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Analytic functors between presheaf categories over groupoids ☆

Marcelo Fiore [1]

University of Cambridge, Computer Laboratory, 15 JJ Thomson Avenue, Cambridge CB3 0FD, UK

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Dedicated to Glynn Winskel on the occasion of his 60th birthday

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ABSTRACT

The paper studies analytic functors between presheaf categories. Generalising results of A. Joyal [11] and R. Hasegawa [9] for analytic endofunctors on the category of sets, we give two characterisations of analytic functors between presheaf categories over groupoids: (i) as functors preserving filtered colimits, quasi-pullbacks, and cofiltered limits; and (ii) as functors preserving filtered colimits and wide quasi-pullbacks. The development establishes that small groupoids, analytic functors between their presheaf categories, and quasi-cartesian natural transformations between them form a 2-category.

1. Introduction

The concept of multivariate analytic functor on the category $\mathcal{S}et$ of sets and functions was introduced by A. Joyal in [11] to provide a conceptual basis for his theory of combinatorial species of structures [10,1].

A species of structures is a functor from the category of finite sets and bijections to $\mathcal{S}et$. These can be equivalently presented as functors from the category of finite cardinals and permutations to $\mathcal{S}et$, or as symmetric sequences

$$P = \{P_n \times \mathfrak{S}_n \to P_n : (p, \sigma) \mapsto p \cdot_P \sigma\}_{n \in \mathbb{N}}$$

given by families of set-theoretic representations of the symmetric groups. Here, the sets P_n are thought of as a species of combinatorial structures P on an n-element set, while the symmetric-group representations induce isomorphism types that correspond to their unlabelled version. In general, for a species P and a set of labels X, the set of X-labelled P-structures is given by

$$\widetilde{P}X \stackrel{\text{def}}{=} \sum_{n \in \mathbb{N}} P_n \underset{\mathfrak{S}_n}{\times} X^n \quad (X \in \mathcal{S}et) \tag{1}$$

where $P_n \underset{\mathfrak{S}_n}{\times} X^n$ denotes the quotient of $P_n \times X^n$ by the equivalence relation identifying $(p, (x_{\sigma 1}, \ldots, x_{\sigma n}))$ with $(p \cdot_P \sigma, (x_1, \ldots, x_n))$ for all $\sigma \in \mathfrak{S}_n$, $p \in P_n$, and $x_1, \ldots, x_n \in X$. In particular, the set $\widetilde{P}1$ for a singleton set 1 corresponds to that of unlabelled P-structures.

An endofunctor on $\mathcal{S}et$ is said to be analytic if it has a Taylor series development as in (1) above; that is, if it is naturally isomorphic to $\widetilde{P}$ for some species P. One respectively regards species of structures and analytic functors as combinatorial versions of formal exponential power series and exponential generating functions. A. Joyal characterised the analytic

☆ A write-up of [3].

E-mail address: Marcelo.Fiore@cl.cam.ac.uk.

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endofunctors on $\mathfrak{Set}$ as those that preserve filtered colimits, cofiltered limits, and quasi-pullbacks (equivalently, weak pullbacks).

In [10], A. Joyal also introduced the notion of a linear species as a functor from the category of finite linear orders and monotone bijections to $\mathfrak{Set}$; equivalently, an $\mathbb{N}$-indexed family of sets. Every linear species L freely induces a species $L \times \mathfrak{S}$ as follows

$$(L_n \times \mathfrak{S}_n) \times \mathfrak{S}_n \to (L_n \times \mathfrak{S}_n) : \big((\ell, \sigma), \sigma'\big) \mapsto \big(\ell, \sigma \cdot \sigma'\big) \quad (n \in \mathbb{N})$$

Its associated analytic endofunctor $\widetilde{L \times \mathfrak{S}}$ on $\mathfrak{Set}$ is of the form

$$\widetilde{L \times \mathfrak{S}}(X) \cong \sum_{n \in \mathbb{N}} L_n \times X^n \quad (X \in \mathfrak{Set}) \tag{2}$$

Thus, one respectively regards linear species and their induced analytic functors as combinatorial versions of formal power series and generating functions.

Independently of the above considerations, the multivariate version of functors on $\mathfrak{Set}$ of the form (2) was introduced by J.-Y. Girard in [6] also under the name of analytic functors. These he characterised as those that preserve filtered colimits, wide pullbacks, and equalisers. In [12], P. Taylor tightens this characterisation remarking that the preservation of equalisers was redundant. R. Hasegawa revisited the characterisation of Joyal's analytic endofunctors on $\mathfrak{Set}$ in this light in [9], observing that they can be also characterised as those preserving filtered colimits and weak wide pullbacks (equivalently, wide quasi-pullbacks). The development of J.-Y. Girard put this line of work in the context of categorical stable domain theory (as so did explicitly the subsequent work of P. Taylor) and was a preliminary step leading to linear logic [7].

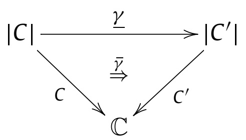
A bicategorical framework for the above body of work was put forward by G.L. Cattani and G. Winskel in [2] from the perspective of presheaf models for concurrency and by M. Fiore, N. Gambino, M. Hyland and G. Winskel in [5] from the viewpoint of species of structures. The work reported here supplements the latter one. Indeed, we generalise the aforementioned characterisations of analytic endofunctors on $\mathfrak{Set}$ to analytic functors between presheaf categories over groupoids (Theorem 6.8); and, in this context, exhibit an equivalence of categories between generalised species of structures and natural transformations, and analytic functors and quasi-cartesian natural transformations (Corollary 5.14). This leads to the 2-category of small groupoids, analytic functors between their presheaf categories, and quasi-cartesian natural transformations between them (Corollary 6.9), placing the subject in the context of categorical stable domain theory and providing 2-dimensional models of a rich variety of computational structures (Remark 6.10).

The paper contributes thus to one of the many fundamental structures researched by Glynn Winskel in his work on the mathematical understanding and modelling of processes.

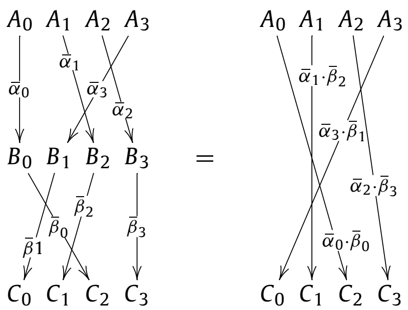
2. Free symmetric strict monoidal completion

We let ! be the left adjoint to the forgetful functor from the category of symmetric strict monoidal small categories and strong monoidal functors to the category $\mathfrak{Cat}$ of small categories and functors. For a small category $\mathbb{C}$, the unit of this adjunction is denoted $\langle[\_]\rangle : \mathbb{C} \to !\mathbb{C}$.

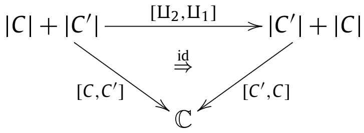
The category $!\mathbb{C}$ can be explicitly described by the Grothendieck construction [8] applied to the functor $\mathbb{C}^{(-)} : \mathbf{P} \to \mathfrak{Cat} : n \mapsto \mathbb{C}^n$ for $\mathbf{P}$ the category of finite cardinals and permutations. That is, $!\mathbb{C}$ has objects given by functions $C : |C| \to \mathbb{C}$ with $|C|$ in $\mathbf{P}$ and morphisms $\gamma = (\underline{\gamma}, \overline{\gamma}) : C \to C'$ given as in the following diagram



with $\underline{\gamma}$ in $\mathbf{P}$. Identities are given by the maps $(\mathrm{id}_{|C|}, \mathrm{id}_C)$, while diagrammatic composition is given by $\alpha \cdot \beta \stackrel{\text{def}}{=} (\underline{\alpha} \cdot \underline{\beta}, \overline{\alpha} \cdot \overline{\beta}_{\underline{\alpha}})$. Thus, maps and their composition can be visualised as follows

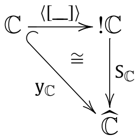


The strict symmetric monoidal structure of $!\mathbb{C}$ has as unit object the empty function $0 \to \mathbb{C}$, as tensor product $\oplus$ the construction $[C, C'] : |C| + |C'| \to \mathbb{C}$, and as symmetry the maps



where $+$ denotes the sum of cardinals, with injections $\amalg_1$ and $\amalg_2$, and copairing $[\_, =]$.

