h, s_0x)$ to obtain $(s_0d_0h, x)(h, x)^{-1}(h, *) \in R$, whence also $(h, x)^{-1}(h, *) \in R$. Next take $u = (s_0k, y)$ to obtain $(ky, d_0y)(k, d_1y)^{-1}(s_0d_1k, d_2y) \in R$, whence also $(ky, d_0y)(k, d_1y) \in R$. Thus all $(g, y) \in R$, so $\pi_1(EK) = 0$.

(3) There is contracting homotopy D for $(C_*(E'K), \partial)$. For each $u = \langle x_1, \dots, x_{2k+1}\rangle$, let for $0 \leqslant i \leqslant n$, $1 \leqslant j \leqslant 2k+1$,

$$u_{i,j} = \langle s_{i+1}x_1, \dots, s_{i+1}x_{j-1}, s_i \cdots s_0d_1 \cdots d_nx_j\rangle \qquad \text{for } j \text{ odd},$$
$$u_{i,j} = \langle s_{i+1}x_1, \dots, s_{i+1}x_j, s_i \cdots s_0d_1 \cdots d_nx_j\rangle \qquad \text{for } j \text{ even}.$$

Let $D_nu = \sum_{i,j}(-1)^{i+j+1}u_{i,j}$. Then a straightforward calculation shows that

$$\partial_{n+1}D_n \pm D_{n-1}\partial_n = \text{identity on } C_*(E'K).$$

Thus $H_n(EK) = 0$ for all n, and EK is contractible.

Hurewicz Homomorphism (3.18). For a simplicial group G, let $\Gamma_2 G = [G, G]$ be the commutator subgroup of G (in every dimension). Then the fibration

$$\Gamma_2 G \xrightarrow{i} G \xrightarrow{p} G/\Gamma_2 G$$

yields a long exact sequence in homotopy. Take $G = GK$, where K is a reduced Kan complex. Then it is not difficult to verify that there is a commutative diagram

$$\begin{tikzcd}
 & \pi_q(K) \arrow[r, "h_q"] \arrow[d] & H_q(K) \arrow[d] & \\
\cdots \arrow[r] \pi_{q-1}(\Gamma_2 G) \arrow[r, "i_*"] & \pi_{q-1}(G) \arrow[r, "p_*"] & \pi_{q-1}(G/\Gamma_2 G) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Kan [K1], gives the following reformulation of the Hurewicz Theorem, which we state without proof.

Proposition (3.19). *Let F be a simplicial group with each F_n free. Then*

(1) $\pi_0(F) \to \pi_0(F/\Gamma_2 F)$ *is onto and has for kernel the commutator subgroup* $[\pi_0(F), \pi_0(F)]$;

(2) *If* $\pi_i(F) = 0$ *for all* $i < n$ $(n \geqslant 1)$, *then* $\pi_i(\Gamma_2 F) = 0$ *for all* $i \leqslant n$.

If we take $F = GK$, where K is a reduced Kan complex, and the LES in π of the fibration $\Gamma_2 F \to F \to F/\Gamma_2 F$, then (3.19) produces the usual form of the Hurewicz theorem. For simply connected K, the groups $\pi_*(\Gamma_2 F)$ are isomorphic to J. H. C. Whitehead's Γ groups, and the LES in π becomes the "certain exact sequence."

Definition (3.20). If G is a simplicial group, then define a simplicial set $\overline{W}G$ by

$$\begin{aligned}
(\overline{W}G)_n &= \{(g_{n-1}, \ldots, g_0) : g_i \in G_i\}, \\
d_0(g_{n-1}, \ldots, a_0) &= (g_{n-2}, \ldots, g_0), \\
d_i(g_{n-1}, \ldots, g_0) &= (d_{i-1}g_{n-1}, \ldots, d_0 g_{n-i} \cdot g_{n-i-1}, g_{n-i-2}, \ldots, g_0), \\
s_i(g_{n-1}, \ldots, g_0) &= (s_{i-1}g_{n-1}, \ldots, s_0 g_{n-i}, g_{n-i-1}, \ldots, g_0).
\end{aligned}$$

Then it is not hard to verify the following statements:

(1) $\overline{W}$ is a Kan complex;

(2) $t : \overline{W} \to G$ given by $t(g_{n-1}, \ldots, g_0) = g_{n-1}$ is a twisting function;

(3) The complex $G \times {}_t \overline{W}G$ is contractible;

(4) Let (G, t, K) be a principle fibration. Then there is a simplicial map $f : K \to \overline{W}G$ where

$$f(x) = (t(x), t(d_0 x), \ldots, t(d_0^{n-1} x))$$

and the fibration over K with fibre G, induced from $p : G \times {}_t \overline{W}G \to \overline{W}(G)$, is just $G \times {}_t K$.

(5) For a reduced Kan complex K, the simplicial map $f : K \to \overline{W}GK$ defined by

$$f(x) = (x, *, \ldots, *)$$

is a homotopy equivalence $K \simeq \overline{W}GK$.

Remark (3.21). If A is any simplicial Abelian group, then $\overline{W}A$ will also be a simplicial Abelian group. This is related to the cone and suspension functors in the category of simplicial Abelian groups as follows. Let CA be defined by

$$(CA)_n = A_n \oplus A_{n-1} \cdots \oplus A_0$$

with face and degeneracy operators as in (1.8); then $SA = CA/A$ becomes

$$(SA)_n \approx A_{n-1} \oplus \cdots \oplus A_0 \,.$$

It can be seen easily that $\overline{W}A \approx SA$.

For example, if π is any group, let $K(\pi, 0)$ be the simplicial group which is π in every dimension, all face and degeneracy operators being the identity. Then $\overline{W}(K(\pi, 0))$ is a minimal complex for $K(\pi, 1)$. If π is an Abelian group, then the n-fold iterate $\overline{W}^{(n)}(K(\pi, 0))$ is a minimal complex for $K(\pi, n)$. The fact that any other potential minimal complex for $K(\pi, n)$ must be isomorphic to this one follows from (3.09).

4. Free Simplicial Groups

A free simplicial group is one which is free in each dimension and the bases can be chosen stable under the degeneracy operators. Just as free groups have some special properties, so do free simplicial groups. Kan's construction gives a free simplicial group GK, as does Milnor's FK which is isomorphic to GSK.

We shall use (special cases of) three purely group-theoretical devices, viz., the Nielsen–Schreier–Reidemeister method for describing generators of subgroups of free groups, the Tietze transformations, which replace one set of free generators by another, and the collection process of P. Hall.

These are technically useful in working with free simplicial groups; as an easy example we prove a theorem of Hopf which describes for any complex K, the quotient of $H_2(K)$ by the subgroup of spherical cycles. Other examples can be found in Kan's papers, e.g., in computing $\pi_3(S^2)$. These group-theoretic devices are used especially in the Hilton–Milnor theorem; we give a proof closely following Ref. [Mi2].

We write e for the identity of a (multiplicative) group, and (x, y) for the commutator $x^{-1}y^{-1}xy$.

The references for Section 4 are [C2], [H], [K2], [Mi2].

Definition (4.1). A simplicial group F is called a free simplicial group if

(1) For each $n \geqslant 0$, F_n is a free group with a given basis;

(2) The bases are stable under all degeneracy operators; that is, if $x \in F_n$ is a basis element, then $s_i x \in F_{n+1}$ is also a basis element for each $0 \leqslant i \leqslant n$.

Remark. GK is a free simplicial group.

Group Homotopies (4.2). If $f, g : G \to H$ are simplicial homomorphisms of simplicial groups G, H, then we call f group homotopic to g if there is a homotopy $F = \{F_t\} : G \to H$ between f and g, and for which each F_t is a homomorphism. It is not in general true that group homotopy is an equivalence relation, but it is an equivalence relation if the domain G is a free simplicial group. Thus if $f, g, h : G \to H$ with $f \simeq g$ and $g \simeq h$ by group homotopies, then a group homotopy $f \simeq h$ can be constructed by a suitable choice for each nondegenerate generator. We leave the details to the reader.

Schreier Systems (4.3). We interpolate here the Nielsen–Schreier–Reidemeister technique for describing subgroups of free groups; also see, for example, Ref. [H]. To begin with, consider just groups, not simplicial groups. Let $X = \{x_i\}$ be a set, let $F = FX$ be the free (non-Abelian) group generated by X, and let H be a normal subgroup of F.

DEFINITION (4.4). A Schreier system S for the cosets of H in F is a set of elements in F with the following three properties:

(1) Each coset of F/H contains exactly one element of S;

(2) The identity $e \in S$;

(3) If the reduced word $x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k} \in S$ then so is every shorter word $x_{i_h}^{\epsilon_h} \cdots x_{i_j}^{\epsilon_j} \in S$ for $1 \leqslant h \leqslant j \leqslant k$.

Remark (4.5). We have defined what is usually called a two-sided Schreier system for the cosets of H in F. If H is only a subgroup of F, not necessarily normal, then there is a similar definition of a Schreier system for the right cosets of H in F, except that (3) is changed to read $h = 1$.

If S is a Schreier system for the normal subgroup H in F, let ϕ be the function from F to S which takes each coset to its representative in S. The fundamental result is the following.

THEOREM (4.6). *H is freely generated by those reduced words*

$$yx(\phi(yx))^{-1},$$

where $y \in S$, $x \in X$, and $yx(\phi(yx))^{-1} \not\equiv e$.

We shall frequently be concerned with a simpler situation. Let Z and Y be sets, let $f : FZ \to FY$ be a homomorphism of the free groups, and suppose there is a function $s : Y \to Z$, extended homomorphically to $s : FY \to FZ$ such that $f \circ s = 1_{FY}$. Then the following is immediate from (4.6).

COROLLARY (4.7). *In this situation,* $\ker f$ *is freely generated by those*

$$s(w)\, zsf(z)^{-1}\, s(w)^{-1},$$

where $s \in FY$, $z \in Z - s(Y)$.

TIETZE TRANSFORMATIONS (4.8). Let $F = FW$ be the free group generated by a set W and let $a \in W$. Then F is also freely generated by the set

$$W' = (W - \{a\}) \cup \{a^{-1}\}.$$

If $a, b \in W$, then F is also freely generated by the set

$$W'' = (W - \{a\}) \cup \{b \cdot a\}.$$

These are the (elementary) Tietze transformations which replace one set of free generators by another. We denote them by $T(a \to a^{-1})$ and $T(a \to ba)$, respectively.

More generally, let $u, v \in F(W - \{a\})$ and let $T(a \to ua^{\pm 1}v)$ be the Tietze transformation which replaces the generator a by $ua^{\pm 1}v$. Such a T can be obtained by a finite sequence of elementary ones. Under some conditions, an infinite sequence of transformations can still be applied to obtain a free generating set; see, for example, Ref. [C2].

Collection Process (4.9). Let X be a set, and let $w \in FX$ be a reduced word. Suppose the symbol x occurs in w, and that the element standing next to the left of x is y. Then use the identity $yx = xy(y, x)$ to change the expression for $w = \cdots yx \cdots$ to

$$w = \cdots xy(y, x) \cdots,$$

and the occurence of x has been shifted one place to the left. Also yx^{-1}, $y^{-1}x$, and $y^{-1}x^{-1}$ are handled by similar identities, but with slight complications; see, for example, Ref. [H], p. 165.

We look for the leftmost occurence of x, and "collect" it all the way to the left by repeated applications of this process. In this way, the expression for w will become

$$w = x^q v,$$

where v is an expression in the remaining symbols of X and some new ones of the form (y, x), $((y, x), x)$, etc., created in the collecting process.

We next choose another symbol, collect it to the left, and repeat. Eventually the expression for w becomes

$$w = x_1^{q_1} \cdots x_m^{q_m} \cdot r_m,$$

where we have collected $x_1, \ldots, x_m$ (in this order), and r_m is expression involving symbols created in the collecting process.

Basic Commutators (4.10). Let $X = \{x_1, \ldots, x_r\}$ be a set, and call the x_i the basic commutators of weight one, ordered by the indexing set. Inductively suppose defined the basic commutators of weight $\leqslant n - 1$; let them be $x_1, \ldots, x_{f(n-1)}$, ordered by the indexing set. Then let the basic commutators of weight n be all expressions (x_i, x_j) such that

(1) weight x_i + weight $x_j = n$;
(2) $i > j$;
(3) If $x_i = (x_h, x_k)$, then $k \leqslant j$.

Let the basic commutators of weight $\leqslant n$ be $x_1, \ldots, x_{f(n)}$, ordered by the already chosen order on those of weight $\leqslant n$, and taking those of weight n (in any order) after those of weight $< n$.

Remark. Let g be the integer-valued function on basic commutators by $g(x_i) = 0$ for $1 \leqslant i \leqslant r$, and $g(x_i, x_j) = j$. Then the conditions above read that (x_i, x_j) is to be counted as a basic commutator if (1), (2), and (3'): $g(x_i) \leqslant j$.

Remark (4.11). It is unnecessarily restrictive to collect the elements singly. Let $A \subset X$ be any subset, and collect the part of w in FA to obtain $w = w_1 v$, where $w_1 \in FA$ and v is obtained in the collecting process. For example, in $F(A \vee B)$, we can collect first all $a \in A$, then all $b \in B$, then all $(b, a) \in B \wedge A$, then all $(b, a, a) \in ((B \wedge A) \wedge A)$, etc., to obtain

$$w = w_1 w_2 \cdots w_m r_m .$$

Here each $w_i \in F(C)$ where C is a "basic smash product" of the sets A, B, and r_m is some remainder term. This is the process underlying the Hilton–Milnor theorem.

EXAMPLE (4.12). Let $G = GK$ be the free simplicial group where K is a reduced simplicial set. The above techniques can be used to describe $\pi_1(K) \approx \pi_0(G)$ in the form F/R, where F is free on K_1, and R is the normal subgroup generated by all $(d_0x)(d_1x)^{-1}(d_2x)$ for $x \in K_2$ nondegenerate.

Let us do even more. Let $\Gamma_2 G$ be the commutator subgroup of G (in every dimension). We describe $\pi_0(\Gamma_2 G)$ and $\pi_0(G)$, and thereby deduce a theorem of Hopf.

We do this in three steps.

(1) First apply corollary (4.7) to

$$G_1 \underset{d_0}{\overset{s_0}{\rightleftarrows}} G_0$$

and find that $NG_1 = \ker d_1$ is freely generated by those

$$s_0 g_0 z (s_0 d_1 z)^{-1} (s_0 g_0)^{-1}$$

where $g_0 \in G_0$ and z is a nondegenerate generator of G_1. Let $s_0 g_0 = g^{-1}$, and $z(s_0 d_1 z)^{-1} = y$. Thus NG_1 is freely generated by

$$\{y, g^{-1}yg\}.$$

Some Tietze transformations, $T(g^{-1}yg \to y^{-1}g^{-1}yg)$, show that NG_1 is freely generated by

$$\{y, (g, y)\}.$$

(2) Next observe that

$$N(\Gamma_2 G)_1 = \Gamma_2 G_1 \cap NG_1 = G_1 \circ NG_1,$$

where for A, B subgroups of a group, $A \circ B$ is the smallest subgroup containing all (a, b) for $a \in A$, $b \in V$. To see this, first note that

$$G_1 \circ NG_1 \subset \Gamma_2 G_1 \cap NG_1.$$

Next, suppose $w \in NG_1$. Then by (1), w must be a word in y, (g, y), where the y are in one-one correspondence with nondegenerate generators z of G_1. An easy collecting process argument shows that if w is also to be in $\Gamma_2 G$, w must be a product of elements of the form (x, y) and (g, y), where the $x \in G_1$ are obtained in the collecting process. Thus $w \in G_1 \circ NG$, so (2) holds.

(3) Let $F = G_0$, and $d_0(NG_1) = R \subset F$, so that $\pi_0(G) \approx F/R$. Then

$$d_0(G_1 \circ NG_1) = F \circ R \subset \Gamma_2(F).$$

Thus $\pi_0(\Gamma_2 G) \approx \Gamma_2(F)/F \circ R$. The fibration

$$\Gamma_2 G \to G \to G/\Gamma_2 G$$

yields a long exact sequence in homotopy,

$$\begin{tikzcd}
\pi_2(K) \arrow[r, "h_2"] & H_2(K) \arrow[r, "\partial"] & \gamma_1 \arrow[r, "i_1"] \arrow[d, equal] & \pi_1(K) \arrow[r, "h_1"] \arrow[d, "\wr"] & H_1(K) \\
& & \pi_0(\Gamma_2 G) \arrow[r] \arrow[d, "\wr"] & \pi_0(F) \arrow[d, "\wr"] & \\
& & \Gamma_2(F)/F \circ R \arrow[r] & F/R &
\end{tikzcd}$$

From the exactness, coker $h_2 \approx \ker i_1$, and we have proven the following:

THEOREM (Hopf). *If K is a complex with $\pi_1(K) \approx F/R$, then*

$$H_2(K)/h_2(\pi_2(K)) \approx \Gamma_2(F) \cap R/F \circ R.$$

EXAMPLE (4.13). It is illuminating to see how Kan (Ref. [K1], p. 308) computes $\pi_3(S^2)$ from the free simplicial group $G = GS^2$. First find free generators for each of the groups of the sequence

$$(NG)_3 \xrightarrow{\partial_3} (NG)_2 \xrightarrow{\partial_2} (NG)_1 .$$

Then $\pi_3(S^2) \approx \ker \partial_2/\text{image } \partial_3$ is seen to be infinite-cyclic on one generator $\eta = (s_1\sigma, s_0\sigma)$.

THE FK CONSTRUCTION (4.14) (Milnor [Mi2]). For each simplicial set K with base point $*$, let FK be the result of applying the free group functor to K, with the single relation $* =$ identity e. The free product of two group complexes $G * H$ is defined by $(G * H)_n = G_n * H_n$. Then $F(K \vee L) \approx FK * FL$.

PROPOSITION (4.15). *There is a canonical isomorphism $FK \approx GSK$, where GSK is the result of applying Kan's construction G to the reduced suspension SK of K.*

Proof. For each $x \in K_n$, there is $(x, 1) \in (SK)_{n+1}$, and $\overline{(x, 1)}$ a generator of $(GSK)_n$. The map $x \to \overline{(x, 1)}$ is easily seen to give an isomorphism of the simplicial groups.

Let A and B be simplicial sets with base point. The idea is to decompose $F(A \vee B)$ into a Cartesian product

$$FA \times FB \times F(B \wedge A) \times \cdots$$

by a collecting process.

PROPOSITION (4.16). *For simplicial sets A and B, the simplicial set $F(A \vee B)$ is isomorphic to the simplicial set $FA \times F(B \vee (B \wedge FA))$; that is, ignoring the group structure.*

Proof. The collapsing map $A \vee B \to A$ induces a homomorphism $f : F(A \vee B) \to FA$. Then (4.7) and (4.8) show that $\ker f$ is freely generated by the set $\{b, (b, w)\}$ for $b \in B$, $w \in FA$. Identify (b, w) with

$(b \wedge w) \in B \wedge FA$, and the kernel becomes isomorphic with $F(B \vee (B \wedge FA))$. There is short exact sequence

$$e \to F(B \vee (B \vee FA)) \to F(A \vee B) \to FA \to e$$

which is split by $i : FA \to F(A \vee B)$, i.e., $f \circ i = i_{FA}$. All this is natural in maps in A and B, which thus commute with the face and degeneracy operators, and the proposition follows.

PROPOSITION (4.17). *The simplicial group $F(B \wedge FA)$ is isomorphic to $F((B \wedge A) \vee (B \wedge A \wedge FA))$.*

Proof. The inclusion $A \to FA$ induces a homomorphism

$$F(B \wedge A) \to F(B \wedge FA).$$

A homomorphism

$$F(B \wedge A \wedge FA) \to F(B \wedge FA)$$

is defined by $(b \wedge a \wedge w) \to (b \wedge a)^{-1} (b \wedge w)^{-1} (b \wedge aw)$; note the group identity $((b, a), w) = (b, a)^{-1} (b, w)^{-1} (b, aw)$. Combining these, we obtain a homomorphism

$$F(B \wedge A) * F(B \wedge A \wedge FA) \to F(B \wedge BA)$$

which is an isomorphism. For this, use Tietze transformations (4.8) to see that in $F(A \vee B)$, the subgroup freely generated by $\{(b, w)\}$ is also freely generated by $\{(b, a), ((b, a), w)\}$, for $b \in B$, $a \in A$, $w \in FA$.

Let $A^m = A \wedge A \wedge \cdots \wedge A$, m copies. Then an induction on m, using (4.17), shows the following:

PROPOSITION (4.18). *For each $m \geqslant 1$, the group complex $F(B \wedge FA)$ is isomorphic to*

$$F(B \wedge A) * F(B \wedge A^2) * \cdots * F(B \wedge A^m) * F(B \wedge A^m \wedge FA).$$

THEOREM (4.19). *If A and B are simplicial sets with A connected, then there is an inclusion*

$$G : F\left(\bigvee_{i=0}^{\infty} B \wedge A^i\right) \to F(B \wedge FA)$$

which is an homotopy equivalence.

