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Optimal robust fault detection

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OPTIMAL ROBUST FAULT DETECTION

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To My Parents

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Notation and Symbols

$\mathbf{A_{m\times n}}$: m-row and n-column matrix

$\mathbf{A^{-1}}$: Inverse of $\mathbf{A}$

$\mathbf{A}'$: Transpose of $\mathbf{A}$

$\mathbf{A}^*$: Complex conjugate transpose of $\mathbf{A}$

$\mathbf{I_n}$: Identity matrix of size $n \times n$

$\mathbf{0_{m,n}}$: Zero matrix of size $m \times n$

$\underline{\sigma}(\mathbf{A})$: Smallest singular value of $\mathbf{A}$

$\bar{\sigma}(\mathbf{A})$: Largest singular value of $\mathbf{A}$

$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$: Abbreviation of the transfer matrix $D + C(sI - A)^{-1}B$

$\|\boldsymbol{G}\|_2$: $\mathscr{H}_2$ norm of $\boldsymbol{G}$

$\|\boldsymbol{G}\|_\infty$: $\mathscr{H}_\infty$ norm of $\boldsymbol{G}$

$\|\boldsymbol{G}\|_-$: $\mathscr{H}_-$ index of $\boldsymbol{G}$

List of Acronyms

FDI	fault detection and isolation
LMI	linear matrix inequality
LTI	linear time invariant
LCF	left coprime factorization
ARE	algebraic Riccati equation

Abstract

This dissertation gives complete, analytic, and optimal solutions to several robust fault detection problems for both continuous and discrete linear systems that have been considered in the research community in the last twenty years. It is shown that several well-recognized robust fault detection problems, such as $\mathscr{H}_-/\mathscr{H}_\infty$, $\mathscr{H}_2/\mathscr{H}_\infty$ and $\mathscr{H}_\infty/\mathscr{H}_\infty$ problems, have a very simple optimal solution in an observer form by solving a standard algebraic Riccati equation. The optimal solutions to some other robust fault detection problems, such as $\mathscr{H}_-/\mathscr{H}_2$ and $\mathscr{H}_2/\mathscr{H}_2$ problems are also given. In addition, it is shown that some well-studied and seeming sensible optimization criteria for fault detection filter design could lead to (optimal but) useless fault detection filter designs.

Chapter 1

Introduction

This chapter gives a brief introduction to robust fault detection problems and the motivation for studying them. Some basic concepts and important terminologies in this area are provided. Previous works in the literature are reviewed and an overview of this dissertation is given.

1.1 Fault Detection for Modern Dynamic Systems

During the past several decades, modern dynamic systems have become very complex with large numbers of components and functional units. It is not realistic to assume that all these components and units can always work perfectly well, under all kinds of environmental changes and conditions. The malfunctions of these components and units, for example, can be measurement error of sensors or loss of efficiency of actuators. In this dissertation, we use the terminology fault (or failure) to represent this kind of malfunctions.

Faults in modern complex systems, such as aircrafts and petrochemical plants, can lead to very serious consequences if appropriate measure is not taken in time. These consequences may include significant performance degradation, serious damage of the system, catastrophic disaster and loss of human lives. Therefore, it is critically important to design our control system that can deal with possible faults and failures. The control system with this ability is called fault tolerant control system. There are two main approaches for fault tolerant control system design: passive approach and active approach. Passive fault tolerant control system uses one fixed design to accommodate all possible faults and failures. Since there may be many different faults in a system, the design of passive fault tolerant control system is rather difficult or even impossible in some cases. On the contrary, active fault tolerant control system takes different measures corresponding to different faults and is a better way for fault tolerant design. However, active fault tolerant control requires fast diagnosis and judgement after fault occurs. This fault diagnosis and judgement process, defined as fault detection, is thus a crucial issue and the main topic to be addressed in this dissertation.

1.2 Model-based Fault Detection

Most fault detection techniques and approaches can be fitted in two categories: model-free fault detection and model-based fault detection. Model-free approach detects fault without knowing system model information. In general, this approach employs data driven techniques such as Artificial Neural Network and Data Mining, and uses large amount of system measurements as training data to develop fault detection logic. Model-free fault detection can be effective in many cases, especially when model information is not available, but has relatively high computational complexity and is not suitable for online fault detection. On the other hand, model-based approach makes use of priori information of mathematic system model to detect fault. When system model information is available, model-based approaches are generally faster and more accurate than model-free ones. In the last twenty some years, fault diagnosis of dynamic systems has received much attention and significant progress has been made in searching for both data-driven and model-based diagnosis techniques, see [4, 7, 29, 30, 37, 38] and the references therein.

1.3 Robust Fault Detection Problem

The objective of model-based fault detection is to design a detection mechanism that generates fault indicating signals. These signals, called residual signals, are compared with given thresholds to judge whether a fault occurs or not. For this purpose, many model-based fault detection techniques have been studied in the last two decades [4]. One of the particular interesting techniques among all the model-based techniques is observer-based fault detection filter design [7]. Observer-based fault detection filter is not only easy to implement in practical systems, but also has been shown in many theoretical studies and applications that suitably designed observer-based fault detection filters can be very effective in detecting sensors, actuators, and system components faults.

Since known/unknown disturbances, noise, and model uncertainties are unavoidable for any practical systems, it is essential in the design of any fault detection filter to take these effects into consideration so that fault detection can be done reliably and robustly. Nevertheless, finding systematic design methods for systems subject to unknown disturbances and model uncertainties have proven to be difficult. Many robust filter design techniques, such as $\mathscr{H}_\infty$ optimization, LMI, and μ design techniques, have been applied to fault detection filter design with limited success [6, 18, 32, 33, 46]. The reason is that a fault detection filter design is really a multi-objective design task. It needs not only rejecting disturbance, noise and being insensitive to model uncertainties, but also being as sensitive as possible to potential faults so that early detection of faults is possible. Unfortunately, these two design objectives are almost always conflicting with each other. Hence a design tradeoff between these two objectives is unavoidable and needs to be addressed explicitly in the design process. To do that, some suitable design criteria for both objectives have to be defined. It has been widely accepted in the field that $\mathscr{H}_2$ norm and $\mathscr{H}_\infty$ norm of the transfer matrix from disturbances to fault detection residuals are good candidates for measuring up the disturbance rejection capability of a fault detection system. In some cases, $\mathscr{H}_2$ norm of the transfer matrix from faults to fault detection residual signals is also suitable for evaluating the fault detection system's sensitivity to faults. It has also been recognized that the $\mathscr{H}_-$ index (which is used to be called $\mathscr{H}_-$ norm too), first introduced by Hou and Patten [12] and further extended by Liu et al [22], seems to be a very appropriate measure of the fault detection sensitivity [4, 7, 30]. With such defined performance objectives, several fault detection design problems have been formulated as multi-objective optimization problems by minimizing the effects of disturbances and maximizing the fault sensitivity, for example, $\mathscr{H}_-/\mathscr{H}_\infty$ problem, $\mathscr{H}_\infty/\mathscr{H}_\infty$ problem, $\mathscr{H}_2/\mathscr{H}_\infty$ problem, $\mathscr{H}_-/\mathscr{H}_2$ problem, and $\mathscr{H}_2/\mathscr{H}_2$ problem. In particular, the $\mathscr{H}_-/\mathscr{H}_\infty$ problem has attracted a great deal of attention recently, [9, 10, 15, 22, 31, 34–36, 41]. However, most of the results obtained in the existing literature are either conservative or complicate to apply. Furthermore, they are usually not guaranteed to be optimal. A notable exception is the result by Ding et al [5], where optimal solutions to some formations of $\mathscr{H}_-/\mathscr{H}_\infty$ and $\mathscr{H}_\infty/\mathscr{H}_\infty$ problems are given.

Since most (continuous) dynamical systems are nowadays controlled by digital devices, it is also important to understand those theoretical development in the digital (sampled-data) setting. Furthermore, it has been shown in [13] that sample-data fault detection problem can be converted to equivalent discrete time detection problem using certain discretization method and thus discrete time fault detection is of great importance and most nature for modern digital implementation. There are significant amount of works addressing discrete time fault detection problem using Kalman filter related techniques [2, 16, 39]. Similar to continuous time fault detection, robustness in discrete time fault detection is also a very important issue but difficult to handle. Many robust filter design techniques, such as $\mathscr{H}_\infty$ optimization, LMI, parity space, and eigen-structure assignment, have been applied to discrete time fault detection filter design with limited success since we have similar multi-objective design problem like in continuous case [21, 28, 41, 45]. With $\mathscr{H}_2$ norm, $\mathscr{H}_\infty$ norm and $\mathscr{H}_-$ index similarly defined for discrete time systems, several discrete time fault detection design problems can be formulated as multiple objective optimization problems by minimizing the effects of disturbances and maximizing the fault sensitivity, for example, $\mathscr{H}_-/\mathscr{H}_\infty$ problem, $\mathscr{H}_\infty/\mathscr{H}_\infty$ problem, $\mathscr{H}_2/\mathscr{H}_\infty$ problem, $\mathscr{H}_-/\mathscr{H}_2$ problem, and $\mathscr{H}_2/\mathscr{H}_2$ problem. These problems have been studied intensively, [13–15, 17, 34, 42, 43]. Like the continuous problems, most of the results obtained in the existing literature are conservative to some extend and they are usually not optimal. However, an optimal solution to $\mathscr{H}_\infty/\mathscr{H}_\infty$ problem for linear discrete time periodic systems is given recently by Zhang et al [44].

1.4 Dissertation Overview

In this dissertation we propose a new technique to solve the above mentioned problems. In fact, we shall give complete solutions to all robust fault detection problems mentioned above, in both the continuous time and discrete time cases. For those problems that optimal solutions exist, we provide analytic and optimal solutions to them. It turns out that our solutions are surprising simple once the problems are suitably formulated. For those problems that have no analytic optimal solutions, we also derived effective algorithms to get good approximate solutions.

This dissertation is organized as follows: Chapter 2 introduces the notations and summarizes some key facts that will be used in the later parts. Chapter 3 gives the solutions to continuous time fault detection problems. Analytic and optimal solutions to $\mathscr{H}_-/\mathscr{H}_\infty$ problem, $\mathscr{H}_2/\mathscr{H}_\infty$ problem and $\mathscr{H}_\infty/\mathscr{H}_\infty$ problem are given in this chapter. It is also shown that no rational optimal solutions to $\mathscr{H}_-/\mathscr{H}_2$ problem and $\mathscr{H}_2/\mathscr{H}_2$ problem exists and effective approaches of finding good approximate solutions are provided. Chapter 4 gives the solutions to discrete time fault detection problems. We give analytic and optimal solutions to discrete $\mathscr{H}_-/\mathscr{H}_\infty$ problem, $\mathscr{H}_2/\mathscr{H}_\infty$ problem and $\mathscr{H}_\infty/\mathscr{H}_\infty$ problem. Surprisingly, we are able to give an optimal analytic solution to one case of discrete $\mathscr{H}_-/\mathscr{H}_2$ problem, while the analytic solution for corresponding continuous problem doesn't exist. Effective approaches for finding good approximate solutions to discrete $\mathscr{H}_2/\mathscr{H}_2$ problem and the other cases of discrete $\mathscr{H}_-/\mathscr{H}_2$ problem are also given in this chapter. Some numerical examples of our fault detection designs and comparison with existing results are shown in Chapter 5. Finally, some conclusions are given in Chapter 6.

Chapter 2

Preliminary Results

In this chapter some important notations and preliminary results are given. We first introduce the notations used in this dissertation in section 2.1. After that, important definitions such as transfer matrix and norms are defined in section 2.2 and 2.3. Some useful facts about singular value inequality, Riccati equation and transfer matrix factorization are presented in section 2.4. Since these preliminary results are well-known and contained in standard robust control theory, we state the results directly without giving proofs.

2.1 Matrix Notations

The set of m by n real (complex) matrices is denoted as $\mathscr{R}^{m\times n}$ ($\mathscr{C}^{m\times n}$). For a matrix $A \in \mathscr{C}^{m\times n}$ we use A' to denote its transpose and A^* for its complex conjugate transpose. In the case of $m = n$ this matrix is called a square matrix. For a Hermitian matrix $A = A' \in \mathscr{C}^{n\times n}$, $\bar{\lambda}(A)$ represents the largest eigenvalue of A and $\underline{\lambda}(A)$ represents the smallest eigenvalue of A. For any $A \in \mathscr{C}^{m\times n}$, $\bar{\sigma}(A) = \sqrt{\bar{\lambda}(AA')} = \sqrt{\bar{\lambda}(A'A)}$ denotes the largest singular value of A and $\underline{\sigma}(A) = \sqrt{\underline{\lambda}(AA')}$ ($\sqrt{\underline{\lambda}(A'A)}$) denotes the smallest singular value of A if $m \leq n$ ($m \geq n$). A Hermitian matrix A is said to be positive semi-definite, i.e., $A \geq 0$, if $\mathrm{Re}[x'Ax] \geq 0$ for any vector x. For $A \geq 0$, $A^{\frac{1}{2}}$ is a matrix such that $A^{\frac{1}{2}} \times A^{\frac{1}{2}} = A$. The $n \times n$ identity matrix is denoted as I_n and the $m \times n$ zero matrix is denoted as $0_{m,n}$, with the subscripts dropped if they can be inferred from context.

2.2 Transfer Matrix

2.2.1 Continuous Time Case

We use $\mathscr{RL}_\infty^{\mathrm{m}\times\mathrm{n}}$ to denote the set of all $m \times n$ real rational proper transfer matrices with no poles on the imaginary axis. The superscripts for dimensions will usually be dropped when they are either not important or clear from context. $\mathscr{RH}_\infty$ is a subset of $\mathscr{RL}_\infty$ with all stable transfer matrices. Similarly $\mathscr{RH}_2$ is the set of all real rational strictly proper stable transfer matrices. Transfer functions and Laplace transforms of signals are represented using bold characters and sometimes in dependence of the Laplace variable s. A state space realization of a transfer matrix $\boldsymbol{G}(s)$ is denoted as

$$\boldsymbol{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

such that $\boldsymbol{G}(s) = D + C(sI - A)^{-1}B$. Let $\boldsymbol{G}^\sim(s) := \boldsymbol{G}(-s)^T$ be the para-Hermitian complex conjugate transpose of $\boldsymbol{G}$ and $\boldsymbol{G}^{-1}(s)$ be the inverse of $\boldsymbol{G}$ if $\boldsymbol{G}(s)$ is square and invertible. Now suppose $\boldsymbol{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is square and D is nonsingular, then we have from [47]

$$\boldsymbol{G}^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right].$$

2.2.2 Discrete Time Case

In discrete time case, $\mathscr{R}\mathscr{L}_2^{\mathrm{m}\times\mathrm{n}}$ is used to denote the set of $m \times n$ real rational proper transfer matrices with no poles on the unit circle. The superscripts for dimensions will usually be dropped when they are either not important or clear from context. It turns out in discrete time, $\mathscr{R}\mathscr{H}_\infty$ is the same as $\mathscr{R}\mathscr{L}_2$, which consists of all stable proper transfer matrices.

Discrete transfer matrices and $\mathcal{Z}$-transforms of signals are represented using bold characters and sometimes in dependence of the variable z. A state space realization of a transfer matrix $\boldsymbol{G}(z)$ is denoted as

$$\boldsymbol{G}(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

such that $\boldsymbol{G}(z) = D + C(zI - A)^{-1}B$.

We define $\boldsymbol{G}^{\sim}(z) := \boldsymbol{G}^T(z^{-1})$ and denote $\boldsymbol{G}^{-1}(z)$ as the inverse of $\boldsymbol{G}$ if $\boldsymbol{G}(z)$ is square and invertible. Now suppose $\boldsymbol{G}(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is square and D is nonsingular, then we have from [47]

$$\boldsymbol{G}^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right].$$

2.3 Definitions of Norms

2.3.1 Continuous Time Case

For $\boldsymbol{G} \in \mathscr{R}\mathscr{H}_2$ we define the $\mathscr{H}_2$ norm of $\boldsymbol{G}$ as

$$\|\boldsymbol{G}\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Trace}\{\boldsymbol{G}^{\sim}(\mathrm{j}\omega)\boldsymbol{G}(\mathrm{j}\omega)\}\mathrm{d}\omega}.$$

For $\boldsymbol{G} \in \mathscr{R}\mathscr{H}_\infty$ we define the $\mathscr{H}_\infty$ norm of $\boldsymbol{G}$ as

$$\|\boldsymbol{G}\|_\infty = \sup_{\omega\in\mathscr{R}} \bar{\sigma}(\boldsymbol{G}(j\omega)).$$

Similarly the $\mathscr{H}_-$ index of $\boldsymbol{G}$ over all frequency is defined as [31]

$$\|\boldsymbol{G}\|_-^{[\infty]} = \inf_{\omega \in \mathscr{R}} \underline{\sigma}(\boldsymbol{G}(j\omega)).$$

The $\mathscr{H}_-$ index of $\boldsymbol{G}$ over a finite frequency range $[\omega_1, \omega_2]$ is defined as [22]

$$\|\boldsymbol{G}\|_-^{[\omega_1,\omega_2]} = \inf_{\omega \in [\omega_1,\omega_2]} \underline{\sigma}(\boldsymbol{G}(j\omega)).$$

In particular the $\mathscr{H}_-$ index defined at zero frequency [12] is

$$\|\boldsymbol{G}\|_-^{[0]} = \underline{\sigma}(\boldsymbol{G}(0)).$$

If no superscript is added to the $\mathscr{H}_-$ symbol, such as $\|\boldsymbol{G}\|_-$, then it represents all possible $\mathscr{H}_-$ definitions.

It should be noted that $\mathscr{H}_-$ index is sometimes called $\mathscr{H}_-$ norm in the literature although it does not satisfy the property of a norm.

2.3.2 Discrete Time Case

For $\boldsymbol{G}(z) \in \mathscr{R}\mathscr{H}_2$ we define the $\mathscr{H}_2$ norm of $\boldsymbol{G}$ as

$$\|\boldsymbol{G}\|_2 = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Trace}[\boldsymbol{G}^{\sim}(e^{j\theta})\boldsymbol{G}(e^{j\theta})]d\theta}.$$

For $\boldsymbol{G} \in \mathscr{R}\mathscr{H}_\infty$ we define the $\mathscr{H}_\infty$ norm of $\boldsymbol{G}$ as

$$\|\boldsymbol{G}\|_\infty = \sup_{\theta \in [0,2\pi]} \bar{\sigma}[\boldsymbol{G}(e^{j\theta})].$$

Similar to the $\mathscr{H}_-$ definitions of continuous time system, we define the $\mathscr{H}_-$ index of a discrete transfer matrix $\boldsymbol{G}$ on the whole unit circle as

$$\|\boldsymbol{G}\|_-^{[0,2\pi]} = \inf_{\theta \in [0,2\pi]} \underline{\sigma}(\boldsymbol{G}(e^{j\theta})).$$

The $\mathscr{H}_-$ index of $\boldsymbol{G}$ over a finite frequency range $[\theta_1, \theta_2]$ is defined as

$$\|\boldsymbol{G}\|_-^{[\theta_1,\theta_2]} = \inf_{\theta\in[\theta_1,\theta_2]} \underline{\sigma}(\boldsymbol{G}(e^{j\theta})).$$

In particular the $\mathscr{H}_-$ index defined at $\theta = 0$ is

$$\|\boldsymbol{G}\|_-^{[0]} = \underline{\sigma}(\boldsymbol{G}(1)).$$

If no superscript is added to the $\mathscr{H}_-$ symbol, such as $\|\boldsymbol{G}\|_-$, then it represents all possible $\mathscr{H}_-$ definitions.

2.4 Preliminary Results

It is easy to show that we have the following result by the definition of matrix singular value [11].

Lemma 1 *Let $A \in \mathscr{C}^{m\times n}$ and $B \in \mathscr{C}^{n\times p}$ be two matrices with appropriate dimensions, then $\underline{\sigma}(AB) \leq \bar{\sigma}(A)\underline{\sigma}(B)$.*

The following equations are important in this dissertation and can be found in [48].

Let A, Q, and R be real $n \times n$ matrices with Q and R symmetric. Then a (continuous) algebraic Riccati equation is the following matrix equation:

$$A'X + XA + XRX + Q = 0.$$

A square matrix $X = X'$ satisfying the algebraic Riccati equation is said to be a stabilization solution if $A + RX$ is stable.

Let A, Q, and G be real $n \times n$ matrices with Q and G symmetric. Then a discrete algebraic Riccati Equation is the following matrix equation:

$$A'(I + XG)^{-1}XA - X + Q = 0.$$

A square matrix $X = X'$ satisfying the discrete algebraic Riccati equation is said to be a stabilization solution if $(I + XG)^{-1}A$ is stable.

The following transfer matrix factorizations will be frequently used in this dissertation and can be found in [48].

Lemma 2 *(Left Coprime Factorization) Let $\boldsymbol{P}(s)$ or $\boldsymbol{P}(z)$ be a proper real rational transfer matrix. A left coprime factorization (LCF) of $\boldsymbol{P}$ is a factorization*

$$\boldsymbol{P} = \boldsymbol{M}^{-1}\boldsymbol{N}$$

where $\boldsymbol{N}$ and $\boldsymbol{M}$ are left-coprime over $\mathscr{RH}_\infty$. Let

$$\boldsymbol{P} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

be a detectable state-space realization of $\boldsymbol{P}$ and L be a matrix with appropriate dimensions such that $A + LC$ is stable, then a left coprime factorization of $\boldsymbol{P}$ is given by

$$\left[\begin{array}{cc} \boldsymbol{M} & \boldsymbol{N} \end{array}\right] = \left[\begin{array}{c|cc} A + LC & L & B + LD \\ \hline C & I & D \end{array}\right].$$

Lemma 3 *(Spectral Factorization: Continuous Case) Let $\boldsymbol{G}(s)$ be a proper real rational transfer matrix and*

$$\boldsymbol{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

be a detectable realization of $\boldsymbol{G}$*. Suppose* D *has full row rank and* $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ *has full row rank for all* $\omega \in \mathscr{R}$*. Let* $R := DD' > 0$ *and let* $Y \geq 0$ *be the stabilizing solution to the following algebraic Riccati equation*

$$(A - BD'R^{-1}C)X + X(A - BD'R^{-1}C)' - XC'R^{-1}CX + B(I - D'R^{-1}D)B' = 0$$

such that $A - BD'R^{-1}C - XC'R^{-1}C$ *is stable. Then the following spectral factorization holds*

$$\boldsymbol{W}\boldsymbol{W}^{\sim} = \boldsymbol{G}\boldsymbol{G}^{\sim}$$

where $\boldsymbol{W}^{-1} \in \mathscr{RH}_\infty$ *and*

$$\boldsymbol{W} = \left[\begin{array}{c|c} A & (BD' + XC')R^{-1/2} \\ \hline C & R^{1/2} \end{array}\right].$$

Lemma 4 *(Spectral Factorization: Discrete Case) Let* $\boldsymbol{G}(z)$ *be a proper real rational transfer matrix and*

$$\boldsymbol{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

be a detectable realization of $\boldsymbol{G}$*. Suppose* D *has full row rank and* $\begin{bmatrix} A - e^{j\theta} I & B \\ C & D \end{bmatrix}$ *has full row rank for all* $\theta \in [0, 2\pi]$*. Let* $P \geq 0$ *be the stabilizing solution to the following algebraic Riccati equation*

$$APA' - P - (APC' + BD')(DD' + CPC')^{-1}(DB' + CPA') + BB' = 0$$

such that $A - (APC' + BD')(DD' + CPC')^{-1}C$ *is stable and let* $R := DD' + CPC'$*. Then the following spectral factorization holds*

$$\boldsymbol{W}\boldsymbol{W}^{\sim} = \boldsymbol{G}\boldsymbol{G}^{\sim}$$

where $\boldsymbol{W}^{-1} \in \mathscr{R}\mathscr{H}_\infty$ *and*

$$\boldsymbol{W} = \left[\begin{array}{c|c} A & (APC' + BD')R^{-1/2} \\ \hline C & R^{1/2} \end{array}\right].$$

Chapter 3

Solutions to Continuous Robust Fault Detection Problems

In this chapter we derive complete solutions to robust fault detection problems in continuous time [23, 24]. In section 3.1, We first formulate the robust fault detection problems to be addressed in this chapter and a key technique that converts these problems to simple equivalent problems is presented. The analytic and optimal solutions to $\mathscr{H}_-/\mathscr{H}_\infty$ problem, $\mathscr{H}_2/\mathscr{H}_\infty$ problem and $\mathscr{H}_\infty/\mathscr{H}_\infty$ problem are provided in sections 3.2-3.3. An effective bisection algorithm for computing $\mathscr{H}_-$ index and the solutions to $\mathscr{H}_-/\mathscr{H}_2$ problem are given in section 3.4. Finally, the solution to $\mathscr{H}_2/\mathscr{H}_2$ problem is presented in section 3.5.

3.1 Problem Formulation

Consider a linear continuous time invariant system (LTI) with disturbance and possible faults as:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d d(t) + B_f f(t) \tag{3.1}$$

$$y(t) = Cx(t) + Du(t) + D_d d(t) + D_f f(t) \tag{3.2}$$

where $x(t) \in \mathscr{R}^n$ is the state vector, $y(t) \in \mathscr{R}^{n_y}$ is the output measurement, $d(t) \in \mathscr{R}^{n_d}$ represents the unknown/uncertain disturbance and measurement noise, and $f(t) \in \mathscr{R}^{n_f}$ denotes the process, sensor or actuator fault vector. $f(t)$ and $d(t)$ can be modeled as different types of signals, depending on specific situations under consideration. See Chapter 4 of [48] for some detailed discussions. Two frequently used assumptions on $d(t)$ and $f(t)$ are:

(i) unknown signal with bounded energy or bounded power;

(ii) white noise.

Different assumptions on $d(t)$ and $f(t)$ will lead to different fault detection problem formulations and the solutions to all these problems will be discussed in this dissertation.

Remark 1 *It is important to note that we assume that all knowledge about the disturbances and faults have been built into the system model. Hence there will be no further modeling (or weighting functions) on the disturbance vector d and faulty vector f.*

All coefficient matrices in equations (3.1) and (3.2) are assumed to be known constant matrices. Furthermore, the following assumptions are made:

Assumption 1 (A, C) *is detectable.*

This is a standard assumption for all fault detection problems.

