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Sobriety for equilogical spaces ☆

Anna Bucalo [a], Giuseppe Rosolini [b]

[a] Dipartimento di Matematica "Federigo Enriques", Università di Milano, via Cesare Saldini 50, 20133 Milano, Italy
[b] DIMA, via Dodecaneso 35, 16146 Genova, Italy

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ABSTRACT

The category of equilogical spaces, as well as the exact completions of the category of T_0-spaces and of the category of topological spaces, offers locally cartesian closed extensions of the category of topological spaces. Hence in any one of such categories, it is straightforward to consider spaces of continuous functions without bothering about ensuring that they be topological spaces.
We test this fact with the notion of sober topological space, producing a synthetic characterization of those topological spaces which are sober in terms of a construction on equilogical spaces of functions.

1. Introduction

Natural categories in modeling computation consist of topological spaces obtained from directed-complete partial orders endowed with the topology of directed sups (often called the Scott topology). These are cartesian closed, but need not be closed under various useful constructions like subspaces or quotients which are otherwise natural in the approach using (one of the various notions of) filter spaces. In an attempt to reunify the views, Dana Scott proposed the category ***Equ*** of equilogical spaces: An *equilogical space* is a triple $E = (S_E, \tau_E, \equiv_E)$ where (S_E, τ_E) is a topological T_0-space and $\equiv_E \subseteq S_E \times S_E$ is an equivalence relation. A map $f : E \to E'$ between equilogical spaces is a function between the quotient sets $f : S_E/{\equiv_E} \to S_{E'}/{\equiv_{E'}}$ which has a *continuous* choice function $g : S_E \to S_{E'}$ tracking it on the representatives. In other words, for some appropriate continuous function g

$$f\left([x]_{\equiv_E}\right) = \left[g(x)\right]_{\equiv_{E'}}, \quad \text{all } x \in S_E.$$

In [1], Dana Scott had already noticed that these data form a cartesian closed category, and that proposal was reaffirmed in [2].

The construction has been compared to that of the category ***Mod*** of the modest sets in the effective topos, see [3,4]; in fact, in both situations, one can explain local cartesian closure based on the same general facts, see [5,6]. And the crucial construction involved is that of the exact completion of a category with finite limits. In that perspective, a thorough comparison between various cartesian closed extensions of the category of topological spaces was conducted in [7].

In Section 2 we recall the basic constructions of the various categories starting from that of topological spaces and explaining the underlying intuition. In Section 3 we recall a synthetic description of sobriety, as proposed in [8,7] and also

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E-mail addresses: anna.bucalo@unimi.it (A. Bucalo), rosolini@unige.it (G. Rosolini).

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considered for possibly different purposes in [9–11] and state the characterization theorem of sober spaces as topological spaces satisfying an intrinsic property in ***Equ***. In Section 4 we produce the proof of the characterization theorem.

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2. Viewing topological spaces as abstract sets

The intuition about expanding the notion of set as a collection of points *without any further structure* to include objects where some kind of cohesion among the points is described quite vividly in [12–14]. And the first approximation to such an intuition that one may take is to consider topological spaces and continuous functions and the category ***Top*** that they form.

One immediately notices that, in order to consider topological spaces (and continuous functions) *as sets*, it is crucial to have the construction of quotient by an equivalence relation. And the definition of exact category singles out precisely the properties of the standard construction of a set of equivalence classes, see [15]: given an equivalence relation $\sim$ on an object S, the quotient $S/\sim$ is the smallest solution to the problem of defining transformations which preserve the identification on arguments induced by the equivalence relation. The property (that one then checks in proving the factorization theorem for set-functions) that

$$[x]_\sim = \left[x'\right]_\sim \quad \Longleftrightarrow \quad x \sim x'$$

can be restated category-theoretically as saying that the kernel equivalence relation induced by the canonical surjection $S \twoheadrightarrow S/\sim$ coincides with the given equivalence relation $\sim$. This makes a quotient of sets *effective*. Finally, it is a property of the logic that gives *stability*: any renaming of the equivalence classes $g : X \to S/\sim$ is (in bijection with) the classes for an equivalence relation on $\{(x, s) \mid g(x) = [s]_\sim\}$.

