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Anatomy of a domain of continuous random variables I

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ABSTRACT

In this paper we study the family of thin probability measures on the domain A^∞ of finite and infinite words over a finite alphabet A. This structure is inspired by work of Jean Goubault-Larrecq and Daniele Varacca, who recently proposed a model of continuous random variables over bounded complete domains. Their presentation leaves out many details, and also misses some motivations. In this and a related paper we attempt to fill in some of these details, and in the process, we reveal some features of their model. Our approach to constructing the thin probability measures uses domain theory, and we show the family forms a bounded complete algebraic domain over A^∞. In the second paper in this series, we explore using the thin probability measures to reconstruct the bounded complete domain of continuous random variables over any bounded complete domain due originally to Goubault-Larrecq and Varacca.

1. Introduction

Domains are perhaps the most widely-used models of computational processes (cf. [2,4,24]). This is due to the remarkably simple basis for their structure – a partial order closed under directed suprema and supporting an approximation relation – that allows an equally simple description of the relevant morphisms – maps that preserve the order and that also preserve suprema of directed sets. There is a wealth of Cartesian closed categories of domains, the maximal ones of which have been charted in the seminal work of Achim Jung [13]. The approach proposed by Moggi [21] in which computational effects [11,22] such as continuations, nondeterminism, etc., should be modeled as monads has largely been successful, with CCCs of domains demonstrating that various combinations of these monads all can be accommodated "under one roof".

The singular exception to this program has been models of probabilistic choice. To be sure, there is a natural model for probabilistic choice over a domain – the family of (Borel) probability measures over the domain is again a domain (cf. [23, 12]), where one uses the Scott topology to define the Borel sets, and uses the lattice of Scott-open sets to define the partial order on probability measures. But this construct suffers from two flaws, one irreparable, and the other inscrutable:

- While the *probabilistic power domain* (as the family of probability measures over a domain is called) forms a monad on the category of directed complete partial orders, there is no distributive law between this monad and any of the three nondeterminism monads over domains (cf. [26]), so according to Beck's Theorem [3] the composition of the probabilistic power domain and any of the nondeterminism power domains will not be a monad. This means that one must add new laws to form a monad when combining these two effect models, an approach that has been studied in [18,16]. The result is a model in which the nondeterministic choice of processes p and q is generated by the set

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of probabilistic choices $p +_r q$, for $r \in [0, 1]$ (where $p +_r q$ denotes choosing p with probability r and choosing q with probability $1 - r$, for $0 \leqslant r \leqslant 1$).[1]

- Even though the probabilistic power domain leaves the CCC of directed complete posets (dcpos, for short) and Scott-continuous maps invariant, there is no Cartesian closed category of domains – dcpos that satisfy the usual approximation assumption – that is known to be invariant under this construct. The best that is known is that the category of *coherent domains* is invariant under the probabilistic choice monad [14], but this category is not Cartesian closed.

In response to the irreparable flaw that there is no distributive law between the probabilistic power domain and any of the power domains for nondeterminism, Varacca and Winskel [26,27] explored weakening the laws of probabilistic choice, and discovered three monads for probabilistic choice based on weakened laws – $p \leqslant p +_r p$; $p \geqslant p +_r p$; and last p and $p +_r p$ are unrelated – which they called *indexed valuation monads* (because probability measures over domains can be viewed equally as continuous valuations on the lattice of Scott-open subsets of the underlying domain). Moreover, each of these monads enjoys a distributive law with respect to the monads for nondeterminism (at least over Set).

This author took this work a bit further, showing in [19] that one could use one of the indexed valuation models to define a monad of finite random variables over either the domain RB or the domain FS, the latter of which is a maximal CCC of domains, and both of which are closed under all three nondeterminism monads. More recently, Goubault-Larrecq and Varacca proposed a monad of *continuous* random variables over the CCC of bounded complete domains [9]. Bounded complete domains are more general forms of Scott domains, the category used by Dana Scott in devising the first model of the untyped lambda calculus [24]. While BCD is a CCC, it is not closed under the convex power domain monad, and it also is not a maximal CCC. The results of Goubault-Larrecq and Varacca inspired the work we report here.

1.1. The model of Goubault-Larrecq and Varacca

In a nutshell, the model of continuous random variables proposed by Goubault-Larrecq and Varacca is based on a simple premise: By restricting probability measures to one particular domain $\mathcal{C}$, and then modeling probabilistic choice on an arbitrary domain D as the family of (Scott) continuous maps $f : \operatorname{supp}\mu \to D$, where $\mu \in \mathsf{Prob}(\mathcal{C})$ is a probability measure on $\mathcal{C}$, one could achieve a better behaved model for probabilistic choice. For $\mathcal{C}$, they choose the *Cantor tree*, which is the ideal completion of the rooted full binary tree. Then given a bounded complete domain D, they define

$$RV(D) = \bigl\{(\mu, f) \bigm| \mu \in \mathsf{Prob}(\mathcal{C})\ \&\ f : \operatorname{supp}\mu \to D \text{ Scott continuous}\bigr\},$$

ordered by

$$(\mu, f) \leqslant (\nu, g) \quad \text{iff} \quad \pi_{\operatorname{supp}\mu}(\nu) = \mu \ \wedge\ f \circ \pi_{\operatorname{supp}\mu} \leqslant g.$$

They then restrict their attention to

$$\Theta RV(D) = \bigl\{(\mu, f) \bigm| (\mu, f) \in RV(D)\ \&\ \operatorname{supp}\mu \text{ a Lawson-compact antichain}\bigr\}$$

in the inherited order. This is the family of *continuous random variables over D*. At the heart of the model is the family of *thin probability measures over $\mathcal{C}$* – those probability measures $\mu \in \mathsf{Prob}(\mathcal{C})$ that are supported on Lawson-compact antichains in $\mathcal{C}$. This is the structure that we focus on in this paper. In another paper [17], we examine the rest of the construction of continuous random variables over a bounded complete domain D.

1.2. Our contribution

In this and a succeeding paper [17], we elaborate the construction devised by Goubault-Larrecq and Varacca. For example, the Cantor tree $\mathcal{C}$ is the order ideal completion of the full binary tree, from which it follows that $\mathsf{Prob}(\mathcal{C})$ is a bounded complete domain (cf. [14]), but this is not the order that is used in $\Theta RV(D)$. Explaining this relies on a fundamental example of probabilistic computation – the model of trace distributions generated by a probabilistic automaton. The order induced from this model informs the order on $\Theta RV(D)$ for bounded complete domains D.

In their presentation, Goubault-Larrecq and Varacca first include all probability measures on the Cantor tree in their construction, but then they impose the restriction that the only simple measures – affine combinations of finitely many point masses – in $\Theta RV(D)$ are those supported on antichains. They then define $\Theta RV(D)$ to be the least subset of $RV(D)$ containing these measures in the first component, and also satisfying the property that $\Theta RV(D)$ is closed under directed suprema; in effect, they give a basis for the allowable measures, and capture the rest by taking directed suprema.

Clarifying which probability measures qualify for the family of thin measures requires a completely different presentation of $\Theta RV(D)$ from the one given in [9]. We show that their definition of thin measures is the same thing as defining thin measures in the model to be those that are supported on Lawson closed antichains, but we need Stone duality to prove this

[1] To be more precise, the angelic choice of p and q is the supremum of $\{p +_r q \mid r \in [0, 1]\}$, the demonic choice is the infimum of $\{p +_r q \mid r \in [0, 1]\}$, and the convex choice of p and q is the closed, order-convex hull of $\{p +_r q \mid r \in [0, 1]\}$.

result. Further, our results show that the order on the thin measures allows one to show all measures in the model have the form $\pi_X(\mu)$ where $X \subseteq \mathcal{C}$ is a Lawson-closed subset and μ is a probability measure that is supported on a Lawson-closed subset of $\mathrm{Max}(\mathcal{C})$, the Cantor set which forms the set of maximal elements of $\mathcal{C}$. To complete the picture, we justify this order by showing it arises naturally on probabilistic automata.

Our results also are more general than those in [9], since ours hold for A^∞ for an arbitrary finite alphabet A, whereas they restrict themselves to the case $A = \{0, 1\}$. Our hope is that this may allow researchers in modeling probabilistic process calculi to use the model being devised here and in [17].