Assumption 2 D_d *has full row rank.*

This means that $n_y \leq n_d$ and every measurement of the output signals is either affected by some disturbance or corrupted with some measurement noise. We argue that this assumption in some sense can be made without loss of any generality in many applications since it is impossible to take perfect measurement in any practical system and furthermore it is reasonable to assume that the measurement noises are independent of each other. So it is reasonable to assume that D_d has full row rank. In the case of some simplified model where D_d does not have full row rank, we can simply add some columns to make it full row rank. For example, suppose D_d is not full row rank, then let

$$\tilde{d} = \begin{bmatrix} d \\ d_\epsilon \end{bmatrix}, \; \tilde{B}_d = \begin{bmatrix} B_d & 0_{n \times n_y} \end{bmatrix}, \; \tilde{D}_d = \begin{bmatrix} D_d & \epsilon I_{n_y} \end{bmatrix}$$

for a small $\epsilon > 0$. Then $\tilde{D}_d$ has full row rank.

Of course, this assumption may be restrictive in some applications where the external disturbances and measurement noises are significantly different classes of signals. Hence the disturbance and measurement noise effects cannot be measured in the same metric. Further research to extend our results in this dissertation to those cases in highly desirable and important to this field.

Assumption 3 $\begin{bmatrix} A - j\omega I & B_d \\ C & D_d \end{bmatrix}$ *has full row rank for all* $\omega \in \mathscr{R}$*. Or, equivalently, the transfer function matrix* $\boldsymbol{G}_d := \left[\begin{array}{c|c} A & B_d \\ \hline C & D_d \end{array}\right]$ *has no transmission zero on the imaginary axis.*

This assumption can be removed and the detailed procedure is given in Appendix A.

Assumption 4 $n_y \geq n_f$.

Remark 2 *This assumption will not be used explicitly in the following development. All filter design methods in this dissertation still hold if this assumption is not true. However, if* *this assumption is not true, it is possible that some combinations of faults may fall into the null space of the transfer function from fault to the measurement and may not be detectable by any filter including ours.*

Remark 3 *We want to point out that in several recent works on fault detection problems [15, 22, 34, 35], it is assumed that D_f has full column rank. We believe that this assumption is extremely restrictive. The assumption implies that measurement y contains directly the information on the fault f. In particular, this implies that D_f cannot be zero which is usually not the case when there is only actuator/system component fault and no sensor fault. Furthermore, we believe that the fault detection for sensor fault is relatively easier than that for actuator/system fault.*

By taking Laplace transform of equations (3.1) and (3.2) we have the system input/output equation

$$\boldsymbol{y} = \boldsymbol{G}_u\boldsymbol{u} + \boldsymbol{G}_d\boldsymbol{d} + \boldsymbol{G}_f\boldsymbol{f} \tag{3.3}$$

where $\boldsymbol{G}_u$, $\boldsymbol{G}_d$, and $\boldsymbol{G}_f$ are $n_y \times n_u$, $n_y \times n_d$ and $n_y \times n_f$ transfer matrices respectively and their state-space realizations are

$$\begin{bmatrix} \boldsymbol{G}_u & \boldsymbol{G}_d & \boldsymbol{G}_f \end{bmatrix} = \left[\begin{array}{c|ccc} A & B & B_d & B_f \\ \hline C & D & D_d & D_f \end{array}\right]. \tag{3.4}$$

Since the state-space realization of $\boldsymbol{G}_u$, $\boldsymbol{G}_d$ and $\boldsymbol{G}_f$ share the same A and C matrices, applying Lemma 2 we can find a LCF for the system (3.4)

$$\begin{bmatrix} \boldsymbol{G}_u & \boldsymbol{G}_d & \boldsymbol{G}_f \end{bmatrix} = \boldsymbol{M}^{-1}\begin{bmatrix} \boldsymbol{N}_u & \boldsymbol{N}_d & \boldsymbol{N}_f \end{bmatrix} \tag{3.5}$$

where

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{N}_u & \boldsymbol{N}_d & \boldsymbol{N}_f \end{bmatrix} = \left[\begin{array}{c|cccc} A+LC & L & B+LD & B_d+LD_d & B_f+LD_f \\ \hline C & I & D & D_d & D_f \end{array}\right] \tag{3.6}$$

and L is a matrix such that $A+LC$ is stable.

It has been shown in [7] that, without loss of generality, the fault detection filter can take the following general form

$$\boldsymbol{r} = \boldsymbol{Q}(\boldsymbol{M}\boldsymbol{y} - \boldsymbol{N}_u\boldsymbol{u}) = \boldsymbol{Q}\begin{bmatrix} \boldsymbol{M} & -\boldsymbol{N}_u \end{bmatrix}\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{u} \end{bmatrix} \tag{3.7}$$

where $\boldsymbol{r}$ is the residual vector for detection, $\boldsymbol{Q} \in \mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ is a free stable transfer matrix to be designed. The filter structure is shown in Figure 3.1.

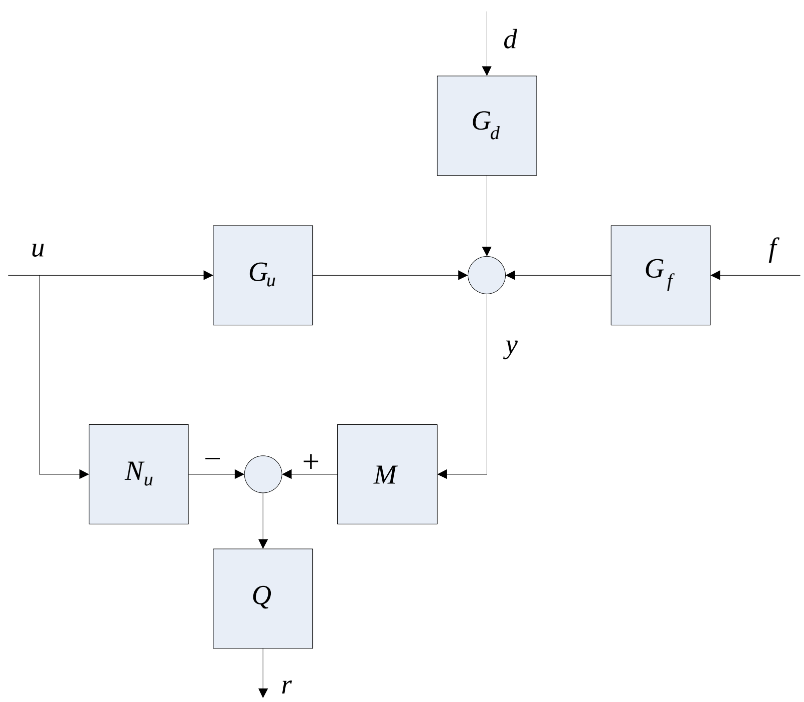

Figure 3.1: General fault detection filter structure

Remark 4 *In general* $\boldsymbol{Q}$ *can be any system in* $\mathscr{RH}_\infty^{\mathrm{p}\times\mathrm{n_y}}$ *for any* $p > 0$*. However, it is not hard to see that there is no advantage for taking* $p > n_y$ *and there may be significant* *restrictions on the filter for* $p < n_f$*. Furthermore,* $p = n_y$ *case includes all cases of* $n_y > p$ *as special cases by filling the last* $n_y - p$ *rows of* $\boldsymbol{Q}$ *with zeros. Hence there is no loss of generality in assuming* $p = n_y$*.*

Replacing $\boldsymbol{y}$ in (3.7) by right-hand sides of (3.3) and (3.5) we have

$$\boldsymbol{r} = \boldsymbol{Q}(\boldsymbol{M}\boldsymbol{y} - \boldsymbol{N}_u\boldsymbol{u}) = \boldsymbol{Q}[\boldsymbol{M}(\boldsymbol{G}_u\boldsymbol{u} + \boldsymbol{G}_d\boldsymbol{d} + \boldsymbol{G}_f\boldsymbol{f}) - \boldsymbol{N}_u\boldsymbol{u}]$$

$$= \boldsymbol{Q}[(\boldsymbol{N}_u\boldsymbol{u} + \boldsymbol{N}_d\boldsymbol{d} + \boldsymbol{N}_f\boldsymbol{f}) - \boldsymbol{N}_u\boldsymbol{u}] = \boldsymbol{Q}\boldsymbol{N}_d\boldsymbol{d} + \boldsymbol{Q}\boldsymbol{N}_f\boldsymbol{f}.$$

Thus

$$\boldsymbol{r} = \boldsymbol{Q}[\ \boldsymbol{N}_d \quad \boldsymbol{N}_f\ ]\begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{f} \end{bmatrix} = \boldsymbol{Q}\boldsymbol{N}_d\boldsymbol{d} + \boldsymbol{Q}\boldsymbol{N}_f\boldsymbol{f}. \tag{3.8}$$

Denote the transfer matrices from $\boldsymbol{d}$ and $\boldsymbol{f}$ to $\boldsymbol{r}$ by $\boldsymbol{G}_{rd}$ and $\boldsymbol{G}_{rf}$, respectively, then

$$\boldsymbol{G}_{rd} = \boldsymbol{Q}\boldsymbol{N}_d, \quad \boldsymbol{G}_{rf} = \boldsymbol{Q}\boldsymbol{N}_f. \tag{3.9}$$

As we have discussed in Chapter 1, a good fault detection filter needs to make a trade-off between two conflicting performance objectives: robustness to disturbance rejection and sensitivity to faults. To achieve good robustness to disturbance, the influence of disturbance must be minimized at the output of the residual signals. On the other hand, the residual signal should be as sensitive as possible to the faults. Since an $\mathscr{H}_-$ index is a good measurement for a transfer function's smallest gain, $\|\boldsymbol{G}_{rf}\|_-$ is a reasonable performance criterion for measuring fault detection sensitivity if $f(t)$ is modeled as unknown energy or power bounded signals. If $d(t)$ is modeled as unknown energy or power bounded signals, then the $\mathscr{H}_\infty$ norm of the corresponding transfer function is a widely accepted worst case measure and $\|\boldsymbol{G}_{rd}\|_\infty$

is a good indicator of disturbance rejection performance. On the other hand, if $d(t)$ and/or $f(t)$ are white noise, the $\mathscr{H}_2$ norms of $\boldsymbol{G}_{rd}$ and/or $\boldsymbol{G}_{rf}$ seem to be more suitable criteria. See [4, 48] for more detailed discussions and motivations on various performance measures.

We shall now formulate several fault detection filter design problems.

Problem 1 *($\mathscr{H}_-/\mathscr{H}_\infty$ Problem) Let an uncertain system be described by equations (3.1)-(3.4) and let $\gamma > 0$ be a given disturbance rejection level. Find a stable transfer matrix $\boldsymbol{Q} \in \mathscr{R}\mathscr{H}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ in (3.7)-(3.9) such that $\|\boldsymbol{G}_{rd}\|_\infty \leq \gamma$ and $\|\boldsymbol{G}_{rf}\|_-$ is maximized, i.e.*

$$\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_- : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \leq \gamma\}$$

Problem 2 *($\mathscr{H}_2/\mathscr{H}_\infty$ Problem) Let an uncertain system be described by equations (3.1)-(3.4) and let $\gamma > 0$ be a given disturbance rejection level. Find a stable transfer matrix $\boldsymbol{Q} \in \mathscr{R}\mathscr{H}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ in (3.7)-(3.9) such that $\|\boldsymbol{G}_{rd}\|_\infty \leq \gamma$ and $\|\boldsymbol{G}_{rf}\|_2$ is maximized, i.e.*

$$\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_2 : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \leq \gamma\}$$

Problem 3 *($\mathscr{H}_-/\mathscr{H}_2$ Problem) Let an uncertain system be described by equations (3.1)-(3.4) and let $\gamma > 0$ be a given disturbance rejection level. Find a stable transfer matrix $\boldsymbol{Q} \in \mathscr{R}\mathscr{H}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ in (3.7)-(3.9) such that $\|\boldsymbol{G}_{rd}\|_2 \leq \gamma$ and $\|\boldsymbol{G}_{rf}\|_-$ is maximized, i.e.*

$$\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_- : \|\boldsymbol{Q}\boldsymbol{N}_d\|_2 \leq \gamma\}$$

Problem 4 *($\mathscr{H}_2/\mathscr{H}_2$ Problem) Let an uncertain system be described by equations (3.1)-(3.4) and let $\gamma > 0$ be a given disturbance rejection level. Find a stable transfer matrix $\boldsymbol{Q} \in \mathscr{R}\mathscr{H}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ in (3.7)-(3.9) such that $\|\boldsymbol{G}_{rd}\|_2 \leq \gamma$ and $\|\boldsymbol{G}_{rf}\|_2$ is maximized, i.e.*

$$\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_2 : \|\boldsymbol{Q}\boldsymbol{N}_d\|_2 \leq \gamma\}$$

Remark 5 *Some more conventional formulations of the above problems are to optimize the following*

$$\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}} \frac{\|\boldsymbol{Q}\boldsymbol{N}_f\|_X}{\|\boldsymbol{Q}\boldsymbol{N}_d\|_Y}$$

where X and Y can be 2, ∞, or $-$. The problem of $X = Y = 2$ is a classical one and the optimal solution is available [7]. The case for $X = -, \infty$ and $Y = \infty$ has been solved in [5]. We include Problem 1 and Problem 4 here to demonstrate our unified solution framework. It is appropriate to point out here that the so-called $\mathscr{H}_\infty/\mathscr{H}_\infty$ problem

$$\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_\infty : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \leq \gamma\}$$

and any variations of the problem with $X = \infty$ (and any Y) are not very useful in practical fault detection. This is because large $\|\boldsymbol{Q}\boldsymbol{N}_f\|_\infty$ does not in general imply good sensitivity to faults. It is merely sensitive to some combinations (or directions) of faults (i.e., when faults are aligned in the direction of the largest singular vector of $\boldsymbol{Q}\boldsymbol{N}_f$) and can be extremely insensitive to other combinations (or directions) of faults (i.e., when faults are aligned in the direction of the smallest singular vector of $\boldsymbol{Q}\boldsymbol{N}_f$). Hence we shall not consider specifically the $\mathscr{H}_\infty/\mathscr{H}_\infty$ problem in this dissertation although our solutions to other cases also provide an optimal solution to this problem as will be pointed later.

Before we proceed to the solutions to the above problems, we shall first establish some preliminary results.

Lemma 5 *Suppose Assumption 3 is satisfied and let* $\boldsymbol{G}_d = \boldsymbol{M}^{-1}\boldsymbol{N}_d$ *be any left coprime factorization over* $\mathscr{RH}_\infty$*. Then* $\boldsymbol{N}_d$ *has no transmission zero on imaginary axis. Or, equivalently, for any appropriately dimensioned matrix* L*,*

$$\begin{bmatrix} A + LC - j\omega I & B_d + LD_d \\ C & D_d \end{bmatrix}$$

has full row rank for all $\omega \in \mathscr{R}$*.*

Proof The result follows by noting that

$$\begin{bmatrix} A + LC - j\omega I & B_d + LD_d \\ C & D_d \end{bmatrix} = \begin{bmatrix} I & L \\ 0 & I \end{bmatrix} \begin{bmatrix} A - j\omega I & B_d \\ C & D_d \end{bmatrix}$$

and the fact that all coprime factors have the same unstable transmission zeros [48]. □

An immediate consequence of the above result is the following spectral factorization formula.

Lemma 6 *Suppose Assumptions 1-3 are satisfied and let* $\boldsymbol{G}_d = \boldsymbol{M}^{-1}\boldsymbol{N}_d$ *be any left coprime factorization over* $\mathscr{RH}_\infty$*. Then there is a square transfer matrix* $\boldsymbol{V} \in \mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ *such that* $\boldsymbol{V}^{-1} \in \mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ *and*

$$\boldsymbol{V}\boldsymbol{V}^{\sim} = \boldsymbol{N}_d\boldsymbol{N}_d^{\sim}. \tag{3.10}$$

In particular, if a state space representation of $\boldsymbol{N}_d$ *is given as in equation (3.6), then a state space representation of* $\boldsymbol{V}$ *is given by*

$$\boldsymbol{V} = \left[\begin{array}{c|c} A + LC & (L - L_0)R_d^{1/2} \\ \hline C & R_d^{1/2} \end{array}\right] \tag{3.11}$$

with

$$\boldsymbol{V}^{-1} = \left[\begin{array}{c|c} A + L_0C & L_0 - L \\ \hline R_d^{-1/2}C & R_d^{-1/2} \end{array}\right] \tag{3.12}$$

where $R_d := D_dD_d' > 0$ *and* $Y \geq 0$ *is the stabilizing solution to the Riccati equation*

$$(A - B_dD_d'R_d^{-1}C)Y + Y(A - B_dD_d'R_d^{-1}C)' - YC'R_d^{-1}CY + B_d(I - D_d'R_d^{-1}D_d)B_d' = 0 \tag{3.13}$$

such that $A - B_dD_d'R_d^{-1}C - YC'R_d^{-1}C$ *is stable and*

$$L_0 := -(B_dD_d' + YC')R_d^{-1}. \tag{3.14}$$

Proof Since Assumptions 1-3 are satisfied, Lemma 3 and Lemma 5 can be applied to $\boldsymbol{N}_d$ to get $\boldsymbol{V}\boldsymbol{V}^{\sim} = \boldsymbol{N}_d\boldsymbol{N}_d^{\sim}$ where $Y \geq 0$ satisfies the following Riccati equation

$$A_{LD}Y + YA_{LD}' - YC'R_d^{-1}CY + (B_d + LD_d)(I - D_d'R_d^{-1}D_d)(B + LD_d)' = 0$$

and

$$A_{LD} := A + LC - (B_d + LD_d)D_d'R_d^{-1}C.$$

Since

$$A_{LD}Y + YA_{LD}' = [A + LC - (B_d + LD_d)D_d'R_d^{-1}C]Y + Y[A + LC - (B_d + LD_d)D_d'R_d^{-1}C]'$$

$$= (A - B_dD_d'R_d^{-1}C)Y + LCY - LCY + Y(A - B_dD_d'R_d^{-1}C)' + Y(LC)' - Y(LC)'$$

$$= (A - B_dD_d'R_d^{-1}C)Y + Y(A - B_dD_d'R_d^{-1}C)'$$

and

$$(B_d + LD_d)(I - D_d'R_d^{-1}D_d)(B + LD_d)'$$

$$= B_d(I - D_d'R_d^{-1}D_d)B_d' + LD_d(I - D_d'R_d^{-1}D_d)(B_d + LD_d)' + B_d(I - D_d'R_d^{-1}D_d)(LD_d)'$$

$$= B_d(I - D_d' R_d^{-1} D_d) B_d' + L D_d B_d' - L D_d B_d' + L D_d D_d' L' - L D_d D_d' L' + B_d D_d' L' - B_d D_d' L$$

$$= B_d(I - D_d' R_d^{-1} D_d) B_d',$$

the algebraic Riccati equation

$$A_{LD} Y + Y A_{LD}' - Y C' R_d^{-1} C Y + (B_d + L D_d)(I - D_d' R_d^{-1} D_d)(B + L D_d)' = 0$$

can be simplified to equation (3.13). If $\boldsymbol{N}_d$ has the state space representation given in equation (3.6), then $\boldsymbol{V}$ has the form given in equation (3.11) by Lemma 3. Therefore

$$\boldsymbol{V}^{-1} = \left[\begin{array}{c|c} A + LC - (L - L_0) R_d^{1/2} R_d^{-1/2} C & -(L - L_0) R_d^{1/2} R_d^{-1/2} \\ \hline R_d^{-1/2} C & R_d^{-1/2} \end{array}\right]$$

$$= \left[\begin{array}{c|c} A + L_0 C & L_0 - L \\ \hline R_d^{-1/2} C & R_d^{-1/2} \end{array}\right].$$

$\square$

The following Lemma is the key to the solutions of all the above problems.

Lemma 7 *Suppose Assumptions 1-3 are satisfied. Let* $\boldsymbol{V}, \boldsymbol{V}^{-1} \in \mathscr{RH}_\infty$ *be defined as in equation (3.10). Let*

$$\boldsymbol{Q} = \Psi \boldsymbol{V}^{-1}$$

for $\Psi \in \mathscr{RH}_\infty$ *and denote* $\tilde{\boldsymbol{N}}_f = \boldsymbol{V}^{-1} \boldsymbol{N}_f \in \mathscr{RH}_\infty$*. Then the fault detection Problems 1-4 are equivalent to Problems 1e-4e below, respectively:*

Problem 1e

$$\max_{\Psi \in \mathscr{RH}_\infty^{\mathrm{ny} \times \mathrm{ny}}} \{ \left\| \Psi \tilde{\boldsymbol{N}}_f \right\|_- : \|\Psi\|_\infty \leq \gamma \}$$

Problem 2e

$$\max_{\Psi\in\mathscr{RH}_\infty^{\mathrm{ny}\times\mathrm{ny}}}\{\left\|\Psi\tilde{\boldsymbol{N}}_f\right\|_2 : \|\Psi\|_\infty \le \gamma\}$$

Problem 3e

$$\max_{\Psi\in\mathscr{RH}_2^{\mathrm{ny}\times\mathrm{ny}}}\{\left\|\Psi\tilde{\boldsymbol{N}}_f\right\|_- : \|\Psi\|_2 \le \gamma\}$$

Problem 4e

$$\max_{\Psi\in\mathscr{RH}_2^{\mathrm{ny}\times\mathrm{ny}}}\{\left\|\Psi\tilde{\boldsymbol{N}}_f\right\|_2 : \|\Psi\|_2 \le \gamma\}$$

Proof We shall first show that Problem 1 and Problem 2 are equivalent to Problem 1e and Problem 2e, respectively.

Note that by Lemma 6 there exists $\boldsymbol{V} \in \mathscr{RH}_\infty$ such that $\boldsymbol{V}\boldsymbol{V}^\sim = \boldsymbol{N}_d\boldsymbol{N}_d^\sim$ and $\boldsymbol{V}^{-1} \in \mathscr{RH}_\infty$.

Therefore

$$\|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty^2 = \sup_\omega \bar{\lambda}(\boldsymbol{Q}(j\omega)\boldsymbol{N}_d(j\omega)\boldsymbol{N}_d^\sim(j\omega)\boldsymbol{Q}^\sim(j\omega))$$

$$= \sup_\omega \bar{\lambda}(\boldsymbol{Q}(j\omega)\boldsymbol{V}(j\omega)\boldsymbol{V}^\sim(j\omega)\boldsymbol{Q}^\sim(j\omega)) = \|\boldsymbol{Q}\boldsymbol{V}\|_\infty^2$$

i.e.,

$$\|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty = \|\boldsymbol{Q}\boldsymbol{V}\|_\infty .$$

We can, therefore without loss of generality, take $\boldsymbol{Q}$ in the form of $\boldsymbol{Q} = \Psi\boldsymbol{V}^{-1}$ for some $\Psi \in \mathscr{RH}_\infty$. Hence $\|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty = \|\boldsymbol{Q}\boldsymbol{V}\|_\infty = \|\Psi\|_\infty$ so that $\|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \le \gamma$ is equivalent to $\|\Psi\|_\infty \le \gamma$. Moreover, $\boldsymbol{Q}\boldsymbol{N}_f = \Psi\boldsymbol{V}^{-1}\boldsymbol{N}_f = \Psi\tilde{\boldsymbol{N}}_f$, hence Problem 1 is equivalent to Problem 1e and Problem 2 is equivalent to Problem 2e.

Next we show that Problem 3 and Problem 4 are equivalent to Problem 3e and Problem 4e, respectively. Note that in Problem 3 and Problem 4, we have $\boldsymbol{Q}\boldsymbol{N}_d \in \mathscr{R}\mathscr{H}_2$. Hence

$$\|\boldsymbol{Q}\boldsymbol{N}_d\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Trace}\{\boldsymbol{Q}(\mathrm{j}\omega)\boldsymbol{N}_{\mathrm{d}}(\mathrm{j}\omega)\boldsymbol{N}_{\mathrm{d}}^{\sim}(\mathrm{j}\omega)\boldsymbol{Q}^{\sim}(\mathrm{j}\omega)\}\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Trace}\{\boldsymbol{Q}(\mathrm{j}\omega)\boldsymbol{V}(\mathrm{j}\omega)\boldsymbol{V}^{\sim}(\mathrm{j}\omega)\boldsymbol{Q}^{\sim}(\mathrm{j}\omega)\}\mathrm{d}\omega = \|\boldsymbol{Q}\boldsymbol{V}\|_2^2$$

such that $\|\boldsymbol{Q}\boldsymbol{N}_d\|_2 = \|\boldsymbol{Q}\boldsymbol{V}\|_2$. Since $\boldsymbol{Q}\boldsymbol{V} \in \mathscr{R}\mathscr{H}_2$ and $\boldsymbol{V}, \boldsymbol{V}^{-1} \in \mathscr{R}\mathscr{H}_\infty$, we can let $\boldsymbol{Q} = \Psi\boldsymbol{V}^{-1}$ for some $\Psi \in \mathscr{R}\mathscr{H}_2$. Therefore $\|\boldsymbol{Q}\boldsymbol{N}_d\|_2 = \|\boldsymbol{Q}\boldsymbol{V}\|_2 = \|\Psi\|_2$ so that $\|\boldsymbol{Q}\boldsymbol{N}_d\|_2 \le \gamma$ is equivalent to $\|\Psi\|_2 \le \gamma$. Moreover, $\boldsymbol{Q}\boldsymbol{N}_f = \Psi\boldsymbol{V}^{-1}\boldsymbol{N}_f = \Psi\tilde{\boldsymbol{N}}_f$, so Problem 3 is equivalent to Problem 3e and Problem 4 is equivalent to Problem 4e. □

We shall provide analytic and optimal solutions for each of the above problems in the following sections.

3.2 $\mathscr{H}_-/\mathscr{H}_\infty$ Fault Detection Filter Design

In this section we give a complete solution to $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection filter design problem, i.e., Problem 1 or Problem 1e.