Proof. Each element of $F(\bigvee_{i=1}^{\infty} B \wedge A^i)$ is already contained in $F(\bigvee_{i=1}^{m} B \wedge A^i)$ for some m. This provides a homomorphism G completing the diagram

$$\begin{array}{ccc} F\left(\bigvee\limits_{i=1}^{\infty} B \wedge A^i\right) & \xrightarrow{G} & F(B \wedge FA) \\ \uparrow & & \uparrow \\ F\left(\bigvee\limits_{i=1}^{m} B \wedge A^i\right) & \longrightarrow & F\left(\bigvee\limits_{i=1}^{m} B \wedge A^i\right) * F(B \wedge A^m \wedge FA) \end{array}$$

The "remainder" terms $F(B \wedge A^m \wedge FA)$ and $F(B \wedge A^i)$ for $i > m$ have trivial homology in dimensions less than m. Thus G induces isomorphism of all homotopy groups, so in a homotopy equivalence.

Combining (4.16) and (4.19) gives the following:

COROLLARY (4.20). *If A and B are simplicial sets with A connected, there is a homotopy equivalence*

$$FA \times F\left(\bigvee_{i=0}^{\infty} B \wedge A^i\right) \to F(A \vee B).$$

Let $A_1, \ldots, A_r$ be simplicial sets. Construct a family of basic complexes (each is a simplicial set) in strict analogy with the basic commutators (4.10), by the substitutions basic complex A_i for basic commutator x_i, and $(A_i \wedge A_j)$ for (x_i, x_j).

THEOREM (4.21) (Hilton–Milnor). *Let $A_1, \ldots, A_r$ be connected simplicial sets. Then $F(A_1 \vee \cdots \vee A_r)$ has the same homotopy type as the weak infinite Cartesian product $\prod_{i=1}^{\infty} FA_i$, where the A_i are the basic complexes.*

Proof. This results by iterating the previous decomposition. For each $m \geqslant 1$, let $R_m = F(\bigvee A_k)$, where the wedge is taken over all $k \geqslant m$, with $g(A_k) < m$. Then (4.20), applied to $F(A_m \vee B)$ where $B = \bigvee A_k$, the wedge taken for $k < m$, gives a homotopy equivalence

$$F(A_m) \times R_{m+1} \to R_m .$$

By induction there follows a homotopy equivalence

$$\left(\prod_{i=1}^{m} FA_i\right) \times R_{m+1} \xrightarrow{\simeq} R_1 = F(A_1 \vee \cdots \vee A_r).$$

This defines an inclusion of the weak infinite Cartesian product

$$\prod_{i=1}^{\infty} FA \to F(A_1 \vee \cdots \vee A_r).$$

As the $A_1, ..., A_r$ are connected, the "remainder" terms R_m are n-connected, where $n \to \infty$ as $m \to \infty$. The above inclusion map induces isomorphisms in all homotopy groups, and so it is a homotopy equivalence.

Another consequence of (4.19) is the following theorem of James.

THEOREM (4.22). *If A is a connected simplicial set, the complex SFA has the same homotopy type as $\bigvee_{i=1}^{\infty} SA^i$.*

Proof. Observe that $S^0 \wedge K \cong K$. Take $B = S^0$ in (4.19), and thus

$$F\left(\bigvee_{i=1}^{\infty} A^i\right) \to FFA$$

is a homotopy equivalence. Apply $\overline{W}(\cdot)$ to both sides, and as $SK \to \overline{W}FK$ is a homotopy equivalence, the theorem follows.

5. SIMPLICIAL ABELIAN GROUPS

A simplicial Abelian group is the analogue of a generalized Eilenberg–MacLane space (GEM), i.e., a product of $K(\pi, n)$'s. Let $\mathbf{S}_A$ be the category of simplicial Abelian groups, and $\mathbf{C}_A$ be the category of Abelian chain complexes. It is a result of Dold and Kan that there are functors $N : \mathscr{S}_A \to \mathscr{C}_A$ and $K : \mathscr{C}_A \to \mathscr{S}_A$ which are inverses and thus provide an isomorphism of categories $\mathscr{S}_A \approx \mathscr{C}_A$.

Under this isomorphism, group homotopies in $\mathscr{S}_A$ (i.e., $\{F_t\}$, where each F_t is a homomorphism) are in one-one correspondence with chain homotopies in $\mathscr{C}_A$.

Tensor products are defined in each of the categories $\mathscr{S}_A$, $\mathscr{C}_A$, and a simple form of the Eilenberg–Zilber theorem is proven.

It is a standard fact that an Abelian chain complex of free Abelian groups is determined to within chain homotopy equivalence by its homology groups. The above correspondences show that a simplicial free Abelian group is determined to within group homotopy equivalence by its homotopy groups. Using the principle that functors preserve homotopy, we conclude that if A and B are simplicial free Abelian groups with $\pi_*(A) \approx \pi_*(B)$, and T is a functor from the category of Abelian groups to another category, then $\pi_*(TA) \approx \pi_*(TB)$.

The references for Section 5 are [D], [D, P], [Mac].

THE FUNCTORS N AND K (5.1). For each simplicial Abelian group A, recall the chain complex (NA, ∂), where

$$(NA)_q = \bigcap_{i \neq 0} \ker d_i ,$$
$$\partial_q = d_0 \text{ (restricted to } (NA)_q).$$

There is a functor K which is inverse to N, provided by the following:

DEFINITION (5.2). For each Abelian (and nonnegatively graded) chain complex (C, ∂), let KC be the simplicial Abelian group

$$(KC)_n = \bigoplus_a (C_{n-p}, s_a)$$

where s_a ranges over all iterated p-fold degeneracies $(0 \leqslant p \leqslant n)$,

$$s_a = s_{i_p} \cdots s_{i_1} ,$$

where $n > i_p \geqslant \cdots \geqslant i_1 \geqslant 0$. That is, the direct sum includes as many copies of C_{n-p} as there are different p-fold degeneracies; the summand C_n is included, corresponding to the empty degeneracy. The face and degeneracy operators in KC are given by the rules

(1) If $d_i s_a = s_b$, then d_i is to map $(C_{n-p}, s_a) \to (C_{n-p}, s_b)$ isomorphically, and be 0 into the other factors.

(2) If $d_i s_a = s_b d_0$, then d_i is to map $(C_{n-p}, s_a) \to (C_{n-p-1}, s_b)$ as the homomorphism ∂_{n-p}, and be 0 into the other factors.

(3) If $d_i s_a = s_b d_j$, $j > 0$, then $d_i(C_{n-p}, s_a) = 0$.

(4) If $s_i s_a = s_b$, then s_i is to map $(C_{n-p}, s_a) \to (C_{n-p}, s_b)$ isomorphically, and be 0 into the other factors.

THEOREM (5.3). *The functors N and K provide natural isomorphisms between the categories $\mathscr{S}_A$ and $\mathscr{C}_A$.*

Proof. For each $C \in \mathscr{C}_A$, let

$$\Phi_C : C \to NKC$$

be the map of C_n to the summand $C_n \subset (NKC)_n$.

For each $A \in \mathscr{S}_A$, let

$$\Psi_A : KNA \to A$$

be defined on the summand $((NA)_{n-p}, s_a)$ as the composite

$$((NA)_{n-p}, s_a) \xrightarrow{\text{inclusion}} A_{n-p} \xrightarrow{\quad s_a \quad} A_n .$$

It is not difficult to verify that Φ_C and Ψ_A are natural isomorphisms. We also leave to the reader the verification that under N and K,

$$\hom_{\mathbf{S}_A}(\ ,\ ) \approx \hom_{\mathbf{C}_A}(\ ,\ ).$$

Tensor Products (5.4). If A and B are in $\mathscr{S}_A$, the tensor product $A \otimes B$ in $\mathscr{S}_A$ is described as follows:

$$\begin{aligned} (A \otimes B)_n &= A_n \otimes B_n , \\ d_i(x \otimes y) &= (d_i x \otimes d_i y), \qquad 0 \leqslant i \leqslant n, \\ s_i(x \otimes y) &= (s_i x \otimes s_i y), \qquad 0 \leqslant i \leqslant n. \end{aligned}$$

If (C, ∂) and (C', ∂) are in $\mathscr{C}_A$, their tensor product $(C \otimes C', \partial)$ in $\mathscr{C}_A$ is as follows:

$$\begin{aligned} (C \otimes C') &= \bigoplus_{p+q=n} C_p \otimes C_q', \\ \partial(x \otimes y) &= \partial x \otimes y + (-1)^{\dim x}\, x \otimes \partial y. \end{aligned}$$

Homotopies (5.5). A homotopy $\{F_t\} : A \to B$ of simplicial (Abelian) groups is called a group homotopy if each F_t is a homomorphism. The simplicial map $F : A \times I \to B$ is not very enlightening; more useful is the simplicial homomorphism

$$\theta : A \otimes Z(1) \to B,$$

where $Z(1) = Z(\Delta[1])$; θ is defined by

$$\theta(x \otimes t) = F_t(x).$$

A chain homotopy D between two chain maps $f, g : C \to C'$ is a homomorphism raising dimension by one and satisfying

$$\partial_{n+1} D_n + D_{n-1} \partial_n = f - g.$$

Such a D provides a chain map

$$\bar{\theta} : C \otimes N(1) \to C',$$

where $N(1) = NZ(\Delta[1])$; thus $N(1)$ has for basis $v_0, v_1 \in N(1)_0$ and $w \in N(1)_1$, and $\partial_w = v_0 - v_1$. Then for each $x \in C$,

$$\begin{aligned} \bar{\theta}(x, w) &= D(x), \\ \bar{\theta}(x, v_0) &= f(x), \\ \bar{\theta}(x, v_1) &= g(x). \end{aligned}$$

It is convenient to work with such maps θ and $\bar{\theta}$, coming from F and D; we call θ and $\bar{\theta}$ tensor simplicial homotopy and tensor chain homotopy, respectively.

Let A and B be simplicial Abelian groups. Then the shuffle (∇) and Alexander–Witney (f) maps

$$(A, \partial) \otimes (B, \partial) \xrightarrow{\nabla} (A \otimes B, \partial) \xrightarrow{f} (A, \partial) \otimes (B, \partial)$$

are defined by

$$\nabla(x \otimes y) = \Sigma \pm (s_b x \otimes s_a y),$$

where for $x \in A_p$, $y \in B_q$ the sum ranges over all "(p, q) shuffles"

$$(a; b) = (a_1, \ldots, a_p ; b_1, \ldots, b_q)$$

whose $p + q$ integers $(a_1, \ldots, a_p, b_1, \ldots, b_q)$ are a permutation of $(0, \ldots, (p + q - 1))$ satisfying $a_1 < \cdots < a_p$ and $b_1 < \cdots < b_q$, $\pm$ is the sign of the permutation, and $s_a = s_{a_p} \cdots s_{a_1}$, $s_b = s_{b_q} \cdots s_{b_1}$.

For $x \otimes y \in A_n \otimes B_n = (A \otimes B)_n$,

$$\begin{aligned} f(x \otimes y) &= \Sigma_{p+q=n} (\tilde{d})^p (x) \otimes (d_0)^q (y), \\ (\tilde{d})^p (x) &= d_{n-p+1} \cdots d_{n-1} d_n x, \\ (d_0)^q (y) &= d_0 \cdots d_0 y. \end{aligned}$$

PROPOSITION (5.6). *The maps ∇ and f each induce isomorphisms of the homology groups. The maps ∇ and f are also defined on the normalized chain complexes*

$$NA \otimes NB \xrightarrow{\nabla} N(A \otimes B) \xrightarrow{f} NA \otimes NB$$

and induce isomorphisms of the homology groups.

The proof is a standard exercise, not too difficult; see also Ref. [Mac], p. 238.

Let $\theta : A \otimes Z(1) \to B$ be a tensor simplicial homotopy. Then the composite

$$NA \otimes N(1) \xrightarrow{\nabla} N(A \otimes Z(1)) \xrightarrow{N\theta} NB$$

will be a chain homotopy.

Let $\theta : NA \otimes N(1) \to NB$ be a tensor chain homotopy. Then the composite

$$\begin{tikzcd}
A \otimes Z(1) \arrow[rrr, dashed] \arrow[d, "\approx"] & & & B \arrow[d, "\approx"] \\
KN(A \otimes Z(1)) \arrow[r, "Kf"] & K(NA \otimes N(1)) \arrow[rr, "K\bar{\theta}"] & & KNB
\end{tikzcd}$$

will be a tensor simplicial homotopy. From this we can conclude the following:

THEOREM (5.7). *There is a one-one correspondence between simplicial group homotopies in $\mathscr{S}_A$ and chain homotopies in $\mathscr{C}_A$.*

It is easy to see that if C and C' are chain complexes of free Abelian groups with $H_*(C) \approx H_*(C')$ then there are chain maps $f : C \to C'$ and $g : C' \to C$ such that the composites $g \circ f$ and $f \circ g$ are chain homotopic to 1_C and $1_{C'}$, respectively.

THEOREM (5.8). *If A and B are simplicial free Abelian groups with $\pi_*(A) \approx \pi_*(B)$, then there are simplicial homomorphisms $f : A \to B$ and $g : B \to A$ such that the composites $g \circ f$ and $f \circ g$ are simplicial group homotopic to 1_A and 1_B, respectively.*

THEOREM (5.9). *Let A and B be simplicial free Abelian groups with $\pi_*(A) \approx \pi_*(B)$ and T a functor from Abelian groups to another category. Then $\pi_*(TA) \approx \pi_*(TB)$.*

The category of Abelian groups can be replaced by modules over P.I.D., or other Abelian category where the above is valid.

EXAMPLE (5.10). Let SP^n be the n-fold symmetric product functor. If X and Y are simplicial sets with $H_*(X) \approx H_*(Y)$, then also $H_*(SP^nX) \approx H_*(SP^nY)$. For the simplicial free Abelian groups $C_*(X)$, $C_*(Y)$ have isomorphic homotopy, and $H_*(SP^nC_*(X)) \approx H_*(SP^n(X))$, etc. This is the example Dold has in mind [D].

SUSPENSION (5.11). For each simplicial Abelian group A, let $\overline{W}A$ be as in (3.20) and $WA = A_t\overline{W}A$. Then $\overline{W}A$ and WA are again simplicial Abelian groups, and the principal fibration

$$A \xrightarrow{i} WA \xrightarrow{p} \overline{W}A$$

is a short exact sequence of simplicial Abelian groups which splits dimension-wise.

Let T be a functor from Abelian groups to another Abelian category. Then T_p is onto, and there is a fibration

$$\ker Tp \longrightarrow TWA \xrightarrow{Tp} T\overline{W}A.$$

Since WA is contractible, so is $T\overline{W}A$ (functors preserve homotopy), and from the LES in π,

$$\partial : \pi_{q+1}(T\overline{W}A) \xrightarrow{\approx} \pi_q(\ker Tp).$$

Then the composite $\sigma = \partial^{-1} \circ (Ti)_*$,

$$\pi_q(TA) \xrightarrow{(Ti)_*} \pi_q(\ker Tp) \xrightarrow{\partial^{-1}} \pi_{q+1}(T\overline{W}A),$$

is called the suspension homomorphism. For more details, see Refs. [D, P], where WA is called CA, and $\overline{W}A$ is called SA.

SIMPLICIAL VECTOR SPACES (5.12). The functors N and K also give isomorphisms between the categories of simplicial vector spaces and differential vector spaces. Let F be a field, and for each integer $n \geqslant 0$, let $C(F, n)$ be the differential vector space concentrated in dimension n; that is, $C(F, n)_n \approx F$ and $C(F, n)_q = 0$ for $q \neq n$. Then any differential vector space C is a chain homotopy equivalent to a direct sum of such simple ones,

$$C \simeq \bigoplus_i C(F, n_i).$$

By (5.3), a simplicial vector space V will be equally simple; that is,

$$\begin{aligned} V &\approx KNV \\ &\simeq K\left(\bigoplus_i C(F, n_i)\right) \\ &\simeq \bigoplus_i K(F, n_i) \end{aligned}$$

where $K(F, n) = K(C(F, n))$.

The suspension σ in the category of simplicial vector spaces takes $K(F, n)$ to $\overline{W}(K(F, n)) \approx K(F, n+1)$, with $\sigma i_n = i_{n+1}$.

For a simple example, let $T^2(\cdot)$ be the two-fold tensor power; then

$$\sigma : \pi_n T^2(V) \to \pi_{n+1} T^2(\overline{W}V)$$

is the 0 homomorphism. Later, we shall need the following, slightly more general fact. Let $i_m \underline{\otimes} i_n$ be the generator of $\pi_{m+n}(K(m) \otimes K(n))$ given by

$$i_m \underline{\otimes} i_n = \sum_{(a;b)} \pm (s_b i_m \otimes s_a i_n),$$

where $(a; b)$ varies over all (m, n) shuffles. Under suspension, $i_m \underline{\otimes} i_n$ goes to 0; that is, there must be a (universal) formula $f_{m,n}(\ ,\ )$ with

$$\partial f_{m,n}(i_{m+1}, i_{n+1}) = \sum_{(a;b)} \pm (s_b i_{m+1} \otimes s_a i_{n+1}).$$

This is related to the homology suspension in the following way. For any simplicial set X, let $Z_p(X)$ be the simplicial vector space generated by X over the field of p elements. Then $\pi_n(Z_p(X)) \approx H_n(X; Z_p)$. The diagonal $\Delta : X \to X \times X$, where $\Delta(x) = (x, x)$, induces a homomorphism $\Delta : Z_p(X) \otimes Z_p(X)$, and thence a coproduct

$$\begin{array}{ccc} \pi_n(Z_p(X)) & \xrightarrow{\Delta_*} & \pi_n(Z_p(X) \otimes Z_p(X)) \\ \wr & & \wr \\ H_n(X; Z_p) & \xrightarrow{\Delta_*} & \bigoplus_{p+q=n} H_p(X : Z_p) \otimes H_q(X : Z_p) \end{array}$$

Via suspension, $\pi_n(\overline{W}Z_p(X)) \approx H_n(SX; Z_p)$. Thus, for suspended spaces, the coproduct in homology is 0; dually, in cohomology, the cup-product is 0.

6. Fibre Bundles

How can we classify fibration? The answer is given by Refs. [Mo2] and [B, G, M]. The first step is to replace a given fibration by a deformation retract of it which is a minimal fibration, and then to show that a minimal fibration is a fibre bundle. Next, each fibre bundle can be considered as a regular twisted Cartesian product (RTCP) of its base B and fibre Y. Each RTCP is determined by a twisting function $t : B \to G$, where G is a subgroup of the automorphism group of Y. Finally, the equivalence class of such RTCP's is in one-one correspondence with homotopy classes of maps of B into $\overline{W}(G)$, the classifying complex of G.

For convenience, we postpone until the last the retraction of a fibration to a minimal one, and take up fibre bundles first.

The reference for Section 6 is [B, G, M].

Definition (6.1). A fibre map $p : E \to B$ is called a *fibre bundle map* if p is onto, and if for each $b \in B_n$, the representing map for $b, f_b : \Delta[n] \to B$ induces a fibration $p' : E' \to \Delta[n]$ which is isomorphic to the fibration $F \times \Delta[n] \to \Delta[n]$, where F is some fixed simplicial set called the fibre.

Fibre bundles occur plentifully, as shown by the following, which will be proven later.

Proposition (6.2). *Let $p : E \to B$ be a fibre map onto, where B is connected and p is minimal. Then p is a fibre bundle map with fibre $F = p^{-1}(*)$.*

Thus the fibrations in the Postnikov system of a minimal complex are all fibre bundles. If $p : E \to B$ is any fibre map onto a connected base B, then we will also show later that there is a "retraction" of p onto a minimal fibre map, which is thus a fibre bundle.

Fibre bundles also occur as twisted Cartesian products. Suppose we have two simplicial sets, B and Y, and want to make a twisted Cartesian product with base B and fibre Y. To do this, we need to consider simplicial groups acting on Y, in particular the automorphism group of Y, or a subgroup of it.

Simplicial Automorphisms (6.3). Let X and Y be simplicial sets, and recall the function complex Y^X where

$$(Y^X)_n = \{f : X \times \Delta[n] \to Y\}.$$

Each $f \in (Y^X)_n$ gives a commutative diagram

$$\begin{tikzcd} X \times \Delta[n] \arrow[rr, "g"] \arrow[dr] & & Y \times \Delta[n] \arrow[dl] \\ & \Delta[n] & \end{tikzcd}$$

where $g(x, t) = (f(x, t), t)$. It will sometimes be convenient to regard $(Y^X)_n$ as the set of such commutative diagrams.

For X, Y, and W simplicial sets, there is a simplicial map

$$X^W \times Y^X \to Y^W,$$

where $(f, g) \to g \circ f$. Thus Y^Y acts on the right of Y^W and on the left of Z^Y. Also Y^Y becomes a simplicial monoid (i.e., has an associate multiplication). Let

$$\mathrm{aut}(Y) \subset Y^Y$$

be the subset of invertible elements, (i.e., in every dimension); $\mathrm{aut}(Y)$ is a simplicial group.