Formally (and in a nutshell), given a category $\boldsymbol{C}$, an *equivalence relation in* $\boldsymbol{C}$ is a pair of arrows $A_1 \underset{r_2}{\overset{r_1}{\rightrightarrows}} A_0$ which is jointly monic, reflexive, symmetric, and transitive. A *quotient* of such an equivalence relation is a coequalizer $f : A_0 \twoheadrightarrow A$ of the parallel pair. And the quotient is said to be *effective* if its kernel pair is isomorphic to the given equivalence relation. Moreover it is *stable* if any pullback of it is a quotient.

A category $\boldsymbol{C}$ is *exact* if it has finite limits, quotients of equivalence relations, and these are effective and stable under pullbacks. We refer the interested reader to [15–17,7] for a thorough discussion of the notions involved and examples.

A weaker notion is that of a *regular* category which is a category with finite limits and with quotients of kernel pairs which are effective and stable.

The category of topological spaces and continuous functions has quotients of equivalence relations, but these are *not* in general effective or stable. In other words, those quotients are *wrong* if one wants to think in terms of "spaces as sets".

Again, category theory offers a solution for free: since the notion of exact category is algebraic over categories with finite limits, there is always the free exact category generated by a category with finite limits. Moreover, as presented in a well-known paper [18], such a free construction can be obtained in finitary terms over the given category with finite limits. In the case of the free exact category $\boldsymbol{Top_0}_{\mathrm{ex}}$ over the category of T_0-spaces, one obtains an exact category which is also locally cartesian closed and extensive (in other words, a locally cartesian closed pretopos), extending the original result of [1,2], see [7].

Relating the notion of exact category to that of regular category in the sense of [15], Aurelio Carboni also produced a finitary presentation of the free regular category over a category with finite limits in [16]. And all this provides a square of product-preserving reflections

$$\begin{tikzcd}
\boldsymbol{Top}_0 \arrow[r, hook] \arrow[r, phantom, shift left=3, "\perp"] \arrow[d, hook] \arrow[d, phantom, shift right=3, "\vdash"] & \boldsymbol{Equ} = (\boldsymbol{Top}_0)_{\mathrm{reg}} \arrow[l, bend right] \arrow[r, hook] \arrow[r, phantom, shift left=3, "\perp"] \arrow[d] \arrow[d, phantom, shift right=3, "\vdash"] & (\boldsymbol{Top}_0)_{\mathrm{ex}} \arrow[l, bend right] \arrow[d, hook] \arrow[d, phantom, shift right=3, "\vdash"] \\
\boldsymbol{Top} \arrow[u, bend right] \arrow[r, hook] \arrow[r, phantom, shift left=3, "\perp"] & \boldsymbol{Top}_{\mathrm{reg}} \arrow[l, bend right] \arrow[u, bend right] \arrow[r, hook] \arrow[r, phantom, shift left=3, "\perp"] & \boldsymbol{Top}_{\mathrm{ex}} \arrow[l, bend right] \arrow[u, bend right]
\end{tikzcd}$$

The categories in the last column are locally cartesian closed pretoposes, those in the middle column are locally cartesian closed quasitoposes, see [19]. And the inclusions preserve finite limits and all existing exponentials.

The characterization which follows can be considered in any of the completions in the diagram above. But, since we are interested in characterizing topological spaces which are T_0-spaces using a condition on exponentials, we can restrict the computation to ***Equ***, the smallest of the cartesian closed categories in the diagram above, but the final result can be extended to all of the other cartesian closed categories in the diagram above.

3. A characterization of sobriety

Let $\Sigma = (\{0, 1\}, \varsigma)$ be the Sierpinski topological space with two points: one open, one closed. The space Σ is actually an algebraic lattice endowed with its Scott topology, see Section 4 for the general definition, but recall that an algebraic lattice is a complete lattice where every element is the sup of the compact elements below it.

In the category ***Top***, the space Σ classifies open subsets as the set $\boldsymbol{Top}(S,\Sigma)$ is in bijection with the set $\mathcal{O}(S)$ of open subsets of S. The exponential Σ^S must be computed in ***Equ***, as in general it is not a topological space. As explicitly suggested by Paul Taylor in a series of papers, the monad on the double exponential

$$\Sigma^{(\Sigma^{(-)})} : \boldsymbol{Equ} \longrightarrow \boldsymbol{Equ}$$

should be considered as enclosing the properties of the spaces which can be expressed in the "logic of Σ", see [8,20–24]. In particular, a suggestion for abstract sobriety, which had already been considered in similar context, see *e.g.* [10,7], was put forward as follows.