1.3. The plan of the paper

In the next section, we review some background material from domain theory and other areas we need. The latter includes a version of Stone duality and some results about the monad of probability measures on various categories, including the category of compact Hausdorff spaces and continuous maps. Section 3 contains some technical results about Lawson-compact antichains in A^∞ for a finite alphabet A. A motivating example that informs the order we use to define our model of thin probability measures occupies Section 4. This example is one of the most fundamental from computer science, that of a (probabilistic) automaton: we show the natural order on the so-called trace distributions of a probabilistic automation that reflects how the computation evolves over time is the same as the order on thin measures. The final Section 5 constitutes the main part of the paper, where we develop the main results, which culminate in a proof that the thin measures over A^∞ form a bounded complete domain. In Section 6 we summarize our results and pose some questions for future research, most of which will be addressed in the second paper in this series.

2. Background

In this section we present the background material we need for our main results.

2.1. Domains

Our results rely fundamentally on domain theory, an area that arose from Dana Scott's models of the untyped lambda calculus. Most of the results that we quote below can be found in [2] or [6]; we give specific references for those that are not in these references.

To start, a *poset* is a partially ordered set. Antichains play a major role in our development: a subset $A \subseteq P$ of a poset is an *antichain* if any two distinct elements in A are incomparable in the order.

A poset is *directed complete* if each of its directed subsets has a least upper bound; here a subset S is *directed* if each finite subset of S has an upper bound in S. A directed complete partial order is called a *dcpo*. The relevant maps between dcpos are the monotone maps that also preserve suprema of directed sets; these maps are usually called *Scott continuous*.

These notions can be presented from a purely topological perspective: a subset $U \subseteq P$ of a poset is *Scott open* if (i) $U = \uparrow U \equiv \{x \in P \mid (\exists u \in U)\ u \leqslant x\}$ is an upper set, and (ii) if $\sup S \in U$ implies $S \cap U \neq \emptyset$ for each directed subset $S \subseteq P$. It is routine to show that the family of Scott-open sets forms a topology on any poset; this topology satisfies $\downarrow x \equiv \{y \in P \mid y \leqslant x\} = \overline{\{x\}}$ is the closure of a point, so the Scott topology is always T_0, but it is T_1 iff P is a flat poset. In any case, a mapping between dcpos is Scott continuous in the order-theoretic sense iff it is continuous with respect to the Scott topologies on its domain and range. We let DCPO denote the category of dcpos and Scott-continuous maps; DCPO is a Cartesian closed category.

If P is a poset, and $x, y \in P$, then x *approximates* y iff for every directed set $S \subseteq P$, if $\sup S$ exists and if $y \leqslant \sup S$, then there is some $s \in S$ with $x \leqslant s$. In this case, we write $x \ll y$ and we let $\twoheaddownarrow y = \{x \in P \mid x \ll y\}$. A *basis* for a poset P is a family $B \subseteq P$ satisfying $\twoheaddownarrow y \cap B$ is directed and $y = \sup(\twoheaddownarrow y \cap B)$ for each $y \in P$. A *continuous poset* is one that has a basis, and P is a *domain* if P is a continuous dcpo. An element $k \in P$ is *compact* iff $x \ll x$, and P is *algebraic* iff $KP = \{k \in P \mid k \ll k\}$ forms a basis. Domains are sober spaces in the Scott topology.

We let DOM denote the category of domains and Scott continuous maps; this is a full subcategory of DCPO, but it is not Cartesian closed. Nevertheless, DOM has several Cartesian closed full subcategories. Two of particular interest to us are the full subcategory SDOM of Scott domains, and BCD its continuous analog. Precisely, a *Scott domain* is an algebraic domain for which KP is countable and that also satisfies the property that every non-empty subset of P has a greatest lower bound. An equivalent statement to the last condition is that every subset of P with an upper bound has a least upper bound. A domain is *bounded complete* iff it satisfies this last property that every non-empty subset has a greatest lower bound; BCD denotes the category of bounded complete domains and Scott-continuous maps.

Example 2.1. A prototypical example of a bounded complete domain is the free monoid $A^\infty = A^* \cup A^\omega$ of finite and infinite words over a finite alphabet A, where we use the prefix order on words: $s \leqslant t \in A^\infty$ iff $(\exists w \in A^\infty)\, sw = t$. Two words compare iff one is a prefix of the other, and the infimum of any set of words is their longest common prefix. As a domain, $KA^\infty = A^*$, so A^∞ is a Scott domain if A is finite.

Note that this same reasoning applies to any *Scott-closed subset* of A^∞ – examples here are the traces from a finite state automaton, where the "alphabet" is the product $S \times Act$ of the set of states and the set of actions.

Domains also have a Hausdorff refinement of the Scott topology which will play a role in our work. The *weak lower topology* on P has the sets of the form $O = P \setminus \uparrow F$ as a basis, where $F \subset P$ is a finite subset. The *Lawson topology* on a domain P is the common refinement of the Scott and weak lower topologies on P. This topology has the family

$$\{U \setminus \uparrow F \mid U \text{ Scott open } \& \ F \subseteq P \text{ finite}\}$$

as a basis. The Lawson topology on a domain is always Hausdorff.

A domain is *coherent* if its Lawson topology is compact. We denote the closure of a subset $X \subseteq P$ of a coherent domain in the Lawson topology by $\overline{X}^{\Lambda}$.

Example 2.2. A basic example of a coherent domain is A^∞ for A finite. If P is an algebraic domain, then the family $\{\uparrow k \setminus \uparrow F \mid k \in KP \ \& \ F \subseteq KP \text{ finite}\}$ is a base for the Lawson topology (cf. Exercise III-1.14 and proof in [6]), so this holds for the case $P = A^\infty$ in particular. The fact that $\uparrow k$ is clopen in the Lawson topology for each compact element k implies that the Lawson topology on an algebraic domain is totally disconnected.

A non-algebraic example is the unit interval; here $x \ll y$ iff $x = 0$ or $x < y$. The Scott topology on the $[0,1]$ has basic open sets $[0,1]$ together with $\twoheaduparrow x = (x, 1]$ for $x \in (0,1)$. Since DOM has finite products, $[0,1]^n$ is a domain in the product order, where $x \ll y$ iff $x_i \ll y_i$ for each i; a basis of Scott-open sets is formed by the sets $\twoheaduparrow x$ for $x \in [0,1]^n$ (this last is true in any domain).

The Lawson topology on $[0,1]$ has basic open sets $(x,1] \setminus [y,1]$ for $x < y$ – i.e., sets of the form (x, y) for $x < y$, which is the usual topology. Thus, the Lawson topology on $[0,1]^n$ is the product topology from the usual topology on $[0,1]$.

Since $[0,1]$ has a least element, the same results apply for any power of $[0,1]$, where $x \ll y$ in $[0,1]^J$ iff $x_j = 0$ for almost all $j \in J$, and $x_j \ll y_j$ for all $j \in J$. Thus, every power of $[0,1]$ is a coherent domain.

While coherent domains having least elements are closed under arbitrary products, the category COH of coherent domains and Scott continuous maps is not Cartesian closed. There is an inclusion of the category of coherent domains and Lawson continuous monotone maps into the category of compact ordered spaces and continuous monotone maps that is obtained by equipping coherent domains with the Lawson topology. In this case, the Lawson topology on the family of closed subsets of the domain is the topology the family inherits from the Vietoris topology on the family of compact subsets of the underlying space. For a compactum X, the Vietoris topology has a subbasis consisting of the sets $\Box U = \{C \subseteq X \mid C \subseteq U\}$ and $\Diamond U = \{C \subseteq X \mid C \cap U \neq \emptyset\}$; these correspond to the Scott-open and lower-open subsets in case X is a domain. This and a related adjunction are detailed in Examples VI-3.8 and VI-3.10 of [6].

Finally, we need some results related to power domains, the convex power domain in particular. Details for the following can be found in [20]. For a coherent domain D, the *convex power domain* consists of the family

$$\mathcal{P}_C(D) = \{X \subseteq D \mid \emptyset \neq X = \downarrow X \cap \uparrow X \text{ is Lawson closed}\}$$

under the *Egli–Milner order:*

$$X \leqslant Y \quad \text{iff} \quad X \subseteq \downarrow Y \ \& \ Y \subseteq \uparrow X.$$

$\mathcal{P}_C(D)$ is a coherent domain if D is one, and in this case,

$$X \ll Y \quad \text{iff} \quad (\exists F \subseteq D \text{ finite}) \ X \leqslant \langle F \rangle \leqslant Y \ \& \ Y \subseteq \twoheaduparrow F = (\uparrow F)^\circ, \tag{1}$$

where $\langle F \rangle = \downarrow F \cap \uparrow F$.