If G is a simplicial group, and Y is a simplicial set, a (right) action of G on Y is to be a simplicial map

$$\phi : Y \times G \to Y$$

with the usual properties. Such an action ϕ determines a simplicial homomorphism

$$\tilde{\phi} : G \to \mathrm{aut}(Y)$$

by $\tilde{\phi}(g)(y, t) = \phi(y, f_g(t))$, where $f_g : \Delta[n] \to G$ is the representing map for $g \in G_n$. That is, for each $g \in G_n$, $\tilde{\phi}(g)$ is the composite

$$Y \times \Delta[n] \xrightarrow{1_Y \times f_g} Y \times G \xrightarrow{\phi} Y.$$

On the other hand, such a map $\tilde{\phi} : G \to \mathrm{aut}(Y)$ determines an action of G on Y by reversing the above. We call the action ϕ effective if $\tilde{\phi}$ is one-one, and from now on we restrict to this case (by factoring G by $\ker \tilde{\phi}$ if necessary). We will use ϕ and $\tilde{\phi}$ interchangeably. Also, write $y \circ g$ for $\phi(y, g)$.

For example, $\mathrm{aut}(Y)$ acts on Y by

$$\phi : Y \times \mathrm{aut}(Y) \to Y,$$

where $\phi(y, f) = f(y, i_n)$ for $y \in Y_n$ and $f : Y \times \Delta[n] \to Y$. The corresponding $\tilde{\phi} : \mathrm{aut}(Y) \to \mathrm{aut}(Y)$ is the identity isomorphism.

Hereafter, assume that the (potential) fibres Y are Kan complexes; better yet for the applications, assume Y a minimal complex.

TWISTED CARTESIAN PRODUCTS (6.4). We continue with our attempt to make a fibration with fibre Y, base B. We try for total complex E, where

$$E_n = Y_n \times B_n ,$$

and suppose that the face and degeneracy operators in E satisfy

$$\begin{aligned} s_i(y, b) &= (s_i y, s_i b) && \text{for all} \quad i \geqslant 0, \\ d_i(y, b) &= (d_i y, d_i b) && \text{for all} \quad i \geqslant 0, \\ d_0(g, b) &= (d_0 y \circ t(b), d_0 b), \end{aligned}$$

where $t : B_n \to \operatorname{aut}(Y)_{n-1}$ is some function. It is an easy consequence of the simplicial identities that $Y \to E \to B$ will be a fibration $\Leftrightarrow$ t satisfies the following:

$$\text{(6.4a)} \qquad \begin{aligned} d_i t(b) &= t(d_{i+1} b) && \text{for} \quad i \geqslant 0, \\ d_0 t(b) &= t(d_1 b) \circ t(d_0 b)^{-1}, \\ s_i t(b) &= t(s_{i+1} b) && \text{for} \quad i \geqslant 0, \\ * &= t(s_0 b). \end{aligned}$$

A function t satisfying (6.4a) is called a twisting function, and the simplicial set $E = F \times {}_t B$ is called a *regular twisted Cartesian product* (RTCP). It looks somewhat special to suppose t "twists" only the face d_0, but we are going to show that any fibre bundle is such an RTCP, so allowing "twists" of all faces and degeneracies would be unnecessary.

ATLAS OF A FIBRE BUNDLE (6.5). Let $p : E \to B$ be a fibre bundle with fibre f. Thus for each $b \in B_n$, there is a commutative diagram

$$\begin{array}{ccccc} F \times \Delta[n] & \xrightarrow[\approx]{\alpha(b)} & E' & \xrightarrow{f_b} & E \\ & & \downarrow & & \downarrow \\ & & \Delta[n] & \xrightarrow{f_b} & B \end{array}$$

where $\alpha(b)$ is some isomorphism. A choice $\{\alpha(b)\}$ of this isomorphism for each $b \in B$ is called an atlas for the bundle.

We investigate the possibility of changing, and simplifying, an atlas by different choices of these isomorphisms. Note that we do not change the fibre bundle, i.e., the quadruple (F, E, p, B), in any way. If $\{\alpha(b)\}$, $\{\beta(b)\}$ are two atlases for the fibre bundle, then for each $b \in B_n$,

$$(\alpha(b))^{-1} \circ \beta(b) = \gamma(b),$$

where $\gamma(b) \in \operatorname{aut}(F)_n$. Conversely, given such an atlas $\{\alpha(b)\}$, and for each $b \in B_n$ an element $\gamma(b) \in \operatorname{aut}(F)_n$, the collection

$$\beta(b) = \alpha(b) \circ \gamma(b)$$

determines a new atlas.

Let $b \in B_n$, and consider a degeneracy s_i. Let $\eta_i : \Delta[n+1] \to \Delta[n]$ be the corresponding map, and there is a commutative diagram

$$\begin{tikzcd}
F\times\Delta[n+1] \arrow[r, "1\times\eta_i"] & F\times\Delta[n] \arrow[r, "\overset{\alpha(b)}{\approx}"] & E' \arrow[r, "\bar f_b"] \arrow[dd] & E \arrow[dd] \\
F\times\Delta[n+1] \arrow[r, "\overset{\alpha(s_i b)}{\approx}"] & E'' \arrow[ur, "\bar\eta_i"'] \arrow[dd] & & \\
& & \Delta[n] \arrow[r, "f_b"] & B \\
& \Delta[n+1] \arrow[ur, "\eta_i"] \arrow[urr, "f_{s_i b}"'] & &
\end{tikzcd}$$

Regarding $\alpha(b) \in ((E')^F)_n$, we have $s_i\alpha(b) = \alpha(b) \circ (1 \times \eta_i)$, and it may not in general be true that

$$s_i\alpha(b) = \bar\eta_i \circ \alpha(s_i b).$$

By redefining $\alpha(s_i b)$ on degenerate elements $s_i b \in B$, we can replace an atlas by a normalized one, i.e., one for which this is true. From now on assume all atlases normalized.

What about the face operators? Let d_i be a face operator, and let $\epsilon_i : \Delta[n-1] \to \Delta[n]$ be the corresponding map. For $b \in B_n$, we have a commutative diagram

$$\begin{tikzcd}
F\times\Delta[n-1] \arrow[r, "1\times\epsilon_i"] & F\times\Delta[n] \arrow[r, "\overset{\alpha(b)}{\approx}"] & E' \arrow[r, "\bar f_b"] \arrow[dd, "p'"] & E \arrow[dd, "p"] \\
F\times\Delta[n-1] \arrow[r, "\overset{\alpha(d_i b)}{\approx}"] & E'' \arrow[ur, "\bar\epsilon_i"'] \arrow[dd, "p''"'] & & \\
& & \Delta[n] \arrow[r, "f_b"] & B \\
& \Delta[n-1] \arrow[ur, "\epsilon_i"] \arrow[urr, "f_{d_i b}"'] & &
\end{tikzcd}$$

Then $d_i\alpha(b) = \alpha(b) \circ (1 \times \epsilon_i)$ and again it may not in general happen that

$$d_i\alpha(b) = \bar\epsilon_i \circ \alpha(d_i b).$$

We will disregard the monomorphism $\bar{\epsilon}_i$ since the two maps in question have the same image in E'. Then let

$$\xi^i(b) = (\alpha(d_i b))^{-1} \circ d_i \alpha(b),$$

where $\xi^i(b) \in \operatorname{aut}(F)_{n-1}$ for $b \in B_n$. The $\xi^i(b)$ are called the transformation elements of the atlas. If all $\xi^i(b) \in G \subset \operatorname{aut}(F)$ we call $\alpha(b)$ a G atlas and the bundle a G bundle. Two G atlases are said to be G-equivalent if

$$\beta(b) = \alpha(b)\, \gamma(b),$$

where $\gamma(b) \in G$, all $b \in B$.

THEOREM (6.6). *In every G-equivalence class of atlases, there is (at least) one atlas for which*

$$\xi^i(b) = e\ (= \text{identity}) \qquad \text{for} \quad i > 0.$$

Proof. Let $\{\alpha(b)\}$ be an atlas with transformation elements $\xi^i(b) \in G \subset \operatorname{aut}(F)$. We can specify a new atlas on the nondegenerate elements, and the normalizing process will take care of the degenerate ones.

First, let $b \in B_1$ be nondegenerate. Since G is a simplicial group, G satisfies the extension condition, so let $\gamma \in G_1$ with $d_1\gamma = \xi^1(b)$. Put $\beta(b) = \alpha(b) \circ \gamma^{-1}$; then

$$\begin{aligned} d_1\beta(b) &= d_1(\alpha(b)) \circ d_1\gamma^{-1} \\ &= \alpha(d_1 b) \circ \xi^1(b) \circ (\xi^1(b))^{-1} \\ &= \alpha(d_1 b). \end{aligned}$$

Next, suppose inductively that $\{\alpha(b)\}$ satisfies $\xi^i(b) = e$ for $i > 0$, and dimension $b \leqslant n - 1$, and let $b \in B_n$ be nondegenerate. Then from the induction hypothesis,

$$d_i\xi^j(b) = d_{j-1}\xi^i(b) \qquad \text{for} \quad 0 < i < j.$$

Hence, using the extension condition on G, there is $\gamma \in G_n$ with $d_i\gamma = \xi^i(b)$ for all $i > 0$. Put $\beta(b) = \alpha(b) \circ \gamma^{-1}$; then for $i > 0$,

$$\begin{aligned} d_i\beta(b) &= d_i(\alpha(b)) \circ d_i\gamma^{-1} \\ &= \alpha(d_i b) \circ \xi^i(b) \circ (\xi^i(b))^{-1} \\ &= \alpha(d_i b). \end{aligned}$$

Thus if we replace $\alpha(b)$ by $\beta(b)$ for each nondegenerate $b \in B_n$, the new transformation elements satisfy $\xi^i(b) = e$ for $i > 0$.

An atlas $\{\alpha(b)\}$ is called *regular* if $\xi^i(b) = e$ for $i > 0$. The above shows that in every G-equivalence class of atlases, there is at least one regular one.

THEOREM (6.7). *Let $p : E \to B$ be a fibre bundle with a regular atlas. Then the transformation elements $\xi^0(b)$ determine a twisting function*

$$\xi^0 = t : B_n \to \operatorname{aut}(F)_{n-1}$$

and thereby $F \times {}_tB$ becomes an RTCP. *Furthermore, there is an isomorphism of fibre bundles,*

$$\begin{tikzcd} \varphi : F \times {}_tB \arrow[rr, "\approx"] \arrow[dr] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

Proof. The map is given by $\varphi(x, b) = \bar{f}_b \circ \alpha(b)(x, i_n)$.

PRINCIPAL BUNDLES (6.8). Let G be a simplicial group. A G bundle $p : E \to B$ with fibre G is called a principal G bundle. This does not conflict with the previous definition of principal G bundles via twisting functions (3.14) because any fibre bundle can be considered an RTCP by (6.7). Conversely, it is not hard to show that an RTCP is a fibre bundle, in particular, a fibration. Observe that G acts on the left of $G \times {}_tB$ by

$$g \circ (h, b) = (gh, b).$$

Equivalently, if E is a principal G bundle with regular atlas $\{\alpha(b)\}$, G acts on the left of E by

$$g \circ x = \bar{f}_b \circ \alpha(b)(gh, i_n),$$

where $g \in G_n$, $x \in E_n$, $b = p(x) \in B_n$, and

$$\alpha(b) \circ \bar{f}_b(h, i_n) = x.$$

This action is independent of the choice of atlas in its G-equivalence class. In a similar way, G acts on the right of E.

Any bundle induced from a principal G bundle is another principal G bundle. If $G \times {}_tB$ is the total complex and $f : A \to B$, then the composite $u = t \circ f$ is a twisting function, and the bundle over A induced by f is $G \times {}_uA$. For example, the bundles $GK \times {}_tK$ and $G \times {}_t\overline{W}(G)$ are principal bundles, and so are any bundles induced from them.

EXAMPLE (6.9). Another example of principal bundles is that of covering complexes. Let X be a reduced Kan complex, $\pi = \pi_1(X)$, and let $K(\pi, 0)$ be the simplicial group which is π in every dimension and all face and degeneracy operators are the identity. Then let the twisting function

$$t : X_n \to \pi = K(\pi, 0)_{n-1}$$

be given by $t(x) = [d_2 \cdots d_n x] \in \pi_1(X)$. Let

$$\hat{X} = K(\pi, 0) \times {}_tX$$

and $p : \hat{X} \to X$ is called the universal cover of X.

EXAMPLE (6.9a). More appealing and sometimes useful is the following construction. There is a natural homomorphism $f : GX \to K(\pi, 0)$, and let $\tilde{G}X = \ker(f)$. Then $\tilde{G}X$ is a free simplicial group representing the "loops on $\hat{X}$," and there is a fibration

$$\tilde{G}X \longrightarrow GX \xrightarrow{f} K(\pi, 0).$$

Hence also $\hat{X} \simeq \overline{W}\tilde{G}X$, and $H_*(\hat{X}) \approx \pi_{*-1}(\tilde{G}X/\Gamma_2\tilde{G}X)$.

ASSOCIATED BUNDLE (6.10). Let G be a simplicial group acting on the right of a simplicial set Y, and define the associated bundle (E^*, p^*, B) by $E^* = Y \times {}_GE$, obtained from $Y \times E$ by identifying all (yg, x) with (y, gx) for $y \in Y_n$, $g \in G_n$, $x \in E_n$. It is easily verified that (E^*, p^*, B) is a G bundle with fibre Y.

On the other hand, if $p : D \to B$ is a G bundle with fibre Y, where F is a subgroup complex of $\operatorname{aut}(Y)$, we can construct a principal G bundle over B by using the same twisting function $t : B_n \to G_{n-1}$. Thus we have the following.

PROPOSITION (6.11). *Given a principal G bundle and an action of G on Y, i.e., $G \subset \operatorname{aut}(Y)$, there is a unique associated G bundle with fibre Y.*

Given a G bundle with fibre Y, it is associated to a unique principal G bundle.

G Equivalence (6.12). Let $p : E \to B$ and $p^1 : E^1 \to B$ be two fibre bundles with the same base and fibre, and suppose $\{\alpha(b)\}$ and $\{\alpha^1(b)\}$ are atlases for p, p^1 whose transformation elements are in G. Then a map u,

$$\begin{tikzcd} E \arrow[rr, "u"] \arrow[dr, "p"'] & & E^1 \arrow[dl, "p_1"] \\ & B & \end{tikzcd}$$

is called a G equivalence if the atlases are related, for each $b \in B$, by

$$u \circ \bar{f}_b \circ \alpha(b) = \bar{f}_b \circ \alpha^1(b) \circ \gamma(b)$$

for some $\gamma(b) \in G$.

Equivalently, if E and E^1 are expressed as RTCP's $F \times {}_tB$ and $F \times {}_{t_1}B$, respectively, then a map u,

$$\begin{tikzcd} F \times {}_tB \arrow[rr, "u"] \arrow[dr] & & F \times {}_{t_1}B \arrow[dl] \\ & B & \end{tikzcd}$$

is called a G equivalence if

$$f(x, b) = (x \circ \gamma(b), b)$$

where $\gamma(b) \in G$.

Theorem (6.13). *Let $p : E \to B$ be a G bundle, and let $f \simeq g : A \to B$. Then the G bundles over A induced by f and g are G equivalent.*

Proof. It is sufficient to consider $p : E \to B$, a principal G bundle. Let $F : A \times I \to B$ be the homotopy $F : f \simeq g$. Let $p^0 : E^0 \to A$, $p^1 : E^1 \to A$, and $\bar{p} : \bar{E} \to A \times I$ be the bundles induced by f, g, and F, respectively. Slightly modify the proof of (2.11) to construct an equivariant homotopy $F^0 : E^0 \times I \to \bar{E}$; F^0 is constructed by skeletons of E^0, and having chosen $F^0(x)$ for one simplex of $(p^0 \times 1)^{-1}(a, t)$,

F^0 is then defined on the G orbit by equivariance. From F^0 we obtain a map $u : E^0 \to E^1$ which is a G equivalence of the two bundles.

Classifying Bundle (6.14). For each simplicial group G, the construction of (3.20) provides a classifying space $\overline{W}(G)$. The total complex $W(G)$ is defined by

$$(WG)_n = G_n \times G_{n-1} \times \cdots \times G_0 ,$$

$$d_i(g_n ,..., g_0) = (d_i g_n ,..., d_0 g_{n-i} \cdot g_{n-i-1} , g_{n-i-2} ,..., g_0),$$

$$s_i(g_n ,..., g_0) = (s_i g_n ,..., s_0 g_{n-i} , e, g_{n-i-1} ,..., g_0),$$

and G acts (freely) on the left of $W(G)$ by

$$g \cdot (g_n ,..., g_0) = (g \cdot g_n ,..., g_0)$$

for $g \in G_n$, $(g_n ,..., g_0) \in W(G)_n$. Then $(\overline{W}G) = G \times {}_G W(G)$, i.e.,

$$\begin{aligned} \overline{W}(G)_n &\approx (e) \times G_{n-1} \times \cdots \times G_0 \\ &\approx G_{n-1} \times \cdots \times G_0 . \end{aligned}$$

We call $p : W(G) \to \overline{W}(G)$ the classifying G bundle, which as an RTCP is $W(G) \approx G \times {}_t\overline{W}(G)$, with twisting function

$$t(g_{n-1} ,..., g_0) = g_{n-1} .$$

Note the following (cf. (3.20)):

(1) $W = W(G)$ is a Kan complex: use the methods of (3.1).

(2) W is contractible:

(i) In the LES in π of the fibration $g \to W \to \overline{W}$, $\partial : \pi_1(\overline{W}) \to \pi_0(G)$ is onto;

(ii) $\pi_1(W) = (e)$;

(iii) There is a contracting homotopy D for $(C(W), \partial)$ where

$$D(g_n ,..., g_0) = (e, g_n ,..., g_0).$$

(3) If $p : E \to B$ is a principal G bundle, express $E = G \times {}_tB$, and let $f_t : B \to \overline{W}(G)$ be

$$f_t(b) = (t(b), t(d_0 b),..., t(d_0^{n-1} b)).$$

Then the bundle over B induced from the classifying bundle by f has twisting function t.

THEOREM (6.15). *The assignment $t \to f_t$ sets up a one-one correspondence between G-equivalence classes of principal G bundles with base B, and* $[B \to \overline{W}(G)]$.

Proof. We have shown (6.13) that homotopic maps induce G-equivalent bundles. Conversely, let $f, g : B \to \overline{W}(G)$ induce the principal bundles $p^0 : E^0 \to B$ and $p^1 : E^1 \to B$ which are G-equivalent by a $u : E^0 \to E^1$. Then consider the two maps

$$\bar{f}, \bar{g} \circ u : E^0 \to W(G).$$

We want to define $F^0 : E^0 \times I \to W(G)$, an equivariant homotopy $F^0 : \bar{f} \simeq \bar{g} \circ u$. F^0 is defined by skeletons of E^0, using the contractibility. Suppose F^0 has been defined for $(E^0)^{n-1} \times I$, and take $e_0 \in E_n{}^0$. Then our hand is forced on $e_0 \times 0$ by $\bar{f}$, on $e_0 \times 1$ by $\bar{g} \circ u$, and on $e_0 \times I$ by F^0, and by the contractibility of $W(G)$, F^0 can be "filled-in" on $e_0 \times I$. Then extend F^0 to the G orbit equivariantly.

Now let a homotopy $F : f \simeq g$ be defined by

$$F(b, t) = F^0((p^0)^{-1}(b), t),$$

which is independent of the choice of $(p^0)^{-1}(b)$ by equivariance.

COROLLARY (6.16). *Let Y be a complex on which G operates. Then there is a one-one correspondence between* $[B \to \overline{W}(G)]$ *and G-equivalence classes of G bundles with base B and fibre Y.*

The special case where $G = \operatorname{aut}(Y)$ is the simplicial analogue of a theorem of Stasheff. Note especially that if Y is (a minimal complex for) the stable sphere, $\overline{W}\operatorname{aut}(Y) = B_H$ is a classifying space for sphere bundles.

MINIMAL FIBRATIONS (6.17). In order that these results apply more generally, i.e., to fibrations, we sketch the retraction of any fibration onto a minimal fibration, which is unique to within isomorphism. Since this minimal fibration is a fibre bundle, the previous theorems apply to it.

First we reformulate the minimality condition.

PROPOSITION (6.18). *Let $p : E \to B$ be a fibre map and suppose there is a commutative diagram*

$$\begin{tikzcd}
\Delta[n] \times (0) \cup \dot{\Delta}[n] \times I \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
\Delta[n] \times I \arrow[r, "g"] \arrow[ur, dashed, "f'"] & B
\end{tikzcd}$$

Then p is a minimal fibre map $\Leftrightarrow$ the fill-in f' is uniquely determined on $\Delta[n] \times (1)$.

Proof. This can be proven by the usual prismatic arguments.

If $p : E \to B$ is a fibre map, two simplices $x, y \in E_n$ are called p-compatible if $px = py$, and two p-compatible simplices x, y are called p-homotopic if there is a homotopy between the representing maps

$$\theta : f_x \simeq f_y : \Delta[n] \to E \qquad (\mathrm{rel}\, \dot{\Delta}[n])$$

so that $p \circ \theta$ is a constant homotopy in B.

