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Discrete coherent states for higher Landau levels

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ABSTRACT

We consider the quantum dynamics of a charged particle evolving under the action of a constant homogeneous magnetic field, with emphasis on the discrete subgroups of the Heisenberg group (in the Euclidean case) and of the $SL(2, \mathbb{R})$ group (in the Hyperbolic case). We investigate completeness properties of discrete coherent states associated with higher order Euclidean and hyperbolic Landau levels, partially extending classic results of Perelomov and of Bargmann, Butera, Girardello and Klauder. In the Euclidean case, our results follow from identifying the completeness problem with known results from the theory of Gabor frames. The results for the hyperbolic setting follow by using a combination of methods from coherent states, time-scale analysis and the theory of Fuchsian groups and their associated automorphic forms.

1. Introduction

In this paper we consider the quantum dynamics of a charged particle evolving under the action of a constant homogeneous magnetic field, first in the Euclidean and then in the hyperbolic setting. The goal is to construct discrete coherent states associated with the evolution of the particle when higher Landau levels are formed and to obtain conditions on the completeness of such coherent states. This extends well known results of Perelomov [1] and of Bargmann et al. [2]. In the first part of the paper,

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we consider a constant magnetic field acting on the Euclidean space realized as the complex plane $\mathbb{C}$, leading to the formation of a discrete spectrum known as the Euclidean Landau Levels. In the second part of the paper, we let a constant magnetic field act on the open hyperbolic plane realized as the Poincaré upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C}, \mathrm{Im} z > 0\}$, leading to the formation of a mixed spectrum, with a *discrete part* corresponding to *bound states* (*hyperbolic Landau levels*) and a continuous part corresponding to *scattering states*.

The concept of a set of states on a lattice in phase space was first considered by J. von Neumann in the Euclidean case [3]. It became physically very attractive because it contains the fundamental commutation relations of quantum mechanics. Indeed, lattices have an underlying unit cell (fundamental domain) related to the size of the Planck constant (see Fig. 1).

In his treatment of quantum mechanics [3], J. von Neumann raised the question of completeness of coherent states indexed by a lattice. The question turned out to be nontrivial from a mathematical point of view and, so far, it has only been fully understood for some special coherent states. This is the case of the coherent states associated with the first Landau Level. The situation has been clarified in [2] and [1], because it can be related to the structure of zeros of analytic functions, where classical methods from complex analysis can be used. However, in higher Landau Levels, even the case of the Euclidean Landau levels is not yet fully understood. In both the Euclidean and Hyperbolic setting, one has to deal with spaces of polyanalytic functions [4–9]. Since polyanalytic functions have a much more complicated structure of zeros [10], several essential tools from complex analysis cannot be applied. However, in recent years, important progress has been made by combining analytic function theory with methods from time–frequency analysis [11,4,12]. The purpose of the first part of this paper is to translate these results from time–frequency analysis to the setting of coherent states attached to higher Landau Levels. This has a twofold purpose: to bring the results to the attention of the physics community and to motivate the results on the hyperbolic setting of the subsequent section, where time-scale (wavelet) theory replaces time–frequency (Gabor) analysis. Indeed, our main object of study in the paper is the quantum dynamics of a charged particle evolving on the open hyperbolic plane under the action of a constant magnetic field. While previous work on this problem has been concerned with the spectral properties of the corresponding Landau Hamiltonian [13,14] and their associated continuous coherent states [15], the investigation of the associated discrete coherent states labeled by discrete subgroups of $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I\}$ seems to have been overlooked. The discrete coherent states are relevant for the understanding of the hyperbolic setting because the nontrivial dynamics is induced by the tesselation of the Poincaré plane by discrete subgroups of $PSL(2, \mathbb{R})$, which are called *Fuchsian groups*. Important examples of Fuchsian groups are provided by the modular group $PSL(2, \mathbb{Z})$ and by the congruence groups of order n. Some background and examples of Fuchsian groups are given in the last section. This is a remarkable instance of the usefulness of analytic number theory in a physical problem. The idea of using Fuchsian groups as a replacement for the Euclidean lattices seems to have first been used by Perelomov, who provides a full analysis of the first hyperbolic Landau level in [16, Chapter 14], where the analysis is done in the disc. In the present paper we make the corresponding analysis for the higher hyperbolic Landau levels. As the unit cell of the model one considers a fundamental domain for the group. For instance, the set

$$D = \left\{ z \in \mathbb{C}^+ : |z| \geq 1 \text{ and } |\mathrm{Re} z| \leq \frac{1}{2} \right\}$$

is a fundamental domain for the modular group $PSL(2, \mathbb{R})$.

The shadow area in Fig. 2 represents the fundamental domain D.

The following terminology will be used [17]. A functional Hilbert space $\mathcal{H}$ has a system $\{f_g\}$ of coherent states, labeled by elements g of a locally compact group G if:

(i) There is a representation $T : g \to T_g$ of G labeled by unitary operators T_g on $\mathcal{H}$

(ii) There is a vector $f_0 \in \mathcal{H}$ such that for $f_g = T_g [f_0]$ and for arbitrary $f \in \mathcal{H}$ we have:

$$\langle f, f \rangle_{\mathcal{H}} = \int_G \left| \langle f, f_g \rangle \right|^2 d\nu(g),$$

where $d\nu$ stands for the left Haar measure of G.

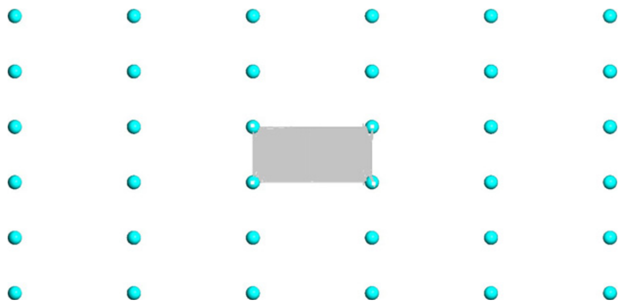

Fig. 1. An Euclidean lattice and a fundamental domain. See Section 2.3.

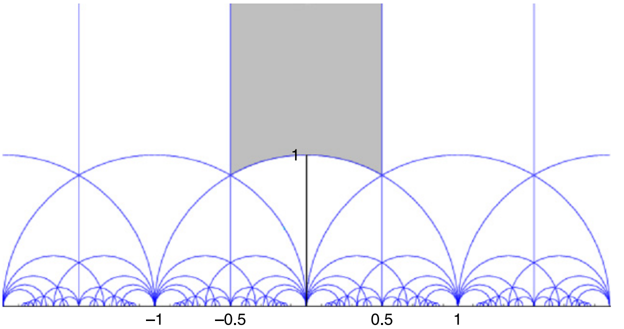

Fig. 2. The modular group $PSL(2, \mathbb{Z})$. See Section 3.5.

The core of the paper is organized in two sections and an Appendix with the more technical proofs. Section 2 deals with Euclidean Landau levels and Section 3 with their hyperbolic analogues. In each of the sections, after providing some background on the mathematical and physical model, we first construct the coherent states associated with the higher levels and then investigate their discrete counterparts. We finish with a short conclusion including some remarks about the theoretical methodology, highlighting the interaction between physical and signal analysis which has made possible the investigations carried out in this paper.