We write $\widehat{\mathbb{C}}$ for the *presheaf category* $\mathcal{S}\boldsymbol{et}^{\mathbb{C}^\circ}$ over a small category $\mathbb{C}$. By the universal property of $!\mathbb{C}$, the Yoneda embedding $\mathrm{y}_\mathbb{C} : \mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$ extends as a (strong symmetric monoidal) *sum functor* $\mathrm{S}_\mathbb{C} : !\mathbb{C} \to \widehat{\mathbb{C}}$ (with respect to the coproduct symmetric monoidal structure of $\widehat{\mathbb{C}}$) as follows



where

$$\mathrm{S}_\mathbb{C}(C) \stackrel{\mathrm{def}}{=} \sum_{i \in |C|} \mathrm{y}_\mathbb{C}(C_i) \quad (C \in !\mathbb{C})$$

Examining the sum functor, one notes that, for $A, B \in !\mathbb{C}$,

$$\widehat{\mathbb{C}}[SA, SB] \cong \prod_{i \in |A|} \widehat{\mathbb{C}}\big[\mathrm{y}(A_i), SB\big] \cong \prod_{i \in |A|} SB(A_i) \cong \prod_{i \in |A|} \sum_{j \in |B|} \mathbb{C}[A_i, B_j] \cong \sum_{\varphi \in |B|^{|A|}} \prod_{i \in |A|} \mathbb{C}[A_i, B_{\varphi i}] \tag{3}$$

In other words, the full subcategory of $\widehat{\mathbb{C}}$ determined by the set of objects $\{SC \in \widehat{\mathbb{C}} \mid C \in !\mathbb{C}\}$ is the free finite coproduct completion of $\mathbb{C}$.

By means of the projection map

$$\sum_{\varphi \in |B|^{|A|}} \prod_{i \in |A|} \mathbb{C}[A_i, B_{\varphi i}] \to \mathcal{S}\boldsymbol{et}\big(|A|, |B|\big) : \big(\varphi, \langle f_i \rangle_{i \in |A|}\big) \mapsto \varphi$$

the isomorphism (3) induces a map

$$\widehat{\mathbb{C}}[SA, SB] \to \mathcal{S}\boldsymbol{et}\big(|A|, |B|\big)$$

that associates an *underlying function* $|A| \to |B|$ to every morphism $SA \to SB$ in $\widehat{\mathbb{C}}$.

Definition 2.1. For $A, B \in !\mathbb{C}$, we say that $SA \to SB$ in $\widehat{\mathbb{C}}$ is *injective, surjective, or bijective on indices* whenever its underlying function $|A| \to |B|$ is.

Proposition 2.2.

(i) *The sum functor is faithful.*
(ii) *If $f : SA \to SB$ in $\widehat{\mathbb{C}}$ is bijective on indices then there exists a (necessarily unique) $\gamma : A \to B$ in $!\mathbb{C}$ such that $S\gamma = f$. Hence, the sum functor is conservative.*

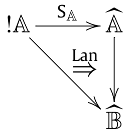
Proposition 2.3.

(i) *For a small category $\mathbb{A}$ and $A, A' \in !\mathbb{A}$, every epi (resp. iso) $SA \to SA'$ in $\widehat{\mathbb{A}}$ is surjective (resp. bijective) on indices.*
(ii) *For a small groupoid $\mathbb{G}$ and $G, G' \in !\mathbb{G}$, every mono $SG \to SG'$ in $\widehat{\mathbb{G}}$ is injective on indices.*

3. Analytic functors

We recall the notion of analytic functor between presheaf categories introduced in [5]. These analytic functors generalise the ones previously introduced by A. Joyal between categories of indexed sets and sets [11, §1.1], and are the central structure of study in the paper.

Definition 3.1. A functor $\widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ is said to be *analytic* if it appears in a left Kan extension as follows



for some functor $!\mathbb{A} \to \widehat{\mathbb{B}}$.

That is, analytic functors between presheaf categories are those naturally isomorphic to the functors $\widetilde{P} : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ given by the following coend

$$\widetilde{P}\,X\,b \stackrel{\text{def}}{=} \int^{A \in !\mathbb{A}} P\,A\,b \times \widehat{\mathbb{A}}\big[\mathrm{S}_{\mathbb{A}}(A), X\big] \quad \big(X \in \widehat{\mathbb{A}},\ b \in \mathbb{B}^{\circ}\big) \tag{4}$$

for some $P : !\mathbb{A} \to \widehat{\mathbb{B}}$.

Notation. For a functor $F : \mathcal{C} \to \widehat{\mathbb{C}}$ it will be convenient to use the following notational conventions. For morphisms $f : A \to B$ in $\mathcal{C}$ and $g : c \to d$ in $\mathbb{C}$, and for an element $x \in F\,A\,d$, we set $x \cdot_F f \stackrel{\text{def}}{=} (F\,f)_d(x) \in F\,B\,d$; $g \cdot_F x \stackrel{\text{def}}{=} F\,A\,g\,(x) \in F\,A\,c$; and $g \cdot_F x \cdot_F f \stackrel{\text{def}}{=} g \cdot_F (x \cdot_F f) = (g \cdot_F x) \cdot_F f \in F\,B\,c$.

Henceforth, we will use the following explicit description of the coend (4):

$$\Big(\sum_{A \in !\mathbb{A}} P\,A\,b \times \widehat{\mathbb{A}}\big[\mathrm{S}_{\mathbb{A}}(A), X\big]\Big)_{/\approx} \quad \big(X \in \widehat{\mathbb{A}},\ b \in \mathbb{B}^{\circ}\big)$$

where $\approx$ is the equivalence relation generated by

$$\big(A, p, \mathrm{S}_{\mathbb{A}}(\alpha) \cdot x\big) \sim \big(A', p \cdot_P \alpha, x\big) \tag{5}$$

for all $\alpha : A \to A'$ in $!\mathbb{A}$, $p \in P\,A\,b$, and $x : \mathrm{S}_{\mathbb{A}}(A') \to X$ in $\widehat{\mathbb{A}}$. Further, we write $p \underset{A'}{\otimes} x$ for the equivalence class of (A, p, x). Under this convention, the identification (5) amounts to the identity

$$p \underset{A}{\otimes} \big(\mathrm{S}_{\mathbb{A}}(\alpha) \cdot x\big) = (p \cdot_P \alpha) \underset{A'}{\otimes} x$$

and the functorial action of $\widetilde{P}$ is given by

$$\beta \cdot_{\widetilde{P}} (p \underset{A}{\otimes} x) \cdot_{\widetilde{P}} f \stackrel{\text{def}}{=} (\beta \cdot_P p) \underset{A}{\otimes} (x \cdot f)$$

for all $(p \underset{A}{\otimes} x) \in \widetilde{P}\,X\,b$, $f : X \to X'$ in $\widehat{\mathbb{A}}$ and $\beta : b' \to b$ in $\mathbb{B}$.

Notation. For categories $\mathcal{A}$ and $\mathcal{B}$, we let $\mathfrak{CAT}[\mathcal{A}, \mathcal{B}]$ denote the category of functors $\mathcal{A} \to \mathcal{B}$ and natural transformations between them.