Theorem 1 *Suppose Assumptions 1-3 are satisfied. Let*

$$\begin{bmatrix} \boldsymbol{G}_u & \boldsymbol{G}_d & \boldsymbol{G}_f \end{bmatrix} = \boldsymbol{M}^{-1}\begin{bmatrix} \boldsymbol{N}_u & \boldsymbol{N}_d & \boldsymbol{N}_f \end{bmatrix}$$

be any left coprime factorization over $\mathscr{R}\mathscr{H}_\infty$ and let $\boldsymbol{V} \in \mathscr{R}\mathscr{H}_\infty$ be a square transfer matrix such that $\boldsymbol{V}^{-1} \in \mathscr{R}\mathscr{H}_\infty$ and $\boldsymbol{V}\boldsymbol{V}^{\sim} = \boldsymbol{N}_d\boldsymbol{N}_d^{\sim}$. Then

$$\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{ny}\times\mathrm{ny}}}\{\|\boldsymbol{Q}\boldsymbol{N}_f\|_- : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \le \gamma\} = \gamma\left\|\boldsymbol{V}^{-1}\boldsymbol{N}_f\right\|_-$$

and an optimal fault detection filter for Problem 1 is given by

$$\boldsymbol{r} = \boldsymbol{Q}_{opt} \begin{bmatrix} \boldsymbol{M} & -\boldsymbol{N}_u \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{u} \end{bmatrix}$$

where

$$\boldsymbol{Q}_{opt} = \gamma \boldsymbol{V}^{-1}.$$

Proof Note that by Lemma 7, we only need to solve Problem 1e:

$$\max_{\Psi \in \mathscr{R}\mathscr{H}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{ \left\| \Psi \tilde{\boldsymbol{N}}_f \right\|_- : \|\Psi\|_\infty \leq \gamma \}.$$

From Lemma 1 we know that for every frequency $\omega \in \mathscr{R}$,

$$\underline{\sigma}(\Psi(j\omega)\tilde{\boldsymbol{N}}_f(j\omega)) \leq \bar{\sigma}(\Psi(j\omega))\underline{\sigma}(\tilde{\boldsymbol{N}}_f(j\omega))$$

so that

$$\left\| \Psi \tilde{\boldsymbol{N}}_f \right\|_- \leq \|\Psi\|_\infty \left\| \tilde{\boldsymbol{N}}_f \right\|_- \leq \gamma \left\| \tilde{\boldsymbol{N}}_f \right\|_- .$$

By letting $\Psi = \gamma I$, we have $\|\Psi\|_\infty = \gamma$ and $\left\| \Psi \tilde{\boldsymbol{N}}_f \right\|_- = \gamma \left\| \tilde{\boldsymbol{N}}_f \right\|_-$, which means that $\Psi = \gamma I$ is an optimal solution achieving the maximum. □

Remark 6 *The optimal fault detection filter given in Theorem 1 does not depend on B_f and D_f matrices. But this does not imply that the filter does not depend on $\boldsymbol{G}_f$ (i.e., the fault in the system) since A and C matrices already include parts of $\boldsymbol{G}_f$. On the other hand, $\|\boldsymbol{Q}\boldsymbol{N}_f\|_-^{[\infty]}$ depends critically on the rank of D_f. For example, if D_f is not full column rank, then $\|\boldsymbol{Q}\boldsymbol{N}_f\|_-^{[\infty]} = 0$ for any $\boldsymbol{Q}$. Hence in this case, the optimization does not make any sense while other $\mathscr{H}_-$ indices still make sense. In general, we believe $\|\boldsymbol{Q}\boldsymbol{N}_f\|_-^{[\infty]}$ is not a suitable design criterion for fault detection.*

Remark 7 *It is also noted that if* $\boldsymbol{N}_f(j\omega_0)$ *loses column rank at any frequency* $\omega_0 \in [\omega_1, \omega_2]$, *then* $\left\| \boldsymbol{V}^{-1} \boldsymbol{N}_f \right\|_-^{[\omega_1,\omega_2]} = 0$. *This would imply that the optimization of Problem 1 is meaningless. However, this does not imply that our filter is useless. In fact, our filter is still the best possible filter in the sense of maximizing* $\underline{\sigma}(\boldsymbol{Q}(j\omega)\boldsymbol{N}_f(j\omega))$ *for all other frequencies. It just implies that no filter can detect the faulty signal at that frequency* ω_0.

Remark 8 *Notice that the solution given in the above theorem does not depend on the specific definition of* $\mathscr{H}_-$ *index. Hence the solution provided here is an optimal solution for all* $\mathscr{H}_-$ *indices. However, it should be pointed out that this optimal filter is not the only optimal solution for some* $\mathscr{H}_-$ *index criterion. For example, let* $\boldsymbol{Q} = \gamma \boldsymbol{L}(s)\, \boldsymbol{V}^{-1}$ *where* $\boldsymbol{L} \in \mathscr{R}\mathscr{H}_\infty$ *is a low pass filter so that* $\|\boldsymbol{L}(s)\|_\infty = 1$ *and* $\boldsymbol{L}(0) = I$. *Then this* $\boldsymbol{Q}$ *is also an optimal solution for*

$$\max_{\boldsymbol{Q} \in \mathscr{R}\mathscr{H}_\infty^{\mathrm{ny} \times \mathrm{ny}}} \{ \|\boldsymbol{Q}\boldsymbol{N}_f\|_-^{[0]} : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \leq \gamma \}$$

even though this is obviously a bad fault detection filter because the low pass filter $L(s)$ *would make the residuals much less sensitive to faults.*

Note also that the solution given in the above theorem is completely general and is not dependent on specific state space representation of those coprime factorizations and spectral factorizations, which may be necessary in some fault tolerant control application [27, 49]. This point can be shown by a simple example.

Consider a system model

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t) + d(t) + f(t) \qquad (3.15) \\ y(t) &= x(t) + d(t). \end{aligned}$$

Then

$$\boldsymbol{G}_u = \frac{1}{s+1}, \boldsymbol{G}_d = \frac{s+2}{s+1}, \boldsymbol{G}_f = \frac{1}{s+1}.$$

One left coprime factorization for these transfer functions is given by

$$\begin{bmatrix} \boldsymbol{G}_u & \boldsymbol{G}_d & \boldsymbol{G}_f \end{bmatrix} = \boldsymbol{M}^{-1} \begin{bmatrix} \boldsymbol{N}_u & \boldsymbol{N}_d & \boldsymbol{N}_f \end{bmatrix} = \frac{s+3}{s+1} \begin{bmatrix} \frac{1}{s+3} & \frac{s+2}{s+3} & \frac{1}{s+3} \end{bmatrix}.$$

Pick $\boldsymbol{V} = \boldsymbol{N}_d$, from Theorem 1 we have

$$\max_{\boldsymbol{Q} \in \mathscr{RH}_\infty^{\mathrm{ny} \times \mathrm{ny}}} \{ \|\boldsymbol{Q}\boldsymbol{N}_f\|_- : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \leq \gamma \}$$

$$= \gamma \left\| \boldsymbol{V}^{-1}\boldsymbol{N}_f \right\|_- = \gamma \left\| \frac{s+3}{s+2} \frac{1}{s+3} \right\|_- = \gamma \left\| \frac{1}{s+2} \right\|_- .$$

If we choose another left coprime factorization

$$\begin{bmatrix} \boldsymbol{G}_u & \boldsymbol{G}_d & \boldsymbol{G}_f \end{bmatrix} = \boldsymbol{M}^{-1} \begin{bmatrix} \boldsymbol{N}_u & \boldsymbol{N}_d & \boldsymbol{N}_f \end{bmatrix} = \frac{s+4}{s+1} \begin{bmatrix} \frac{1}{s+4} & \frac{s+2}{s+4} & \frac{1}{s+4} \end{bmatrix},$$

from Theorem 1 we also have

$$\max_{\boldsymbol{Q} \in \mathscr{RH}_\infty^{\mathrm{ny} \times \mathrm{ny}}} \{ \|\boldsymbol{Q}\boldsymbol{N}_f\|_- : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \leq \gamma \}$$

$$= \gamma \left\| \boldsymbol{V}^{-1}\boldsymbol{N}_f \right\|_- = \gamma \left\| \frac{s+4}{s+2} \frac{1}{s+4} \right\|_- = \gamma \left\| \frac{1}{s+2} \right\|_- .$$

The solutions we obtained using different factorizations are exactly the same.

On the other hand, if those specific state space coprime and spectral factorizations in the previous sections are used, the optimal filter can be written in a very simple form.

Theorem 2 *Suppose Assumptions 1-3 are satisfied. Let* $R_d := D_d D_d' > 0$ *and let* $Y \geq 0$ *be the stabilizing solution to the Riccati equation*

$$(A - B_d D_d' R_d^{-1} C)Y + Y(A - B_d D_d' R_d^{-1} C)' - YC' R_d^{-1} CY + B_d(I - D_d' R_d^{-1} D_d)B_d' = 0 \quad (3.16)$$

such that $A - B_d D_d' R_d^{-1} C - Y C' R_d^{-1} C$ *is stable. Define*

$$L_0 = -(B_d D_d' + Y C') R_d^{-1}.$$

Then

$$\max_{\boldsymbol{Q} \in \mathscr{R}\mathscr{H}_\infty^{\mathrm{ny} \times \mathrm{ny}}} \{ \| \boldsymbol{Q} \boldsymbol{N}_f \|_- : \| \boldsymbol{Q} \boldsymbol{N}_d \|_\infty \leq \gamma \} = \gamma \left\| \boldsymbol{V}^{-1} \boldsymbol{N}_f \right\|_-$$

and an optimal $\mathscr{H}_-/\mathscr{H}_\infty$ *fault detection filter has the following state space representation*

$$\boldsymbol{r} = \boldsymbol{Q}_{opt} \begin{bmatrix} \boldsymbol{M} & -\boldsymbol{N}_u \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{u} \end{bmatrix}$$

where

$$\boldsymbol{Q}_{opt} \begin{bmatrix} \boldsymbol{M} & -\boldsymbol{N}_u \end{bmatrix} = \gamma \left[\begin{array}{c|cc} A + L_0 C & -L_0 & B + L_0 D \\ \hline -R_d^{-1/2} C & R_d^{-1/2} & -R_d^{-1/2} D \end{array} \right]$$

and

$$\boldsymbol{V}^{-1} \boldsymbol{N}_f = \left[\begin{array}{c|c} A + L_0 C & B_f + L_0 D_f \\ \hline R_d^{-1/2} C & R_d^{-1/2} D_f \end{array} \right].$$

In other words, the optimal $\mathscr{H}_-/\mathscr{H}_\infty$ *fault detection filter is the following observer:*

$$\dot{\hat{x}}(t) = (A + L_0 C)\hat{x}(t) - L_0 y(t) + (B + L_0 D) u(t)$$

$$r(t) = \gamma R_d^{-1/2} \left(y(t) - C\hat{x} - D u(t) \right).$$

Proof Note that

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{N}_u \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & L & B + LD \\ \hline C & I & D \end{array} \right]$$

where L is a matrix with appropriate dimensions such that $A + LC$ is stable. Note also from Theorem 1 that

$$\boldsymbol{Q}_{opt} = \gamma \boldsymbol{V}^{-1} = \gamma \left[\begin{array}{c|c} A + L_0 C & L_0 - L \\ \hline R_d^{-1/2} C & R_d^{-1/2} \end{array} \right].$$

Then

$$\boldsymbol{Q}_{opt}\left[\begin{array}{cc}\boldsymbol{M} & -\boldsymbol{N}_u\end{array}\right] = \gamma\left[\begin{array}{c|c} A+L_0C & L_0-L \\ \hline R_d^{-1/2}C & R_d^{-1/2}\end{array}\right]\left[\begin{array}{c|cc} A+LC & L & -(B+LD) \\ \hline C & I & -D\end{array}\right]$$

$$= \gamma\left[\begin{array}{cc|cc} A+L_0C & L_0C-LC & L_0-L & -(L_0-L)D \\ 0 & A+LC & L & -(B+LD) \\ \hline R_d^{-1/2}C & R_d^{-1/2}C & R_d^{-1/2} & -R_d^{-1/2}D\end{array}\right]$$

$$= \gamma\left[\begin{array}{cc|cc} A+L_0C & 0 & L_0 & -(B+L_0D) \\ 0 & A+LC & L & -(B+LD) \\ \hline R^{-1/2}C & 0 & R_d^{-1/2} & -R_d^{-1/2}D\end{array}\right]$$

$$= \gamma\left[\begin{array}{c|cc} A+L_0C & L_0 & -(B+L_0D) \\ \hline R_d^{-1/2}C & R_d^{-1/2} & -R_d^{-1/2}D\end{array}\right] = \gamma\left[\begin{array}{c|cc} A+L_0C & -L_0 & B+L_0D \\ \hline -R_d^{-1/2}C & R_d^{-1/2} & -R_d^{-1/2}D\end{array}\right].$$

Similarly, we have

$$\boldsymbol{V}^{-1}\boldsymbol{N}_f = \left[\begin{array}{c|c} A+L_0C & L_0-L \\ \hline R_d^{-1/2}C & R_d^{-1/2}\end{array}\right]\left[\begin{array}{c|c} A+LC & B_f+LD_f \\ \hline C & D_f\end{array}\right]$$

$$= \left[\begin{array}{cc|c} A+L_0C & (L_0-L)C & (L_0-L)D_f \\ 0 & A+LC & B_f+LD_f \\ \hline R_d^{-1/2}C & R_d^{-1/2}C & R_d^{-1/2}D_f\end{array}\right]$$

$$= \left[\begin{array}{cc|c} A+L_0C & 0 & B_f+L_0D_f \\ 0 & A+LC & B_f+LD_f \\ \hline R_d^{-1/2}C & 0 & R_d^{-1/2}D_f\end{array}\right] = \left[\begin{array}{c|c} A+L_0C & B_f+L_0D_f \\ \hline R_d^{-1/2}C & R_d^{-1/2}D_f\end{array}\right].$$

□

Remark 9 *The optimal solution in Theorem 2 is a particular example of the solution set given in Theorem 1 and is independent of the choice of L matrix. It turns out this solution* *is exactly the same as the one given by Ding et al in [5] under the following optimization criterion:*

$$\max_{\boldsymbol{Q}} \frac{\|\boldsymbol{Q}\boldsymbol{N}_f\|_-}{\|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty}.$$

Remark 10 *It is easy to see that our optimal filter given in Theorem 1 and Theorem 2 is also optimal for the so-called $\mathscr{H}_\infty/\mathscr{H}_\infty$ problem:*

$$\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_\infty : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \leq \gamma\}.$$

3.3 $\mathscr{H}_2/\mathscr{H}_\infty$ Fault Detection Filter Design

In this section we give an optimal solution to $\mathscr{H}_2/\mathscr{H}_\infty$ problem stated in section 3.1 as Problem 2. Similar to the solution to $\mathscr{H}_-/\mathscr{H}_\infty$ problem given in Theorem 1 and Theorem 2, we have the following parallel results for $\mathscr{H}_2/\mathscr{H}_\infty$ problem.

Theorem 3 *Suppose Assumptions 1-3 are satisfied. Let*

$$\begin{bmatrix} \boldsymbol{G}_u & \boldsymbol{G}_d & \boldsymbol{G}_f \end{bmatrix} = \boldsymbol{M}^{-1} \begin{bmatrix} \boldsymbol{N}_u & \boldsymbol{N}_d & \boldsymbol{N}_f \end{bmatrix}$$

be any left coprime factorization over $\mathscr{R}\mathscr{H}_\infty$ and let $\boldsymbol{V} \in \mathscr{R}\mathscr{H}_\infty$ be a square transfer matrix such that $\boldsymbol{V}^{-1} \in \mathscr{R}\mathscr{H}_\infty$ and $\boldsymbol{V}\boldsymbol{V}^\sim = \boldsymbol{N}_d\boldsymbol{N}_d^\sim$. Then

$$\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_2 : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \leq \gamma\} = \gamma \left\|\boldsymbol{V}^{-1}\boldsymbol{N}_f\right\|_2$$

and the optimal fault detection filters for Problem 1 given in Theorem 1 and Theorem 2 are also the optimal filters for this problem.

Proof Note that by Lemma 7, we only need to solve Problem 2e:

$$\max_{\Psi \in \mathscr{RH}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{\left\|\Psi \tilde{\boldsymbol{N}}_f\right\|_2 : \|\Psi\|_\infty \leq \gamma\}.$$

Note that

$$\left\|\Psi \tilde{\boldsymbol{N}}_f\right\|_2 \leq \|\Psi\|_\infty \left\|\tilde{\boldsymbol{N}}_f\right\|_2 \leq \gamma \left\|\tilde{\boldsymbol{N}}_f\right\|_2.$$

By letting $\Psi = \gamma I$, we have $\|\Psi\|_\infty = \gamma$ and $\left\|\Psi \tilde{\boldsymbol{N}}_f\right\|_2 = \gamma \left\|\tilde{\boldsymbol{N}}_f\right\|_2$, which means that $\Psi = \gamma I$ is an optimal solution achieving the maximum. □

Remark 11 *Note that if $D_f \neq 0$, then*

$$\max_{\boldsymbol{Q} \in \mathscr{RH}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_2 : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \leq \gamma\} = \gamma \left\|\boldsymbol{V}^{-1}\boldsymbol{N}_f\right\|_2 = \infty.$$

This means that this optimal filter (as well as almost any other filter) is very sensitive to sensor faults.

3.4 $\mathscr{H}_-/\mathscr{H}_2$ Fault Detection Filter Design

From Lemma 7 we know that the $\mathscr{H}_-/\mathscr{H}_2$ problem is equivalent to Problem 3e, i.e.

$$\max_{\Psi \in \mathscr{RH}_2^{\mathrm{ny}\times\mathrm{ny}}} \{\left\|\Psi \tilde{\boldsymbol{N}}_f\right\|_- : \|\Psi\|_2 \leq \gamma\}.$$

Unlike $\mathscr{H}_-/\mathscr{H}_\infty$ problem and $\mathscr{H}_2/\mathscr{H}_\infty$ problem we studied in section 3.2 and section 3.3, we have different solutions for $\mathscr{H}_-/\mathscr{H}_2$ problem if different $\mathscr{H}_-$ definitions are considered. In this section we shall illustrate this point and give solutions for all cases.

Theorem 4 *Suppose Assumptions 1-3 are satisfied. Then*

$$\max_{\boldsymbol{Q} \in \mathscr{RH}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_-^{[\infty]} : \|\boldsymbol{Q}\boldsymbol{N}_d\|_2 \leq \gamma\} = 0.$$

Proof Note that the equivalent Problem 3e in this case is

$$\max_{\Psi \in \mathscr{RH}_2^{\mathrm{ny} \times \mathrm{ny}}} \{ \left\| \Psi \tilde{\boldsymbol{N}}_f \right\|_-^{[\infty]} : \|\Psi\|_2 \leq \gamma \}.$$

Since $\Psi \in \mathscr{RH}_2$,

$$\lim_{\omega \to \infty} \Psi(j\omega) = 0.$$

Hence

$$\lim_{\omega \to \infty} \Psi(j\omega) \tilde{\boldsymbol{N}}_f(j\omega) = 0.$$

As a result, $\left\| \Psi \tilde{\boldsymbol{N}} \right\|_-^{[\infty]}$ is always zero. □

Hence it is concluded that this design criterion is not suitable for fault detection design.

Theorem 5 *Suppose Assumptions 1-3 are satisfied. Then*

$$\sup_{\boldsymbol{Q} \in \mathscr{RH}_\infty^{\mathrm{ny} \times \mathrm{ny}}} \{ \| \boldsymbol{Q} \boldsymbol{N}_f \|_-^{[0]} : \| \boldsymbol{Q} \boldsymbol{N}_d \|_2 \leq \gamma \} = \infty.$$

Furthermore, for any given $\alpha > 0$, let $\epsilon > 0$ and

$$\boldsymbol{Q}_{sub} = \frac{\gamma \sqrt{2\epsilon}}{s + \epsilon} \boldsymbol{V}^{-1}.$$

Then

$$\{ \| \boldsymbol{Q}_{sub} \boldsymbol{N}_f \|_-^{[0]} > \alpha \quad \text{and} \quad \| \boldsymbol{Q}_{sub} \boldsymbol{N}_d \|_2 \leq \gamma \}$$

is satisfied for a sufficiently small $\epsilon > 0$.

Proof Again note that the equivalent Problem 3e in this case is

$$\max_{\Psi \in \mathscr{RH}_2^{\mathrm{ny} \times \mathrm{ny}}} \{ \left\| \Psi \tilde{\boldsymbol{N}}_f \right\|_-^{[0]} : \|\Psi\|_2 \leq \gamma \}.$$

Take $\Psi = \frac{\gamma\sqrt{2\epsilon}}{s+\epsilon} I$ such that $\epsilon > 0$. Then $\Psi(0) = \gamma\sqrt{\frac{2}{\epsilon}} I$ and $\|\Psi\|_2 = \gamma$. Let $\epsilon \to 0$, then $\Psi(0) \to \infty$, so that

$$\left\|\Psi \tilde{\boldsymbol{N}}_f\right\|_-^{[0]} = \underline{\sigma}(\Psi(0)\tilde{\boldsymbol{N}}_f(0)) = \gamma\sqrt{\frac{2}{\epsilon}}\underline{\sigma}(\tilde{\boldsymbol{N}}_f(0)) \to \infty.$$

□

Remark 12 *We should point out that an optimal filter designed using Theorem 5 is not necessarily good for fault detection since this optimal filter can be extremely narrow-banded near* 0 *frequency so that any higher frequency component of fault may not be detected.*

When Problem 3 is considered with the $\mathscr{H}_-$ index defined over a finite frequency range $[\omega_1, \omega_2]$, the solution becomes much more complicated. We shall now state this as a separate problem as below.

Problem 5 *(Interval $\mathscr{H}_-/\mathscr{H}_2$ Problem) Let an uncertain system be described by equations (3.1)-(3.4) and let $\gamma > 0$ be a given disturbance rejection level. Find a stable transfer matrix $\boldsymbol{Q} \in \mathscr{R}\mathscr{H}_\infty^{\mathrm{n_y}\times \mathrm{n_y}}$ in (3.7)-(3.9) such that $\|\boldsymbol{G}_{rd}\|_2 \le \gamma$ and $\|\boldsymbol{G}_{rf}\|_-^{[\omega_1,\omega_2]}$ is maximized, i.e.*

$$\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{n_y}\times \mathrm{n_y}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_-^{[\omega_1,\omega_2]} : \|\boldsymbol{Q}\boldsymbol{N}_d\|_2 \le \gamma\}$$

Or, equivalently, let $\boldsymbol{Q} = \gamma\Psi \boldsymbol{V}^{-1}$ and solve

$$\max_{\Psi\in\mathscr{R}\mathscr{H}_2^{\mathrm{n_y}\times \mathrm{n_y}}} \left\{\left\|\Psi \tilde{\boldsymbol{N}}_f\right\|_-^{[\omega_1,\omega_2]} : \|\Psi\|_2 \le \gamma\right\}.$$

Remark 13 *It is not hard to see that there is no rational function solution to the above problem. This is because an optimal Ψ must satisfy $\Psi(j\omega) = 0$ almost everywhere for any*

$\omega \notin [\omega_1, \omega_2]$. Hence an analytic optimal solution seems to be impossible. Nevertheless, it is intuitively feasible to find some rational approximations so that a rational Ψ has the form of a bandpass filter with the pass-band close to $[\omega_1, \omega_2]$ and $\|\Psi\|_2 = \gamma$.

In the following, we shall describe an optimization approach to find a good rational approximation. To do that, we shall need a state space parametrization of a stable rational function with a given $\mathscr{H}_2$ norm [3].

Lemma 8 *Let $\Psi = \left[\begin{array}{c|c} A_\psi & B_\psi \\ \hline C_\psi & 0 \end{array}\right] \in \mathscr{RH}_2^{n_y \times n_y}$ be an n_ψ-th order strictly proper stable transfer matrix. Then the state space parameters of Ψ can be expressed as $A_\psi = A_{\psi s} + A_{\psi k}$ for some*

$$A_{\psi k} = -A'_{\psi k} = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n_\psi} \\ -a_{12} & 0 & a_{23} & \cdots & a_{2n_\psi} \\ -a_{13} & -a_{23} & 0 & \cdots & a_{3n_\psi} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{1n_\psi} & -a_{2n_\psi} & -a_{3n_\psi} & \cdots & 0 \end{bmatrix}$$

and $A_{\psi s} = -\frac{1}{2}C'_\psi C_\psi$. Furthermore, $\|\Psi\|_2^2 =$ Trace$(B'_\psi B_\psi)$.

It is obvious that A_ψ may not be stable for some given $A_{\psi k}$. From [3] (C_ψ, A_ψ) satisfies the Lyapunov equation $A_\psi + A'_\psi + C_\psi C'_\psi = 0$. Denote the eigenvalue of A_ψ as λ, then for nontrivial vector x,

$$x' A_\psi x + x' A'_\psi x + x' C_\psi C'_\psi x = x'(\lambda + \lambda^*)x + \|C_\psi x\|^2 = 0.$$

Since $\|C_\psi x\|^2 \geq 0$, $x'(\lambda + \lambda^*)x \leq 0$, i.e. A_ψ has all eigenvalues in closed left half plane. Let λ_0 be an eigenvalue of A_ψ on the $j\omega$ axis, then $x'(\lambda_0 + \lambda_0^*)x = 0$ so that $C_\psi x = 0$. Therefore we have $\lambda_0 x = A_\psi x$ and $C_\psi x = 0$, by PBH test [47] λ_0 is an unobservable eigenvalue of

(C_ψ, A_ψ). That is, any eigenvalue of A_ψ on the $j\omega$ axis is an unobservable eigenvalue of (C_ψ, A_ψ), so Ψ can always be made to be stable after eliminating unobservable eigenvalues.

If we use directly the elements of A_ψ, B_ψ and C_ψ as optimization variables the total number of variables is $n_\psi^2 + 2n_y n_\psi$. However, from Lemma 8 $A_{\psi s}$ can be computed from C_ψ so the elements of $A_{\psi k}$, B_ψ and C_ψ are all the optimization variables. Using this technique the total number of optimization variables is $n_\psi(n_\psi - 1)/2 + 2n_y n_\psi$ and the reduction is $n_\psi(n_\psi + 1)/2$.

In order to carry out the subsequent optimization effectively, we need an effective method of computing $\mathscr{H}_-$ index fast and exactly. Computing $\|\boldsymbol{G}\|_-^{[0]}$ is easy, we need only check the singular value of $\boldsymbol{G}(0)$. But computing the other two $\mathscr{H}_-$ indices are much harder. Enlighten by the bi-section method of computing $\mathscr{H}_\infty$ norm of a transfer matrix [1], we now present a bi-section algorithm to compute the $\mathscr{H}_-$ index defined over all frequencies or over a frequency range. In the following $\mathscr{H}_-/\mathscr{H}_2$ optimization we shall use this algorithm to compute the $\mathscr{H}_-$ index defined over a frequency range. The following result shows the main idea used in our algorithm.