2. Euclidean Landau levels

2.1. Definitions

The Hamiltonian operator describing the dynamics of a particle of charge e and mass m_* on the Euclidean xy-plane, while interacting with a perpendicular constant homogeneous magnetic field, is given by the operator

$$H := \frac{1}{2m_*}\left(i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2, \tag{2.1}$$

where $\hbar$ denotes Planck's constant, c is the light speed and i the imaginary unit. Denote by $B > 0$ the strength of the magnetic field and select the symmetric gauge

$$\mathbf{A} = -\frac{\mathbf{r}}{2} \times \mathbf{B} = \left(-\frac{B}{2}y, \frac{B}{2}x\right),$$

where $\mathbf{r} = (x, y) \in \mathbb{R}^2$. For simplicity, we set $m_* = e = c = \hbar = 1$ in (2.1), leading to the Landau Hamiltonian

$$H_B^L := \frac{1}{2}\left(\left(i\frac{\partial}{\partial x} - \frac{B}{2}y\right)^2 + \left(i\frac{\partial}{\partial y} + \frac{B}{2}x\right)^2\right) \tag{2.2}$$

acting on the Hilbert space $L^2\left(\mathbb{R}^2, dxdy\right)$. The spectrum of the Hamiltonian H_B^L consists of infinite number of eigenvalues with infinite multiplicity of the form

$$\epsilon_n^B = \left(n + \frac{1}{2}\right)B, \quad n = 0, 1, 2, \ldots. \tag{2.3}$$

These eigenvalues are called *Euclidean Landau levels*. Denote the eigenspace of H_B^L corresponding to the eigenvalue ϵ_n^B in (2.3) by

$$\mathcal{A}_{B,n}\left(\mathbb{R}^2\right) = \left\{\varphi \in L^2\left(\mathbb{R}^2, dxdy\right), H_B^L[\varphi] = \epsilon_n^B\varphi\right\}. \tag{2.4}$$

The following functions form an orthogonal basis for $\mathcal{A}_{B,n}(\mathbb{C})$ [5]:

$$\begin{cases} e_{i,n}^1(z) = \sqrt{\dfrac{n!}{(n-i)!}}B^{\frac{i+1}{2}}z^i L_n^{(i)}(B|z|^2), & 0 \le i \\ e_{j,n}^2(z) = \sqrt{\dfrac{j!}{(j+n)!}}B^{\frac{n-1}{2}}\bar{z}^n L_j^{(n)}(B|z|^2), & 0 \le j, \end{cases} \tag{2.5}$$

where the Laguerre polynomial is defined as

$$L_n^{(\alpha)}(t) = \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{t^k}{k!}, \quad \alpha > -1.$$

Remark 1. In his book [16, pag. 35], Perelomov points out that the basis (2.5) had been used by Feynman and Schwinger in a somewhat different form in order to obtain an explicit expression for the matrix elements of the displacement operator. The functions (2.5) are also related to the complex Hermite polynomials [18]. They occur naturally in several problems and different representations are used. For instance, they have recently found applications in quantization [19–21], time–frequency analysis [4], partial differential equations [22] and planar point processes [5]. In the next section we recall a characterization theorem of the eigenspace $\mathcal{A}_{B,n}\left(\mathbb{R}^2\right)$ as the range of a suitable coherent state transform of the Hilbert space $L^2(\mathbb{R})$, originally obtained in [23].

2.2. Coherent states for Euclidean Landau levels

Define the Heisenberg group $\mathbb{H}$ as the Lie group whose underlying manifold is $\mathbb{R}^3$ together with the group operation

$$(x, y, r)(x\prime, y\prime, r\prime) = \left(x + x\prime, y + y\prime, r + r\prime + \frac{1}{2}(xy\prime - x\prime y)\right).$$

The continuous unitary irreducible representations of $\mathbb{H}$ are well known [24]. Here we consider the Schrödinger representation T_B of $\mathbb{H}$ on the Hilbert space $L^2(\mathbb{R}, dt)$ [25] defined as

$$T_{B,(x,y,t)}[\psi](t) = \exp\left(i\left(Bt - \sqrt{B}y\xi + \frac{B}{2}xy\right)\right)\psi\left(t - \sqrt{B}x\right)$$

for $(x, y, r) \in \mathbb{H}$, $B > 0$, $\psi \in L^2(\mathbb{R}, dt)$ and $t \in \mathbb{R}$. This representation is square integrable modulo the center $\mathbb{R}$ of $\mathbb{H}$ and the Borel section σ_0 of $\mathbb{H}$ over $\mathbb{H}/\mathbb{R} = \mathbb{R}^2$ which is given by $\sigma_0(x, y) = (x, y, 0)$. Further, the following identity holds

$$\int_{\mathbb{R}^2} \left\langle \psi_1, T_{B,\sigma_0(x,y)}[\phi_1]\right\rangle \left\langle T_{B,\sigma_0(x,y)}[\phi_2], \psi_2\right\rangle d\mu(x, y) = \langle \psi_1, \psi_2\rangle \langle \phi_1, \phi_2\rangle \tag{2.6}$$

for all $\psi_1, \psi_2, \phi_1, \phi_2 \in L^2(\mathbb{R})$. Displacing the reference state

$$\langle t \mid 0\rangle_{B,n} = \left(\sqrt{\pi}2^n n!\right)^{-\frac{1}{2}} e^{-\frac{1}{2}t^2} H_n(t), \quad t \in \mathbb{R},$$

where $H_n(.)$ is the Hermite polynomial

$$H_n(t) = \sum_{k=0}^{[n/2]} \frac{n!(-1)^k (2t)^{n-2k}}{k!(n-2k)!},$$

via the representation operator $T_{B,\sigma_0(x,y)}$, one obtains a set of coherent states denoted by the kets vectors $|(x, y), B, n\rangle$, with wave functions

$$\langle t \mid (x, y), B, n\rangle = \left(\sqrt{\pi}2^n n!\right)^{-\frac{1}{2}} \exp\left(-i\sqrt{B}ty + i\frac{B}{2}xy - \frac{1}{2}\left(t - \sqrt{B}x\right)^2\right) H_n\left(t - \sqrt{B}x\right). \tag{2.7}$$

The following resolution of the identity

$$\mathbf{1}_{L^2(\mathbb{R})} = \int_{\mathbb{R}^2} |(x, y), B, n\rangle \langle (x, y), B, n| \, d\mu(x, y)$$

holds as a consequence of (2.6). Thus the construction of coherent states is justified by the square integrability of representation T_B modulo the subgroup $\mathbb{R}$ and the section σ_0. For $n = 0$ (the lowest Euclidean Landau level), the states $|(x, y), B, 0\rangle$ coincide with the canonical coherent states of the harmonic oscillator. The coherent states (2.7) are associated with the coherent state transform

$$V_{B,n} : L^2(\mathbb{R}) \to L^2\left(\mathbb{R}^2, dxdy\right)$$

such that, given $\varphi \in L^2(\mathbb{R})$,

$$V_{B,n}[\varphi](x, y) := \int_{\mathbb{R}} \overline{\langle t \mid (x, y), B, n\rangle} \varphi(t)\, dt.$$

Thanks to the square integrability of T_B, the transform $V_{B,n}$ is an isometrical map. Since $V_{B,n}$ maps the Hermite functions (an orthogonal basis of $L^2(\mathbb{R})$) to the basis (2.5) (see [4] for details) its range is exactly the eigenspace in (2.4):

$$V_{B,n}\left[L^2(\mathbb{R})\right] = \mathcal{A}_{B,n}\left(\mathbb{R}^2\right).$$

Another realization of this eigenspace can be obtained by intertwining the Landau Hamiltonian (2.2) as follows

$$\Delta_B := e^{\frac{B}{2}z\bar{z}} \left(\frac{1}{2}H^L_{2B} - \frac{B}{2}\right) e^{-\frac{B}{2}z\bar{z}} = -\frac{\partial^2}{\partial z \partial \bar{z}} + B\bar{z}\frac{\partial}{\partial \bar{z}}.$$