Proposition 3.2. *The functor* $\widetilde{(\_)} : \mathfrak{CAT}\big[!\mathbb{A}, \widehat{\mathbb{B}}\big] \to \mathfrak{CAT}\big[\widehat{\mathbb{A}}, \widehat{\mathbb{B}}\big]$ *is faithful.*

This is a consequence of the following.

Lemma 3.3. *Let* $P : !\mathbb{A} \to \widehat{\mathbb{B}}$. *For* $\alpha_0 : A_0 \to A$ *in* $!\mathbb{A}$, *and* $p_0 \underset{A_0}{\otimes} \mathrm{S}(\alpha_0)$ *and* $p_1 \underset{A_1}{\otimes} x_1$ *in* $\widetilde{P}(\mathrm{S}A)(b)$, *if* $p_0 \underset{A_0}{\otimes} \mathrm{S}(\alpha_0) = p_1 \underset{A_1}{\otimes} x_1$ *then there exists (a necessarily unique)* $\alpha_1 : A_1 \to A$ *in* $!\mathbb{A}$ *such that* $x_1 = \mathrm{S}(\alpha_1)$ *and* $p_0 \cdot_P \alpha_0 = p_1 \cdot_P \alpha_1$.

Proof. It is enough to establish the lemma in the following two cases.

- When there exists $\alpha : A_1 \to A_0$ in $!\mathbb{A}$ such that $p_1 \cdot_P \alpha = p_0$ and $\mathrm{S}(\alpha) \cdot \mathrm{S}(\alpha_0) = x_1$. In which case, taking $\alpha_1 = \alpha \cdot \alpha_0$ we are done.
- When there exists $\alpha : A_0 \to A_1$ in $!\mathbb{A}$ such that $p_0 \cdot_P \alpha = p_1$ and $\mathrm{S}\alpha \cdot x_1 = S(\alpha_0)$. In which case, $x_1 : \mathrm{S}(A_1) \to \mathrm{S}(A)$ in $\widehat{\mathbb{A}}$ is bijective on indices and hence, by Proposition 2.2(ii), there exists $\alpha_1 : A_1 \to A$ in $!\mathbb{A}$ such that $\mathrm{S}(\alpha_1) = x_1$. Further, by Proposition 2.2(i), we have that $\alpha_0 = \alpha \cdot \alpha_1$ and hence also that $p_0 \cdot_P \alpha_0 = p \cdot_P (\alpha \cdot \alpha_1) = (p \cdot_P \alpha) \cdot_P \alpha = p_1 \cdot_P \alpha_1$. □

Corollary 3.4. *For* $P : !\mathbb{A} \to \widehat{\mathbb{B}}$, *if* $p \underset{A}{\otimes} \mathrm{id}_{\mathrm{S}A} = p' \underset{A}{\otimes} \mathrm{id}_{\mathrm{S}A}$ *in* $\widetilde{P}(\mathrm{S}A)(b)$ *then* $p = p'$ *in* $P\,A\,b$.

4. Coefficients functors

Via the canonical natural isomorphisms

$$\widehat{\mathbb{A}}[SA, X] \cong \prod_{i \in |A|} \widehat{\mathbb{A}}\big[\mathrm{y}(A_i), X\big] \cong \prod_{i \in |A|} X(A_i) \quad (A \in !\mathbb{A}, X \in \widehat{\mathbb{A}})$$

every analytic functor $F : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ admits a *Taylor series development* as follows

$$F\,X\,b \cong \left(\sum_{n \in \mathbb{N}} \sum_{a_1,\ldots,a_n \in \mathbb{A}} P\left(\bigoplus_{i=1}^{n} \langle [a_i] \rangle \right)(b) \times \prod_{i=1}^{n} X(a_i) \right)_{/\approx} \quad \big(X \in \widehat{\mathbb{A}},\ b \in \mathbb{B}^\circ\big) \tag{6}$$

for some *coefficients functor* $P : !\mathbb{A} \to \widehat{\mathbb{B}}$ (referred to as an $(\mathbb{A}, \mathbb{B})$-*species of structures* in [4,5]). The representation of analytic functors (6) for $\mathbb{A}$ a finite discrete category and $\mathbb{B}$ the one-object category directly exhibits them as the multivariate analytic functors of A. Joyal [11, §1.1].

The coefficients functors of an analytic functor are unique up to isomorphism.

Proposition 4.1. *The functor* $\widetilde{(\_)} : \mathcal{CAT}\big[!\mathbb{A}, \widehat{\mathbb{B}}\big] \to \mathcal{CAT}\big[\widehat{\mathbb{A}}, \widehat{\mathbb{B}}\big]$ *is conservative. That is, for* $P, Q : !\mathbb{A} \to \widehat{\mathbb{B}}$*, if* $\widetilde{P} \cong \widetilde{Q} : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ *then* $P \cong Q$*.*

This result is a corollary of Proposition 4.3 below, for which we need to recall that a natural transformation is said to be *quasi-cartesian* whenever all its naturality squares are quasi-pullbacks, where a *quasi-pullback* is a commutative square for which the unique mediating morphism from its span to the pullback of its cospan is an epimorphism.

The notion of quasi-pullback in presheaf categories is given pointwise.

Lemma 4.2. *For a small category* $\mathbb{C}$*, a commutative square in* $\widehat{\mathbb{C}}$ *as on the left below*

$$\begin{array}{ccc} Q & \xrightarrow{k} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array} \qquad\qquad \begin{array}{ccc} Q\,c & \xrightarrow{k_c} & Y\,c \\ {\scriptstyle h_c}\downarrow & & \downarrow{\scriptstyle g_c} \\ X\,c & \xrightarrow[f_c]{} & Z\,c \end{array}$$

is a quasi-pullback iff so are the commutative squares in $\mathcal{S}\boldsymbol{et}$ *as on the right above for every* $c \in \mathbb{C}$*.*

Proof. Follows from the facts that in presheaf categories limits and colimits are given pointwise and that the functors that evaluate presheaves at an object preserve them. □

Proposition 4.3. *Let* $P, Q : !\mathbb{A} \to \widehat{\mathbb{B}}$ *and* $\varphi : \widetilde{P} \Rightarrow \widetilde{Q} : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$*. For the following statements:*

(i) *the natural transformation* φ *is quasi-cartesian,*
(ii) *for every* $A \in !\mathbb{A}$*,* $b \in \mathbb{B}^\circ$*, and* $p \in P\,A\,b$ *there exists (a necessarily unique)* $q \in Q\,A\,b$ *such that* $\varphi(p \underset{A}{\otimes} \mathrm{id}_{SA}) = q \underset{A}{\otimes} \mathrm{id}_{SA}$*,*
(iii) *there exists a (necessarily unique) natural transformation* $\phi : P \Rightarrow Q : !\mathbb{A} \to \widehat{\mathbb{B}}$ *such that* $\varphi = \widetilde{\phi}$*,*

we have that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)*.*

Proof. (i) $\Rightarrow$ (ii) For $p \in P\,A\,b$, let

$$\varphi_{SA,b}(p \underset{A}{\otimes} \mathrm{id}_{SA}) = \big(q \underset{A'}{\otimes} \mathrm{id}_{SA'}\big) \cdot_{\widetilde{P}} s$$

for $q \in Q(A')(b)$ and $s : SA' \to SA$ in $\widehat{\mathbb{A}}$.

