Theoretical Computer Science 546 (2014) 132–144

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

# Elements of a theory of algebraic theories

J.M.E. Hyland

*DPMMS, CMS, Wilberforce Road, Cambridge CB1 2BX, UK*

ARTICLE INFO

Available online 6 March 2014

*Keywords:*
Algebraic theories
Operads
Kleisli bicategories

ABSTRACT

Kleisli bicategories are a natural environment in which the combinatorics involved in various notions of algebraic theory can be handled in a uniform way. The setting allows a clear account of comparisons between such notions. Algebraic theories, symmetric operads and nonsymmetric operads are treated as examples.



## 1. Introduction

This paper has its genesis in Glynn Winskel's use of presheaf categories and profunctors in the foundations of concurrency. His basic theory is laid out in [4] with Cattani, and particular cases of Kleisli bicategories appear there. A preordered set version, providing a model for linear logic, is already in [24]. Kleisli bicategories are both a rich source of models and a context in which to understand subtle theory. Their value was recognised by a group of us in Cambridge and we set about preparing an exposition [8] of the general theory. Around the same time John Power realised the significance of the key pseudo-distributivities in connection with extensions of Edinburgh work [9] on variable binding. The paper [6] shows the common interest and in Edinburgh a thesis [22] and papers (for example [23]) quickly followed. By contrast the Cambridge exposition remains unfinished, and there is just one paper [7] which gives some sense of our preferred approach. That is my fault and I have written this paper for Glynn Winskel by way of apology. It is not intended as a substitute for the unfinished paper. Rather, it sketches applications to algebraic theories and operads, which I have presented in talks over the years. In developing the ideas, I have profited from discussions on with Richard Garner and John Power. Recently Garner and I have made progress on coalgebraic aspects, and a substantial theory is emerging. Here I focus on just one strand of ideas, and leave details and the wider perspective for other occasions.

The paper is organised as follows. In Sections 2, I describe and give elementary properties of the basic construction, that of the Kleisli bicategory $\mathrm{Kl}(P)$ of a Kleisli structure $P$. This is in my view a good way to understand the bicategories with which we shall be concerned, and I explain how even the basic bicategory **Prof** of profunctors, which underlies the whole paper, can be considered from this perspective. Section 3 is concerned with distributive laws, composite Kleisli structures and features of the corresponding Kleisli categories. The idea of a pseudo-distributive law at roughly the level we need is old, see [14]. Perhaps because there were no compelling applications, the details were not worked through. The first complete account seems to be [19]. The use here of Kleisli structures creates a new focus but there are no great surprises.

In Section 4, I describe my approach to algebraic theories as monads in Kleisli bicategories. A concrete categorical treatment of essentially the same point of view is in [5]. The value of an abstract treatment becomes more apparent with very recent work but even at the level of this paper I hope readers will appreciate the smooth treatment of categories generated by theories. In the final Section 5, I use the general setting to give a treatment of comparisons between notions of algebraic theory. I hope inter alia to encourage sensitivity to some subtleties in the notions of symmetric and non-symmetric operads.

*E-mail address:* m.hyland@dpmms.cam.ac.uk.

http://dx.doi.org/10.1016/j.tcs.2014.03.005

I make a small remark about notation. I have decided to use $P$ both when describing a general Kleisli structure and for the presheaf Kleisli structure. I hope to avoid confusion by the following convention. In the general case I use lower case letter for the objects of a bicategory. In the special case when the objects are themselves categories I use upper case letters.

These introductory remarks make clear that I have discussed material with many people. But on this occasion I thank in particular Glynn Winskel. We go back a long way and it has been continually stimulating to discuss logic and computer science with him over many years. I very much appreciate his intellectual honesty and openness, and his talent for grounding abstract mathematics in the modelling of computational phenomena. This paper derives from work of his and he was the first person with whom I discussed Kleisli bicategories. I dedicate the paper to him with affection and best wishes for the future.

## 2. Kleisli bicategories

The Kleisli formulation of a monad on a category $\mathbb{C}$, given in [18], has not played a prominent role in mathematics, but it is familiar in the programming language community from the computational $\lambda$-calculus [20] and premonoidal categories [21]. The Kleisli presentation gives for each object $c$ in a category $\mathbb{C}$ a unit $c \to Tc$ in $\mathbb{C}$ and for each $f : c \to Td$ in $\mathbb{C}$ a lift $f^\sharp : Tc \to Td$. This data is required to satisfy evident equations. The virtue of the formulation is that it makes trivial the definition and basic properties of the Kleisli category of a monad. Now one can still define a Kleisli category when structure is only given for some subcollection of the objects of $\mathbb{C}$ and generalisations of this kind have been identified, for example in [1]. The phrase relative monad is in use, but for the cases considered here I prefer to say restricted.

### 2.1. Kleisli structures

A Kleisli structure is a 2-dimensional version of a restricted monad. The starting point is a bicategory $\mathcal{K}$ equipped with a sub-bicategory $\mathcal{A} \hookrightarrow \mathcal{K}$.

**Definition 2.1.** A *Kleisli structure* $P$ on $\mathcal{A} \hookrightarrow \mathcal{K}$ is the following.

- A choice for each object $a \in \mathcal{A}$, of an arrow $y_a : a \to Pa$ in $\mathcal{K}$.
- For each pair $a, b \in \mathcal{A}$ of objects, a functor

$$\mathcal{K}(a, Pb) \longrightarrow \mathcal{K}(Pa, Pb) \qquad f \longmapsto f^\sharp$$

- Families of invertible 2-cells

$$\eta_f : f \to f^\sharp.y_a \qquad \kappa_a : (y_a)^\sharp \to 1_{Pa} \qquad \kappa_{g,f} : \left(g^\sharp.f\right)^\sharp \to g^\sharp.f^\sharp$$

natural in $f : a \to Pb$ and $g : b \to Pc$ as appropriate, and subject to unit and pentagon coherence conditions.

It is clear from the data that $P$ can be given the structure of a pseudo-functor $P : \mathcal{A} \to \mathcal{K}$. For $f : a \to b$ in $\mathcal{A}$, set $Pf = (y_b f)^\sharp : Pa \to Pb$. The 2-cell structure and its coherence are routine. Then $y_a : a \to Pa$ can be given the structure of a transformation. For $a \xrightarrow{f} b$ the structure 2-cells are $y_b.f \xrightarrow{\eta_{y_b.f}} (y_b.f)^\sharp.y_a = Pf.y_a$.

### 2.2. The Kleisli bicategory

Given a Kleisli structure $P$ on $\mathcal{A} \hookrightarrow \mathcal{K}$ we define its *Kleisli bicategory* $\mathrm{Kl}(P)$ as follows. The objects of $\mathrm{Kl}(P)$ are the objects of $\mathcal{A}$. For objects $a, b$, set $\mathrm{Kl}(P)(a, b) = \mathcal{K}(a, Pb)$. The identities of $\mathrm{Kl}(P)$ are the $y_a : a \to Pa$, from the Kleisli structure. The Kleisli composition of $f : a \to Pb$ and $g : b \to Pc$, is $g \cdot f = g^\sharp.f : a \to Pc$. This extends to 2-cells so we have composition functors. To obtain a bicategory, it remains to define coherent unit and associativity isomorphisms $\lambda_f : y_b \cdot f \to f$, $\rho_f : f \to f \cdot y_a$ and $\alpha_{h,g,f} : (h \cdot g) \cdot f \to h \cdot (g \cdot f)$. The unit isomorphisms $\lambda_f$ and $\rho_f$ are given by $(y_b)^\sharp f \xrightarrow{\kappa_b f} 1_{Pb} f \cong f$ and $f \xrightarrow{\eta_f} f^\sharp y_a$ respectively, while the associativity $\alpha_{h,g,f}$ is $(h^\sharp g)^\sharp f \xrightarrow{\kappa_{h,g} f} h^\sharp g^\sharp f \cong h^\sharp (g^\sharp f)$. The coherence axioms follow directly from the coherence conditions of the Kleisli structure.