The space $\mathcal{A}_{B,n}\left(\mathbb{R}^2\right)$ then becomes

$$\mathcal{A}_{B,n}(\mathbb{C}) := \left\{\varphi \in L^2\left(\mathbb{C}, e^{-Bz\bar{z}} d\mu\right), \Delta_B \varphi = nB\varphi\right\}. \tag{2.8}$$

If $B = \pi$ and $n = 0$ the space (2.8) is precisely the Fock–Bargmann space of entire square integrable functions with respect to the Gaussian measure on $\mathbb{C}$. For $n > 0$, the characterization takes the form

$$\widetilde{V}_{2\pi,n}\left[L^2(\mathbb{R})\right] = \mathcal{A}_{\pi,n}(\mathbb{C})$$

where the coherent state transform is given explicitly by

$$\widetilde{V}_{2\pi,n}[\varphi](z) = e^{\frac{1}{2}\pi z\bar{z}} \circ V_{2\pi,n}[\varphi](z) = (-1)^n B_n[\varphi]\left(\sqrt{\pi}z\right)$$

where

$$B_n[\varphi](w) = (-1)^n c_n \int_{\mathbb{R}} \varphi(t) \exp\left(-\frac{1}{2}t^2 + \sqrt{2}tw - \frac{1}{2}w^2\right) H_n\left(t - \frac{w+\overline{w}}{\sqrt{2}}\right) dt.$$

The transform $\widetilde{V}_{2\pi,n}$ is precisely the true polyanalytic Bargmann transform and the space $\mathcal{A}_{\pi,n}(\mathbb{C})$ is the *true-polyanalytic* space of index n, see [26,4,5].

2.3. Completeness properties

We want to understand the completeness properties of the coherent states constructed in the previous section once they are labeled by a lattice $\Lambda \subset \mathbb{C}$. The key observation is the fact that their completeness and basis properties are equivalent to the completeness and basis properties of Gabor systems with Hermite functions [11] and to sampling and uniqueness sets in true-polyanalytic spaces [4]. Consider the lattice

$$\Lambda = \Lambda(\omega_1, \omega_2) := \{m_1\omega_1 + m_2\omega_2; m_1, m_2 \in \mathbb{Z}\} \subset \mathbb{C}$$

spanned by the periods ω_1 and $\omega_2 \in \mathbb{C}$ with $\mathrm{Im}(\omega_1/\omega_2) > 0$. The size of the lattice Λ is the area of the parallelogram spanned by ω_1 and ω_2. Identifying $\mathbb{R}^2$ with $\mathbb{C}$ we can write $\Lambda = \Omega\mathbb{Z}^2$, where $\Omega = [\omega_1, \omega_2]$ is an invertible 2×2 matrix. The size of the lattice can now be defined as $s(\Lambda) = |\det \Omega|$. We say that Λ is a set of sampling for the space $\mathcal{A}_{B,n}(\mathbb{C})$ if there exist constants $C_1, C_2 > 0$ such that for all $F \in \mathcal{A}_{B,n}(\mathbb{C})$,

$$C_1 \|F\|^2_{\mathcal{A}_{B,n}(\mathbb{C})} \leq \sum_{\lambda\in\Lambda} |F(\lambda)|^2 e^{-B|\lambda|^2} \leq C_2 \|F\|^2_{\mathcal{A}_{B,n}(\mathbb{C})}.$$

Given a point (q, p) in the phase space $\mathbb{R}^2$, the corresponding time–frequency shift is

$$\pi_{(q,p)}[f](t) = e^{2\pi ipt} f(t-q), \quad t \in \mathbb{R}.$$

Let $h_n(t)$ denote a Hermite function. The set $G(h_n, \Lambda) := \left\{\pi_{(q,p)}h_n, (q, p) \in \mathbb{R}\right\}$ is a Gabor frame or a Weyl–Heisenberg frame in $L^2(\mathbb{R})$ whenever there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|f\|^2_{L^2(\mathbb{R})} \leq \sum_{(q,p)\in\Lambda} \left|\left\langle f, \pi_{(q,p)}[h_n]\right\rangle_{L^2(\mathbb{R})}\right|^2 \leq C_2 \|f\|^2_{L^2(\mathbb{R})}.$$

It follows from the lower inequality that if $G(h_n, \Lambda)$ is a frame then $G(h_n, \Lambda)$ is complete. For simplicity, we consider the square lattice $\Lambda_\omega := \omega(\mathbb{Z} + i\mathbb{Z})$, $\omega \in \mathbb{R}$. In this case $s(\Lambda_\omega) = \omega^2$. For $B = \pi$, it was proved that the lattice Λ_ω is a set of sampling for $\mathcal{A}_{\pi,n}(\mathbb{C})$ if and only if $G(h_n, \Lambda_\omega)$ is a Gabor frame, see [4]. The following result is a consequence of combining this identification with relatively recent results from time–frequency analysis.

Theorem 1. *Let $(|(x, y), \pi, n\rangle)_{(x,y)\in\mathbb{R}^2}$ be a system of coherent states attached to the nth Landau level defined in (2.7). Then, the following holds:*

(i) *If $\omega^2 < \frac{1}{n+1}$ then the system $(|(x, y), \pi, n\rangle)_{(x,y)\in\Lambda_\omega}$ is complete.*

(ii) *If $\omega^2 > 1$ then the system $(|(x, y), \pi, n\rangle)_{(x,y)\in\Lambda_\omega}$ is incomplete.*

Proof. The completeness property (i) follows from the fact that if $\omega^2 < \frac{1}{n+1}$, then $G(h_n, \Lambda)$ is a Gabor frame [11], therefore complete. The property (ii) is a consequence of the fact that, if $\omega^2 > 1$, then a Gabor system cannot be complete [27].

Remark 2. In the case $n = 0$ it is a classical result [1,2] that the systems are complete if $\omega^2 \leq 1$ and incomplete if $\omega^2 > 1$. The above result is an extension of these results to coherent states attached to higher Euclidean Landau levels ϵ_n^π, $n = 1, 2, 3, \ldots$.

Remark 3. For $n > 0$ there is still a considerable gap between conditions (i) and (ii). Finding a whole description of the completeness and frame properties of Gabor systems indexed by lattices is a highly non-trivial problem which has been subject of study since [11]. The very recent preprint [28] seems to answer the question in the case of rational lattices.

Remark 4. The Landau Hamiltonian arises in the two-dimensional quantized Hall effect. A coherent states formalism for the study of this problem has been developed by projecting the higher order states in the lowest Landau Level, which can be modeled by analytic functions [29]. It is reasonable to expect that the discrete coherent states associated with higher Landau Levels may provide an alternative formalism.

3. Hyperbolic Landau levels

3.1. Hyperbolic Landau levels

In the hyperbolic setting, the configuration space is now the Poincaré upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C}, \operatorname{Im} z > 0\}$. It is a complete two-dimensional simply connected Riemannian manifold of constant negative curvature $R = -1$, endowed with the metric $ds^2 = y^{-2}\left(dx^2 + dy^2\right)$, where $z = x + iy$. A constant homogeneous magnetic field on $\mathbb{C}^+$ is given by a 2-form $d\mu_B$ defined as

$$d\mu_B = \frac{2B}{y^2} dxdy$$

where B is the field intensity. The form $d\mu_B$ is exact and any 1-form A such that $d\mu_B = dA$ is called a vector potential related to $d\mu_B$. For our purposes it is convenient to choose $A = 2By^{-1}dx$. In suitable units and up to an additive constant, the Schrödinger operator describing the dynamics of a charged particle moving on $\mathbb{C}^+$ under the action of the magnetic field B is given by [13]:

$$H_B := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2iBy\frac{\partial}{\partial x}.$$

Different aspects of the spectral analysis of the operator H_B have been studied by many authors, (see [14,13] or, for a more mathematical approach, [30]). We list here the following important properties.