Since φ is quasi-cartesian, there exists $(p_0 \underset{A_0}{\otimes} s_0) \in \widetilde{P}(SA')(b)$ such that

$$p_0 \underset{A_0}{\otimes} (s_0 \cdot s) = p \underset{A}{\otimes} \mathrm{id}_{SA} \tag{7}$$

and

$$\varphi_{SA',b}(p_0 \underset{A_0}{\otimes} s_0) = q \underset{A'}{\otimes} \mathrm{id}_{SA'}$$

From (7), by Lemma 3.3, there exists $\alpha_0 : A_0 \to A$ in $!\mathbb{A}$ such that

$$s_0 \cdot s = \mathrm{S}(\alpha_0) \tag{8}$$

and $p_0 \cdot_P \alpha_0 = p$. In particular, thus, $s_0 : \mathrm{S}(A_0) \to \mathrm{S}(A')$ in $\widehat{\mathbb{A}}$ is injective on indices.

Now, let

$$\varphi_{S(A_0),b}(p_0 \underset{A_0}{\otimes} \mathrm{id}_{S(A_0)}) = q_1 \underset{A_1}{\otimes} s_1$$

for $q_1 \in Q(A_1)(b)$ and $s_1 : S(A_1) \to S(A_0)$ in $\widehat{\mathbb{A}}$. By naturality of φ, we have that

$$q_1 \underset{A_1}{\otimes} (s_1 \cdot s_0) = q \underset{A'}{\otimes} \mathrm{id}_{S(A')}$$

and, by Lemma 3.3, that there exists $\alpha_1 : A_1 \to A'$ in $!\mathbb{A}$ such that $s_1 \cdot s_0 = S(\alpha_1)$ and $q_1 \cdot_Q \alpha_1 = q$. In particular, thus, s_0 is surjective, and hence bijective, on indices. It then follows from (8) that also s is bijective on indices and hence that there exists $\alpha : A' \to A$ such that $S\alpha = s$.

Thus, $\varphi_{SA,b}(p \underset{A}{\otimes} \mathrm{id}_{SA}) = (q \cdot_Q \alpha) \underset{A}{\otimes} \mathrm{id}_{SA}$.

(ii) ⇒ (iii) The family of mappings $\phi_{A,b} : P\,A\,b \to Q\,A\,b$ ($A \in !\mathbb{A}$, $b \in \mathbb{B}^\circ$) associating $p \in P(A)(b)$ with the unique $q \in Q(A)(b)$ such that $\varphi_{SA,b}(p \underset{A}{\otimes} \mathrm{id}_{SA}) = (q \underset{A}{\otimes} \mathrm{id}_{SA})$ determine a natural transformation $\phi : P \Rightarrow Q$ with the desired property. □

It is interesting to note that not every natural transformation in the image of $\widetilde{(\_)} : \mathcal{CAT}[!\mathbb{A}, \widehat{\mathbb{B}}] \to \mathcal{CAT}[\widehat{\mathbb{A}}, \widehat{\mathbb{B}}]$ is quasi-cartesian. Indeed, for $\boldsymbol{\Sigma} \stackrel{\mathrm{def}}{=} (\bot \to \top)$, $P \stackrel{\mathrm{def}}{=} !\boldsymbol{\Sigma}[\langle[\top]\rangle, \_]$, and $\phi : P \Rightarrow 1 : !\boldsymbol{\Sigma} \to \mathcal{S}\boldsymbol{et}$, the naturality square associated to $\widetilde{\phi} : \widetilde{P} \Rightarrow \widetilde{1} : \widehat{\boldsymbol{\Sigma}} \to \mathcal{S}\boldsymbol{et}$ induced by $\mathrm{y}(\bot) \to \mathrm{y}(\top)$ in $\widehat{\boldsymbol{\Sigma}}$ is not a quasi-pullback. However, we have the following result.

Proposition 4.4. *For $\phi : P \Rightarrow Q : !\mathbb{G} \to \widehat{\mathbb{C}}$ where $\mathbb{G}$ is a small groupoid, the natural transformation $\widetilde{\phi} : \widetilde{P} \Rightarrow \widetilde{Q} : \widehat{\mathbb{G}} \to \widehat{\mathbb{C}}$ is quasi-cartesian.*

Proof. For $f : X \to Y$ in $\widehat{\mathbb{G}}$, let $(p \underset{G}{\otimes} y) \in \widetilde{P}\,Y\,b$ and $(q \underset{G'}{\otimes} x) \in \widetilde{Q}\,X\,b$ be such that

$$\phi_{G,b}(p) \underset{G}{\otimes} y = \widetilde{\phi}_{Y,b}(p \underset{G}{\otimes} y) = (q \underset{G'}{\otimes} x) \cdot_{\widetilde{Q}} f = q \underset{G'}{\otimes} (x \cdot f)$$

Then, as $\mathbb{G}$ is a groupoid, it follows that there exists $\sigma : G \to G'$ in $!\mathbb{G}$ such that $\phi_{G,b}(p) \cdot_Q \sigma = q$ and $y = S(\sigma) \cdot x \cdot f$.

Since, for $p \underset{G}{\otimes} (S\sigma \cdot x) = (p \cdot_P \sigma) \underset{G'}{\otimes} x$ in $\widetilde{P}\,X\,b$ we have that $(p \underset{G}{\otimes} (S\sigma \cdot x)) \cdot_{\widetilde{P}} f = p \underset{G}{\otimes} (S\sigma \cdot x \cdot f) = p \underset{G}{\otimes} y$ and $\widetilde{\phi}_{X,b}((p \cdot_P \sigma) \underset{G'}{\otimes} x) = (\phi_{G',b}(p \cdot_P \sigma)) \underset{G'}{\otimes} x = (\phi_{G,b}(p) \cdot_P \sigma) \underset{G'}{\otimes} x = q \underset{G'}{\otimes} x$ we are done. □

Quasi-cartesian natural transformations are closed under vertical composition and we are naturally led to introduce the following.

Definition 4.5. For small categories $\mathbb{A}$ and $\mathbb{B}$, we let $\boldsymbol{\mathcal{AF}}[\mathbb{A}, \mathbb{B}]$ be the subcategory of $\mathcal{CAT}\big[\widehat{\mathbb{A}}, \widehat{\mathbb{B}}\big]$ consisting of analytic functors and quasi-cartesian natural transformations between them.

Corollary 4.6. *For $\mathbb{G}$ a small groupoid, the functor $\widetilde{(\_)} : \mathcal{CAT}[!\mathbb{G}, \widehat{\mathbb{C}}] \to \mathcal{CAT}[\widehat{\mathbb{G}}, \widehat{\mathbb{C}}]$ restricts to an essentially surjective, full and faithful functor*

$$\widetilde{(\_)} : \mathcal{CAT}[!\mathbb{G}, \widehat{\mathbb{C}}] \to \boldsymbol{\mathcal{AF}}[\mathbb{G}, \mathbb{C}] \tag{9}$$

5. Generic coefficients functor

We proceed to construct a quasi-inverse to (9) when the small category $\mathbb{C}$ is a groupoid. The central notion isolated by A. Joyal for this purpose is that of generic element [11, Appendice, Définition 2].

Definition 5.1. For $F : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$, we say that $x \in F\,X\,b$ is *generic* if for every cospan $f : X \to Z \leftarrow Y : g$ in $\widehat{\mathbb{A}}$ and $y \in F\,Y\,b$ such that $x \cdot_F f = y \cdot_F g$ there exists $h : X \to Y$ in $\widehat{\mathbb{A}}$ such that $f = h \cdot g$ and $x \cdot_F h = y$.

For instance, it follows from the proposition below that for $P : !\mathbb{G} \to \widehat{\mathbb{C}}$ with $\mathbb{G}$ a small groupoid, $G \in !\mathbb{G}$, and $c \in \mathbb{C}$, the generic elements in $\widetilde{P}(SG)(c)$ are of the form $p \underset{G}{\otimes} \mathrm{id}_{SG}$ for $p \in P\,G\,c$.