**Theorem 2.2.** *Let $P$ be a Kleisli structure on $\mathcal{A} \hookrightarrow \mathcal{K}$. Then $\mathrm{Kl}(P)$ is a bicategory.*

For simplicity in what follows, I shall adopt standard notation and often write $a \to\!\!\!\!\!\to b$ instead of $a \to PB$ for maps in $\mathrm{Kl}(P)(a, b)$.

In traditional category theory, the Kleisli construction is one universal way to associate an adjunction with a monad. In the 2-dimensional setting of Kleisli structures we get a restricted (pseudo)adjunction as follows. There is a 'forgetful' pseudofunctor $\mathrm{Kl}(P) \to \mathcal{K}$ taking $f : a \to Pb$ in $\mathrm{Kl}(P)$ to $f^\sharp : Pa \to Pb$ in $\mathcal{K}$. And there is a pseudofunctor $F : \mathcal{A} \to \mathrm{Kl}(P)$, taking $f : a \to b$ in $\mathcal{A}$ to $y_b.f : a \to Pb$ considered as a map from $a$ to $b$ in $\mathrm{Kl}(P)$. I omit the 2-dimensional structure which is routine, but note that the fact that $F$ is a restricted left pseudoadjoint is immediate from the identification $\mathrm{Kl}(P)(a, b) = \mathcal{K}(a, Pb)$.

### 2.3. Presheaves and profunctors

The basic example of a Kleisli structure is the *presheaf Kleisli structure* arising from the presheaf construction. This gives a Kleisli structure on $\mathbf{Cat} \hookrightarrow \mathbf{CAT}$, the inclusion of the 2-category of small categories in that of locally small categories. First recall my notational convention: in general bicategories objects are lower case; but now the objects are themselves categories and I use upper case.

Here are the details of the presheaf Kleisli structure. For a small category $A$, take $PA = [A^{op}, \mathbf{Sets}]$, the category of presheaves over $A$, with the usual Yoneda embedding $y_A : A \to PA$. For $f : A \to PB$, we have $f^\sharp : PA \to PB$, a given choice of left Kan extension of $f$ along the Yoneda embedding. The 2-dimensional structure arises as part of the story of why presheaves give a Kleisli structure.

Let **COC** be the 2-category of locally small cocomplete categories, cocontinuous functors and natural transformations. There is an evident forgetful 2-functor $U : \mathbf{COC} \to \mathbf{CAT}$. Consider the diagram

$$
\begin{tikzcd}
 & \mathbf{COC} \arrow[d, "U"] \\
\mathbf{Cat} \arrow[ur, "\hat{P}"] \arrow[r, hook] & \mathbf{CAT}
\end{tikzcd}
$$

where $\hat{P}$ is the presheaf construction thought of as a pseudo-functor $\mathbf{Cat} \to \mathbf{COC}$. We have then for $A \in \mathbf{Cat}$ and $\mathcal{B} \in \mathbf{COC}$ an adjoint equivalence

$$
\begin{tikzcd}
\mathbf{CAT}(A, U\mathcal{B}) \arrow[r, shift left, "(-)^\dagger"] \arrow[r, phantom, "\perp"] & \mathbf{COC}(FA, \mathcal{B}) \arrow[l, shift left, "(-)y_A"]
\end{tikzcd}
$$

where $(-)^\dagger$ is left Kan extension thought of as landing in **COC**. It is straightforward to extract Kleisli structure from such a situation. We have $P = U\hat{P}$ as pseudo-functor $\mathbf{Cat} \to \mathbf{CAT}$ and the Yoneda left Kan extension is just $f^\sharp = U(f^\dagger)$. Now we can see how the 2-cells in the Kleisli structure arise. For $f : A \to PB = U\hat{P}B$, $\eta_f : f \to f^\sharp y_A = U(f^\dagger)y_A$ is the unit of the adjunction above. The 2-cell $\kappa_A : (y_A)^\sharp = U({y_A}^\dagger) \to 1_{PA}$ is

$$U\big({y_A}^\dagger\big) \cong U\big(\big(U(1_{FA})y_A\big)^\dagger\big) \xrightarrow{G(\varepsilon_{1_{FA}})} U(1_{FA}) \cong 1_{PA}$$

using the counit of the adjunction. The 2-cell $\kappa_{g,f} : (g^\sharp f)^\sharp \to g^\sharp f^\sharp$ is the composite of the two lines displayed here.

$$\big(g^\sharp f\big)^\sharp = G\big(G\big(g^\dagger\big)f\big)^\dagger \xrightarrow{G(G(g^\dagger)\eta_f)^\dagger} G\big(G\big(g^\dagger\big)G\big(f^\dagger\big)y_A\big)^\dagger \cong G\big(G\big(g^\dagger f^\dagger\big)y_A\big)^\dagger;$$
$$G\big(G\big(g^\dagger \cdot f^\dagger\big) \cdot y_A\big)^\dagger \xrightarrow{G(\varepsilon_{g^\dagger \cdot f^\dagger})} G\big(g^\dagger \cdot f^\dagger\big) \cong G\big(g^\dagger\big) \cdot G\big(f^\dagger\big).$$

With these definitions it is routine to check the Kleisli structure coherence.

Consider now the Kleisli bicategory $\mathrm{Kl}(P)$ of the presheaf Kleisli structure. It has objects the small categories, and for $A, B \in \mathbf{Cat}$ we have

$$\mathrm{Kl}(P)(A, B) = \mathbf{CAT}(A, PB) \cong \big[B^{\mathrm{op}} \times A, \mathbf{Sets}\big] = \mathbf{Prof}(A, B).$$

So the hom-categories can be identified with the familiar ones in the definition of the bicategory **Prof**, and then the Kleisli composition corresponds to the standard composition in **Prof**, and the structure 2-cells are as expected. So from Theorem 2.2 we deduce the following.

**Proposition 2.3.** *With structure as usually defined* **Prof** *is a bicategory.*

This outline deduction is not as pointless as may appear. Where in the literature can one find a proof? The associativity may be made explicit, but the reader is left to check the coherence conditions. Another point is that the biequivalence of **Prof** with the explicit sub-2-category of **COC** on presheaf categories is usually given as extra information. For example it is proved carefully in [4]. However the moral of the treatment here is that **Prof** is a bicategory in the first place just because it is biequivalent to **COC**.

## 3. Distributivity

Distributive laws between monads were introduced by Beck who established the connection between lifting and composition of monads. The equally elementary connection with extensions seems less well known but often arises in semantics (see [12]). From [10], I recall features of the general situation.

**Theorem 3.1.** *Let $P$ and $S$ be monads on a category $\mathcal{C}$. The following forms of data determine each other.*

- *A distributive law $SP \to PS$, that is, a natural transformation preserving the units and multplications of $P$ and $T$.*
- *A lifting of $P$ along the forgetful functor $S$-Alg $\to \mathcal{C}$ to a monad $P^S$ on the category $S$-Alg.*
- *An extension of $S$ along the free functor $\mathcal{C} \to \mathrm{Kl}(P)$ to a monad $S_P$ on the Kleisli category $\mathrm{Kl}(P)$.*

*Moreover such data determines a monad structure on the composite $PS$ with the category $PS$-Alg of $PS$-algebras isomorphic to $P^S$-Alg and the Kleisli category for $\mathrm{Kl}(PS)$ isomorphic to $\mathrm{Kl}(S_P)$.*

The subject of pseudo-distributive laws between 2-monads or more generally between pseudo-monads was already discussed by Kelly [14] and has recently been the subject of renewed attention, in [19] and [22] for example. Work is required but unsurprisingly versions of the basic categorical results go through. Here I adapt the ideas to Kleisli structures to the extent needed to arrive at Kleisli categories $\mathrm{Kl}(PS)$ of composed Kleisli structures.

### 3.1. Lifts of Kleisli structures

The situation which arises for us is the following. We start with $\mathcal{A} \hookrightarrow \mathcal{K}$ a sub-2-category of a 2-category. We have on the one hand a Kleisli structure $P$ on $\mathcal{A} \hookrightarrow \mathcal{K}$ and on the other hand a 2-monad $(S, e, m)$ on $\mathcal{K}$ which restricts to a 2-monad on $\mathcal{A}$. (The less standard notation avoids confusion with an earlier $\eta$.)