2.2. Stone duality

In modern parlance, Marshall Stone's seminal result states that the category of Stone spaces – compact Hausdorff totally disconnected spaces – and continuous maps is dually equivalent to the category of Boolean algebras and Boolean algebra maps. The dual equivalence sends a Stone space to the Boolean algebra of its compact-open subsets; dually, a Boolean algebra is sent to the set of prime ideals, endowed with the hull-kernel topology. This dual equivalence was used to great effect by Abramsky [1] where he showed how to extract a logic from a domain constructed using Moggi's monadic approach, so that the logic was tailor made for the domain used to build it.

Our approach to Stone duality is somewhat unconventional, but one that also has been utilized in recent work by Gehrke [7,8]. The idea is to realize a Stone space as a projective limit of finite spaces, a result which follows from Stone duality, as we now demonstrate.

Theorem 2.3 *(Stone Duality). Each Stone space X can be represented as a projective limit $X \simeq \varprojlim_{\alpha \in A} X_\alpha$, where X_α is a finite space. In fact, each X_α is a partition of X into a finite cover by clopen subsets, and the projection $X \twoheadrightarrow X_\alpha$ maps each point of X to the element of X_α containing it.*

Proof. If X is a Stone space, then $\mathcal{B}(X)$, the family of compact-open subsets of X is a Boolean algebra. Clearly $\mathcal{B}(X) \simeq \varinjlim_{\alpha\in A}\mathcal{B}_\alpha$ is the injective limit of its family $\{\mathcal{B}_\alpha \mid \alpha \in A\}$ of finite Boolean subalgebras. For a given $\alpha \in A$, we let X_α denote the finite set of atoms of $\mathcal{B}_\alpha$. Then $\mathcal{B}_\alpha \hookrightarrow \mathcal{B}(X)$ implies $\mathcal{B}_\alpha$ is a family of clopen subsets of X, and the set of atoms of $\mathcal{B}_\alpha$ are pairwise disjoint, and their sup – i.e., union – is all of X, so X_α forms a partition of X into clopen subsets. Thus there is a continuous surmorphism $X \twoheadrightarrow X_\alpha$ sending each element of X to the unique atom in X_α containing it. The family $\{\mathcal{B}_\alpha \mid \alpha \in A\}$ is an injective system, since given $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$, the Boolean subalgebra they generate is again finite. Dually the family $\{X_\alpha \mid \alpha \in A\}$ is a projective system, and since $\mathcal{B}(X) \simeq \varinjlim_{\alpha\in A}\mathcal{B}_\alpha$, it follows that $X \simeq \varprojlim_{\alpha\in A}X_\alpha$. $\square$

We note that a corollary of this result says that it is enough to have a basis for the family of finite Boolean subalgebras of $\mathcal{B}(X)$ in order to realize X as a projective limit of finite spaces, where by a *basis*, we mean a directed family whose union generates all of $\mathcal{B}(X)$. The following example illustrates this point.

Example 2.4. Let C denote the middle third Cantor set from the unit interval. This is a Stone space, and so it can be realized as a projective limit of finite spaces $C \simeq \varprojlim_{\alpha\in A}C_\alpha$. But since C is second countable, we can define a countable family of finite spaces C_n for which $C \simeq \varprojlim_n C_n$. Indeed, we can use the construction of C from $[0,1]$ to define these finite spaces:

- $C_0 = [0,1]$ is the entire space.
- $C_1 = \{[0,\frac{1}{3}],[\frac{2}{3},1]\}$ is the result of deleting the middle third from $[0,1]$.

  $\vdots$
- $C_n = \{[0,\frac{1}{3^n}],\ldots,[\frac{3^n-1}{3^n},1]\}$.

  $\vdots$

Note that C_n has 2^n elements – this is the "top down" approach to building C, as opposed the "bottom up" approach obtained by viewing C as the set of maximal elements of the Cantor tree.

While the example considers the simplest non-degenerate case of a two-element alphabet $\{0,1\}$ to produce the Cantor tree, in fact the same argument applies to any finite alphabet A to show the set of infinite words over the alphabet is a Stone space. In both instances, Stone duality shows that the "bottom-up" co-algebraic view of A^ω as the colimit of the finite sets A^n also can be realized by taking the (projective) limit of A^ns. In categorical parlance, the approach via Stone duality realizes A^ω as an F-algebra, whereas the "bottom-up" approach realizes A^ω as a (final) F-coalgebra, where F is the functor that sends a space X to $X \dot\cup \cdots \dot\cup X$, which takes the disjoint union of $|A|$-copies of X. We will make use of these ideas in Section 3 which lay the basis for the main results of the paper.

2.3. The Prob monad on Comp and DCPO

It is well known that the family of probability measures on a compact Hausdorff space is the object level of a functor which defines a monad on Comp, the category of compact Hausdorff spaces and continuous maps (Theorem 2.13 of [5]). As outlined in [10], this monad gives rise to several related monads:

- On Comp, it associates to a compact Hausdorff space X the free *barycentric algebra* over X, the name deriving from the counit $\epsilon : \mathsf{Prob}(S) \to S$ which assigns to each measure μ on a probabilistic algebra S its barycenter $\epsilon(\mu)$ (cf. Theorem 5.3 of [15], which references [25]).
- A *compact affine monoid* is a compact monoid S for which there also is a continuous mapping $\cdot : [0,1] \times S \times S \to S$ satisfying the property that translations by elements of S are affine maps (cf. Section 1.1ff. of [10]). On the category CompMon of compact monoids and continuous monoid homomorphisms, Prob gives rise to a monad that assigns to a compact monoid S the free compact affine monoid over S (cf. Corollary 7.4 of [10]).
- On the category CompGrp of compact groups and continuous homomorphisms, Prob assigns to a compact group G the free compact affine monoid over G; in this case the right adjoint sends a compact affine monoid to its group of units, as opposed to the inclusion functor, which is the right adjoint in the first two cases (cf. Theorem 7.5 of [10]).

As we have already commented, Prob also defines a monad on DCPO. In this case, probability measures are viewed as *valuations:* maps from the lattice of Scott-open sets of the dcpo into the non-negative reals, and the order is then pointwise: $\mu \leqslant \nu$ iff $\mu(U) \leqslant \nu(U)$ ($\forall U$ Scott open).

Remark 2.5. Theorem 2.3 gives a powerful tool for the constructions we will devise in Sections 3 and 5. Theorem 2.3 shows that any Stone space arises as an inverse limit of finite spaces, which allows us to conclude that A^ω is a Stone space, and to apply the constructions in Section 3 to A^∞ and its approximation via finite sets. We will see that some standard domain-theoretic arguments then show that the family of probability measures supported on a Lawson-closed antichain X in A^∞ can be written as the inverse limit of the measures supported on finite subsets $\pi_n(X)$ (cf. Theorem 3.7); this follows by showing that $X = \sup_n \pi_n(X)$ and the fact (quoted from [20]) that the Lawson topology on the family of antichains is the same as the Vietoris topology, which coincides with the topology used to form the inverse limit.

3. Lawson compact antichains in A^∞

We now develop some results about Lawson-closed sets and Lawson-closed antichains in coherent domains. We then use these results to show the family of Lawson-closed antichains in A^∞ is a Scott domain for a finite alphabet A, and this in turn is used in developing the model of thin measures over A^∞.