Proposition 5.2. *For $P : !\mathbb{G} \to \widehat{\mathbb{C}}$ with $\mathbb{G}$ a small groupoid, $(p \underset{G}{\otimes} x) \in \widetilde{P}\,X\,c$ is generic iff $x : SG \to X$ in $\widehat{\mathbb{G}}$ is an isomorphism.*

Proof. (⇒) Let $(p \underset{G}{\otimes} x) \in \widetilde{P}\,X\,c$ be generic. As $(p \underset{G}{\otimes} x) = (p \underset{G}{\otimes} \mathrm{id}_{SG}) \cdot_{\widetilde{P}} x$, there exists $h : X \to SG$ such that $h \cdot x = \mathrm{id}_X$ and $p \underset{G}{\otimes} (x \cdot h) = (p \underset{G}{\otimes} x) \cdot_{\widetilde{P}} h = (p \underset{G}{\otimes} \mathrm{id}_{SG})$. The latter identity implies that $x \cdot h$ is an automorphism on SG, and we are done.

($\Leftarrow$) Let $(p \underset{G}{\otimes} x) \in \widetilde{P}\,X\,c$ with x an isomorphism. Consider a cospan $f : X \to Z \leftarrow Y : g$ and $(q \underset{H}{\otimes} y) \in \widetilde{P}\,Y\,c$ with $(p \underset{G}{\otimes} (x \cdot f)) = ((p \underset{G}{\otimes} x) \cdot_{\widetilde{P}} f) = ((q \underset{H}{\otimes} y) \cdot_{\widetilde{P}} g) = (q \underset{H}{\otimes} (y \cdot g))$. Then, there exists $\sigma : G \to H$ in $!\mathbb{G}$ such that $p \cdot_P \sigma = q$ and $x \cdot f = (\mathrm{S}\sigma) \cdot y \cdot g$; and the map $x^{-1} \cdot (\mathrm{S}\sigma) \cdot y : X \to Y$ has the desired properties. □

Lemma 5.3. *Let $F : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$. For every $x \in F\,X\,b$ generic, $y \in F\,Y\,b$, and $f : Y \to X$ in $\widehat{\mathbb{A}}$ such that $y \cdot_F f = x$, one has that f is split epi.*

Proof. Because the hypotheses imply the existence of $h : X \to Y$ such that $x \cdot_F h = y$ and $h \cdot f = \mathrm{id}_X$. □

We now explain how analytic functors from presheaf categories over groupoids are engendered by their compact generic elements uniquely up to isomorphism.

Definition 5.4. A functor $F : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ is said to be engendered by its (compact) generic elements whenever for every $x \in F\,X\,b$ there exists a generic element $x_0 \in F(X_0)(b)$ (with $X_0 = \mathrm{S}A$ for $A \in !\mathbb{A}$) and $f : X_0 \to X$ in $\widehat{\mathbb{A}}$ such that $x_0 \cdot_F f = x$.

Proposition 5.5. *Let $F : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$. For $x \in F(\mathrm{S}A)(b)$ and $x' \in F(\mathrm{S}A')(b)$ both generic, and for $f : \mathrm{S}A \to X$ and $f' : \mathrm{S}A' \to X$ in $\widehat{\mathbb{A}}$ such that $x \cdot_F f = x' \cdot_F f'$, there exists a split epi $\alpha : A \to A'$ in $!\mathbb{A}$ such that $x \cdot_F \mathrm{S}(\alpha) = x'$ and $f = \mathrm{S}(\alpha) \cdot f'$.*

Proof. Since x is generic, there exists $g : \mathrm{S}(A) \to \mathrm{S}(A')$ in $\widehat{\mathbb{A}}$ such that $x \cdot_F g = x'$ and $g \cdot f' = f$. Further, since x' is generic, by Lemma 5.3, g is split epi. Analogously, since x' is generic, there exists $g' : \mathrm{S}(A') \to \mathrm{S}(A)$ in $\widehat{\mathbb{A}}$ such that $x' \cdot_F g' = x$ and $g' \cdot f = f'$. Further, since x is generic, by Lemma 5.3, g' is split epi.

As $g : \mathrm{S}(A) \to \mathrm{S}(A')$ and $g' : \mathrm{S}(A') \to \mathrm{S}(A)$ in $\widehat{\mathbb{A}}$ are both surjective, and hence bijective, on indices, there exist $\alpha : A \to A'$ and $\alpha' : A' \to A$ in $!\mathbb{A}$ such that $\mathrm{S}\alpha = g$ and $\mathrm{S}\alpha' = g'$. Moreover, a section $\mathrm{S}(A') \to \mathrm{S}(A)$ of $\mathrm{S}\alpha$ in $\widehat{\mathbb{A}}$ is necessarily bijective on indices and hence of the form $\mathrm{S}\sigma$ for $\sigma : A' \to A$ in $!\mathbb{A}$. Finally, by Proposition 2.2(i), the identity $\mathrm{S}(\sigma \cdot \alpha) = \mathrm{id}_{\mathrm{S}(A')}$ implies that σ is a section of α. □

Proposition 5.6. *Every analytic functor $\widehat{\mathbb{G}} \to \widehat{\mathbb{C}}$ with $\mathbb{G}$ a small groupoid is engendered by its compact generic elements uniquely up to isomorphism.*

Proof. It is enough to consider $\widetilde{P} : \widehat{\mathbb{G}} \to \widehat{\mathbb{C}}$ for $P : !\mathbb{G} \to \widehat{\mathbb{C}}$. In which case, for every $(p \underset{G}{\otimes} x) \in \widetilde{P}\,X\,c$ one has $(p \underset{G}{\otimes} \mathrm{id}_{\mathrm{S}G}) \cdot_{\widetilde{P}} x$. □

Most importantly, generic elements of functors between presheaf categories over groupoids are invariant under the functorial action.

Lemma 5.7. *Let $F : \widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$ for $\mathbb{G}$ and $\mathbb{H}$ small groupoids. If $x \in F(\mathrm{S}G)(h)$ is generic then so is the element $(\xi \cdot_F x \cdot_F \mathrm{S}\sigma) \in F(\mathrm{S}G')(h')$ for all $\sigma : G \to G'$ in $!\mathbb{G}$ and $\xi : h' \to h$ in $\mathbb{H}$.*

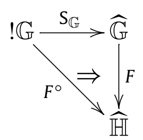
Proof. We first show that $(x \cdot_F \mathrm{S}\sigma) \in F(\mathrm{S}G')(h)$ is generic. So, consider a cospan $f : \mathrm{S}G' \to Z \leftarrow Y : g$ in $\widehat{\mathbb{G}}$ and $y \in F\,Y\,h$ such that $(x \cdot_F \mathrm{S}\sigma) \cdot_F f = y \cdot_F g$ in $F\,Z\,h$. As x is generic, there exists $k : \mathrm{S}G \to Y$ in $\widehat{\mathbb{G}}$ such that $\mathrm{S}(\sigma) \cdot f = k \cdot g$ and $x \cdot_F k = y$. Then, $(\mathrm{S}\sigma^{-1}) \cdot k : \mathrm{S}G' \to Y$ exhibits $x \cdot_F \mathrm{S}\sigma$ as generic.