From $S$ we get a range of 2-categories: there are $S\text{-Alg}_{\mathcal{A}}$ and $S\text{-Alg}_{\mathcal{K}}$ the 2-categories of (strict) $S$-algebras and pseudo-maps over $\mathcal{A}$ and $\mathcal{B}$; and $S\text{-PsAlg}_{\mathcal{A}}$ and $S\text{-PsAlg}_{\mathcal{K}}$, the corresponding 2-categories of pseudo-algebras. The classic reference for material relating to this kind of situation is [3]. In case $S$ is a flexible monad the choices here have little importance, but our leading examples are not flexible and so it makes best sense to consider the inclusion $S\text{-Alg}_{\mathcal{A}} \hookrightarrow S\text{-PsAlg}_{\mathcal{K}}$. I display the obvious forgetful functors together with the left pseudo-adjoints given by taking free $S$-algebras.

$$\begin{tikzcd}
S\text{-Alg}_{\mathcal{A}} \arrow[r, hook] \arrow[d, shift left, "U"] & S\text{-PsAlg}_{\mathcal{K}} \arrow[d, shift left, "U"] \\
\mathcal{A} \arrow[r, hook] \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"] & \mathcal{K} \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

The details of all this are completely routine.

Now we consider *lifts* of the Kleisli structure $P$ on $\mathcal{A} \hookrightarrow \mathcal{K}$ to a Kleisli structure $P^S$ on $S\text{-Alg}_{\mathcal{A}} \hookrightarrow S\text{-PsAlg}_{\mathcal{K}}$ By this we should presumably mean that the forgetful 2-functors $U$ preserve the Kleisli structures up to coherent isomorphism. We can skip the details of the coherence as in all our examples, $U$ will preserve the structure on the nose.

To give a lift of $P$ amounts in outline to the following. For each $S$-algebra structure on $a$ we give an $S$-pseudoalgebra structure on $Pa$ together with the structure of a pseudomap on $y_a : a \to Pa$; and for each pseudomap with underlying 1-cell $f : a \to Pb$ we give a pseudomap with underlying 1-cell $f^\sharp : Pa \to Pb$, in such a way that the 2-cells $\eta_f$, $\kappa_A$ and $\kappa_{g,f}$ are pseudoalgebra 2-cells.

### 3.2. Lifting presheaves

Our applications involve the presheaf Kleisli structure $P$ on **Cat** $\hookrightarrow$ **CAT**. To give a lift of a 2-monad $S$ is to give the following. First for every $S$-algebra on a category $A$ we need to equip the presheaf category $PA$ with the structure of an $S$-algebra in such a way that the Yoneda preserves the structure in the up to coherent isomorphism sense. Secondly given $S$-algebras on categories $A$ and $B$ and an $S$-algebra pseudomap $f : A \to PB$ to the induced $S$-algebra $PB$, we equip the left Kan extension $f^\sharp$ with the structure of an $S$-algebra pseudomap so that we have an equality $f = y_A \cdot f^\sharp$ of pseudomaps.

It is as well to appreciate that none of this is automatic. If $S$ gives structure which presheaves do not have then evidently $P$ cannot lift. Biproducts gives an obvious example of that. Also even if $S$ gives structure which presheaves do have that structure may not be preserved by Yoneda. Initial objects, coproducts, indeed most colimits are of this kind. Finally even if $S$ satisfies the Yoneda condition, there may be a problem with the condition that appropriate left Kan extensions $f^\sharp : PA \to PB$ preserve $S$-structure. For example if $S$ is the monad which adds just equalizers or adds just pullbacks then the left Kan extension condition fails. In connection with the last point, I conjecture that there are very few classes of finite limits which extend. A precise form of this is the subject of a projected PhD thesis of Marie Bjerrum.

Fortunately there are many 2-monads for which $P$ does lift. Examples include 2-monads for some familiar classes of limits, terminal object, products and finite limits. The work of Im and Kelly [13] provides additional examples: the 2-monads for monoidal categories and for symmetric monoidal categories, together with minor variants. Finally there are a range of miscellaneous examples: the 2-monad equipping a category with an endofunctor; the 2-monad equipping a category with factorization. These last can be treated directly. I expand a little on the examples which will be of most interest for the remainder of this paper.

1. The monad for a strict monoidal category. Given a strict monoidal category $A$, $PA$ is monoidal by the Day convolution tensor product and so readily acquires the structure of an $S$-pseudoalgebra. The Yoneda preserves the tensor product in an up to coherent isomorphism sense, and the left Kan extension condition is essentially in [13].
2. The case of the monad for a symmetric strict monoidal category follows readily from the strict monoidal case, following the lines of [13].
3. The monad for strictly associative products could be another application of [13], but can be handled directly. Presheaf categories have products and the Yoneda preserves them. Direct calculation shows that if $f : A \to PB$ preserves products then so does $f^\sharp : PA \to PB$. (For $X \in PA$ the functor $X \times -$ preserves colimits.)

### 3.3. Composed Kleisli structures

I return to the general setting and suppose that $S$ is a 2-monad and $P$ lifts from $\mathcal{A} \hookrightarrow \mathcal{K}$ to $P^S$ on $S$-$\mathrm{Alg}_{\mathcal{A}} \hookrightarrow S$-$\mathrm{PsAlg}_{\mathcal{K}}$. The aim of this section is to construct a Kleisli structure on $\mathcal{A} \hookrightarrow \mathcal{K}$ with the composite $PS$ as the basic operation on objects.

Applying our lifted Kleisli $P^S$ to free $S$-algebras $m_b : S^2b \to Sb$ gives pseudoalgebras with $\sigma_b : SPSb \to PSb$ say as structure map. In addition we have a pseudoalgebra map as in the diagram

$$\begin{tikzcd} S^2B \arrow[r, "Sy_{Sb}"] \arrow[d, "m_b"'] \arrow[dr, phantom, "\cong"] & SPSb \arrow[d, "\sigma_b"] \\ SB \arrow[r, "y_{Sb}"'] & PSb \end{tikzcd}$$

Now I define the unit structure $y^S$ and extension structure $(-)^{\sharp^S}$ for a Kleisli structure on $PS$. For each $a$ take as unit

$$\big(y^S_a : a \to PSa\big) = \big(a \xrightarrow{e_a} Sa \xrightarrow{y_{Sa}} PSa\big).$$

For $f : a \to PSb$ take as extension

$$\big(f^{\sharp^S} : PSa \to PSb\big) = \big(SA \xrightarrow{Sf} SPSb \xrightarrow{\sigma_b} PSb\big)^\sharp.$$

Now for the structure 2-cells. Given $f : a \to PSb$ let $\eta^S_f : f \to f^{\sharp^S}.y^S_a$ be the composite of the 2-cells isomorphisms

$$f \cong \sigma_b.e_{PSb}.f \cong \sigma_b.Sf.e_a \cong (\sigma_b.Sf)^\sharp.y_{Sa}.e_a \cong f^{\sharp^S}.y^S_a$$

To define $\kappa^S_a : (y^S_a)^{\sharp^S} \to 1_{PSa}$ take the composite of the 2-call isomorphisms

$$\big(y^S_a\big)^{\sharp^S} \cong (\sigma_a.Sy_{Sa}.Se_a)^\sharp \cong (y_{Sa}.m_a.Se_a)^\sharp \cong (y_{Sa})^\sharp \cong 1_{PSa}.$$

Finally given $f : a \to PSb$ and $g : b \to PSc$, define $\kappa^S_{g,f} : (g^{\sharp^S}.f)^{\sharp^S} \to g^{\sharp^S}.f^{\sharp^S}$ as the composite of the 2-cell isomorphisms indicated.

$$\big(g^{\sharp^S}.f\big)^{\sharp^S} \cong \big(\sigma_c.S(\sigma_c.Sg)^{\sharp^S}.Sf\big)^{\sharp^S} \cong \big((\sigma_c.Sg)^{\sharp^S}.\sigma_c.Sf\big)^{\sharp^S};$$

$$\big((\sigma_c.Sg)^{\sharp^S}.\sigma_c.Sf\big)^{\sharp^S} \cong (\sigma_c.Sg)^{\sharp^S}.(\sigma_c.Sg)^{\sharp^S} = g^{\sharp^S}.f^{\sharp^S}.$$

In all cases the isomorphism in question is either structural, comes from the Kleisli structure $P$ or is in the diagram above. I did not make explicit what are the coherence isomorphisms for a Kleisli structure and the reader will have to take on trust my assertion that checking them is straightforward.