Recall that a topological space S is sober when every closed, irreducible subset of S is the closure of a unique point. Note that the condition of irreducibility for a closed subset $C \subseteq S$ is equivalent to the fact that the map

$$k_C : U \mapsto \{0 \mid C \cap U \neq \emptyset\} : \mathcal{O}(S) \to \mathcal{P}\big(\{0\}\big)$$

from the complete lattice of open subsets $\mathcal{O}(S)$ to the (complete) lattice $\mathcal{P}(\{0\})$ of subsets of the singleton $\{0\}$ is a frame homomorphism, *i.e.* it preserves finite infs and arbitrary sups, see [25].

Replace that by performing the same construction within ***Equ***: say that an object X in ***Equ*** is *abstractly sober* if the diagram

$$X \xrightarrow{\ \eta_X\ } \Sigma^{(\Sigma^X)} \mathrel{\substack{\xrightarrow{\ \eta_{\Sigma^{(\Sigma^X)}}\ } \\ \xrightarrow[\ \Sigma^{(\Sigma^{(\eta_X)})}\ ]{}}} \Sigma^{(\Sigma^{(\Sigma^{(\Sigma^X)})})}$$

induced by the exponential adjunction is an equalizer.

Note that it is immediate that the full subcategory of ***Equ*** on abstractly sober objects is reflexive as the equalizer

$$[\Sigma^Z, \Sigma] \xrightarrow{\ e\ } \Sigma^{(\Sigma^Z)} \mathrel{\substack{\xrightarrow{\ \eta_{\Sigma^{(\Sigma^Z)}}\ } \\ \xrightarrow[\ \Sigma^{(\Sigma^{(\eta_Z)})}\ ]{}}} \Sigma^{(\Sigma^{(\Sigma^{(\Sigma^Z)})})}$$

of the two maps gives the reflection, see [7].

3.1. Remark. The definition of abstract sobriety and the reflection can be performed, in a general cartesian closed category with finite limits, with respect to a fixed object A. An interesting instance of this is the case of j-sheaves in an elementary topos for a Lawvere–Tierney topology j, see [26,27].

We have the following results

3.2. Theorem. *Let S be a* T_0*-space.*

(i) *S is sober if and only if it is abstractly sober in* ***Equ***.
(ii) $[\Sigma^S, \Sigma]$ *is a topological space (and the sober reflection of S in* $\boldsymbol{Top}_0$*).*

From the square of product-preserving reflections, one derives immediately the following.

3.3. Corollary. *Let S be a topological space.*

(i) *S is sober if and only if it is abstractly sober in* $\boldsymbol{Top}_{\mathrm{ex}}$.
(ii) $[\Sigma^S, \Sigma]$ *is a topological space (and the sober reflection of S in* ***Top***).

4. Proof of the characterization theorem

It is useful to recall an equivalent presentation of the category ***Equ*** as the category ***PEqu***, see [2]. Before that, recall that the Scott topology $\mathcal{O}_{\mathrm{Sc}}(|A|, \leqslant_A)$ on the algebraic lattice $(|A|, \leqslant_A)$ consists of those subsets $U \subseteq |A|$ which are

upward-closed: for every $x \in U$ and every $y \geqslant_A x$, also $y \in U$
inaccessible by directed sups: for every directed subset $D \subseteq |A|$ such that $\bigvee D \in U$, there is an element $x \in D$ such that $x \in U$.

An object in ***PEqu*** is a triple $A = (|A|, \leqslant_A, \sim_A)$ where $(|A|, \leqslant_A)$ is an algebraic lattice and $\sim_A \subseteq |A| \times |A|$ is a partial equivalence relation. In order to avoid confusion, we follow [17] and we say that $\sim_A$ is *entire* if its domain is all of $|A|$. A map $f : A \longrightarrow A'$ in ***PEqu*** is a function between the quotient sets $f : \mathrm{dom}(\sim_A)/\sim_A \longrightarrow \mathrm{dom}(\sim_{A'})/\sim_{A'}$ which has a Scott-continuous function $f' : (|A|, \leqslant_A) \longrightarrow (|A'|, \leqslant_{A'})$ on the lattices which induces it by its action on the representatives. The Extension Theorem and the Embedding Theorem which state respectively that algebraic lattices endowed with the Scott topology are injective in $\boldsymbol{Top}_0$ and that every T_0-space is a subspace of an algebraic lattice allow to show that the categories ***Equ*** and ***PEqu*** are equivalent, see [2].