Second, let us see that $(\xi \cdot_F x) \in F(\mathrm{S}G)(h')$ is generic. To this end, consider a cospan $f : \mathrm{S}G \to Z \leftarrow Y : g$ in $\widehat{\mathbb{G}}$ and $y \in F\,Y\,h'$ such that $(\xi \cdot_F x) \cdot_F f = y \cdot_F g$ in $F\,Z\,h'$. Then, $x \cdot_F f = (\xi^{-1} \cdot_F y) \cdot_F g$ and since x is generic, there exists $k : \mathrm{S}G \to Y$ in $\widehat{\mathbb{G}}$ such that $f = k \cdot g$ and $x \cdot_F k = \xi^{-1} \cdot_F y$; so that $(\xi \cdot_F x) \cdot_F k = y$. □

For $F : \widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$, define

$$F^\circ(G)(h) \overset{\mathrm{def}}{=} \{x \in F(\mathrm{S}G)(h) \mid x \text{ is generic}\} \quad (G \in !\mathbb{G},\ h \in \mathbb{H}^\circ)$$

By Lemma 5.7, for $\mathbb{G}$ and $\mathbb{H}$ small groupoids, we have a functor $F^\circ : !\mathbb{G} \to \widehat{\mathbb{H}}$ with action, for σ in $!\mathbb{G}$ and ξ in $\mathbb{H}$, given by $F^\circ(\sigma)(\xi) \overset{\mathrm{def}}{=} F(\mathrm{S}\sigma)(\xi)$. As F° is a subfunctor of the restriction of F along $\mathrm{S}_{\mathbb{G}}$, we have the following situation



from which, by the universal property of left Kan extensions, we obtain a canonical natural transformation $\eta^F : \widetilde{F^\circ} \Rightarrow F : \widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$ explicitly given by

$$\int^{G \in !\mathbb{G}} F^{\circ}(G)(h) \times !\mathbb{G}[SG, X] \xrightarrow{\eta^F{}_{X,h}} F(X)(h)$$
$$p \underset{G}{\otimes} x \mapsto p \cdot_F x$$

These mappings will be now shown to be injective. To this end, we need consider an important minimality property of generic elements (see [11, Appendice, Définition 5]).

Definition 5.8. For $F : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$, we say that $x \in F\,X\,b$ is *minimal* if for every $y \in F\,Y\,b$ and $f : Y \to X$ in $\widehat{\mathbb{A}}$, $y \cdot_F f = x$ implies f epi.

Proposition 5.9. *For $P : !\mathbb{G} \to \widehat{\mathbb{C}}$ with $\mathbb{G}$ a small groupoid, $(p \underset{G}{\otimes} x) \in \widetilde{P}\,X\,c$ is minimal iff x is epi.*

Proof. ($\Rightarrow$) Follows from the definition of minimality using that $(p \underset{G}{\otimes} \mathrm{id}_{SG}) \cdot_{\widetilde{P}} x = p \underset{G}{\otimes} x$.

($\Leftarrow$) Let $(q \underset{G'}{\otimes} y) \in \widetilde{P}\,Y\,c$ and $f : Y \to X$ in $\widehat{\mathbb{G}}$ be such that $q \underset{G'}{\otimes} (y \cdot f) = (q \underset{G'}{\otimes} y) \cdot_{\widetilde{P}} f = (p \underset{G}{\otimes} x)$. It follows that there exists an isomorphism $\sigma : G' \to G$ in $!\mathbb{G}$ such that $(S\sigma) \cdot x = y \cdot f$. Thus, if x is epi then so is f. $\square$

Proposition 5.10. *The generic elements of a functor between presheaf categories are minimal.*

Proof. By Lemma 5.3. $\square$

Proposition 5.11. *For every $F : \widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$ with $\mathbb{G}$ and $\mathbb{H}$ small groupoids, its associated natural transformation η^F is a monomorphism.*

Proof. Let $p \underset{G}{\otimes} x$ and $q \underset{G'}{\otimes} y$ in $\widetilde{F^{\circ}}\,X\,h$ be such that $p \cdot_F x = q \cdot_F y$.

Since $p \in F(SG)(h)$ is generic and $q \in F(SG')(h)$ is minimal, there exists an epimorphism $f : SG \to SG'$ in $\widehat{\mathbb{G}}$ such that $p \cdot_F f = q$ and $f \cdot y = x$. Analogously, since $q \in F(SG')(h)$ is generic and $p \in F(SG)(h)$ is minimal, there exists an epimorphism $g : SG' \to SG$ in $\widehat{\mathbb{G}}$ such that $q \cdot_F g = p$ and $g \cdot x = y$.

By Proposition 2.3(i), f and g are bijective on indices and hence there exist $\sigma : G \to G'$ and $\tau : G' \to G$ in $!\mathbb{G}$ such that $S\sigma = f$ and $S\tau = g$.

It follows that

$$\begin{aligned} p \underset{G}{\otimes} x &= p \underset{G}{\otimes} (f \cdot y) = p \underset{G}{\otimes} \big(\mathrm{S}(\sigma) \cdot y\big) \\ &= (p \cdot_{\widetilde{F^{\circ}}} \sigma) \underset{G'}{\otimes} y = (p \cdot_F S\sigma) \underset{G'}{\otimes} y \\ &= (p \cdot_F f) \underset{G'}{\otimes} y = q \underset{G'}{\otimes} y \qquad \square \end{aligned}$$

Thus, a functor between presheaf categories over groupoids is analytic iff it is engendered by its compact generic elements.

Corollary 5.12. *A functor $F : \widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$ with $\mathbb{G}$ and $\mathbb{H}$ small groupoids is analytic iff its associated natural transformation $\eta^F : \widetilde{F^{\circ}} \Rightarrow F$ is an epimorphism.*

In particular, the coefficients functor of an analytic functor between presheaf categories over groupoids is characterised by its generic elements. Furthermore, since quasi-cartesian natural transformations between such analytic functors are precisely those that preserve generic elements, this correspondence extends to an equivalence of categories between coefficient functors (and natural transformations) and analytic functors (and quasi-cartesian natural transformations).

Proposition 5.13.

(i) *Quasi-cartesian natural transformations between functors $\widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ preserve generic elements.*
(ii) *If a natural transformation between analytic functors $\widehat{\mathbb{G}} \to \widehat{\mathbb{C}}$ with $\mathbb{G}$ a small groupoid preserves generic elements then it is quasi-cartesian.*

Corollary 5.14. *For small groupoids $\mathbb{G}$ and $\mathbb{H}$, the functors*

$$\mathcal{CAT}[!\mathbb{G}, \widehat{\mathbb{H}}] \underset{(\_)^{\circ}}{\overset{\widetilde{(\_)}}{\rightleftarrows}} \mathcal{AF}[\mathbb{G}, \mathbb{H}]$$

establish an equivalence of categories.

6. Characterisation of analytic functors

We conclude the paper with two characterisations of analytic functors between presheaf categories over groupoids by means of preservation properties. As a first step in this direction, we leave the verification of the following to the reader.

Proposition 6.1. *Analytic functors $\widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ preserve filtered colimits. For $\mathbb{A}$ a groupoid, they further preserve wide quasi-pullbacks and cofiltered limits.*

Recall that a *wide quasi-pullback* is a commutative diagram $(\, Q \longrightarrow D_i \longrightarrow D \,)_{i \in I}$ for an indexing set I such that the unique mediating morphism from the cone $(\, Q \longrightarrow D_i \quad D \,)_{i \in I}$ to a limiting cone of the diagram $(\, D_i \longrightarrow D \,)_{i \in I}$ is an epimorphism.