PROPOSITION (6.19). *A fibre map $p : E \to B$ is minimal $\Leftrightarrow$ whenever x is p-homotopic to y, then $x = y$.*

PROPOSITION (6.20). *Let $p : E \to B$ and $p' : E' \to B$ be fibre maps with p minimal. Let f, g be maps of p to p',*

$$\begin{tikzcd}
E' \arrow[r, "f,g"] \arrow[d, "p'"] & E \arrow[d, "p"] \\
B \arrow[r, "1_B"] & B
\end{tikzcd}$$

and suppose $f \simeq g$ by a strong homotopy. Then if f is an isomorphism, so is g (but g may be a different isomorphism).

Proof. Not too difficult.

THEOREM (6.21). *Let $p : E \to B$ be a fibre map. Then there is a minimal fibration $p' : E' \to B$ which is a strong deformation retract of $p : E \to B$, and any two such are isomorphic.*

Proof. Similar to (1.23), as follows. E_n' is chosen by induction on dimension n. If E_{n-1}' has been chosen, let E_n' consist of one representative from each equivalence class of p-compatible, p-homotopic simplices of E_n all of whose faces lie in E_{n-1}', choosing a degenerate one if possible.

More generally, if $p : E \to B$ is a fibre map onto a Kan complex B, we can find a minimal fibre map $p' : E' \to B'$ where B' is also minimal, and p' is a strong deformation retract of p. Start by choosing a minimal subcomplex $B' \subset B$, etc.

Proposition (6.22). *Let $p : E \to B$ be a minimal fibre map, and let $f : A \to B$ be a simplicial map. Then the induced fibration is minimal. If $f \simeq g : A \to B$ then the two fibrations induced by f and g are isomorphic.*

The first part is immediate from the definitions. The second part is not too difficult (cf. (2.11) and (6.20)).

Proposition (6.23). *Let $p : E \to B$ be a minimal fibre map onto with fibre $F = p^{-1}(*)$, and suppose B is contractible. Then $F \times B \approx E$ and there is a commutative diagram*

$$\begin{array}{ccc} F \times B & \xrightarrow{\approx} & E \\ \downarrow{\scriptstyle p_B} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{1_B} & B \end{array}$$

where p_B is the natural projection and f is an isomorphism.

Proof. The identity map 1_B and the constant map $c : B \to * \subset B$ are homotopic.

Corollary (6.24). *If $p : E \to B$ is a minimal fibration onto a connected base, then p is a fibre bundle.*

This concludes the demonstration that if $p : E \to B$ is a fibre map onto, then it can be retracted to a fibre bundle. Then the previous RTCP structure and the classifying theorems apply.

7. The Lower Central Series

To study a group G, it is useful to filter G in such a way that the quotients are Abelian, e.g., the filtration on G by its lower central series (LCS). Furthermore, if G is a free group, the quotients of the LCS form the free Lie algebra generated by the free Abelian group $G/\Gamma_2 G$. The same considerations apply when G is a free simplicial group, e.g., when $G = GK$. In this way there arises a spectral sequence $E^i(K)$, whose E^1 terms are homology invariants of K and which converges to $\pi_*(K)$.

A similar situation occurs when the filtration is by the mod-p restricted lower central series (mod-p RLCS); now the quotients form the free restricted Lie algebra over Z_p generated by $(G/\Gamma_2 G) \otimes Z_p$. Apply this filtration to GK and there arises a spectral sequence which (suitably speeded up and reindexed) becomes a sort of Adams spectral sequence for K. We describe $(E^1(S^n), d^1)$, calling it $\Lambda(n)$, and $\Lambda = \bigcup_n \Lambda(n)$, which becomes a ring under composition.

The references for Section 7 are [6A], [C2], [R].

THE LOWER CENTRAL SERIES (7.1). Let G be a simplicial group. The lower central series (LCS) filtration of G is obtained by taking (in every dimension of G)

$$\Gamma_r G = \{(x_1, \ldots, x_n) : x_i \in G\},$$

where $(x, y) = x^{-1}y^{-1}xy$ and $(x_1, \ldots, x_r)$ is the iterated commutator $(\ldots, (x_1, x_2), \ldots, x_r)$; here $\{\ \}$ denotes the subgroup generated by.

THEOREM (7.2). *If G is a connected free simplicial group with $\pi_i(G) = 0$ for $i \leqslant n$, then $\pi_i(\Gamma_r G) = 0$ for $i \leqslant \{n + \log_2 r\}$, where $\{a\}$ denotes the least integer $\geqslant a$.*

Comments. We will not prove (7.2) here; see Ref. [C2]. What is important is that connectivity $\Gamma_r G \to \infty$ as $r \to \infty$. The proof is complicated and involves the following steps.

(1) It is sufficient to consider the special case $G = GX$ where X is a finite wedge of S^2, for some general arguments then show that $\Gamma_r(\ )$ raises the connectivity of an arbitrary free G by as much as it does for this one.

(2) For $G = GX$, the techniques of Section 4 give a free basis for $\Gamma_r GX$, which in the case of $X =$ wedge of S^2, becomes a product of simplicial sets involving inductively calculable connectivity. The techniques here are similar to, and greatly influenced by, those of Milnor [Mi2] and Kan [K1].

Now let $G = GK$ be the free simplicial group resulting from Kan's construction applied to a connected and simply connected simplicial set K. Let GK be filtered (in every dimension) by its LCS, and let the homotopy exact couple give rise to a spectral sequence whose terms we call $E^i(K)$.

THEOREM (7.3). (1) *The* $E^i(K)$ *converge to* $E^\infty(K)$, *and* $\bigoplus_r E^\infty_{r,q}(K)$ *is the graded group associated with the filtration on* $\pi_q(GK) = \pi_{q+1}(K)$.

(2) *The groups* $E^1(K)$,

$$E^1_{r,q}(K) = \pi_q(\Gamma_r GK/\Gamma_{r+1}GK),$$

are homology invariants of K.

Proof. (1) The spectral sequence converges by (7.2).

(2) We will shortly show that

$$\Gamma_r GK/\Gamma_{r+1}GK \approx L_r(GK/\Gamma_2 GK)$$

where L_r is the r-th component of the free Lie ring functor. Thus by (5.9) its homotopy depends only on $\pi_*(GK/\Gamma_2 GK)$, which is $H_*(K)$ with a dimension shift.

FREE LIE RING (7.4). Let A be an Abelian group. Let $T(A)$ be the tensor ring of A, made into a Lie ring by $[x, y] = x \otimes y - y \otimes x$. Let $L^u(A) \subset T(A)$ be the sub-Lie ring of $T(A)$ generated by A. $L^u(A)$ is called the free (unrestricted) Lie ring generated by A. $L^u(A)$ is universal, viz., if $A \to R$ is any homomorphism of A into (the underlying Abelian group of) a Lie ring R, then there is a unique factorisation $A \to L^u(A) \to R$.

Let G be a group; then the LCS filtration of G,

$$\cdots \to \Gamma_{r+1}G \to \Gamma_r G \to \cdots \to \Gamma_2 G \to \Gamma_1 G = G,$$

yields quotients $\Gamma_r G/\Gamma_{r+1}G$ which are Abelian. Furthermore,

$$\bigoplus_r \Gamma_r G/\Gamma_{r+1}G = \mathfrak{L}(G)$$

becomes a Lie ring, where for $u \in \Gamma_r/\Gamma_{r+1}$, represented by $\bar{u} \in \Gamma_r$, $v \in \Gamma_s/\Gamma_{s+1}$, represented by $\bar{v} \in \Gamma_s$,

$$[u, v] = (\bar{u}, \bar{v}) \quad \mod \quad \Gamma_{r+s+1}.$$

THEOREM (7.5) (Witt). *There is a natural homomorphism*

$$\theta : L^u(G/\Gamma_2 G) \approx \mathfrak{L}(G)$$

which is an isomorphism if G is a free group.

The homomorphism θ comes from the universality of $L^u(\ )$. For the isomorphism, see, for example, Ref. [H, p. 175].

Since θ is natural with respect to homomorphisms (of G), the same constructions, applied in every dimension and to the face and degeneracy operators, work for free simplicial groups. Taking the r-th component $L_r{}^u(\ )$ as a functor from Abelian groups to Abelian groups proves (7.3 (2)).

In order to get Adams-type spectral sequences, and unstable versions thereof, we modify the filtration of G to make the quotients vector spaces, rather than free Abelian groups. This leads us to the mod-p restricted lower central series (RLCS).

THE MOD-p RLCS (7.5). Let p be a fixed prime number, and let G be a group (or work in each dimension of a simplicial group). Then the mod-p RLCS filtration on G is defined by

$$\Gamma_r^{(p)} G = \{(x_1, \ldots, x_s)^{p^t} : sp^t \geqslant r, x_i \in G\}.$$

The quotients $\Gamma_r^{(p)} G / \Gamma_{r+1}^{(p)} G$ are vector spaces, and their direct sum becomes a restricted Lie algebra

$$\bigoplus \Gamma_r^{(p)} G / \Gamma_{r+1}^{(p)} G = \mathfrak{L}^{(p)} G,$$

where the bracket is as before, and the p-th power map is $x \to x^p$. When a prime p has been fixed, we drop the superscript (p); hereafter Γ_r refers to the mod-p RLCS, and $\Gamma_r{}^u$ is the usual (i.e., integral) LCS.

Let V be a vector space and let $T(V)$ be the tensor algebra of V. Then under the bracket $[\ ,\ ]$ and p-th tensor power map $x \to x^p$, $T(V)$ becomes a restricted Lie algebra; let $L(V) \subset T(V)$ be the sub-restricted Lie algebra in $T(V)$ generated by V; $L^u(V)$ will denote the free unrestricted Lie algebra. The analogue of (7.5) is the following.

THEOREM (7.6). *Let G be a group and $V = G/\Gamma_2 G$. Then there is a homomorphism $\theta : L(V) \to \mathfrak{L}(G)$ which is an isomorphism if G is a free group.*

From here on, let $E^i(K)$ be the terms of the spectral sequence arising from GK filtered by its mod-p RLCS.

The following was proven by Rector, Ref [*R*]

THEOREM (7.7). *Let K be a connected and simply connected simplicial set. Then*

(1) *The $E^i(K)$ converge to $E^\infty(K)$, which is the graded group associated to the filtration on $\pi_*(K)$ modulo the subgroup of elements of finite order prime to p;*

(2) *The groups $E^1(K)$, where*

$$E^1_{r,q}(K) = \pi_q(\Gamma_r GK/\Gamma_{r+1}GK),$$

are invariants of $H_*(K; Z_p)$.

We proceed to describe $E^1(K)$; take $p = 2$ (for simplicity) and let $H_*(K)$ stand for $H_*(K; Z_2)$. Also let $K(n)$ stand for the Eilenberg–MacLane complex $K(Z_2, n)$.

First consider the special case $K = S^n$, the n sphere. Then

$$GS^n/\Gamma_2 GS^n \approx K(n-1),$$

so also

$$E^1(S^n) \approx \pi_* L(K(n-1)).$$

To handle this, we need two operations, composition and suspension.

COMPOSITION (7.8). Let $x \in \pi_{n+q} L_r(K(n))$. Then there is a (unique to within homotopy) simplicial homomorphism

$$f_x : K(n+q) \to L_r(K(n))$$

with $[f_x(i_{n+q})] = x$. If also $y \in \pi_{n+q+q'} L_t(K(n+q))$, represented by f_y, then the composite

$$K(n+q+q') \xrightarrow{f_y} L_t(K(n+1)) \xrightarrow{L_t(f_x)} L_r L_t(K(n)) \to L_{rt}(K(n))$$

determines an element $x \circ y \in \pi_{n+q+q'} L_{rt}(K(n))$.

SUSPENSION (7.9). Recall that the suspension σ is a homomorphism

$$\sigma : \pi_* L_r(K(n)) \to \pi_{*+1} L_r(K(n+1)).$$

Here we consider $A = K(n)$, $\overline{W}A = K(n+1)$ in the category of simplicial vector spaces, and $L_r(\ )$ is the functor (cf. Section 5 or Ref. [D, P]).

Suspension and composition are related by a commutative diagram:

$$\begin{array}{ccc} \pi_{n+q+q'+1}L_t(K(n+q+1)) & \xrightarrow{\text{composition}} & \pi_{n+q+q'+1}L_{rt}(K(n+1)) \\ \uparrow{\scriptstyle\sigma} & & \uparrow{\scriptstyle\sigma} \\ \pi_{n+q+q'}L_tK(n+q) & \xrightarrow{\text{composition}} & \pi_{n+q+q'}L_{rt}(K(n)) \end{array}$$

SHUFFLES (7.10). Let V be a simplicial vector space, and let $y \in V_p$, $z \in V_q$. Then let $y \underline{\otimes} z \in (V \otimes V)_{p+q}$ be given by

$$y \underline{\otimes} z = \sum_{(a;b)} s_b y \otimes s_a z,$$

where $(a; b)$ vary over all (p, q) shuffles; recall (5.6). For later use, define

$$[y, z] = y \underline{\otimes} z - z \underline{\otimes} y.$$

THE RING Λ. We now describe $E^1(S^n)$, which is $\pi_* L(K(n-1))$, and which we also call $\Lambda(n)$. There will be monomorphisms $\Lambda(n) \to \Lambda(n+1)$, and the union $\Lambda = \bigcup_n \Lambda(n)$ becomes a ring. For each $m \geqslant 0$, let

$$\lambda_m \in \pi_{2m} L_2(K(m))$$

be the homotopy class of $i_m \underline{\otimes} i_m$; also λ_m will stand for any of its suspensions.

A sequence of compositions $\lambda_{i_1} \circ \cdots \circ \lambda_{i_s}$ will be abbreviated by λ_I, where $I = (i_1, ..., i_s)$. A sequence $I = (i_1, ..., i_s)$ of nonnegative integers will be called *allowable* if $2i_j \geqslant i_{j+1}$ for all $1 \leqslant j \leqslant s-1$, or *allowable with respect to* n if also $i_1 < n$.

THEOREM (7.11). *The compositions* λ_I, *for* $I = (i_1, ..., i_s)$ *allowable with respect to* n, *form an additive basis of* $\Lambda(n)$; *the empty composition, denoted by* 1, *is to be included.*

The proof will be given in the next section.

Thus we have that $\pi_* L_r(K(n-1)) = 0$ for $r \neq 2^s$, while for $r = 2^s$, it has for basis all compositions λ_I of length s and allowable with respect to n. For this reason, speed up the filtration of GS^n by

$$G \supset \Gamma_2 \supset \Gamma_4 \supset \cdots \supset \Gamma_{2^s} \supset \Gamma_{2^{s+1}} \supset \cdots.$$

The new spectral sequence has isomorphic E^1 groups, but reindexed, and with faster-acting differentials.

THEOREM (7.12). (1) *On* $E^1(S^n) = \Lambda(n)$, d^1 *is the derivation for which*

$$d^1\lambda_m = \sum_{j>0} \binom{m-j}{j} \lambda_{m-j}\lambda_{j-1} .$$

(2) *The suspension* $\Lambda(n-1) \to \Lambda(n)$ *is a monomorphism. The union* $\Lambda = \bigcup_n \Lambda(n)$ *is a ring (under composition) with* 1, *and with generators* λ_i *for each integer* $i \geqslant 0$. *The relations in* Λ *are generated by*

$$\lambda_i\lambda_{2i+1+m} = \sum_{j\geqslant 0} \binom{m-1-j}{j} \lambda_{i+m-j}\lambda_{2i+1+j} .$$

This will be proven in Section 8.

8. SIMPLICIAL LIE ALGEBRAS

We want to describe the homotopy of simplicial Lie algebras. In particular, if $L(V)$ is the simplicial free restricted Lie algebra generated by the simplicial vector space V, then $\pi_*L(V)$ is given by the formula (8.9). This requires some preliminary techniques, especially: (1) A Whitehead lemma for simplicial Lie algebras; (2) a decomposition formula for $\pi_*L(K(Z_2, n-1)) = \Lambda(n)$; (3) the operations of $\Lambda(n)$ on $\pi_*L(V)$. We then describe the E^1 term of the mod-2 RLCS spectral sequence, and obtain the proof of (7.12).

The reference for Section 8 is [B, C].

We consider simplicial Lie algebras R; that is, each R_n is a Lie algebra, and all the d_i, s_i are homomorphisms. R is called a free simplicial Lie algebra if each R_n is the free Lie algebra $L^u(M_n)$, and the M_n are stable under degeneracies, i.e., $s_iM_n \subset M_{n+1}$. Similar considerations hold for simplicial restricted Lie algebras.

FREE LIE ALGEBRAS (8.1). Recall the definition (7.4) of the free Lie algebra $L^u(V)$, generated by a vector space V. Let V and W be two vector spaces, and $V \oplus W \to W$ be the homomorphism which maps V to 0 and W isomorphically. Then $L^u(V \oplus W) \to L^u(W)$ has for kernel $L^u(U)$, where U is the vector space

$$U = W \oplus (W \otimes V) \oplus \cdots \oplus (W \otimes V^n) \oplus \cdots.$$

$V^n = V \otimes \cdots \otimes V$, n copies, and $L^u(U) \to L^u(V \oplus W)$ is defined by

$$w \otimes v_1 \otimes \cdots \otimes v_n \to [\ldots, [w, v_1], \ldots, v_n].$$

If $V_1, V_2, \ldots, V_r$ are vector spaces, let the basic tensor products V_i for $1 \leqslant i \leqslant \infty$ be obtained from $V_1, \ldots, V_r$, just as the basic commutators in (4.10). The analogous result is

$$L^u\left(\bigoplus_{i=1}^{r} V_i\right) \approx \bigoplus_{i=1}^{\infty} L^u(V_i).$$

The lower central series of a (simplicial) Lie algebra R is defined by $\Gamma_1 R = R$, and $\Gamma_r R = [\Gamma_{r-1} R, R]$. The Abelianization of R is $\mathrm{Ab}\, R = R/\Gamma_2 R$.

LEMMA (8.2). *Let $f: R \to R'$ be a homomorphism of connected free simplicial Lie algebras. Then if* $(\mathrm{Ab} f)_*$ *is an isomorphism* (*in homotopy*), *so is* f_*.

This is the analogue of the Whitehead lemma for simplicial Lie algebras. The proof uses two sublemmas.

SUBLEMMA (8.3). *If R is a connected free simplicial Lie algebra, then $\Gamma_r R$ is* $\log_2 r$ *connected, i.e.,* $\pi_q(\Gamma_r R) = 0$ *for* $q < \log_2 r$.

A proof can be found in Ref. [C1]. What is important here is that as $r \to \infty$, connectivity $\Gamma_r(R) \to \infty$.

SUBLEMMA (8.4). *Let $f: R \to R'$ be a homomorphism of free simplicial Lie algebras. Then if* $(\mathrm{Ab} f)_*$ *is an isomorphism, so is*

$$(\Gamma_r f / \Gamma_{r+1} f)_*.$$

This follows from the isomorphism

$$\Gamma_r R / \Gamma_{r+1} R \approx L_r{}^u(\mathrm{Ab}\, R)$$

and the results of Section 5.

The proof of (8.2) now follows from (8.3) and (8.4) by iterated application of the five-lemma.

Proof of (7.11). We use the following two reductions.

(1) For any vector space V, the inclusion $L^u(V) \to L(V)$ and the

squaring map $L(V) \to L(V)$ by $a \to a \otimes a$ induce a decomposition as sets,

$$L^u(V) \times L(V) \approx L(V).$$

This composition is natural with respect to maps, so holds for a simplicial vector space V. Thus

$$\pi_* L_s^u(V) \oplus \pi_* L_{2s}(V) \approx \pi_* L_s(V),$$

where the squaring map induces $(\ ) \circ \lambda_0$, i.e., composition on the right with λ_0.

(2) The suspension σ and composition on the left with λ_n induce

$$\pi_* L_r^u K(n) \approx \pi_{*-1} L_r^u K(n-1), \qquad r \text{ odd},$$

$$\pi_* L_r^u K(n) \approx \pi_{*-1} L_r^u K(n-1) \oplus \pi_* L_{r/2}^u(K(2n)), \qquad \text{for} \quad r \text{ even}.$$

Here is an outline of a proof of (2); cf. Ref. [6A]. Let W be the simplicial Lie algebra freely generated by x in dimension $n-1$, y in dimension n, and z in dimension $2n$, with $d_n y = x$, all other $d_i x = 0$, $d_i y = 0$, $d_i z = 0$. Thus, dimension-wise,

$$W \approx L^u(K(n-1) \oplus K(n) \oplus K(2n)).$$

Let

$$f : W \to L^u(K(n))$$

by the homomorphism defined by $f(x) = 0, f(y) = i_n, f(z) = i_n \underline{\otimes} i_n$. Then by (8.1), $\ker f$ is dimension-wise isomorphic to the free Lie algebra generated by

$$V = \left(\bigoplus_{r=0}^{\infty} X \otimes Y^r\right) \oplus \left(\bigoplus_{r=0}^{\infty} Z' \otimes Y^r\right)$$

where $Z' \approx K(Z, 2n)$ with generator $y \underline{\otimes} y - z$. Then by a straightforward calculation

$$d_i(x \underline{\otimes} y \underline{\otimes} \cdots \underline{\otimes} y) = 0 \qquad \text{for all } i,$$

$$d_i(z' \underline{\otimes} y \underline{\otimes} \cdots \underline{\otimes} y) = 0 \qquad \text{for} \quad i \neq rn,$$

$$d_{rn}(z' \underline{\otimes} y \underline{\otimes} \cdots \underline{\otimes} y) = (x \underline{\otimes} y \underline{\otimes} \cdots \underline{\otimes} y).$$

Thus

$$\pi_i(\mathrm{Ab}\, L^u(V)) = \begin{cases} Z_2, & i = n-1, \\ 0, & i \neq n-1. \end{cases}$$

Consider the diagram

$$\begin{tikzcd} L^u(K(n-1)) \arrow[r, "g"] & \ker f \arrow[r] & W \arrow[r, "h"] \arrow[d, "f"] & L^uK(2n) \\ & & L^uK(n) & \end{tikzcd}$$

where $g(i_{n-1}) = x$; $h(x) = 0$, $h(y) = 0$, $h(z) = i_{2n}$. Then g_* and f_* are isomorphisms by (8.2). The decomposition formula of (2) follows, with a little perseverence to see that

$$\pi_* L^uK(n) \xrightarrow{\partial} \pi_{*-1} L^uK(n-1)$$

is onto and has for inverse (on one side) the suspension σ.