**Theorem 3.2.** *Suppose that $S$ is a 2-monad and $P$ a Kleisli structure on $\mathcal{A} \hookrightarrow \mathcal{K}$ with $P$ lifting to $P^S$ on $S$-$\mathrm{Alg}_{\mathcal{A}} \hookrightarrow S$-$\mathrm{PsAlg}_{\mathcal{K}}$. Then $PS$ with structure above is a Kleisli structure on $\mathcal{A} \hookrightarrow \mathcal{K}$.*

We do not need much more than this for applications so I say little about 2-dimensional versions of other points in Theorem 3.1. From a lift $P^S$ of $P$ we easily get a distributive law $\lambda : SP \to PS$ with components composites

$$SPb \xrightarrow{SPe_b} SPSb \xrightarrow{\sigma_b} PSb = SPb \xrightarrow{\lambda_b} PSb$$

Preservation of structure by Yoneda specifies a natural isomorphism

$$Sb \xrightarrow{y_{Sb}} PSb \cong Sb \xrightarrow{Sy_b} SPb \xrightarrow{\lambda_b} PSb$$

which is part of the essential structure of a pseudo-distributivity. I omit the rest of the structure. Also there is an extension of $S$ to $S_P$ on $\mathrm{Kl}(P)$. In principle it can be defined by composition with $\lambda$ as in the ordinary case. A slicker equivalent approach is to note that the forgetful $\mathrm{Kl}(U) : \mathrm{Kl}(P^S) \to \mathrm{Kl}(P)$ has a left biadjoint. The $S_P$ can be taken to be the resulting pseudomonad on $\mathrm{Kl}(P)$. If we do this carefully we get an isomorphism $\mathrm{Kl}(S_P) \cong \mathrm{Kl}(PS)$.

### *3.4. Extensions to profunctors*

The bicategory **Prof** is very special and I give details of the extension of $S$ on $\mathbf{Cat} \hookrightarrow \mathbf{CAT}$ to $S_P$ on **Prof**. Extension is along $F : \mathbf{Cat} \to \mathbf{Prof}$ as in Section 2.2. This pseudofunctor takes $f : A \to B$ in **Cat** to $f_* : A \nrightarrow B$ in **Prof**, where $f_*(b, a) = B(b, fa)$. The first crucial feature of the situation is that the arrow $f_*$ is a left adjoint in **Prof** with right adjoint $f^* : B \nrightarrow A$ given by $f^*(a, b) = B(fa, b)$. Now extension gives for $f : A \to B$ a specified natural isomorphism between

$$(Sf)_* = SA \xrightarrow{Sf} SB \xrightarrow{y_{SB}} PSB$$

and

$$S_P(f_*) = SA \xrightarrow{Sf} SP \xrightarrow{Sy_B} SPB \xrightarrow{\lambda_B} PSB$$

Since we have specified adjunctions $(Sf)_* \dashv (Sf)^*$ and $S(f_*) \dashv S(f^*)$, we get specified natural isomorphisms $(Sf)^* \cong S_P(f^*)$.

In **Prof**, there is a factorization of arrows analogous to the factorization of relations through the graph of the relation as the opposite of a function followed by a function. Given $M : A \nrightarrow B$ in **Prof**, there is a category $E(M)$ of elements of $M$ and functors $p : E(M) \to A$ and $q : E(M) \to B$, which come together with an isomorphism $M \cong q_* p^*$: in other words up to isomorphism every arrow $M : A \nrightarrow B$ in **Prof** is a composite $A \xrightarrow{p^*} E(M) \xrightarrow{q_*} B$. Note that this means that the extension $S_P$ is determined by the requirement that $S_P(f_*) \cong (Sf)_*$. For that requirement determines $S_P(f^*)$, and taken together that determine the extension $S_P$.

## 4. Algebraic theories

For the remainder of this paper I shall work in the setting provided by the presheaf Kleisli structure $P$ on $\mathbf{Cat} \hookrightarrow \mathbf{CAT}$. I consider notions of algebraic theory determined by 2-monads $S$ for which we have a lift $P^S$ of $P$. The action takes place in the Kleisli bicategory $\mathrm{Kl}(PS)$. I shall write its maps as $A \nrightarrow SB$ which makes visible the identification with $\mathrm{Kl}(S_P)$. (At a few points I shall make active use of the extension $S_P$ of $S$.)

Since in many modern applications one needs enriched notions of algebraic theory I note that the theory described enriches readily. The background needed for that is contained in [15]. There are details to check but no essential difficulties appear. That of course is to say that the basic enriched theory is already apparent in what I present.

### *4.1. Kleisli objects in profunctors*

My approach to algebraic theories uses monads in a Kleisli bicategory $\mathrm{Kl}(PS)$ so to prepare the ground I say a little about monads in **Prof**. A monad is given by a profunctor $M : A \nrightarrow A$ with a unit 2-cell $I_A \Longrightarrow M$ and composition 2-cell $M \cdot M \Longrightarrow M$ satisfying the usual equations. For **Prof** the Eilenberg–Moore and Kleisli objects for a monad coincide, but it is the Kleisli which is important here. The universal diagram is of the form

$$\begin{tikzcd} A \arrow[rr, "M", "|" marking] \arrow[dr, "k_*"', "|" marking] & \arrow[d, phantom, "\Leftarrow"] & A \arrow[dl, "k_*", "|" marking] \\ & C(M) & \end{tikzcd}$$

where the 2-cell satisfies the conditions for the algebra for a monad. Concretely the Kleisli object $C(M)$ has the same objects as $A$ and its arrows $C(M)(b, a) = M(b, a)$ are given by the elements of $M$. Composition and units come from the corresponding structure on $M$, and the comparison arrow $k_* : A \to C(M)$ corresponds to an identity on objects functor $k : A \to C(M)$.

Every functor can be factorised as an identity on objects followed by a full and faithful functor. This well known factorisation system on **Cat** can be derived from the Kleisli construction. For let $f : A \to B$ be a functor. As $f_* \dashv f^*$, the composite $f^* \cdot f_* : A \nrightarrow A$ is a monad. Moreover the counit of the adjunction gives a 2-cell

$$\begin{tikzcd} A \arrow[rr, "f^* \cdot f_*", "|" marking] \arrow[dr, "f_*"', "|" marking] & \arrow[d, phantom, "\Leftarrow"] & A \arrow[dl, "f^*", "|" marking] \\ & B & \end{tikzcd}$$

satisfying the algebra conditions. So by the universality of the Kleisli construction, $A \xrightarrow{f} B$ factors uniquely as $A \xrightarrow{f} C(f^* \cdot f_*) \longrightarrow B$.

### 4.2. Intuition on composed Kleisli bicategories

Let $S$ be a 2 monad with $P$ lifting so that we have the composed Kleisli bicategory $\mathrm{Kl}(PS) \cong \mathrm{Kl}(S_P)$ with arrows $A \nrightarrow SB$. I shall use $\odot$ to denote the composition in $\mathrm{Kl}(PS)$: that is more or less in accord with practice in the operads community.