Lemma 3.1. *Let A be a finite alphabet. If $X \subseteq A^\infty$ is a Lawson-compact subset, then $\downarrow X$ is Scott closed. Moreover there is a canonical map $\pi_{\downarrow X} : A^\infty \to \downarrow X$ that is both Scott- and Lawson continuous.*

Proof. The first claim is a corollary of Lemma 6.6.20 of [2], but we include a proof for completeness sake. By definition, $\downarrow X = \{y \in D \mid (\exists x \in X) y \leqslant x\}$ is a lower set, so we only need to show $\downarrow X$ is closed under directed suprema. If $S \subseteq \downarrow X$ is a directed set, then $\uparrow s \cap X \neq \emptyset$ for each $s \in S$. Moreover, the set $\uparrow s \cap X$ is closed in X, so $\{\uparrow s \cap X \mid s \in S\}$ is a filterbasis of nonempty closed subsets of the compact space X, so the intersection is nonempty. If $x \in \bigcap_{s\in S}(\uparrow s \cap X)$, then clearly $s \leqslant x$ for all $s \in S$, so $\sup S \leqslant x$; i.e., $\sup S \in \downarrow X$.

We next show the mapping $\pi_C : A^\infty \to C$ is Scott- and Lawson continuous for each Scott-closed subset $C \subseteq A^\infty$: Indeed, since C is Scott closed, each $s \in A^\infty$ has a longest prefix in C, which means π_C is well-defined. The map is clearly monotone, so if $S \subseteq A^\infty$ is directed, then $\pi_C(\sup S) \geqslant \sup \pi_C(S)$. Conversely, if $\sup \pi_C(S) \in A^\omega$, then $\sup \pi_C(S) = \sup S$ since A^ω consists of maximal elements. On the other hand, if $\sup \pi_C(S) \in A^*$, then $\sup \pi_C(S) = \pi_C(s)$ for some $s \in S$, and then $\pi_C(s') = \pi_C(s)$ for all $s' \geqslant s$, whence $\pi_C(\sup S) = \pi_C(s)$ as well. Thus π_C is Scott continuous.

Since we have just shown that π_C is Scott continuous, the proof that π_C is Lawson continuous is complete if we show $\pi_C^{-1}(\uparrow_C x) = \uparrow F$ for some finite $F \subseteq A^\infty$, for each $x \in C$. But $\pi_C^{-1}(\uparrow_C x) = \uparrow_{A^\infty} x$.

The first part shows that $\downarrow X$ is Scott closed if X is Lawson compact, so $\pi_{\downarrow X}$ is Scott- and Lawson continuous. □

Corollary 3.2. *Let A be a finite alphabet. If $X \subseteq A^\infty$ is a Lawson-compact antichain, then there is a Lawson compact subset $Y \subseteq A^\omega$ (which is necessarily an antichain) for which $\pi_{\downarrow X}(Y) = X$.*

Proof. Of course, $A^\omega = \bigcap_n \uparrow A^n$ is the intersection of a filterbasis of Scott-compact saturated sets, each of which is therefore also Lawson compact, so their intersection is as well.

Since $\pi_{\downarrow X}$ is Lawson continuous, $\pi_{\downarrow X}^{-1}(X) \subseteq A^\infty$ is Lawson closed, and so the same is true of $Y = \pi_{\downarrow X}^{-1}(X) \cap A^\omega$. Now, for any word $x \in X$, there is an infinite word $x' \in A^\omega$ satisfying $\pi_{\downarrow X}(x') = x$, and so $\pi_{\downarrow X}(A^\omega) \supseteq X$. Thus $\pi_{\downarrow X}(Y) = X$. □

For the following, we let

$$AC\big(A^\infty\big) = \big\{X \subseteq A^\infty \bigm| \emptyset \neq X = \overline{X}^\Lambda \text{ is an antichain}\big\}$$

denote the family of non-empty Lawson-closed antichain in A^∞, endowed with the Egli–Milner order inherited from $\mathcal{P}_C(A^\infty)$.

Lemma 3.3. *$AC(A^\infty)$ is a dcpo. In fact, if $\{X_i\}_{i\in I} \subseteq AC(A^\infty)$ is a directed family of Lawson-compact antichains in the A^∞, then $\sup_i X_i = \bigcap_i(Y \cap \uparrow X_i)$, where $Y = \overline{\bigcup_i \downarrow X_i}^\sigma$ be the Scott-closure of the union of the lower sets of the X_i's.*

Proof. Before we begin the proof, we note that we are assuming that the directed family $\{X_i\}_{i\in I}$ satisfies the property that I is directed and that the mapping $i \mapsto X_i$ is monotone.

If we define $Z \equiv \bigcap_i(Y \cap \uparrow X_i)$, where $Y = \overline{\bigcup_i \downarrow X_i}^\sigma$, then $Z = \sup_{\mathcal{P}_C(A^\infty)} X_i$ by Proposition 4.45 of [20]. To conclude the proof, we show $Z \in AC(A^\infty)$.

Suppose that $x, y \in A^\infty$ with $y < x$. Then $y \in KA^\infty$, so $\uparrow y$ is Scott open. If $x \in Z$, then Proposition 4.47 of [20] implies that $\lim_i X_i = Z$, where the limit is taken in the Lawson topology on $\mathcal{P}_C(A^\infty)$, which is the Vietoris topology from the Lawson topology on A^∞. Thus, $X_i \in \Diamond\uparrow y$, or equivalently $y \in \downarrow X_i$, for residually many i.

If $y \in Z$ also holds, then $y \in \uparrow X_i$ for residually many i, by definition of Z. So $y \in \uparrow X_i \cap \downarrow X_i = X_i$ for residually many i.

Now since $y < x$, there is some $z \in A^*$ with $y < z \leqslant x$, which implies $\uparrow z$ is Scott open. Then $y < z$ implies $z \in \uparrow X_i$, since $y \in X_i$, for residually many i. On the other hand $x \in \Uparrow z$ and $x \in Z$ implies $z \in \downarrow X_i$ for residually many i, just as in the case of y. So $z \in \uparrow X_i \cap \downarrow X_i = X_i$ for residually many i. But then we have $y < z$ and $y, z \in X_i$ for residually many i, which contradicts X_i being an antichain.

We conclude that at most one of x and y is in $Z = \sup X_i$. This shows $Z \in AC(A^\infty)$, so $AC(A^\infty)$ is a subdcpo of $\mathcal{P}_C(A^\infty)$. □

Proposition 3.4. *Let A be a finite alphabet. Then $X \subseteq A^\infty$ is Scott closed iff* Max X *is Lawson closed and $X = \downarrow(\text{Max}\, X)$.*

Proof. We showed in Lemma 3.1 that $\operatorname{Max} X$ Lawson closed implies $\downarrow \operatorname{Max} X = X$ is Scott closed. Conversely, suppose that X is Scott closed. Then $\downarrow \operatorname{Max} X = X$ by Zorn's Lemma. So we only need to show that $\operatorname{Max} X$ is Lawson closed. Since X is Scott closed, it also is Lawson closed, so $\overline{\operatorname{Max} X}^{\Lambda} \subseteq X$. Let $y \in \overline{\operatorname{Max} X}^{\Lambda}$. If $y \in X \cap A^{\omega}$, then $y \in \operatorname{Max} A^{\infty}$, so $y \in \operatorname{Max} X$. On the other hand, if $y \in X \cap A^*$, then $y \in KA^{\infty}$, and since A is finite, we have $\{y\} = \uparrow y \setminus \uparrow\{ya \mid a \in A\}$ is Lawson open. So, $y \in \overline{\operatorname{Max} X}^{\Lambda}$ implies $\operatorname{Max} X \cap \{y\} \neq \emptyset$, so $y \in \operatorname{Max} X$. Thus $\overline{\operatorname{Max} X}^{\Lambda} \subseteq \operatorname{Max} X$, so they are equal. □

Corollary 3.5. *If A is a finite alphabet, then for all $X, Y \in AC(A^{\infty})$, if X and Y have an upper bound in $AC(A^{\infty})$, then $X \vee Y = Max(X \cup Y)$.*

Proof. Since X and Y are Lawson-closed antichains, the proposition implies $\downarrow X$ and $\downarrow Y$ are Scott-closed sets, so the same is true of $\downarrow(X \cup Y) = \downarrow X \cup \downarrow Y$. Then $\operatorname{Max}\downarrow(X \cup Y) = \operatorname{Max}(X \cup Y)$ is also a Lawson-closed antichain.

If $X, Y \leqslant Z \in AC(A^{\infty})$, then $\downarrow(X \cup Y) \subseteq \downarrow Z$, and this implies $\operatorname{Max}(X \cup Y) \leqslant Z$. So we only have to show that $\operatorname{Max}(X \cup Y)$ is an upper bound for X and Y in $AC(A^{\infty})$.