The proof of (7.11) now follows by two inductions, using (1) and (2).

SIMPLICIAL RESTRICTED LIE ALGEBRAS (8.5). Let R be a simplicial RLA (over Z_2), and let Λ be the algebra of (7.11). Then $\pi_*(R)$ becomes a restricted Lie algebra, and elements of Λ operate on the right of R as follows.

(1) The bracket $[x, y]$ for $x, y \in \pi_*(R)$ is defined as the composite

$$\pi_*(R) \otimes \pi_*(R) \xrightarrow{\Delta} \pi_*(R \otimes R) \xrightarrow{m_*} \pi_*(R),$$

where ∇ is the "shuffle" and m is the Lie multiplication in R.

(2) For $x \in \pi_n(R)$, $\lambda_i \in \Lambda$ with $0 \leqslant i \leqslant n$, define $(x)\lambda_i$ as the composite

$$K(n+i) \xrightarrow{\lambda_i} L_2K(n) \xrightarrow{L_2(f_x)} L_2(R) \xrightarrow{m} R$$

where $f_x : K(n) \to R$ represents x.

PROPOSITION (8.6). *For R a simplicial RLA over Z_2, the operations $[\cdot, \cdot]$ and $(\cdot)\lambda_i$ in $\pi_*(R)$ satisfy*

(1) $[x, x] = 0$;

(2) $[x, y, z] + [y, z, x] + [z, x, y] = 0$;

(3) *For* $x, y \in \pi_n(R)$,

$$(x + y)\,\lambda_i = (x)\,\lambda_i + (y)\,\lambda_i \qquad \textit{for} \quad i < n,$$

$$(x + y)\,\lambda_n = (y)\,\lambda_n + (y)\,\lambda_n + [x, y];$$

(4) *If* $x \in \pi_m(R)$, $y \in \pi_n(R)$,

$$[x, (y)\,\lambda_i] = 0 \qquad \textit{for} \quad i < n,$$

$$[x, (y)\,\lambda_n] = [x, y, y],$$

(5) *The squaring map in* $\pi_*(R)$ *is given by* $x \to (x)\lambda_n$ *for* $x \in \pi_n(R)$;

(6) $\pi_*(R)$ *is a RLA.*

The proofs of these statements are exercises in simplicial formulas. If $x \in \pi_q(R)$, $y \in \pi_q(R)$, represented by $\bar{x} \in R_p$, $\bar{y} \in R_q$, then

$$[x, y] = \sum_{(a;b)} [s_b\bar{x}, s_a\bar{y}],$$

where $(a; b)$ vary over all (p, q) shuffles. For example, to show (2), let $x \in \pi_p(R)$, $y \in \pi_q(R)$, $z \in \pi_r(R)$ be represented by $\bar{x}$, $\bar{y}$, $\bar{z}$. Then

$$\begin{aligned}
[x, y, z] &= \sum_{(c;d)} \Big[s_d \Big(\sum_{(a;b)} [s_b\bar{x}, s_a\bar{y}]\Big), s_c\bar{z}\Big] \\
&= \sum_{\substack{(a;b)\\(c;d)}} [s_d s_b\bar{x}, s_d s_a\bar{y}, s_c\bar{z}] \\
&= \sum_{(c;c';c'')} [s_{c''}\bar{x}, s_{c'}\bar{y}, s_c\bar{z}],
\end{aligned}$$

where the $(c; c'; c'')$ vary over subsets whose complements $(\hat{c}; \hat{c}'; \hat{c}'')$ form a partition of $\{0, 1, \ldots, p + q + r - 1\}$. Similar expressions obtain for $[y, z, x]$ and $[z, x, y]$, and the Jacobi law (2) follows by summing and using the Jacobi law in R.

For (3): The expression for $(x)\lambda_i$ is

$$(x)\,\lambda_i = \sum_{(a;b)} [s_b\bar{x}, s_a\bar{x}]$$

where $(a; b)$ vary over (i, i) shuffles with $b_i = 2i - 1$. Thus

$$\begin{aligned}
(x + y)\,\lambda_i &= \sum_{(a;b)} [s_b(\bar{x} + \bar{y}), s_a(\bar{x} + y)] \\
&= (x)\,\lambda_i + (y)\,\lambda_i + \sum_{(a;b)} [s_b\bar{x}, s_a\bar{y}] + \sum_{(a;b)} [s_b\bar{y}, s_a\bar{x}] \\
&= (x)\,\lambda_i + (y)\,\lambda_i + \sum_{(c;d)} [s_d\bar{x}, s_c\bar{y}],
\end{aligned}$$

where $(c; d)$ vary over *all* (i, i) shuffles, and formula (3) follows. The verification of the other formulas is left as an exercise.

For each integer $n \geqslant 0$, let there be given a vector space W_n, and let $W = \bigoplus_n W_n$; consider n as dimension. Then the tensor algebra $T(W)$ inherits a dimension by

$$\dim(w_1 \otimes \cdots \otimes w_k) = \sum \dim(w_i),$$

and so $L^u(W)$ and $L(W)$ also inherit a dimension.

Let V be a simplicial vector space over the field Z_2; then $\pi(V) = \bigoplus_n \pi_n(V)$ is such a vector space with dimension. Form the direct sum

$$\bigoplus_n L(\pi(V))_n \otimes \Lambda(n+1)$$

with relations (generated by), for $u \in L(\pi(V))_n$,

$$u \otimes \lambda_n = u^{(2)},$$

and call this $L(\pi(V)) \mathbin{\widehat{\otimes}} \Lambda$. For this, and for later use, we define, for each graded vector space M,

$$M \mathbin{\widetilde{\otimes}} \Lambda = \bigoplus_n M_n \otimes \Lambda(n-1),$$

$$M \mathbin{\widehat{\otimes}} \Lambda = \bigoplus_n M_n \otimes \Lambda(n).$$

Notice that additively,

$$L(\pi(V)) \mathbin{\widehat{\otimes}} \Lambda \approx \bigoplus_n L(\pi(V))_n \otimes \Lambda(n+1)/u \otimes \lambda_n = u^{(2)}.$$

Let $\varphi : L(\pi(V)) \to \pi_* L(V)$ be the restricted Lie algebra map which extends

$$L_1(\pi(V)) = \pi_*(V) \subset \pi_* L(V),$$

and let also

$$\varphi : L(\pi(V)) \mathbin{\widehat{\otimes}} \Lambda \to \pi_* L(V)$$

be defined by $\varphi(u) \otimes \lambda_I = (u)\lambda_I$.

THEOREM (8.8). *This construction yields a natural isomorphism*

$$\varphi : L(\pi(V)) \mathbin{\widehat{\otimes}} \Lambda \to \pi_* L(V).$$

Proof. Let $\Lambda^+ \subset \Lambda$ be generated by all allowable $\lambda_I = \lambda_{i_1} \circ \cdots \circ \lambda_{i_s}$ with $i_s > 0$ (as $2i_j \geqslant i_{j+1}$, so also all $i_j > 0$). Then by means of the decomposition of 8.4 (proof of 7.11) it will be sufficient to show that the analogous

$$\varphi^+ : L^u(\pi(V)) \mathbin{\widehat{\otimes}} \Lambda^+ \to \pi_* L^u(V)$$

is an isomorphism. We may as well assume $V = \bigoplus_a V(a)$, where each $V(a)$ is a $K(n_a)$. Then (8.1) shows that

$$L^u(V) \approx \bigoplus_b L^u(V(b)),$$

where the $V(b)$ vary over the basic tensor products of the $V(a)$ according to (4.10), and each $V(b) \simeq K(n_b)$. Thus

$$L^u(V) \simeq \bigoplus_b L^u(K(n_b))$$

and as $\pi_* L^u(K(n_b)) \approx \Lambda^+(n_b + 1)$,

$$\pi_* L^u(V) \approx \bigoplus_b (i_{n_b}) \otimes \Lambda^+(n_b + 1).$$

Construct also a basis u_b for $L^u(\pi(V))$ in a similar way (i.e., by basic products) from a basis u_a for $\pi(V)$. Then $\varphi(u_b) = i_{n_b}$, and the isomorphism follows.

PROPOSITION (8.9). *For the decomposition of* (8.8) *for* $\pi_* L(V)$ *and* $\pi_* L(\overline{W}V)$, *we have*

$$\begin{array}{ccc} L(\pi(V)) \mathbin{\widehat{\otimes}} \Lambda & \xrightarrow[\approx]{\varphi} & \pi_* L(V) \\ \big\downarrow{\scriptstyle \hat{\sigma}} & & \big\downarrow{\scriptstyle q} \\ L(\pi(\overline{W}V)) \mathbin{\widehat{\otimes}} \Lambda & \xrightarrow[\approx]{\varphi} & \pi_{*+1} L(\overline{W}V) \end{array}$$

where $\varphi^{-1} \circ \sigma \circ \varphi = \hat{\sigma}$ *satisfies*

(1) $\tilde{\sigma} : \pi_*(V) \mathbin{\widehat{\otimes}} \Lambda \to \pi_{*+1}(\overline{W}V)$ *is a monomorphism*;

(2) *For* $u = [x_1, \ldots, x_r] \in L_r(\pi(V))$, *with* $r \geqslant 2$,

$$\hat{\sigma}(u \otimes \lambda_I) = 0;$$

(3) *For* $w = v^{(2^s)} \in L_{2^s}(\pi(V))$, *with* $v \in \pi_n(V)$,

$$\hat{\sigma}(w \otimes \lambda_I) = \hat{\sigma}(v)\, \lambda_n \lambda_{2n} \cdots \lambda_{2^{s-1}n} \lambda_I .$$

Proof (sketch). For (1), we take first the special case $V = K(n)$; then $\hat{\sigma}$ is a monomorphism by (7.11), considered as the suspension $\sigma : \Lambda(n) \to \Lambda(n+1)$. For the general case, use $V \simeq \oplus_a K(n_a)$.

For (2), the methods of (5.12) show that

$$\sigma[x_1, \ldots, x_r] = 0.$$

For example, if $x \in \pi_p(v)$ and $y \in \pi_q(v)$ are represented by $\bar{x}$ and $\bar{y}$, respectively, then

$$\sigma[\bar{x}, \bar{y}] = \partial f_{p,q}(\bar{x}, \bar{y}),$$

etc. The details are left to the reader.

THEOREM (8.10). *Let K be a connected and simply connected simplicial set. Then in the* (*unreindexed*) mod-2 *RLCSSS for GK*,

$$E^1(K) \approx L(H(K)) \mathbin{\widehat{\otimes}} \Lambda,$$

and the differential satisfies

(i) *For* $x \in \bar{H}_n(K) \subset L(\bar{H}(K)) \subset E^1$

$$d^1 x = \partial^\Delta(x) + \sum_{i=1}^{[n/2]} (x)\, Sq^i \otimes \lambda_{i-1} ;$$

(ii) *For* $x \in L_k(\bar{H}(K))$, $k \geqslant 2$,

$$d^1 x = \partial^\Delta(x).$$

Proof. (i) The map

$$\begin{array}{ccc} \pi_*(GK/\Gamma_2 GK) & \xrightarrow{d_1} & \pi_*(\Gamma_2 GK/\Gamma_3 GK) \\ \wr\!\wr & & \wr\!\wr \\ \bar{H}_*(K) & \longrightarrow & (\bar{H}(K) \otimes \Lambda^1) \oplus L_2(\bar{H}(K)) \end{array}$$

is natural in K. It follows that

$$d^1 x = \sum_{i>0} (x)\, T^i \otimes \lambda_{i\ 1} + \partial^\Delta(x),$$

where T^i are elements of degree i in the mod-2 Steenrod algebra A_2, considered as acting on the *right* of $\bar{H}_*(K; Z_2)$; the term $\partial^\Delta(x) \in L_2(\bar{H}_*(K))$ will be described in Section 10, and it turns out that ∂^Δ is the dual of the cup product. We want to show that $T^i = Sq^i$, for which we need the following facts.

FACTS (8.11). The dual $(A_2)_*$ of the Steenrod algebra is the polynomial algebra $Z_2[\xi_0, ..., \xi_i, ...]$ with $\xi_0 = 1$. Let $Sq = \sum Sq^n$; then

$$(\xi_i)\, Sq = \xi_i + \xi_{i-1},$$
$$(\xi\xi')\, Sq = ((\xi)\, Sq)(\xi')\, Sq).$$

Also, $Sq^i \in A_2$ is the only nonzero element of degree i which vanishes on $H_*(K(i-1))$. Finally, $H_*(K(N)) \approx (A_2)_*$ for a stable range.

Continuation of proof. Take $x \in H_{2i-1}(K(i-1))$. Then $(x)T^i = 0$ since expressions of the form $y \otimes \lambda_{i-1}$, where $y \in H_{i-1}(K)$, do not occur on the right side. It remains to show $T^i \neq 0$. Take N large. If T^1 were 0, then $1 \otimes \lambda_0$ would persist to $E^\infty(K(N))$, contradicting the convergence of the spectral sequence. Similarly, if T^2 were 0, either $1 \otimes \lambda_1$ or $1 \otimes \lambda_1 + \xi_1 \otimes \lambda_0$ would persist to E^∞. Thus $T^1 = Sq^1$, $T^2 = Sq^2$.

Assume inductively that $T^i = Sq^i$ for $i \leqslant 2k$, and suppose $T^{2k+1} = 0$; then an easy calculation using (8.11) shows that $d^1d^1(\xi_1\xi_2{}^k)$ is a polynomial in the ξ_i with constant term $\lambda_k\lambda_{2k-1} \neq 0$. But as $d^1d^1 = 0$, it must be that $T^{2k+1} = Sq^{2k+1}$. Similarly, $\xi_1{}^2\xi_2{}^k$ can be used to show that $T^{2k+2} = Sq^{2k+2}$. Part (ii) will be proven in Section 10.

Proof of (7.12). The identity $d^1d^1 = 0$ shows that for any $x \in \bar{H}_*(K)$,

$$0 = \sum_{i,j>0} xSq^iSq^j \otimes \lambda_{j-1}\lambda_{i-1} + \sum_{i>0} xSq^i \otimes d\lambda_{i-1}.$$

The Adem relations in A_2 for $0 < a < 2b$,

$$Sq^aSq^b = \sum_{c=0}^{[a/2]} \binom{b-1-c}{a-2c} Sq^{a+b-c}Sq^c,$$

when substituted into the above identity, quickly imply the differential and the relations for the λ's.

REFERENCE (8.12). More can be said about the differentials in $E^r(K)$. In particular, Ref. [B, C] shows that

(iii) If $x \otimes \lambda_I \in \bar{H}_*(K) \mathbin{\hat{\otimes}} \Lambda^s$, with $s > 0$, then

$$d^r(x \otimes \lambda_I) = 0 \qquad \text{for} \quad i \leqslant r < 2^s,$$

$$d^{2^s}(x \otimes \lambda_I) = x \otimes d\lambda_I + (\partial^A x)\, \lambda_I + \sum_{i=1}^{m} xSq^i \otimes \lambda_{i-1}\lambda_I,$$

where $m = [\deg x/2]$.

(iv) If $x \otimes \lambda_I \in L_k(\bar{H}_*(k)) \hat{\otimes} \Lambda^s$, with $k > 1$ and $s \geqslant 0$, then

$$d^r(x \otimes \lambda_I) = 0 \qquad \text{for} \quad 1 \leqslant r < 2^s,$$
$$d^{2^s}(x \otimes \lambda_I) = (\partial^\Delta x) \otimes \lambda_I .$$

If we assume that the coalgebra H_*K $(= H_*(K; Z_2))$ is *nice* in the sense of Ref. [B], then the effect of the differentials in the speeded-up and reindexed mod-2 RLCSSS for GK is to cancel out so many of the cycles (i.e., they become boundaries), that $E^2(K)$ becomes accessible. This happens when H^*K is isomorphic to a polynomial algebra modulo a Borel ideal; in particular, when K is a sphere or a loop space.

For reference, we describe a chain complex $W(H_*K, \delta)$ which serves as an E^1 term for the spectral sequence when K is nice.

The diagonal $K \to K \times K$ induces

$$\Delta : H_*K \to H_*K \otimes H_*K$$

and hence also,

$$\tilde{H}_*K \xrightarrow{\Delta_1} \tilde{H}_*K \otimes \tilde{H}_*K \xrightarrow{\Delta_2} \tilde{H}_*K \otimes \tilde{H}_*K \otimes \tilde{H}_*K \xrightarrow{\Delta_3} \cdots,$$

where $\Delta_1 = \Delta$, $\Delta_2 = 1 \otimes \Delta + \Delta \otimes 1$, etc.

Define $\Psi(H_*K)$ as the kernel of the composition

$$L_2(\tilde{H}_*K) \to \tilde{H}_*K \otimes \tilde{H}_*K \to \tilde{H}_*K \otimes \tilde{H}_*K \otimes \tilde{H}_*K.$$

The right action of A_2 on $H_*(K \times K) \approx H_*K \otimes H_*K$ induces an action of A_2 on $\Psi(H_*K)$, and the map

$$\Delta : \tilde{H}_*K \to \Psi(H_*K)$$

is a right A_2 map of degree -1. Let

$$\sigma : \Psi(H_*K) \to \tilde{H}_*K$$

be the restriction of the homomorphism which sends $[x, y] \to 0$ and $x \otimes x \to x$ for x and y in $\bar{H}_*K$. Finally, define, for each $s \geqslant 0$,

$$W^s(H_*K) = (\tilde{H}_*K \tilde{\otimes} \Lambda^s) \oplus \Psi(H_*K) \tilde{\otimes} \Lambda^{s-1},$$

then the differential $\delta : W^s \to W^{s+1}$ has the following components:

(i) For $x \otimes \lambda_I \in H_nK \tilde{\otimes} \Lambda^s$

$$\delta(x \otimes \lambda_I) = (x \otimes d\lambda_I) + \sum_{i=1}^{m} (xSq^i \otimes \lambda_{i-1}\lambda_I) \oplus (\Delta x \otimes \lambda_I),$$

where $m = [(n - 1)/2]$;

(ii) For $x \otimes \lambda_I \in \Psi(H_*K) \tilde{\otimes} \Lambda^{s-1}$,

$$\delta(x \otimes \lambda_I) = (0) + \Big(x \otimes d\lambda_I + \sum_{i \geqslant 1} xSq^i \otimes \lambda_{i-1}\lambda_I\Big)$$

where degree $x = 2m + 1$, and

$$\delta(x \otimes \lambda_I) = \Big(\sigma x \otimes (d\lambda_m)\, \lambda_I + \sum_{i \geqslant 1} (\sigma x)\, Sq^i \otimes (\lambda_{i-1}\lambda_m + \lambda_{m-i}\lambda_{2i-1})\, \lambda_I\Big)$$

$$\oplus \Big(x \otimes \partial\lambda_I + \sum_{i=1}^{m} xSq^i \otimes \lambda_{i-1}\lambda_I\Big)$$

where degree $x = 2m + 2$.

The main result of Ref. [B, C] is that for nice space K,

$$E^2(K) \approx H^*(W^*(H(K)), \delta).$$

For the indexing, an element $x \otimes \lambda_{i_1} \cdots \lambda_{i_s} \in \tilde{H}_n(K) \tilde{\otimes} \Lambda^s$ will have filtration s and dimension

$$n - 1 + \Sigma^{i}{}_{j} \qquad \text{and} \qquad y \otimes \lambda_{i_1} \cdots \lambda_{i_{s-1}} \in \Psi_n(H_*K) \tilde{\otimes} \Lambda^{s-1}$$

will have filtration s and dimension $n - 2 + \sum i_j$. Thus δ raises filtration by one and decreases dimension by one. The dimension refers to dimension q in $\pi_q(GK)$. For a more homological indexing, see Section 9.