Corollary 6.2. *Analytic endofunctors on presheaf categories over groupoids have both initial algebra and final coalgebra.*

We will now consider the following properties of functors between presheaf categories over groupoids:

(1) preservation of filtered colimits,
(2) preservation of epimorphisms,
(3) preservation of quasi-pullbacks,
(4) preservation of wide quasi-pullbacks,
(5) preservation of cofiltered limits,
(6) being engendered by compact minimal elements,
(7) being engendered by compact generic elements (i.e. analytic),

and show

Proposition 6.4: $(1) \,\&\, (2) \Rightarrow (6)$

Proposition 6.6: $(4) \,\&\, (6) \Rightarrow (7)$

Proposition 6.7: $(3) \,\&\, (5) \,\&\, (6) \Rightarrow (7)$

so that, since $(4) \Rightarrow (3) \Rightarrow (2)$, we have that

$$(1) \,\&\, (4) \Rightarrow (7) \quad \text{and} \quad (1) \,\&\, (3) \,\&\, (5) \Rightarrow (7)$$

Definition 6.3. A functor $F : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ is said to be engendered by its (compact) minimal elements whenever for every $x \in F\,X\,b$ there exists a minimal element $x_0 \in F(X_0)(b)$ (with $X_0 = SA$ for $A \in !\mathbb{A}$) and $f : X_0 \to X$ in $\widehat{\mathbb{A}}$ such that $x_0 \cdot_F f = x$.

Proposition 6.4. *Every functor $\widehat{\mathbb{G}} \to \widehat{\mathbb{C}}$, with $\mathbb{G}$ a small groupoid, preserving filtered colimits and epimorphisms is engendered by its compact minimal elements.*

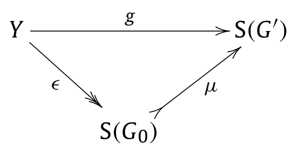
Proof. Let $F : \widehat{\mathbb{G}} \to \widehat{\mathbb{C}}$ be a functor, with $\mathbb{G}$ a small groupoid, preserving filtered colimits and epimorphisms, and let $x \in F\,X\,c$.

Since $X \in \widehat{\mathbb{G}}$ is a filtered colimit of finitely presentable objects, there exist a finitely presentable object $X_0 \in \widehat{\mathbb{G}}$, an element $x_0 \in F(X_0)(c)$, and a morphism $f : X_0 \to X$ in $\widehat{\mathbb{G}}$ such that $x_0 \cdot_F f = x$.

Further, since finitely presentable objects in $\widehat{\mathbb{G}}$ are quotients of finite coproducts of representables, there exist an object $G \in !\mathbb{G}$, an element $x_1 \in F(SG)(c)$, and an epimorphism $q : SG \twoheadrightarrow X_0$ in $\widehat{\mathbb{G}}$ such that $x_1 \cdot_F q = x_0$.

Let $G' \in !\mathbb{G}$, $x' \in F(SG')(c)$, and $m : \mathrm{S}(G') \rightarrowtail \mathrm{S}(G)$ a monomorphism in $\widehat{\mathbb{G}}$ be such that $x' \cdot_F m = x_1$ with $|G'|$ chosen minimally. We have that x' is a compact element engendering x, and we now show that it is minimal.

Indeed, consider $y \in F\,Y\,c$ and $g : Y \to \mathrm{S}(G')$ in $\widehat{\mathbb{G}}$ such that $y \cdot_F g = x'$. Note that the epi-mono factorisation of g is of the form



because, as $\mathbb{G}$ is a groupoid,

$$\mathrm{Sub}_{\widehat{\mathbb{G}}}(SG) = \left\{ \sum_{i \in I} \mathrm{y}(G_i) \;\middle|\; I \subseteq |G| \right\} \quad (G \in !\mathbb{G})$$

Thus we have $G_0 \in !\mathbb{G}$, $y \cdot_F \epsilon \in F(\mathrm{S}(G_0))(c)$, and the monomorphism $\mu \cdot m : \mathrm{S}(G_0) \rightarrowtail \mathrm{S}(G)$ in $\widehat{\mathbb{G}}$ satisfying $(y \cdot_F \epsilon) \cdot_F (\mu \cdot m) = x_1$, from which it follows by the minimality of $|G'|$ that $|G'| \subseteq |G_0|$. Hence, the monomorphism μ is bijective on indices and therefore (since $\mathbb{G}$ is a groupoid) an isomorphism, establishing that g is epi. □

Lemma 6.5.

(i) *For $F : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$, if $x \in F\,X\,b$ is generic then for every minimal $y \in F\,Y\,b$ and $f : Y \to X$ in $\widehat{\mathbb{A}}$, $y \cdot_F f = x$ implies f iso.*

(ii) *Let $F : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ be a functor engendered by its (compact) minimal elements and preserving quasi-pullbacks. For $x \in F\,X\,b$, if for every (compact) minimal element $y \in F\,Y\,b$ (with $Y = \mathrm{S}A$ for $A \in {!\mathbb{A}}$) and $f : Y \to X$ in $\widehat{\mathbb{A}}$, $y \cdot_F f = x$ implies f iso, then x is generic.*

Proof. (i) Assume the hypotheses. By Lemma 5.3, f has a section $g : X \to Y$ in $\widehat{\mathbb{A}}$ such that $g \cdot_F x = y$. Since y is minimal, g is epi and hence an iso, and then so is f.

(ii) Let $x \in F\,X\,b$ satisfy the hypothesis of the statement, and let the cospan $f : X \to Z \leftarrow Y : g$ in $\widehat{\mathbb{A}}$ and $y \in F\,Y\,b$ be such that $x \cdot_F f = y \cdot_F g$ in $F\,Z\,b$.

Consider a pullback square

$$\begin{array}{ccc} P & \xrightarrow{\;q\;} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[\;f\;]{} & Z \end{array}$$

in $\widehat{\mathbb{A}}$. Since F preserves quasi-pullbacks, there exists $z \in F\,P\,b$ such that $z \cdot_F p = x$ and $z \cdot_F q = y$. Further, since F is engendered by its (compact) minimal elements, there exists $Z_0 \in \widehat{\mathbb{A}}$ (with $Z_0 = \mathrm{S}A$ for $A \in {!\mathbb{A}}$), $z_0 \in F(Z_0)(b)$ minimal, and $h : Z_0 \to Z$ in $\widehat{\mathbb{A}}$ such that $z_0 \cdot_F h = z$.

By hypothesis then, as $z_0 \cdot_F (h \cdot p) = x$, we have that $h \cdot p : Z_0 \to X$ in $\widehat{\mathbb{A}}$ is an isomorphism. We thus have $(h \cdot p)^{-1} \cdot h \cdot q : X \to Y$ in $\widehat{\mathbb{A}}$ such that

$$\big((h \cdot p)^{-1} \cdot h \cdot q\big) \cdot g = (h \cdot p)^{-1} \cdot h \cdot p \cdot f = f$$

and

$$x \cdot_F \big((h \cdot p)^{-1} \cdot h \cdot q\big) = z_0 \cdot_F (h \cdot q) = z \cdot_F q = y$$

showing that x is generic. □

Proposition 6.6. *Every functor $\widehat{\mathbb{G}} \to \widehat{\mathbb{C}}$, with $\mathbb{G}$ a small groupoid, engendered by its compact minimal elements and preserving wide quasi-pullbacks is engendered by its compact generic elements.*

Proof. Let $F : \widehat{\mathbb{G}} \to \widehat{\mathbb{C}}$, with $\mathbb{G}$ a small groupoid, be a functor engendered by its compact minimal elements and preserving wide quasi-pullbacks.

For $x \in F\,X\,b$ consider the wide cospan

$$\nabla = \big\langle \nabla_{(x_0,f)} = f : \mathrm{S}G \to X \text{ in } \widehat{\mathbb{G}} \;\big|\; x_0 \in F(\mathrm{S}G)(b) \text{ is minimal and } x_0 \cdot_F f = x \big\rangle$$

and let $\pi : P \dot{\to} \nabla$ be a limiting cone in $\widehat{\mathbb{G}}$. (Note that, as F is engendered by its compact minimal elements, ∇ is non-empty.)