Lemma 9 *Suppose* $\boldsymbol{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathscr{RL}_\infty$ *and* $\Omega \subseteq \mathscr{R} \cup \{\infty\}$, *then*

$$\min_{\omega \in \Omega} \underline{\sigma}(\boldsymbol{G}(j\omega)) > \beta$$

if and only if $\underline{\sigma}(D) > \beta$ *and*

$$H = \begin{bmatrix} A + BR^{-1}D'C & BR^{-1}B' \\ -C(I + DR^{-1}D')C' & -(A + BR^{-1}D'C) \end{bmatrix}$$

has no eigenvalues in $j\Omega$ *where* $R = \beta^2 I - D'D$.

In Lemma 9, if $\mathscr{H}_-$ index over a frequency range $[\omega_1, \omega_2]$ is being computed, then $\Omega = [\omega_1, \omega_2]$; if the $\mathscr{H}_-$ index over all frequency is being computed, then $\Omega = \mathscr{R} \cup \{\infty\}$. The detailed procedure of our algorithm for computing $\mathscr{H}_-$ index is summarized below.

Algorithm for Computing $\mathscr{H}_-$ Index

1. Give an initial guess on lower bound and upper bound such that

   $0 \leq \beta_1 \leq \min_{\omega \in \Omega} \underline{\sigma}(\boldsymbol{G}(j\omega)) \leq \beta_2$ and give a tolerance $\epsilon > 0$.

2. Let $\beta = \frac{1}{2}(\beta_1 + \beta_2)$. Compute the eigenvalues of

$$H = \begin{bmatrix} A + BR^{-1}D'C & BR^{-1}B' \\ -C(I + DR^{-1}D')C' & -(A + BR^{-1}D'C) \end{bmatrix}.$$

3. If H has no eigenvalue on $j\Omega$, which means $\min_{\omega \in \Omega} \underline{\sigma}(\boldsymbol{G}(j\omega)) > \beta$ is true, then let $\beta_1 = \beta$; else if H has any eigenvalue on $j\Omega$, which means $\min_{\omega \in \Omega} \underline{\sigma}(\boldsymbol{G}(j\omega)) \leq \beta$ is true, then let $\beta_2 = \beta$.

4. Repeat steps 2 and 3 until $\beta_2 - \beta_1 < \epsilon$ is satisfied. And the approximate value of $\min_{\omega \in \Omega} \underline{\sigma}(\boldsymbol{G}(j\omega))$ is given as $\frac{1}{2}(\beta_1 + \beta_2)$, with tolerance ϵ.

The Matlab implementation for this algorithm is included in Appendix B.

With the state space parametrization of Ψ on $\mathscr{R}\mathscr{H}_2$ space and our bi-section algorithm for computing $\mathscr{H}_-$ index, the optimization process for solving Problem 3

$$\max_{\|\Psi\|_2 \leq \gamma} \left\| \Psi \tilde{\boldsymbol{N}}_f \right\|_-^{[\omega_1, \omega_2]}$$

can be performed as:

$$\max_{A_{\psi k}, B_\psi, C_\psi, \mathrm{Trace}(B_\psi' B_\psi) \leq \gamma^2} \left\| \left[\begin{array}{c|c} A_{\psi k} - \frac{1}{2} C_\psi' C_\psi & B_\psi \\ \hline C_\psi & 0 \end{array} \right] \tilde{\boldsymbol{N}}_f \right\|_-^{[\omega_1, \omega_2]}$$

Furthermore, we introduce a penalty function $\Theta(B_\psi)$ to ensure $\mathrm{Trace}(\mathrm{B}'_\psi \mathrm{B}_\psi) < \gamma^2$. Θ is defined as

$$\Theta(B_\psi) = \begin{cases} C, & \text{if } \mathrm{Trace}(\mathrm{B}'_\psi \mathrm{B}_\psi) > \gamma^2; \\ 0, & \text{if } \mathrm{Trace}(\mathrm{B}'_\psi \mathrm{B}_\psi) \leq \gamma^2. \end{cases}$$

where C is a large positive number. Therefore, the new optimization scheme is:

$$\max_{A_{\psi k}, B_\psi, C_\psi} \left\| \left[\begin{array}{c|c} A_{\psi k} - \frac{1}{2} C'_\psi C_\psi & B_\psi \\ \hline C_\psi & 0 \end{array} \right] \tilde{\boldsymbol{N}}_f \right\|_{-}^{[\omega_1, \omega_2]} - \Theta(B_\psi)$$

For this optimization scheme we have developed a two-stage optimization algorithm [21] which is a combination of genetic algorithm [8] and Nelder-Mead simplex method [19]. Genetic algorithm is good at searching for the right direction for global optimum but has slow convergence, while Nelder-Mead simplex method is good at searching for small neighborhood. So the result obtained by genetic algorithm is used as the starting point for the second step optimization by Nelder-Mead simplex method, the latter gives the final results of the optimization process.

Theoretically, Ψ can be a transfer matrix of any order. However, in practice we try to find a Ψ with low degree. Thus we run the optimization process as follows: first set Ψ with a given starting order, searching for the optimal value; then increase the order of Ψ, run the searching algorithm again and compare the results with the former one; if higher degree Ψ gives a better performance and the Ψ's degree doesn't exceed the pre-defined limit then keep increasing the degree of Ψ and re-do the searching process; else the optimization process ends. An example is given in Chapter 5 to show the effectiveness of this algorithm and the illustrating Matlab code is shown in Appendix B.

3.5 $\mathscr{H}_2/\mathscr{H}_2$ Fault Detection Filter Design

$\mathscr{H}_2/\mathscr{H}_2$ problem has been extensively studied in the literature, see [7] and references therein. For completeness and comparison, an explicit solution to this problem under current problem formulation is given in this section.

Theorem 6 *Suppose Assumptions 1-3 are satisfied. Then*

$$\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{ny}\times\mathrm{ny}}}\{\|\boldsymbol{Q}\boldsymbol{N}_f\|_2 : \|\boldsymbol{Q}\boldsymbol{N}_d\|_2 \leq \gamma\} = \gamma\left\|\boldsymbol{V}^{-1}\boldsymbol{N}_f\right\|_\infty.$$

Proof Note that by Lemma 7 we only need to solve Problem 4e:

$$\max_{\Psi\in\mathscr{R}\mathscr{H}_2^{\mathrm{ny}\times\mathrm{ny}}}\{\left\|\Psi\tilde{\boldsymbol{N}}_f\right\|_2 : \|\Psi\|_2 \leq \gamma\}.$$

Now suppose $\left\|\tilde{\boldsymbol{N}}_f\right\|_\infty = \bar{\sigma}(\tilde{\boldsymbol{N}}_f(j\omega_0))$ for some $\omega_0 \in \mathscr{R}\cup\{\infty\}$. We shall first consider the case where $0 \leq \omega_0 < \infty$. Let the singular value decomposition of $\tilde{\boldsymbol{N}}_f(j\omega_0)$ be $U(j\omega_0)\Sigma(j\omega_0)V'(j\omega_0)$. Let $u_1(j\omega_0) \in \mathscr{C}^{n_y}$ be the left singular vector corresponding to the largest singular value $\bar{\sigma}(\tilde{\boldsymbol{N}}_f(j\omega_0))$, i.e., $u_1(j\omega_0)$ is the first column of $U(j\omega_0)$.

Write $u_1'(j\omega_0)$ as

$$u_1'(j\omega_0) = \left[\begin{array}{cccc} \alpha_1 e^{j\theta_1} & \alpha_2 e^{j\theta_2} & \cdots & \alpha_{n_y} e^{j\theta_{n_y}} \end{array}\right]$$

such that $\alpha_i \in \mathscr{R}$ and $\theta_i \in (-\pi, 0]$. Let $\beta_i > 0$ be such that

$$\theta_i = \angle\left(\frac{\beta_i - j\omega_0}{\beta_i + j\omega_0}\right)$$

(take $\beta_i = \infty$ if $\theta_i = 0$) and define

$$\hat{\mathbf{u}}(s) := \left[\begin{array}{cccc} \alpha_1\frac{\beta_1 - s}{\beta_1 + s} & \alpha_2\frac{\beta_2 - s}{\beta_2 + s} & \cdots & \alpha_{n_y}\frac{\beta_{n_y} - s}{\beta_{n_y} + s} \end{array}\right].$$

Then

$$\hat{\mathbf{u}}(j\omega_0) = u_1'(j\omega_0).$$

Next let $\phi(s) \in \mathscr{RH}_2$ be such that

$$|\phi(j\omega)| = \begin{cases} \gamma\sqrt{\frac{\pi}{\epsilon}},\ \omega \in [\omega_0 - \epsilon, \omega_0 + \epsilon] \\ 0,\ \omega \notin [\omega_0 - \epsilon, \omega_0 + \epsilon] \end{cases}$$

and ϵ is a small positive number.

Finally, let

$$\Psi(s) = \phi(s) e_1 \hat{\mathbf{u}}(s)$$

where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathscr{R}^{n_y}.$$

Then

$$\left\|\Psi \tilde{\boldsymbol{N}}_f\right\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Trace}\{\Psi(\mathrm{j}\omega)\tilde{\boldsymbol{N}}_{\mathrm{f}}(\mathrm{j}\omega)\tilde{\boldsymbol{N}}_{\mathrm{f}}^{\sim}(\mathrm{j}\omega)\Psi^{\sim}(\mathrm{j}\omega)\}\mathrm{d}\omega}$$

$$= \sqrt{\frac{2}{2\pi}\int_{\omega_0-\epsilon}^{\omega_0+\epsilon} |\phi(j\omega)|^2 \hat{\mathbf{u}}(j\omega)\tilde{\boldsymbol{N}}_f(j\omega)\tilde{\boldsymbol{N}}_f^{\sim}(j\omega)\hat{\mathbf{u}}^{\sim}(j\omega)d\omega}$$

$$\approx \sqrt{\hat{\mathbf{u}}(j\omega_0)\tilde{\boldsymbol{N}}_f(j\omega_0)\tilde{\boldsymbol{N}}_f^{\sim}(j\omega_0)\hat{\mathbf{u}}^{\sim}(j\omega_0)\ \frac{2}{2\pi}\int_{\omega_0-\epsilon}^{\omega_0+\epsilon} |\phi(j\omega)|^2 d\omega}$$

$$= \bar{\sigma}(\tilde{\boldsymbol{N}}_f(j\omega_0))\sqrt{\frac{2}{2\pi}\int_{\omega_0-\epsilon}^{\omega_0+\epsilon} |\phi(j\omega)|^2 d\omega} = \bar{\sigma}(\tilde{\boldsymbol{N}}_f(j\omega_0))\,\|\phi(s)\|_2 = \gamma\left\|\tilde{\boldsymbol{N}}_f\right\|_\infty$$

as $\epsilon \to 0$.

If $\omega_0 = \infty$, then the above procedure can be used to construct a sequence of solutions as $\omega_0 \to \infty$. □

In practical the ideal narrow bandpass filter $\phi(s)$ does not exist and we need to approximate it with a bandpass filter. The procedure of obtaining an approximate design is summarized below.

$\mathscr{H}_2/\mathscr{H}_2$ **Fault Detection Filter Design Procedure:**

- Find ω_0 such that $\left\|\tilde{\boldsymbol{N}}_f\right\|_\infty = \bar{\sigma}(\tilde{\boldsymbol{N}}_f(j\omega_0))$.
- Compute $u_1(j\omega_0)$ from the singular value decomposition of

$$\tilde{\boldsymbol{N}}_f(j\omega_0) = U(j\omega_0)\Sigma(j\omega_0)V'(j\omega_0).$$

- Write $u_1'(j\omega_0)$ as

$$u_1'(j\omega_0) = \left[\begin{array}{cccc} \alpha_1 e^{j\theta_1} & \alpha_2 e^{j\theta_2} & \cdots & \alpha_{n_y} e^{j\theta_{n_y}} \end{array}\right]$$

  such that $\alpha_i \in \mathscr{R}$ and $\theta_i \in (-\pi, 0]$. Let $\beta_i > 0$ be such that

$$\theta_i = \angle\left(\frac{\beta_i - j\omega_0}{\beta_i + j\omega_0}\right)$$

  (take $\beta_i = \infty$ if $\theta_i = 0$) and define

$$\hat{\mathbf{u}}(s) := \left[\begin{array}{cccc} \alpha_1 \frac{\beta_1 - s}{\beta_1 + s} & \alpha_2 \frac{\beta_2 - s}{\beta_2 + s} & \cdots & \alpha_{n_y} \frac{\beta_{n_y} - s}{\beta_{n_y} + s} \end{array}\right].$$

- Design a strictly proper bandpass filter $\boldsymbol{F}(s) \in \mathscr{R}\mathscr{H}_2$ which has pass-band around ω_0;
- Let $\phi(s) = \frac{\gamma \boldsymbol{F}(s)}{\|\boldsymbol{F}(s)\|_2}$;
- Let $\Psi(s) = \phi(s) e_1 \hat{\mathbf{u}}(s)$ and $\boldsymbol{Q} = \Psi \boldsymbol{V}^{-1}$.

Remark 14 *It is easy to see that the optimal filter constructed above will be only sensitive to a fault signal near ω_0 frequency. Hence it is clear that such filter is not very useful for fault detection. This will be illustrated in Chapter 5.*

Chapter 4

Solutions to Discrete Robust Fault Detection Problems

In Chapter 3, complete solutions to continuous robust fault detection problems have been developed. In this chapter, we will carry out the parallel development for the discrete time problems [25, 26]. Although there are considerable similarities between the continuous time and discrete time solutions, there are also significant differences in some cases. For example, we can give analytic and optimal solution to some discrete time problem, but it cannot be done for continuous time case. In addition, explicit discrete time solutions have their own merits in applications.

This chapter is organized as below: we first formulate the discrete robust fault detection problems to be addressed in this chapter in section 4.1. The analytic and optimal solutions to the $\mathscr{H}_-/\mathscr{H}_\infty$ problem, $\mathscr{H}_2/\mathscr{H}_\infty$ problem and $\mathscr{H}_\infty/\mathscr{H}_\infty$ problem are provided in sections 4.2-4.3. The solutions to three different cases of $\mathscr{H}_-/\mathscr{H}_2$ problem are given in sections 4.4-4.6. Finally, the solution to $\mathscr{H}_2/\mathscr{H}_2$ problem is presented in section 4.7. [1]

[1]The main results in this chapter were published as [25]. See Appendix C for authorization letter.

4.1 Problem Formulation

Consider a discrete time invariant system with disturbance and possible faults as:

$$x(k+1) = Ax(k) + Bu(k) + B_d d(k) + B_f f(k) \tag{4.1}$$

$$y(k) = Cx(k) + Du(k) + D_d d(k) + D_f f(k) \tag{4.2}$$

where $x(k) \in \mathscr{R}^n$ is the state vector, $y(k) \in \mathscr{R}^{n_y}$ is the output measurement, $d(k) \in \mathscr{R}^{n_d}$ represents the unknown/uncertain disturbance and measurement noise, and $f(k) \in \mathscr{R}^{n_f}$ denotes the process, sensor or actuator fault vector. $f(k)$ and $d(k)$ can be modeled as different type of signals, depending on specific situations under consideration. See Chapters 4 and 8 of [48] for some detailed discussions. Two frequently used assumptions on $d(k)$ and $f(k)$ are:

(i) unknown signal with bounded energy or bounded power;

(ii) white noise.

Different assumptions on $d(k)$ and $f(k)$ will lead to different fault detection problem formulations and the solutions for all these problems will be discussed in this Chapter.

All coefficient matrices in equations (4.1) and (4.2) are assumed to be known constant matrices. Furthermore, the following assumptions are made:

Assumption 5 (A, C) *is detectable.*

This is a standard assumption for all fault detection problems.

Assumption 6 D_d *has full row rank.*

This assumption is similar to Assumption 2 of the continuous case and it is for the similar consideration. See section 3.1 for related discussion.

Assumption 7 $\begin{bmatrix} A - e^{j\theta}I & B_d \\ C & D_d \end{bmatrix}$ *has full row rank for all* $\theta \in [0, 2\pi]$. *Or, equivalently, the transfer function matrix* $\boldsymbol{G}_d := \left[\begin{array}{c|c} A & B_d \\ \hline C & D_d \end{array}\right]$ *has no transmission zero on the unit circle.*

This assumption can be relaxed in the same way as in the continuous time case [23, 24].

By taking $\mathscr{Z}$-transform of equations (4.1) and (4.2) we have the system input/output equation

$$\boldsymbol{y} = \boldsymbol{G}_u\boldsymbol{u} + \boldsymbol{G}_d\boldsymbol{d} + \boldsymbol{G}_f\boldsymbol{f} \tag{4.3}$$

where $\boldsymbol{G}_u$, $\boldsymbol{G}_d$, and $\boldsymbol{G}_f$ are $n_y \times n_u$, $n_y \times n_d$ and $n_y \times n_f$ transfer matrices respectively and their state-space realizations are

$$\begin{bmatrix} \boldsymbol{G}_u & \boldsymbol{G}_d & \boldsymbol{G}_f \end{bmatrix} = \left[\begin{array}{c|ccc} A & B & B_d & B_f \\ \hline C & D & D_d & D_f \end{array}\right] \tag{4.4}$$

Since the state-space realizations of $\boldsymbol{G}_u$, $\boldsymbol{G}_d$ and $\boldsymbol{G}_f$ share the same A and C matrices, by applying Lemma 2 a LCF for the system can be found as (4.4)

$$\begin{bmatrix} \boldsymbol{G}_u & \boldsymbol{G}_d & \boldsymbol{G}_f \end{bmatrix} = \boldsymbol{M}^{-1} \begin{bmatrix} \boldsymbol{N}_u & \boldsymbol{N}_d & \boldsymbol{N}_f \end{bmatrix} \tag{4.5}$$

where

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{N}_u & \boldsymbol{N}_d & \boldsymbol{N}_f \end{bmatrix} = \left[\begin{array}{c|cccc} A + LC & L & B + LD & B_d + LD_d & B_f + LD_f \\ \hline C & I & D & D_d & D_f \end{array}\right] \tag{4.6}$$

and L is a matrix such that $A + LC$ is stable.

It has been shown in [7] that, without loss of generality, the fault detection filter can take the following general form

$$\boldsymbol{r} = \boldsymbol{Q}(\boldsymbol{M}\boldsymbol{y} - \boldsymbol{N}_u\boldsymbol{u}) = \boldsymbol{Q}\begin{bmatrix} \boldsymbol{M} & -\boldsymbol{N}_u \end{bmatrix}\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{u} \end{bmatrix} \tag{4.7}$$

where $\boldsymbol{r}$ is the residual vector for detection, $\boldsymbol{Q} \in \mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ is a free stable transfer matrix to be designed. The filter structure, which is the same as that of the continuous case, is shown in Figure 3.1. Replacing $\boldsymbol{y}$ in (4.7) by right-hand sides of (4.3) and (4.5) we have

$$\boldsymbol{r} = \boldsymbol{Q}[\ \boldsymbol{N}_d \quad \boldsymbol{N}_f\ ]\begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{f} \end{bmatrix} = \boldsymbol{Q}\boldsymbol{N}_d\boldsymbol{d} + \boldsymbol{Q}\boldsymbol{N}_f\boldsymbol{f}, \tag{4.8}$$

which is also the same as the equation in continuous case. Denote the transfer matrices from $\boldsymbol{d}$ and $\boldsymbol{f}$ to $\boldsymbol{r}$ by $\boldsymbol{G}_{rd}$ and $\boldsymbol{G}_{rf}$, respectively, then

$$\boldsymbol{G}_{rd} = \boldsymbol{Q}\boldsymbol{N}_d, \quad \boldsymbol{G}_{rf} = \boldsymbol{Q}\boldsymbol{N}_f. \tag{4.9}$$

For discrete fault detection filter design we also need to make tradeoff between robustness to disturbance and fault detection sensitivity, and choose certain performance criteria for both objectives. Similar to the continuous case, we use $\mathscr{H}_\infty$, $\mathscr{H}_-$(see Chapter 2 for definitions) for disturbance rejection and $\mathscr{H}_2$, $\mathscr{H}_\infty$ for fault detection sensitivity. Thus several fault detection filter design problems are formulated in discrete time.

Problem 6 *(Discrete $\mathscr{H}_-/\mathscr{H}_\infty$ Problem) Let an uncertain system be described by equations (4.1)-(4.4) and let $\gamma > 0$ be a given disturbance rejection level. Find a stable transfer matrix $\boldsymbol{Q} \in \mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ in (4.7)-(4.9) such that $\|\boldsymbol{G}_{rd}\|_\infty \leq \gamma$ and $\|\boldsymbol{G}_{rf}\|_-$ is maximized, i.e.*

$$\max_{\boldsymbol{Q}\in\mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_- : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \leq \gamma\}$$

Problem 7 *(Discrete $\mathscr{H}_2/\mathscr{H}_\infty$ Problem) Let an uncertain system be described by equations (4.1)-(4.4) and let $\gamma > 0$ be a given disturbance rejection level. Find a stable transfer matrix $\boldsymbol{Q} \in \mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ in (4.7)-(4.9) such that $\|\boldsymbol{G}_{rd}\|_\infty \le \gamma$ and $\|\boldsymbol{G}_{rf}\|_2$ is maximized, i.e.*

$$\max_{\boldsymbol{Q}\in\mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}} \{\|\boldsymbol{QN}_f\|_2 : \|\boldsymbol{QN}_d\|_\infty \le \gamma\}$$

Problem 8 *(Discrete $\mathscr{H}_-/\mathscr{H}_2$ Problem) Let an uncertain system be described by equations (4.1)-(4.4) and let $\gamma > 0$ be a given disturbance rejection level. Find a stable transfer matrix $\boldsymbol{Q} \in \mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ in (4.7)-(4.9) such that $\|\boldsymbol{G}_{rd}\|_2 \le \gamma$ and $\|\boldsymbol{G}_{rf}\|_-$ is maximized, i.e.*

$$\max_{\boldsymbol{Q}\in\mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}} \{\|\boldsymbol{QN}_f\|_- : \|\boldsymbol{QN}_d\|_2 \le \gamma\}$$

Problem 9 *(Discrete $\mathscr{H}_2/\mathscr{H}_2$ Problem) Let an uncertain system be described by equations (4.1)-(4.4) and let $\gamma > 0$ be a given disturbance rejection level. Find a stable transfer matrix $\boldsymbol{Q} \in \mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ in (4.7)-(4.9) such that $\|\boldsymbol{G}_{rd}\|_2 \le \gamma$ and $\|\boldsymbol{G}_{rf}\|_2$ is maximized, i.e.*

$$\max_{\boldsymbol{Q}\in\mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}} \{\|\boldsymbol{QN}_f\|_2 : \|\boldsymbol{QN}_d\|_2 \le \gamma\}$$

Before we proceed to the solutions to the above problems, we shall first establish some preliminary results.

Lemma 10 *Suppose Assumption 7 is satisfied and let $\boldsymbol{G}_d = \boldsymbol{M}^{-1}\boldsymbol{N}_d$ be any left coprime factorization over $\mathscr{RH}_\infty$. Then $\boldsymbol{N}_d$ has no transmission zero on the unit circle. Or, equivalently, for any appropriately dimensioned matrix L,*

$$\begin{bmatrix} A + LC - e^{j\theta}I & B_d + LD_d \\ C & D_d \end{bmatrix}$$

has full row rank for all $\theta \in [0, 2\pi]$.

Proof The result follows by noting that

$$\begin{bmatrix} A + LC - e^{j\theta} I & B_d + LD_d \\ C & D_d \end{bmatrix} = \begin{bmatrix} I & L \\ 0 & I \end{bmatrix} \begin{bmatrix} A - e^{j\theta} I & B_d \\ C & D_d \end{bmatrix}$$

and the fact that all coprime factors have the same unstable transmission zeros [48]. □

An immediate consequence of the above result is the following spectral factorization formula.

Lemma 11 *Suppose Assumptions 5-7 are satisfied and let $\boldsymbol{G}_d = \boldsymbol{M}^{-1}\boldsymbol{N}_d$ be any left coprime factorization over $\mathscr{RH}_\infty$. Then there is a square transfer matrix $\boldsymbol{V} \in \mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ such that $\boldsymbol{V}^{-1} \in \mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ and*

$$\boldsymbol{V}\boldsymbol{V}^{\sim} = \boldsymbol{N}_d\boldsymbol{N}_d^{\sim}. \tag{4.10}$$

In particular, if a state space representation of $\boldsymbol{N}_d$ is given as in equation (4.6), then a state space representation of $\boldsymbol{V}$ is given by

$$\boldsymbol{V} = \left[\begin{array}{c|c} A + LC & [(A+LC)PC' + (B_d + LD_d)D_d']R_d^{-1/2} \\ \hline C & R_d^{1/2} \end{array}\right] \tag{4.11}$$

with

$$\boldsymbol{V}^{-1} = \left[\begin{array}{c|c} A - (APC' + B_dD_d')R_d^{-1}C & -(APC' + B_dD_d')R_d^{-1} - L \\ \hline R_d^{-1/2}C & R_d^{-1/2} \end{array}\right] \tag{4.12}$$

where $P \geq 0$ is the stabilizing solution to the Riccati equation

$$APA' - P - (APC' + B_dD_d')(D_dD_d' + CPC')^{-1}(D_dB_d' + CPA') + B_dB_d' = 0 \tag{4.13}$$

such that $A - (APC' + B_dD_d')(D_dD_d' + CPC')^{-1}C$ is stable and $R_d := D_dD_d' + CPC'$.

Proof Since Assumptions 5-7 are satisfied, Lemma 4 and Lemma 10 can be applied to $\boldsymbol{N}_d$ to get $\boldsymbol{V}\boldsymbol{V}^{\sim} = \boldsymbol{N}_d\boldsymbol{N}_d^{\sim}$ where $P \geq 0$ satisfies the following Riccati equation

$$A_{LC}PA'_{LC} - P - (A_{LC}PC' + B_{LD}D'_d)R_d^{-1}(D_dB'_{LD} + CPA'_{LC}) + B_{LD}B'_{LD} = 0$$

and

$$A_{LC} := A + LC, \;\; B_{LD} := B + LD$$

It is easy to show that the above algebraic Riccati equation can be simplified to equation (4.13). The rest of the proof follows the same algebraic manipulations as shown in Lemma 6. □

The following Lemma is the key to the solutions of all the above problems.

