Theoretical Computer Science 546 (2014) 32–51

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Branching cells for asymmetric event structures

Samy Abbes

PPS/Université Paris Diderot, Bâtiment Sophie Germain, Avenue de France, 75 013 Paris, France

ARTICLE INFO

Keywords:
Petri net
Contextual net
Read arc net
Asymmetric event structure
Branching cell
Choice

ABSTRACT

This paper introduces branching cells as elementary units of independent choices in the model of Asymmetric Event Structures (AES), extending a previous work on branching cells for Prime Event Structures. Branching cells consist of subAES of the surrounding AES. Their maximal configurations are shown to tile any maximal configuration of the surrounding AES in a dynamic way.

Branching cells for AES are developed in order to allow the analysis of an optimization procedure in the context of QoS management of web services, presented in a companion paper. Other applications of branching cells include the ability to add a probabilistic layer to AES in a natural fashion where concurrency meets probabilistic independence of choices in distinct and parallel branching cells.

0. Introduction

Asymmetric Event Structures (AES) introduced in the 1990s [1,2] are a model for computational processes involving concurrency, which extends the model of Prime Event Structures (PES) [3]. They depart from PES mainly by the fact that an event has not only a set of mandatory causes, but also some *possible* causes, modeled by a new type of causality called *asymmetric conflict* (see the references in [2] for earlier models with similar aims). The history of an event has a *locality* property in AES. Indeed, the actual history of an event will differ according to the given computation that involves it (configuration, in the event structures language), since the possible causes of the event may or may not be present in the given computation. AES are shown to unfold so-called *contextual nets* or *nets with read arcs* [4,1,2,5–7] in a non-interleaving semantics, just as PES unfold safe Petri nets. Contexual nets differ from usual safe Petri nets in that the firing of a transition depends not only on the presence of tokens in *resource places*, and which are to be consumed, but also on the mere presence of tokens in a set of *contextual places*, and which are not to be consumed by the firing of the transition. It is this very feature that induces the asymmetry of conflict in the unfolding AES.

As for PES, the computational processes associated with an AES are captured by *configurations*. Since configurations are conflict free, it is natural to see a configuration as obtained by different choices, consisting in the resolution of conflicts. The concurrency features of the model, as well as the *confusion* that might appear in it, as described in [3], make it however non-trivial to isolate independent choices. Motivated by probabilistic applications, we have introduced branching cells for PES in this purpose in an earlier work [8]. This paper extends the notion of branching cells to AES, motivated this time by applications in QoS management and orchestration of composite services; this is the topic of the companion paper [9] in this issue.

DOI of original article: http://dx.doi.org/10.1016/j.tcs.2014.02.043.
E-mail address: samy.abbes@univ-paris-diderot.fr.

http://dx.doi.org/10.1016/j.tcs.2014.02.044
0304-3975/

The results we obtain are quite similar to the previous ones: we show that branching cells decompose in a dynamic way any maximal configuration of a finite AES, providing a decomposition of the configuration as a partial order of independent choices. These independent choices consist precisely in resolving conflicts *inside* each branching cell in order to select a maximal configuration of the branching cell. Branching cells attached to a given configuration are a tiling, in the sense that they are disjoint. However the entire collection of branching cells of the AES are not disjoint in general, excepted for particular cases such as *trees* (without concurrency) and *confusion free event structures* (restricted concurrency). In contrast, branching cells of a general AES *dynamically tile* configurations. The extension of branching cells from PES to AES is non-trivial mainly for two reasons, that we explain now.

Firstly, the partial order structure on configurations of AES is more subtle than the mere set theoretic subset relation between configurations. This implies than the simple notion of down-closed subset with respect to the causality relation is no longer adequate to provide an "initial view" of the computational process. We therefore introduce choice-complete prefixes (CC-prefix es), which have two keys properties. If C is any configuration, and if U is a CC-prefix, by putting $C_U = C \cap U$ we have that:

1. C_U is indeed an initial sub-configuration of C (which would not hold in general if U was only down closed for the causality, contrasting with PES); and
2. No more choices are pending at the end of the execution of C_U inside U. Informally speaking, the "future" of C_U is entirely disjoint from U.

The above informal mention of the "future" of a configuration is actually made rigorous in the core of the paper, and proves to be an essential tool for the subsequent analysis.

Secondly, in AES, conflict is *asymmetric* and not necessarily binary, whereas in PES, conflict is both symmetric and binary. The fine analysis of choices that we aim at leads us to introduce sources of conflict for AES, generalizing the minimal conflict relation introduced for PES. Sources of conflict are a non-binary relation, but basically play the same role for AES than the minimal conflict relation plays for PES. It is mainly a technicality, and the intuition about minimal conflict relation translates without difficulty into the sources of conflict of AES.

Organization of the paper. Asymmetric Event Structures are presented in Section 1. Our contributions start in Section 2, where we introduce the sources of conflict for AES and the different kinds of prefixes that will be needed in the sequel. Branching cells are introduced in Section 3, first informally described on a simple example. The notion of future of a configuration is introduced afterward; then we state the main result of the paper, which establishes the existence and uniqueness (up to their order) of the decomposition of a maximal configuration through its covering branching cells. Section 4 is devoted to examples illustrating the different notions introduced earlier. In particular, the dynamic character of branching cells is explained on an example. Finally the proof of the main result is the topic of Section 5. After our concluding section (Section 6), two additional sections form an appendix at the end of the paper. In Appendix A, we investigate the applications of branching cells theory to infinite AES, and more specifically to the case of so-called locally finite AES. The topic of Appendix B is to recall through examples the relationship between contextual nets and AES *via* the unfolding theory. It is intended to help the reader in view of the applications presented in companion paper [9].

1. Asymmetric event structures

In this section we follow the presentation of Asymmetric Event Structures (AES) from [2]. In view of our application, we restrict ourselves to *finite* AES; the extension to a class of infinite AES is discussed in Appendix A.

For any set X, we denote by $\mathcal{P}_{\text{fin}}(X)$ the set of finite subsets of X. A *relation* on finite sets of X is a subset $\mathscr{R} \subseteq \mathcal{P}_{\text{fin}}(X)$, with the intuition that a finite subset A of elements of X are $\mathscr{R}$-related if $A \in \mathscr{R}$.

If $(E, \leqslant)$ is a partially ordered set we put $\lfloor x \rfloor = \{y \in E \mid y \leqslant x\}$ for any element $x \in E$, and more generally $\lfloor A \rfloor = \bigcup_{x \in A} \lfloor x \rfloor$ for $A \subseteq E$. We say that a subset $U \subseteq E$ is *$\leqslant$-left closed* if $x \in U \Rightarrow \lfloor x \rfloor \subseteq U$, or equivalently if $\lfloor U \rfloor = U$.

Let $(E, \leqslant, \nearrow)$ be a triple such that $(E, \leqslant)$ is a partially ordered set and $\nearrow$ is a binary relation on E. A relation $\mathscr{R}$ on finite sets of E is said to be a *conflict* relation for $(E, \leqslant, \nearrow)$ if:

1. *($\mathscr{R}$ is $\leqslant$-inherited)*:
$$\forall A \in \mathcal{P}_{\text{fin}}(E)\ \forall x, y \in E \quad \big(A \cup \{x\} \in \mathscr{R}\big) \wedge (x \leqslant y) \quad \Rightarrow \quad A \cup \{y\} \in \mathscr{R}.$$
2. *($\mathscr{R}$ contains the $\nearrow$-cycles)* For any integer $n \geqslant 1$ and for any elements $x_1, \dots, x_n \in E$:
$$x_1 \nearrow x_2 \nearrow \cdots \nearrow x_n \nearrow x_1 \quad \Rightarrow \quad \{x_1, \dots, x_n\} \in \mathscr{R}.$$

If $(\mathscr{R}_i)_{i \in I}$ is any nonempty family of conflict relations, then $\bigcap_{i \in I} \mathscr{R}_i$ is obviously a conflict relation. Since $\mathcal{P}_{\text{fin}}(E)$ is itself a conflict relation, it follows that there exists a smallest conflict relation, that we call *the* conflict relation associated to

$(E, \leqslant, \nearrow)$, denoted by $\sharp$. For a finite set $A \in \mathcal{P}_{\text{fin}}(E)$, we have that $A \in \sharp$ if and only if for some integer $n \geqslant 1$:

$$\exists e_1, \ldots, e_n \in A\ \exists x_1 \in \lfloor e_1 \rfloor, \ldots, \exists x_n \in \lfloor e_n \rfloor \quad x_1 \nearrow x_2 \nearrow \cdots \nearrow x_n \nearrow x_1. \tag{1}$$

Note that $E = \emptyset$ is allowed, and that $\emptyset \notin \sharp$ in all cases.

Asymmetric Event Structures are then defined as follows:

Definition 1.1 *(AES).* (See [2].) Let $G = (E, \leqslant, \nearrow)$ be a triple such that $(E, \leqslant)$ is a partially ordered set and $\nearrow$ is a binary relation on E. We say that G is an *Asymmetric Event Structure* (AES) if it satisfies the following conditions, denoting by $\sharp$ the conflict relation associated with G:

1. For all $e \in E$, $\lfloor e \rfloor$ is finite.
2. For all $e, e' \in E$: $e < e' \Rightarrow e \nearrow e'$, where $e < e'$ means as usual $e \leqslant e'$ and $e \neq e'$.
3. For all $e \in E$, $\nearrow \cap (\lfloor e \rfloor \times \lfloor e \rfloor)$ is acyclic.
4. For any $e, e' \in E$: $\{e, e'\} \in \sharp \Rightarrow e \nearrow e' \nearrow e$.

Elements of E are called *events*, $\leqslant$ is called the *causality relation* and $\nearrow$ is called the *asymmetric conflict* relation. If the context makes the causality and the asymmetric conflict relations clear, we will identify G and the set E.

In Definition 1.1 the relations $\leqslant$ and $\nearrow$ have the following intuitive meanings. On the one hand, $e \leqslant e'$ means that e is a *mandatory* cause of e': event e' must be preceded by e in any computation involving e'. Condition 1 is therefore natural, and is a copy of the equivalent condition for PES. On the other hand, $e \nearrow e'$ means that e is a *possible* cause of e', the precise meaning of which will be clarified when considering configurations and their order below. But in the meantime, we observe that Condition 2 is natural: mandatory causes are some particular cases of possible causes. A cycle of the form $e_1 \nearrow \cdots \nearrow e_n \nearrow e_1$ with all $e_i \in \lfloor e \rfloor$ would imply that e_1 should precede itself, which prevents e from ever being reached. This explains Condition 3. Finally, Condition 4 is a technical condition: if G satisfies Conditions 1–3, one can always complete $\nearrow$ while preserving $\leqslant$ and $\sharp$ to reach Condition 4 (see [2, §2] for details). It will be justified when considering configurations of the AES.

Observe that relation $\nearrow$ is not reflexive, and actually the relation $e \nearrow e$ never holds, otherwise there would be singletons $\{e\} \in \sharp$, contradicting Condition 3. Furthermore, note that $\nearrow$ is *not* assumed to be transitive; the presence of cycles together with the impossibility of having $e \nearrow e$ is anyway an obstruction to transitivity. Observe also that one should not think of asymmetric conflict as a "special case" of conflict as in a PES; since two events in asymmetric conflict might very well be compatible in an AES; indeed, one is supposed to be a possible cause of the other.

As in PES, the notion of computational process is captured by *configurations*, defined as follows (since we consider finite AES, the definition is slightly simpler than in [2]).

Definition 1.2 *(Configurations).* (See [2].) Let $(E, \leqslant, \nearrow)$ be a finite AES. A set of events $A \subseteq E$ is called a *configuration of* E if:

1. The set A contains all the mandatory causes of all its events; formally: $\forall e \in A\ \forall e' \in E\ e' \leqslant e \Rightarrow e' \in A$;
2. $\nearrow \cap (A \times A)$ is acyclic.

We denote by $\mathsf{Conf}(E)$ the set of configurations of E.

It is worth to observe that, in presence of Condition 1, Condition 2 is equivalent to saying that no finite subset of A belongs to the conflict relation $\sharp$, meeting the usual intuition from PES that configurations are conflict free and $\leqslant$-left closed subsets of events.

Note also that $\mathsf{Conf}(E) \neq \emptyset$, since $\emptyset \in \mathsf{Conf}(E)$ even if $E = \emptyset$, and that $\{\emptyset\} \subsetneq \mathsf{Conf}(E)$ as soon as $E \neq \emptyset$. Indeed, pick any minimal event in $(E, \leqslant)$ if $E \neq \emptyset$, then $\{e\} \in \mathsf{Conf}(E)$.

Finally, Condition 4 in Definition 1.1 can now be explained as follows. If e and e' are two events such that $e \nearrow e'$ and $\neg(e' \nearrow e)$ both hold, then it intuitively means that e is a possible cause of e'; one would then expect that they belong to some configuration C, representing a computation where both events occur. And indeed, putting $C = \lfloor e \rfloor \cup \lfloor e' \rfloor$, then C is $\leqslant$-left closed and conflict free, otherwise one would have $\{e, e'\} \in \sharp$ and thus $e \nearrow e' \nearrow e$ according to Condition 4, contrary to the assumption $\neg(e' \nearrow e)$. Hence Condition 4 reinforces the interpretation of $(e \nearrow e') \wedge \neg(e' \nearrow e)$ as e being a possible cause of e'.