Since F preserves wide quasi-pullbacks, there exists $p \in F\,P\,b$ such that, for all minimal $x_0 \in F(\mathrm{S}G)(b)$ and $f : \mathrm{S}G \to X$ in $\widehat{\mathbb{G}}$ with $x_0 \cdot_F f = x$, we have that $p \cdot_F \pi_{(x_0,f)} = x_0$. Thus, the cone π consists of epimorphims.

We now show the following general property:

$$\text{For all minimal } y \in F\big(\mathrm{S}\big(G'\big)\big)(b) \text{ and } g : \mathrm{S}\big(G'\big) \to P \text{ in } \widehat{\mathbb{G}} \text{ such that } y \cdot_F g = p, \text{ it follows that } g \text{ is split mono.} \tag{10}$$

Indeed, with respect to any minimal $x_0 \in F(\mathrm{S}G)(b)$ and $f : \mathrm{S}G \to X$ in $\widehat{\mathbb{G}}$ with $x_0 \cdot_F f = x$, we have the endomorphism

$$\mathrm{S}(G') \xrightarrow{\;g\;} P \xrightarrow{\;\pi_{(y,\,g\cdot\pi_{(x_0,f)}\cdot f)}\;} \mathrm{S}(G'), \qquad e_{(x_0,f)} : \mathrm{S}(G') \to \mathrm{S}(G')$$

(since y is minimal and $y \cdot_F (g \cdot \pi_{(x_0,f)} \cdot f) = p \cdot_F (\pi_{(x_0,f)} \cdot f) = x_0 \cdot_F f = x$) satisfying

$$y \cdot_F \big(g \cdot \pi_{(y,\,g\cdot\pi_{(x_0,f)}\cdot f)}\big) = p \cdot_F \pi_{(y,\,g\cdot\pi_{(x_0,f)}\cdot f)} = y$$

which, by the minimality of y, is then an epimorphism. Thus, $e_{(x_0,f)}$ is bijective on indices and, as $\mathbb{G}$ is a groupoid, an isomorphism; which makes g a split mono.

As F is engendered by its compact minimal elements it follows from (10) that there exists $G_0 \in {!\mathbb{G}}$, $p_0 \in F(\mathrm{S}(G_0))(b)$, and a section $m : \mathrm{S}(G_0) \rightarrowtail P$ in $\widehat{\mathbb{G}}$ such that $p_0 \cdot_F m = p$. Since such a p_0 engenders x (as $p_0 \cdot_F (m \cdot \pi_{(x_0,f)} \cdot f) = p \cdot_F (\pi_{(x_0,f)} \cdot f) = x_0 \cdot_F f = x$), we conclude the proof by showing that it further satisfies the hypothesis of Lemma 6.5(ii). Indeed, let

$y \in F(\mathrm{S}(G'))(b)$ be minimal and $f : \mathrm{S}(G') \to \mathrm{S}(G_0)$ in $\widehat{\mathbb{G}}$ be such that $y \cdot_F f = p_0$. Since p_0 is minimal, f is epi. Further, since y is minimal and $y \cdot_F (f \cdot m) = p_0 \cdot_F m = p$, we have from (10) that $f \cdot m$ is split mono. It follows that f is split mono, and thus an iso. □

Proposition 6.7. *Every functor $\widehat{\mathbb{G}} \to \widehat{\mathbb{C}}$, with $\mathbb{G}$ a small groupoid, engendered by its compact minimal elements, and preserving quasi-pullbacks and cofiltered limits is engendered by its compact generic elements.*

Proof. Let F be a functor as in the hypothesis.

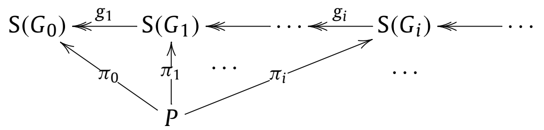
We first show that

Every infinite cochain

$$x_0 \xleftarrow{g_1} x_1 \longleftarrow \cdots \xleftarrow{g_i} x_i \longleftarrow \cdots \quad (i \in \mathbb{N}) \tag{11}$$

with $x_i \in F(\mathrm{S}(G_i))(c)$ minimal and $g_i : \mathrm{S}(G_{i+1}) \twoheadrightarrow \mathrm{S}(G_i)$ in $\widehat{\mathbb{G}}$ such that $x_{i+1} \cdot_F g_{i+1} = x_i$ for all $i \in \mathbb{N}$, stabilises; i.e. there exists $i_0 \in \mathbb{N}$ such that g_i is an iso for all $i \geqslant i_0$.

Indeed, let



$(i \in \mathbb{N})$

be limiting in $\widehat{\mathbb{G}}$. As F preserves cofiltered limits there exists $p \in F\,P\,c$ such that $p \cdot_F \pi_i = x_i$ for all $i \in \mathbb{N}$. Further, since F is engendered by its compact minimal elements, there exist $x \in F(SG)(c)$ minimal and $f : SG \to P$ in $\widehat{\mathbb{G}}$ such that $x \cdot_F f = p$. Thus, as $x \cdot_F (f \cdot \pi_i) = x_i$ is minimal, we have epimorphisms $f \cdot \pi_i : \mathrm{S}(G) \twoheadrightarrow \mathrm{S}(G_i)$ in $\widehat{\mathbb{G}}$ for all $i \in \mathbb{N}$. It follows that $|G_i| \subseteq |G|$ for all $i \in \mathbb{N}$ and hence, since $|G_i| \subseteq |G_{i+1}|$ $(i \in \mathbb{N})$ that there exists $i_0 \in \mathbb{N}$ such that $|G_i| = |G_{i+1}|$ for all $i \geqslant i_0$. Thus, for all such i, we have that g_i is bijective on indices and consequently, as $\mathbb{G}$ is a groupoid, an iso.

Now, for $x \in F\,X\,c$, consider the set $\widehat{\widehat{x}}$ of finite cochains

$$x \xleftarrow{e} x_0 \xleftarrow{e_1} x_1 \longleftarrow \cdots \xleftarrow{e_n} x_n \quad (n \in \mathbb{N})$$

with $x_i \in F(\mathrm{S}(G_i))(c)$ minimal for all $0 \leqslant i \leqslant n$, $e : \mathrm{S}(G_0) \to X$ in $\widehat{\mathbb{G}}$ such that $x_0 \cdot_F e = x$, and proper epis (i.e. not isos) $e_i : \mathrm{S}(G_{i+1}) \twoheadrightarrow \mathrm{S}(G_i)$ in $\widehat{\mathbb{G}}$ such that $x_{i+1} \cdot_F e_i = x_i$ for all $1 \leqslant i \leqslant n$. Since F is engendered by its compact minimal elements, $\widehat{\widehat{x}}$ is non-empty. Further, by (11) above, every chain in $\widehat{\widehat{x}}$ under the prefix order is finite; hence the set of maximal elements of $\widehat{\widehat{x}}$ is non-empty. Finally, since for every maximal cochain $(x \twoheadleftarrow x_0 \twoheadleftarrow \cdots \twoheadleftarrow x_n)$ in $\widehat{\widehat{x}}$, we have that x_n engenders x and satisfies the hypothesis of Lemma 6.5(ii) we are done. □

We have thus established the following characterisation result.

Theorem 6.8. *For a functor between presheaf categories over groupoids the following are equivalent.*

(i) *The functor is analytic (i.e. engendered by its compact generic elements).*
(ii) *The functor preserves filtered colimits and wide quasi-pullbacks.*
(iii) *The functor preserves filtered colimits, quasi-pullbacks, and cofiltered limits.*

Corollary 6.9. *Small groupoids, analytic functors between their presheaf categories, and quasi-cartesian natural transformations between them form a 2-category $\mathcal{AF}$.*

Remark 6.10. The 2-category $\mathcal{AF}$ provides 2-dimensional models of the typed and untyped lambda calculus and of the typed and untyped differential lambda calculus (cf. [4,5]).

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