It is clear that $X, Y \subseteq \downarrow \operatorname{Max}(X \cup Y)$, so we only need to show that $\operatorname{Max}(X \cup Y) \subseteq \uparrow X \cap \uparrow Y$. Let $m \in \operatorname{Max}(X \cup Y)$. If $m \in X$, then clearly $m \in \uparrow X$. And if $m \notin \uparrow X$, then $m \in Y \cap \operatorname{Max}(X \cup Y)$. Now $m \in \downarrow Z$, so there is some $z \in Z$ with $m \leqslant z$, and then $X \leqslant Z$ implies there is some $x \in X$ with $x \leqslant z$. Then m and x are prefixes of z, so they must compare. But $m \leqslant x$ is impossible, since X is an antichain would then imply $m \in X$. Hence $x \leqslant m$. This shows $Y \cap \operatorname{Max}(X \cup Y) \subseteq \uparrow X$, so $\operatorname{Max}(X \cup Y) \subseteq \uparrow X$. The argument for Y is similar, so $\operatorname{Max}(X \cup Y)$ is an upper bound for the pair $\{X, Y\}$, which completes the proof that $\operatorname{Max}(X \cup Y) = X \vee Y$. □

Theorem 3.6. *Let A be a finite alphabet and consider the domain A^{∞} in the prefix order, and let $(AC(A^{\infty}), \leqslant)$ denote the family of Lawson-compact antichains in A^{∞} endowed with the Egli–Milner order. Then $AC(A^{\infty})$ is a subdomain of $\mathcal{P}_C(A^{\infty})$ that also is a Scott domain, and $KAC(A^{\infty}) = \{F \subseteq K(A^{\infty}) \mid F \text{ a finite antichain}\}$.*

Proof. Lemma 3.3 shows that $AC(A^{\infty})$ is a dcpo. To show $AC(A^{\infty})$ is an algebraic domain, we first show that given $X \in AC(A^{\infty})$ and a finite set F satisfying $\langle F\rangle \ll X$, there is a finite antichain $G \subseteq \langle F\rangle$ consisting of compact elements from A^{∞} with $G = \langle G\rangle \ll X$:

Indeed, Eq. (1) implies $\langle F\rangle \ll X$ iff $F \subseteq \downarrow X$ and $X \subseteq \Uparrow F$, which in turn implies that $X \subseteq \uparrow(F \cap KA^{\infty})$ since KA^{∞} is a lower set. We can then select a subset $G_0 \subseteq F \cap KA^{\infty}$ with $X \subseteq \uparrow G_0$ and $G_0 \subseteq \downarrow X$. From this subset G_0 we can then select an antichain G as desired, and then $G = \langle G\rangle$ and $G \ll X$.

Next, given two finite antichains $F, G \ll X$, Corollary 3.5 implies $F \vee G = \operatorname{Max}(F \cup G)$, which clearly is in $AC(A^*)$, and obviously $F \vee G \ll X$ by the observations above.

Finally, $AC(A^{\infty}) \subseteq \mathcal{P}_C(A^{\infty})$ and the latter is a domain, so the results just shown imply that

$$X = \sup\{F \mid \emptyset \neq F \subseteq KA^{\infty} \text{ a finite antichain } \& \ F \ll X\}.$$

Thus $\{F \in AC(A^*) \mid F \ll X\}$ is directed and its supremum is X, so $KAC(A^{\infty}) = \{F \mid F \subseteq A^* \text{ a finite antichain}\}$ is a basis for $AC(A^{\infty})$, which proves $AC(A^{\infty})$ is algebraic.

Since Corollary 3.5 shows $AC(A^{\infty})$ is closed under sups of bounded pairs, $AC(A^{\infty})$ is bounded complete, and since A is finite, $KAC(A^{\infty})$ is countable, so $AC(A^{\infty})$ is in fact a Scott domain. □

The proof of Lemma 3.3 relies on some results from [20]: Proposition 4.47 of [20] implies that the Lawson topology on $\mathcal{P}_C(A^{\infty})$ is the same as the topology $\mathcal{P}_C(A^{\infty})$ inherits from the Vietoris topology on the family of compact subsets of A^{∞}, when A^{∞} is endowed with the Lawson topology. Since $\mathcal{P}_C(A^{\infty})$ is coherent if D is, directed sets in $\mathcal{P}_C(A^{\infty})$ converge to their suprema in the Lawson topology. This applies in particular to a directed family of Lawson compact antichains, which we showed is closed under directed suprema in $\mathcal{P}_C(A^{\infty})$ for A finite. These results can be applied further to deduce the following.

Theorem 3.7. *Let A be a finite set, and for each n, let $\pi_n : A^{\infty} \to A^{\leqslant n} \equiv \{s \in A^* \mid |s| \leqslant n\}$ be the projection onto the set of words of length at most n. Then π_n is continuous for each n, where we endow A^{∞} and $A^{\leqslant n}$ with either the Scott or Lawson topologies. Moreover,*

1. *Each Lawson-compact antichain $X \subseteq A^{\infty}$ satisfies $\{\pi_n(X)\}_n$ is a directed family of finite antichains satisfying $\sup_n \pi_n(X) = X$.*
2. *Conversely, each directed family of finite antichains $F_n \subseteq A^{\leqslant n}$ satisfies $\sup_n F_n = X$ is a Lawson-compact antichain in A^{∞} satisfying $\pi_n(X) = F_n$ for each n.*

Proof. That π_n is Scott- and Lawson continuous follows from Lemma 3.1 by observing that $A^{\leqslant n} = \downarrow A^n$ is a Scott-closed set.

For (i), we first note that for a word $s \in A^{\leqslant n}$ and any word $t \in A^{\infty}$, $\pi_n(t) = s$ implies $s \leqslant t$; in fact, $s \leqslant t$ iff $\pi_{|s|}(t) = s$. From this it follows that if X is an antichain, then any two words $s, t \in X$ are incomparable, so for each n, $\pi_n(s) = \pi_n(t)$ or else $\pi_n(s)$ and $\pi_n(t)$ are incomparable. Hence $\pi_n(X)$ is an antichain for each n, and since $A^{\leqslant n}$ is finite, so is $\pi_n(X)$. If $m \leqslant n$, there is a projection $\pi_{mn} : A^{\leqslant n} \to A^{\leqslant m}$ which satisfies $\pi_m = \pi_{mn} \circ \pi_n$. It follows that $\{\pi_n(X)\}_n$ is a directed family satisfying $\pi_n(X) \leqslant X$ for each n. In fact $\sup_n \pi_n(X) = X$ since each word $w \in A^{\infty}$ satisfies $w = \sup_n \pi_n(w)$. This proves part (i).

For part (ii), if $\{F_n\}_n$ is a directed sequence of finite antichains with $F_n \subseteq A^{\leqslant n}$ for each n, then $\sup_n F_n$ exists and is an antichain by Theorem 3.6. The arguments in the previous part apply again to show that $\pi_m(\sup F_n) = F_m$ for each m. $\square$

Remark 3.8. The gist of the last few results is that Lawson-closed antichains in A^∞ are closed under directed suprema, and that each can be approximated by its projections to the family $\{A^{\leqslant n} \mid n \geqslant 0\}$. The last result shows further that every Lawson-closed antichain is the projection of a Lawson-closed subset of A^∞, thus accounting for all such antichains.

Our interest in Lawson-closed antichains will become clear in our construction of the model of thin measures over A^∞ – as we shall see, they form the (Lawson) support of such measures.

4. A motivating example

In this section we present an example that provides motivation for how we define the order on the thin measures over A^∞. The example is one of the most fundamental for computer science – that of a (probabilistic) automaton.

Definition 4.1. A *probabilistic automaton* is a tuple (S, A, q_0, D) where S is a finite set of states, A a finite set of actions, $q_0 \in S$ a start state, and $D \subseteq S \times \mathsf{Prob}(A \times S)$ a transition relation that assigns to each state s_0 a probability distribution $\sum_{A\times S} r_{(s_0,(a,s))}\delta_{(a,s)}$ on $A \times S$.

Remark 4.2. This is a very restrictive notion of a probabilistic automaton, but it suffices for our purposes. More general notions include transition relations D that are truly relational, rather than being functional, as our definition requires. There also is a dichotomy of such automata into generative and reactive automata, which we are eliding. But, our goal simply is to provide a motivating example for the order on probability measures we define later, and this is accomplished most easily without the distractions of the many possible nuances of the large variety of probabilistic automata in the literature.