Let me give an intuitive syntactic reading of the arrows and their composition in $\mathrm{Kl}(PS)$. Consider $F : A \nrightarrow SB$. Write $a \in A$ for objects of $A$ and $\mathbf{b} \in SB$ for objects of $SB$. Then an element $f \in M(\mathbf{b}, a)$ should be thought of as a formal function or function symbol with input arity $\mathbf{b}$ and output arity $a$. (The choice of $S$ determines the nature of the input arities.) Now suppose we have $F : A \nrightarrow SB$ and $G : B \nrightarrow SC$. By definition $G \odot F : A \nrightarrow SC$ is the composite $A \xrightarrow{F} SB \xrightarrow{SG} S^2C \xrightarrow{m_*} SC$. What is an element of $G \odot F(\mathbf{c}, a)$? In the language of function symbols the intuition is that it is determined by a function symbol $f \in F(\mathbf{b}, a)$ together with a certain $S$-arities worth of functions symbols $g_i \in G(\mathbf{c}_i, b_i)$, with the $b_i$ making up $\mathbf{b}$; the $g_i$ should be thought of as substituted into the input arity $\mathbf{b}$ and the arities $\mathbf{c}_i$ composed together by the multiplication in $S$ to give the single arity $\mathbf{c}$. The overall idea is that the 2-monad $S$ has determined a general notion of substitution.

### 4.3. Theories as monads

In this section I explain the idea that a many-sorted $S$-algebraic theory or simply an $S$-multicategory, can be represented as a monad $M : A \nrightarrow SA$ in $\mathrm{Kl}(PS)$, or equivalently as monoids in the endocategory $\mathrm{Kl}(PS)(A, A)$. The familiar single-sorted theories correspond to the case $A = 1$. I shall use the term monad here though the operads community often says monoid.

A monad in $\mathrm{Kl}(PS)$ consists of a profunctor $M : A \nrightarrow SA$ with a unit and composition 2-cells $\eta_{A*} \Longrightarrow M$ and $M \odot M \Longrightarrow M$ satisfying the usual equations. In terms of the syntactic reading, the identity provides identity functions of input arity $e(a)$ and output arity $a$. This is usually represented in syntax by a variable. Composition gives the interpretation of formal composites of $f \in M(\mathbf{a}, a)$ with an $S$-indexed family $\mathbf{g} \in SM(\alpha, \mathbf{a})$, the interpretation being given in $M(m(\alpha), a)$. So it provides the interpretation of composite symbols. That is in general terms exactly what we expect of an algebraic theory. I shall tighten up this idea shortly but for the moment let us assume that an $S$-algebraic theory is just a monad in $\mathrm{Kl}(PS)$

To define a map between $S$-algebraic theories we need to exploit the pseudofunctor $F : \mathbf{Cat} \to \mathbf{Prof}$: we cannot simply take monad maps in $\mathrm{Kl}(PS)$ as they evidently do not give what we want.

**Definition 4.1.** A *map of theories* from $M : A \nrightarrow SA$ to $N : B \nrightarrow SB$ consists of a functor $f : A \to B$ together with a 2-cell

$$\begin{tikzcd} A \arrow[r, "M", "|" marking] \arrow[d, "f_*"', "|" marking] \arrow[dr, phantom, "\Downarrow"] & SA \arrow[d, "Sf_*", "|" marking] \\ B \arrow[r, "N"', "|" marking] & SB \end{tikzcd}$$

which satisfies commutation conditions expressing compatibility with the unit and composition 2-cells. A 2-*cell between maps* is given by a natural transformation between the functors compatible with the 2-cells in the obvious sense. Taken together this data gives us a 2-category $S$-Mult of $S$-*algebraic theories*.

I need immediately to refine that definition. It is almost right except for the fact that $A \overset{M}{\nrightarrow} SA$ carries not just the data of the multicategory but data corresponding to the category $A$ and its action on the multicategory. This data is unwelcome and there are two ways to suppress it.

1. One can either insist that the categories $A$ are discrete so that there is no information either in $A$ or its action; This is the evident choice.
2. One can require that the category $A$ is exactly the underlying category of the multicategory $M$, i.e. it is *normal* in Australian terminology.

For the theory developed here the second is the more natural choice so let us assume normal in the definition above. For clarity I explain precisely what it means for a monoid $A \overset{M}{\nrightarrow} SA$ to be normal. The unit 2-cell $\eta_{A*} \Longrightarrow M$ gives by transpose a cell $I_A \Longrightarrow \eta_A^* \cdot M$, and we require that this be an isomorphism. The discussion in Section 4.1 shows that this what is intended. The point is that $\eta_A^* \cdot M$ becomes a monad in **Prof** and so gives an identity on objects functor from $A$ to what is the underlying category of $M$; and we want this to be an isomorphism.

I do not want to make a big deal about the choice here. The rest of the paper can be read without much change with the other choice. In any case the existence of two distinct embeddings, one discrete, the other chaotic, is a common situation to which Lawvere has frequently drawn attention.

### *4.4. Models of algebraic theories*

This is a rich topic and I just give the basic definition and make a few remarks about the monadic approach to algebra. First I recall an argument from [11] in favour of the Lawvere theory approach to algebraic theories over the monadic. The monadic approach gives immediate access only to what are called algebras for the monad, that is, models in the ambient category of the monad. The Lawvere theory approach from [16] gives models in any category with products. In line with the general practice of categorical logic, models are product preserving functors. How does the bicategorical approach to algebraic theories compare?

The simple view is that models of an $S$-algebraic theory $A \overset{M}{\to} SA$ can be taken in any $S$-algebraic theory. A category of models in $B \overset{N}{\to} SB$ is just the category of maps of algebraic theories from $A \overset{M}{\to} SA$ to $B \overset{N}{\to} SB$. That is the fundamental notion, but perhaps it will leave readers unsatisfied. What about models in categories, or rather, taking the lead from [16], in categories with $S$-structure up to isomorphism, that is to say in $S$-pseudoalgebras? Happily there is an elegant account of that.

First suppose that we are given an $S$-algebra $a : SA \to A$. This gives rise to an $S$-multicategory in a ridiculously simple way. One takes the profunctor $A \overset{a^*}{\nrightarrow} SA$ and equips it with a monoid structure. The unit $\eta_* \Longrightarrow a^*$ is the transpose of the isomorphism (identity) $a_*\eta_* \Longrightarrow 1_A$. The multiplication $a^* \odot a^* \Longrightarrow a^*$, that is, $\mu_*(Sa^*)a^* \Longrightarrow a^*$ is the transpose of the isomorphism $(Sa^*)a^* \Longrightarrow \mu^* a^*$. It is straightforward to check that this makes $A \overset{a^*}{\nrightarrow} SA$ a normal $S$-multicategory. This construction extends to a 2-functor $S$-Alg to $S$-Mult. We shall consider it further in the next section.

However nothing depended on the supposition that $a : SA \to A$ was a strict as opposed to a pseudoalgebra; and nothing beyond a question of extending definitions on the supposition that $A$ was small. Certainly for any $x : SX \to X$ in $S$-PsAlg$_{\mathbf{CAT}}$, we can formally construct a large $S$-multicategory $X \overset{x^*}{\to} SX$. Then by definition a model of $S$-algebraic theory $A \overset{M}{\to} SA$ in an $S$-pseudoalgebra $x : SX \to X$ is just a map of theories $A \overset{M}{\to} SA$ to $X \overset{x^*}{\to} SX$, and we get a (possibly large) category of models.

I close this section with observations from the monadic point of view. It is clear that a monad in a bicategory acts by composition (on either side) as a monad on suitable hom-categories. In the case of our leading examples, this gives an immediate explanation of the monad on **Sets** generated by a single-sorted algebraic theory $1 \overset{M}{\to} S1$. In $\mathrm{Kl}(S_P)$, $M$ is a monad on 1, and so we obviously have an action by composition in particular on $\mathrm{Kl}(S_P)(1,0)$. But in each case we have $S0 \cong 1$, so that $\mathrm{Kl}(S_P)(1,0)$ is isomorphic to **Sets**. Thus we get the usual monad on **Sets**.

From the bicategorical point of view the choice of $\mathrm{Kl}(S_P)(1,0)$ is rather arbitrary. There is at least one other compelling case which is $\mathrm{Kl}(S_P)(1,1)$. I note that when $S$ is symmetric strict monoidal categories 2-monad, this choice gives the notion of twisted (French tordue) algebra. Clearly there is more to be said to reconcile the different points of view on models.