Lemma 12 *Suppose Assumptions 5-7 are satisfied. Let* $\boldsymbol{V}, \boldsymbol{V}^{-1} \in \mathscr{RH}_\infty$ *be defined as in equation (4.10). Let*

$$\boldsymbol{Q} = \Psi\boldsymbol{V}^{-1}$$

for $\Psi \in \mathscr{RH}_\infty$ *and denote* $\tilde{\boldsymbol{N}}_f = \boldsymbol{V}^{-1}\boldsymbol{N}_f \in \mathscr{RH}_\infty$*. Then the fault detection Problems 6-9 are equivalent to Problems 6e-9e below, respectively:*

Problem 6e

$$\max_{\Psi \in \mathscr{RH}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{ \left\| \Psi\tilde{\boldsymbol{N}}_f \right\|_- : \|\Psi\|_\infty \leq \gamma \}$$

Problem 7e

$$\max_{\Psi \in \mathscr{RH}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{ \left\| \Psi\tilde{\boldsymbol{N}}_f \right\|_2 : \|\Psi\|_\infty \leq \gamma \}$$

Problem 8e

$$\max_{\Psi \in \mathscr{RH}_2^{\mathrm{ny}\times\mathrm{ny}}} \{ \left\| \Psi \tilde{\boldsymbol{N}}_f \right\|_- : \|\Psi\|_2 \le \gamma \}$$

Problem 9e

$$\max_{\Psi \in \mathscr{RH}_2^{\mathrm{ny}\times\mathrm{ny}}} \{ \left\| \Psi \tilde{\boldsymbol{N}}_f \right\|_2 : \|\Psi\|_2 \le \gamma \}$$

Proof We shall first show that Problem 6 and Problem 7 are equivalent to Problem 6e and Problem 7e, respectively.

Note that by Lemma 11 there exists $\boldsymbol{V} \in \mathscr{RH}_\infty$ such that $\boldsymbol{V}\boldsymbol{V}^\sim = \boldsymbol{N}_d\boldsymbol{N}_d^\sim$ and $\boldsymbol{V}^{-1} \in \mathscr{RH}_\infty$. Therefore

$$\|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty^2 = \sup_{\theta\in[0,2\pi]} \bar{\sigma}(\boldsymbol{Q}(e^{j\theta})\boldsymbol{N}_d(e^{j\theta})\boldsymbol{N}_d^\sim(e^{j\theta})\boldsymbol{Q}^\sim(e^{j\theta}))$$

$$= \sup_{\theta\in[0,2\pi]} \bar{\sigma}(\boldsymbol{Q}(e^{j\theta})\boldsymbol{V}(e^{j\theta})\boldsymbol{V}^\sim(e^{j\theta})\boldsymbol{Q}^\sim(e^{j\theta})) = \|\boldsymbol{Q}\boldsymbol{V}\|_\infty^2$$

i.e., $\|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty = \|\boldsymbol{Q}\boldsymbol{V}\|_\infty$. We can, therefore without loss of generality, take $\boldsymbol{Q}$ in the form of $\boldsymbol{Q} = \Psi\boldsymbol{V}^{-1}$ for some $\Psi \in \mathscr{RH}_\infty$. Hence $\|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty = \|\boldsymbol{Q}\boldsymbol{V}\|_\infty = \|\Psi\|_\infty$ so that $\|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \le \gamma$ is equivalent to $\|\Psi\|_\infty \le \gamma$. Moreover, $\boldsymbol{Q}\boldsymbol{N}_f = \Psi\boldsymbol{V}^{-1}\boldsymbol{N}_f = \Psi\tilde{\boldsymbol{N}}_f$, hence Problem 6 is equivalent to Problem 6e and Problem 7 is equivalent to Problem 7e.

Next we show that Problem 8 and Problem 9 are equivalent to Problem 8e and Problem 9e, respectively. Note that in Problem 8 and Problem 9, we have $\boldsymbol{Q}\boldsymbol{N}_d \in \mathscr{RH}_2$. Hence

$$\|\boldsymbol{Q}\boldsymbol{N}_d\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Trace}\{\boldsymbol{Q}(e^{j\theta})\boldsymbol{N}_d(e^{j\theta})\boldsymbol{N}_d^\sim(e^{j\theta})\boldsymbol{Q}^\sim(e^{j\theta})\}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Trace}\{\boldsymbol{Q}(e^{j\theta})\boldsymbol{V}(e^{j\theta})\boldsymbol{V}^\sim(e^{j\theta})\boldsymbol{Q}^\sim(e^{j\theta})\}d\theta = \|\boldsymbol{Q}\boldsymbol{V}\|_2^2$$

such that $\|\boldsymbol{Q}\boldsymbol{N}_d\|_2 = \|\boldsymbol{Q}\boldsymbol{V}\|_2$. Since $\boldsymbol{Q}\boldsymbol{V} \in \mathscr{RH}_2$ and $\boldsymbol{V}$, $\boldsymbol{V}^{-1} \in \mathscr{RH}_\infty$, we can let $\boldsymbol{Q} = \Psi\boldsymbol{V}^{-1}$ for some $\Psi \in \mathscr{RH}_2$. Therefore $\|\boldsymbol{Q}\boldsymbol{N}_d\|_2 = \|\boldsymbol{Q}\boldsymbol{V}\|_2 = \|\Psi\|_2$ so that $\|\boldsymbol{Q}\boldsymbol{N}_d\|_2 \le \gamma$

is equivalent to $\|\Psi\|_2 \leq \gamma$. Moreover, $\boldsymbol{Q}\boldsymbol{N}_f = \Psi \boldsymbol{V}^{-1}\boldsymbol{N}_f = \Psi\tilde{\boldsymbol{N}}_f$, hence Problem 8 is equivalent to Problem 8e and Problem 9 is equivalent to Problem 9e. $\square$

We shall provide optimal solutions to each of the above problems in the following sections.

4.2 $\mathscr{H}_-/\mathscr{H}_\infty$ Fault Detection Filter Design

In this section we give a complete solution for $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection filter design problem, i.e., Problem 6 or Problem 6e.

Theorem 7 *Suppose Assumptions 5-7 are satisfied. Let*

$$\begin{bmatrix} \boldsymbol{G}_u & \boldsymbol{G}_d & \boldsymbol{G}_f \end{bmatrix} = \boldsymbol{M}^{-1} \begin{bmatrix} \boldsymbol{N}_u & \boldsymbol{N}_d & \boldsymbol{N}_f \end{bmatrix}$$

be any left coprime factorization over $\mathscr{RH}_\infty$ and let $\boldsymbol{V} \in \mathscr{RH}_\infty$ be a square transfer matrix such that $\boldsymbol{V}^{-1} \in \mathscr{RH}_\infty$ and $\boldsymbol{V}\boldsymbol{V}^\sim = \boldsymbol{N}_d\boldsymbol{N}_d^\sim$. Then

$$\max_{\boldsymbol{Q}\in\mathscr{RH}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_- : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \leq \gamma\} = \gamma \left\| \boldsymbol{V}^{-1}\boldsymbol{N}_f \right\|_-$$

and an optimal fault detection filter for Problem 6 is given by

$$\boldsymbol{r} = \boldsymbol{Q}_{opt} \begin{bmatrix} \boldsymbol{M} & -\boldsymbol{N}_u \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{u} \end{bmatrix}$$

where

$$\boldsymbol{Q}_{opt} = \gamma \boldsymbol{V}^{-1}.$$

Proof Note that by Lemma 12, we only need to solve Problem 6e:

$$\max_{\Psi\in\mathscr{RH}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{\left\|\Psi\tilde{\boldsymbol{N}}_f\right\|_- : \|\Psi\|_\infty \leq \gamma\}.$$

From Lemma 1 we know that for every frequency $\theta \in [0, 2\pi]$,

$$\underline{\sigma}(\Psi(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta})) \leq \bar{\sigma}(\Psi(e^{j\theta}))\underline{\sigma}(\tilde{\boldsymbol{N}}_f(e^{j\theta}))$$

so that

$$\left\|\Psi\tilde{\boldsymbol{N}}_f\right\|_- \leq \|\Psi\|_\infty \left\|\tilde{\boldsymbol{N}}_f\right\|_- \leq \gamma \left\|\tilde{\boldsymbol{N}}_f\right\|_- .$$

By letting $\Psi = \gamma I$, we have $\|\Psi\|_\infty = \gamma$ and $\left\|\Psi\tilde{\boldsymbol{N}}_f\right\|_- = \gamma\left\|\tilde{\boldsymbol{N}}_f\right\|_-$, which means that $\Psi = \gamma I$ is an optimal solution achieving the maximum. □

Remark 15 *Note that the remarks for continuous $\mathscr{H}_-/\mathscr{H}_\infty$ problem in section 3.2 also apply to this problem.*

Note also that the solution given in the above theorem is completely general and it is not dependent on specific state space representation of those coprime factorization and spectral factorization. However, if specific state space coprime and spectral factorizations are used, the optimal filter can be written in a very simple form.

Theorem 8 *Suppose Assumptions 5-7 are satisfied. Let $P \geq 0$ be the stabilizing solution to the Riccati equation*

$$APA' - P - (APC' + B_dD_d')(D_dD_d' + CPC')^{-1}(D_dB_d' + CPA') + B_dB_d' = 0 \tag{4.14}$$

such that $A - (APC' + B_dD_d')(D_dD_d' + CPC')^{-1}C$ is stable and let $R_d = D_dD_d' + CPC'$. Define

$$L_0 = -(APC' + B_dD_d')R_d^{-1}.$$

Then

$$\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_- : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \le \gamma\} = \gamma \left\| \boldsymbol{V}^{-1}\boldsymbol{N}_f \right\|_-$$

and an optimal $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection filter has the following state space representation

$$\boldsymbol{r} = \boldsymbol{Q}_{opt} \begin{bmatrix} \boldsymbol{M} & -\boldsymbol{N}_u \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{u} \end{bmatrix}$$

where

$$\boldsymbol{Q}_{opt} \begin{bmatrix} \boldsymbol{M} & -\boldsymbol{N}_u \end{bmatrix} = \gamma \left[\begin{array}{c|cc} A + L_0C & -L_0 & B + L_0D \\ \hline -R_d^{-1/2}C & R_d^{-1/2} & -R_d^{-1/2}D \end{array}\right]$$

and

$$\boldsymbol{V}^{-1}\boldsymbol{N}_f = \left[\begin{array}{c|c} A + L_0C & B_f + L_0D_f \\ \hline R_d^{-1/2}C & R_d^{-1/2}D_f \end{array}\right].$$

In other words, the optimal $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection filter is the following observer:

$$\hat{x}(k+1) = (A + L_0C)\hat{x}(k) - L_0y(t) + (B + L_0D)u(k) \tag{4.15}$$

$$r(k) = \gamma R_d^{-1/2} \left(y(k) - C\hat{x}(k) - Du(k)\right). \tag{4.16}$$

Proof Note that

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{N}_u \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & L & B + LD \\ \hline C & I & D \end{array}\right]$$

where L is a matrix with appropriate dimensions such that $A + LC$ is stable. Note from Theorem 7 and Lemma 11 that

$$\boldsymbol{Q}_{opt} = \gamma \boldsymbol{V}^{-1} = \gamma \left[\begin{array}{c|c} A + L_0C & L_0 - L \\ \hline R_d^{-1/2}C & R_d^{-1/2} \end{array}\right].$$

Then

$$\boldsymbol{Q}_{opt} \begin{bmatrix} \boldsymbol{M} & -\boldsymbol{N}_u \end{bmatrix} = \gamma \left[\begin{array}{c|c} A + L_0C & L_0 - L \\ \hline R_d^{-1/2}C & R_d^{-1/2} \end{array}\right] \left[\begin{array}{c|cc} A + LC & L & -(B + LD) \\ \hline C & I & -D \end{array}\right]$$

$$
= \gamma \left[\begin{array}{cc|cc} A+L_0C & L_0C-LC & L_0-L & -(L_0-L)D \\ 0 & A+LC & L & -(B+LD) \\ \hline R_d^{-1/2}C & R_d^{-1/2}C & R_d^{-1/2} & -R_d^{-1/2}D \end{array}\right]
$$

$$
= \gamma \left[\begin{array}{cc|cc} A+L_0C & 0 & L_0 & -(B+L_0D) \\ 0 & A+LC & L & -(B+LD) \\ \hline R^{-1/2}C & 0 & R_d^{-1/2} & -R_d^{-1/2}D \end{array}\right]
$$

$$
= \gamma \left[\begin{array}{c|cc} A+L_0C & L_0 & -(B+L_0D) \\ \hline R_d^{-1/2}C & R_d^{-1/2} & -R_d^{-1/2}D \end{array}\right] = \gamma \left[\begin{array}{c|cc} A+L_0C & -L_0 & B+L_0D \\ \hline -R_d^{-1/2}C & R_d^{-1/2} & -R_d^{-1/2}D \end{array}\right].
$$

Similarly, we have

$$
\boldsymbol{V}^{-1}\boldsymbol{N}_f = \left[\begin{array}{c|c} A+L_0C & L_0-L \\ \hline R_d^{-1/2}C & R_d^{-1/2} \end{array}\right] \left[\begin{array}{c|c} A+LC & B_f+LD_f \\ \hline C & D_f \end{array}\right]
$$

$$
= \left[\begin{array}{cc|c} A+L_0C & (L_0-L)C & (L_0-L)D_f \\ 0 & A+LC & B_f+LD_f \\ \hline R_d^{-1/2}C & R_d^{-1/2}C & R_d^{-1/2}D_f \end{array}\right]
$$

$$
= \left[\begin{array}{cc|c} A+L_0C & 0 & B_f+L_0D_f \\ 0 & A+LC & B_f+LD_f \\ \hline R_d^{-1/2}C & 0 & R_d^{-1/2}D_f \end{array}\right] = \left[\begin{array}{c|c} A+L_0C & B_f+L_0D_f \\ \hline R_d^{-1/2}C & R_d^{-1/2}D_f \end{array}\right].
$$

□

Remark 16 *It is easy to see that our optimal filters given in Theorem 7 and Theorem 8 are also optimal for the so-called discrete $\mathscr{H}_\infty/\mathscr{H}_\infty$ problem:*

$$
\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_\infty : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \le \gamma\}
$$

4.3 $\mathscr{H}_2/\mathscr{H}_\infty$ Fault Detection Filter Design

In this section we give an optimal solution for the discrete $\mathscr{H}_2/\mathscr{H}_\infty$ problem stated in Section 4.1 as Problem 7. Similar to the solution for the $\mathscr{H}_-/\mathscr{H}_\infty$ problem given in Theorem 7 and Theorem 8, we have the following parallel results for $\mathscr{H}_2/\mathscr{H}_\infty$ problem.

Theorem 9 *Suppose Assumptions 5-7 are satisfied. Let*

$$\begin{bmatrix} \boldsymbol{G}_u & \boldsymbol{G}_d & \boldsymbol{G}_f \end{bmatrix} = \boldsymbol{M}^{-1} \begin{bmatrix} \boldsymbol{N}_u & \boldsymbol{N}_d & \boldsymbol{N}_f \end{bmatrix}$$

be any left coprime factorization over $\mathscr{R}\mathscr{H}_\infty$ and let $\boldsymbol{V} \in \mathscr{R}\mathscr{H}_\infty$ be a square transfer matrix such that $\boldsymbol{V}^{-1} \in \mathscr{R}\mathscr{H}_\infty$ and $\boldsymbol{V}\boldsymbol{V}^\sim = \boldsymbol{N}_d\boldsymbol{N}_d^\sim$. Then

$$\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_2 : \|\boldsymbol{Q}\boldsymbol{N}_d\|_\infty \le \gamma\} = \gamma \left\| \boldsymbol{V}^{-1}\boldsymbol{N}_f \right\|_2$$

and the optimal fault detection filters for Problem 6 given in Theorem 7 and Theorem 8 are also the optimal filters for this problem.

Proof Note that by Lemma 12, we only need to solve Problem 7e:

$$\max_{\Psi\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{\left\|\Psi\tilde{\boldsymbol{N}}_f\right\|_2 : \|\Psi\|_\infty \le \gamma\}.$$

Note that

$$\left\|\Psi\tilde{\boldsymbol{N}}_f\right\|_2 \le \|\Psi\|_\infty \left\|\tilde{\boldsymbol{N}}_f\right\|_2 \le \gamma \left\|\tilde{\boldsymbol{N}}_f\right\|_2.$$

By letting $\Psi = \gamma I$, we have $\|\Psi\|_\infty = \gamma$ and $\left\|\Psi\tilde{\boldsymbol{N}}_f\right\|_2 = \gamma\left\|\tilde{\boldsymbol{N}}_f\right\|_2$, which means that $\Psi = \gamma I$ is an optimal solution achieving the maximum. □

Remark 17 *Note that the remarks for continuous $\mathscr{H}_2/\mathscr{H}_\infty$ problem in section 3.3 also apply to this problem.*

4.4 $\mathscr{H}_-/\mathscr{H}_2$ Fault Detection Filter Design: Case 1

By Lemma 12 the discrete $\mathscr{H}_-/\mathscr{H}_2$ problem is equivalent to Problem 8e, i.e.

$$\max_{\Psi \in \mathscr{RH}_2^{\mathrm{ny}\times\mathrm{ny}}} \{ \left\| \Psi \tilde{\boldsymbol{N}}_f \right\|_- : \|\Psi\|_2 \leq \gamma \}.$$

Unlike the $\mathscr{H}_-/\mathscr{H}_\infty$ problem studied in section 4.2, we have different solutions for the $\mathscr{H}_-/\mathscr{H}_2$ problem if different $\mathscr{H}_-$ definitions are considered. In this section and the next two sections we shall illustrate this point and give solutions for all cases.

Theorem 10 *Suppose Assumptions 5-7 are satisfied. Then*

$$\sup_{\boldsymbol{Q} \in \mathscr{RH}_\infty^{\mathrm{ny}\times\mathrm{ny}}} \{ \| \boldsymbol{Q}\boldsymbol{N}_f \|_-^{[0]} : \| \boldsymbol{Q}\boldsymbol{N}_d \|_2 \leq \gamma \} = \infty.$$

Furthermore, for any given $\alpha > 0$*, let* $2 > \epsilon > 0$ *and*

$$\boldsymbol{Q}_{sub} = \frac{\gamma\sqrt{\epsilon(2-\epsilon)}}{z-1+\epsilon} \boldsymbol{V}^{-1}.$$

Then $\{ \| \boldsymbol{Q}_{sub}\boldsymbol{N}_f \|_-^{[0]} > \alpha \ \text{ and } \ \| \boldsymbol{Q}_{sub}\boldsymbol{N}_d \|_2 \leq \gamma \}$ *is satisfied for a sufficiently small* $\epsilon > 0$.

Proof Again note that the equivalent Problem 8e in this case is

$$\sup_{\Psi \in \mathscr{RH}_2^{\mathrm{ny}\times\mathrm{ny}}} \{ \left\| \Psi \tilde{\boldsymbol{N}}_f \right\|_-^{[0]} : \|\Psi\|_2 \leq \gamma \}.$$

Take $\Psi(z) = \frac{\gamma\sqrt{\epsilon(2-\epsilon)}}{z-1+\epsilon} I$ such that $2 > \epsilon > 0$. Then $\Psi(1) = \gamma\sqrt{\frac{2-\epsilon}{\epsilon}} I$ and $\|\Psi\|_2 = \gamma$. Let $\epsilon \to 0$, then $\Psi(1) \to \infty$, so that

$$\left\| \Psi \tilde{\boldsymbol{N}}_f \right\|_-^{[0]} = \underline{\sigma}(\Psi(1)\tilde{\boldsymbol{N}}_f(1)) = \gamma\sqrt{\frac{2-\epsilon}{\epsilon}} \underline{\sigma}(\tilde{\boldsymbol{N}}_f(1)) \to \infty.$$

□

Remark 18 *We should point out that an optimal filter designed using Theorem 10 is not necessarily good for fault detection since this optimal filter can be extremely narrow-banded near* 0 *frequency so that any higher frequency component of fault may not be detected.*

4.5 $\mathscr{H}_-/\mathscr{H}_2$ Fault Detection Filter Design: Case 2

In this section, we shall consider another special case where the $\mathscr{H}_-$ index is defined for all frequency but with D_f full column rank. As we have mentioned before, this is a very restrictive case. We are interested in this case because an analytic solution is possible. Notice that the corresponding continuous time problem has no analytic solution in this case.

Lemma 13 *Suppose D_f has full column rank. Then an optimal solution Ψ_{opt} to the Problem 8e*

$$\max_{\Psi\in\mathscr{R}\mathscr{H}_2^{\mathrm{n_y}\times\mathrm{n_y}}}\{\left\|\Psi\tilde{\boldsymbol{N}}_f\right\|_-^{[0,2\pi]}:\|\Psi\|_2\leq\gamma\}$$

has the form $\Psi_{opt} = U\Psi_o$ and

$$\Psi_o\tilde{\boldsymbol{N}}_f = \begin{bmatrix} \alpha I_{n_f} \\ 0_{(n_y-n_f)\times n_f} \end{bmatrix}$$

where α is a positive scalar and U is a $n_y \times n_y$ all-pass stable transfer matrix.

Proof We shall first show

$$\underline{\sigma}(\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta})) = C \text{ for every } \theta\in[0,2\pi], \text{ where C is a positive scalar.}$$

Suppose there exists a Ψ_{opt} such that $\underline{\sigma}(\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta})) = C$ doesn't hold. Let Θ_1 denotes the set of all θ values such that $\underline{\sigma}(\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta})) = \left\|\Psi_{opt}\tilde{\boldsymbol{N}}_f\right\|_-^{[0,2\pi]}$ is achieved. Let

$\Theta_2 := [0, 2\pi] - \Theta_1$ such that

$$\underline{\sigma}(\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta}))_{\theta\in\Theta_1} < \underline{\sigma}(\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta}))_{\theta\in\Theta_2}.$$

Then there exists a weighting function $\boldsymbol{W} \in \mathscr{R}\mathscr{H}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ such that

$$\|\boldsymbol{W}\Psi_{opt}\|_2 = \|\Psi_{opt}\|_2$$

and

$$\underline{\sigma}(\Psi_{opt}(e^{j\theta})\boldsymbol{N}_f(e^{j\theta}))_{\theta\in\Theta_1} < \underline{\sigma}(\boldsymbol{W}(e^{j\theta})\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta}))_{\theta\in\Theta_1}$$

$$\leq \underline{\sigma}(\boldsymbol{W}(e^{j\theta})\Psi_{opt}(e^{j\theta})\boldsymbol{N}_f(e^{j\theta}))_{\theta\in\Theta_2}$$

Therefore $\left\|\boldsymbol{W}\Psi_{opt}\tilde{\boldsymbol{N}}_f\right\|_-^{[0,2\pi]} > \left\|\Psi_{opt}\tilde{\boldsymbol{N}}_f\right\|_-^{[0,2\pi]}$ and Ψ_{opt} is not an optimal solution. Hence it is true that $\underline{\sigma}(\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta})) = C$ for every $\theta \in [0, 2\pi]$.

Next we show that

$$\bar{\sigma}(\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta})) = \underline{\sigma}(\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta})) \text{ for every } \theta \in [0, 2\pi].$$

Suppose there exists a Ψ_{opt} such that $\bar{\sigma}(\Psi_{opt}(e^{j\theta_1})\tilde{\boldsymbol{N}}_f(e^{j\theta_1})) \neq \underline{\sigma}(\Psi_{opt}(e^{j\theta_1})\tilde{\boldsymbol{N}}_f(e^{j\theta_1}))$ for some θ_1, i.e.

$$\bar{\sigma}(\Psi_{opt}(e^{j\theta_1})\tilde{\boldsymbol{N}}_f(e^{j\theta_1})) > \underline{\sigma}(\Psi_{opt}(e^{j\theta_1})\tilde{\boldsymbol{N}}_f(e^{j\theta_1})) = \left\|\Psi_{opt}\tilde{\boldsymbol{N}}_f\right\|_-^{[0,2\pi]}.$$

Then a Ψ_1 can be selected such that

$$\underline{\sigma}(\Psi_1(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta})) = \underline{\sigma}(\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta})), \text{for all } \theta \in [0, 2\pi]$$

$$\bar{\sigma}(\Psi_1(e^{j\theta_1})\tilde{\boldsymbol{N}}_f(e^{j\theta_1})) < \bar{\sigma}(\Psi_{opt}(e^{j\theta_1})\tilde{\boldsymbol{N}}_f(e^{j\theta_1}))$$

and

$$\|\Psi_1\|_2 < \|\Psi_{opt}\|_2.$$

Since $\underline{\sigma}(\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta})) = C$ for every $\theta \in [0, 2\pi]$, $\left\|\Psi_1 \tilde{\boldsymbol{N}}_f\right\|_-^{[0,2\pi]} = \left\|\Psi_{opt}\tilde{\boldsymbol{N}}_f\right\|_-^{[0,2\pi]}$. Let $\Psi_2 = \frac{\|\Psi_{opt}\|_2}{\|\Psi_1\|_2}\Psi_1$ then $\|\Psi_2\|_2 = \|\Psi_{opt}\|_2$ and

$$\left\|\Psi_2 \tilde{\boldsymbol{N}}_f\right\|_-^{[0,2\pi]} = \frac{\|\Psi_{opt}\|_2}{\|\Psi_1\|_2}\left\|\Psi_1 \tilde{\boldsymbol{N}}_f\right\|_-^{[0,2\pi]} = \frac{\|\Psi_{opt}\|_2}{\|\Psi_1\|_2}\left\|\Psi_{opt} \tilde{\boldsymbol{N}}_f\right\|_-^{[0,2\pi]} > \left\|\Psi_{opt} \tilde{\boldsymbol{N}}_f\right\|_-^{[0,2\pi]}.$$

Therefore Ψ_{opt} is not optimal and by contradiction the assumption is false. So

$$\bar{\sigma}(\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta})) = \underline{\sigma}(\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta}))$$

holds for every $\theta \in [0, 2\pi]$.