If we start such an automaton in its start state – which amounts to assigning it the starting distribution δ_{q_0}, and follow the automaton as it evolves, then we see a sequence of *global trace distributions* that describe the step-by-step evolution of the automaton:

1. δ_{q_0},
2. $\sum_{(a_1,s_1)\in A\times S} r_{(q_0,(a_1,s_1))}\delta_{q_0 a_1 s_1}$,
3. $\sum_{(a_1,s_1)\in A\times S} r_{(q_0,(a_1,s_1))}(\sum_{(a_2,s_2)\in A\times S} r_{(s_1,(a_2,s_2))}\delta_{q_0 a_1 s_1 a_2 s_2})$,

$\vdots$

If we strip away the probabilities, we have a nondeterministic finite state automaton (albeit one without final states), and the resulting automaton generates a language that is a subset of $(S \times A)^\infty$. This automaton generates the sequence

$$\{q_0\}, \{q_0 s_1 a_1 \mid r_{(q_0,(s_1,a_1))} \neq 0\}, \{q_0 s_1 a_1 s_2 a_2 \mid r_{(q_0,(s_1,a_1))} \neq 0 \neq r_{(s_1,(a_2,s_2))}\}, \ldots.$$

Note that the sequence of sets of states this automaton generates is a family of finite antichains, which we showed in Section 2 is a Scott subdomain of $\mathcal{P}_C((S \times A)^\infty)$ under the Egli–Milner order. Moreover, the projections $\pi_{mn} : (S \times A)^{\leqslant n} \to (S \times A)^{\leqslant m}$ for $m \leqslant n$ map the antichain of possible states at the nth stage to those at the mth stage, by truncation.

Since Prob is a monad on Comp, the mappings π_{mn} lift to mappings $\mathsf{Prob}(\pi_{mn}) : \mathsf{Prob}((S \times A)^{\leqslant n}) \to \mathsf{Prob}((S \times A)^{\leqslant m})$. Using the mappings π_{mm+1}, we see that each succeeding distribution is projected onto the previous distribution. For example, the second distribution $\sum_{(a_1,s_1)\in A\times S} r_{(q_0,(a_1,s_1))}\delta_{q_0 a_1 s_1}$ collapses to δ_{q_0}, and the third distribution $\sum_{(a_1,s_1)\in A\times S} r_{(q_0,(a_1,s_1))}(\sum_{(a_2,s_2)\in A\times S} r_{(s_1,(a_2,s_2))}\delta_{q_0 a_1 s_1 a_2 s_2})$ collapses to the second. Thus, Prob lifts the order on $AC((S \times A)^\infty)$ to $\mathsf{Prob}(AC((S \times A)^\infty))$, and it is this order we will use in defining the order on the family of thin probability measures (and eventually on the domain of continuous random variables over a bounded complete domain). Our next goal is to make this observation precise.

Remark 4.3. We thank one of the anonymous referees for pointing out that a similar example is given in [27].

5. A bounded complete domain of measures

In this section we develop the main results of the paper, which are a detailed examination of the order used by Goubault-Larrecq and Varacca on the thin probability measures that are used in their model of continuous random variables over a bounded complete domain. Their presentation only sketches their model, and here we have more space to develop the ideas in depth. Throughout this section we assume that the alphabet A which we use to form A^∞ is finite. We begin with a fundamental notion for our approach.

Definition 5.1. If Y is a compact Hausdorff space, $X \subseteq Y$ is a compact subspace of Y and $\mu \in \mathsf{Prob}(Y)$, then we say μ has *full support on* X if $\operatorname{supp}\mu = X$.[2] We denote by $\mathsf{Prob}^{\dagger}(X)$ the family of $\mu \in \mathsf{Prob}(Y)$ having full support on X.

Definition 5.2. For a finite alphabet A, we define

$$\Theta\mathsf{Prob}\big(A^{\infty}\big) \equiv \Bigg(\bigoplus_{X \in AC(A^{\infty})} \mathsf{Prob}^{\dagger}(X), \leqslant\Bigg)$$

to be the direct sum of the family of probability measures in $\mathsf{Prob}^{\dagger}(X)$ as X ranges over $AC(A^{\infty})$, ordered by $\mu \leqslant \nu$ iff $\pi_{\downarrow(\operatorname{supp}\mu)}(\nu) = \mu$. These are the *thin probability measures* on A^{∞}, those that are fully supported on Lawson-closed antichains in A^{∞}.

The next series of results are about the structure of $\Theta\mathsf{Prob}(A^{\infty})$. We begin with a simple result about mapping supports of measures.

Lemma 5.3. *If $f : X \to Y$ is a continuous map between compacta, then $f(\mu) = \nu$ implies $f(\operatorname{supp}\mu) = \operatorname{supp}\nu$.*

Proof. Indeed, $f(\operatorname{supp}\mu)$ is a compact, hence closed subset of Y. Let $C = \operatorname{supp}\nu \cup f(\operatorname{supp}\mu)$. If $y \in \operatorname{supp}\nu \,\Delta\, f(\operatorname{supp}\mu)$ (the symmetric difference), then there is an open set U containing y and satisfying $U \cap C \subseteq \operatorname{supp}\nu \setminus \operatorname{supp} f(\mu)$ or $U \cap C \subseteq f(\operatorname{supp}\mu) \setminus \operatorname{supp}\nu$. This means either $\nu(U) > 0$ and $f(\mu)(U) = 0$, or $\nu(U) = 0$ and $f(\mu)(U) > 0$. In either case, we conclude $f(\mu) \neq \nu$ if $f(\operatorname{supp}\mu) \neq \operatorname{supp}\nu$. $\square$

Lemma 5.4. *If A is a finite alphabet, then the mapping $\operatorname{supp} : \Theta\mathsf{Prob}(A^{\infty}) \to AC(A^{\infty})$ sending each measure μ to its support in the Lawson topology is monotone.*

Proof. The mapping $\mu \mapsto \operatorname{supp}\mu$ clearly is well-defined and assigns to each measure a Lawson-closed antichain in $AC(A^{\infty})$, by definition of $\Theta\mathsf{Prob}(A^{\infty})$.

If $\mu \leqslant \nu$, then $\pi_{\operatorname{supp}\mu}(\nu) = \mu$, and it follows that $\pi_{\operatorname{supp}\mu}(\operatorname{supp}\nu) = \operatorname{supp}\mu$ by Lemma 5.3. This in turn implies $\operatorname{supp}\mu \leqslant \operatorname{supp}\nu$ in $AC(A^{\infty})$, so the support map is monotone. $\square$

Proposition 5.5. *If A is a finite alphabet, then the family $(\Theta\mathsf{Prob}(A^{\infty}), \leqslant)$ is a dcpo.*

Proof. Recall that $\mu \leqslant \nu$ iff $\pi_{\downarrow(\operatorname{supp}\mu)}(\nu) = \mu$. It is clear that this relation is reflexive. The relation also is antisymmetric: Indeed, if $\mu \leqslant \nu \leqslant \mu$, then $\pi_{\downarrow(\operatorname{supp}\mu)}(\nu) = \mu$ and $\pi_{\downarrow(\operatorname{supp}\nu)}(\mu) = \nu$. Then Lemma 5.3 implies that $\pi_{\downarrow(\operatorname{supp}\mu)}(\operatorname{supp}\nu) = \operatorname{supp}\mu$ and $\pi_{\downarrow(\operatorname{supp}\nu)}(\operatorname{supp}\mu) = \operatorname{supp}\nu$. But the mappings $\pi_{\downarrow(\operatorname{supp}\mu)}$ and $\pi_{\downarrow(\operatorname{supp}\nu)}$ are projections, and their composition in either order must be the identity on $\operatorname{supp}\mu$ and $\operatorname{supp}\nu$, respectively. This implies $\operatorname{supp}\mu = \operatorname{supp}\nu$, and then both projections are the identity map on $\mathsf{Prob}^{\dagger}(\operatorname{supp}\mu)$. Hence $\mu = \nu$.