### *4.5. The free S-algebra*

In the previous section we saw the simple functor $S$-Alg $\to$ $S$-Mult giving the $S$-algebraic theory corresponding to an $S$-algebra. We expect a left adjoint, that is, given an $S$-multicategory we expect to be able to construct freely from it an $S$-algebra. We can do that very generally and I sketch the essential point. Suppose that we have a multicategory $M : A \nrightarrow SA$ and $S$-algebra $b : SB \to B$. Then there is a correspondence between diagrams of the two forms

$$\begin{tikzcd}
A \arrow[r, "M", "|" marking] \arrow[d, "f_*"', "-" marking] \arrow[dr, phantom, "\Downarrow"] & SA \arrow[d, "Sf_*", "-" marking] \\
B \arrow[r, "b^*"', "|" marking] & SB
\end{tikzcd}
\qquad
\begin{tikzcd}
SA \arrow[r, "S_P M", "|" marking] \arrow[dr, "g_*"', "/" marking] & S^2A \arrow[r, "(m_A)_*", "|" marking] \arrow[d, phantom, "\Leftarrow"] & SA \arrow[dl, "g^*", "\backslash" marking] \\
& B &
\end{tikzcd}$$

where predictably $g : SA \to B$ and $f : A \to B$ determine each other by $g = b.Sf$ and $f = g.e_A$. In the diagram on the left we take the structure of a map of multicategories from $M : A \nrightarrow SA$ to $b^* : B \nrightarrow SB$, while on the left we take a Kleisli cone in a prima facie non-evident bicategory *of lax S-profunctors* in which objects are $S$-algebras and arrows profunctors preserving $S$-algebra structure in a lax sense. (This precise formulation is Richard Garner's.) It follows that quite generally the free $S$-algebra on an $S$-multicategory is given by a special Kleisli construction.

**Theorem 4.2.** *If Kleisli objects exist in the bicategory of lax $S$-profunctors, then there is a left adjoint $S$-Mult $\to$ $S$-Alg to $S$-Alg $\to$ $S$-Mult.*

As with many colimits the existence of Kleisli objects is shown by an iterative construction. (In view of Proposition 4.5 it is not surprising that this is very close to a construction in [3].) However in our leading examples there is a much simpler approach. All satisfy the condition that the 2-monad $S$ preserves bijective on objects functors. It is easy to check the following basic observation.

**Proposition 4.3.** *Let $S$ be a 2-monad on* **Cat** $\hookrightarrow$ **CAT** *with extension $S_P$ on* **Prof**. *Then $S$ preserves bijective on objects functors if and only if $S_P$ preserves Kleisli objects.*

Now suppose we are in the situation above. Let $M : A \nrightarrow SA$ be an $S$-multicategory and consider the composite

$$\begin{tikzcd} SA \arrow[r, "SM", "|" marking] & S^2A \arrow[r, "\mu_*", "|" marking] & SA \end{tikzcd}$$

which is certainly a monad in Prof. We can construct its Kleisli object $C(M)$ as in the diagram

$$\begin{tikzcd} SA \arrow[r, "S_P M", "|" marking] \arrow[dr, "h_*"', "|" marking] & S^2A \arrow[r, "(m_A)_*", "|" marking] \arrow[d, phantom, "\Leftarrow"] & SA \arrow[dl, "h_*", "|" marking] \\ & C(M) & \end{tikzcd}$$

and exploit the basic fact that we have maps where indicated. Applying $S_P$ to this diagram we get another Kleisli object and so by the universality explained earlier a map $SC(M) \to C(M)$. Easily that makes $C(M)$ into an $S$-algebra and $h : SA \to C(M)$ a strict map of $S$-algebras. It is now routine to check the following.

**Proposition 4.4.** *Suppose $S$ is a 2-monad on* **Cat** $\hookrightarrow$ **CAT** *extending to $S_P$ on* Prof, *and suppose further that $S$ preserves bijective on objects functors. Then for every monad $M : A \nrightarrow SA$ in* Kl$(PS)$, *its Kleisli object is given by $C(M)$ with the structure above.*

### *4.6. Lawvere theories and PROPs*

I record what the construction of the free $S$-algebra amounts to in our leading special examples from Section 3.2, which I now take in reverse order.

First take $S$ to be the monad for strict products. An $S$-algebraic theory $1 \overset{M}{\nrightarrow} S1$ corresponds exactly to what is usually called an algebraic theory: my formulation is a fancy way to describe the notion of abstract clone from universal algebra. If $M$ is an algebraic theory then the construction of $C(M)$ gives exactly the corresponding Lawvere theory. More general $S$-algebraic theories $A \overset{M}{\nrightarrow} SA$ might naturally be called many-sorted algebraic theories. I prefer and shall use the precise terminology cartesian theories. For $M$ a cartesian theory $C(M)$ gives a category with strict products. By Section 4.5 the models of $C(M)$ in the usual sense of categorical logic are equivalent to the models of the cartesian theory.

Now take $S$ to be the monad for symmetric strict monoidal categories. An $S$-algebraic theory $1 \overset{M}{\nrightarrow} S1$ corresponds exactly to a symmetric operad. In this case the construction of $C(M)$ gives the corresponding *PROP*, in effect symmetric monoidal category generated by one object. A more general $S$-algebraic theory $A \overset{M}{\nrightarrow} SA$ is what is called a coloured operad. For those $C(M)$ gives a symmetric strict monoidal category whose models in the sense of symmetric monoidal categories are models for the coloured operad.

Finally take $S$ to be the monad for strict monoidal categories. An $S$-algebraic theory $1 \overset{M}{\nrightarrow} S1$ corresponds exactly to a non-symmetric operad. In this case the construction of $C(M)$ gives the corresponding non-symmetric *PROP*. (That is a slight contradiction of terminology!) A more general $S$-algebraic theory $A \overset{M}{\nrightarrow} SA$ is a non-symmetric coloured operad. For those $C(M)$ gives a strict monoidal category whose models in the sense of monoidal categories are models for the coloured non-symmetric operad.

All the above is quite straightforward, but there is a significant difference between the first case and the other two. This is most easily explained by making precise a connection with [3]. For a 2-monad $S$, one has 2-categories, $S$-Alg$_{str}$, $S$-Alg and $S$-Alg$_{lax}$ of strict $S$-algebras with strict, pseudo and lax maps respectively. One has forgetful functors $S$-Alg$_{str}$ $\to$ $S$-Alg $\to$ $S$-Alg$_{lax}$. A major achievement of [3] is the construction of left adjoints

$$(-)' : S\text{-Alg} \to S\text{-Alg}_{str} \quad \text{and} \quad (-)^\dagger : S\text{-Alg}_{lax} \to S\text{-Alg}_{str}.$$

**Proposition 4.5.** *Suppose $S$ is a 2-monad with extension $S_P$. Then for every $S$-algebra $a : SA \to A$, the free $S$-algebra generated by the $S$-algebraic theory $A \xrightarrow{a^*} SA$ gives the left adjoint $(A)^\dagger$.*

Another way to say the same thing is that the 2-comonad on $S\text{-Alg}_{str}$ generated by the adjunction of Theorem 4.2 coincides with that given in [3]. The following is then immediate.

**Proposition 4.6.** *The 2-category of $S$-Mult of $S$-algebraic theories embeds in the 2-category of $(-)^\dagger$-coalgebras.*

Now suppose in our standard setting that the 2-monad $S$ is colax idempotent. Then lax $S$-algebra maps are automatically pseudo and we can identify the 2-functors $(-)^\dagger$ and $(-)'$. Now by [3] each $S$-algebra $A$ is equivalent to $A'$ which is a free $(-)'$-coalgebra, equivalently a free $(-)^\dagger$-coalgebra. But the category of free $(-)^\dagger$-coalgebra is just $S\text{-Alg}_{lax}$. Hence every $S$-algebraic theory is equivalent to one arising from an $S$-algebra. Since the 2-monad for strict products is colax idempotent, it follows that up to equivalence Lawvere theories are just algebraic theories. In the many object case, categories with products can be identified with cartesian theories.