Since $\bar{\sigma}(\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta})) = \underline{\sigma}(\Psi_{opt}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta})) = C$ for every $\theta \in [0, 2\pi]$, and that D_f has full column rank implies $n_y \geq n_f$, $\Psi_{opt}\tilde{\boldsymbol{N}}_f$ has the form

$$\Psi_{opt}\tilde{\boldsymbol{N}}_f = U \begin{bmatrix} \alpha I_{n_f} \\ 0_{(n_y-n_f)\times n_f} \end{bmatrix},$$

where U is an all-pass stable transfer matrix and α is a positive scalar. Let $\Psi_{opt} = U\Psi_o$ then

$$\Psi_o\tilde{\boldsymbol{N}}_f = \begin{bmatrix} \alpha I_{n_f} \\ 0_{(n_y-n_f)\times n_f} \end{bmatrix}.$$

□

Lemma 14 *Suppose D_f has full column rank. Then Problem 8e*

$$\max_{\Psi \in \mathscr{RH}_2^{\mathrm{n_y}\times \mathrm{n_y}}} \{\left\|\Psi \tilde{\boldsymbol{N}}_f\right\|_-^{[0,2\pi]} : \|\Psi\|_2 \leq \gamma\}$$

is equivalent to

Problem 8a

$$\min_{\tilde{\boldsymbol{N}}_f^+ \in \mathscr{RH}_2^{\mathrm{n_f}\times \mathrm{n_y}}} \{\left\|\tilde{\boldsymbol{N}}_f^+\right\|_2 : \tilde{\boldsymbol{N}}_f^+ \tilde{\boldsymbol{N}}_f = I\}.$$

Proof From Lemma 13 we know that the optimal solution to Problem 8e has the form $\Psi_{opt} = U\Psi_o$ and

$$\Psi_o \tilde{\boldsymbol{N}}_f = \begin{bmatrix} \alpha I_{n_f} \\ 0_{(n_y - n_f)\times n_f} \end{bmatrix}.$$

Let $\Psi_o = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$, where Ψ_1 is $n_f \times n_y$ and Ψ_2 is $(n_y - n_f) \times n_y$. Then $\begin{bmatrix} \Psi_1 \tilde{\boldsymbol{N}}_f \\ \Psi_2 \tilde{\boldsymbol{N}}_f \end{bmatrix} = \begin{bmatrix} \alpha I \\ 0 \end{bmatrix}$ so $\Psi_2 = 0$ and $\|\Psi_o\|_2 = \|\Psi_1\|_2$. Since Problem 8e needs to maximize $\left\|\Psi \tilde{\boldsymbol{N}}_f\right\|_-$ with the constraint $\|\Psi\|_2 \leq \gamma$, it is equivalent to find a Ψ_1 with smallest $\mathscr{H}_2$ norm such that $\Psi_1 \tilde{\boldsymbol{N}}_f = I$. Denote $\Psi_1 = \tilde{\boldsymbol{N}}_f^+$, then Problem 8e is equivalent to Problem 8a. □

In [20] the solution to a dual problem of Problem 8a is given. Similarly we have the solution to Problem 8a given by the following lemma.

Lemma 15 *Assume* $\boldsymbol{H}(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is strictly minimum phase and* D *has full column rank. Let* $D^+ = D(D'D)^{-1}$, $D_\perp$ *is chosen such that* $D'_\perp D_\perp = I - D(D'D)^{-1}D'$ *and* $A_0 = A - B(D^+)'C$*, then the optimal solution to problem*

$$\min_{\boldsymbol{H}^+(z)\in\mathscr{R}\mathscr{H}_2^{\mathrm{n_f}\times\mathrm{n_y}}} \left\{\left\|\boldsymbol{H}^+(z)\right\|_2 : \ \boldsymbol{H}^+(z)\boldsymbol{H}(z) = I\right\}$$

is given by

$$\boldsymbol{H}^+(z)_{opt} = \left[\begin{array}{c|c} A + KC & K \\ \hline RC & R \end{array}\right],$$

where $Q \geq 0$ *is the solution to the algebraic Riccati equation*

$$Q = A_0'QA_0 - A_0'QC'D'_\perp(I + D_\perp CQC'D'_\perp)^{-1}D_\perp CQA_0 + B(D'D)^{-1}B',$$

$$K = -B(D^+)' - A_0QC'D'_\perp(I + D_\perp CQC'D'_\perp)^{-1}D_\perp$$

and

$$R = (D^+)'(I + CQC'D'_\perp D_\perp)^{-1}.$$

Proof The equation $\boldsymbol{H}^{+}(z)\boldsymbol{H}(z) = I$ is equivalent to $\boldsymbol{H}^{T}(z)(\boldsymbol{H}^{+}(z))^{T} = I$, so Problem 8a is equivalent to finding a $\boldsymbol{H}^{(rinv)}(z)$ with smallest $\mathscr{H}_2$ norm such that $\boldsymbol{H}^{T}(z)\boldsymbol{H}^{(rinv)}(z) = I$. Hence the conclusion in [20] can be applied to $\boldsymbol{H}^{T}(z)$ to get the optimal $\boldsymbol{H}^{(rinv)}(z)_{opt}$. $\boldsymbol{H}^{+}(z)_{opt}$ is then obtained by taking transpose of $\boldsymbol{H}^{(rinv)}(z)_{opt}$. □

Theorem 11 *Suppose Assumptions 5-7 are satisfied. Let $\boldsymbol{G}_f$ have all zeros inside the unit circle and D_f have full column rank. Let*

$$\left[\begin{array}{ccc} \boldsymbol{G}_u & \boldsymbol{G}_d & \boldsymbol{G}_f \end{array}\right] = \boldsymbol{M}^{-1}\left[\begin{array}{ccc} \boldsymbol{N}_u & \boldsymbol{N}_d & \boldsymbol{N}_f \end{array}\right]$$

be any left coprime factorization over $\mathscr{RH}_\infty$ and let $\boldsymbol{V} \in \mathscr{RH}_\infty$ be a square transfer matrix such that $\boldsymbol{V}^{-1} \in \mathscr{RH}_\infty$ and $\boldsymbol{V}\boldsymbol{V}^{\sim} = \boldsymbol{N}_d\boldsymbol{N}_d^{\sim}$. Let $(\tilde{\boldsymbol{N}}_f^{+})_{opt} = (\boldsymbol{V}^{-1}\boldsymbol{N}_f)^{+}_{opt}$ be the optimal solution to Problem 8a. Then

$$\max_{\boldsymbol{Q}\in\mathscr{RH}_2^{\mathrm{n_y}\times\mathrm{n_y}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_{-}^{[0,2\pi]} : \|\boldsymbol{Q}\boldsymbol{N}_d\|_2 \le \gamma\} = \frac{\gamma}{\left\|(\boldsymbol{V}^{-1}\boldsymbol{N}_f)^{+}_{opt}\right\|_2}$$

and an optimal fault detection filter is given by

$$\boldsymbol{r} = \boldsymbol{Q}_{opt}\left[\begin{array}{cc} \boldsymbol{M} & -\boldsymbol{N}_u \end{array}\right]\left[\begin{array}{c} \boldsymbol{y} \\ \boldsymbol{u} \end{array}\right]$$

where

$$\boldsymbol{Q}_{opt} = \frac{\gamma}{\left\|(\boldsymbol{V}^{-1}\boldsymbol{N}_f)^{+}_{opt}\right\|_2}\left[\begin{array}{c} (\boldsymbol{V}^{-1}\boldsymbol{N}_f)^{+}_{opt}\boldsymbol{V}^{-1} \\ 0 \end{array}\right].$$

Proof Note that by Lemma 12, we only need to solve Problem 8e:

$$\max_{\Psi\in\mathscr{RH}_2^{\mathrm{n_y}\times\mathrm{n_y}}} \{\left\|\Psi\tilde{\boldsymbol{N}}_f\right\|_{-}^{[0,2\pi]} : \|\Psi\|_2 \le \gamma\}.$$

Since G_f has all zeros inside the unit circle and $\boldsymbol{V}^{-1} \in \mathscr{RH}_\infty$, $\tilde{\boldsymbol{N}}_f$ is strictly minimum phase. From Lemma 13-15 we know that the optimal solution to Problem 8e is given by

$$\Psi_{opt} = U \frac{\gamma}{\left\| (\tilde{\boldsymbol{N}}_f^+)_{opt} \right\|_2} \begin{bmatrix} (\tilde{\boldsymbol{N}}_f^+)_{opt} \\ 0 \end{bmatrix}$$

where $(\tilde{\boldsymbol{N}}_f^+)_{opt} = (\boldsymbol{V}^{-1}\boldsymbol{N}_f)^+_{opt}$ is the optimal solution to Problem 8a and U is a unitary matrix. Take $U = I$ then an optimal solution is given by

$$\boldsymbol{Q}_{opt} = \frac{\gamma}{\left\| (\tilde{\boldsymbol{N}}_f^+)_{opt} \right\|_2} \begin{bmatrix} (\tilde{\boldsymbol{N}}_f^+)_{opt} \boldsymbol{V}^{-1} \\ 0 \end{bmatrix} = \frac{\gamma}{\| (\boldsymbol{V}^{-1}\boldsymbol{N}_f)^+_{opt} \|_2} \begin{bmatrix} (\boldsymbol{V}^{-1}\boldsymbol{N}_f)^+_{opt} \boldsymbol{V}^{-1} \\ 0 \end{bmatrix}.$$

□

Again the solution given in the above theorem is general and it does not dependent on specific state space representation of those coprime factorization and spectral factorization. If specific state space coprime and spectral factorization in the previous section are used, the optimal filter can be written in an explicit form.

Theorem 12 *Suppose Assumptions 5-7 are satisfied. Let $\boldsymbol{G}_f$ have all zeros inside the unit circle and D_f have full column rank. Let $P \geq 0$ be the stabilizing solution to the Riccati equation*

$$APA' - P - (APC' + B_d D_d')(D_d D_d' + CPC')^{-1}(D_d B_d' + CPA') + B_d B_d' = 0 \qquad (4.17)$$

such that $A - (APC' + BD')(DD' + CPC')^{-1}C$ is stable. Let $R_d = D_d D_d' + CPC'$ and define

$$L_0 = -(APC' + B_d D_d')R_d^{-1}.$$

Let $D^+ = R_d^{-1/2} D_f (D_f' R_d^{-1} D_f)^{-1}$, $D_\perp$ is chosen such that

$$D_\perp' D_\perp = I - R_d^{-1/2} D_f (D_f' R_d^{-1} D_f)^{-1} D_f' R_d^{-1/2},$$

and $A_0 = A + L_0C - (B_f + L_0D_f)(D^+)'R_d^{-1/2}C$. *Let* $Q \geq 0$ *be the solution to the algebraic Riccati equation*

$$Q = A_0'QA_0' - A_0'QC'D_\perp'(I + D_\perp CQC'D_\perp')^{-1}D_\perp CQA_0$$

$$+(B_f + L_0D_f)(D_f'R_d^{-1}D_f)^{-1}(B_f + L_0D_f)'$$

and define

$$K_0 = -(B_f + L_0D_f)(D^+)' - A_0QC'R_d^{-1/2}D_\perp'(I + D_\perp R_d^{-1/2}CQC'R_d^{-1/2}D_\perp')^{-1}D_\perp,$$

$$R_0 = (D^+)'(I + R_d^{-1/2}CQC'R_d^{-1/2}D_\perp'D_\perp)^{-1}.$$

Then

$$\max_{\boldsymbol{Q}\in\mathscr{RH}_2^{\mathrm{ny}\times\mathrm{ny}}} \{\|\boldsymbol{Q}\boldsymbol{N}_f\|_-^{[0,2\pi]} : \|\boldsymbol{Q}\boldsymbol{N}_d\|_2 \leq \gamma\} = \frac{\gamma}{\left\|(\boldsymbol{V}^{-1}\boldsymbol{N}_f)_{opt}^+\right\|_2}$$

where

$$(\boldsymbol{V}^{-1}\boldsymbol{N}_f)_{opt}^+ = \left[\begin{array}{c|c} A + K_0C & K_0 - L_0 \\ \hline R_0C & R_0 \end{array}\right]$$

and an optimal $\mathscr{H}_-/\mathscr{H}_2$ *fault detection filter has the following state space representation*

$$\boldsymbol{r} = \boldsymbol{Q}_{opt}\left[\begin{array}{cc} \boldsymbol{M} & -\boldsymbol{N}_u \end{array}\right]\left[\begin{array}{c} \boldsymbol{y} \\ \boldsymbol{u} \end{array}\right]$$

where

$$\boldsymbol{Q}_{opt} = \frac{\gamma}{\left\|(\boldsymbol{V}^{-1}\boldsymbol{N}_f)_{opt}^+\right\|_2}\left[\begin{array}{c} (\boldsymbol{V}^{-1}\boldsymbol{N}_f)_{opt}^+\boldsymbol{V}^{-1} \\ 0 \end{array}\right]$$

and

$$\boldsymbol{Q}_{opt}\left[\begin{array}{cc} \boldsymbol{M} & -\boldsymbol{N}_u \end{array}\right]$$

$$= \frac{\gamma}{\left\|(\boldsymbol{V}^{-1}\boldsymbol{N}_f)_{opt}^+\right\|_2}\left[\begin{array}{c}\left[\begin{array}{cc|cc} A + L_0C & 0 & -L_0 & B + L_0D \\ -(K_0 - L_0)R_d^{-1/2}C & A + K_0C & (K_0 - L_0)R_d^{-1/2} & -(K_0 - L_0)R_d^{-1/2}D \\ \hline -R_0R_d^{-1/2}C & R_0C & R_0R_d^{-1/2} & -R_0R_d^{-1/2}D \end{array}\right] \\ 0 \end{array}\right].$$

Proof Note that

$$\left[\begin{array}{cc} \boldsymbol{M} & \boldsymbol{N}_u \end{array}\right] = \left[\begin{array}{c|cc} A+LC & L & B+LD \\ \hline C & I & D \end{array}\right]$$

where L is a matrix with appropriate dimensions such that $A+LC$ is stable. From Theorem 7

$$\boldsymbol{V}^{-1} = \left[\begin{array}{c|c} A+L_0C & L_0 - L \\ \hline R_d^{-1/2}C & R_d^{-1/2} \end{array}\right].$$

From Theorem 8

$$\boldsymbol{V}^{-1}\boldsymbol{N}_f = \left[\begin{array}{c|c} A+L_0C & B_f + L_0D_f \\ \hline R_d^{-1/2}C & R_d^{-1/2} \end{array}\right]$$

and

$$\boldsymbol{V}^{-1}\left[\begin{array}{cc} \boldsymbol{M} & -\boldsymbol{N}_u \end{array}\right] = \left[\begin{array}{c|cc} A+L_0C & -L_0 & B+L_0D \\ \hline -R_d^{-1/2}C & R_d^{-1/2} & -R_d^{-1/2}D \end{array}\right].$$

From Lemma 15

$$(\boldsymbol{V}^{-1}\boldsymbol{N}_f)^+_{opt} = \left[\begin{array}{c|c} A+K_0C & K_0 - L_0 \\ \hline R_0C & R_0 \end{array}\right].$$

Therefore

$$\boldsymbol{Q}_{opt}\left[\begin{array}{cc} \boldsymbol{M} & -\boldsymbol{N}_u \end{array}\right] = \frac{\gamma}{\left\|(\boldsymbol{V}^{-1}\boldsymbol{N}_f)^+_{opt}\right\|_2}\left[\begin{array}{c} (\boldsymbol{V}^{-1}\boldsymbol{N}_f)^+_{opt} \\ 0 \end{array}\right]\boldsymbol{V}^{-1}\left[\begin{array}{cc} \boldsymbol{M} & -\boldsymbol{N}_u \end{array}\right]$$

$$= \frac{\gamma}{\left\|(\boldsymbol{V}^{-1}\boldsymbol{N}_f)^+_{opt}\right\|_2}\left[\begin{array}{c} \left[\begin{array}{c|c} A+K_0C & K_0 - L_0 \\ \hline R_0C & R_0 \end{array}\right] \\ 0 \end{array}\right]\left[\begin{array}{c|cc} A+L_0C & -L_0 & B+L_0D \\ \hline -R_d^{-1/2}C & R_d^{-1/2} & -R_d^{-1/2}D \end{array}\right]$$

$$= \frac{\gamma}{\left\|(\boldsymbol{V}^{-1}\boldsymbol{N}_f)^+_{opt}\right\|_2}\left[\begin{array}{c} \left[\begin{array}{cc|cc} A+L_0C & 0 & -L_0 & B+L_0D \\ -(K_0-L_0)R_d^{-1/2}C & A+K_0C & (K_0-L_0)R_d^{-1/2} & -(K_0-L_0)R_d^{-1/2}D \\ \hline -R_0R_d^{-1/2}C & R_0C & R_0R_d^{-1/2} & -R_0R_d^{-1/2}D \end{array}\right] \\ 0 \end{array}\right].$$

□

Remark 19 *Note that the optimal fault detection filter* $\boldsymbol{Q}_{opt}\left[\begin{array}{cc}\boldsymbol{M} & -\boldsymbol{N}_u\end{array}\right]$ *is independent of the choice of* L *matrix.*

Remark 20 *Note that the strictly minimum phase assumption for* $\boldsymbol{G}_f$ *(or* $\tilde{\boldsymbol{N}}_f$*) is not needed. In general, if* $\tilde{\boldsymbol{N}}_f$ *does not have any zeros on the unit circle, one can always factorize* $\tilde{\boldsymbol{N}}_f = \tilde{\boldsymbol{N}}_f^{\,min}\tilde{\boldsymbol{N}}_f^{\,a}$ *so that* $\tilde{\boldsymbol{N}}_f^{\,min}$ *is strictly minimum phase and* $\tilde{\boldsymbol{N}}_f^{\,a}$ *is a stable all-pass matrix. Then the solution can be computed by using* $\tilde{\boldsymbol{N}}_f^{\,min}$ *in place of* $\tilde{\boldsymbol{N}}_f$*. In the case when* $\tilde{\boldsymbol{N}}_f$ *has zeros on the unit circle, approximation factorization can also be carried out to obtain an approximation solution.*

4.6 $\mathscr{H}_-/\mathscr{H}_2$ Fault Detection Filter Design: Case 3

When Problem 8 is considered with the $\mathscr{H}_-$ index defined over a finite frequency range $[\theta_1, \theta_2]$, the solution becomes much more complicated. We shall now state this as a separate problem as below.

Problem 10 *(Discrete Interval* $\mathscr{H}_-/\mathscr{H}_2$ *Problem) Let an uncertain system be described by equations (4.1)-(4.4) and let* $\gamma > 0$ *be a given disturbance rejection level. Find a stable transfer matrix* $\boldsymbol{Q} \in \mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}$ *in (4.7)-(4.9) such that* $\|\boldsymbol{G}_{rd}\|_2 \leq \gamma$ *and* $\|\boldsymbol{G}_{rf}\|_-^{[\theta_1,\theta_2]}$ *is maximized, i.e.*

$$\max_{\boldsymbol{Q}\in\mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}}\{\|\boldsymbol{Q}\boldsymbol{N}_f\|_-^{[\theta_1,\theta_2]} : \|\boldsymbol{Q}\boldsymbol{N}_d\|_2 \leq \gamma\}$$

Or, equivalently, let $\boldsymbol{Q} = \Psi\boldsymbol{V}^{-1}$ *and solve*

$$\max_{\Psi\in\mathscr{RH}_2^{\mathrm{n_y}\times\mathrm{n_y}}}\left\{\left\|\Psi\tilde{\boldsymbol{N}}_f\right\|_-^{[\theta_1,\theta_2]} : \|\Psi\|_2 \leq \gamma\right\}.$$

Remark 21 *It is not hard to see that there is no rational function solution to the above problem. This is because an optimal* Ψ *must satisfy* $\Psi(e^{j\theta}) = 0$ *almost every where for any* $\theta \notin [\theta_1, \theta_2]$*. Hence an analytic optimal solution seems to be impossible. Nevertheless, it is intuitively feasible to find some rational approximations so that a rational* Ψ *has the form of a bandpass filter with the pass-band close to* $[\theta_1, \theta_2]$ *and* $\|\Psi\|_2 = \gamma$*.*

Remark 22 *When the condition that* D_f *has full column rank is not satisfied, the rational optimal solution to the problem*

$$\max_{\boldsymbol{Q}\in\mathscr{RH}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}} \{\|\boldsymbol{QN}_f\|_-^{[0,2\pi]} : \|\boldsymbol{QN}_d\|_2 \leq \gamma\}$$

may not exist. In this case we also need to find some rational approximate solutions. Moreover, this problem is a special case of Problem 10 by letting $\theta_1 = 0$ *and* $\theta_2 = 2\pi$*, we shall only consider the solution to Problem 10.*

In the following, we shall describe an optimization approach to find a good rational approximation for the two cases above. To do that, we shall need a state space parametrization of a stable rational function with a given $\mathscr{H}_2$ norm.

Lemma 16 *Let* $\Psi = \left[\begin{array}{c|c} A_\psi & B_\psi \\ \hline C_\psi & D_\psi \end{array}\right] \in \mathscr{RH}_2^{n_y\times n_y}$ *be an* n_ψ*-th order proper stable transfer matrix. Then the state space parameters of* Ψ *can be expressed as* $A_\psi = (I + A_{\psi k})(I - A_{\psi k})^{-1}(I - C_\psi' C_\psi)^{1/2}$ *for some*

$$A_{\psi k} = -A_{\psi k}' = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n_\psi} \\ -a_{12} & 0 & a_{23} & \cdots & a_{2n_\psi} \\ -a_{13} & -a_{23} & 0 & \cdots & a_{3n_\psi} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{1n_\psi} & -a_{2n_\psi} & -a_{3n_\psi} & \cdots & 0 \end{bmatrix}$$

and some C_ψ satisfies $\|C_\psi\| \leq 1$. Furthermore, $\|\Psi\|_2^2 = Trace(D'_\psi D_\psi + B'_\psi B_\psi)$.

Proof Assume that $\Psi = \left[\begin{array}{c|c} \hat{A}_\psi & \hat{B}_\psi \\ \hline \hat{C}_\psi & \hat{D}_\psi \end{array}\right] \in \mathscr{RH}_2$ is a n_ψth order observable realization, then the observability Gramian L_o satisfies

$$\hat{A}'_\psi L_o \hat{A}_\psi - L_o + \hat{C}'_\psi \hat{C}_\psi = 0.$$

Since $L_o > 0$, there exists a Cholesky factorization of $L_o = T'T$ where T is invertible. Perform a similarity transformation on Ψ such that

$$\Psi = \left[\begin{array}{c|c} T\hat{A}_\psi T^{-1} & T\hat{B}_\psi \\ \hline \hat{C}_\psi T^{-1} & \hat{D}_\psi \end{array}\right] = \left[\begin{array}{c|c} A_\psi & B_\psi \\ \hline C_\psi & D_\psi \end{array}\right].$$

Thus $A'_\psi A_\psi - I + C'_\psi C_\psi = 0$ so that $A_\psi = O(I - C'_\psi C_\psi)^{1/2}$ where O is an orthogonal matrix and $I - C'_\psi C_\psi$ is nonnegative definite. Since an orthogonal matrix O with no eigenvalue equals -1 can be represented as $A = (I + A_{\psi k})(I - A_{\psi k})^{-1}$, where $A_{\psi k} = -A'_{\psi k}$ is a skew-symmetric matrix, we have

$$A_\psi = (I + A_{\psi k})(I - A_{\psi k})^{-1}(I - C'_\psi C_\psi)^{1/2}$$

and $\|C_\psi\| \leq 1$. Consequently $\|\Psi\|_2^2 = Trace(D'_\psi D_\psi + B'_\psi B_\psi)$. □

If we use directly the elements of A_ψ, B_ψ, C_ψ and D_ψ as optimization variables the total number of variables is $n_\psi^2 + 2n_y n_\psi + n_y^2$. However, from Lemma 16 A_ψ can be computed from C_ψ and $A_{\psi k}$ so the elements $A_{\psi k}$, B_ψ, C_ψ and D_ψ are all the optimization variables. Using this technique the total number of optimization variables is $n_\psi(n_\psi - 1)/2 + 2n_y n_\psi + n_y^2$ and the reduction is $n_\psi(n_\psi + 1)/2$.

In order to carry out the subsequent optimization, we need an effective method of computing $\mathscr{H}_-$ index fast and exactly. Following the similar idea of computing $\mathscr{H}_-$ index for continuous system, we now present a bi-section algorithm to compute the $\mathscr{H}_-$ index defined over $[\theta_1, \theta_2]$.

The following result shows the main idea used in our algorithm.

Lemma 17 *Suppose* $\boldsymbol{G}(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathscr{RH}_\infty$ *and* $\theta \in [\theta_1, \theta_2]$*, then*

$$\min_{\theta} \underline{\sigma}[\boldsymbol{G}(e^{j\theta})] > \beta$$

if and only if

$$\underline{\sigma}[D + C(I - A)^{-1}] > \beta,$$

$R := \beta^2 I - D'D$*, and* $S :=$

$$\begin{bmatrix} A + BR^{-1}D'C - BR^{-1}B'(A' + C'DR^{-1}B')^{-1}C'\beta^2R^{-1}C & BR^{-1}B'(A' + C'DR^{-1}B')^{-1} \\ -(A' + C'DR^{-1}B')^{-1}C'\beta^2R^{-1}C & (A' + C'DR^{-1}B')^{-1} \end{bmatrix}$$

has no eigenvalues on the segment of unit circle between $\theta = \theta_1$ *and* $\theta = \theta_2$.

The detailed procedure of our algorithm for computing $\mathscr{H}_-$ index is summarized in below.

Algorithm for Computing $\mathscr{H}_-$ index: Discrete Time

1. Give an initial guess on lower bound and upper bound such that

$$0 \le \beta_1 \le \min_{\theta \in [\theta_1, \theta_2]} \underline{\sigma}(\boldsymbol{G}(e^{j\theta})) \le \beta_2$$

   and give a tolerance $\epsilon > 0$.

2. Let $\beta = \frac{1}{2}(\beta_1 + \beta_2)$. Compute the eigenvalues of S.

3. If S has no eigenvalue on the segment of unit circle between $\theta = \theta_1$ and $\theta = \theta_2$, which means

$$\min_{\theta \in [\theta_1, \theta_2]} \underline{\sigma}(\boldsymbol{G}(e^{j\theta})) > \beta$$

is true, then let $\beta_1 = \beta$; else let $\beta_2 = \beta$.

4. Repeat steps 2 and 3 until $\beta_2 - \beta_1 < \epsilon$ is satisfied. The approximate value of

$$\min_{\theta \in [\theta_1, \theta_2]} \underline{\sigma}(\boldsymbol{G}(e^{j\theta}))$$

is given by $\frac{1}{2}(\beta_1 + \beta_2)$ with tolerance ϵ.