In PES, the order on configurations is given by the mere set theoretic inclusion of subsets. This order however does not capture for AES the distinction between mandatory causes ($\leqslant$) and possible causes ($\nearrow$) of events. Indeed, the fact that an event e may have some possible, non-mandatory causes, implies that e may have *different* histories. However, a natural requirement is that its "history cannot change after the event has occurred" [2, §3]. In particular, if a configuration A contains some event e, one cannot accept as an extension of the computation represented by A, a configuration B that would contain a possible cause e' of e, and that would not already be present in A. Whence the following definition for the order $\sqsubseteq_E$ on configurations—the verification that $(\mathsf{Conf}(E), \sqsubseteq_E)$ is indeed a partial order is straightforward.

Definition 1.3 *(Extension of configurations).* (See [2].) The *extension* relation $\sqsubseteq_E$ on configurations of an AES $(E, \leqslant, \nearrow)$ is defined by:

$$A \sqsubseteq_E B \iff \begin{cases} A \subseteq B, & \text{and} \\ \forall e \in A\ \forall e' \in B \quad e' \nearrow e \Rightarrow e' \in A. \end{cases}$$

This definition enlightens the role of the asymmetric conflict relation $e' \nearrow e$. On the one hand, if e' is *not* a mandatory cause of e, then one can find computations of e without e' in general ($\lfloor e \rfloor$ for instance, as long as $e' \notin \lfloor e \rfloor$). But on the other hand, if in some computation, both events e and ed appear, then this is no coincidence, this is really because e has used some resource attached with e'. In particular, a possible cause of e cannot appear after e has occurred. This leads to the following other interpretation of $e' \nearrow e$: event e has a preemption action on its possible causes. Since, once e has fired in a computation represented by a configuration C, all of the possible causes of e not already present in C are definitively ruled out for the extensions of C. This preemption interpretation is specially relevant when AES are constructed as unfolding of contextual nets (see Appendix B).

Two configurations A, B are said to be *compatible*, which is denoted by $A \uparrow B$, if there exists a configuration C such that $A \sqsubseteq_E C$ and $B \sqsubseteq_E C$. The least upper bound (*lub*) of two compatible configurations is given as follows, according to [2, Lemma 3.2]:

$$A \uparrow B \quad \Rightarrow \quad A \vee B = A \cup B. \tag{2}$$

Note however that $A \cup B \in \mathrm{Conf}(E)$ does not imply the compatibility $A \uparrow B$ in general, even if A and B are two configurations. We will use several times the following characterization of compatible configurations.

Lemma 1.4. *Let $A, B \in \mathrm{Conf}(E)$. Then $A \uparrow B$ if and only if:*

$$\forall a \in A\ \forall b \in B \quad (a \nearrow b \Rightarrow a \in B) \wedge (b \nearrow a \Rightarrow b \in A). \tag{3}$$

Proof. *Proof of* (3) $\Rightarrow A \uparrow B$. Let $C = A \cup B$. We first show that C is a configuration. Since C is obviously $\leqslant$-left closed, it suffices to show that C does not contain any $\nearrow$-cycle. Assume for the seek of contradiction that $a_1 \nearrow \cdots \nearrow a_n \nearrow a_1$ are events of C. All a_i do not belong to B since B is a configuration, hence one of them at least, say a_1, belongs to $A \setminus B$. Then it follows from (3), and since $a_1 \nearrow a_2$, that $a_2 \in B \Rightarrow a_1 \in B$, which shows that $a_2 \notin B$, hence $a_2 \in A \setminus B$. We see therefore by induction that all a_i belong to $A \setminus B$, yielding a $\nearrow$-cycle in A and contradicting that A is a configuration. This shows that C is a configuration.

We now check that $A \sqsubseteq_E C$ and $B \sqsubseteq_E C$. Indeed the set theoretic inclusions $A \subseteq C$ and $B \subseteq C$ are obvious. Assume that $a \in A$ and $c \in C$ are such that $c \nearrow a$. Then $c \in B \Rightarrow c \in A$ by (3), and this shows that $A \sqsubseteq_E C$. Similarly, if $b \in B$ and $c \in C$ are such that $c \nearrow b$, then $c \in B$ by (3) and this shows that $B \sqsubseteq_E C$.

Proof of $A \uparrow B \Rightarrow$ (3). If $A \uparrow B$, it follows from Eq. (2) that $A \sqsubseteq_E C$ and $B \sqsubseteq_E C$ with $C = A \cup B$, from which (3) follows. □

2. Different kinds of prefixes for AES

In this section we introduce the notions necessary to the definition of branching cells in Section 3. We first analyze the sources of conflict in an AES, then we introduce two particular classes of $\leqslant$-left closed subsets of an AES and give some of their properties.

2.1. Sources of conflict

Since we target a fine analysis of choice within AES, we need some insight on the conflict relation. In view of generalizing the minimal conflict relation from PES, adapted to *binary* conflicts, we propose the notion of *source* of conflict for AES. We first consider the following *pre-order* on subsets of E:

$$\forall A \in \mathcal{P}(E)\ \forall B \in \mathcal{P}(E) \quad A \precsim B \iff \lfloor A \rfloor \subseteq \lfloor B \rfloor.$$

(Adding the converse relation $\lfloor B \rfloor \subseteq \lfloor A \rfloor$ would lead to the so-called Egli-Milner order.)

Definition 2.1 *(Source of conflict).* A subset X of E is called a source of conflict if:

1. X is a $\nearrow$-cycle; and
2. No strict subset of X is a $\nearrow$-cycle; and
3. If Y is a $\nearrow$-cycle such that $Y \precsim X$, then $X \precsim Y$.

We denote by $\mathscr{S}(E)$ the collection of sources of conflict of E.

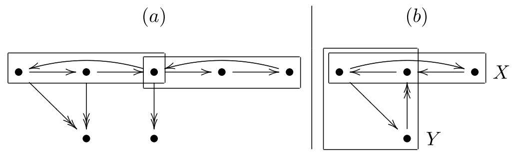

Fig. 1. Examples of sources of conflict. Single arrow arcs depict the $\nearrow$ relation and double arrow arcs depict the immediate successors of relation $<$.

The relevance of this definition comes form the following result.

Lemma 2.2. *For all $A \subseteq E$, we have:*

$$A \in \sharp \quad \iff \quad \exists X \in \mathscr{S}(E) \quad X \precsim A.$$

Proof. The implication ($\Leftarrow$) is obvious. Conversely, let $A \in \sharp$. According to Eq. (1), there exists a $\nearrow$-cycle Y such that $Y \precsim A$. Consider the non-empty set $\mathcal{Y}$ of all $\nearrow$-cycles Z such that $Z \subseteq \lfloor Y \rfloor$, and then a minimal element Z_0 in the finite pre-order $(\mathcal{Y}, \precsim)$; finally pick in Z_0 a minimal $\nearrow$-cycle X. Then Condition 2 is fulfilled since X has been chosen minimal in Z_0, and Condition 3 follows from $\precsim$-minimality of Z_0, and hence of X. $\square$

$\mathscr{S}(E)$ is a relation in the sense of Section 1. It follows from Lemma 2.2 that a $\leqslant$-left closed subset C is a configuration if and only if C does not contain any source of conflict.

Fig. 1 depicts by frames examples of sources of conflict. Note that, if X is a source of conflict, there still can be a $\nearrow$-cycle Y such that $Y \precsim X$ and $X \neq Y$; this is the case in Example (b) of Fig. 1, where X and Y are framed. This is of course due to the lack of antisymmetry of relation $\precsim$.

2.2. SubAES and $\leqslant$-left closed subsets of AES

Since we will consider several subsets of AES, it is worth introducing the notion of subAES.

Definition 2.3 *(SubAES).* An AES $(U, \leqslant_U, \nearrow_U)$ is called a *subAES* of an AES $(E, \leqslant, \nearrow)$ if (a) $U \subseteq E$; (b) $\leqslant_U = \leqslant \cap (U \times U)$; and (c) $\nearrow_U = \nearrow \cap (U \times U)$.

Evidently, any subset U of E can be equipped with the relations $\leqslant_U = \leqslant \cap (U \times U)$ and $\nearrow_U = \nearrow \cap (U \times U)$, making $(U, \leqslant_U, \nearrow_U)$ a subAES of $(E, \leqslant, \nearrow)$. We identify therefore the subAES and the subset U. The conflict relation $\sharp_U$ of AES U is defined as in Section 1 accordingly to $\leqslant_U$ and $\nearrow_U$. Obviously, one always has $A \in \sharp_U \Rightarrow A \in \sharp$ for any $A \in \mathcal{P}_{\mathrm{fin}}(U)$. Therefore, if C is any configuration of E, if we put $C_U = C \cap U$, no finite subset $A \subseteq C_U$ belongs to $\sharp_U$. Since C_U is also $\leqslant_U$-left closed, it is actually a configuration of U, whence a mapping:

$$\phi_U : \mathsf{Conf}(E) \to \mathsf{Conf}(U), \qquad C \mapsto C_U = C \cap U. \tag{4}$$

It is immediate to observe that $\phi_U : (\mathsf{Conf}(E), \sqsubseteq_E) \to (\mathsf{Conf}(U), \sqsubseteq_U)$ is actually a morphism of partial orders, where $\sqsubseteq_U$ is the order on $\mathsf{Conf}(U)$ defined according to Definition 1.3. Observe however that $C_U \sqsubseteq_E C$ does not hold in general, since there could be some events $e \in C_U$ and $e' \in C \setminus C_U$ such that $e' \nearrow e$.

A particular case where $\sharp_U$ coincides exactly with the restriction of $\sharp$ to U is when U is $\leqslant$-left closed, as stated in the following lemma.

Lemma 2.4. *If U is a $\leqslant$-left closed subset of E, then for any finite subset A of U one has $A \in \sharp_U \iff A \in \sharp$.*

Proof. Let $A \in \mathcal{P}_{\mathrm{fin}}(U)$. As already observed, one has $A \in \sharp_U \Rightarrow A \in \sharp$. Assume conversely that $A \in \sharp$. Then we pick for each $a_i \in A$ some $e_i \in \lfloor a \rfloor$ such that $e_1 \nearrow \cdots \nearrow e_n \nearrow e_1$. Since U is left closed w.r.t. $\leqslant$, the e_i belong to U and therefore $e_1 \nearrow_U \cdots \nearrow_U e_n \nearrow_U e_1$ and $e_i \leqslant_U a_i$, which implies that $\{e_1, \ldots, e_n\} \in \sharp_U$ and finally that $A \in \sharp_U$. $\square$

It follows from Lemma 2.4 that if U is a $\leqslant$-left closed subset of E, and if C is a configuration of U, then C is also $\leqslant$-left closed in E and conflict free in E, and thus C is a configuration of E. This defines a mapping:

$$\psi_U : \mathsf{Conf}(U) \to \mathsf{Conf}(E), \qquad C \mapsto \psi_U(C) = C. \tag{5}$$

One easily checks furthermore that $\psi_U : (\mathsf{Conf}(U), \sqsubseteq_U) \to (\mathsf{Conf}(E), \sqsubseteq_E)$ is a morphism of partial orders.

2.3. Introducing S-prefixes

Left closure w.r.t. $\leqslant$ is not enough however to capture the notion of an initial view of the execution of computational processes. Indeed, even if U is $\leqslant$-left closed, one still does not have $C_U \sqsubseteq_E C$ for $C \in \mathsf{Conf}(E)$ in general since, as we have already observed, there could be some events $e \in C_U$ and $e' \in C \setminus C_U$ such that $e' \nearrow e$, preventing the relation $C_U \sqsubseteq_E C$ from holding. Therefore such an event e' should be included in U. This motivates the introduction of the notion of S-prefix (Strong prefix) for AES defined as follows.

Definition 2.5 *(S-prefix).* A S-prefix of an AES E is a subset $U \subseteq E$ such that:

1. $\forall x \in U\ \forall y \in E\ (y \nearrow x) \wedge \neg(x \nearrow y) \Rightarrow y \in U$
2. $\forall X \in \mathscr{S}(E)\ X \cap U \neq \emptyset \Rightarrow X \subseteq U$.

The first condition corresponds to the former explanation: if $x \in U$ and $y \nearrow x$ and $\neg(x \nearrow y)$, then y is thought of as a possible cause of x. If both $x \nearrow y$ and $y \nearrow x$ hold however, then $\{x, y\} \in \sharp$ and the situation is a bit more delicate. The purpose of the second condition is to include an element $y \in U$ in this case only if it belongs to some source of conflict involving x; since there is no need to include events y conflicting with elements of U by inheritance of the conflict.

Let us state in a separate proposition that S-prefixes are in particular $\leqslant$-left closed, so that Lemma 2.4 applies to S-prefixes.

Proposition 2.6. *Any S-prefix U is $\leqslant$-left closed. Therefore $\sharp_U$ is the restriction of $\sharp$ to U and both morphisms ϕ_U and ψ_U are well defined:*

$$\mathsf{Conf}(U) \underset{\phi_U}{\overset{\psi_U}{\rightleftarrows}} \mathsf{Conf}(E)\ .$$

Proof. Let $e \in U$ and $e' \in E$ such that $e' < e$. Then $e' \nearrow e$ thanks to point 2 of Definition 1.1. The relation $e \nearrow e'$ does not hold, otherwise $\lfloor e \rfloor$ would contain the cycle $e \nearrow e' \nearrow e$, contradicting point 3 of Definition 1.1. Therefore $e' \in U$ since U is a S-prefix. $\square$

We show below in Proposition 2.8 that, if U is a S-prefix of E and if C is any configuration of E, then $C_U = \phi_U(C)$ is an approximation of C, *i.e.*, $C_U \sqsubseteq_E C$. The proposition is easily derived from the following lemma, which makes use of the notion of adjunction pair between morphisms of partial orders (the Galois connections of [10, Ch. O-3]).