For transitivity, suppose that $\mu \leqslant \nu \leqslant \rho$. Then $\pi_{\downarrow(\operatorname{supp}\mu)}(\nu) = \mu$ and $\pi_{\downarrow(\operatorname{supp}\nu)}(\rho) = \nu$. Again, Lemma 5.3 implies $\pi_{\downarrow\operatorname{supp}\mu}(\operatorname{supp}\nu) = \operatorname{supp}\mu$ and $\pi_{\downarrow\operatorname{supp}\nu}(\operatorname{supp}\rho) = \operatorname{supp}\nu$. But $\operatorname{supp}\mu, \operatorname{supp}\nu$ and $\operatorname{supp}\rho$ are Lawson-compact antichains, and $\pi_{\downarrow\operatorname{supp}\mu}(\operatorname{supp}\nu) = \operatorname{supp}\mu$ is equivalent to $\operatorname{supp}\mu \leqslant \operatorname{supp}\nu$ in $AC(A^{\infty})$. Likewise $\operatorname{supp}\nu \leqslant \operatorname{supp}\rho$, and so $\operatorname{supp}\mu \leqslant \operatorname{supp}\rho$, which implies $\pi_{\downarrow\operatorname{supp}\mu}(\operatorname{supp}\rho) = \operatorname{supp}\mu$. Recalling that $\mathsf{Prob}(\pi_{\downarrow\operatorname{supp}\mu})(\nu) = \pi_{\downarrow\operatorname{supp}\mu}(\nu)$ and applying the functoriality of Prob shows $\pi_{\downarrow\operatorname{supp}\mu}(\rho) = (\pi_{\downarrow\operatorname{supp}\mu} \circ \pi_{\downarrow\operatorname{supp}\nu})(\rho) = \mu$, so the relation is transitive.

This shows $(\Theta\mathsf{Prob}(A^{\infty}), \leqslant)$ is a partial order. To show it is a dcpo, let $\{\mu_i\}_{i \in I}$ is a directed family in $\Theta\mathsf{Prob}(A^{\infty})$. Then $\mathsf{Prob}(A^{\infty})$ is a dcpo, so there is a $\mu \in \mathsf{Prob}(A^{\infty})$ with $\mu = \sup_i \mu_i$. To complete the proof, we must show that $\operatorname{supp}\mu \in AC(A^{\infty})$, and for that we apply Lemma 3.3. Indeed, Lemma 5.4 implies $\nu \mapsto \operatorname{supp}\nu$ is a monotone map, so $\{\operatorname{supp}\mu_i\}_{i \in I} \subseteq AC(A^{\infty})$ is directed and then Lemma 3.3 implies $\sup_i \operatorname{supp}\mu_i = \bigcap_i (Y \cap \uparrow\operatorname{supp}\mu_i) \in AC(A^{\infty})$, where $Y = \overline{\bigcup_i \downarrow\operatorname{supp}\mu_i}^{\sigma}$ is the Scott-closed set generated by $\{\operatorname{supp}\mu_i\}_i$. Now Y is Scott closed and $\operatorname{supp}\mu_i \subseteq Y$ for each i, so $\mu_i(A^{\infty} \setminus Y) = 0$ for each i, from which it follows that $\mu(A^{\infty} \setminus Y) = 0$. This implies $\operatorname{supp}\mu \subseteq Y$.

Further, given $i \in I$, if $j \geqslant i$, then $\operatorname{supp}\mu_j \subseteq Y \cap \uparrow(\operatorname{supp}\mu_i) \subseteq \uparrow\operatorname{supp}\mu_i$, so $\mu_j(U) = 1$ for each Scott-open subset U containing $\operatorname{supp}\mu_i$. Since $\mu = \sup_{j \geqslant i}\mu_j$, it follows that $\mu(U) = 1$ as well, and since U is arbitrary we conclude that $\operatorname{supp}\mu \subseteq \uparrow\operatorname{supp}\mu_i$. Now i is arbitrary, which means $\operatorname{supp}\mu \subseteq \bigcap_i \uparrow\operatorname{supp}\mu_i$.

Thus $\operatorname{supp}\mu \subseteq \bigcap_i (Y \cap \uparrow(\operatorname{supp}\mu_i)) = \sup_i \operatorname{supp}\mu_i$, where the supremum is taken in $AC(A^{\infty})$. Then Lemma 3.3 implies $\sup_i \operatorname{supp}\mu_i$ is an antichain, so the same is true of $\operatorname{supp}\mu$, and so $\operatorname{supp}\mu \in AC(A^{\infty})$. Thus $\mu = \sup_i \mu_i \in \Theta\mathsf{Prob}(A^{\infty})$, so $\Theta\mathsf{Prob}(A^{\infty})$ is a dcpo. $\square$

Proposition 5.6. *If A is a finite alphabet, then the mapping $\operatorname{supp} : \Theta\mathsf{Prob}(A^{\infty}) \to AC(A^{\infty})$ sending each measure μ to its support in the Lawson topology is Scott continuous.*

[2] Recall that $\operatorname{supp}\mu$ is the smallest closed set who complement has μ-measure 0.

Proof. We showed in Lemma 5.4 that the mapping is well-defined and monotone.

If $\{\mu_i\}_{i\in I} \subseteq \Theta\mathrm{Prob}(A^\infty)$ is a directed family, then the previous proposition implies there is $\mu \in \Theta RV(A^\infty)$ with $\sup_i \mu_i = \mu$. Moreover, as in its proof, $Y = \sup_i \operatorname{supp} \mu_i$ satisfies $\operatorname{supp}\mu \subseteq Y$.

Claim: $\operatorname{supp}\mu = Y$.

To start, note that Proposition 4.47 of [20] implies the Lawson topology on $\mathcal{P}_C(A^\infty)$ is the same as the Vietoris topology. So, if $y \in Y$, then there is a net $\{x_k\}_{k\in K} \subseteq \bigcup_i \operatorname{supp}\mu_i$ with $\lim_k x_k = y$ in the Lawson topology. But $\downarrow \operatorname{supp}\mu$ is Scott-, hence Lawson closed, and for each k, $x_k \in \downarrow \operatorname{supp}\mu$ because $\pi_{\operatorname{supp}\mu_i}(\operatorname{supp}\mu) = \operatorname{supp}\mu_i$ for each $i \in I$. Since $\operatorname{supp}\mu$ is compact, there is some $z \in \operatorname{supp}\mu$ with $y \leqslant z$. But Y is an antichain and $\operatorname{supp}\mu \subseteq Y$, so $y \in Y$, so $y = z \in \operatorname{supp}\mu$. This proves the claim.

The claim implies that $\operatorname{supp}\mu = Y = \sup_{i\in I} \operatorname{supp}\mu_i \in AC(A^\infty)$. So supp is Scott continuous. $\square$

Proposition 5.7. *If A is a finite alphabet, and if $\mu \in \Theta\mathrm{Prob}(A^\infty)$ and $F \subseteq A^*$ is a finite antichain with $\pi_F(\operatorname{supp}\mu) = F$, then $\pi_F(\mu) \ll \mu$ in $\Theta\mathrm{Prob}(A^\infty)$.*

Proof. If $F \subseteq A^*$ is a finite antichain with $\pi_F(\operatorname{supp}\mu) = F$, then we know from Theorem 3.6 that $F \ll \operatorname{supp}\mu$ in $AC(A^\infty)$. Now, suppose $S \subseteq \Theta\mathrm{Prob}(A^\infty)$ is directed with $\mu = \sup S$. Then $\operatorname{supp}\mu = \sup_{\sigma\in S} \operatorname{supp}\sigma$ by Proposition 5.6, and $\pi_F(\operatorname{supp}\mu) = F$ implies $\operatorname{supp}\mu \subseteq \Uparrow F$. Since $\Uparrow F$ is Scott open, $(\exists \sigma_0 \in S)$ $\operatorname{supp}\sigma \subseteq \Uparrow F$ for $\sigma_0 \leqslant \sigma$.