9. The Unstable Adams Spectral Sequence

The previous sections indicate that the E^2 term of the RLCSSS for suitably nice spaces is a sort of ext group. In this section we present a construction of Bousfield and Kan [B, K], which leads directly to a spectral sequence for each space K, converging to the homotopy groups of K, and for which E^2 is identifiable as the derived functors of hom(·, ·), as defined by Andre [An].

Throughout this section, assume $\pi_1(K) = 0$ and K has finite type (i.e., $\pi_q(K)$ is finitely generated for all q). Also assume that a prime p has been chosen, and H_*K will mean $H_*(K; Z_p)$.

THEOREM (9.1). *To each simplicial set K, there is associated a spectral sequence $E_r(K)$, for $2 \leqslant r \leqslant \infty$, which is natural in K, and for which*

(1) *$E_2(K)$ depends only on H_*K as a coalgebra over the Steenrod algebra A_p;*

(2) *The $E_r(K)$ converge to $E_\infty(K)$, which is a graded group associated with $\pi_*(K)$ modulo* non-p-torsion.

Remark (9.2). The proof of (2) will not be given here; it is proven by homological methods in Ref. [B, K], to which we refer the reader. By a roundabout way, it is equivalent to the convergence statements of (7.2) and Refs. [Cu2] and [R].

Remark (9.3). Of course corresponding statements hold (via the functors S and R) for each topological space X. In fact the techniques here used are not essentially simplicial. The interested reader can supply a topological proof of (1) by letting $Z_p(X)$ be the vector space generated by the points of X, with the topology which makes the vector space operations continuous, and the natural map $X \to Z_p(X)$ continuous and open. Then $Z_p(X)$ is a generalized Eilenberg–MacLane space, and

$$\pi_q(Z_p(X)) \approx H_q(X; Z_p),$$

which is the Dold–Thom theorem.

Before beginning the proof of (9.1), part (1), we make some observations which follow from Section 5.

Observation (9.4). Any simplicial Abelian group is a generalized Eilenberg–MacLane complex (GEM), i.e., a product of $K(\pi, n)$'s. If all of the π's which occur in a space which is a GEM are Z_p, we call the space (or any homotopy equivalent one) a GEM(p). Thus any simplicial vector space over Z_p is a GEM(p). In particular, each $Z_p(K)$ is a GEM(p). Also, as in the topological case,

$$\pi_*(Z_p(K)) \approx H_*(K; Z_p),$$

this time by (3.10). The functor we really want to use is not $Z_p(\cdot)$ itself, but

$$V_pK = Z_p(K)/Z_p(*),$$

which is also a GEM(p). Notice that for a connected K,

$$\pi_n(V_p(K)) \approx \tilde{H}_n(K; Z_p),$$

the *reduced* homology groups of K.

THE CONSTRUCTION (9.5). For each simplicial set K, let $i : K \to V_p K$ be the natural inclusion. Let

$$E(V_p K) \to V_p K$$

be the path space fibration, as in (3.16). Then take $q_1 : D_1 K \to K$ as the induced fibre map, so that there is a map of fibrations

$$\begin{tikzcd} D_1(K) \arrow[r] \arrow[d, "q_1"] & E(V_p K) \arrow[d] \\ K \arrow[r, "i"] & V_p(K) \end{tikzcd}$$

Notice that $D_1(\cdot)$ is a functor from simplicial sets to simplicial sets.

Inductively define $q_s : D_s(K) \to D_{s-1}(K)$ as the fibre map induced from the path fibration $E(D_{s-1}(V_p K)) \to D_{s-1}(V_p K)$ by the map $D_{s-1}(i) : D_{s-1}(K) \to D_{s-1}(V_p K)$, so that there is a map of fibrations

$$\begin{tikzcd} D_s(K) \arrow[r] \arrow[d, "q_s"] & E(D_{s-1}(V_p K)) \arrow[d] \\ D_{s-1}(K) \arrow[r, "D_{s-1}(i)"] & D_{s-1}(V_p K) \end{tikzcd}$$

In this way there occurs a sequence of fibre maps

$$\cdots \longrightarrow D_s(K) \xrightarrow{q_s} D_{s-1}(K) \longrightarrow \cdots \longrightarrow D_1(K) \xrightarrow{q_1} K.$$

Consider the homotopy exact couple which arises, and label the terms of the ensuing spectral sequence $E_r(K)$; more specifically,

$$E_1^{s,t}(K) = \pi_{t-s}(D_s(V_p K)),$$
$$E_r^{s,t}(K) = H_*(E_{r-1}, d^{r-1}).$$

We proceed with the proof of (9.1), which is to identify $E_2^{*,*}(K)$.

LEMMA (9.6). *Each $D_s(V_p K)$ is a $GEM(p)$.*

Proof. By induction on s. The maps i and j,

$$V_p K \xrightarrow{i} V_p(V_p K) \xrightarrow{j} V_p K,$$

where i is the inclusion (not the homomorphism), and j adds linear combinations of linear combinations, satisfy $j \circ i =$ identity. Since $D_s(V_pK)$ is constructed as the induced fibre map

$$\begin{tikzcd} D_s(V_pK) \arrow[r] \arrow[d] & E(D_{s-1}(V_pV_pK)) \arrow[d] \\ D_{s-1}(V_pK) \arrow[r] & D_{s-1}(V_pV_pK) \end{tikzcd}$$

it follows (from $D_{s-1}(j) \circ D_{s-1}(i) =$ identity) that

$$D_{s-1}(V_pV_pK) \simeq D_{s-1}(V_pK) \times \text{(another factor)}$$

and this other factor must also be a GEM(p). As

$$D_{s-1}(V_pK) \simeq \Omega \;\; \text{(this other factor)},$$

so $D_{s-1}(V_pK)$ is also a GEM(p).

LEMMA (9.7). *If M is a GEM(p), the spectral sequence collapses; that is,*

$$E_2^{s,t}(M) = 0, \qquad s > 0,$$
$$E_2^{0,*}(M) \approx \pi_*(M).$$

Proof. We may assume $M = V_pK$. Then the sequence of maps

$$\begin{tikzcd} D_s(V_pK) \arrow[d, "q_s"] & \\ D_{s-1}(V_pK) \arrow[r, "D_{s-1}(i)"] & D_{s-1}(V_pV_pK) \end{tikzcd}$$

yields a LES in homotopy (part of the homotopy exact couple). Since $D_{s-1}(i)_*$ is one-one, $(q_s)_*$ must be 0, which shows

$$E_2^{s,t}(V_pK) = 0, \qquad s > 0,$$
$$E_2^{0,t}(V_pK) \approx \pi_*(V_pK).$$

We shall use the notation

$$V_p^{(n)}K = \overbrace{V_p \cdots V_p}^{n}K.$$

Consider next that while there is *one* map $i : K \to V_pK$, there are *two* maps $V_pK \rightrightarrows V_pV_pK$, which are $V_p(i)$, and the natural inclusion, which we may still call i. There are $n + 1$ maps

$$V_p^{(n)}K \ \vdots\ V_p^{(n+1)}K$$

which are $f_0 = i$, $f_1 = V_p(i),\ldots, f_n = V_p^{(n)}(i)$.

Remark. The family

$$\{K, V_pK, V_p^{(2)}K,\ldots, V_p^{(n)}K,\ldots\}$$

is a cosimplicial object (where the objects are themselves simplicial sets). That is, the family can be regarded as a covariant functor from $\mathcal{O}$ to the category of simplicial sets. The use we make of this is that $\delta \circ \delta = 0$ in the following.

LEMMA (9.8). *For any K,*

$$E_2(K) \approx \ker \delta/\mathrm{im}\ \delta$$

of the sequence

$$(*) \qquad \pi_*(V_pK) \xrightarrow{\delta^1} \pi_*(V_pV_pK) \xrightarrow{\delta^2} \pi_*(V_p^{(3)}K) \xrightarrow{\delta^3} \cdots,$$

where $\delta^n = \sum_{i=0}^{n} (-1)^i (f_i)_*$.

Proof. Consider the double chain complex

$$\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow d^1 & & \uparrow d^1 \\
E_1^{1,*}(V_pK) & \xrightarrow{\delta^1} & E_1^{1,*}(V_pV_pK) \xrightarrow{\delta^2} \cdots \\
\uparrow d^1 & & \uparrow d^1 \\
E_1^{0,*}(V_pK) & \xrightarrow{\delta^1} & E_1^{0,*}(V_pV_pK) \xrightarrow{\delta^2} \cdots
\end{array}$$

First take vertical homology, when all that remains is $\ker d^1 \subset$ lowest row; then the horizontal homology must yield

$$H_* \ (\textit{total complex}) \approx \ker \delta/\mathrm{im}\ \delta \ \textit{of}\ (*).$$

Next take homology in the other order. The sequence

$$K \xrightarrow{f_0} V_pK \xrightarrow{f_0,f_1} V_pV_pK \xrightarrow{f_0,f_1,f_2} \cdots$$

yields, after application of the functor $V_p(\cdot)$, a sequence of homomorphisms

$$V_pK \xrightarrow{f_1} V_pV_p(K) \xrightarrow{f_1,f_2} V_pV_pV_pK \xrightarrow{f_1,f_2,f_3} \cdots$$

which is canonically acyclic. Now apply the functor $\pi_{t-s}D_s(\cdot)$ which produces the double complex above. Since $\pi_{t-s}D_s(\cdot)$ preserves the acyclicity, taking homology of the double complex horizontally first, produces 0, except for the left column; there we are left with

$$\begin{array}{c}
\vdots \quad \approx \quad \vdots \\
\uparrow d^1 \\
E_1^{1,*}(K) \approx \ker \delta^1 \\
\uparrow d^1 \\
E_1^{0,*}(K) \approx \ker \delta^1
\end{array}$$

Thus also, $E_2(K) \approx H_*$ (total complex), and the lemma (3) follows.

Finally, to identify $E_2(K)$ as a derived functor, consider

$$\begin{array}{ccccccc}
 & & \overbrace{\hphantom{H_*(V_pK) \longrightarrow H_*(V_pV_pK) \longrightarrow}}^{\text{resol.}} & & & & \cdots \\
H_*(K) & \longrightarrow & H_*(V_pK) & \longrightarrow & H_*(V_pV_pK) & \longrightarrow & \cdots \\
\wr & & \wr & & \wr & & \\
\pi_*(V_pK) & \longrightarrow & \pi_*(V_pV_pK) & \longrightarrow & \pi\varphi(V_p^{(3)}K) & \longrightarrow & \cdots
\end{array}$$

We consider $H_*(K)$, $H_*(V_pK)$,..., etc., to be in the category $\mathscr{C}A$ of unstable coalgebras over the Steenrod algebra A_p. That is, each $C \in \mathscr{C}A$ is simultaneously a connected cocommutative coalgebra over Z_p, and a right module over A_p, where the structures are compatible concerning the Cartan formula and the (dual of the) p-th power map. Even though $\mathscr{C}A$ is not an Abelian category, we may still take derived functors (of hom$(\cdot, \cdot)$ for example), as in Andre [An]. The part of the sequence indicated by resol. forms a resolution of $H_*(K)$ by $H_*(\mathrm{GEM}(p))$'s, i.e., by models. Next apply the functor

$$\bigoplus_t \hom_{\mathscr{C}A}(H_*(S^t), \cdot)$$

to the sequence, with the observation that, for each $s > 1$,

$$\hom_{\mathscr{C}A}(H_*(S^t), H_*(V_p^{(s)}K)) \approx \pi_*(V_p^{(s)}K).$$

Thus

$$E_2^{s,t} \approx \operatorname{ext}_{\mathscr{C}A}^{s,t}(Z_p, H_*(K))$$

considered as derived functors of $\hom_{\mathscr{C}A}(\cdot, \cdot)$ in the category $\mathscr{C}A$, which concludes part (1) of (9.1).

10. The Cobar Construction

Let X be a simplicial set, and take GX. Recall from (3.18) that

(10.1a) $$\pi_*(GX/\Gamma_2 GX) \approx H_{*+1}(X).$$

On the other hand $Z(GX)$ is a simplicial free Abelian group with

(10.1b) $$\pi_*(Z(GX) \approx H_*(GX).$$

The cobar construction of Adams [A2] provides a spectral sequence relating these, but the approach here is that of Ref. [B, C]. The results are more conveniently stated for homology with Z_p coefficients where p is a prime; from now on let $H_*(\ )$ stand for $H_*(\ ; Z_p)$. Thus

$$G/\Gamma_2^{(p)}G \approx (G/\Gamma_2 G) \otimes Z_p$$

and the analogues of (9.1) are

(10.2a) $$\pi_*(GX/\Gamma_2^{(p)}GX) \approx H_{*+1}(X),$$

(10.2b) $$\pi_*(Z_p(GX)) \approx H_*(GX).$$

For any group G, let $Z_p(G)$ be filtered by powers of the augmentation ideal I

$$Z_p(G) = I^0 \supset I^1 \supset \cdots \supset I^n \supset \cdots,$$

where $I = \{\sum a_i g_i : \sum a_i = 0\} \subset Z_p(G)$. Then $\bigoplus_{n \geqslant 0} (I^n/I^{n+1})$ becomes an associative algebra.

PROPOSITION (10.3). *There is a natural homomorphism*

$$\psi : T(G/\Gamma_2^{(p)}G) \to \bigoplus_{n \geqslant 0} (I^n/I^{n+1})$$

where $T(\ )$ is the tensor algebra. If G is free, ψ is an isomorphism.

Proof. Let $\theta : G \to I$ be the *function* $\theta(g) = g - e$ for all $g \in G$. Then θ defines a homomorphism,

$$\theta : G/\Gamma_2^{(p)}G \to I/I^2.$$

Let ψ be the unique extension of θ to the tensor algebra. The fact that if G is a free group, ψ is an isomorphism follows from the remarks in Ref. [Mac, p. 122].

THEOREM (10.4). *Let X be a connected and simply connected simplicial set. Let $\bar{E}^i(X)$ be the spectral sequence arising from $Z_p(GX)$ filtered by the powers of I. Then*

(1) *Each $(\bar{E}^i(X), d^i)$ is an associative differential algebra, with d^i a derivation*;

(2) *The $\bar{E}^i(X)$ converge to $\bar{E}^\infty(X)$, which is the graded group associated with a filtration on $H_*(GX)$;*

(3) $\bar{E}^1(X) \approx T(H_*(X))$;

(4) *Let $\Delta : X \to X \times X$ be the diagonal and suppose*

$$\Delta_* : H_*(X) \to H_*(X) \otimes H_*(X)$$

is given by $\Delta_*(x) = \sum x_k' \otimes x_k''$.
Then

$$d^1(x) = \sum + \bar{x}_k' \otimes \bar{x}_k'',$$

where $y \in H_q(X)$ and $\bar{y} \in \pi_{q-1}(GX/\Gamma_2^{(p)}(GX))$ denote classes corresponding under the isomorphism (10.2a).

There is not much difficulty here; for details see Ref. [B, C]. (2) uses a technique of Ref. [C2], but simpler; (4) is a straightforward calculation.

We make more use of the function $\theta : G \to I$.

PROPOSITION (10.5). (1) *$\theta(\Gamma_n G) \subset I^n$ for each $n \geqslant 1$.*

(2) *$\theta : (\Gamma_n G/\Gamma_{n+1}G) \to I^n/I^{n+1}$ is a homomorphism.*

(3) *There is a natural commutative diagram*

$$\begin{tikzcd} L(G/\Gamma_2 G) \arrow[r, "i"] \arrow[d] & T(G/\Gamma_2 G) \arrow[d] \\ \bigoplus_{n\geqslant 1} (\Gamma_n G/\Gamma_{n+1} G) \arrow[r, "\theta"] & \bigoplus_{n\geqslant 0} (I^n/I^{n+1}) \end{tikzcd}$$

where $i : L(\cdot) \to T(\cdot)$ is the inclusion of the free RLA in the TA; the vertical maps are the natural extensions, which are isomorphisms if G is a free group.

We again consider the mod-p RLCS spectral sequence $E^i(X)$, not speeded-up, not reindexed.

PROPOSITION (10.6). *The function $\theta : G \to Z_p(G)$ induces maps of spectral sequences*

$$\theta^i : E^i(X) \to \bar{E}^i(X)$$

with

(1) $\theta^i[x, y] = [\theta^i(x), \theta^i(y)]$;

(2) $\theta^i d^i = \bar{d}^i \theta^i$;

(3) $\theta^\infty : E^\infty(X) \to \bar{E}^\infty(X)$ *is induced by the Hurewicz homomorphism* $h : \pi_*(GX) \to H_*(GX)$;

(4) *For* $x \otimes \lambda_I \in E^1(X)$,

$$\theta^1(x \otimes \lambda_I) = \begin{cases} x & \textit{for} \quad \lambda_I = 1, \\ 0 & \textit{for} \quad \lambda_I \neq 1. \end{cases}$$

11. SOME APPLICATIONS

We describe the mod-2 RLCS spectral sequence for some X, e.g., $X =$ sphere, unitary group, etc. Also, there are homomorphisms between the various spectral sequences, like the J homomorphism and the maps of the EHP sequence. For H there is a simplicial formula like James' combinatorial map. Using the description of $E^2(S)$ as $H_*(\Lambda)$, we give a proof of Adams' result on the nonexistence of elements of Hopf invariant one. We conclude with Kan's simplicial formula for the Whitehead product.

The references for Section 11 are [B], [B, K], and [C3].

$E^2(S^n)$ (11.1). There is a short exact sequence of differential groups,

$$0 \longrightarrow \Lambda(n) \xrightarrow{i} \Lambda(n+1) \xrightarrow{h} \Lambda(2n+1) \longrightarrow 0,$$

where i is the inclusion and h is the map, given on the allowable basis by

$$h(\lambda_i \lambda_I) = \begin{cases} \lambda_I & \text{for } i = n, \\ 0 & \text{for } i < n. \end{cases}$$

Deriving this, we obtain the LES in homology,

$$\longrightarrow E^2(S^n) \xrightarrow{\sigma} E^2(S^{n+1}) \xrightarrow{h} E^2(S^{2n+1}) \xrightarrow{\partial} \cdots.$$

Later we will show that h is induced by the Hopf invariant H.

From this sequence, it is straightforward to calculate $E^2(S^n)$ inductively. Table I describes $E^2(S^n)$ to dimension $n + 16$. The basis of allowable monomials λ_I is ordered lexicographically from the left, and hence also becomes ordered. Homology classes in Λ are listed by the leading term of a minimal representative. $E^2(S^n)$ has for basis those $i_1 \cdots \mid j$ for which $i_1 < n \leqslant j$ (or no j, meaning in $E^2(S)$). The notation $i_1 \cdots \mid j$ means that there is a cycle created in $E^2(S^{i_1+1})$ which becomes homologous to something smaller (or to 0) in $E^2(S^{j+1})$. For example, $71 \mid 9$ is created on S^8, and becomes homologous to 53 on S^{10}, and this class then persists to the stable sphere. The higher differentials which affect this part of the table are $d^2(15) = 653$, $d^3(14, 1) = 51233$, $d^3(13, 1, 1) = 344111$, and $124333 = d^2(8333)$.

$E^2(SU(n))$ (11.2). The results of Section 8 can be used to calculate $E^2(SU(n))$, where $(SU(n))$ is the special unitary group. Recall that $H_*(SU(n))$ is an exterior algebra generated by classes $e_{2i-1} \in H_{2i-1}(SU(n))$ for $2 \leqslant i \leqslant n$. The speeded-up RLCS spectral sequence, *simplified by cancelling some cycles that are obviously boundaries*, shows that $E^2(SU(n))$ will be the homology of

$$\bigoplus_{i=2}^{i=n} e_{2i-1} \otimes \Lambda(2i-1),$$

$$d^1(e_{2i-1} \otimes \lambda_I) = \Sigma e_{2i-1} Sq^j \otimes \lambda_{j-1}\lambda_I + e_{2i-1} \otimes d^1\lambda_I.$$

We still call this simplified differential group $E^1(SU(n))$.