Lemma 2.7. *If U is a S-prefix of an AES E, then (ψ_U, ϕ_U) is an adjunction pair between morphisms of partial orders. In other words:*

$$\forall A \in \mathsf{Conf}(U)\ \forall B \in \mathsf{Conf}(E) \quad \psi_U(A) \sqsubseteq_E B \iff A \sqsubseteq_U \phi_U(B).$$

Proof. Let $A \in \mathsf{Conf}(U)$ and $B \in \mathsf{Conf}(E)$, and assume that $\psi_U(A) \sqsubseteq_E B$. Then $A \subseteq B$ and since $A \subseteq U$ this implies that $A \subseteq \phi_U(B)$ on the one hand. On the other hand, assume that $e \in A$ and $e' \in \phi_U(B)$ are such that $e' \nearrow_U e$. Then $e \in \psi_U(A)$, $e' \in B$ and $e' \nearrow e$, and therefore $e' \in \psi_U(A)$ since $\psi_U(A) \sqsubseteq_E B$, and so $e' \in A$. This shows that $A \sqsubseteq_U \phi_U(B)$.

Conversely, assume that $A \sqsubseteq_U \phi_U(B)$. Then clearly the set inclusion $\psi_U(A) \subseteq B$ holds. Assume that $e \in \psi_U(A)$ and $e' \in B$ are such that $e' \nearrow e$. We distinguish two cases.

1. *Case* $\neg(e \nearrow e')$. Then, since U is a S-prefix, we derive from Condition 1 in Definition 2.5 that $e' \in U$, and therefore $e' \nearrow_U e$. But now $e \in A$ and $e' \in U \cap B = \phi_U(B)$, and since $A \sqsubseteq_U \phi_U(B)$ this implies that $e' \in A = \psi_U(A)$.
2. *Case* $(e \nearrow e')$. Then $\{e, e'\} \in \sharp$. According to Lemma 2.2, there exists a source of conflict, say X, such that $X \precsim \{e, e'\}$, or put differently: $X \subseteq \lfloor e \rfloor \cup \lfloor e' \rfloor$. Since $\lfloor e' \rfloor$ is conflict free, we must have $X \cap \lfloor e \rfloor \neq \emptyset$, and since $\lfloor e \rfloor \subseteq U$ it follows that $X \cap U \neq \emptyset$ and thus $X \subseteq U$ by Condition 2 of Definition 2.5. Write $X = \{x_1, \ldots, x_n\}$ with $x_1 \nearrow \cdots \nearrow x_n \nearrow x_1$. Without loss of generality, we may assume that $x_1 \in \lfloor e \rfloor$, hence $x_1 \in A$. From $x_n \nearrow_U x_1$, $x_1 \in A$ and $A \sqsubseteq_U \phi_U(B)$ follow the implication
$$x_n \in \phi_U(B) \quad \Rightarrow \quad x_n \in A,$$
from which we derive $x_n \in A$. Proceeding by induction, we obtain in the same way that $x_{n-1} \in A, \ldots, x_2 \in A$, and thus A contains the $\nearrow$-cycle X entirely, which is a contradiction. Hence this case can actually not occur.

Since we have obtained that $e' \in \psi_U(A)$, we have shown that $\psi_U(A) \sqsubseteq_E B$, as expected. $\square$

Proposition 2.8. *Let U be a S-prefix of an AES E. Then $A \cap U \sqsubseteq_E A$ for any configuration $A \in \mathsf{Conf}(E)$.*

Proof. Since (ψ_U, ϕ_U) is an adjunction pair according to Lemma 2.7, the following inequality hods: $\psi_U \circ \phi_U \sqsubseteq_E \mathrm{Id}_{\mathsf{Conf}(E)}$. Applying it to any $A \in \mathsf{Conf}(E)$ yields the desired result since $\psi_U \circ \phi_U(A) = A \cap U$. $\square$

Note that the converse inequality for adjunction pairs, which is $\mathrm{Id}_{\mathsf{Conf}(U)} \sqsubseteq_U \phi_U \circ \psi_U$ yields here the trivial inequality $A \sqsubseteq_U A$ for all $A \in \mathsf{Conf}(U)$.

2.4. Introducing CC-prefixes

The decomposition of configurations that we target aims at capturing the elementary choices made during a computational process. If e, e' are two events of some AES related by $e \nearrow e'$, we have already seen that the choice of including e' in a computation entails the choice regarding e as well. Conversely, the choice of including e in a computation also has consequences regarding whether e' should be included in the computation, specifically if e has the status of a possible but non-mandatory cause of e'. However such an event e' may not belong to some S-prefix containing e. Therefore, a S-prefix does not include in general the complete choices surrounding its events, which motivates the introduction of the stronger notion of CC-prefix (Choice-Complete prefix).

Definition 2.9 *(CC-prefix).* A CC-prefix of an AES E is any S-prefix U satisfying the following additional condition:

$$\forall x \in U\ \forall y \in E \quad (x \nearrow y) \wedge \neg(x \leqslant y) \wedge \neg(y \nearrow x) \Rightarrow y \in U.$$

CC-prefixes share several properties with stopping prefixes defined for PES in [8]. They are based on the same idea of gathering all local choices, so that the evolution of the computational process after the local execution in the CC-prefix is entirely independent of the rest of the CC-prefix. This intuition is captured by Proposition 2.11 below which analyzes the action of ϕ_U on maximal configurations of E, for U a CC-prefix. It is shown on an example in Section 4.3 that this smooth property of CC-prefixes on maximal configurations does not hold in general for S-prefixes, justifying *a posteriori* the introduction of CC-prefixes.

Lemma 2.10. *Let U be a CC-prefix of an AES E. Let $C \in \mathsf{Conf}(E)$ and let $Z \in \mathsf{Conf}(U)$. Assume that $Z \uparrow_U \phi_U(C)$, where $\uparrow_U$ denotes the compatibility relation between configurations of U. Then $\psi_U(Z) \uparrow C$.*

Proof. Considering Z and C as in the statement, we first show the following claim: *$Z \cup C$ is a configuration*. Since $Z \cup C$ is clearly $\leqslant$-left closed, it suffices to show that $Z \cup C$ does not contain any source of conflict. For the seek of contradiction, assume that $X \subseteq Z \cup C$ is a source of conflict. Then $X \cap Z \neq \emptyset$, otherwise $X \subseteq C$, contradicting that C is a configuration. Since U is in particular a S-prefix, and since X is a source of conflict, this implies that $X \subseteq U$. Write $X = \{x_1, \ldots, x_n\}$ with $x_1 \nearrow \cdots \nearrow x_n \nearrow x_1$. We assume without loss of generality that $x_1 \in Z$. The compatibility relation $Z \uparrow_U \phi_U(C)$ yields the following implication:

$$x_n \in C \quad \Rightarrow \quad x_n \in Z,$$

from which we derive that $x_n \in Z$. Proceeding inductively, we obtain that all x_i belong to Z, and since X is a $\nearrow$-cycle this contradicts that Z is a configuration. We have thus shown the above claim.

Now to show $\psi_U(Z) \uparrow C$, we prove the two implications stated in Lemma 1.4. For this, pick $a \in \psi_U(Z)$ and $b \in C$, and keep in mind that $a \in U$ in particular.

1. Assume $a \nearrow b$. If $a \leqslant b$ then obviously $a \in C$, hence we may assume without loss of generality that $\neg(a \leqslant b)$ holds. Furthermore $\neg(b \nearrow a)$ holds, otherwise $\{a, b\} \in \sharp$, contradicting the above claim. Since $a \in U$, and since U is a CC-prefix, this implies that $b \in U$ and therefore $b \in \phi_U(C)$. We now have $a \in Z$, $b \in \phi_U(C)$, $a \nearrow_U b$ and $Z \uparrow_U \phi_U(C)$. This implies according to Lemma 1.4 that $a \in \phi_U(C)$ and thus $a \in C$.
2. Assume $b \nearrow a$. Then again $\neg(a \nearrow b)$ holds thanks to the above claim, and therefore $b \in U$ using this time the S-prefix property of U. We now have $a \in Z$, $b \in \phi_U(C)$, $b \nearrow_U a$ and $Z \uparrow_U \phi_U(C)$. This implies according to Lemma 1.4 that $b \in Z$, *i.e.*, $b \in \psi_U(Z)$.

The proof is complete. $\square$

Proposition 2.11. *Let U be a CC-prefix of an AES E. Then for any maximal element W of $(\mathsf{Conf}(E), \sqsubseteq_E)$, $\phi_U(W)$ is a maximal element of $(\mathsf{Conf}(U), \sqsubseteq_U)$.*

Proof. Let $Z \in \mathsf{Conf}(U)$ such that $\phi_U(W) \sqsubseteq_U Z$. Then in particular $Z \uparrow_U \phi_U(W)$ and therefore, applying Lemma 2.10, $\psi_U(Z) \uparrow W$. By maximality of W in $(\mathsf{Conf}(E), \sqsubseteq_E)$, this implies that $\psi_U(Z) \sqsubseteq_E W$. Applying morphism ϕ_U to both sides of this inequality, and taking into account $\phi_U \circ \psi_U = \mathrm{Id}_{\mathsf{Conf}(U)}$ we obtain $Z \sqsubseteq_U \phi_U(W)$ and thus $Z = \phi_U(W)$. This shows that $\phi_U(W)$ is maximal in $(\mathsf{Conf}(U), \sqsubseteq_U)$. $\square$

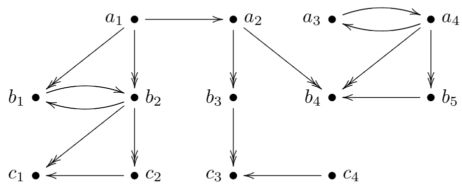

Fig. 2. An example of AES.

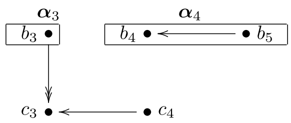

Fig. 3. Illustrating the *future* of $W_U = \{a_2, a_4\}$ in AES from Fig. 2. The branching cells $\boldsymbol{\alpha}_3$ and $\boldsymbol{\alpha}_4$ are depicted by frames.

3. Branching cells for AES

Branching cells aim at decomposing a computational processes through its elementary choices. First introduced for PES in [8,11], we study their counterpart for AES. After having informally described the recursive construction of branching cells on an example, we introduce the notion of future of a configuration in an AES, and then we give the formal definition of branching cells. We then state the main theorem of this section which allows to decompose a maximal configuration through elementary choices, captured by branching cells. We then give small examples illustrating the different notions introduced so far. The proof of the main theorem is postponed to next section.

3.1. Informal description of branching cells on an example

A first notion of choice is related to the conflict relation of an AES: if some events are in (symmetric) conflict, only one of them at most shall belong to a configuration. Consider for example the AES E depicted in Fig. 2. Events $\{a_3, a_4\}$ are in conflict, and therefore a choice between a_3 and a_4 must be made.

A second notion of choice is related to asymmetric conflict. In the same example, consider for instance the events a_1 and a_2 related by $a_1 \nearrow a_2$. This time, the choice is between both a_1 and a_2 on the one hand, and a_2 only on the other hand. Since, if a_2 has been chosen at some stage without a_1, then a_1 cannot belong to any of its later extensions (in the sense of Definition 1.3). Hence the choice has been made inside the subAES $\{a_1, a_2\}$ once and for all. Observe that we do not consider the possibility of a_1 alone, because then nothing will prevent a_2 to fire at some later stage, and hence a_2 will eventually fire.

Furthermore, since the two subAES $\boldsymbol{\alpha}_1 = \{a_1, a_2\}$ and $\boldsymbol{\alpha}_2 = \{a_3, a_4\}$ are not related with one another neither through causality nor through asymmetric conflict, it is obvious on this example that the choices made inside $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are independent from one another. The two subAES $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are called the *initial branching cells* of E. We observe that $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are the two minimal non-empty CC-prefixes of E. According to Proposition 2.11, if W is any maximal configuration of E, then $W_{\boldsymbol{\alpha}_1} = \boldsymbol{\alpha}_1 \cap W$ and $W_{\boldsymbol{\alpha}_2} = \boldsymbol{\alpha}_2 \cap W$ are two maximal configurations of $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ respectively, that correspond to the choices made inside $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$. Therefore $W_{\boldsymbol{\alpha}_1}$ ranges over $\{\{a_2\}, \{a_1, a_2\}\}$ and $W_{\boldsymbol{\alpha}_2}$ ranges over $\{\{a_3\}, \{a_4\}\}$.

Continuing with this example, assume for instance that $W_{\boldsymbol{\alpha}_1} = \{a_2\}$ and $W_{\boldsymbol{\alpha}_2} = \{a_4\}$. Put $U = \boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2$. Since U is in particular a S-prefix, we have that $W_U \sqsubseteq_E W$ (Proposition 2.8). Therefore the events of $W \setminus W_U$ range over the events of E that belong to some continuation of W_U. These events form a subAES that we call the *future* AES of W_U. We depict it in Fig. 3, by removing from E events either in W_U or not compatible with W_U. It is important to notice that, since W_U is maximal, whatever W will eventually be, we are sure that no new event of W will appear in U; this expresses that the choices made in U have been made once and for all, and there is no possible going back. It implies that events of U are automatically ruled out from the future of W_U. This property of CC-prefixes is essential in our construction.