(i) H_B is an elliptic densely defined operator on the Hilbert space $L^2\left(\mathbb{C}^+, d\mu_B\right)$, with a unique self-adjoint realization that we denote also by H_B.

(ii) The spectrum of H_B in $L^2\left(\mathbb{C}^+, d\mu_B\right)$ consists of two parts: a continuous part $[1/4, +\infty[$, corresponding to *scattering states* and a finite number of eigenvalues with infinite degeneracy (*hyperbolic Landau levels*) of the form

$$\epsilon_n^B := (B - n)(1 - B + n), \quad n = 0, 1, 2, \ldots, \left\lfloor B - \frac{1}{2} \right\rfloor. \tag{3.1}$$

The finite part of the spectrum exists provided $2B > 1$. The notation $\lfloor a \rfloor$ stands for the greatest integer not exceeding a.

(iii) For each fixed eigenvalue ϵ_n^B, we denote by

$$\mathcal{E}_n^B\left(\mathbb{C}^+\right) = \left\{\Phi \in L^2\left(\mathbb{C}^+, d\mu_B\right), H_B\Phi = \epsilon_n^B \Phi\right\} \tag{3.2}$$

the corresponding eigenspace. Its reproducing kernel is given by

$$K_{n,B}(z, \zeta) = \frac{(-1)^n \Gamma(2B - n)}{n! \Gamma(2B - 2n)} \left(\frac{\left|z - \overline{\zeta}\right|^2}{4 \operatorname{Im} z \operatorname{Im} \zeta}\right)^{-B+m} \left(\frac{\zeta - \overline{z}}{z - \overline{\zeta}}\right)^B \times {}_2F_1\left(-2B - m, -m, 2B - 2m, \frac{4 \operatorname{Im} z \operatorname{Im} \zeta}{\left|z - \overline{\zeta}\right|^2}\right)$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Remark 5. The condition $2B > 1$ ensuring the existence of the discrete eigenvalues means that the magnetic field has to be strong enough to capture the particle in a closed orbit. If this condition is not fulfilled the motion will be unbounded and the particle will escape to infinity. More precisely, the orbit of the particle will intercept the upper half-plane boundary whose points stand for points at infinity [13, pg. 189]. To the eigenvalues in (3.1) correspond eigenfunctions which are called *bound states*. This terminology is due to the fact that the particle in such a bound state cannot leave the system without additional energy.

3.2. Bergman spaces

For $n = 0$, the reproducing kernel of $\mathcal{E}_0^B\left(\mathbb{C}^+\right)$ reduces to

$$K_{0,B}\left(z,\zeta\right) = e^{i\pi B}4^B \frac{\left(\mathrm{Im}z\ \mathrm{Im}\zeta\right)^B}{\left(z-\overline{\zeta}\right)^{2B}}.$$

This is the reproducing kernel for the $(2B-2)$-weighted Bergman space $A_{2B-1}\left(\mathbb{C}^+\right)$, constituted by analytic functions f on the upper half-plane with finite norm

$$\|f\|_{A_{2B-1}(\mathbb{C}^+)} = \int_{\mathbb{C}^+} |f\left(z\right)|^2 y^{2B-2} dxdy < +\infty.$$

Thus, $\mathcal{E}_0^B\left(\mathbb{C}^+\right)$ coincides with $A_{2B-2}\left(\mathbb{C}^+\right)$.

An important fact to be used in the Appendix proof of the main results is the following. Note also that for a general weight ν, the Bergman space $A_\nu\left(\mathbb{C}^+\right)$ is connected to the space $L^2\left(\mathbb{R}^+, t^{-1}dt\right)$ by the integral transform defined as

$$Ber_\nu\left[h\right]\left(z\right) = \int_0^{+\infty} t^{\frac{\nu+3}{2}} h\left(t\right) e^{izt} dt \tag{3.3}$$

see, for instance [31,32]. This provides an isometric isomorphism

$$Ber_\nu : L^2\left(\mathbb{R}^+, t^{-1}dt\right) \to A_\nu\left(\mathbb{C}^+\right).$$

The transform is onto because one can deduce from the special function formula

$$\int_0^\infty t^\alpha L_m^\alpha(t) e^{-tu} dt = \frac{\Gamma(m+1+\alpha)}{m!}\left(\frac{u-1}{u}\right)^m \frac{1}{u^{\alpha+1}} \tag{3.4}$$

that the Laguerre functions are mapped to a basis of $A_\nu\left(\mathbb{C}^+\right)$ formed by rational functions which are further mapped to the unit disc by a conformal map. Some details and remarks about these calculations are given in [33] and [34].

3.3. The affine group acting on the Poincaré half-plane

For our purposes we will recall a characterization theorem of $\mathcal{E}_n^B\left(\mathbb{C}^+\right)$ as the range under a suitable coherent state transform $W_{B,n}$ defined on the Hilbert space $\mathcal{H} := L^2\left(\mathbb{R}^+, t^{-1}dt\right)$. We start with the identification of the Poincaré upper half-plane $\mathbb{C}^+$ with the affine group $\mathbf{G} = \mathbb{R} \times \mathbb{R}^+$, by setting $z = x+iy \equiv (x, y)$. The group law of $\mathbb{G}$ is $(x, y) . (x\prime, y\prime) = (x + yx\prime, yy\prime)$. $\mathbf{G}$ is a locally compact nonunimodular group with the left Haar measure $d\mu\left(x, y\right) = y^{-2}dxdy$ and modular function $\Delta\left(x, y\right) = y^{-1}$. By this identification the space $L^2\left(\mathbf{G}, d\mu\right)$ coincides with the space $L^2\left(\mathbb{C}^+, d\mu_B\right)$. We shall consider one of the two inequivalent infinite dimensional unitary irreducible representations of the affine group $\mathbf{G}$, denoted π_+, realized on the Hilbert space $\mathcal{H}$ as

$$\pi_+\left(x, y\right)\left[\varphi\right]\left(t\right) := \exp\left(ixt/2\right)\varphi\left(yt\right), \quad \varphi \in \mathcal{H},\ t \in \mathbb{R}^+.$$

This representation is square integrable since it is easy to find a vector $\phi_0 \in \mathcal{H}$ such that the function $(x, y) \mapsto \langle \pi_+ (x, y) [\phi_0], \phi_0 \rangle_{\mathcal{H}}$ belongs to $L^2 (\mathbf{G}, d\mu)$. This condition can also be expressed by saying that the self-adjoint operator $K : \mathcal{H} \to \mathcal{H}$ defined as $K [\psi] (.) = (.)^{-\frac{1}{2}} \psi (.)$ gives

$$\int_{\mathbf{G}} d\mu (x, y) \langle \varphi_1, \pi_+ (x, y) [\psi_1] \rangle \langle \pi_+ (x, y) \psi_2, [\varphi_2] \rangle = \langle \varphi_1, \varphi_2 \rangle \left\langle K^{\frac{1}{2}} [\psi_1], K^{\frac{1}{2}} [\psi_2] \right\rangle,$$

for all $\psi_1, \psi_2, \varphi_1, \varphi_2 \in \mathcal{H}$. The operator K is unbounded because $\mathbf{G}$ is not unimodular [35]. We will also use the notation

$$\pi_+^1 (x, y) [\varphi] (t) := y^{\frac{1}{2}} \exp (ixt/2) \varphi (yt), \quad \varphi \in \mathcal{H}, \ t \in \mathbb{R}^+$$

such that

$$\pi_+ (x, y) \left[(.)^{\frac{1}{2}} \varphi (.) \right] (t) = t^{\frac{1}{2}} \pi_+^1 (x, y) [\varphi] (t) \tag{3.5}$$

and also, for functions Φ such that their Fourier transform belongs to $L^2 \left(\mathbb{R}^+\right)$ (this is essentially the Hardy space where the wavelet transformation is often defined),

$$\pi^{wav} (x, y) [\Phi] (t) = y^{-\frac{1}{2}} \Phi \left(y^{-1} (t - x) \right).$$

For shortness of notations, in some situations we will represent the point (x, y) by the complex number $z = x + iy$, often with no explicit mention.