We next show that $F \subset \downarrow \operatorname{supp}\sigma$ for residually many $\sigma \in S$.[3] To start, let $x \in F$ and consider $U = \uparrow x$. Since $F \subseteq A^*$ is an antichain, U is a Scott-open set containing only x from F. Since $\pi_F(\operatorname{supp}\mu) = F$, $(\exists y \in \operatorname{supp}\mu)\, \pi_F(y) = x \in U$, so $y \in U$. But $\lim_{\sigma\in S} \operatorname{supp}\sigma = \operatorname{supp}\mu$ then implies $\operatorname{supp}\sigma \cap U \neq \emptyset$ for residually many $\sigma \in S$.[4] For these σ, we have $x \in \pi_F(\operatorname{supp}\sigma)$. Iterating this process for the finitely many elements of F implies $\pi_F(\operatorname{supp}\sigma) = F$, so $F \subseteq \downarrow \operatorname{supp}\sigma$ for residually many $\sigma \in S$, as we wanted.

The last two results imply that $F \leqslant \operatorname{supp}\sigma$ in $AC(A^\infty)$ for residually many $\sigma \in S$. To complete the proof, we note that $F \leqslant \operatorname{supp}\sigma$ implies $\pi_F(\operatorname{supp}\sigma) = F$, and then

$$\pi_F(\sigma) = \pi_F\big(\pi_{\operatorname{supp}\sigma}(\mu)\big) = \pi_F \circ \pi_{\operatorname{supp}\sigma}(\mu) = \pi_F(\mu)$$

which implies $\pi_F(\mu) \leqslant \sigma$ for residually many $\sigma \in S$. This implies $\pi_F(\mu) \ll \mu$. $\square$

The main result of the paper is the following:

Theorem 5.8. *If A is a finite alphabet, then $\Theta\mathrm{Prob}(A^\infty)$ is a bounded complete algebraic domain.*

Proof. If $\mu \in \Theta\mathrm{Prob}(A^\infty)$, then Proposition 5.7 implies that $\pi_F(\mu) \ll \mu$ for each finite antichain $F \subseteq A^*$ satisfying $F \ll \operatorname{supp}\mu$. According to Theorem 3.6, $AC(A^\infty)$ is a bounded complete algebraic domain in which $F \vee G = \mathrm{Max}(F \cup G)$ for any pair $F, G \in AC(A^*)$ of finite antichains with an upper bound. Given $F, G \ll \operatorname{supp}\mu$, we now show how to form $\pi_F(\mu) \vee \pi_G(\mu)$ using this result.

In fact, $\mathrm{Max}(F \cup G) = F \vee G$ in $AC(A^\infty)$ by the result just cited. Since F and G are finite we can write $\pi_F(\mu) = \sum_{a\in F} r_a \delta_a$ and $\pi_G(\mu) = \sum_{b\in G} s_b \delta_b$, where $\sum_{a\in F} r_a = 1 = \sum_{b\in G} s_b$. If $\nu \in \Theta\mathrm{Prob}(A^\infty)$ satisfies $\pi_F(\mu), \pi_G(\mu) \leqslant \nu$, then $\pi_F(\nu) = \pi_F(\mu)$ and $\pi_G(\nu) = \pi_G(\mu)$. Hence $\pi_F(\nu) = \sum_{a\in F} r_a \delta_a$ and $\pi_G(\nu) = \sum_{b\in G} s_b \delta_b$. If we define $H \equiv \mathrm{Max}(F \cup G)$ and let $H_F = H \cap F$ and $H_G = H \cap G$, then by Corollary 3.5, $F, G \ll \operatorname{supp}\mu$ implies that

$$(\forall a \in F \setminus H)(\exists b \in H_G)\, a \leqslant b \quad \text{and} \quad (\forall b \in G \setminus H)(\exists a \in H_F)\, b \leqslant a.$$

Now, for $a \in F \setminus H$, we let $H_G(a) = \uparrow a \cap H_G$, and we note that each $b \in H_G$ can belong to $H_G(a)$ for at most one $a \in F \setminus H$ since A^* is a tree and F is an antichain. Similarly, for $b \in G \setminus H$, we let $H_F(b) = \uparrow b \cap H_F$.

Next, H is finite, so $\pi_H(\mu)$ is a simple measure, which means

$$\pi_H(\mu) = \sum_{x\in H} t_x \delta_x = \sum_{a\in H_F} r_a \delta_a + \sum_{b\in H_G\setminus F} s_b \delta_b,$$

where $H_G \setminus F$ in the second summand avoids double-counting $F \cap G$. Since $F \leqslant H$, we have $\pi_F = \pi_F \circ \pi_H$, so we conclude that

$$\sum_{a\in F} r_a \delta_a = \pi_F(\mu) = \pi_F \circ \pi_H(\mu)$$

[3] That is, $(\exists \sigma_0 \in S)\, \sigma \geqslant \sigma_0 \;\Rightarrow\; F \subseteq \downarrow \operatorname{supp}\sigma$.

[4] Recall that $\Diamond U = \{C \mid C \cap U \neq \emptyset\}$ is open in the Vietoris, hence also the Lawson topology.

$$
\begin{aligned}
&= \pi_F\Big(\sum_{a\in H_F} r_a\delta_a + \sum_{b\in H_G\setminus F} s_b\delta_b\Big)\\
&= \pi_F\Big(\sum_{a\in H_F} r_a\delta_a\Big) + \pi_F\Big(\sum_{b\in H_G\setminus F} s_b\delta_b\Big)\\
&= \sum_{a\in F} r_a\delta_a + \pi_F\Big(\sum_{b\in H_G\setminus F} s_b\delta_b\Big)\\
&= \sum_{a\in F} r_a\delta_a + \sum_{b\in H_G\setminus F} s_b\delta_{\pi_F(b)}\\
&= \sum_{a\in F} r_a\delta_a + \sum_{a\in F\setminus H}\Big(\sum_{b\in H_G(a)} s_b\delta_a\Big),
\end{aligned}
$$

where the second line follows because π_F is a convex map, and the third because π_F is a projection. From this, we conclude that

$$(\forall a\in F\setminus H)\, r_a = \sum_{b\in H_G(a)} s_b \quad \text{and similarly} \quad (\forall b\in G\setminus H)\, s_b = \sum_{a\in H_F(b)} r_a.$$

Since $\pi_F(\nu)=\pi_F(\mu)=\sum_{a\in F} r_a\delta_a$ and $\pi_G(\nu)=\pi_G(\mu)=\sum_{b\in G} s_b\delta_b$, we can write

$$\pi_F(\nu) = \sum_{a\in H_F} r_a\delta_a + \sum_{a\in F\setminus H} r_a\delta_a \quad \text{and} \quad \pi_G(\nu) = \sum_{b\in H_G\setminus F} s_b\delta_b + \sum_{b\in G\cap F} s_b\delta_b + \sum_{b\in G\setminus H} s_b\delta_b.$$

Since $F, G \leqslant \operatorname{supp}\nu$ it follows that $H \leqslant \operatorname{supp}\nu$. Then $H = H_F \,\dot\cup\, H_G \setminus F$, which implies

$$\pi_H(\nu) = \sum_{a\in H_F} r_a\delta_a + \sum_{b\in H_G\setminus F} s_b\delta_b = \pi_H(\mu)$$

so $\pi_H(\mu) \leqslant \nu$. This shows $\pi_H(\mu) = \pi_F(\mu) \vee \pi_G(\mu)$, which proves the result. □

6. Summary and future work

In this paper we have used domain theory and Stone duality, as well as other components, to give a detailed construction of the bounded complete domain of thin probability measures on A^∞ for a finite alphabet A. This family plays a fundamental role in the domain of continuous random variables over a bounded complete domain, a construction proposed by Goubault-Larrecq and Varacca in [9]. Applying the results we presented here to complete the reconstruction of Goubault-Larrecq's and Varacca's model is the focus of the second paper in this series [17].

Our reconstruction of the thin probability measures over A^∞ relies on the Lawson-closed antichains in A^∞. Our main result is that the family is a bounded complete algebraic domain. While our results require some technical development, the proofs are fairly straightforward, relying on domain theory to single out the compact elements in the model, and using them to prove the order is bounded complete. We also clarified how the order on the thin measures is motivated by the example of trace distributions of a probabilistic automaton.

In addition to completing this line of work elaborating the random variables of Goubault-Larrecq and Varacca, we are also interested in applying these constructions for other applications. We believe the thin probability measures could be useful to model probabilistic automata. We also believe the construction could be used to move from finite and totally disconnected state spaces (e.g., ones that are Stone spaces) to continuous state spaces, including for example the unit interval. Exploring this line will require understanding the mapping from the Cantor set onto the interval, for example.

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