In the future of W_U, we find again two initial branching cells, depicted by frames in Fig. 3 and given by $\boldsymbol{\alpha}_3 = \{b_3\}$ and $\boldsymbol{\alpha}_4 = \{b_4, b_5\}$. The possible choices correspond to the maximal configurations of subAES $\boldsymbol{\alpha}_3$ and $\boldsymbol{\alpha}_4$, which respectively range over $\{\{b_3\}\}$ (there is actually no choice in $\boldsymbol{\alpha}_3$) and over $\{\{b_4\}, \{b_4, b_5\}\}$. Finally, whatever choice is made in $\boldsymbol{\alpha}_4$, the future of the obtained configuration is the following AES, that coincides with its only branching cell $\boldsymbol{\alpha}_5$: $c_3\bullet \longleftarrow \bullet c_4$. The maximal configurations of E that we have described by this way are the four maximal configurations that contain the events a_2, a_4 and b_3. They also contain either b_4 or b_4 and b_5 on the one hand, and either c_3 or c_3 and c_4 on the other hand. It is part of the following theory that every maximal configuration can be described by its decomposition through branching cells.

3.2. Initial branching cells

Assume that $E \neq \emptyset$. It is obvious that CC-prefixes of E are stable by intersection, and that E itself is a non-empty CC-prefix. In other words, CC-prefixes of an AES E form a non-empty semi-lattice, which is finite since E is assumed to be finite. As a consequence, every non-empty CC-prefix of E contains at least a minimal non-empty CC-prefix; whence the following definition.

Definition 3.1 *(Initial branching cells).* The initial branching cells of a non-empty AES E are the minimal non-empty CC-prefixes of E. The empty AES has $\emptyset$ as unique branching cell.

Note that any non-empty AES always has initial branching cells, which are non-empty by definition. In the above example depicted in Fig. 2, the initial branching cells are $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$.

3.3. Future of a configuration

Initial branching cells capture the initial choices made by a process. In order to capture the choices made afterward, we need to formally define the future of a configuration and to study its properties.

Definition 3.2 *(Future of a configuration).* If C is a configuration of an AES E, the future of C is the subAES of E defined by:

$$E^C = \{e \in E \mid \lfloor e \rfloor \uparrow C\} \setminus C.$$

Events of E^C can be characterized as follows, which is simply a rephrasing of Lemma 1.4.

Lemma 3.3. *Let $C \in \mathsf{Conf}(E)$. Then an event e belongs to E^C if and only if:*

$$(e \notin C) \wedge \forall (e', c) \in \lfloor e \rfloor \times C \quad (e' \nearrow c \Rightarrow e' \in C) \wedge (c \nearrow e' \Rightarrow c \in \lfloor e \rfloor). \tag{6}$$

Observe that if $e \in E^C$, then $\lfloor e \rfloor \uparrow C$ and therefore any $e' \in \lfloor e \rfloor$ satisfies $\lfloor e' \rfloor \uparrow C$. Therefore, we have:

$$\forall e \in E^C \ \forall e' \in \lfloor e \rfloor \quad e' \notin C \Rightarrow e' \in E^C. \tag{7}$$

If $C' \in \mathsf{Conf}(E^C)$, we use the special notation $\uplus$ for the *concatenation* of C and C', simply defined by:

$$\forall C' \in \mathsf{Conf}(E^C) \quad C \uplus C' = C \cup C'.$$

Note that futures of configurations "compose" in the following sense:

$$\forall C \in \mathsf{Conf}(E) \ \forall C' \in \mathsf{Conf}(E^C) \quad E^{C \uplus C'} = (E^C)^{C'}. \tag{8}$$

We also consider the following sub-partial orders of $(\mathsf{Conf}(E), \sqsubseteq_E)$:

$$\forall C \in \mathsf{Conf}(E) \quad \mathsf{Conf}(E)^C = \{C' \in \mathsf{Conf}(E) \mid C \sqsubseteq_E C'\}.$$

Point 2 in the following proposition shows that $\mathsf{Conf}(E^C)$ and $\mathsf{Conf}(E)^C$ are two isomorphic partial orders. In other words, the extensions of a configuration C identify with the configurations of the future of C. Note also that point 1 in the proposition is not a consequence of previous Lemma 2.4, since E^C is not a $\leqslant$-left closed subset of E in general; and a similar result concerning the sources of conflict in E^C will be given in Section 5.3, Lemma 5.2.

Proposition 3.4. *Let C be a configuration of an AES E.*

1. *The conflict in E^C is the restriction of $\sharp$ to E^C: for $A \subseteq E^C$, we have $A \in \sharp_{E^C} \iff A \in \sharp$.*
2. *The formula $\theta^C(C') = C \uplus C'$ defines a mapping*

$$\theta^C : \mathsf{Conf}(E^C) \to \mathsf{Conf}(E)^C,$$

which is an isomorphism of partial orders. In particular, we have:

$$\forall C \in \mathsf{Conf}(E) \ \forall C' \in \mathsf{Conf}(E^C) \quad C \sqsubseteq_E C \uplus C'. \tag{9}$$

Proof. 1. As for any subAES, we already have that $A \in \sharp_{E^C} \Rightarrow A \in \sharp$. Conversely, assume that $A = \{a_1, \ldots, a_n\}$ satisfies $A \subseteq E^C$ and $A \in \sharp$. Pick for each a_i some event $a'_i \in \lfloor a_i \rfloor$ such that $a'_1 \nearrow \cdots \nearrow a'_n \nearrow a'_1$. We show that $a'_i \in E^C$ for all i. Seeking a contradiction, assume for instance $a'_1 \notin E^C$. Then, using the property (7) observed above, since $a'_1 \uparrow C$ this implies that $a'_1 \in C$. We also have $\lfloor a'_n \rfloor \uparrow C$ since $\lfloor a_n \rfloor \uparrow C$ and $a'_n \nearrow a'_1$ together with $a'_1 \in C$: therefore $a'_n \in C$ (Lemma 1.4). Proceeding inductively, it follows that $a'_i \in C$ for all $i = 1, \ldots, n$, and this contradicts that C is a configuration. This implies that $A \in \sharp_{E^C}$, as expected.

2. We show that $C \uplus C' \in \mathsf{Conf}(E)$ for $C \in \mathsf{Conf}(E)$ and $C' \in \mathsf{Conf}(E^C)$. For this, we first show that $C \uplus C'$ is $\leqslant$-left closed. Let $e \in C \uplus C'$ and let $e' \in E$ such that $e' \leqslant e$. If $e \in C$, obviously then $e' \in C \subseteq C \uplus C'$. Otherwise we have $e \in C'$. Observe that $\lfloor e' \rfloor \uparrow C$ and therefore we either have $e' \in C$ or $e' \in E^C$. If $e' \in C$ then $e' \in C \uplus C'$ and we are done. And if $e' \in E^C$ then $e' \in C'$ since $C' \in \mathsf{Conf}(E^C)$, and we are done. This shows that $C \uplus C'$ is $\leqslant$-left closed.

To see that $C \uplus C' \in \mathsf{Conf}(E)$ it now suffices to show that $C \uplus C'$ does not contain a $\nearrow$-cycle $a_1 \nearrow \cdots \nearrow a_n \nearrow a_1$. Assume for the seek of contradiction that it does, and put $A = \{a_1, \ldots, a_n\}$. Since C is conflict free, we do not have $A \subseteq C$. All elements of A do not belong to C' either, otherwise by the first point of the proposition, C' would not be conflict free for $\#_{E^C}$. Therefore there are some elements $a, b \in A$ with $a \in C$, $b \in C'$ and $b \nearrow a$. But then $b \in E^C$ and therefore $b \in C$ by Lemma 3.3, which is a contradiction. Hence $C \uplus C' \in \mathsf{Conf}(E)$.

It remains to show that $C \sqsubseteq_E C \uplus C'$. For this, let $e \in C$ and $e' \in C \uplus C'$ such that $e' \nearrow e$. Assume that $e' \notin C$. Then $e' \in C' \subseteq E^C$, and since $e' \nearrow e$ with $e \in C$ this implies that $e' \in C$ according to Lemma 3.3, which is a contradiction. Hence θ^C is well defined $\mathsf{Conf}(E^C) \to \mathsf{Conf}(E)^C$.

It is easy to check that θ^C is a morphism of partial orders, and that $(\theta^C)^{-1}$, given by $(\theta^C)^{-1}(D) = D \setminus C$ is also a morphism of partial orders, completing the proof. □

3.4. Branching cells

Initial branching cells have been defined above as the minimal non-empty CC-prefixes of an AES. In order to define branching cells in general, we consider other CC-prefixes and their associated maximal configurations.

Definition 3.5 *(CC-configurations).* We denote by $\mathsf{CC\text{-}Conf}(E)$ the class of CC-config*urations* of an AES E, which is defined as the smallest class of configurations such that:

1. $\emptyset \in \mathsf{CC\text{-}Conf}(E)$; and
2. for every $C \in \mathsf{CC\text{-}Conf}(E)$, for every CC-prefix $U \subseteq E^C$, and for every maximal configuration C' of U: $C \uplus C' \in \mathsf{CC\text{-}Conf}(E)$.

In other words, CC-configurations are obtained by recursively concatenating maximal configurations of CC-prefixes, where each CC-prefix is chosen in the future of the configuration already constructed. Hence any CC-configuration C is obtained as the last element in an increasing sequence of configurations $\emptyset = C_0 \sqsubseteq_E C_1 \sqsubseteq_E \cdots \sqsubseteq_E C_n$, where $C_{j+1} = C_j \uplus W_{j+1}$, and W_{j+1} is a maximal configuration of some CC-prefix U_{j+1} of E^{C_j}.

Branching cells are then defined as follows.

Definition 3.6 *(Branching cells).* Let E be an AES.

1. The branching cells enabled at C, for $C \in \mathsf{CC\text{-}Conf}(E)$, are the initial branching cells of E^C.
2. The branching cells of E are the collection of all branching cells enabled at C, for C ranging over $\mathsf{CC\text{-}Conf}(E)$.
3. **Notation:** we reserve the symbols $\boldsymbol{\alpha}, \boldsymbol{\beta}$ to denote branching cells, and we write $C \vdash_E \boldsymbol{\alpha}$ to denote that $\boldsymbol{\alpha}$ is a branching cell of E enabled at $C \in \mathsf{CC\text{-}Conf}(E)$.

By definition, each branching cell is thus an AES. Note that the *initial* branching cells of E are the branching cells enabled at $\emptyset$ since $E^\emptyset = E$. If $C \in \mathsf{CC\text{-}Conf}(E)$, we have by definition:

$$C \vdash_E \boldsymbol{\alpha} \iff \emptyset \vdash_{E^C} \boldsymbol{\alpha}.$$

This generalizes using Eq. (8) to the following:

$$\forall C \in \mathsf{CC\text{-}Conf}(E)\ \forall C' \in \mathsf{CC\text{-}Conf}\big(E^C\big) \quad C \uplus C' \vdash_E \boldsymbol{\alpha} \iff C' \vdash_{E^C} \boldsymbol{\alpha}.$$

This property implies that any branching cell of E^C, for $C \in \mathsf{CC\text{-}Conf}(E)$, is also a branching cell of E. Finally, we will see in Lemma 5.1 that $E^W = \emptyset$ if W is a maximal configuration of E, and therefore $W \vdash_E \emptyset$, hence $\emptyset$ is always a branching cell of any AES.

3.5. Covering of a maximal configuration by its branching cells

Before reviewing some examples illustrating the properties of branching cells, we state the main theorem about branching cells. It will be convenient to adopt a special notation for the set of maximal configurations of an AES.

Notation. If E is an AES, we denote by $\overline{\mathsf{Conf}(E)}$ the set of maximal configurations of E, that is to say, the set of maximal elements of $(\mathsf{Conf}(E), \sqsubseteq_E)$.

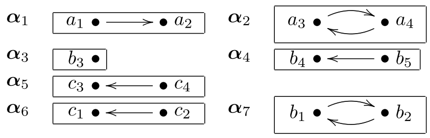

Fig. 4. All non-empty branching cells of the AES depicted in Fig. 2.

It follows in particular from Definition 3.5 that:

$$\forall C \in \text{CC-Conf}(E)\ \forall \boldsymbol{\alpha}\ \forall W \in \overline{\text{Conf}(\boldsymbol{\alpha})} \quad C \vdash_E \boldsymbol{\alpha} \Rightarrow C \uplus W \in \text{CC-Conf}(E). \tag{10}$$

Theorem 3.7. *Let E be an AES. For each $W \in \overline{\text{Conf}(E)}$, there is an integer $n \geqslant 0$ and a sequence of pairwise distinct and non-empty branching cells $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n)$ of E, and for each $\boldsymbol{\alpha}_i$ some $W_i \in \overline{\text{Conf}(\boldsymbol{\alpha}_i)}$ such that, if $(C_i)_{0 \leqslant i \leqslant n}$ are the configurations of E defined by:*

$$C_0 = \emptyset, \qquad C_{i+1} = C_i \uplus W_{i+1} \quad (\textit{for } i = 0, \dots, n-1), \tag{11}$$

we have:

$$\forall i = 0, \dots, n-1 \quad C_i \vdash_E \boldsymbol{\alpha}_{i+1}, \quad \textit{and} \quad \emptyset \sqsubseteq_E C_1 \sqsubseteq_E \cdots \sqsubseteq_E C_n = W.$$

The integer n is > 0 if and only if $E \neq \emptyset$. Furthermore, in any such a decomposition, the branching cells $\boldsymbol{\alpha}_i$ are pairwise disjoint and the W_i's are necessarily given by:

$$\forall i = 1, \dots, n \quad W_i = W \cap \boldsymbol{\alpha}_i.$$

Finally, the sequence of branching cells that appear in the above decomposition is unique up to their order of appearance, and therefore so are the W_i's.