3.4. Coherent states for higher hyperbolic Landau levels

Now, as in [15], we consider a set of coherent states denoted by the ket vectors $|(x, y), B, n\rangle$ and obtained by displacing, via the representation operator $\pi_+ (x, y)$, the reference state vector $|0\rangle_{B,n}$ in the Hilbert space $\mathcal{H}$ with wave function given by

$$\langle t \mid 0 \rangle_{B,n} = \left(\frac{\Gamma (2B - n)}{n!} \right)^{-\frac{1}{2}} t^{B-n} e^{-\frac{1}{2} t} L_n^{(2B-2n-1)} (t).$$

Precisely,

$$|(x, y), B, n\rangle := \pi_+ (x, y) |0\rangle_{B,n}. \tag{3.6}$$

The wave functions of the coherent states (3.6) are given by

$$\langle t \mid (x, y), B, n \rangle = \left(\frac{\Gamma (2B - n)}{n!} \right)^{-\frac{1}{2}} (ty)^{B-n} e^{-\frac{1}{2} t (y - ix)} L_n^{(2B-2n-1)} (ty). \tag{3.7}$$

These coherent states are completely justified by the square integrability of the unitary irreducible representation π_+ and if follows from the special function formula (3.4) that we have a resolution of the identity for the space $\mathcal{H} = L^2 \left(\mathbb{R}^+, t^{-1} dt\right)$:

$$\mathbf{1}_{\mathcal{H}} = c_{B,n}^{-1} \int_{\mathbf{G}} d\mu (x, y) |(x, y), B, n\rangle \langle (x, y), B, n|,$$

where $c_{B,n} = (2 (B - n) - 1)^{-1}$. The coherent states (3.6) are associated with the coherent state transform

$$W_{B,n} [\varphi] (x, y) = c_{B,n}^{-\frac{1}{2}} \int_{\mathbb{R}^+} \overline{\langle t \mid (x, y), B, n \rangle} \varphi (t) \frac{dt}{t}. \tag{3.8}$$

The range of the map $W_{B,n} : L^2 \left(\mathbb{R}^+, t^{-1} dt\right) \to L^2 \left(\mathbb{C}^+, d\mu_B\right)$ is the eigenspace (3.2):

$$W_{B,n} \left[L^2 \left(\mathbb{R}^+, t^{-1} dt\right) \right] = \mathcal{E}_n^B \left(\mathbb{C}^+\right)$$

for every $n \in \mathbb{Z}_+ \cap \left[0, B - \frac{1}{2}\right]$ provided that $2B > 1$.

Remark 6. Note that, for $n = 0$, the lowest hyperbolic Landau level, the states $|(x, y), B, 0\rangle$ coincide with the well known *affine* coherent states [36].

3.5. Wavelet transforms with Laguerre functions

In this subsection we write the coherent states of the previous section in terms of wavelet transforms with analyzing wavelets Φ_n^α defined via the Fourier transforms in terms of Laguerre polynomials L_n^α as

$$\mathcal{F}\Phi_n^\alpha(t) = t^{\frac{\alpha+1}{2}} e^{-t} L_n^\alpha(2t). \tag{3.9}$$

Some of the structural properties of Φ_n^α that will be key in our approach are a consequence of their explicit formula, which displays Φ_n^α as linear combinations of $\{\Phi_n^{\alpha+2k}\}_{k=0}^n$:

$$\Phi_n^\alpha(t) = \sum_{k=0}^{n} \frac{(-2)^k}{k!} \binom{n+\alpha}{n-k} \Phi_0^{\alpha+2k}(t).$$

Now, let $\varphi \in L^2\left(\mathbb{R}^+, t^{-1}dt\right)$. Combining (3.8) and (3.7) gives

$$W_{B,n}[\varphi](x,y) = c_{B,n}^{-\frac{1}{2}} \left(\frac{\Gamma(2B-n)}{n!}\right)^{-\frac{1}{2}} \int_{\mathbb{R}^+} (ty)^{B-n} e^{-\frac{1}{2}t(y+ix)} L_n^{(2B-2n-1)}(ty)\,\varphi(t)\,\frac{dt}{t}.$$

With $z = x + iy$, we have that $-\frac{1}{2}t(y+ix) = \overline{\frac{1}{2}\xi iz}$. Set $\gamma_{B,n} = c_{B,n}(n!)^{-1}\Gamma(2B-n)$ and rewrite the above as

$$W_{B,n}[\varphi](x,y) = \gamma_{B,n}^{-\frac{1}{2}} \int_{\mathbb{R}^+} \varphi(t)\,\overline{\left((ty)^{B-n} e^{\frac{1}{2}\xi iz} L_n^{(2B-2n-1)}(ty)\right)}\frac{dt}{t}. \tag{3.10}$$

Since $\pi_+(x,y)\left[(.)^{\frac{1}{2}} l_n^{2B-2n-1}(.)\right](t) = (ty)^{B-n} e^{\frac{1}{2}tiz} L_n^{(2B-2n-1)}(ty)$, then (3.10) becomes

$$\begin{aligned} W_{B,n}[\varphi](x,y) &= \gamma_{B,n}^{-\frac{1}{2}} \int_{\mathbb{R}^+} \varphi(t)\,\overline{\left(\pi_+(x,y)\left[(.)^{\frac{1}{2}} l_n^{2B-2n-1}(.)\right]\right)}(t)\,\frac{dt}{t} \\ &= \gamma_{B,n}^{-\frac{1}{2}} \left\langle \varphi, \pi_+(x,y)\left[(.)^{\frac{1}{2}} l_n^{2B-2n-1}(.)\right]\right\rangle_{L^2\left(\mathbb{R}^+,\frac{dt}{t}\right)}. \end{aligned} \tag{3.11}$$

Since $\pi_+(x,y)\left[(.)^{\frac{1}{2}}\phi(.)\right](t) = t^{\frac{1}{2}}\pi_+^1(x,y)[\phi(.)](t)$, then (3.11) becomes

$$W_{B,n}[\varphi](x,y) = \gamma_{B,n}^{-\frac{1}{2}} \left\langle (.)^{-\frac{1}{2}}\varphi(.), \pi_+^1(z)\left[l_n^{2B-2n-1}\right](.)\right\rangle_{L^2(\mathbb{R}^+,dt)}.$$

If $\varphi \in L^2\left(\mathbb{R}^+, t^{-1}dt\right)$, then $\mathcal{F}^{-1}\left(t^{-\frac{1}{2}}\varphi\right)$ is in $H^2\left(\mathbb{C}^+\right)$ and the scalar product above may also be written as

$$\left\langle (.)^{-\frac{1}{2}}\varphi(.), \pi_1^+(z)\left[l_n^{2B-2n-1}\right](.)\right\rangle_{L^2(\mathbb{R}^+,dt)} = \mathcal{W}_{\Phi_n^{2(B-n)-1}}\left[\mathcal{F}^{-1}\left((.)^{-\frac{1}{2}}\varphi\right)\right](z),$$

where $\mathcal{W}_{\Phi_n^{2(B-n)-1}}$ stands for the *wavelet transformation* [31], defined as

$$\mathcal{W}_\Phi[\varphi](x,y) = \langle \varphi, \pi_z\Phi\rangle_{L^2(\mathbb{R})}, \quad z = x+iy,\ y > 0,$$

where $\mathcal{F}\Phi \in L^2\left(\mathbb{R}^+, t^{-1}dt\right)$. The two transforms are related as follows

$$W_{B,n}[\varphi](x,y) = \gamma_{B,n}^{-\frac{1}{2}}\,\mathcal{W}_{\Phi_n^{2(B-n)-1}}\left[\mathcal{F}^{-1}\left((.)^{-\frac{1}{2}}\varphi(.)\right)\right](x,y). \tag{3.12}$$

Remark 7. This also means that we have another realization of the bound states space $\mathcal{E}_n^B\left(\mathbb{C}^+\right)$ in (3.2) as the image of the Hardy space $H\left(\mathbb{C}^+\right)$ under the wavelet transform $\mathcal{W}_{\Phi_n^{2(B-n)-1}}$.