Let (T, τ) be a fixed T_0-space for the rest of the section. As an equilogical space, it is the triple (T, τ, Δ_T) where Δ_T is the diagonal relation on the set T. The corresponding object in ***PEqu*** is $(\mathcal{P}(\tau), \subseteq, \delta_T)$ where $\mathcal{P}(\tau)$ is the collection of all sets of open subsets of (T, τ) and $V \ \delta_T \ W$ exactly when V and W are the collection of open neighborhoods of the same point of T, *i.e.* writing $\mathcal{N}_x := \{U \in \tau \mid x \in U\}$ for the collection of open neighborhoods of x in T,

$$V \ \delta_T \ W \leftrightarrow \exists_{x \in T} [V = W = \mathcal{N}_x].$$

Because of the Extension Theorem, an object $A = (|A|, \leqslant_A, \sim_A)$ in ***PEqu*** is isomorphic to one of the form $(\mathcal{P}(\tau), \subseteq, \delta_T)$ if and only if $\sim_A \subseteq |A| \times |A|$ is contained in the diagonal relation. Again we follow [17] and call such a relation *coreflexive*. The following elementary result is thus useful for our future purposes.

4.1. Lemma. *Suppose $A = (|A|, \leqslant_A, \sim_A)$ is an object in* ***PEqu***. *Then Σ^A can be represented by the triple $(\mathcal{O}_{Sc}(|A|, \leqslant_A), \subseteq, \approx_A)$ where $\mathcal{O}_{Sc}(|A|, \leqslant_A)$ is the Scott topology on the algebraic lattice $(|A|, \leqslant_A)$ and*

$$M \approx_A N \leftrightarrow \left[\forall_{a,b \in |A|}\left[a \sim_A b \to (a \in M \leftrightarrow b \in N)\right]\right]$$

for $M, N \in \mathcal{O}_{Sc}(|A|, \leqslant_A)$. Moreover

(i) *if $\sim_A \subseteq |A| \times |A|$ is coreflexive, then $\approx_A \subseteq \mathcal{O}_{Sc}(|A|, \leqslant_A) \times \mathcal{O}_{Sc}(|A|, \leqslant_A)$ is entire,*
(ii) *if $\sim_A \subseteq |A| \times |A|$ is entire, then $\approx_A \subseteq \mathcal{O}_{Sc}(|A|, \leqslant_A) \times \mathcal{O}_{Sc}(|A|, \leqslant_A)$ is coreflexive.*

The following is a slight generalization of a result in [28].

4.2. Proposition. *The exponential $\Sigma^{(T,\tau)}$ in* ***PEqu*** *can be chosen as the triple*

$$\big(\mathcal{O}_{Sc}\big(\mathcal{P}(\tau), \subseteq\big), \subseteq, \equiv_T\big)$$

where $\mathcal{O}_{Sc}(\mathcal{P}(\tau), \subseteq)$ is the Scott topology on the algebraic lattice $(\mathcal{P}(\tau), \subseteq)$, and $M \equiv_T N$ exactly when

$$\bigcup_{\substack{F \in M \\ F \text{ finite}}} \bigcap_{U \in F} U = \bigcup_{\substack{G \in N \\ G \text{ finite}}} \bigcap_{V \in G} V$$

Proof. The exponential of $(\mathcal{P}(\tau), \subseteq, \delta_T)$ and $(\Sigma, \leqslant, =)$ is computed by first taking the exponential algebraic lattice with base $(\Sigma, \leqslant)$ and exponent $(\mathcal{P}(\tau), \subseteq)$ which is isomorphic to $(\mathcal{O}_{Sc}(\mathcal{P}(\tau), \subseteq), \subseteq)$. Next the partial equivalence relation is given by requiring that, for M, N Scott-open subsets in the algebraic lattice $(\mathcal{P}(\tau), \subseteq)$, it is $M \equiv_T N$ when

$$\forall_{x \in T} [\mathcal{N}_x \in M \leftrightarrow \mathcal{N}_x \in N].$$

The conclusion follows since $\mathcal{N}_x \in M$ if and only if $x \in \bigcup_{\substack{F \in M \\ F \text{ fin.}}} \bigcap_{U \in F} U$. □

This gives immediately the first part 3.2(i) of the main theorem.