In the above theorem, observe that the configurations C_i are CC-configurations of E. For, proceeding by induction, $C_0 = \emptyset$ is a CC-configuration, and if $C_{i-1} \in \text{CC-Conf}(E)$, then we deduce from $C_{i-1} \vdash_E \boldsymbol{\alpha}_i$, $W_i \in \overline{\text{Conf}(\boldsymbol{\alpha}_i)}$ and $C_i = C_{i-1} \uplus W_i$ that $C_i \in \text{CC-Conf}(E)$ as in (10). In other words, Theorem 3.7 says that maximal configurations can be decomposed through increasing approximations by CC-configurations (the C_i's), whose increments (the W_i's) are elementary at the grain of branching cells. The theorem also states the uniqueness of such a decomposition up to the order of appearance of the branching cells.

A note on the trivial case $E = \emptyset$: then $W = \emptyset$ and $n = 0$ in Theorem 3.7 and the *sequence* of non-empty branching cells is empty; the statement is true.

The statement in the theorem regarding that the *distinct* branching cells of the decomposition are actually *disjoint* is non-trivial: indeed, in general distinct branching cells may not be disjoint, as illustrated below in Section 4.5. Theorem 3.7 states that the branching cells associated to a given maximal configuration, however, are indeed pairwise disjoint.

The proof of Theorem 3.7 is postponed to Section 5, after reviewing some illustrating examples.

4. Illustrating branching cells through examples

In this section we review some properties of branching cells and of CC-configurations through a few examples, which cover the following topics: *determination of branching cells*, *uniqueness of the decomposition of a maximal configuration through branching cells* (up to the order), *need for considering CC-prefixes*, *need for considering CC-configurations*, and finally the *dynamic behavior of branching cells*.

4.1. Determining branching cells

We keep exploring the example depicted above in Fig. 2, now equipped with rigorous definitions for branching cells. For the seek of completeness, we depict all non-empty branching cells in Fig. 4. They are obtained as follows, by feeding point 1 with E as initial data:

1. Data: H an AES. Find and record all initial branching cells $\boldsymbol{\alpha}_i$ of H, and consider the CC-prefix $U = \bigcup_i \boldsymbol{\alpha}_i$.
2. Compute all maximal configurations of U; for each $W \in \overline{\text{Conf}(U)}$, go to Step 1 with data H^W.

4.2. Uniqueness of the decomposition through branching cells

Let $W \in \overline{\text{Conf}(E)}$. The uniqueness property of the decomposition of W through branching cells has as a consequence that the following non-deterministic algorithm is valid to find the decomposition of W; note that it exactly follows the steps of the proof of existence given in Section 5.1.

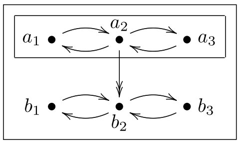

Fig. 5. An AES with two non-empty CC-prefixes which are framed. The configuration $\{a_1, a_3, b_1\}$ is a CC-configuration.

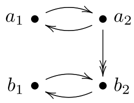

Fig. 6. An AES to illustrate overlapping branching cells.

1. Data: (E, W) where E is an AES and $W \in \overline{\mathsf{Conf}(E)}$.
   (a) Pick any initial branching cell $\boldsymbol{\alpha}$ of E.
   (b) Compute $W_{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \cap W$, the future $E^{W_{\boldsymbol{\alpha}}}$ and the queue $W' = W \setminus W_{\boldsymbol{\alpha}}$.
2. Go to Step 1 with data $(E^{W_{\boldsymbol{\alpha}}}, W')$.

The branching cells encountered are the branching cells of the decomposition of W. Thanks to the uniqueness property stated in Theorem 3.7, although the order of branching cells will change because of the non-determinism of the algorithm, we will always find the same set of branching cells $\boldsymbol{\alpha}$, and also the same local configurations $W_{\boldsymbol{\alpha}}$.

4.3. Need for considering CC-prefixes

We have shown in Proposition 2.11 that CC-prefixes have the following property: $W \in \overline{\mathsf{Conf}(E)} \Rightarrow \phi_U(W) \in \overline{\mathsf{Conf}(U)}$, for U a CC-prefix. This property has proved to be essential in the construction of branching cells.

Although there might be other subAES with this property, we show here on an example that S-prefixes for instance do not have this property in general. Consider the simple AES consisting of two events a and b only related by $a \nearrow b$. Then $U = \{a\}$ is a S-prefix of E, and $W = \{b\}$ is a maximal configuration of E. However $\phi_U(W) = \emptyset$ is not maximal in U.

4.4. Need for considering CC-configurations

Branching cells decompose maximal configurations as a concatenation of configurations maximal in CC-prefixes. Therefore considering the class of CC-configurations, precisely obtained as concatenations of such configurations, is natural. One might wander however if it is really necessary. For instance, maybe for each CC-configuration C, isn't there some CC-prefix U such that $C \in \overline{\mathsf{Conf}(U)}$? The answer is negative, basically because CC-prefixes may split in the future of configurations. More precisely, it is shown in Section 5.3, Lemma 5.4 that $U \cap E^C$ is a CC-prefix of E^C if U is a CC-prefix of E and if $C \in \mathsf{CC\text{-}Conf}(E)$; however $U \cap E^C$ might split in E^C more than U did in E.

We illustrate the previous discussion on an example. Consider the AES E depicted in Fig. 5. It contains only two non-empty CC-prefixes, which are pictured by frames. The configuration $C = \{a_1, a_3, b_1\}$ is thus not a maximal configuration in any CC-prefix of E, since otherwise it should be maximal in E itself, but $C \sqsubseteq_E C \cup \{b_3\}$. However, C is a CC-configuration of E, as shown by the decomposition $C = W_1 \uplus W_2$, with $W_1 = \{a_1, a_3\}$ and $W_2 = \{b_1\}$. Indeed, W_1 is maximal in the initial branching cells $\boldsymbol{\alpha}_1 = \{a_1, a_2, a_3\}$. The future E^{W_1} consists of the two events b_1 and b_2, not related by any relation. Hence $\boldsymbol{\alpha}_2 = \{b_1\}$ is a branching cell of E, initial in E^{W_1}, and $W_2 \in \overline{\mathsf{Conf}(\boldsymbol{\alpha}_2)}$. This shows that $C \in \mathsf{CC\text{-}Conf}(E)$, whereas it is not maximal in any CC-prefix of E.

As this example shows, this property is not related to the asymmetric character of the conflict. And in fact, the very same holds also for PES.

4.5. Branching cells are dynamic

In general, branching cells are not pairwise disjoint. Hence, the fact that the branching cells that tile a maximal configuration are indeed disjoint, as stated in Theorem 3.7, is non-trivial.

This particular feature can be interpreted as a dynamic behavior, caused by concurrency and more specifically by the *confusion* in the sense of [3] found in event structures. Since the overlapping feature of branching cells already holds for PES, it is natural to find it also for AES; indeed any PES can be coded as an AES, and the branching cells for PES correspond to branching cells for AES.

Consider the AES E depicted in Fig. 6. The unique initial branching cell $\boldsymbol{\alpha}$ consists of events a_1 and a_2. Therefore both $C_1 = \{a_1\}$ and $C_2 = \{a_2\}$ are CC-configurations of E. The future AES of C_1 and C_2 respectively are given by $E^{C_1} = \{b_1\}$ and $E^{C_2} = \{b_1, b_2\}$, with unique branching cells respectively $\boldsymbol{\beta}_1 = \{b_1\}$ and $\boldsymbol{\beta}_2 = \{b_1, b_2\}$. Branching cells $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are distinct, yet they overlap.

5. Proof of Theorem 3.7

We decompose the proof of Theorem 3.7 in four statements, of which the third one requires the longest proof:

1. *Existence of the decomposition* (Section 5.1): we prove the existence of branching cells $(\boldsymbol{\alpha}_i)_{1\leqslant i\leqslant n}$ and associated $W_i \in \overline{\mathsf{Conf}(\boldsymbol{\alpha}_i)}$ for $i=1,\dots,n$ satisfying the statement of the theorem.
2. *Characterization of the W_i's* (Section 5.2): in case the branching cells $\boldsymbol{\alpha}_i$ and the configurations $W_i \in \overline{\mathsf{Conf}(\boldsymbol{\alpha}_i)}$ exist as in the statement of the theorem, then we show that $W_i = W \cap \boldsymbol{\alpha}_i$ for $i=1,\dots,n$.
3. *Uniqueness of the decomposition* (Section 5.3): if a sequence of branching cells $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n)$ exists such that the W_i given by $W_i = \boldsymbol{\alpha}_i \cap W$ indeed satisfy the statement of the theorem, then any other sequence of branching cells with the same property can be obtained by switching the order of occurrences of the branching cells in the sequence $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n)$.
4. *The branching cells of the decomposition are disjoint* (Section 5.3): this is proved directly (Lemma 5.9), and used as an auxiliary result in the proof of uniqueness.

5.1. Existence of the decomposition

We begin with a lemma.

Lemma 5.1. *Let $C \in \mathsf{Conf}(E)$.*

1. *Let $W \in \mathsf{Conf}(E^C)$. Then $W \in \overline{\mathsf{Conf}(E^C)} \iff C \uplus W \in \overline{\mathsf{Conf}(E)}$.*
2. *$E^C = \emptyset$ if and only if $C \in \overline{\mathsf{Conf}(E)}$.*

Proof.

1. Indeed, since $\theta^C : \mathsf{Conf}(E^C) \to \mathsf{Conf}(E)^C$ defined by $\theta^C(C') = C \uplus C'$ is an isomorphism of partial orders according to Proposition 3.4.
2. Assume that $E^C = \emptyset$, and let $C' \in \mathsf{Conf}(E)$ such that $C \sqsubseteq_E C'$. Considering the inverse mapping of θ^C, we have that $C' \setminus C$ is a configuration of E^C. But $E^C = \emptyset$, hence $C' \setminus C = \emptyset$ and therefore $C = C'$, showing that C is maximal in $\mathsf{Conf}(E)$.
   Conversely assume that C is maximal in $\mathsf{Conf}(E)$. Let C' be a configuration of E^C. Then $C \uplus C'$ is a configuration of E that satisfies $C \sqsubseteq_E C \uplus C'$ according to Eq. (9) in Proposition 3.4. Since C is maximal, it implies that $C' = \emptyset$. Hence E^C is an AES such that $\mathsf{Conf}(E^C) = \{\emptyset\}$. But $\emptyset$ is the only AES with this property, and thus $E^C = \emptyset$.

The proof of the lemma is complete. □

We now prove the existence part of Theorem 3.7. Let $W \in \overline{\mathsf{Conf}(E)}$. We construct by induction on the integer $i \geqslant 0$ a sequence of branching cells $(\boldsymbol{\alpha}_j)_{1\leqslant j\leqslant i}$ and of subsets $(W_j)_{1\leqslant j\leqslant i}$ of E, with associated CC-configurations $(C_j)_{0\leqslant j\leqslant i}$, and with the following four properties:

1. $C_{j-1} \vdash_E \boldsymbol{\alpha}_j$ and $\boldsymbol{\alpha}_j \neq \emptyset$ for $j=1,\dots,i$;
2. $W_j \in \overline{\mathsf{Conf}(\boldsymbol{\alpha}_j)}$ for $j=1,\dots,i$;
3. $C_0 = \emptyset$, and $C_{j+1} = C_j \uplus W_{j+1}$ for $j=0,\dots,i-1$;
4. $C_0 \sqsubseteq_E \cdots \sqsubseteq_E C_i \sqsubseteq_E W$.

No bound is given *a priori* on the sequence thus constructed; but actually we will see that the construction eventually stops.

We put $C_0 = \emptyset$. If $E = \emptyset$, the construction stops and we put a STOP mark. If not, we pick $\boldsymbol{\alpha}_1$ an initial branching cell of E: $C_0 \vdash_E \boldsymbol{\alpha}_1$. We also put $W_1 = \boldsymbol{\alpha}_1 \cap W$, and we have $W_1 \in \overline{\mathsf{Conf}(\boldsymbol{\alpha}_1)}$ by Proposition 2.11. Putting finally $C_1 = C_0 \uplus W_1 = W_1$, we obviously have $C_0 \sqsubseteq_E C_1$, and we furthermore have $C_1 \sqsubseteq_E W$ by Proposition 2.8. Hence, if the construction has not stopped already, points 1–4 are satisfied for $i=1$.

Assume that $\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_i$ and $W_1,\dots,W_i$ have been constructed satisfying the four items above, together with the associated $C_0,\dots,C_i$ for some integer $i \geqslant 1$. We put $W' = W \setminus C_i$ and $E' = E^{C_i}$. Note that $\theta^{C_i}(W') = W$. If $E' = \emptyset$, we stop the construction and we put a STOP mark.