With the help of the transform Ber_ν in (3.3), we will be able to express the transform $W_{B,n}[f]$ of any function f in $L^2\left(\mathbb{R}^+, t^{-1}dt\right)$ as a combination of derivatives of an analytic function.

Proposition 1. *If $f \in L^2\left(\mathbb{R}^+, t^{-1}dt\right)$, then*

$$W_{B,n}[f](z) = \gamma_{B,n}^{-\frac{1}{2}} \sum_{k=0}^{n} \frac{(2i)^k}{k!} \binom{2B-n-1}{n-k} y^{B-n-\frac{1}{2}+k} F^{(k)}(z),$$

where $F(z) = Ber_{2(B-n)-1}[f](z)$ belongs to the weighted Bergman space $A_{2(B-n)-1}\left(\mathbb{C}^+\right)$.

Proof. Take $f \in L^2\left(\mathbb{R}^+, t^{-1}dt\right)$. Then the function $u = \mathcal{F}^{-1}\left(t^{-\frac{1}{2}}f\right) \in H^2\left(\mathbb{C}^+\right)$. Write $F := Ber_\nu[f]$, where $\nu = 2(B-n)-1$. In [32, pg. 256], it is shown that the wavelet transform of u decomposes in terms of derivatives of the analytic function $F \in A_{2(B-n)-1}\left(\mathbb{C}^+\right)$ as

$$\widetilde{W}_{\Phi_n^{2(B-n)-1}} u(z) = \sum_{k=0}^{n} \frac{(2i)^k}{k!} \binom{2B-n-1}{n-k} y^{B-n-\frac{1}{2}+k} F^{(k)}(z). \tag{3.13}$$

Recalling the relation (3.12) between the two transforms, we may rewrite (3.13) as

$$W_{B,n}f(x,y) = \gamma_{B,n}^{-\frac{1}{2}} \sum_{k=0}^{n} \frac{(2i)^k}{k!} \binom{2B-n-1}{n-k} y^{B-n-\frac{1}{2}+k} F^{(k)}(z). \tag{3.3.24}$$

This completes the proof.

3.6. *Fuchsian groups and their automorphic forms*

Let I_2 be the identity matrix. Since one can identify the Poincaré half-plane $\mathbb{C}^+$ with the quotient group

$$PSL(2,\mathbb{R}) := SL(2,\mathbb{R}) / \{\pm I_2\},$$

also known as the group of Möbius transformations, the subgroups of $PSL(2,\mathbb{R})$, known as *Fuchsian groups*, describe the isometries of the hyperbolic metric of $\mathbb{C}^+$. Since the nontrivial dynamics of a particle in the upper half-plane is induced by its tesselation by discrete subgroups, we want to understand the completeness properties of the coherent states introduced in the previous section, once they are labeled by Fuchsian groups. Thus, we need to recall some basic facts about Fuchsian groups and their associated automorphic forms. Consider the group $SL(2,\mathbb{R})$ of real 2×2 matrices with determinant one, acting on $\mathbb{C}^+$ according to the rule

$$g.z = \frac{az+b}{cz+d}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}).$$

Notice that g and $-g$ have the same action on $\mathbb{C}^+$.

A *Fuchsian group* G is a discrete subgroup of $PSL(2,\mathbb{R})$. The most important example is the modular group $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z}) / \{\pm I_2\}$, where

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

An important class is provided by the congruence groups of order n, $G(n)$,

$$G(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm I(\mathrm{mod}\, n) \right\}.$$

Further terminology will be required. The G-orbit Gz of a point $z \in \mathbb{C}^+$ under the action of the group G is

$$Gz = \{gz : g \in G\}.$$

A *fundamental domain* for a Fuchsian group G is a closed set $D \subset \mathbb{C}^+$ such that D is the closure of its interior D^0, no two points of D^0 lie in the same G-orbit and the images of D under G cover $\mathbb{C}^+$. For instance, a fundamental domain for $G = PSL(2, \mathbb{Z})$ is given by

$$D = \left\{ z \in \mathbb{C}^+ : |z| \geq 1 \text{ and } |\mathrm{Re} z| \leq \frac{1}{2} \right\}.$$

In the hyperbolic model, the orbit of an element $z \in D$ will replace the role of the lattice $\Lambda(\omega_1, \omega_2)$ in the Euclidean model of the previous section, while the fundamental domain D replaces the role of the parallelogram spanned by ω_1 and ω_2. We will restrict to Fuchsian groups such that D has finite hyperbolic area. In this case, D can be chosen as a polygon with *an even number* $2k$ *of sides*. The sides, grouped in pairs, are equivalent with respect to the action of G. The vertices of the polygon are joined in cycles of vertices which are equivalent to each other. If the region is a polygon with vertices lying on the boundary of $\mathbb{C}^+$, the cycle is called *parabolic* (often referred to in the literature as *cusps*), otherwise it is called *elliptic*. Let r be the total number of cycles and $e_1, \ldots, e_r$ be the orders of the inequivalent elliptic points of G. Joining equivalent vertices and cycles, leads to the construction of the Riemann surface $G \setminus \mathbb{C}^+$, whose *genus* $\mathcal{g}$ is given by $2\mathcal{g} = 1 + k - r$. The set $(\mathcal{g}, r, e_1, \ldots, e_r)$ is called the signature of the group G. It contains information to compute the area S_G of the fundamental domain D:

$$S_G = 2\pi \left[2\mathcal{g} - 2 + \sum_{l=1}^{r} \left\lfloor 1 - \frac{1}{e_l} \right\rfloor \right]. \tag{3.14}$$

Now we introduce the notion of an automorphic form associated with G. For all $m \in \mathbb{Z}$, $z \in \mathbb{C}^+$ and any function f with domain $\mathbb{C}^+$, let

$$\left(f \,|_m\, g\right)(z) = (cz + d)^{-2m} f(g.z), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).$$

An *automorphic form* of weight m with respect to a Fuchsian group G is a meromorphic function f on $\mathbb{C}^+$ such that

$$\left(f \,|_m\, g\right) = f,$$

for all $g \in G$. The number N of zeros of f inside the fundamental domain D of the group G is given by Poincaré's formula

$$N = m \frac{S_G}{2\pi}. \tag{3.15}$$

The set of all automorphic forms of weight m is denoted by $\Omega_G^m\left(\mathbb{C}^+\right)$. Consider also $\mathfrak{Hol}_G^m\left(\mathbb{C}^+\right)$, the set of functions $f \in \Omega_G^m\left(\mathbb{C}^+\right)$ holomorphic on $\mathbb{C}^+$ (including all cusps of G). We write $\mathfrak{C}_G^m\left(\mathbb{C}^+\right)$ for the set of functions $f \in \Omega_G^m\left(\mathbb{C}^+\right)$ which are zero at all cusps of G (the so-called cusp forms). The inclusions among these spaces are the following:

$$\mathfrak{C}_G^m\left(\mathbb{C}^+\right) \subset \mathfrak{Hol}_G^m\left(\mathbb{C}^+\right) \subset \Omega_G^m\left(\mathbb{C}^+\right).$$