TABLE I

<table>
<tr><td></td><td>↑</td><td>↑ | 2</td><td></td><td>↑ | 4</td><td></td><td>↑ | 6</td><td></td><td>↑ | 8</td><td></td><td>↑ | 10</td><td>124111</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>↑ | 10</td><td>34111 | 4</td></tr>
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<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>↑ | 8</td><td></td><td>6111 | 8</td><td>3511 | 4</td></tr>
<tr><td></td><td>↑</td><td>↑ | 2</td><td></td><td>↑ | 4</td><td></td><td>↑ | 6</td><td></td><td>4111</td><td>5111 | 6</td><td>12333</td><td>22333 | 4</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>611 | 8</td><td>↑ | 10</td><td>721 | 8</td></tr>
<tr><td></td><td></td><td></td><td></td><td>↑ | 4</td><td>211 | 4</td><td>↑ | 6</td><td></td><td>| 8</td><td>521 | 6</td><td>711 | 9</td><td>433 | 10</td></tr>
<tr><td></td><td>↑</td><td>↑ | 2</td><td></td><td>111</td><td>121 | 2</td><td>311 | 5</td><td>321 | 4</td><td>511</td><td>233</td><td>333</td><td>361 | 4</td></tr>
<tr><td></td><td></td><td></td><td></td><td>30 | 4</td><td></td><td></td><td></td><td>70 | 8</td><td>71 | 9</td><td></td><td></td></tr>
<tr><td></td><td>↑</td><td>10 | 2</td><td>11</td><td>21</td><td>31 | 5</td><td>50 | 6</td><td>33</td><td>61</td><td>53</td><td>90 | 10</td><td>73 | 11</td></tr>
<tr><td>σ_n^0</td><td></td><td>1</td><td></td><td>3</td><td></td><td></td><td></td><td>7</td><td></td><td></td><td></td></tr>
<tr><td>σ_n</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

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<tr><td>↑ | 12</td><td></td><td>↑ | 14</td><td>121124111 | 2</td><td>↑ | 16</td><td></td></tr>
<tr><td></td><td></td><td>↑ | 14</td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td>21124111 | 5</td><td></td><td>↑ | 16</td><td></td></tr>
<tr><td>↑ | 12</td><td>11124111 | 2</td><td>12124111 | 2</td><td>31124111 | 4</td><td>41124111</td><td>51124111 | 6</td></tr>
<tr><td></td><td></td><td>↑ | 14</td><td></td><td></td><td>6124111 | 8</td></tr>
<tr><td>↑ | 12</td><td>2124111 | 4</td><td>3124111 | 5</td><td></td><td>↑ | 16</td><td>5224111 | 6</td></tr>
<tr><td>1124111</td><td>1224111 | 2</td><td>1211233 | 2</td><td>3224111 | 4</td><td>5124111</td><td>2344111</td></tr>
<tr><td>↑ | 12</td><td>324111 | 6</td><td></td><td></td><td>↑ | 16</td><td>724111 | 10</td></tr>
<tr><td>224111</td><td>211233 | 4</td><td>↑ | 14</td><td></td><td>624111</td><td>544111 | 8</td></tr>
<tr><td>111233 | 2</td><td>121233 | 2</td><td>311233 | 4</td><td>344111</td><td>123333 | 2</td><td>124333</td></tr>
<tr><td>↑ | 13</td><td>54111 | 6</td><td></td><td></td><td>↑ | 16</td><td></td></tr>
<tr><td>44111</td><td>36111 | 4</td><td></td><td>74111 | 8</td><td>84111</td><td>94111 | 10</td></tr>
<tr><td>21233 | 4</td><td>31233 | 5</td><td>↑ | 14</td><td>51233</td><td>61233 | 8</td><td>71233 | 9</td></tr>
<tr><td>12233 | 2</td><td>12333 | 2</td><td>32233 | 4</td><td>23333 | 4</td><td>24333</td><td>34333 | 4</td></tr>
<tr><td>↑ | 12</td><td></td><td>↑ | 14</td><td>7511 | 8</td><td>↑ | 16</td><td></td></tr>
<tr><td>3611 | 4</td><td>9111 | 10</td><td>10,111 | 12</td><td>6233</td><td>12,111</td><td>13,111 | 14</td></tr>
<tr><td>3233 | 6</td><td>5511 | 6</td><td>4333 | 8</td><td>5333 | 9</td><td>7233 | 10</td><td>9511 | 10</td></tr>
<tr><td>2333 | 4</td><td>3333 | 5</td><td>3433 | 4</td><td>3533 | 5</td><td>6333 | 8</td><td>7333 | 9</td></tr>
<tr><td></td><td></td><td></td><td>11,111 | 12</td><td></td><td></td></tr>
<tr><td>↑ | 12</td><td>10,11 | 12</td><td>↑ | 14</td><td>11,21 | 12</td><td>↑ | 16</td><td>14,11 | 16</td></tr>
<tr><td>533 | 9</td><td>921 | 10</td><td>11,11 | 13</td><td>761 | 8</td><td>13,1,1</td><td>13,2,1 | 14</td></tr>
<tr><td>353 | 5</td><td>561 | 6</td><td>733 | 11</td><td>653</td><td>753 | 9</td><td>961 | 10</td></tr>
<tr><td></td><td></td><td></td><td></td><td>15,0 | 16</td><td>15,1 | 17</td></tr>
<tr><td>11,0 | 12</td><td>11,1 | 13</td><td>13,0 | 14</td><td>77</td><td>14,1</td><td>13,3</td></tr>
<tr><td></td><td></td><td></td><td></td><td>15</td><td></td></tr>
</table>

The fibration

$$SU(n-1) \to SU(n) \to S^{2n-1}$$

gives a short exact sequence of simplified E^1 terms,

$$0 \to E^1(SU(n-1)) \to E^1(SU(n)) \to E^1(S^{2n-1}) \to 0,$$

which in turn give a long exact sequence

$$\cdots \to E^2(SU(n-1)) \to E^2(SU(n)) \to E^2(S^{2n-1}) \to \cdots.$$

This provides also a straightforward inductive method for computing $E^2(SU(n))$. In Table II we give some results of this computation (by hand). Similar calculations can also be made for other classical groups, Stiefel manifolds, etc. $E^2(SU(n))$ has for basis those $e_k \otimes \lambda_I = ki_1 \cdots \mid j$ for which $k < 2n < j$, or no j, meaning in $E^2(SU)$. The relevant differentials are $d^2e_7 = e_3 21$, $d^2e_5 3 = e_3 211$, $d^2e_5 31 = e_3 2111$, $d^2e_{13}0 = e_9 111$, $d^2e_{13}00 = e_5 4111$, and $d^2e_{13}000 = e_3 24111$.

TABLE II

	↑	↑		↑		↑		↑	324111
									54111
	↑	↑		↑	32111 | 9	90000		↑	31233 | 5
				↑			7111 | 9	11,000	
	↑	↑		3211	5111 | 9		5311 | 11	3233 | 9	9111
		↑							
	↑	311 | 5	321	↑	521 | 9	531	721 | 11	533 | 1	
					71 | 9				
30	31 | 5	50		70	53		73 | 11		11,1 | 13
e_3				e_7					

J HOMOMORPHISM (11.3). The J homomorphism of G. W. Whitehead,

$$J : \pi_r(SU(n)) \to \pi_{r+2n}(S^{2n}),$$

is the composite (in the homotopy groups of)

$$SU(n) \xrightarrow{\sigma^{2n}} SU(n) \wedge S^{2n} \xrightarrow{f} S^{2n},$$

where σ^{2n} is the $2n$-fold suspension, and f is the action of $SU(n)$ on S^{2n}.

We define $J^i : E^i(SU) \to E^i(S^{2n})$ induced by J as follows. Since J induces 0 in homology, there is a lifting $\bar{f}$

$$\begin{tikzcd} & F \arrow[d] \\ SU(2n) \wedge S^{2n} \arrow[ur, dashed, "\bar{f}"] \arrow[r, "f"] & S^{2n} \end{tikzcd}$$

where $F \to S^{2n} \to K(Z_2, 2n)$ is the natural fibration. Thus we have

$$\begin{tikzcd} & \Gamma_2 GS^{2n} \arrow[d] & \simeq & \Omega F \arrow[d] \\ G(SU(n) \wedge S^{2n}) \arrow[ur, "\bar{f}"] \arrow[r, "f"] & GS^{2n} \arrow[d] & \simeq & \Omega S^{2n} \arrow[d] \\ & GS^{2n}/\Gamma_2 GS^{2n} & \simeq & \Omega K(Z_2, 2n) \end{tikzcd}$$

Then also

$$\Gamma_r G(SU(n) \wedge S^{2n}) \xrightarrow{\bar{f}} \Gamma_r \Gamma_2 GS^{2n} \longrightarrow \Gamma_{2r} GS^{2n}$$

and define $J^i : E^i(SU(n)) \to E^i(S^{2n})$ by $J^i = g_* \circ \bar{f}_* \circ \sigma^{2n}$, where g_* is induced from

$$(\Gamma_2 i/\Gamma_2 i + 1)(\Gamma_2 GS^{2n}) \xrightarrow{g} (\Gamma_2 i + 1/\Gamma_2 i + 2)(GS^{2n}).$$

Thus J^i raises filtration by one, and takes homotopy dimension in $SU(n)$ to stem dimension in S^{2n}.

Claim. *$J^1 : \check{E}^1(SU(n)) \to \check{E}^1(S^{2n})$ is the map*

$$J^1(e_{2i-1} \otimes \lambda_I) = \lambda_{2i-1}\lambda_I .$$

To see this, use induction on i, first skipping the integers $2i - 1$ which are $= 2^{\text{power}} - 1$.

It is not hard to show that

(1) $E^2(SU(n))_{q=8k-1, s=4k-1}$ contains a single nonzero element, say x, which persists to E^∞.

(2) $E^2(S^{2n})_{q=2n+8k-1, s=4k}$ contains a single nonzero element, say y.

(3) For dimensions $q = 8k - 1 = 2^{\text{power}} - 1$, $J^2(x) = y$.

Hence the element in $\pi_q(S)$ corresponding to y is in the image of J. By Adams' $J(x) - IV$, in particular the e-invariant arguments, this is sufficient to settle the image of J in these dimensions (i.e., order im $J =$ Adams–Kervaire–Milnor number).

For dimensions $q \neq 2^{\text{power}} - 1$, this method does not work; for example, $q = 23$. This technique (using the map of Adams spectral sequence) is similar to Ref. [Mah], and can be similarly extended to more complicated numbers q. But as the number of 1's in the dyadic expansion of $q + 1$ increases, the filtration change between $E(SU)$ and E (sphere) becomes too great to handle computationally.

THE HOPF INVARIANT (11.4). Let K be a simplicial set, and define

$$h : FK \to F(K \wedge K)$$

as follows. First, let $w : FK \to F(K \vee K)$ be the homomorphism $w(x) = x' \cdot x''$ for each generator x in FK, where x' and x'' are the corresponding generators in $F(K \vee K)$. From (4.21) there is a decomposition

$$\prod_i FK_i \xrightarrow{\cong} F(K \vee K),$$

where the K_i vary over all basic complexes in two variables, each a copy of K. Let p be the projection (*not* a homomorphism) of a word in $F(K \vee K)$ onto the part of it in $F(K \wedge K)$, as in Section 4. Then h is to be the composition $h = p \circ w$. The map h can be taken (via a choice of basic complexes) to have the form: If

$$x = x_1^{\epsilon_1} \cdots x_m^{\epsilon_m}, \qquad \epsilon_i = \pm 1,$$

$$h(x) = \prod_{(i,j)} (x_i' \wedge x_j'')^{\epsilon_i + \epsilon_j},$$

where the product is taken over those (i, j) such that

(1) If

$$\epsilon_i = +1 \qquad \text{for all} \quad j > 1,$$

$$\epsilon_i = -1 \qquad \text{for all} \quad j \geqslant 1,$$

(2) The order is $(i, j) < (i', j')$ if $i < i'$; or $i = i'$, $\epsilon_i = +1$, $j > j'$; or $i = i'$, $\epsilon_i = -1$, $j < j'$.

The map h is the simplicial analogue of James' combinatorial map [J].

Even though h is not a homomorphism—and the resolutions of FK and $F(K \wedge K)$ involve the group structures—we can still see that h induces maps of the spectral sequences as follows (observation of S. Priddy). For each basic complex K_i, there is a homomorphism

$$f_i : FK_i \to F(K \vee K)$$

which defines maps of the spectral sequences (for each $r \geqslant 1$)

$$f_i^r : E^r(SK_i) \to E^r(S(K \vee K)).$$

Then there are maps

$$\bigoplus_i E^r(SK_i) \xrightarrow{(\oplus f_i^r)} E^r(S(K \vee K)). \tag{11.5}$$

Using Section 8, it is easily seen that for $r = 1$, the map in (11.5) is an isomorphism, and hence it is an isomorphism for all $r \geqslant 1$. To each f_i^r, there is a projection back onto the corresponding factor. The maps

$$h : E^2(SK) \to E^2(S(K \wedge K))$$

we seek are the composition of the maps induced from the homomorphism $w : FK \to F(K \vee K)$, and the projection onto the factor $F(K \wedge K)$.

For $K = S^n$, we have $FK \approx GS^{n+1}$ and $F(K \wedge K) \simeq GS^{2n+1}$. In this case, a little computation shows that

$$h(\lambda_i \lambda_I) = \begin{cases} \lambda_I & \text{for} \quad i = n, \\ 0 & \text{for} \quad i < n. \end{cases}$$

EXAMPLE (11.6). If the element $\lambda_n \in E^2(S^{n+1})$ survives to E^∞, it represents an element of Hopf invariant one; that is, H (this element) would be an odd multiple of the generator of $\pi_{2n+1}(S^{2n+1})$. Adams shows that this happens only for $n = 1, 3, 7$. As a variant of Adams' proof, we offer a proof using the J homorphism.

First, it is evident from the differential in Λ that λ_n can survive to $E^2(S)$ only if $n = 2^i - 1$. We shall show that

$$\begin{aligned} d^2 \lambda_{2^i - 1} &= \lambda_{2^{i-1}-1} \lambda_{2^{i-1}-2} \lambda_1 + \text{lower terms} \\ &= (\lambda_{2^{i-1}-1})^2 \lambda_0 \quad \text{in} \quad E^2(S) \\ &\neq 0 \quad \text{in} \quad E^2(S) \quad \text{for} \quad i \geqslant 4. \end{aligned}$$

The lower two lines are straightforward algebraic properties of Λ. For the upper line, it would suffice to show in $E^2(SU)$ that

$$d^2 e_{2^i-1} = e_{2^{i-1}-1}\lambda_{2^{i-1}-2}\lambda_1 + \text{lower terms.}$$

Consider the spectral sequence for $BU = \Omega SU$. In $E^1(BU)$,

$$d^i b_{2^i-2} = b_{2^i-4}\lambda_1 + \text{lower terms,}$$

where the $b_{2k} \in H_{2k}(BU)$ are dual to the Chern classes $c_{2k} \in H^{2k}(BU)$. Then under the homology suspension, $\sigma(b_{2^i-2}) = e_{2^i-1}$ and $\sigma(b_{2^i-4}) = 0$ in filtration 0, but "reappears" in filtration 1, with

$$\bar{\sigma}(b_{2^i-4}) = e_{2^{i-1}-1}\lambda_{2^{i-1}-2} + \text{lower terms.}$$

Note that $b_{2^i-4} = (b_{2^{i-1}-2})^2 + \text{lower terms in } H_*(BU)$. In $\Omega G(SU)$, there is a simplex, say $\bar{x}$ (given by a universal simplicial formula which is a product over shuffles), where

$$\bar{x} \xrightarrow{proj.} (b_{2^{i-1}-2})^2 \in \pi_*(\Omega G(SU)/\Gamma_2\Omega G(SU)) \approx H_*(BU),$$

$$\bar{x} \xrightarrow{proj.} (e_{2^{i-1}-1})\,\lambda_{2^{i-1}-2} \in \pi_*(\Gamma_2 G(SU)/\Gamma_3 G(SU)).$$

Then, since $d^1\bar{\sigma} = \sigma d^2$,

$$d^2(e_{2^i-1}) = e_{2^{i-1}-1}\lambda_{2^{i-1}-2}\lambda_1 + \text{lower terms.}$$

THE EHP SEQUENCE (11.7). Here is the EHP sequence of G. W. Whitehead, formulated more like James [J] or Barratt [B].

Make the map $h : FX \to F(X \wedge X)$ into a fibration with fibre Y. Then the LES in π,

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_*(Y) \arrow[r] & \pi_*(FX) \arrow[r, "H"] & \pi_*(F(X \wedge X)) \arrow[r] \arrow[dr, "P"] & \pi_{*-1}(Y) \arrow[r] & \cdots \\
& \pi_*(X) \arrow[u] \arrow[ur, "E"] & & & \pi_{*-1}(X) \arrow[u] &
\end{tikzcd}$$

is called the EHP sequence, especially when P exists; for example, in the stable range, where $\pi_*(X) \to \pi_*(Y)$ is an isomorphism.

For example, if $X = S^n$, $\pi_*(S^n) \to \pi_*(Y)$ is an isomorphism always on the two-component. In a stable range, P is the Whitehead product $P(\ ) = [i_n ,\ ]$.

PROPOSITION (11.8). *The EHP sequence for* $X = S^n$ *is induced at the* E^2 *level by the LES of* (10.1).

COROLLARY (11.9). *Considering the maps E, H, and P of* (10.7),

(1) *If* $x \in \pi_*(S^{n+1})$ *has* AF $\geqslant s$, *then* $H(X) \in \pi_*(S^{2n+1})$ *has* AF $\geqslant s - 1$;

(2) *If* $y \in \pi_*(S^{2n+1})$ *has* AF $\geqslant s$, *then* $P(y) \in \pi_*(S^n)$ *has* AF $\geqslant s + 2$;

(3) *If* $z \in \pi_*(S^n)$ *has* AF $\geqslant s$, *then* $E(z) \in \pi_*(S^{n+1})$ *has* AF $\geqslant s$.

Comments. Here AF means Adams filtration: $x \in \pi_*(GX)$ has AF $\geqslant s$ says that $x \in \operatorname{im} \pi_*(\Gamma_{2^s}) \to \pi_*(G)$. Then (1) follows from the computation for h above. (3) is an evident property of suspension. (2) can be shown, for a stable range at least, by a formula for the Whitehead product, given in the next section.

This corollary can be stated more specifically in the following manner. Suppose $x \in \pi_*(GX)$ is nonzero, and that x can be pulled back to Γ_{2^s} but no further, and projects to $c \neq 0$, as in the diagram

$$\begin{array}{l} x \in \pi_*(\Gamma_{2^s}GX) \to \cdots \to \pi_*(GX) \\ \downarrow \qquad\quad \downarrow \\ c \in \pi_*(\Gamma_{2^s}GX/\Gamma_{2^{s+1}}GX) \end{array}$$

Let us say that x is detected by c. (Warning ★ different x may be detected by the same c). Then for the elements in $\pi_*(S^n)$, detected by elements in $E^2(S^n) = H_*(\Lambda(n-1))$, we have

(1′) If $x \in \pi_*(S^{n+1})$ is detected by $c = \lambda_i\lambda_I$ + lower terms, then $H(x)$ is detected by λ_I if $i = n$ and λ_I persists to E^∞; otherwise $H(x)$ has AF > length I.

(2′) and (3′) similar.

Put in yet another form, if $y \in \pi_*(S^{2n+1})$ with $\mathrm{AF}(y) = s$, and $P(y) = 0$, then there is an $x \in \pi_*(S^{n+1})$ with $H(x) = y$, and $\mathrm{AF}(x) \geqslant s + 1$. If y is detected by λ_I + lower terms, then x is detected by $\lambda_n\lambda_I$ + lower terms, or by a cycle of AF $\leqslant s$.

We are thus led to the following.

METHOD (11.10). *Let* $y \in \pi_q(S^{2n+1})$ *be detected by* $c = \lambda_I$ + *lower terms in* $E^\infty(S^{2n+1})$. *Suppose* $\lambda_n \cdot c \sim 0$ *and that for filtration* $\leqslant s$, *dimension* q, $E^\infty(S^{n+1}) = 0$. *Then* $P(y) \neq 0$.

Many of the known cases of nonzero Whitehead products are obtained by this method.

WHITEHEAD AND SAMELSON PRODUCTS (11.11). First suppose G is a simplicial group, and we want to describe the Samelson product $\langle x, y\rangle \in \pi_{p+q}(G)$ for $x \in \pi_p(G)$, $y \in \pi_q(G)$. We will need to take a product over some shuffles, and the order matters.

Let the set of all $\{(a_1, \ldots, a_p)\} : a_1 < \cdots < a_p$ be ordered (antilexicographically) as follows. Inductively in p, put $(a_1, \ldots, a_p) < (a_1', \ldots, a_p')$ if

(1) $a_p < a_p'$

or

(2) $a_p = a_p'$ and

$$\begin{aligned} (a_1, \ldots, a_{p-1}) &< (a_1', \ldots, a_{p-1}') \quad \text{if} \quad a_p \text{ odd},\\ (a_1, \ldots, a_{p-1}) &> (a_1', \ldots, a_{p-1}') \quad \text{if} \quad a_p \text{ even}. \end{aligned}$$

For $x \in G_p$, $y \in G_q$, put

$$\langle x, y\rangle = \prod_{(a;b)} (s_b x, s_a y)^{\pm},$$

where the product is taken over all (p, q) shuffles $(a; b)$. The sign $\pm$ is the sign of the shuffle; the order is antilexicographical in $a = (a_1, \ldots, a_p)$.

In the verification that $\langle\ ,\ \rangle$ gives (to within $\pm$) the Samelson product, the simplicial identities enter in the form

$$d_i s_a = \begin{cases} s_{a'} d_{i'} & \text{if neither } i \text{ nor } i+1 \in a,\\ s_{a''}, & \text{if } i \text{ or } i+1 \in a. \end{cases}$$

Thus if all $d_i x = e$, $d_i y = e$, then all $d_i\langle x, y\rangle = e$, and $\langle\ ,\ \rangle$ becomes well-defined on homotopy classes.

The Whitehead product of $u \in \pi_{p+1}(X)$, $v \in \pi_{q+1}(X)$ is the element in $\pi_{p+q+1}(X)$

$$[u, v] = (-1)^p\, \partial^{-1}\langle \partial u, \partial v\rangle,$$

where $\partial : \pi_*(X) \to \pi_{*-1}(GX)$ is the isomorphism of (3.18).