Otherwise we repeat the construction already described for $i=1$ with W' in place of W and E' in place of E. Hence we pick some initial branching cell $\boldsymbol{\alpha}_{i+1}$ of E', which is non-empty since $E' \neq \emptyset$. We have thus $C_i \vdash_E \boldsymbol{\alpha}_{i+1}$. Configuration W' is maximal in E' thanks to Lemma 5.1, point 1, and therefore by putting $W_{i+1} = \boldsymbol{\alpha}_{i+1} \cap W'$ we have $W_{i+1} \in \overline{\mathsf{Conf}(\boldsymbol{\alpha}_{i+1})}$ thanks to Proposition 2.11. We put $C_{i+1} = C_i \uplus W_{i+1}$, and we have $C_i \sqsubseteq_E C_{i+1}$ thanks to Eq. (9) in Proposition 3.4. We also have $W_{i+1} \sqsubseteq_{E'} W'$ according to Proposition 2.8. Applying the morphism $\theta^{C_i}(\cdot) = C_i \uplus \cdot$ to the later inequality, we obtain $C_{i+1} \sqsubseteq_E W$, completing the construction by induction.

Since each branching cell $\boldsymbol{\alpha}_i$ is non-empty, and since $W_i \in \overline{\mathsf{Conf}(\boldsymbol{\alpha}_i)}$, in particular $W_i \neq \emptyset$ and therefore C_i contains at least i events. But since $C_i \subseteq W$ for all $i \geqslant 0$ until the construction stops, there must exist an integer i where the construction stops. Let n be the integer i preceding the occurrence of the STOP mark. By construction of n, we have $E^{C_n} = \emptyset$. It follows therefore from point 2 of Lemma 5.1 that C_n is a maximal configuration of E. But $C_n \sqsubseteq_E W$ and thus $C_n = W$. This proves the existence of the decomposition as stated in Theorem 3.7. □

Two remarks about the above construction:

1. First, $n = 0$ if and only if $E = \emptyset$.
2. By construction, the branching cells obtained $(\boldsymbol{\alpha}_i)_{1 \leqslant i \leqslant n}$ are pairwise *disjoint*.

5.2. Characterization of the W_i's in Theorem 3.7

Let $W \in \overline{\mathsf{Conf}(E)}$. Assume that for some integer $n \geqslant 0$ there is a sequence $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n)$ of branching cells of E, and a sequence $W_1, \dots, W_n$ with $W_i \in \overline{\mathsf{Conf}(\boldsymbol{\alpha}_i)}$ for all $i = 1, \dots, n$, such that, by putting $C_0 = \emptyset$ and $C_i = C_{i-1} \uplus W_i$ for $i = 1, \dots, n$, we have:

$$C_{i-1} \vdash_E \boldsymbol{\alpha}_i \quad \text{for } i = 1, \dots, n, \quad \text{and} \quad C_i \sqsubseteq_E W \quad \text{for } i = 0, \dots, n.$$

Then we claim that:

$$W_i = \boldsymbol{\alpha}_i \cap W, \quad \text{for } i = 1, \dots, n. \tag{12}$$

Observe that this setting entails the one of Theorem 3.7, and is even a bit weaker since we do not assume here that $C_n = W$.

Proof of the claim (12). The case $n = 0$ is trivial, hence we assume that $n \geqslant 1$. Let $i \in \{1, \dots, n\}$. We have $W_i \in \mathsf{Conf}(\boldsymbol{\alpha}_i)$ and $\mathsf{Conf}(\boldsymbol{\alpha}_i) \subseteq \mathsf{Conf}(E^{C_{i-1}})$ since $\boldsymbol{\alpha}_i$ is assumed to be an initial branching cell of C_{i-1}, and thus in particular $\boldsymbol{\alpha}_i$ is a CC-prefix of $E^{C_{i-1}}$. Since we have $C_i = C_{i-1} \uplus W_i$, it follows from Eq. (9) in Proposition 3.4 that $C_{i-1} \sqsubseteq_E C_i$, which is equivalent to $C_i \in \mathsf{Conf}(E)^{C_{i-1}}$. We also have $C_{i-1} \sqsubseteq_E W$, and thus $W \in \mathsf{Conf}(E)^{C_{i-1}}$.

We may thus apply the morphism of partial orders

$$\left(\theta^{C_{i-1}}\right)^{-1} : \mathsf{Conf}(E)^{C_{i-1}} \to \mathsf{Conf}\left(E^{C_{i-1}}\right),$$

to both sides of the inequality $C_i \sqsubseteq_E W$, and we obtain: $W_i \sqsubseteq_{E^{C_{i-1}}} W \setminus C_{i-1}$. Since $\boldsymbol{\alpha}_i$ is a CC-prefix of $E^{C_{i-1}}$, we may consider the morphism of partial orders $\phi = \phi_{\boldsymbol{\alpha}_i} : \mathsf{Conf}(E^{C_{i-1}}) \to \mathsf{Conf}(\boldsymbol{\alpha}_i)$, and applying it to the later inequality we get: $\phi(W_i) \sqsubseteq_{\boldsymbol{\alpha}_i} \phi(W \setminus C_{i-1})$. But $\phi(W_i) = W_i$ since W_i is a configuration of $\boldsymbol{\alpha}_i$; it is moreover a maximal configuration of $\boldsymbol{\alpha}_i$, hence the inequality turns into the equality:

$$W_i = \boldsymbol{\alpha}_i \cap (W \setminus C_{i-1}). \tag{13}$$

We also observe that $\boldsymbol{\alpha}_i \cap C_{i-1} = \emptyset$ since, by definition, $\boldsymbol{\alpha}_i \subseteq E^{C_{i-1}}$ and $C_{i-1} \cap E^{C_{i-1}} = \emptyset$. Therefore Eq. (13) rewrites as Eq. (12). □

5.3. Uniqueness of branching cells in Theorem 3.7

We begin with a couple of lemmas. The first lemmas (Lemmas 5.2–5.5) are stated in view of the exchange Lemma 5.6, which is the first key for the uniqueness proved at the end of this subsection. The second key is the fact that branching cells tiling a maximal configuration are pairwise disjoint; this is stated in a separate lemma (Lemma 5.9).

Lemma 5.2 *(Sources of conflict in $\leqslant$-closed left subsets and in the future). We denote by $\mathscr{S}(E)$ as in Definition 2.1 the sources of conflict of E.*

1. *Let U be a $\leqslant$-closed subset of E, and let $X \subseteq U$. Then $X \in \mathscr{S}(U) \iff X \in \mathscr{S}(E)$.*
2. *Let $C \in \mathsf{Conf}(E)$, and let $X \subseteq E^C$. Then $X \in \mathscr{S}(E^C) \iff X \in \mathscr{S}(E)$.*

Proof. 1. Obvious.

2. Obviously, if $X \in \mathscr{S}(E)$ then $X \in \mathscr{S}(E^C)$. Conversely, let X be a source of conflict of E^C. Then X is a $\nearrow$-cycle and it does not contain any $\nearrow$-cycle as a strict subset. Hence to show $X \in \mathscr{S}(E)$ it suffices to show that X is $\precsim$-minimal. For this, let Y be a $\nearrow$-cycle such that $Y \precsim X$. For every $y \in Y$ there exists some $x \in X$ such that $y \leqslant x$, and since $\lfloor x \rfloor \uparrow C$ we have $\lfloor y \rfloor \uparrow C$. Write $Y = \{y_1, \dots, y_n\}$ with $y_1 \nearrow \cdots \nearrow y_n \nearrow y_1$. Assume that $Y \cap C \neq \emptyset$, say for instance $y_1 \in Y \cap C$. Then the relation $y_n \nearrow y_1$ together with the compatibility $\lfloor y_n \rfloor \uparrow C$ imply that $y_n \in C$. Proceeding inductively, we obtain by this way that $Y \subseteq C$, contradicting that C is a configuration. Hence $Y \cap C = \emptyset$, and therefore $Y \subseteq E^C$. By $\precsim$-minimality of X in E^C, we have thus $X \precsim Y$, and this shows that $X \in \mathscr{S}(E)$. □

Lemma 5.3 *(Heredity of S-prefixes and of CC-prefixes). Let U be a S-prefix of E and let $V \subseteq U$.*

1. *Then V is a S-prefix of U if and only if V is a S-prefix of E.*
2. *If furthermore U is a CC-prefix of E, then V is a CC-prefix of U if and only if V is a CC-prefix of E.*

Proof. 1. ($\Rightarrow$) Assume that V is a S-prefix of U.

(a) If $e \in V$ and $e' \in E$ are such that $e' \nearrow e$ and $\neg(e \nearrow e')$, then $e \in U$ since U is a S-prefix of E, and thus $e' \nearrow_U e$ and $\neg(e \nearrow_U e')$, from which follows that $e' \in V$ since V is a S-prefix of U.
(b) If $X \in \mathscr{S}(E)$ and $X \cap V \neq \emptyset$, then $X \cap U \neq \emptyset$ in particular, and since U is a S-prefix of E this implies that $X \subseteq U$. But then $X \in \mathscr{S}(U)$ by Lemma 5.2, point 1, and thus $X \subseteq V$ since $X \cap V \neq \emptyset$ and since V is assumed to be a S-prefix of U.

This shows that V is a S-prefix of E.

($\Leftarrow$) Same type of proof.

2. Assume furthermore that U is a CC-prefix of E.

($\Rightarrow$) If V is a CC-prefix of U, then V is a S-prefix of U, and thus of E by point 1 above. Now if $e \in V$ and $e' \in E$ are such that $e \nearrow e'$ and $\neg(e' \nearrow e)$ and $\neg(e \leqslant e')$, then $e' \in U$ since U is a CC-prefix of E. Hence $e \nearrow_U e'$ and $\neg(e' \nearrow_U e)$ and therefore $e' \in V$ since V is a CC-prefix of U. This shows that V is a CC-prefix of E.

($\Leftarrow$) Same type of proof. $\square$

Lemma 5.4 *(Trace of S-prefixes and of CC-prefixes in a future). Let U be a S-prefix of E, let $C \in \mathsf{Conf}(E)$ and let $V = U \cap E^C$.*

1. *Then V is a S-prefix of E^C.*
2. *Assume furthermore that U is a CC-prefix of E. Then V is a CC-prefix of E^C.*

Proof. 1. Referring to Definition 2.5 of S-prefixes, we check the two following points regarding V.

(a) Let $e \in V$ and $e' \in E$ such that $e' \nearrow_{E^C} e$ and $\neg(e \nearrow_{E^C} e')$. Then $e' \nearrow e$ and $\neg(e \nearrow e')$ and therefore $e' \in U$ since U is a S-prefix of E, and thus $e' \in V$.
(b) Let $X \in \mathscr{S}(E^C)$ be such that $X \cap V \neq \emptyset$. Then, according to Lemma 5.2, point 2, X is a source of conflict in E. Therefore $X \subseteq U$ and thus $X \subseteq V$.

This shows that V is a S-prefix of E^C.

2. If U is a CC-prefix of E, then U is in particular a S-prefix of E, and thus V is a S-prefix of E^C according to point 1 above. If $e \in V$ and $e' \in E^C$ are such that $e \nearrow_{E^C} e'$ and $\neg(e' \nearrow_{E^C} e)$ and $\neg(e \leqslant e')$ then also $e \nearrow e'$ and $\neg(e' \nearrow e)$ and therefore $e' \in U$ since U is a CC-prefix of E, and finally $e' \in V$. Hence V is a CC-prefix of E^C. $\square$

We start gathering the fruit of our efforts.

Lemma 5.5 *(CC-prefixes back to/back from the future). Let U and V be two disjoint CC-prefixes of E. Let $C \in \mathsf{Conf}(U)$, and identify C with $\psi_U(C) = C \in \mathsf{Conf}(E)$.*

1. *Then $V \subseteq E^C$, and V is a CC-prefix of E^C.*
2. *If $V' \subseteq V$ is a CC-prefix of E^C then V' is a CC-prefix of E.*

Proof. 1. We first show that $V \subseteq E^C$. Let $e \in V$. Then $e \notin C$ since $C \subseteq U$ and $U \cap V = \emptyset$. Let us show that $\lfloor e \rfloor \uparrow C$. We have $\lfloor e \rfloor \subseteq V$ since V is in particular $\leqslant$-left closed (Proposition 2.6). Since $U \cap V = \emptyset$, it follows that $\phi_U(\lfloor e \rfloor) = \emptyset$, hence $\phi_U(\lfloor e \rfloor) \uparrow_U C$. Therefore, applying Lemma 2.10, we have that $\lfloor e \rfloor \uparrow \psi_U(C)$, which is our claim. Therefore $e \in E^C$, and this shows that $V \subseteq E^C$.

It follows from point 2 of Lemma 5.4 that $V \cap E^C$ is a CC-prefix of E^C, and since $V \cap E^C = V$ we are done.

2. Let $V' \subseteq V$ be a CC-prefix of E^C. Since V is a CC-prefix of E^C according to point 1 above, then V' is a CC-prefix of V according to point 2 of Lemma 5.3. Since V is also a CC-prefix of E by assumption, it implies that V' is a CC-prefix of E, again by point 2 of Lemma 5.3. $\square$

Lemma 5.6 *(Exchange lemma). Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be two distinct initial branching cells of E, and let $W \in \overline{\mathsf{Conf}(\boldsymbol{\alpha})}$. Then $W \vdash_E \boldsymbol{\beta}$.*

Proof. By minimality, distinct initial branching cells are disjoint. Therefore $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two disjoint CC-prefixes of E. Since $W \subseteq \boldsymbol{\alpha}$, it follows from point 1 of Lemma 5.5 that $\boldsymbol{\beta}$ is a non-empty CC-prefix of E^W. Let us show that $\boldsymbol{\beta}$ is minimal among non-empty CC-prefixes of E^W. For this, let $U \neq \emptyset$ be a CC-prefix of E^W such that $U \subseteq \boldsymbol{\beta}$. Then U is also a non-empty

CC-prefix of E according to point 2 of Lemma 5.5. Since $\boldsymbol{\beta}$ is minimal among non-empty CC-prefixes of E, it follows that $U=\boldsymbol{\beta}$, and thus $\boldsymbol{\beta}$ is an initial branching cell of E^W, *i.e.*, $W \vdash_E \boldsymbol{\beta}$. □

Since we are about to manipulate decompositions with the form stated in Theorem 3.7, it is convenient to introduce a specific terminology.