The dimension $\dim \mathfrak{Hol}_G^m\left(\mathbb{C}^+\right)$ is known explicitly [37, p. 46, Theorem 2.23] in terms of m, the genus $\mathcal{g}$ of the Riemann surface $G \setminus \mathbb{C}^+$, the orders of the inequivalent elliptic points of G. Assuming that all cusps of G are equivalent,

$$\dim \mathfrak{Hol}_G^m\left(\mathbb{C}^+\right) = \begin{cases} (2m-1)(\mathcal{g}-1) + \sum_{l=1}^{r} \left\lfloor m\left(1 - \frac{1}{e_l}\right) \right\rfloor, & m > 1 \\ \mathcal{g}, & m = 1 \\ 1, & m = 0 \\ 0, & m < 0. \end{cases} \tag{3.16}$$

Here $\lfloor x \rfloor$ denotes the largest integer less or equal to x.

3.7. Completeness theorem

The next results (see the Appendix for proofs) provide necessary conditions for the completeness of the discrete coherent states indexed by Fuchsian groups.

Theorem 2. *Let $\{|z, B, n\rangle\}_{z\in\mathbb{C}^+}$ be a system of coherent states attached to the nth hyperbolic Landau level. If the subsystem $\{|g\zeta_0, B, n\rangle\}_{g\in G}$ indexed by the Fuchsian group G associated with the automorphic form F_0 of weight m_0, vanishing at one point $\zeta_0 \in \mathbb{C}^+$ is complete, then*

$$m_0 \geq \frac{1}{2}\frac{B-n}{1+n}.$$

If we can choose the automorphic form of weight $m_0 = \frac{2\pi}{S_G}$, where S_G is the area of the fundamental domain the above theorem can be rephrased as a necessary upper bound on S_G.

Corollary 1. *Let $\{|z, B, n\rangle\}_{z\in\mathbb{C}^+}$ be a system of coherent states attached to the nth hyperbolic Landau level. If the subsystem $\{|g\zeta_0, B, n\rangle\}_{g\in G}$ indexed by the Fuchsian group G vanishing at one point $\zeta_0 \in \mathbb{C}^+$ is complete, then*

$$S_G \leq 4\pi\frac{1+n}{B-n}.$$

Let us consider $\dim \mathfrak{H}ol_G^m\left(\mathbb{C}^+\right) \geq 2$. This guarantees the existence of an automorphic form of weight m vanishing at a given ζ_0, using appropriate linear combinations. When $\mathcal{g} = 0$ and $m \geq 2$, m_0 can be evaluated explicitly in terms of the signature $(0, r, e_1, \ldots, e_r)$ of the group Fuchsian group G.

Corollary 2. *Let G be a group of signature $(0, r, e_1, \ldots, e_r)$, with $\dim \mathfrak{H}ol_G^m\left(\mathbb{C}^+\right) \geq 2$ and $m \geq 2$. If the subsystem $\{|g\zeta_0, B, n\rangle\}_{g\in G}$ indexed by the Fuchsian group G is complete, then*

$$\sum_{l=1}^{r}\left\lfloor 1-\frac{1}{e_l}\right\rfloor - 2 \leq 2\frac{1+n}{B-n}.$$

In particular, if $G = PSL(2, \mathbb{Z})$, then

$$\frac{1}{6} \geq 2\frac{1+n}{B-n}.$$

Remark 8. If we impose the frame condition on the coherent states, the inequality

$$m_0 \geq \frac{1}{2}\frac{B-n}{1+n}$$

is an obvious consequence of Theorem 1 because the frame property is stronger than the completeness property. In the case of the Fuchsian group of dilations, it is possible to use a standard perturbation argument from wavelet theory [38], which assures that small pseudohyperbolic perturbations of the index set of a wavelet frame keep the wavelet frame property and obtain a strict inequality (this has been done in [39] and [32]). However, it is not clear if such a perturbation argument can be adapted to the case of a general Fuchsian group. We leave the problem as a question for the interested reader.

4. Conclusion

We have constructed discrete coherent states associated with the evolution of a particle under the action of a constant magnetic field when higher Landau levels are formed, first in the Euclidean and the in the hyperbolic model. Both in the higher Euclidean and the hyperbolic Landau levels, one can construct discrete coherent states by indexing the continuous ones by the discrete subgroups that reflect the symmetries of the underlying geometry. The main conclusion is that, in both cases, the completeness of the coherent states depend explicitly on the size of the fundamental domain, on the order of the Landau Level and on the intensity of the magnetic field. The analysis of the hyperbolic case is based on the properties of the automorphic form of weight m associated with the Fuchsian group G of the hyperbolic plane. If G admits an automorphic form of weight with a single zero inside D, then $S_G = \frac{2\pi}{m}$ and we can choose the automorphic form of weight $m_0 = \frac{\pi}{2S_G}$, where S_G is the area of the fundamental domain. Then, the following restriction must be imposed for the completeness of the coherent states:

$$m_0 \geq \frac{1}{2}\frac{B-n}{1+n}.$$

In terms of the area S_G of the fundamental domain

$$S_G \leq 4\pi\frac{1+n}{B-n}.$$

The methods used in this paper have their origins in several areas of mathematics, physics and signal analysis. It is not surprising that signal analysis and physics are strongly interrelated, since time–frequency (Gabor) analysis is the counterpart of the standard coherent states and time-scale (wavelet) analysis is the counterpart of affine coherent states and affine integral quantization [20]. But the arithmetic aspects connected to the hyperbolic geometry seem to have been somehow overlooked. Among the possible subgroups, only the Fuchsian group of dilations has been used in signal analysis [39], leading to the standard discretization of the half plane used in wavelet theory. We speculate that the discrete coherent states introduced in this paper may be useful in the analysis of signals, due to the variety of the discrete groups of the upper half-plane. Finally, we would subscribe to the last sentence of the conclusion of [20], since we believe it also applies to the current research: *(...) mutual irrigations between quantum physics and signal analysis deserve a lot more attention in future investigations.*

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Appendix

Proof of Theorem 2. Let $f \in L^2(\mathbb{R}^+, t^{-1}dt)$. Then we can use Proposition 1

$$W_{B,n}[f](z) = \gamma_{B,n}^{-\frac{1}{2}} \sum_{k=0}^{n} \frac{(2i)^k}{k!} \binom{2B-n-1}{n-k} y^{B-n-\frac{1}{2}+k} F^{(k)}(z) \tag{A.1}$$

where

$$F = Ber_{2(B-n)-1}[u] \in A_{2(B-n)-1}\left(\mathbb{C}^+\right).$$

The idea of the proof is the following. Using the theory of automorphic forms, we will construct a function $H \in A_{2(B-n)-1}\left(\mathbb{C}^+\right)$ vanishing at a point $\zeta_0 \in \mathbb{C}^+$ and such that, for $k = 0, \ldots, n$, $H^{(k)}$

vanishes at $G\zeta_0$, the orbit of ζ_0 under the action of G. Then set $F = H$ in (A.1); the surjectivity of $Ber_{2(B-n)-1}$ assures the existence of $f \in L^2(\mathbb{R}^+, t^{-1}dt)$ such that

$$\begin{cases} H = Ber_{2(B-n)-1}\left[f\right] \\ W_{B,n}\left[f\right](z) = 0, \quad \text{if } z \in G\zeta_0. \end{cases} \tag{A.2}$$