4.3. Proposition. *The object $[\Sigma^{(T,\tau)}, \Sigma]$ in the equalizer*

$$\begin{tikzcd}[\Sigma^{(T,\tau)}, \Sigma] \arrow[r, "e"] & \Sigma^{(\Sigma^{(T,\tau)})} \arrow[r, shift left, "\eta_{\Sigma^{(\Sigma^{(T,\tau)})}}"] \arrow[r, shift right, "\Sigma^{(\Sigma^{(\eta_{(T,\tau)})})}"'] & \Sigma^{(\Sigma^{(\Sigma^{(\Sigma^{(T,\tau)})})})}\end{tikzcd}$$

is a sober topological space.

We devote the rest of the paper to complete the proof of 3.2.

4.4. Remark. Note that the relation $\equiv_T$ in 4.2 is entire.

Moreover, given M in $\mathcal{O}_{Sc}(\mathcal{P}(\tau), \subseteq)$, one has that

$$M \equiv_T \Bigg(\bigcup_{\substack{F \in M \\ F \text{ finite}}} \bigcap_{U \in F} U\Bigg)^{\in}$$

where $B^{\in}$ is the collection of all sets in $\mathcal{P}(T)$ to which B belongs. It follows that the set $\mathcal{O}_{\mathrm{Sc}}(\mathcal{P}(\tau),\subseteq)/{\equiv_T}$ is in bijection with the set τ.

4.5. Proposition. *The exponential $\Sigma^{(\Sigma^{(T,\tau)})}$ in **PEqu** can be chosen as the triple*

$$\big(\mathcal{P}(\tau),\subseteq,\Delta_{\mathcal{P}(\tau)}\big)$$

and the map $\eta:(\mathcal{P}(\tau),\subseteq,\delta_T)\longrightarrow(\mathcal{P}(\tau),\subseteq,\Delta_{\mathcal{P}(\tau)})$ is tracked by the identity function on $\mathcal{P}(\tau)$.

Proof. Like before, the exponential of $(\mathcal{O}_{\mathrm{Sc}}(\mathcal{P}(\tau),\subseteq),\subseteq,\equiv_T)$ and $(\Sigma,\leqslant,=)$ is computed by taking the exponential algebraic lattice with base $(\Sigma,\leqslant)$ and exponent $(\mathcal{O}_{\mathrm{Sc}}(\mathcal{P}(\tau),\subseteq),\subseteq)$ which is (isomorphic to) a certain algebraic lattice of Scott-opens. Since in every equivalence class of the partial equivalence relation there is a unique representative of the form $U^{\in}$ for some $U\in\tau$, the conclusion follows. $\square$

4.6. Proposition. *The structure map*

$$\Sigma^{\eta_{(T,\tau)}}:\Sigma^{(\Sigma^{(\Sigma^{(T,\tau)})})}\longrightarrow\Sigma^{(T,\tau)}$$

*in **PEqu** is tracked by the continuous function*

$$\begin{aligned}(\mathcal{P}(\mathcal{O}_{\mathrm{Sc}}(\mathcal{P}(\tau),\subseteq)),\subseteq)&\longrightarrow(\mathcal{O}_{\mathrm{Sc}}(\mathcal{P}(\tau),\subseteq),\subseteq)\\ \mathcal{M}&\longmapsto\bigcup_{M\in\mathcal{M}}\Bigg(\bigcup_{\substack{F\in M\\ F\ \mathit{finite}}}\ \bigcap_{U\in F}U\Bigg)^{\in}\end{aligned}$$

Proof. It is immediate from 4.5 and 4.4. $\square$

The crucial step in the proof of 3.2 is to show that a homomorphism of $\Sigma^{(\Sigma^{(-)})}$-algebras from $\Sigma^{(T,\tau)}$ to Σ determines a frame homomorphism from τ to $\{0,1\}$. Since τ appears as a quotient set from the equivalence relation in $(\mathcal{O}_{\mathrm{Sc}}(\mathcal{P}(\tau),\subseteq),\subseteq,\equiv_T)$, it is useful to introduce the "global section" functor $\Gamma:\textbf{\textit{PEqu}}\longrightarrow\textbf{\textit{Set}}$ which sends an object $A=(|A|,\leqslant_A,\sim_A)$ to the quotient set $\mathrm{dom}(\sim_A)/{\sim_A}$ and a map $f:A\longrightarrow A'$ in **PEqu** to itself seen as a function $f:\mathrm{dom}(\sim_A)/{\sim_A}\longrightarrow\mathrm{dom}(\sim_{A'})/{\sim_{A'}}$.