With the state space parametrization of Ψ on $\mathscr{RH}_2$ space and our bisection algorithm for computing $\mathscr{H}_-$ index, the optimization process for solving Problem 10:

$$\max_{\|\Psi\|_2 \leq \gamma} \left\| \Psi \tilde{\boldsymbol{N}}_f \right\|_-^{[\theta_1, \theta_2]}$$

can be performed as:

$$\max_{\substack{A_{\psi k}, B_\psi, C_\psi, D_\psi, \|C_\psi\| \leq 1, \\ \mathrm{Trace}(\mathrm{D}'_\psi \mathrm{D}_\psi + \mathrm{B}'_\psi \mathrm{B}_\psi) \leq \gamma^2}} \left\| \left[\begin{array}{c|c} (I + A_{\psi k})(I - A_{\psi k})^{-1}(I - C'_\psi C_\psi)^{1/2} & B_\psi \\ \hline C_\psi & D_\psi \end{array} \right] \tilde{\boldsymbol{N}}_f \right\|_-^{[\theta_1, \theta_2]}$$

Furthermore, we introduce a penalty function $\Theta(B_\psi, C_\psi, D_\psi)$ to ensure the conditions $\mathrm{Trace}(\mathrm{D}'_\psi \mathrm{D}_\psi + \mathrm{B}'_\psi \mathrm{B}_\psi) < \gamma^2$ and $\|C_\psi\| \leq 1$. Θ is defined as

$$\Theta(B_\psi, C_\psi, D_\psi) = \begin{cases} C, & \text{if } \mathrm{Trace}(\mathrm{B}'_\psi \mathrm{B}_\psi + \mathrm{D}'_\psi \mathrm{D}_\psi) > \gamma^2 \text{ or } \|\mathrm{C}_\psi\| > 1; \\ 0, & \text{else} \end{cases}$$

where C is a large positive number. Therefore, the new optimization scheme is:

$$\max_{A_{\psi k}, B_\psi, C_\psi, D_\psi} \left\| \left[\begin{array}{c|c} (I + A_{\psi k})(I - A_{\psi k})^{-1}(I - C'_\psi C_\psi)^{1/2} & B_\psi \\ \hline C_\psi & D_\psi \end{array} \right] \tilde{\boldsymbol{N}}_f \right\|_-^{[\theta_1, \theta_2]} - \Theta(B_\psi, C_\psi, D_\psi)$$

Similar to the continuous case in section 3.4, we developed a two-stage optimization algorithm which is a combination of genetic algorithm and Nelder-Mead simplex method. Example in Chapter 5 will demonstrate the effectiveness of this optimization method.

4.7 $\mathscr{H}_2/\mathscr{H}_2$ Fault Detection Filter Design

The discrete $\mathscr{H}_2/\mathscr{H}_2$ problem has been extensively studied in the literature, see [7] and references therein. For completeness and comparison, we shall give an explicit solution to this problem under current problem formulation in this section.

Theorem 13 *Suppose Assumptions 5-7 are satisfied. Then*

$$\max_{\boldsymbol{Q}\in\mathscr{R}\mathscr{H}_\infty^{\mathrm{n_y}\times\mathrm{n_y}}}\{\|\boldsymbol{Q}\boldsymbol{N}_f\|_2 : \|\boldsymbol{Q}\boldsymbol{N}_d\|_2 \le \gamma\} = \gamma\left\|\boldsymbol{V}^{-1}\boldsymbol{N}_f\right\|_\infty .$$

Proof Note that by Lemma 12 we only need to solve Problem 9e:

$$\max_{\Psi\in\mathscr{R}\mathscr{H}_2^{\mathrm{n_y}\times\mathrm{n_y}}}\{\left\|\Psi\tilde{\boldsymbol{N}}_f\right\|_2 : \|\Psi\|_2 \le \gamma\}.$$

Now suppose $\left\|\tilde{\boldsymbol{N}}_f\right\|_\infty = \bar{\sigma}[\tilde{\boldsymbol{N}}_f(e^{j\theta_0})]$ for some $\theta_0 \in [0, 2\pi]$. We shall first consider the case $\theta_0 \ne 2\pi$. Let the singular value decomposition of $\tilde{\boldsymbol{N}}_f(e^{j\theta_0})$ be $U(e^{j\theta_0})\Sigma(e^{j\theta_0})V'(e^{j\theta_0})$. Let $u_1(e^{j\theta_0}) \in \mathscr{C}^{\mathrm{n_y}}$ be the left singular vector corresponding to the largest singular value $\bar{\sigma}[\tilde{\boldsymbol{N}}_f(e^{j\theta_0})]$, i.e., $u_1(e^{j\theta_0})$ is the first column of $U(e^{j\theta_0})$.

Write $u_1'(e^{j\theta_0})$ as

$$u_1'(e^{j\theta_0}) = \begin{bmatrix} \alpha_1 e^{j\theta_1} & \alpha_2 e^{j\theta_2} & \cdots & \alpha_{n_y} e^{j\theta_{n_y}} \end{bmatrix}$$

such that $\alpha_i \in \mathscr{R}$ and $\theta_i \in (-\pi, 0]$. Let $0 \le \beta_i < 1$ be such that

$$\theta_i = \angle\left(\frac{1 - e^{j\theta_0}\beta_i}{e^{j\theta_0} - \beta_i}\right)$$

and define

$$\hat{\boldsymbol{u}}(z) := \begin{bmatrix} \alpha_1\frac{1-z\beta_1}{z-\beta_1} & \alpha_2\frac{1-z\beta_2}{z-\beta_2} & \cdots & \alpha_{n_y}\frac{1-z\beta_{n_y}}{z-\beta_{n_y}} \end{bmatrix}.$$

Then

$$\hat{\mathbf{u}}(e^{j\theta_0}) = u_1'(e^{j\theta_0}).$$

Next let $\phi(z) \in \mathscr{RH}_2$ be such that

$$|\phi(e^{j\theta})| = \begin{cases} \gamma\sqrt{\frac{2\pi}{\epsilon}}, \ \theta \in [\theta_0, \theta_0 + \epsilon] \\ 0, \ \theta \notin [\theta_0, \theta_0 + \epsilon] \end{cases}$$

and ϵ is a small positive number.

Finally, let

$$\Psi(z) = \phi(z) e_1 \hat{\mathbf{u}}(z)$$

where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathscr{R}^{n_y}.$$

Then

$$\left\|\Psi \tilde{\boldsymbol{N}}_f\right\|_2 = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Trace}\{\Psi(\mathrm{e}^{\mathrm{j}\theta})\tilde{\boldsymbol{N}}_{\mathrm{f}}(\mathrm{e}^{\mathrm{j}\theta})\tilde{\boldsymbol{N}}_{\mathrm{f}}^{\sim}(\mathrm{e}^{\mathrm{j}\theta})\Psi^{\sim}(\mathrm{e}^{\mathrm{j}\theta})\}\mathrm{d}\theta}$$

$$= \sqrt{\frac{2}{2\pi}\int_{\theta_0}^{\theta_0+\epsilon} |\phi(e^{j\theta})|^2 \hat{\mathbf{u}}(e^{j\theta})\tilde{\boldsymbol{N}}_f(e^{j\theta})\tilde{\boldsymbol{N}}_f^{\sim}(e^{j\theta})\hat{\mathbf{u}}^{\sim}(e^{j\theta}) d\theta}$$

$$\approx \sqrt{\hat{\mathbf{u}}(e^{j\theta_0})\tilde{\boldsymbol{N}}_f(e^{j\theta_0})\tilde{\boldsymbol{N}}_f^{\sim}(e^{j\theta_0})\hat{\mathbf{u}}^{\sim}(e^{j\theta_0}) \ \frac{2}{2\pi}\int_{\theta_0}^{\theta_0+\epsilon} |\phi(e^{j\theta_0})|^2 d\theta}$$

$$= \bar{\sigma}(\tilde{\boldsymbol{N}}_f(e^{j\theta_0}))\sqrt{\frac{2}{2\pi}\int_{\theta_0}^{\theta_0+\epsilon} |\phi(e^{j\theta_0})|^2 d\theta} = \bar{\sigma}(\tilde{\boldsymbol{N}}_f(e^{j\theta_0}))\left\|\phi(z)\right\|_2 = \gamma\left\|\tilde{\boldsymbol{N}}_f\right\|_\infty$$

as $\epsilon \to 0$.

In the case $\theta_0 = 2\pi$, let

$$|\phi(e^{j\theta})| = \begin{cases} \gamma\sqrt{\frac{2\pi}{\epsilon}}, \ \theta \in [\theta_0 - \epsilon, \theta_0] \\ 0, \ \theta \notin [\theta_0 - \epsilon, \theta_0] \end{cases} .$$

and the above procedure can be performed. $\square$

In practice the ideal narrow bandpass filter $\phi(z)$ does not exist and we need to approximate it with a bandpass filter. The procedure of obtaining an approximate design is summarized below.

Discrete $\mathscr{H}_2/\mathscr{H}_2$ Fault Detection Filter Design Procedure:

- Find θ_0 such that $\left\|\tilde{\boldsymbol{N}}_f\right\|_\infty = \bar{\sigma}(\tilde{\boldsymbol{N}}_f(e^{j\theta}))$.

- Compute $u_1(e^{j\theta})$ from the singular value decomposition of

$$\tilde{\boldsymbol{N}}_f(e^{j\theta_0}) = U(e^{j\theta_0})\Sigma(e^{j\theta_0})V'(e^{j\theta_0}).$$

- Write $u_1'(e^{j\theta_0})$ as

$$u_1'(e^{j\theta_0}) = \left[\begin{array}{cccc} \alpha_1 e^{j\theta_1} & \alpha_2 e^{j\theta_2} & \cdots & \alpha_{n_y} e^{j\theta_{n_y}} \end{array}\right]$$

  such that $\alpha_i \in \mathscr{R}$ and $\theta_i \in (-\pi, 0]$. Let $0 \leq \beta_i < 1$ be such that

$$\theta_i = \angle\left(\frac{1 - e^{j\theta_0}\beta_i}{e^{j\theta_0} - \beta_i}\right)$$

  and define

$$\hat{\boldsymbol{u}}(z) := \left[\begin{array}{cccc} \alpha_1 \frac{1-z\beta_1}{z-\beta_1} & \alpha_2 \frac{1-z\beta_2}{z-\beta_2} & \cdots & \alpha_{n_y} \frac{1-z\beta_{n_y}}{z-\beta_{n_y}} \end{array}\right].$$

- Design a proper bandpass filter $\boldsymbol{F}(z) \in \mathscr{RH}_2$ which has pass-band around θ_0;

- Let $\phi(z) = \frac{\gamma \boldsymbol{F}(z)}{\|\boldsymbol{F}(z)\|_2}$;

- Let $\Psi(z) = \phi(z) e_1 \hat{\mathbf{u}}(z)$ and $\boldsymbol{Q} = \Psi \boldsymbol{V}^{-1}$.

Remark 23 *Note that the remarks for continuous $\mathscr{H}_2/\mathscr{H}_2$ problem in section 3.5 also apply to this problem.*

Chapter 5

Numerical Examples

In this Chapter we give some numerical examples to show the effectiveness of our approaches for solving the fault detection problems. Comparison with existing results are also made. Section 5.1 includes the examples for continuous problems, and examples for discrete problems are shown in section 5.2.

5.1 Numerical Examples for Continuous Fault Detection Problems

5.1.1 Examples for $\mathscr{H}_-/\mathscr{H}_\infty$ Problem

In this section we demonstrate and compare three examples from the existing literature for Problem 1.

Example 1 *We consider Problem 1 with the example from [34]. A fourth order system is given as:*

$$A = \begin{bmatrix} -10 & 0 & 5 & 0 \\ 0 & -5 & 0 & 2.5 \\ 0 & 0 & -2.5 & 0 \\ 0 & 5 & 0 & -3.75 \end{bmatrix}, B_d = \begin{bmatrix} 0.8 & 0.04 \\ -2.4 & 0.08 \\ 1.6 & 0.08 \\ 0.8 & 0.08 \end{bmatrix}, B_f = \begin{bmatrix} 4 \\ 4 \\ 8 \\ -8 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, D_d = \begin{bmatrix} 0.2 & 0.04 \\ 0.4 & 0.06 \end{bmatrix}, D_f = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

For easy comparison we let the pair (γ, β) *represent the performance of an* $\mathscr{H}_-/\mathscr{H}_\infty$ *fault detection filter such that* $\|\boldsymbol{G}_{rd}\|_\infty \leq \gamma$ *and* $\|\boldsymbol{G}_{rf}\|_-^{[\infty]} \geq \beta$. *The best result by Tao and Zhao [34] using LMI approach is:* $\gamma = 0.5435$, $\beta = 2.2361$ *and* $\frac{\beta}{\gamma} = 4.1143$. *Using our approach an optimal fault detection filter has the form in Theorem 2 with*

$$L_0 = \begin{bmatrix} 8 & -6 \\ -44 & 28 \\ 16 & -12 \\ 4 & -4 \end{bmatrix}$$

so that for $\gamma = 0.5435$ *we have* $\beta = \gamma \left\| \tilde{\boldsymbol{N}}_f \right\|_-^{[\infty]} = 2.2721$ *and* $\frac{\beta}{\gamma} = 4.1805$. *In contrast to Tao and Zhao's approach where each filter has to be designed for a fixed* (γ, β) *value, our filter is optimal for all* γ *value and given any* $\gamma > 0$ *the optimal fault detection filter can be computed by simply multiplying a scalar. The singular value plots of* $\boldsymbol{G}_{rd}$ *and* $\boldsymbol{G}_{rf}$ *are shown in Figure 5.1 and Figure 5.2, respectively.*

Example 2 *We consider Problem 1 with an fourth order system from [35] which is given by*

$$A = \begin{bmatrix} -5.2 & 0.65 & 6.5 & 1.3 \\ -1.56 & -2.6 & 0 & 2.6 \\ -1.3 & 0 & -1.3 & 0 \\ -0.26 & 0 & 3.9 & -1.95 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0.5 & 0.03 \\ -1.5 & 0.02 \\ 1.0 & -0.04 \\ 0.5 & 0.01 \end{bmatrix}, \quad B_f = \begin{bmatrix} 2 \\ 2 \\ 4 \\ -4 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.3 & 0.3 & 0 & 0.3 \\ 0.3 & 0 & 0.3 & 0 \end{bmatrix}, \quad D_d = \begin{bmatrix} 0.15 & 0.012 \\ 0.3 & 0.015 \end{bmatrix}, \quad D_f = \begin{bmatrix} 1.6 \\ -0.8 \end{bmatrix}.$$

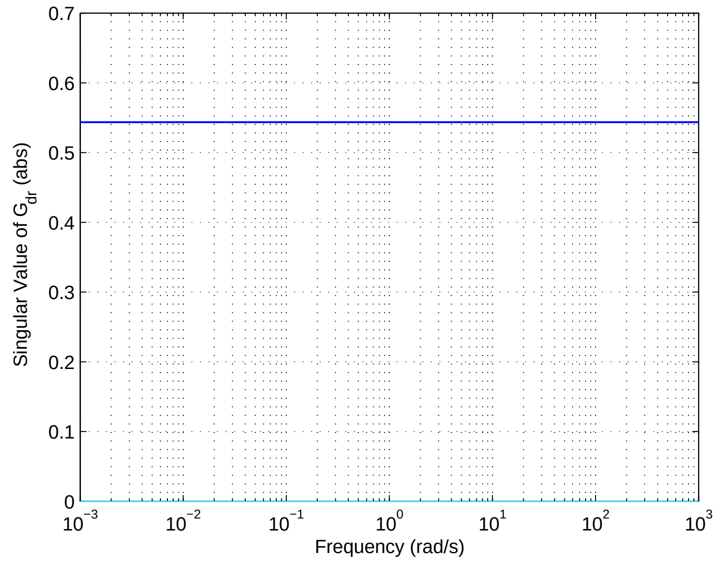

Figure 5.1: Example 1, singular value plot of $\boldsymbol{G}_{rd}$, $\|\boldsymbol{G}_{rd}\|_\infty = 0.5435$

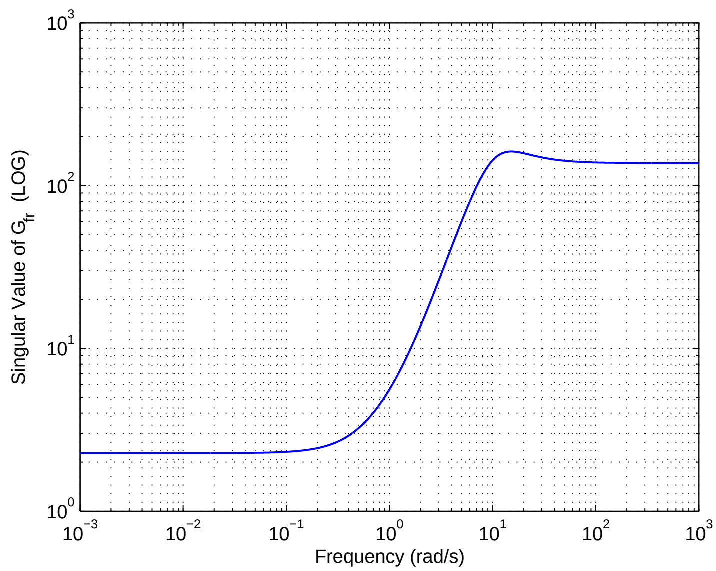

Figure 5.2: Example 1, singular value plot of $\boldsymbol{G}_{rf}$, $\|\boldsymbol{G}_{rf}\|_- = 2.2721$

The results reported in [35] are $\|\boldsymbol{G}_{rd}\|_\infty = 0.34$ *and* $\|\boldsymbol{G}_{rf}\|_\infty = 1.785$. *Since* $\|\boldsymbol{G}_{rf}\|_- < \|\boldsymbol{G}_{rf}\|_\infty$, *it is concluded that* $\gamma = 0.34$, $\beta < 1.785$ *and* $\frac{\beta}{\gamma} < 5.2493$. *For comparison, we also let* $\gamma = 0.34$, *then our approach gives the optimal fault detection filter in the form of Theorem 2 with*

$$L_0 = \begin{bmatrix} -1.1111 & -1.1111 \\ -21.1111 & 15.5556 \\ 20 & -13.3333 \\ 3.3333 & -3.3333 \end{bmatrix}.$$

We can further compute that this optimal filter design has $\gamma = 0.34$ *and* $\beta = 27.0364$ *such that* $\frac{\beta}{\gamma} = 79.5188$. *The singular value plots of* $\boldsymbol{G}_{rd}$ *and* $\boldsymbol{G}_{rf}$ *are shown in Figure 5.3 and Figure 5.4.*

Furthermore, these two detection designs are simulated by taking

$$d(t) = \begin{bmatrix} \sin(2t) & \cos(2t) \end{bmatrix}'$$

and

$$f_1(t) = \begin{cases} 0.5, & 5s \le t < 10s \\ 0, & \text{elsewhere.} \end{cases}$$

The time responses of the residual signals are plotted in Figure 5.5, which shows that our residuals have a much larger amplitude than the reference signal so that they are much more sensitive to fault. With the same $d(t)$ *we redo the simulations for a different*

$$f_2(t) = \begin{cases} 0.2(t-5), & 5s \le t < 7.5s \\ 0.2(10-t), & 7.5s \le t < 10s \\ 0, & \text{elsewhere} \end{cases}$$

and the results are shown in Figure 5.6.

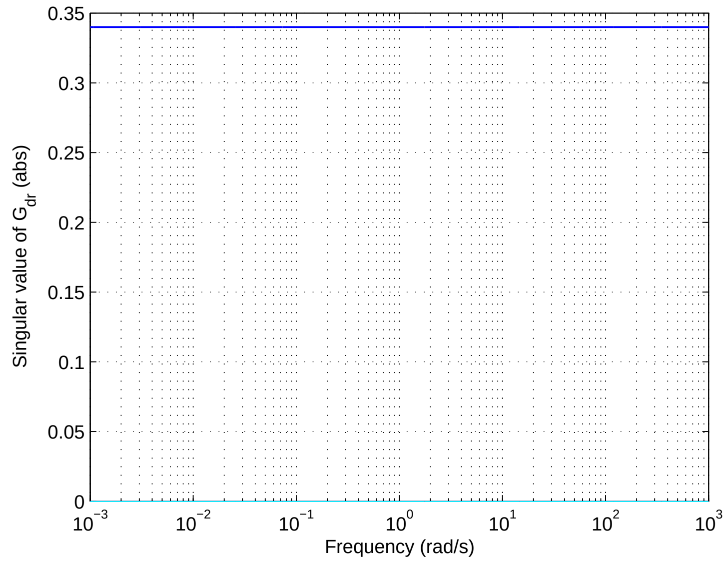

Figure 5.3: Example 2, singular value plot of $\boldsymbol{G}_{rd}$, $\|\boldsymbol{G}_{rd}\|_{\infty} = 0.3400$

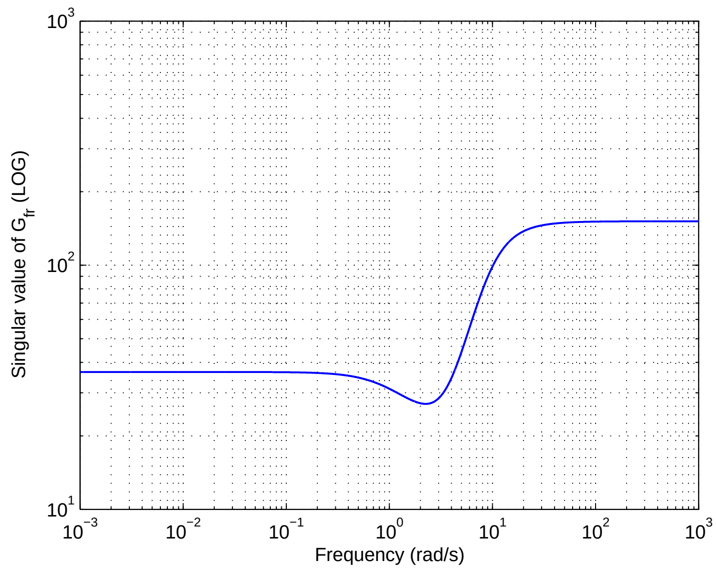

Figure 5.4: Example 2, singular value plot of $\boldsymbol{G}_{rf}$, $\|\boldsymbol{G}_{rf}\|_{-} = 27.0364$

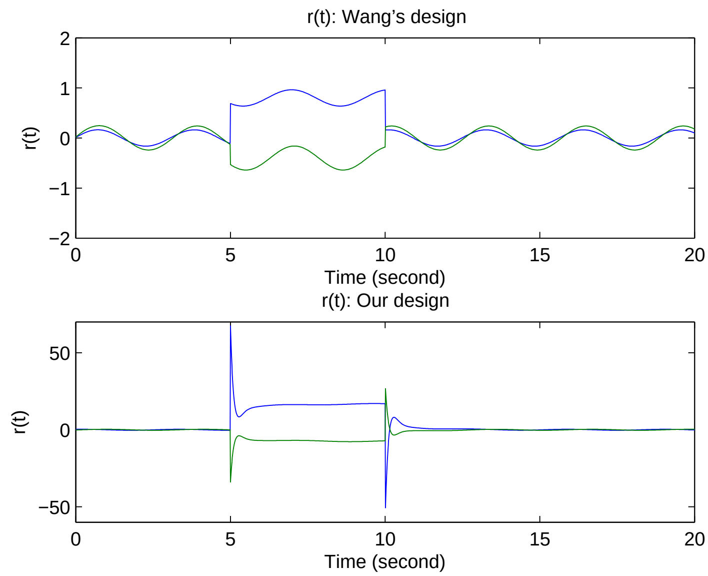

Figure 5.5: Example 2, residual responses with $f(t) = f_1(t)$

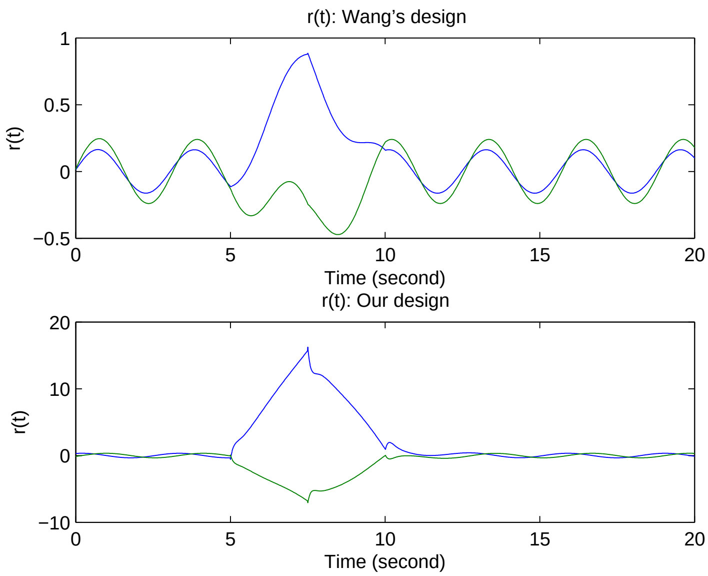

Figure 5.6: Example 2, residual responses with $f(t) = f_2(t)$

Example 3 *We consider Problem 1 for a modified F16XL system which is given in [15]*

$$A = \begin{bmatrix} -0.0674 & 0.0430 & -0.8886 & -0.5587 \\ 0.0205 & -1.4666 & 16.5800 & -0.0299 \\ 0.1377 & -1.6788 & -0.6819 & 0 \\ 0 & 0 & 1.0000 & 0; \end{bmatrix}, B_d = \begin{bmatrix} 0.0430 & -0.1672 \\ -1.4666 & -1.5179 \\ -1.6788 & -9.7842 \\ 0 & 0 \end{bmatrix},$$

$$B_f = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; D_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \; D_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The D_d matrix in this system does not have full row rank so we need to augment additional columns to D_d and B_d such that $\tilde{B}_d = \begin{bmatrix} B_d & 0 \end{bmatrix}$ and $\tilde{D}_d = \begin{bmatrix} D_d & \epsilon I \end{bmatrix}$ has full row rank for a small $\epsilon > 0$. Following our approach we have the optimal fault detection filter in the form of Theorem 2 with

$$L_0 = \begin{bmatrix} -430 & 0 & 0.1672 \\ 14666 & 0 & 1.5179 \\ 16788 & 0 & 9.7842 \\ 0 & 0 & 0 \end{bmatrix}.$$

Furthermore, let $\gamma = 1$, the optimal filter has $\beta = 0.8498$ so that $\frac{\beta}{\gamma} = 0.8498$. This result is the same as the result in [15] but our approach is much simpler. The singular value plots of $\boldsymbol{G}_{rd}$ and $\boldsymbol{G}_{rf}$ are shown in Figure 5.7 and Figure 5.8.