The function H is constructed as follows. Let F_{m_0} be a modular form of weight m_0, that is, a function analytic on the upper-half plane such that

$$F_{m_0}(z) = (cz+d)^{-2m_0} F_{m_0}\left(\frac{az+b}{cz+d}\right). \tag{A.3}$$

If G admits an automorphic form $F_{m_0}(z)$ vanishing at possible cusps and vanishing at a point $\zeta_0 \in \mathbb{C}^+$, the functional equation (A.3) implies that $F_{m_0}(z)$ vanishes at $G\zeta_0$. Since

$$(\mathrm{Im} z)^{-1}\left|\mathrm{Im}\left(\frac{az+b}{cz+d}\right)\right| = |cz+d|^{-2},$$

we have

$$\left|F_{m_0}(z)\right| = (\mathrm{Im} z)^{-m_0}\left|\mathrm{Im}\left(\frac{az+b}{cz+d}\right)\right|^{m_0}\left|F_{m_0}\left(\frac{az+b}{cz+d}\right)\right|.$$

Thus, the function

$$(\mathrm{Im} z)^{m_0}\left|F_{m_0}(z)\right| = \left|\mathrm{Im}\left(\frac{az+b}{cz+d}\right)\right|^{m_0}\left|F_{m_0}\left(\frac{az+b}{cz+d}\right)\right|$$

is non-negative and continuous in the fundamental region D. Moreover, it tends to 0 as $\mathrm{Im} z \to \infty$ (this follows from an argument using q-expansions [40, pg. 94, formula (40)]). Hence, due to its G-invariance, it is bounded in the whole upper half-plane $\mathbb{C}^+$. As a result, the automorphic form $F_{m_0}(z)$ satisfies

$$\left|F_{m_0}(z)\right| \lesssim |\mathrm{Im} z|^{-m_0}, \quad \text{for every } z \in \mathbb{C}^+. \tag{A.4}$$

The above argument is well known in number theory (for instance, it is an important step in the proof of Hecke's bound on Fourier coefficients of cusp forms [40, pg. 94]). Now we argue by contradiction, supposing that $2m_0 < \frac{B-n}{1+n}$. This implies the existence of $\epsilon > 0$ such that $m_0(n+1) = \frac{\alpha+1-\epsilon}{2}$, $\alpha = 2(B-n)-1$. Define

$$H(z) = (z+i)^{-\epsilon}\left[F_{m_0}(z)\right]^{n+1}(z)$$

and observe that $H \neq 0$ and that the derivatives $H^{(k)}(z)$ vanish at $G\zeta_0$. The estimate (A.4) then yields

$$|H(z)| \lesssim |z+i|^{-\epsilon}(\mathrm{Im} z)^{-(n+1)m_0} = |z+i|^{-\epsilon}(\mathrm{Im} z)^{-\frac{\alpha+1-\epsilon}{2}}. \tag{A.5}$$

Now let $w \in \mathbb{D}$. With the change of variables $z = i\frac{w+1}{1-w}$ one can write the integral in the unit disk. The detailed calculation follows

$$z+i = \frac{2i}{1-w}; \qquad \mathrm{Im} z = \frac{(1-|w|^2)}{|1-w|^2}; \qquad (\mathrm{Im} z)^{\alpha}\, d\mu^+(z) = \frac{(1-|w|^2)^{\alpha}}{|1-w|^{2\alpha+2}} d\mu^D(w),$$

where $d\mu^D(w = x+iy \in \mathbb{D}) = dxdy$ is the area measure in the unit disc and $d\mu^+(z)(z \in \mathbb{C}^+) = d(\mathrm{Re} z)\, d(\mathrm{Im} z)$ is the area measure in the upper-half plane. Thus (A.5) becomes

$$\left|\frac{1}{(1-w)^{\alpha+1}} H\left(i\frac{w+1}{1-w}\right)\right| \lesssim (1-|w|^2)^{-\frac{\alpha+1-\epsilon}{2}}.$$

Now, in order to show that $H \in A_{2(B-n)-1}\left(\mathbb{C}^+\right)$, the integral can be estimated as follows.

$$\begin{aligned}\int_{\mathbb{C}^+} |H(z)|^2 \,(\mathrm{Im} z)^\alpha \, d\mu^+(z) &= \int_{\mathbb{D}} \left| \frac{1}{(1-w)^{\alpha+1}} H\left(i\frac{w+1}{1-w}\right)\right|^2 (1-|w|^2)^\alpha d\mu^D(w) \\ &\lesssim \int_{\mathbb{D}} (1-|w|^2)^{-\alpha-1+\epsilon}(1-|w|^2)^\alpha d\mu^D(w) \\ &= \int_{\mathbb{D}} (1-|w|^2)^{-1+\epsilon} d\mu^D(w) < \infty.\end{aligned}$$

The last inequality can easily be verified directly by definition of area measure or using the reproducing kernel equation for Bergman spaces in the unit disc. Thus, $H(z) \in A_{\alpha=2(B-n)-1}(\mathbb{C}^+)$ vanishes on $G\zeta_0$ together with its derivatives and $H(z)$ satisfies (A.2). This is enough to finish the proof, since the existence of a nonzero $f \in L^2(\mathbb{R}^+, t^{-1}dt)$ such that $W_{B,n}[f](z)$ vanishes on the whole orbit $G\zeta_0$ leads to

$$c_{B,n}^{-\frac{1}{2}} \int_{\mathbb{R}^+} \overline{\langle t \mid z, B, n\rangle} f(t)\, \frac{dt}{t} = W_{B,n}[f](z) = 0, \quad z \in G\zeta_0,$$

and contradicts the hypothesis of $\{|g\zeta_0, B, n\rangle\}_{g\in G}$ being complete. Thus, the condition $2m_0 < \frac{B-n}{1+n}$ does not hold. As a result one must have $m_0 \geq \frac{1}{2}\frac{B-n}{1+n}$.

Proof of Corollaries 1 and 2. If $\dim \mathfrak{Hol}_G^m\left(\mathbb{C}^+\right) \geq 2$ one can find an automorphic form of weight m vanishing at a given ζ_0, using appropriate linear combinations. Moreover, if $\mathfrak{g} = 0$ and $m \geq 2$,

$$\dim \mathfrak{Hol}_G^m\left(\mathbb{C}^+\right) = 1 - 2m + 2\sum_{l=1}^{r} \left\lfloor m\left(1-\frac{1}{e_l}\right)\right\rfloor. \tag{A.6}$$

Then, comparing (A.6) with the formula (3.14) for S_G and using Poincaré's formula (3.15), gives:

$$N \geq \dim \mathfrak{Hol}_G^m\left(\mathbb{C}^+\right) - 1 \geq 1,$$

since $\dim \mathfrak{Hol}_G^m\left(\mathbb{C}^+\right) \geq 2$. Thus, the quantity

$$N(m_0) = \frac{m_0 S_G}{2\pi}$$

is minimized when $N(m_0) = 1$, leading to the explicit value of the least weight m_0:

$$m_0 = \frac{2\pi}{S_G} = \left[\sum_{l=1}^{r}\left\lfloor 1-\frac{1}{e_l}\right\rfloor - 2\right]^{-1}.$$

The statement for $G = PSL(2,\mathbb{Z})$ can be obtained by using its signature $(0, 3; 2, 3, \infty)$ or by showing directly that the area of the fundamental domain is $S_G = \frac{\pi}{3}$.

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Further reading

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