Definition 5.7 *(Adapted sequences of branching cells).* Let $W \in \mathsf{CC\text{-}Conf}(E)$. A sequence of branching cells $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n)$ is said to be adapted to W if, by putting $W_i=\boldsymbol{\alpha}_i \cap W$ for $i=1,\dots,n$ and $C_0=\emptyset$ and $C_{i+1}=C_i \uplus W_{i+1}$ for $i=0,\dots,n-1$, we have:

1. $C_{i-1} \vdash_E \boldsymbol{\alpha}_i$ and $W_i \in \overline{\mathsf{Conf}(\boldsymbol{\alpha}_i)}$ for $i=1,\dots,n$;
2. $C_n=W$.

Note that the above definition applies to CC-configurations of E, which contain in particular the maximal configurations of E. Although the result we target deals with maximal configurations, it could actually be similarly stated for CC-configurations, and in the remaining of its proof it will be helpful to have a little more flexibility than we would have by restricting ourselves to maximal configurations only.

Analogously to the composition of futures seen in Eq. (8), adapted sequences of branching cells can be concatenated as follows.

Lemma 5.8. *Let $W \in \mathsf{CC\text{-}Conf}(E)$ and $W' \in \mathsf{CC\text{-}Conf}(E^W)$. Assume that $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n)$ is a sequence of branching cells of E adapted to W, and that $(\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_{n'})$ is a sequence of branching cells of E^W adapted to W'. Then the concatenation $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n,\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_{n'})$ is a sequence of branching cells of E adapted to $W \uplus W'$.*

Proof. Obvious. □

We state in a separate lemma that branching cells of an adapted sequence are necessarily disjoint.

Lemma 5.9. *If $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n)$ is an adapted sequence of branching cells for $W \in \overline{\mathsf{Conf}(E)}$, then $i \neq j \Rightarrow \boldsymbol{\alpha}_i \cap \boldsymbol{\alpha}_j=\emptyset$ for $i,j \in \{1,\dots,n\}$.*

Proof. Assuming that the sequence is non-empty, we first prove that $\boldsymbol{\alpha}_1 \cap \boldsymbol{\alpha}_j=\emptyset$ for all $j=2,\dots,n$. Let $j \in \{2,\dots,n\}$. We have $\boldsymbol{\alpha}_j \subseteq E^{C_{j-1}}$, and $E^{C_{j-1}} \subseteq E^{C_1}$ since $j-1 \geqslant 1$. Therefore $\boldsymbol{\alpha}_1 \cap \boldsymbol{\alpha}_j \subseteq \boldsymbol{\alpha}_1 \cap E^{C_1}$. But $C_1=W_1$ is a maximal configuration of $\boldsymbol{\alpha}_1$, and this implies that $\boldsymbol{\alpha}_1 \cap E^{C_1}=\emptyset$, and thus $\boldsymbol{\alpha}_1 \cap \boldsymbol{\alpha}_j=\emptyset$.

For the general case, we prove that for all $i=1,\dots,n$ we have $\boldsymbol{\alpha}_i \cap \boldsymbol{\alpha}_j=\emptyset$ for $j>i$, which implies the statement of the lemma. Indeed, simply apply the previous case to $W'=\theta^{C_{i-1}}(W)$. □

Lemma 5.10. *Let $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n)$ be a sequence of branching cells of E adapted to $W \in \overline{\mathsf{Conf}(E)}$. Assume that $E \neq \emptyset$ and let $\boldsymbol{\alpha}$ be an initial branching cell of E. Then there is an integer $i \in \{1,\dots,n\}$ such that $\boldsymbol{\alpha}_i=\boldsymbol{\alpha}$.*

Proof. Reasoning by contradiction, assume that $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}_i$ for all $i=1,\dots,n$. We consider the sequence of CC-configurations $C_0,\dots,C_n$ given as in Definition 5.7. Then we claim that $C_i \vdash_E \boldsymbol{\alpha}$ for $i=0,\dots,n$. Proceeding by induction, we have that $C_0=\emptyset \vdash_E \boldsymbol{\alpha}$ since $\boldsymbol{\alpha}$ is an initial branching cell of E. Assume that $C_i \vdash_E \boldsymbol{\alpha}$ for some integer $0 \leqslant i<n$. We also have $C_i \vdash_E \boldsymbol{\alpha}_{i+1}$. Since $\boldsymbol{\alpha}_{i+1} \neq \boldsymbol{\alpha}$ by assumption, $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_{i+1}$ are two disjoint initial branching cells of E^{C_i}. Applying the exchange Lemma 5.6 in AES E^{C_i} and with $W_{i+1} \in \overline{\mathsf{Conf}(\boldsymbol{\alpha}_{i+1})}$ we have thus that $W_{i+1} \vdash_{E^{C_i}} \boldsymbol{\alpha}$, hence $\boldsymbol{\alpha}$ is an initial branching cell of $(E^{C_i})^{W_{i+1}}$. As already observed in Eq. (8), we have

$$\left(E^{C_i}\right)^{W_{i+1}}=E^{C_i \uplus W_{i+1}}=E^{C_{i+1}}.$$

Hence $C_{i+1} \vdash_E \boldsymbol{\alpha}$, and the induction is complete.

We now derive a contradiction by considering C_n. Indeed, we have $C_n \vdash_E \boldsymbol{\alpha}$ thanks to the previous induction, but $C_n=W$ is maximal, and therefore $\boldsymbol{\alpha}=\emptyset$ since $E^C=\emptyset$ by point 2 of Lemma 5.1, a contradiction. □

Lemma 5.11. *Assume that $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n)$ is a sequence of branching cells adapted to $W \in \overline{\mathsf{Conf}(E)}$. Assume that for some integer $i>1$, $\boldsymbol{\alpha}_i$ is an initial branching cell of E. Then the sequence obtained from $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n)$ by switching $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_{i-1}$ is adapted to W.*

Proof. Thanks to Lemma 5.9, we know that the branching cells in the sequence $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n)$ are pairwise disjoint.

We first prove the result for the case $n=i=2$. If $(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)$ is a sequence of branching cells adapted to W, the exchange Lemma 5.6 shows that $(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_1)$ is also adapted to W. The lemma applies since $\boldsymbol{\alpha}_1 \cap \boldsymbol{\alpha}_2=\emptyset$.

The case where $i = 2$ and $n > 2$ follows then from the previous case by considering the concatenation of the two adapted sequences of branching cells, $(\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1)$ on the one hand, adapted to $C_2 = W_1 \uplus W_2 = W_2 \uplus W_1$, and $(\boldsymbol{\alpha}_3, \ldots, \boldsymbol{\alpha}_n)$ on the other hand, branching cells of E^{C_2} that form a sequence adapted to $(\theta^{C_2})^{-1}(W) = W_3 \uplus \cdots \uplus W_n$.

Finally, for the case $i > 2$, we put $W' = (\theta^{C_{i-2}})^{-1}(W)$. Then $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{i-2})$ is a sequence of branching cells of E adapted to C_{i-2}, while $(\boldsymbol{\alpha}_{i-1}, \ldots, \boldsymbol{\alpha}_n)$ is a sequence of branching cells of $E^{C_{i-2}}$ adapted to W'. Applying $(i-2)$ times the exchange Lemma 5.6, which is legitimate since the $\boldsymbol{\alpha}_j$ are pairwise disjoint, yields that $\boldsymbol{\alpha}_i$ is an initial branching cell of $E^{C_1}, E^{C_2}, \ldots, E^{C_{i-2}}$. Hence the case already seen with $i = 2$ applies to W', and thus $(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_{i-1}, \boldsymbol{\alpha}_{i+1}, \ldots, \boldsymbol{\alpha}_n)$ is a sequence of branching cells of $E^{C_{i-2}}$ adapted to W'. Since $W = C_{i-2} \uplus W'$, we recompose again the two sequences of branching cells with Lemma 5.8 to obtain that

$$(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{i-2}, \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_{i-1}, \boldsymbol{\alpha}_{i+1}, \ldots, \boldsymbol{\alpha}_n)$$

is a sequence of branching cells of E adapted to W. □

We now prove the uniqueness property of branching cells in Theorem 3.7. We prove by induction on the integer $n \geqslant 0$ the following claim: *if W is a maximal configuration of an AES E, and if $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n)$ and $(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_m)$ are two sequences of branching cells of E adapted to W, then $m = n$ and $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n\} = \{\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_m\}$.*

If $n = 0$, then $E = \emptyset$ and therefore $m = 0$ as well and we are done.

Assume that $n = 1$. Then it follows from Lemma 5.10 that $\boldsymbol{\alpha}_1$ is the only initial branching cell of E. Hence $m = 1$ and $\boldsymbol{\beta}_1 = \boldsymbol{\alpha}_1$.

Assume the result is true for some integer $n \geqslant 1$. Then $m \geqslant 1$, necessarily. Consider the initial branching cell $\boldsymbol{\alpha}_1$. Then it follows from Lemma 5.10 that there is some integer $1 \leqslant i \leqslant m$ such that $\boldsymbol{\beta}_i = \boldsymbol{\alpha}_1$. Since $\boldsymbol{\beta}_i$ is then an initial branching cell of E, we apply $(i-1)$ times Lemma 5.11 to obtain that the sequence

$$(\boldsymbol{\beta}_i, \boldsymbol{\beta}_1, \ldots, \widehat{\boldsymbol{\beta}_i}, \ldots, \boldsymbol{\beta}_n) = (\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \ldots, \widehat{\boldsymbol{\beta}_i}, \ldots, \boldsymbol{\beta}_n)$$

is adapted to W, where the symbol $\widehat{\boldsymbol{\beta}_i}$ means that $\boldsymbol{\beta}_i$ is missing. Putting $C_1 = W_1 = W \cap \boldsymbol{\alpha}_1$, and $W' = (\theta^{C_1})^{-1}(W)$, we have thus that $(\boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_n)$ and $(\boldsymbol{\beta}_1, \ldots, \widehat{\boldsymbol{\beta}_i}, \ldots, \boldsymbol{\beta}_m)$ are two sequences of branching cells of E^{C_1} which are both adapted to W'. The induction hypothesis implies that $m-1 = n-1$ and that $\{\boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_n\} = \{\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_m\} \setminus \{\boldsymbol{\beta}_i\}$; whence the equalities $m = n$ and $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n\} = \{\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_m\}$. □

6. Conclusion

In this paper we have introduced branching cells for AES. For this purpose, several tools were needed on the way: different kinds of prefixes for AES, and most notably the Choice-Complete prefixes and the future of a configuration as a subAES of the original one. It is worth noting that adjunction pairs of increasing mappings made an appearance, which revealed a nice mathematical hidden structure proving to be much useful.

Branching cells for AES extend the previously introduced branching cells for PES mainly by dealing with the asymmetric conflict, resulting in several technical differences listed in the introduction. Branching cells provide a decomposition of maximal configurations through elementary choice units, where parallel choices are independent. The development of branching cells was motivated by its applications in the orchestration of web services and on-line QoS optimization, presented in a companion paper.

Appendix A. Extension to locally finite AES

We meet several issues when trying to generalize the previous constructions to infinite AES. However, infinite AES are natural objects; for instance the unfolding of a contextual net is an infinite AES as soon as the net contains a loop (see Appendix B for a short review on contextual nets and their unfoldings). Hence it is worth trying to have some insight on infinite AES.

Although the notion of S-prefix and of CC-prefix have straightforward generalization with the very same definitions, the existence of minimal non-empty CC-prefix is not always guaranteed. The same phenomenon is observed for infinite PES in general as noted in [12], where stopping prefixes play the same role as CC-prefixes for AES. The issue here concerns the possibly infinite concurrency width of the AES. Fortunately, the unfolding of a finite contextual net always has a finite concurrency width, *i.e.*, there is a bound on the number of pairwise concurrent events.

In other words, when considering an AES obtained as the unfolding of a contextual net, the existence of non-empty minimal CC-prefixes, that is to say, of initial branching cells, always holds. The remaining issues are the two following:

1. Branching cells might be infinite.
2. The covering of a maximal configuration through branching cells might not be complete. To entirely cover a maximal configuration by branching cells, one might need to index the covering branching cells by an ordinal still countable of course, yet greater than $\boldsymbol{\omega}$. In other words, performing the covering of $W \in \overline{\mathsf{Conf}(E)}$ through branching cells yields in general a sub-configuration $W' \sqsubseteq_E W$; and one might still needs to complete the covering by additional branching cells

for the queue $W \setminus W'$ of W. The latter operation might need to be repeated several times, although only finitely many times for the unfolding of a contextual net.

Nothing special can be done within branching cells theory about the raw fact that brings the first issue. The second issue is less problematic than it first appears, specifically if one adopts a probabilistic point of view; since then and as shown in [11], the maximal configurations with a problematic behavior, that is, with an incomplete covering by branching cells, have a statistical rare occurrence.