For general $S$ however the above analysis does not apply. Specifically, the 2-monad for symmetric strict monoidal categories is certainly not colax idempotent and it is not the case that every symmetric strict monoidal category arises up to equivalence, as the free symmetric monoidal category generated by a coloured operad. For example the PROP for comonoids does not so arise. Why? Because if a theory is given by an operad, then there is a corresponding monad on any symmetric monoidal closed cocomplete category and there is no free comonoid functor on most such categories. In particular there is none in the case of $\mathbf{Vect}_k$, the category of vector spaces over a field $k$ with its standard tensor structure. Why? Because the forgetful does not preserve limits. Why? Well the terminal coalgebra is just the field $k$ with trivial coalgebra structure and $k$ is not terminal in $\mathbf{Vect}_k$.

## 5. Comparing notions of theory

The aim of this section is to develop the theory needed to support the free and forgetful functors linking various notions of theory. Since notions of theory here correspond to certain 2-monads, this depends on understanding maps between them.

### *5.1. Compatible maps of monads*

Suppose now that $S$ and $T$ are 2-monads on $\mathbf{Cat} \hookrightarrow \mathbf{CAT}$ with liftings $P^S$ and $P^T$ of the presheaf Kleisli structure to the categories of algebras. Suppose $k : T \to S$ is a map of 2-monads, inducing maps $S\text{-Alg}_{\mathbf{Cat}} \to T\text{-Alg}_{\mathbf{Cat}}$ and $S\text{-PsAlg}_{\mathbf{CAT}} \to T\text{-PsAlg}_{\mathbf{CAT}}$, which is in addition compatible with taking categories of presheaves in that we have a natural isomorphism

$$\begin{tikzcd} S\text{-Alg}_{\mathbf{Cat}} \arrow[r, "P"] \arrow[d] \arrow[dr, phantom, "\cong"] & S\text{-PsAlg}_{\mathbf{CAT}} \arrow[d] \\ T\text{-Alg}_{\mathbf{Cat}} \arrow[r, "P"'] & T\text{-PsAlg}_{\mathbf{CAT}}. \end{tikzcd}$$

We have the distributivities $\lambda^S : SP \to PS$ and $\lambda^T : TP \to PT$. Evaluating the isomorphism at free $S$-algebras gives isomorphisms between the resulting families of $T$-algebras

$$TPSA \xrightarrow{k_{PSA}} SPSA \xrightarrow{\lambda^S_{SA}} PS^2A \xrightarrow{Pm^S} PSA$$
$$TPSA \xrightarrow{\lambda^T_{SA}} PTSA \xrightarrow{Pk_{SA}} PS^2A \xrightarrow{Pm^S} PSA$$

with underlying category $PS$. That gives a choice of natural isomorphism

$$\begin{tikzcd} TP \arrow[r, "kP"] \arrow[d, "\lambda^T"'] \arrow[dr, phantom, "\cong"] & SP \arrow[d, "\lambda^S"] \\ PT \arrow[r, "Pk"'] & PS \end{tikzcd}$$

It is relatively straightforward to see that conversely this condition implies the compatibility of the maps above with the taking of presheaves.

It will be good at this point to see that we can find simple cases of compatible maps of monads between our leading examples.

Consider first the case when $S$ and $T$ are the 2-monads for categories with strict finite products and strict symmetric monoidal categories respectively. Any category with strict products is symmetric strict monoidal, so we certainly have a map $k : T \to S$ of 2-monads. What about the compatibility with presheaves? Well let $A$ be a category with strict finite products. Going one way round the natural isomorphism square we get first $PA$ as a category with finite products (considered as an $S$-pseudoalgebra) and then $PA$ again and with the products structure simply considered as (symmetric) monoidal structure (as a $T$-pseudoalgebra). Going the other way round we get first $A$ considered now as a symmetric strict monoidal category, and then $PA$ with the induced Day tensor product. Now it is both folklore and an easy computation to show that the Day tensor product obtained from cartesian product structure is itself a cartesian product. So there is an isomorphism given by the unique isomorphism between choices of products in the compatibility square. The uniqueness makes naturality rather evident.

Now what about the case when $S$ and $T$ are the 2-monads for symmetric strict monoidal categories and strict monoidal categories respectively. Here the situation is even easer. One way round we take the Day tensor product and forget the symmetry; the other we forget the symmetry and then take the Day tensor product. That amounts to exactly the same thing, so quite exceptionally the natural isomorphism is the identity. (Note the first case could have been made similarly easy by using the Day tensor product throughout, but that is rather unilluminating.)

*5.2. Comparison of Kleisli bicategories*

At this point it is probably simplest to think in terms of the extended pseudomonads $T_P$ and $S_P$. Take $f : A \nrightarrow B$ in **Prof**, that is, $f : A \to PB$ in **CAT**. The images of $f$ under $T_P$ and $S_P$ lie in a diagram

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "k_A"'] \arrow[dr, phantom, "\cong"] & TPB \arrow[r, "\lambda^T_B"] \arrow[d, "k_{PB}"] \arrow[dr, phantom, "\cong"] & PTB \arrow[d, "Pk_B"] \\ SA \arrow[r, "Sf"'] & SPB \arrow[r, "\lambda^S_B"'] & PSB \end{tikzcd}$$

Rotating for readability that gives isomorphisms

$$\begin{tikzcd} T_PA \arrow[r, "k_*", "\shortmid"{marking}] \arrow[d, "T_Pf"', "\shortmid"{marking}] \arrow[dr, phantom, "\cong"] & S_PA \arrow[d, "S_Pf", "\shortmid"{marking}] \\ T_PB \arrow[r, "k_*"', "\shortmid"{marking}] & S_PB \end{tikzcd}$$

which gives the family $(k_A)_* : TA \nrightarrow SA$ the structure of a strong transformation $k_* : T_P \to S_P$ on **Prof**. It follows automatically, or if you prefer it can be proved directly, that the family of right adjoints $(k_A)^* : SA \nrightarrow TA$ have the structure of a lax transformation $k^* : S_P \to T_P$ on **Prof**. The coherent 2-cells for the lax transformation are as in the diagram

$$\begin{tikzcd} S_PA \arrow[r, "k^*", "\shortmid"{marking}] \arrow[d, "S_Pf"', "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow"] & T_PA \arrow[d, "T_Pf", "\shortmid"{marking}] \\ S_PB \arrow[r, "k^*"', "\shortmid"{marking}] & T_PB \end{tikzcd}$$

Now consider composition with $k_*$ and $k^*$ as operations on the Kleisli bicategories. The former

$$\hat{k}_* : (A \overset{u}{\nrightarrow} TB) \quad \longmapsto \quad (A \overset{u}{\nrightarrow} TB \overset{k_*}{\nrightarrow} SB)$$

gives a pseudofunctor $\hat{k}_* : \mathrm{Kl}(PT) \to \mathrm{Kl}(PS)$, while the latter

$$\hat{k}^* : (A \overset{v}{\nrightarrow} SB) \quad \longmapsto \quad (A \overset{v}{\nrightarrow} SB \overset{k^*}{\nrightarrow} TB)$$

gives a lax functor $\hat{k}^* : \mathrm{Kl}(PS) \to \mathrm{Kl}(PT)$. The adjunction $k_* \dashv k^*$ induces isomorphisms between $\mathrm{Kl}(S_P)(k_*u, v)$ and $\mathrm{Kl}(T_P)(u, k^*v)$ so $\hat{k}^*$ is locally right adjoint to $\hat{k}_*$ in the sense that we have adjunctions

$$\begin{tikzcd} \mathrm{Kl}(PT)(A,B) \arrow[r, shift left, "\hat{k}_*"] \arrow[r, phantom, "\perp"] & \mathrm{Kl}(SPS)(A,B). \arrow[l, shift left, "\hat{k}^*"] \end{tikzcd}$$

### 5.3. Comparison of notions of theory

The simple fact that lax functors between bicategories (morphisms in Benabou's terminology) take monads to monads already plays an important role in the original paper [2]. As the condition of being normal is algebraic it is easy to see that both $\hat{k}_*$ and $\hat{k}^*$ preserve algebraic theories. Maps require more care: despite the importance of lax functors to the Benabou vision, they do not preserve maps of monads. (In a sense that is part of the point of [2].) Also one has to pause because maps of algebraic theories correspond to rather special maps of monad, using the free pseudo-functor $\mathbf{Cat} \to \mathrm{Kl}(PS)$. So there is a little work to do.