4.7. Proposition. *Let $f:\Sigma^{(T,\tau)}\longrightarrow\Sigma$ be any homomorphism of $\Sigma^{(\Sigma^{(-)})}$-algebras in **PEqu** and write $g:\tau\longrightarrow\{0,1\}$ the function obtained by composing f with the bijection from τ to $\mathcal{O}_{\mathrm{Sc}}(\mathcal{P}(\tau),\subseteq)/{\equiv_T}$. The function g is a frame-homomorphism.*

Proof. Since f is continuous with respect to the Scott topology, it preserves directed sups. Since the equivalence relation $\equiv_T$ is closed under directed sups, it follows that g preserves directed sups. To see that it preserves finite infs and sups, consider the commutative diagram

$$\begin{tikzcd}
\Gamma(\Sigma^{(T,\tau)}) \arrow[r, "\Gamma(\eta_{(T,\tau)})"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "\Gamma(f)"'] & \Gamma(\Sigma^{(\Sigma^{(\Sigma^{(T,\tau)})})}) \arrow[r, "\Gamma(\Sigma^{\eta_{(T,\tau)}})"'] \arrow[d, "\Gamma(\Sigma^{(\Sigma^{f})})"'] & \Gamma(\Sigma^{(T,\tau)}) \arrow[r, "q"'] \arrow[d, "\Gamma(f)"] & \tau \arrow[d, "g"] \\
\Gamma(\Sigma) \arrow[r, "\Gamma(\eta_{\Sigma})"] \arrow[rr, bend right, "\mathrm{id}"'] & \Gamma(\Sigma^{(\Sigma^{\Sigma})}) \arrow[r, "\Gamma(\Sigma^{\eta_{\Sigma}})"] & \Gamma(\Sigma) \arrow[r, "p"] & \{0,1\}
\end{tikzcd}$$

Given $U,V\in\tau$, the sets $\{U\}$ and $\{V\}$ are elements in $\mathcal{P}(\tau)$; let $\mathcal{N}_{\{U\}}$ and $\mathcal{N}_{\{V\}}$ be the respective collections of Scott-open neighborhoods in the algebraic lattice $(\mathcal{P}(\tau),\subseteq)$ which are elements of $\mathcal{P}(\mathcal{O}_{\mathrm{Sc}}(\mathcal{P}(\tau),\subseteq))=\Gamma(\Sigma^{(\Sigma^{(\Sigma^{(T,\tau)})})})$. The function

$$\Gamma(\Sigma^{(\Sigma^{(\Sigma^{(T,\tau)})})})\xrightarrow[\Gamma(\Sigma^{\eta_{(T,\tau)}})]{}\Gamma(\Sigma^{(T,\tau)})\xrightarrow[q]{}\tau$$

takes $\mathcal{N}_{\{U\}}\cap\mathcal{N}_{\{V\}}$ to $U\cap V$ and $\mathcal{N}_{\{U\}}\cup\mathcal{N}_{\{V\}}$ to $U\cup V$. The similar result for the function

$$\Gamma(\Sigma^{(\Sigma^{\Sigma})})\xrightarrow{\Gamma(\Sigma^{\eta_{\Sigma}})}\Gamma(\Sigma)\xrightarrow{p}\{0,1\}$$

and the commutativity of the diagram yield the conclusion. $\square$

To complete the proof of 3.2, one notices that the factoring map j in the commutative diagram

$$\begin{tikzcd}
(T,\tau) \arrow[d, dashed, "j"'] \arrow[dr, "\eta_{(T,\tau)}"] & \\
{[\Sigma^{(T,\tau)}, \Sigma]} \arrow[r, "e"] & \Sigma^{(\Sigma^{(T,\tau)})}
\end{tikzcd}$$

is a subspace inclusion since so is $\eta_{(T,\tau)}$, and 4.7 ensures that j is a bijection.

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