There is however a class of infinite PES, with a counterpart in the category of AES, with a smooth behavior where both difficulties simply vanish, since the possibly pathological behavior is shown to actually not occur. This class is the class of *locally finite* AES.

Definition *(Locally finite AES).* Let E be an AES obtained as the unfolding of a contextual net. Then E is said to be locally finite if for every event $e \in E$, there is a finite CC-prefix of E that contains e.

As explained above, the assumption in Definition 2 that E is the unfolding of a contextual net guarantees the existence of branching cells. The same property then necessarily holds for any future E^C, where C ranges over finite configurations of E. The property of local finiteness has the following consequences (where points 3–4 are consequences of Theorem 3.7 applied in finite subAES of E):

1. Every branching cell is finite.
2. Any maximal configuration $W \in \overline{\mathsf{Conf}(E)}$ is obtained as the increasing countable union

$$W = \bigcup_U \uparrow \phi_U(W),$$

where U ranges over finite CC-prefixes of E. Lemma 2.7 could be a basis for interpreting both $\mathsf{Conf}(E)$ and $\overline{\mathsf{Conf}(E)}$ as limits of projective systems of finite sets, as developed for PES in [12] and more generally for bifinite domains in [13].
3. When considering for $W \in \overline{\mathsf{Conf}(E)}$ and for U, U' two finite CC-prefixes of E the decompositions through branching cells of $\phi_U(W)$ and of $\phi_U(W')$, as stated in Theorem 3.7 applied in finite AES U and U', then these decomposition are coherent with one another. In other words, if $U \subseteq U'$, then the covering of $\phi_U(W)$ by branching cells is a subset of the covering of $\phi_{U'}(W)$.
4. Theorem 3.7 extends by considering possibly infinite sequences of non-empty (and finite) branching cells. For $W \in \overline{\mathsf{Conf}(E)}$, the $\sqsubseteq_E$-increasing sequence $(C_i)_{i\geqslant 0}$ of CC-configurations that appears in the statement of the theorem is possibly infinite, in which case it satisfies:

$$W = \bigcup_{i\geqslant 0} \uparrow C_i.$$

Deciding whether a contextual net has a locally finite unfolding is a topic with no settled answer yet.

Appendix B. Contextual nets, unfoldings and AES

Contextual nets or *nets with read arcs* play for AES the role that safe Petri nets play for prime event structure. In particular, the *unfolding* of a contextual net is defined by means of an AES. The unfolding of a contextual net has the universal property that the partial order semantics attached to the contextual net is retrieved through the poset of configurations of its unfolding. For the convenience of the reader looking for the applications found in companion paper [9], we quickly review the definitions for contextual nets and illustrate the procedure for their unfolding on examples.

A *contextual net* is given by a tuple $N = (P, T, F, R, M_0)$ where P is a set of *places*, T is a set of *transitions*, $F \subseteq (P \times T) \cup (T \times P)$ is the *flow relation*, $R \subseteq P \times T$ is the *read relation*, and $M_0 : P \to \mathbb{N}$ is the *initial marking*. It is required that P and T are two disjoint sets, and that $F \cap R = \emptyset$. For any node $x \in P \cup T$, the *preset* of x is ${}^\bullet x = \{y \mid (y, x) \in F\}$ and the *postset* of x is $x^\bullet = \{y \mid (x, y) \in F\}$. Finally, if $t \in T$ is a transition, the *context* of t is the set of places ${}^\circ t = \{x \mid (x, t) \in R\}$. A *marking* is any mapping $M : P \to \mathbb{N}$. A transition t is *enabled* by marking M if $M(p) > 0$ for every $p \in {}^\bullet t \cup {}^\circ t$. *Firing* transition t if it is enabled yields the new marking M' defined by

$$M'(p) = \begin{cases} M(p), & \text{if } p \notin {}^\bullet t \cup t^\bullet, \\ M(p) - 1, & \text{if } p \in {}^\bullet t, \\ M(p) + 1, & \text{if } p \in t^\bullet. \end{cases}$$

(By convention $M'(p) = M(p)$ if $p \in {}^\bullet t \cap t^\bullet$.) In this case we write $M \xrightarrow{t} M'$. Note that, although it is required that $M(p) > 0$ for $p \in {}^\bullet t$ *and* for $p \in {}^\circ t$ for t to be enabled, the only resources consumed by the firing of t are those of ${}^\bullet t$, not those of ${}^\circ t$. A *firing sequence* from marking M is a finite sequence $t_1, \ldots, t_n$ of transitions such that, for some markings $M_1, \ldots, M_n$,

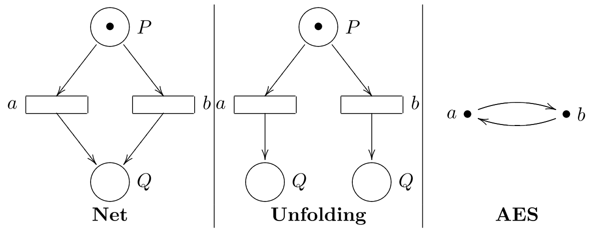

Fig. 7. Illustrating the unfolding of a contextual net: first example (a net, its unfolding and the associated AES). In this example, the read relation is empty, only the flow relation is nonempty. It is depicted by the arcs. Places are depicted by circles and transitions are depicted by rectangles.

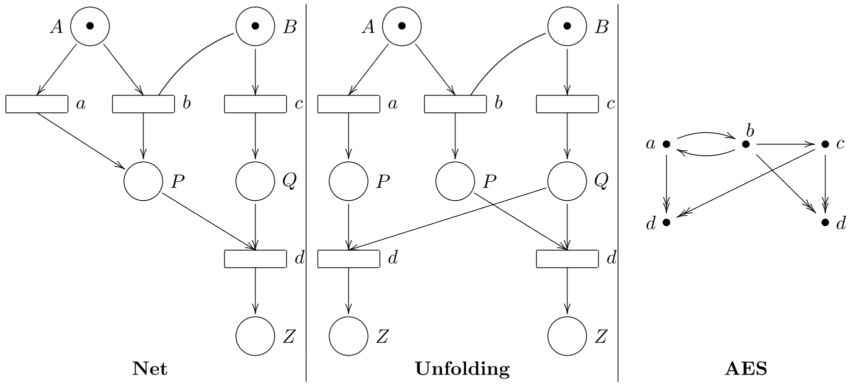

Fig. 8. Illustrating the unfolding of a contextual net: second example (a net, its unfolding and the associated AES). The arrow free arcs depict the read relation R, so that $^{\circ}a = \emptyset$, $^{\circ}b = \{B\}$ and $^{\circ}c = \emptyset$.

the sequence $M \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots \xrightarrow{t_n} M_n$ holds. The markings $M_1, \ldots, M_n$ are then uniquely determined, and we write $M \xrightarrow{t_1 \cdot t_2 \cdots t_n} M_n$. The net is said to be *safe* if, for any firing sequence $t_1, \ldots, t_n$ starting from the initial marking M_0, and for any marking M, whenever $M_0 \xrightarrow{t_1 \cdot t_2 \cdots t_n} M$, we have $M(p) \leqslant 1$ for all $p \in P$. This implies that $M_0(p) \leqslant 1$ for all $p \in P$, since $M_0 \xrightarrow{\emptyset} M_0$.

Let $N = (P, T, F, R, M_0)$ be a contextual net. Define the *causality relation* $\preceq$ as the reflexive and transitive closure of the relation on $P \cup T$ defined by: (a) if $s \in {}^{\bullet}t$ then $s \prec t$; (b) if $t \in s^{\bullet}$, then $s \prec t$; (c) if $t_1^{\bullet} \cap {}^{\circ}t_2 \neq \emptyset$, then $t_1 \prec t_2$. The set of causes $\lfloor x \rfloor$ is defined by $\lfloor x \rfloor = \{y \in P \cup T \mid y \preceq x\}$. Consider the *asymmetric conflict relation* defined for $t \neq t'$ by $t \nearrow t'$ if either (i) $t \prec t'$; or (ii) $^{\circ}t \cap {}^{\bullet}t' \neq \emptyset$; or (iii) $^{\bullet}t \cap {}^{\bullet}t' \neq \emptyset$. Define then the *conflict* relation $\sharp$ associated with $(T, \preceq)$ and $\nearrow$ as in Section 1.

Among contextual nets, the class of *occurrence nets* consists of those nets such that: (i) $\preceq$ is a partial order and $\lfloor x \rfloor$ is finite for every $x \in P \cup T$; and (ii) for each place $p \in P$, $|{}^{\bullet}p| \leqslant 1$; and (iii) $\lfloor t \rfloor$ is conflict free for every $t \in T$; and (iv) $M_0(p) = 1 \iff {}^{\bullet}p = \emptyset$. If $N = (P, T, F, R, M_0)$ is an occurrence net, then $E = (T, \preceq, \nearrow)$, where $\preceq$ and $\nearrow$ are defined as above, is an AES, canonically associated with N.

Consider now a finite net N. Let $\mathcal{F}$ denote the set of firing sequences of N. Put $t_1 \sim t_2$ for two transitions t_1 and t_2 such that (i) $^{\bullet}t_1 \cap {}^{\bullet}t_2 = \emptyset$; and (ii) $^{\bullet}t_1 \cap {}^{\circ}t_2 = \emptyset$; and (iii) $^{\bullet}t_2 \cap {}^{\circ}t_1 = \emptyset$. Observe that, if $t_1 t_2 \sim t_2 t_1$, then $\{t_1, t_2\}$ forms a *step sequence* as defined in [1]. Then consider the smallest congruence on $\mathcal{F}$ with respect to the concatenation of sequences, and that contains all such pairs $(t_1 t_2, t_2 t_1)$. Then the set $\mathcal{C} = \mathcal{F}/\sim$ of equivalence classes is naturally equipped with a concatenation relation and a partial order inherited from those on $\mathcal{F}$.

The unfolding theory of contextual nets [1,2] states the existence of a universal occurrence net O such that, if E denotes the AES associated with O, then the partially ordered set $\mathcal{C}$ is isomorphic to the poset of configurations of E. The nodes of the occurrence net are labeled by the nodes of the original net, in such a way that any firing sequence in the original net can be uniquely lifted to a firing sequence in the unfolding that respects the labels; the lifting commutes with the congruences defined in the original net and in the unfolding.

The correspondence between a net and its associated AES is moreover functorial, provided the appropriate class of morphisms is considered. The existence of the unfolding and of the associated AES is established through the construction of a chain of coreflexions, from nets to occurrence nets and then to AES.

We illustrate the construction of the occurrence net unfolding a given contextual net, and the construction of the associated AES, in Figs. 7 and 8. See [1,2] for a technical description.

References

[1] P. Baldan, A. Corradini, U. Montanari, An event structure semantics for P/T contextual nets: asymmetric event structures, in: M. Nivat (Ed.), Foundations of Software Science and Computation Structures (FOSSACS 1998), in: LNCS, vol. 1378, Springer, 1998, pp. 63–80.

[2] P. Baldan, A. Corradini, U. Montanari, Contextual Petri nets, asymmetric event structures, and processes, Inform. and Comput. 171 (1) (2001) 1–49, http://dx.doi.org/10.1006/inco.2001.3060.
[3] M. Nielsen, G. Plotkin, G. Winskel, Petri nets, event structures and domains, part 1, Theoret. Comput. Sci. 13 (1981) 85–108.
[4] W. Vogler, A. Semenov, A. Yakovlev, Unfolding and finite prefix for nets with read arcs, in: D. Sangiorgi, R. de Simone (Eds.), 9th International Conference on Concurrency Theory (CONCUR), in: LNCS, vol. 1466, Springer, 1998, pp. 501–516.
[5] N. Busi, M. Pinna, Non-sequential semantics for contextual P/T-nets, in: J. Billington, W. Reisig (Eds.), Application and Theory of Petri Nets, in: LNCS, vol. 1091, Springer, 1996, pp. 113–132.
[6] U. Montanari, F. Rossi, Contextual nets, Acta Inform. 32 (1995) 545–596.
[7] S. Christensen, N. Hansen, Coloured Petri nets extended with place capacities, test arcs, and inhibitors arcs, in: M. Ajmone-Marsan (Ed.), Applications and Theory of Petri Nets (ICATPN/APN 1993), in: LNCS, vol. 691, Springer, 1993, pp. 186–205.
[8] S. Abbes, A. Benveniste, Probabilistic true-concurrency models: branching cells and distributed probabilities for event structures, Inform. and Comput. 204 (2) (2006) 231–274.
[9] A. Benveniste, C. Jard, S. Abbes, Application of branching cells to QoS aware service orchestrations, Teoret. Comput. Sci. 546 (2014) 52–62.
[10] G. Gierz, K. Hofmann, K. Keimel, J. Lawson, M. Mislove, D. Scott, Continuous Lattices and Domains, Encyclopedia of Mathematics and Its Applications, vol. 93, Cambridge University Press, 2003.
[11] S. Abbes, A. Benveniste, Concurrency, sigma-algebras and probabilistic fairness, in: L. de Alfaro (Ed.), 12th Conference on Foundations of Software Science and Computation Structures (FOSSACS), in: LNCS, vol. 5504, 2009, pp. 380–394.
[12] S. Abbes, A projective formalism applied to topological and probabilistic event structures, Math. Structures Comput. Sci. 17 (4) (2007) 819–837.
[13] S. Abbes, K. Keimel, Projective topology on bifinite domains and applications, Theoret. Comput. Sci. 365 (3) (2006) 171–183.