The connection between (potential) elements λ_n of Hopf invariant one and the Whitehead product $[i_n, i_n]$ is the following. Let λ_n be lifted to $\bar{\lambda}_n \in \Gamma_2 FS^n$ by

$$\bar{\lambda}_n = \prod_{(a;b)} (s_b i_n, s_a i_n),$$

where the product is taken over all (n, n) shuffles $(a; b) = (a_1, \ldots, a_n; b_1, \ldots, b_n)$ with $b_n = 2n - 1$, and the order is antilexicographic in a. Then

$$d_i \bar{\lambda}_n = \begin{cases} \pm[i_n, i_n] & \text{for } i = 2n - 1, \\ e & \text{for } i \neq 2n - 1. \end{cases}$$

Thus λ_n is a permanent cycle $\Leftrightarrow$ $[i_n, i_n] = 0$.

12. The Simplicial Extension Theorem

The purpose of this section is to prove the following, relative form of the simplicial extension theorem.

THEOREM (12.1). *Let K be a simplicial set, with simplicial subset $A \subset K$, and let L be a Kan simplicial set. Suppose $p : RK \to RL$ is a continuous map and $f : A \to L$ is a simplicial map with $Rf = p$ on RA. Then there is a simplicial map $g : K \to L$ with $g = f$ on A and $Rg \simeq p : RK \to RL$* (rel RA).

We will follow a proof of Barratt [B] and Kan (lecture notes). The method is to use a simplicial approximation theorem, and for this, some careful subdividing. This theorem, although common knowledge, is not well-represented in the literature.

POLYHEDRA (12.2). A simplicial set K is called *polyhedral* if there is a partially ordered set $(V, \leqslant)$, with

$$K_n \subset \{(v_0, \ldots, v_n) : v_0 \leqslant \cdots \leqslant v_n\},$$

$$d_i(v_0, \ldots, v_n) = (v_0, \ldots, \hat{v}_i, \ldots, v_n),$$

$$s_i(v_0, \ldots, v_n) = (v_0, \ldots, v_i, v_i, \ldots, v_n),$$

i.e., K is a subsimplicial set of the simplicial set determined by $(V, \leqslant)$. Observe that if K is polyhedral, then each simplex $x \in K_n$ is uniquely determined by its vertices $v_i(x) = d_0 \cdots \hat{d}_i \cdots d_n x$. Conversely, if K is a simplicial set with the property that each simplex is determined by its vertices, then K is polyhedral. If K is polyhedral and $x \in K_n$ is nondegenerate, then the vertices of x are all distinct, and each face $d_i x$ must also be nondegenerate.

TERMINOLOGY. The geometrical realization of a polyhedral simplicial set is called a *polyhedron*; the old-fashioned term was "simplicial complex," or "geometrical simplicial complex" in Section 1. A *triangulation* of topological space X means a homeomorphism of a polyhedron with X. We show later, following Barratt [B], that the geometrical realization of any simplicial set may be triangulated, though it takes two subdivisions to do so.

BARYCENTRIC COMPLEX (12.3). For each simplicial set K, let $B(K)$ be the polyhedral simplicial set obtained by taking for vertex set $(V, \leqslant)$ the set of all nondegenerate simplices of K, partially ordered by $x \leqslant y$ if x is a face of y, i.e., if $x = d_{i_q} \cdots d_{i_1} y$. Then

$$B(K)_n = \{(x_0, ..., x_n) : x_0 \leqslant \cdots \leqslant x_n\}.$$

If $f : K \to L$ is a simplicial map, then $B(f) : B(K) \to B(L)$ is the simplicial map determined on the vertices of $B(K)$ by $B(f)(x) = y$, where for each nondegenerate $x \in K$, $f(x) = s_{i_q} \cdots s_{i_1} y$ with y nondegenerate in L.

Warning ★. For an arbitrary simplicial set K, $B(K)$ might not help much; consider, for example, $K = S^n$, with only two nondegenerate simplices $*$ and σ_n.

In the case where K is itself a polyhedral simplicial set, $B(K)$ corresponds to the usual barycentric subdivision, and there is a homeomorphism $h : RB(K) \cong RK$. Furthermore, there is a simplicial map $\varphi : B(K) \to K$ determined on the vertices of $B(K)$ by $\varphi(x) =$ last vertex of x, and there is a homotopy

$$R(\varphi) \simeq h : RB(K) \to RK.$$

SUBDIVISION (12.4). For any simplicial set K, let $\bar{K}$ be the simplicial set which is the (disjoint) union

$$\bar{K}_n = \bigcup_{x \in k} (\Delta[\dim x], x),$$

$$\left.\begin{aligned} d_i(t, x) &= (d_i t, x) \\ s_i(t, x) &= (s_i t, x) \end{aligned}\right\} \text{ for each } \quad t \in \Delta[\dim x]_q .$$

Let $(\sim)$ be the equivalence relation on $\bar{K}$ with $(t, x) \sim (u, y)$ if either

$$d_i x = y \qquad \text{and} \qquad \epsilon_i(u) = t$$

or

$$s_i x = y \qquad \text{and} \qquad \eta_i(u) = t.$$

Then there is an equivalence $K \approx \bar{K}/(\sim)$.

To obtain the subdivision of K, first take

$$\overline{SdK} = \bigcup_{x \in k} (B(\Delta[\dim x]), x),$$

$$\left.\begin{aligned} d_i(t, x) &= (d_i t, x) \\ s_i(t, x) &= (s_i\ , x) \end{aligned}\right\} \text{for} \qquad t \in (B(\Delta[\dim x]))_q\,.$$

Let $(\sim)$ be the equivalence relation on $\overline{SdK}$ by $(t, x) \sim (u, y)$ if either

$$d_i x = y \qquad \text{and} \qquad B(\epsilon_i)(u) = t$$

or

$$s_i x = y \qquad \text{and} \qquad B(\eta_i)(u) = t.$$

Finally, the subdivision of K is $SdK = \overline{SdK}/(\sim)$.

This construction SdK is a functor from simplicial sets to simplicial sets, with the following properties:

(1) There is a homeomorphism

$$h : R(SdK) \to RK$$

obtained as follows. For each $x \in K_n$, let

$$R(B\Delta[n]) \xrightarrow{h_x} R(\Delta[n])$$

be the continuous map such that

(i) Let $x = s_{i_q} \cdots s_{i_1} y$, where $y \in K_m$ is nondegenerate. For each vertex j of $\Delta[n]$, let $N(j)$ be the number of vertices of $\Delta[n]$ which have the same image as j under the map

$$\eta_{i_q} \cdots \eta_{i_1} : \Delta[n] \to \Delta[m].$$

Then let

$$h(i_n\, , x) = \left(\left(\frac{1}{(m+1)\, N(0)}, \ldots, \frac{1}{(m+1)\, N(n)}\right), x\right).$$

That is, the "barycenter" of the simplex in $R(SdK)$ represented by x goes to an appropriately weighted point in the corresponding simplex of RK.

(ii) For all $0 \leqslant i \leqslant n$, there is to be a commutative diagram

$$\begin{array}{ccc} R(B(\Delta[n])) & \xrightarrow{h_x} & R(\Delta[n]) \\ {\scriptstyle R(B(\epsilon_i))}\uparrow & & \uparrow{\scriptstyle R(\epsilon_i)} \\ R(B(\Delta[n-1])) & \xrightarrow{h_{d_i x}} & R(\Delta[n-1]) \end{array}$$

(iii) For each $t \in B(\Delta[n])_n$, let $f_t : \Delta[n] \to B(\Delta[n])$ be the representing map. Then the map

$$R(\Delta[n]) \xrightarrow{R(f_t)} R(B(\Delta[n])) \xrightarrow{h_x} R(\Delta[n])$$

is to be linear.

It is straightforward to verify that the map $h : R(SdK) \to RK$ determined by (i), (ii), (iii) is a homeomorphism. Observe also that h is natural with respect to inclusions of subsimplicial sets; h may not be assumed to be natural with respect to all simplicial maps. For a detailed treatment, see Ref. [F, P].

(2) There is a simplicial map $\varphi : SdK \to K$ which is obtained by first defining $\bar{\varphi} : \overline{SdK} \to \bar{K}$ using $\varphi : B(\Delta[n]) \to \Delta[n]$, and passing to the quotient sets.

(3) There is a homotopy

$$R(\varphi) \simeq h : R(SdK) \to RK.$$

Also, there is a simplicial map $\alpha : SdK \to B(K)$, but it might not be possible to factor the map $\varphi : SdK \to K$ through α (consider again S^n).

To get around this difficulty, make an Sd^*K oppositely to SdK as follows. First, let $(V^*, \leqslant)$ be the partially ordered set where $V^* = V$, but with the opposite order. Then obtain functors B^*, Sd^*, a homeomorphism $h^* : R(Sd^*K) \cong RK$, and a simplicial map $\varphi^* : Sd^*K \to K$, with properties similar to (1), (2), (3) above.

By composing the two, we obtain the double subdivision of K.

DEFINITION (12.5). For a simplicial set K, its ***double simplicial subdivision*** is defined by $SD(K) = Sd\, Sd^*K$.

As before, SD is a functor from simplicial sets to simplicial sets with the following properties:

(1) There is a homeomorphism

$$H : R(SD(K)) \cong RK;$$

(2) There is a simplicial map

$$\Phi : SD(K) \to K;$$

(3) There is a homotopy

$$R(\Phi) \simeq H : R(SD(K)) \to RK;$$

(4) Furthermore, for this double subdivision, there are simplicial maps

$$Sd(K) \xrightarrow{\beta} BSd^*K \xrightarrow{\gamma} K$$

with $\Phi = \gamma \circ \beta$. The map $\beta(\ ) = \varphi(Sd^*(\ ))$. The map γ is defined as follows. Let $x \in (Bd\ Sd^*K)_n$; that is, $x = (y_0, ..., y_n)$ where $y_0 \leqslant \cdots \leqslant y_n$ are nondegenerate simplices of Sd^*K. For each y_i , there are representing maps

$$f_{y_i} : \Delta[\dim y_i].$$

Let a_i be the largest integer for which

$$f_{y_n}(a_i) = f_{y_i}(\dim y_i),$$

and then let

$$\gamma(x) = \gamma(y_0, ..., y_n) = \varphi^* f_{y_n}(a_0, ..., a_n).$$

It is straightforward to check that γ is a simplicial map with $\Phi = \beta \circ \gamma$. Because of this factorization of Φ through a polyhedral simplicial set, the (semi-) simplicial approximation can be deduced from the usual simplicial approximation theorem (which will be assumed; see, for example, Ref. [Sp]).

Remark (12.6). The geometrical realization $R(B(Sd^*K))$ is a polyhedron, which is homeomorphic to $R(Sd^*K)$, hence also to RK. This is a theorem of Barratt [B], which asserts that the geometrical realization of a simplicial set can be triangulated. Observe that Sd^*K has the following two properties:

(i) Any two simplices have at most a single face in common;

(ii) For each nondegenerate simplex $x \in Sd^*K$, there is a vertex (the first one) such that any face of x containing this vertex is also nondegenerate.

THEOREM (12.7) (Semisimplicial approximation). *Let K be a simplicial set, $A \subset K$ a subsimplicial set, such that there are only finitely many nondegenerate simplices in $K - A$. Let $f : A \to L$ be a simplicial map (to another simplicial set L), and suppose there is a continuous map $p : RA \to RL$, with $R(f) = p$ on RA. Then there is a nonnegative integer n, and a simplicial map $f' : SD^n(K) \to L$ with $f \circ \Phi^n = f'$ on $SD^n(A)$ and*

$$R(f') \simeq p : RK \to RL \qquad (\text{rel } RA).$$

Consider the diagram

$$\begin{tikzcd} & BSd^*SD(A) \arrow[r, "\gamma"] \arrow[d] & SD(A) \arrow[d] & & \\ SD^2(K) \arrow[r, "\beta"] & BSd^*SD(K) \arrow[r, "\gamma"] & SD(K) \arrow[d] & & \\ & & SD(L) \arrow[r, "\beta"] & BSd^*L \arrow[r, "\gamma"] & L \end{tikzcd}$$

and the diagram for the realizations

$$\begin{tikzcd} & R(BSd^*SD(A)) \arrow[r, "R(\gamma)"] & R(SD(A)) \arrow[d] & & \\ R(SD^2(K)) \arrow[r, "R(\beta)"] & R(BSd^*SD(K)) \arrow[r, "R(\gamma)"] & R(SD(K)) \arrow[d, "p"] & & \\ & & R(SD(L)) \arrow[r, "R(\beta)"] & R(BSd^*L) \arrow[r, "R(\gamma)"] & R(L) \end{tikzcd}$$

Apply the ordinary simplicial approximation to the continuous map $R(\beta) \circ p \circ R(\gamma)$, which yields an integer $n - 2 \geqslant 0$ and a simplicial map

$$SD^{n-2}(BSd^*SD(K)) \xrightarrow{f''} BSd^*(L),$$

so that the composite

$$SD^{n-2}(SD^2(K)) \xrightarrow{SD^{n-2}(\beta)} SD^{n-2}(BSd^*SD(K)) \xrightarrow{f''} BSd^*(L) \xrightarrow{\gamma} L$$

is the desired map f' proving (12.7).

PROPOSITION (12.8). *Let K be a simplicial set with $A \subset K$ a simplicial subset, and let L be a Kan simplicial set. Suppose there are simplicial maps $f: A \to L$ and $f': SD^n \to L$ with $f' = f \circ \Phi^n$ on SD^nA. Then there is a simplicial map g filling in the commutative diagram*

$$\begin{tikzcd} SD^nA \arrow[r] \arrow[d, "\Phi^n"'] & SD^nK \arrow[d, "\Phi^n"'] \arrow[dr, "f'"] & \\ A \arrow[r] & K \arrow[r, "g"] & L \end{tikzcd}$$

and $Rf' \simeq R(g \circ \Phi^n) : R(SD^n(K)) \to RL$ (rel $R(SD^n(A))$).

Proof. By induction; it is sufficient to take $n = 1$, and to consider Sd instead of SD. Consider first the case where K has only one nondegenerate simplex not in A, say x of dimension m.

Let $B \subset SdK \times I$ be the subcomplex

$$B = (SdA \times I) \cup (SdK \times (1)) \cup (\varphi^{-1}(A) \times (0)),$$

and let $\tilde{f}: B \to L$ be the map given by

(1) $\tilde{f}(a, t) = f\varphi(a)$ for $(a, t) \in SdA \times I$,

(2) $\tilde{f}(y, 1) = f^1(y)$ for $(y, 1) \in SdK \times (1)$,

(3) $\tilde{f}(y, 0) = f\varphi(y)$ for $(y, 0) \in \varphi^{-1}(A) \times (0)$.

We use a prism-type argument to extend $\tilde{f}: B \to L$ to a map

$$F: SdK \times I \to L$$

as follows. For each nondegenerate $(m + 1)$ simplex w in $SdK \times I$ but not in B, let

$$w = (s_kv, s_m \cdots \hat{s}_k \cdots s_0i_1),$$

where $k = k(w)$, and let $v = (Bf_x)(z)$ for $z \in \Delta[m]$ and $f_x : \Delta[m] \to K$ be the representing map for x. Then $z = (b_0, ..., b_m)$, and there is a permutation $(a_0, \ldots, a_m)$ of the integers $(0, 1, \ldots, m)$ so that b_i is the simplex spanned by $(a_0, \ldots, a_i)$. Order such w by $w < w'$ if either

(1) $p(w) > p(w')$ in the lexicographic order

or

(2) $p(w) = p(w')$ and $k(w) < k(w')$.

Using the extension property in L, and induction on these w, the map $\bar{f} : B \to L$ can be extended to $F : SdK \times I \to L$.

Finally, let $g : K \to L$ be the simplicial map defined by

$$g(y) = f(y) \qquad \text{for} \quad y \in A,$$
$$g(x) = F(d_{m+1}\bar{w}),$$

where $\bar{w}$ is the "last" w; that is, $p(w) = (0, \ldots, m)$ and $k(w) = m$. Then

$$R(F) : R(f) \simeq R(g) \circ \varphi : R(SdK) \to RL \qquad (\text{rel } R(SdA)).$$

This concludes the argument for such special case; the general case follows by induction on the nondegenerate simplices of $K - A$.

The proof of the simplicial extension theorem (12.1) now follows from (12.7) and (12.8).

THE EQUIVALENCES ϕ AND ψ (12.10). For each simplicial set K, let

$$\phi_K : K \to SRK$$

be defined by $\phi_K = \tilde{id}_{RK}$, where $id_{RK} : RK \to RK$ is the identity and $\tilde{id}_{RK}$ is defined by (1.30). Thus for $x \in K_n$, $\phi_K(x) : t[n] \to RK$ is the realization of f_x, the representing map for x. Similarly, for each topological space X, let

$$\psi_X : RSX \to X$$

be defined by $\psi_X = \tilde{id}_{SX}$. For a point in RSX, represented by (p, f), where $f \in (SX)_n$, $p \in t[n]$, $\psi_X(p, f) = f(p)$. It follows immediately from the definitions that

$$(\psi_{RK}) \circ (R\phi_K) = id_{RK} : RK \to RK$$
$$(S\psi_X) \circ (\phi_{SX}) = id_{SX} : SX \to SX.$$

THEOREM (12.11). (1) *If K is a Kan complex, then $\phi_K : K \to SRK$ is a homotopy equivalence.*

(2) *If X is a realized space (i.e., $X = RL$ for some L), then $\phi_X : RSX \to X$ is a homotopy equivalence.*

Proof (1). The simplicial extension theorem applied to ψ_{RK} produces a simplicial map $g : SRK \to K$ with $Rg \simeq \psi_{RK}$. Then the composites

$$SRK \xrightarrow{g} K \xrightarrow{\phi_K} SRK,$$
$$SRK \xrightarrow{\phi_{SRK}} SRSRK \xrightarrow{SR_g} SRK,$$
$$SRK \xrightarrow{\phi_{SRK}} SRSRK \xrightarrow{S\psi_{RK}} SRK,$$

satisfy

$$\begin{aligned}\phi_K \circ g &= (SR_g) \circ (\phi_{SRK}) \\ &\simeq (S\psi_{RK}) \circ (\phi_{RK}) \\ &= id_{SRK} .\end{aligned}$$

Also, the composites

$$RK \xrightarrow{R\phi_K} RSRK \xrightarrow{R_g} RK$$
$$RK \xrightarrow{R\phi_K} RSRK \xrightarrow{\psi_{RK}} RK$$

satisfy $(Rg) \circ (R\phi_K) \simeq (\psi_{RK}) \circ (R\phi_K) = id_{RK}$. That is, $R(g \circ \phi_K) \simeq id_{RK}$, and another application of the simplicial extension theorem shows that $g \circ \phi_K \simeq id_K$. This completes the proof of (1).

For (2), first make use of (1), which shows that $R\phi_L : RL \to RSRL$ is a homotopy equivalence, and has $RS\psi_{RL}$ for homotopy inverse. Then the composites

$$RSRL \xrightarrow{\psi_{RL}} RL \xrightarrow{R\phi_L} RSRL,$$
$$RSRL \xrightarrow{RSR\phi_L} RSRSRL \xrightarrow{\psi_{RSRL}} RSRL,$$

$$\begin{tikzcd} RSRL \arrow[r, "RSR\phi_L"] & RSRSRL \arrow[r, "RS\psi_{RL}"] & RSRL \arrow[d, "R\phi_{SL}"] \\ & RSRL & RSRSRL \arrow[l, "\psi_{RSRL}"'] \end{tikzcd}$$

satisfy

$$\begin{aligned}(R\phi_L) \circ (\psi_{RL}) &= (\psi_{RSRL}) \circ (RSR\phi_L) \\ &\simeq (\psi_{RSRL}) \circ (R\phi_{SL}) \circ (RS\psi_{RL}) \circ (RSR\phi_L) \\ &= id_{RSRL} .\end{aligned}$$

Hence $(R\phi_L) \circ \psi_{RL} \simeq id_{RSRL}$, and as $(\psi_{RL}) \circ (R\phi_L) = id_{RSRL}$, (2) is proven.

Proof of Equivalence (12.12). For each simplicial set K, Kan complex L,

$$\begin{aligned}[K \to L] &\approx [K \to SRL] \\ &\approx [RK \to RL],\end{aligned}$$

which shows part (1) of (1.35). On the other hand, for X a realized space, Y any space,

$$\begin{aligned}[X \to Y] &\approx [RSX \to X] \\ &\approx [SX \to SY],\end{aligned}$$

which shows part (2) of (1.35) for realized spaces X. For an arbitrary CW space X, it remains only to show that

$$\psi_X : RSX \to X$$

is a homotopy equivalence. For this, (12.12), part (1) shows that

$$S\psi_X : SRSX \to SX$$

is a homotopy equivalence, hence

$$(\psi_X)_* : \pi_*(RSX) \to \pi_*(X)$$

are isomorphisms. Then the Whitehead lemma for CW spaces [Sp, p. 405], shows that ψ_X is a homotopy equivalence.

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