The case of the left adjoint $\hat{k}_*$ is straightforward as it is a pseudo-functor, and its action on maps in $\mathrm{Kl}(PS)$ coming from **Cat** is essentially trivial. So the action of $\hat{k}_*$

$$\begin{tikzcd} A \arrow[r, "M_T", "\shortmid" marking] \arrow[d, "f_*"', "\shortmid" marking] \arrow[dr, phantom, "\Downarrow"] & TA \arrow[d, "Tf_*", "\shortmid" marking] \\ B \arrow[r, "N_T"', "\shortmid" marking] & TB \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A \arrow[r, "M_T", "\shortmid" marking] \arrow[d, "f_*"', "\shortmid" marking] \arrow[dr, phantom, "\Downarrow"] & TA \arrow[r, "k_{A*}", "\shortmid" marking] \arrow[d, "Tf_*"', "\shortmid" marking] \arrow[dr, phantom, "\cong"] & SA \arrow[d, "Sf_*", "\shortmid" marking] \\ B \arrow[r, "N_T"', "\shortmid" marking] & TB \arrow[r, "k_{B*}"', "\shortmid" marking] & SB \end{tikzcd}$$

takes maps of $T$-algebraic theories to maps of $S$-algebraic theories. That action extends to 2-cells so we get a 2-functor $\hat{k}_* : T\text{-Mult} \to S\text{-Mult}$.

Prima facie it looks as if we need more work to deal with the lax functor $\hat{k}^*$, but that is counter intuitive as the local right adjoint $\hat{k}^*$ is forgetful and so should be straightforward. That intuition is correct and one can argue concretely since for $A \overset{M}{\nrightarrow} SA$, we have $\hat{k}(M)(\mathbf{a}, a) \cong M(k\mathbf{a}, a)$. Naturally there is a more abstract approach but either way we get the following.

**Theorem 5.1.** *Composition with $k_*$ and $k^*$ induce a 2-adjunction*

$$\begin{tikzcd} T\text{-}\mathit{Mult} \arrow[r, shift left, "\hat{k}_*"] \arrow[r, phantom, "\perp"] & S\text{-}\mathit{Mult}. \arrow[l, shift left, "\hat{k}^*"] \end{tikzcd}$$

Here the right adjoint is the easy forgetful 2-functor, while the left adjoint is the more subtle free 2-functor. One gets exactly what is expected in the examples. One can forget from an algebraic theory to a symmetric operad, and there is a free algebraic theory generated by such an operad. Similarly one can forget from a symmetric operad to a non-symmetric operad and there is a free symmetric operad on a non-symmetric one. (Of course there are the composites.)

I close with a remark from Tom Leinster about a difference between the two cases. In the first the free functor $\hat{k}^*$ reflects isomorphisms. If a cartesian theory arises from a coloured operad then the operad in question is determined up to isomorphism. Restricting to a single sort, being (symmetric) operadic is thus a property of an algebraic theory. However in the second case the free functor $\hat{k}^*$ does not reflect isomorphisms. An example of non-isomorphic non-symmetric operads which generate isomorphic symmetric operads is given in [17]. Hence there is an issue if one says of a symmetric operad that it is non-symmetric. Probably one should make a choice of the non-symmetric operad in question, in which case being non-symmetric is not a property but additional structure.

## References


[1] T. Altenkirch, J. Chapman, T. Uustalu, Monads need not be endofunctors, Submitted for publication.

[2] J. Benabou, Introduction to bicategories, in: Reports of the Midwest Category Theory Seminar, in: Lecture Notes in Mathematics, vol. 47, Springer, 1967, pp. 1–71.

[3] R. Blackwell, G.M. Kelly, A.J. Power, Two-dimensional monad theory, J. Pure Appl. Algebra 59 (1989) 1–41.

[4] G.L. Cattani, G. Winskel, Profunctors, open maps and bisimulation, Math. Structures Comput. Sci. 15 (2005) 553–614.

[5] P.-L. Curien, Operads, clones, and distributive laws, in: Proceedings of the International Conference on Operads and Universal Algebra, in: Nankai Series in Pure, Applied Mathematics and Theoretical Physics, vol. 9, World Scientific, Singapore, 2012, pp. 25–50.

[6] E. Cheng, M. Hyland, A.J. Power, Pseudo-distributive laws, in: Proceedings 19th Conference on Mathematical Foundations of Programming Semantics, in: Electronic Notes in Theoretical Computer Science, vol. 83, 2003.
[7] M. Fiore, N. Gambino, M. Hyland, G. Winskel, The cartesian closed bicategory of generalised species of structures, J. Lond. Math. Soc. 77 (2008) 203–220.
[8] M. Fiore, N. Gambino, M. Hyland, G. Winskel, Kleisli bicategories, In preparation.
[9] M. Fiore, G. Plotkin, D. Turi, Abstract syntax and variable binding, in: Proceedings 14th Annual IEEE Symposium on Logic in Computer Science, IEEE Press, 1999, pp. 193–202.
[10] M. Hyland, M. Nagayama, J. Power, G. Rosolini, A category-theoretic formulation for Engeler-style models of the untyped lambda-calculus, in: Proceedings MFCSIT 2004, in: Electronic Notes in Theoretical Computer Science, vol. 161, 2006, pp. 43–57.
[11] M. Hyland, J. Power, The category theoretic understanding of universal algebra: Lawvere theories and monads, in: L. Cardelli, M. Fiore, G. Winskel (Eds.), Computation, Meaning and Logic. Articles Dedicated to Gordon Plotkin, in: Electronic Notes in Theoretical Computer Science, vol. 172, 2007, pp. 437–458.
[12] Martin Hyland, Some reasons for generalising domain theory, Math. Structures Comput. Sci. 20 (2010) 239–265.
[13] G.B. Im, G.M. Kelly, A universal property of the convolution monoidal structure, J. Pure Appl. Algebra 43 (1986) 75–88.
[14] G.M. Kelly, Doctrinal adjunction, in: Category Seminar Sydney 1972/73, in: Lecture Notes in Mathematics, vol. 420, Springer-Verlag, 1974, pp. 257–280.
[15] G.M. Kelly, Basic Concepts of Enriched Category Theory, LMS Lecture Note Series, vol. 64, Cambridge University Press, 1982.
[16] F. William Lawvere, Functorial semantics of algebraic theories, PhD Thesis, Columbia University, 1963. Republished with an introduction in Repr. Theory Appl. Categ. 5 (2004) 1–121.
[17] T. Leinster, Are operads algebraic theories?, Bull. Lond. Math. Soc. 38 (2006) 233–238.
[18] E. Manes, Algebraic Theories, Springer, 1976.
[19] F. Marmolejo, Distributive laws for pseudomonads, Theory Appl. Categ. 5 (1999) 91–147.
[20] E. Moggi, Notions of computation and monads, Inform. and Comput. 93 (1989) 55–92.
[21] A.J. Power, E. Robinson, Premonoidal categories and notions of computation, Math. Structures Comput. Sci. 7 (1997) 453–468.
[22] M. Tanaka, Pseudo-distributive laws and a unified framework for variable binding, Ph.D. thesis, Edinburgh, 2005.
[23] M. Tanaka, J. Power, A unified category-theoretic semantics for binding signatures in substructural logics, J. Logic Comput. 16 (2006) 5–25.
[24] G. Winskel, F. Zappa Nardelli, NEW-HOPLA a higher-order process language with name generation, in: Proceedings of TCS 2004, Third IFIP International Conference on Theoretical Computer Science, 2004, pp. 521–534.