Table 5.1: Result comparison for $\mathscr{H}_-/\mathscr{H}_\infty$ problem

	Previous optimal $\frac{\beta}{\gamma}$	Our optimal $\frac{\beta}{\gamma}$
Example 1 (Tao and Zhao)	4.1143	4.1805
Example 2 (Wang et al)	<5.2493	79.5188
Example 3 (Jaimoukha et al)	0.8498	0.8498

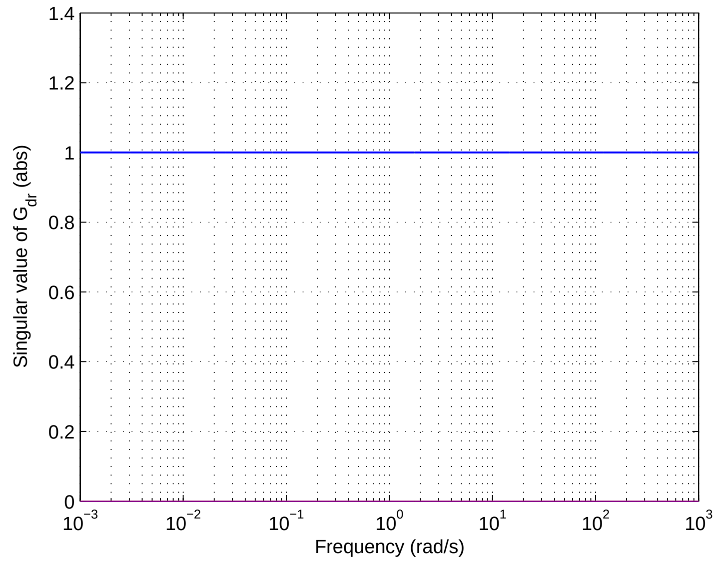

Figure 5.7: Example 3, singular value plot of $\boldsymbol{G}_{rd}$, $\|\boldsymbol{G}_{rd}\|_\infty = 1$

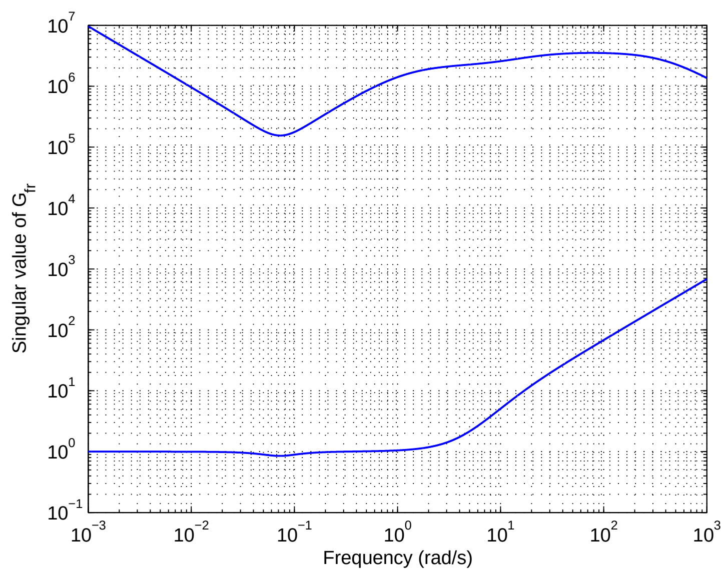

Figure 5.8: Example 3, singular value plot of $\boldsymbol{G}_{rf}$, $\|\boldsymbol{G}_{rf}\|_- = 0.8498$

The comparison between our results and the existing results on $\mathscr{H}_-/\mathscr{H}_\infty$ problem are summarized in Table 5.1.

5.1.2 Example for $\mathscr{H}_2/\mathscr{H}_\infty$ Problem

In this section we will show how to solve the $\mathscr{H}_2/\mathscr{H}_\infty$ problem using our approach through an example.

Example 4 *We consider Problem 2 with the system data given in Example 1 but*

$$D_f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $D_f = 0$, $\boldsymbol{N}_f$ is strict proper. From the conclusion in Chapter 3, we have the optimal fault detection filter for this problem in the form of Theorem 2 with

$$L_0 = \begin{bmatrix} 8 & -6 \\ -44 & 28 \\ 16 & -12 \\ 4 & -4 \end{bmatrix}.$$

If we let $\gamma = 1$ then the maximum of $\mathscr{H}_2/\mathscr{H}_\infty$ problem for this example is $\left\|\tilde{\boldsymbol{N}}_f\right\|_2 = 142.7382$. The singular value plots of $\boldsymbol{G}_{rd}$ and $\boldsymbol{G}_{rf}$ are shown in Figure 5.9 and Figure 5.10.

5.1.3 Example for $\mathscr{H}_-/\mathscr{H}_2$ Problem

In this section we will show how to solve the interval $\mathscr{H}_-/\mathscr{H}_2$ problem using our approach through an example.

Example 5 *We consider Problem 5 for the same system given in Example 1 where $\omega_1 = 0$ and $\omega_2 = 10$.*

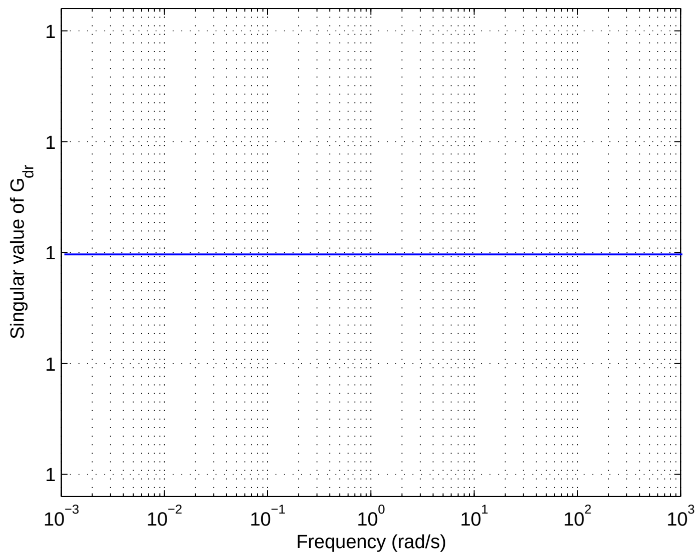

Figure 5.9: Example 4, singular value plot of $\boldsymbol{G}_{rd}$, $\|\boldsymbol{G}_{rd}\|_{\infty} = 1$

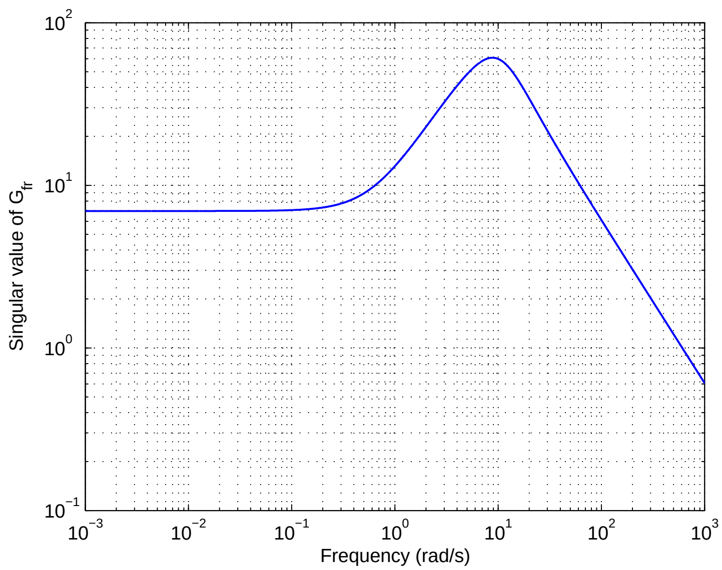

Figure 5.10: Example 4, singular value plot of $\boldsymbol{G}_{rf}$, $\|\boldsymbol{G}_{rf}\|_{2} = 142.7382$

As discussed in Chapter 3 we use optimization method to search for a good solution. Let's denote the maximum of $\| \boldsymbol{G}_{rf} \|_{-}^{[\omega_1,\omega_2]}$ *as* β*. In Table 5.2 the results obtained using our optimization algorithm with different pre-defined* Ψ *orders are given. It is clear that the results improve with the increasing order of* Ψ*.*

Table 5.2: Results for different Ψ's order

Ψ's order	First	Second	Third	Fourth
β	8.3275	9.0071	9.0351	9.1544

In particular, a fourth order Ψ *design achieving* $\beta = 9.1544$ *is given by*

$$\Psi = \left[\begin{array}{cccc|cc} -0.5674 & 0.6757 & -0.0199 & 0.8283 & 0.5688 & -0.2844 \\ 0.3763 & -0.4959 & 2.2232 & 1.2855 & -0.4489 & 0.2392 \\ -1.4427 & -0.9850 & -1.3597 & -1.2312 & -0.5036 & 0.2389 \\ 1.5196 & -3.6904 & 2.6378 & -4.0005 & 0.1608 & -0.0186 \\ \hline 0.8188 & -0.6766 & 1.6396 & -0.5598 & 0 & 0 \\ 0.6815 & -0.7307 & 0.1763 & -2.7727 & 0 & 0 \end{array}\right].$$

The singular value plots of $\boldsymbol{G}_{rd}$ *and* $\boldsymbol{G}_{rf}$ *are shown in Figure 5.11 and Figure 5.12 for a fourth order* Ψ*.*

Figure 5.13 demonstrates how the smallest singular value of $\boldsymbol{G}_{rf}$ *changes in the frequency range of* $[0, 10]$ *with the order of* Ψ*. It is seen that the improvement on the performance with any* Ψ *of higher order than 4 is insignificant.*

It is interesting to note that the Ψ *is trying to invert* $\tilde{\boldsymbol{N}}_f$ *in the frequency interval* $[0, 10]$.

5.1.4 Example for $\mathscr{H}_2/\mathscr{H}_2$ Problem

In this section we will show how to solve the $\mathscr{H}_2/\mathscr{H}_2$ problem using our approach through an example.

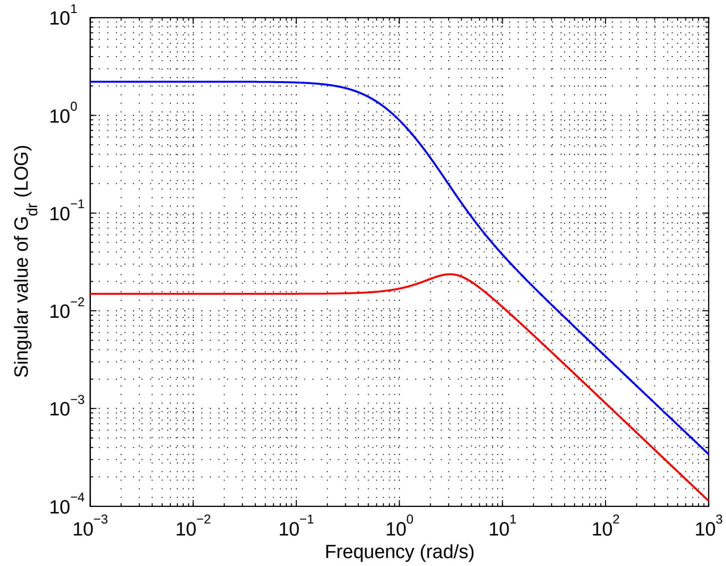

Figure 5.11: Example 5, singular value plot of $\boldsymbol{G}_{rd}$ with a fourth order Ψ, $\|\boldsymbol{G}_{rd}\|_2 = 1$

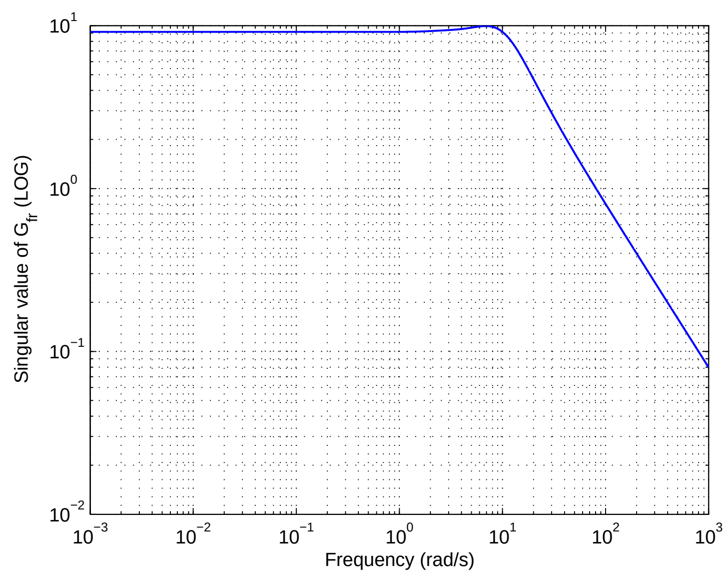

Figure 5.12: Example 5, singular value plot of $\boldsymbol{G}_{rf}$ with a fourth order Ψ, $\|\boldsymbol{G}_{rf}\|_{-}^{[0,10]} = 9.1544$

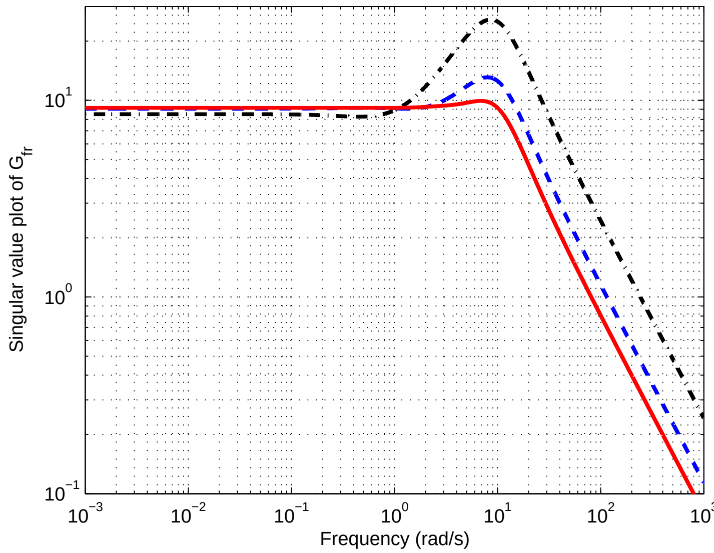

Figure 5.13: Example 5, singular value plot of $\boldsymbol{G}_{rf}$ for different order of Ψ: first order (dash-doted line), second order (dashed line), and fourth order (solid line)

Example 6 *We study Problem 4 for the system given in Example 1.*

Let $\|\boldsymbol{G}_{rd}\|_2 = \gamma$ *and* $\|\boldsymbol{G}_{rf}\|_2 = \beta$. *The result given in [35] for this example is* $\gamma = 0.8$, $\beta = 2.2$ *and* $\frac{\beta}{\gamma} = 2.75$. *So for comparison, we also let* $\gamma = 0.8$. *Since*

$$\left\|\tilde{\boldsymbol{N}}_f\right\|_\infty = \max_\omega\{\bar{\sigma}(\tilde{\boldsymbol{N}}_f(j\omega))\} = 298.1507$$

and the maximum is achieved at $\omega_0 = 15.1938$, *the maximum for Problem 4 is* $298.1507\gamma = 238.5206$. *Following the design procedure after Theorem 6, first we compute*

$$u_1(j\omega_0) = \begin{bmatrix} -0.7650 - j0.4854 \\ 0.3455 + j0.2445 \end{bmatrix}$$

so that

$$u_1'(j\omega_0) = [-0.9060e^{-j0.5654} \; 0.4233e^{-j0.6159}].$$

Then

$$\hat{\mathbf{u}}(s) = \begin{bmatrix} -0.9060\dfrac{52.3059 - s}{52.3059 + s} & 0.4233\dfrac{47.7689 - s}{47.7689 + s} \end{bmatrix}.$$

A second order Butterworth filter is designed as

$$\boldsymbol{F}(s) = \frac{0.3s}{s^2 + 0.3s + 229.5}$$

which is strictly proper and has a pass band around w_0. *Therefore*

$$\phi(s) = \frac{\gamma \boldsymbol{F}(s)}{\|\boldsymbol{F}(s)\|_2} = \frac{0.6197s}{s^2 + 0.3s + 229.5}$$

and

$$\Psi(s) = \phi(s) e_1 \hat{\mathbf{u}}(s) = \left[\begin{array}{cccc|cc} -33.238 & -0.611 & -3.452 & -14.846 & 0.346 & 0.081 \\ 140.281 & -48.689 & -8.285 & -109.065 & -1.450 & 0.666 \\ -20.148 & 22.327 & -1.026 & 12.879 & 0.195 & -0.088 \\ -39.014 & 0.140 & 2.066 & -17.422 & 0.403 & 0.094 \\ \hline -0.013 & -0.226 & 1.246 & -0.013 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Further computation shows that this design has $\gamma = 0.8$ *and* $\beta = 237.848$, *which is close to the maximum. The singular value plots of* $\boldsymbol{G}_{rd}$ *and* $\boldsymbol{G}_{rf}$ *are shown in Figure 5.14 and Figure 5.15.*

5.2 Numerical Examples for Discrete Fault Detection Problems

In this section we give some numerical examples to show the effectiveness of our approaches for solving discrete fault detection problems.

5.2.1 Example for Discrete $\mathscr{H}_-/\mathscr{H}_\infty$ Problem

Example 7 *We consider Problem 6 for a third order system:*

$$A = \begin{bmatrix} -0.1964 & -0.3962 & -0.5884 \\ 1 & 2 & 3 \\ -0.5428 & -1.0879 & -1.6291 \end{bmatrix}, B_d = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \\ 0 & 0 \end{bmatrix}, B_f = \begin{bmatrix} 1 & -3 \\ -0.5 & 1 \\ 0.5 & 0 \end{bmatrix},$$

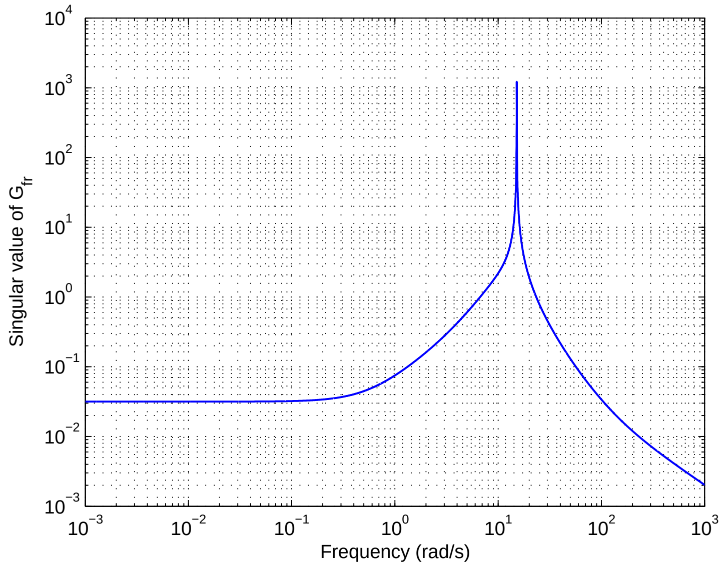

Figure 5.14: Example 6, singular value plot of $\boldsymbol{G}_{rd}$, $\|\boldsymbol{G}_{rd}\|_2 = 0.8$

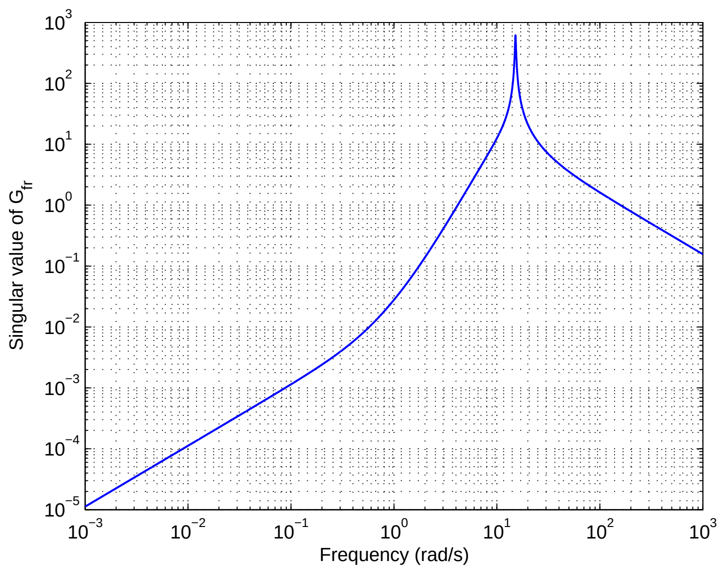

Figure 5.15: Example 6, singular value plot of $\boldsymbol{G}_{rf}$, $\|\boldsymbol{G}_{rf}\|_2 = 237.848$

$$C = \begin{bmatrix} -0.1964 & -0.3962 & -0.5884 \\ -0.3650 & -2.1320 & -3.0951 \end{bmatrix}, D_d = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix}, D_f = \begin{bmatrix} 0 & 0.7 \\ 0 & 1 \end{bmatrix}$$

Let the pair (γ, β) represent the performance of an $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection filter such that $\|\boldsymbol{G}_{rd}\|_\infty \leq \gamma$ and $\|\boldsymbol{G}_{rf}\|_-^{[0,2\pi]} \geq \beta$. Using our approach an optimal fault detection filter has the form in Theorem 8 with

$$L_0 = \begin{bmatrix} -0.05 & 0 \\ 0 & -0.05 \\ 0 & 0 \end{bmatrix}$$

Let $\gamma = 1$ we have the optimal $\beta = \gamma \left\| \tilde{\boldsymbol{N}}_f \right\|_-^{[0,2\pi]} = 0.7632$. The singular value plots of $\boldsymbol{G}_{rd}$ and $\boldsymbol{G}_{rf}$ are shown in Figure 5.16 and Figure 5.17, respectively.

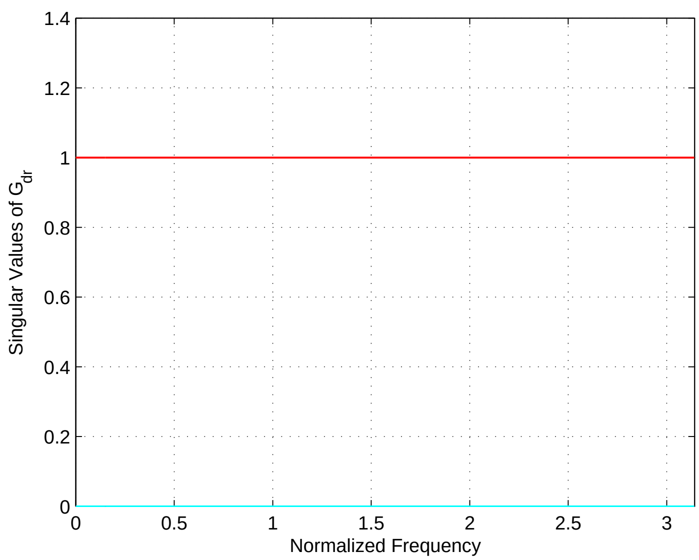

Figure 5.16: Example 7, singular value plot of $\boldsymbol{G}_{rd}$, $\|\boldsymbol{G}_{rd}\|_\infty = 1$

5.2.2 Example for Discrete $\mathscr{H}_2/\mathscr{H}_\infty$ Problem

Example 8 *We consider Problem 7 for the same system in Example 7.*

Let the pair (γ, β) represent the performance of an $\mathscr{H}_2/\mathscr{H}_\infty$ fault detection filter such that

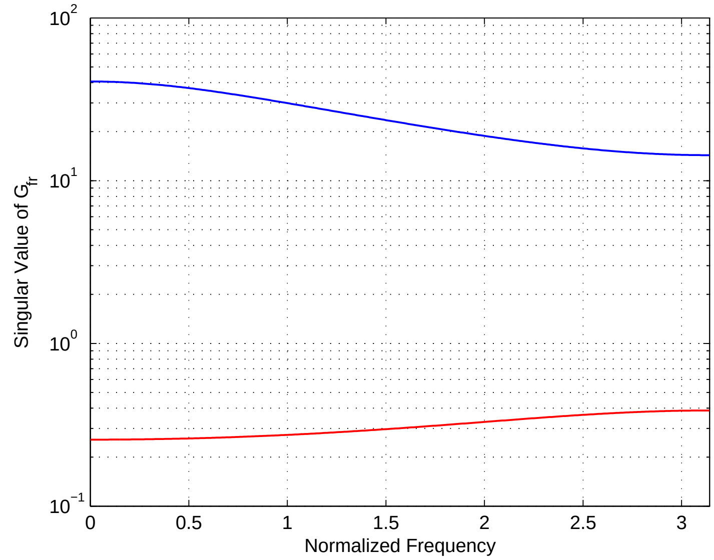

Figure 5.17: Example 7, singular value plot of $\boldsymbol{G}_{rf}$, $\|\boldsymbol{G}_{rf}\|_{-}^{[0,2\pi]} = 0.75$

$\|\boldsymbol{G}_{rd}\|_{\infty} \leq \gamma$ *and* $\|\boldsymbol{G}_{rf}\|_{2}^{[0,2\pi]} \geq \beta$. *From Theorem 9 the optimal fault detection filter in Example 7 is also optimal for this example. Let* $\gamma = 1$, *the optimal* $\beta = \gamma \left\|\tilde{\boldsymbol{N}}_f\right\|_2 = 9.7591$.

Note that if the so-called $\mathscr{H}_\infty/\mathscr{H}_\infty$ *problem is considered for this system, the above fault detection filter is also the optimal* $\mathscr{H}_\infty/\mathscr{H}_\infty$ *filter. Let* $\|\boldsymbol{G}_{rd}\|_{\infty} \leq 1$, *then the optimal* $\|\boldsymbol{G}_{rf}\|_{\infty}$ *is* 11.4598.

5.2.3 Examples for Discrete $\mathscr{H}_-/\mathscr{H}_2$ Problem

Example 9 *We consider Problem 8 for the system:*

$$A = \begin{bmatrix} -0.1964 & -0.3962 & -0.5884 \\ 1 & 2 & 3 \\ -0.5428 & -1.0879 & -1.6291 \end{bmatrix}, B_d = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \\ 0 & 0 \end{bmatrix}, B_f = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0.5 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.1964 & -0.3962 & -0.5884 \\ -0.3650 & -2.1320 & -3.0951 \end{bmatrix}, D_d = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix}, D_f = \begin{bmatrix} 1 & 0.7 \\ 0 & 1 \end{bmatrix}$$

We let the pair (γ, β) represent the performance of an $\mathscr{H}_-/\mathscr{H}_2$ fault detection filter such that $\|\boldsymbol{G}_{rd}\|_2 \leq \gamma$ and $\|\boldsymbol{G}_{rf}\|_-^{[0,2\pi]} \geq \beta$. Since this $\boldsymbol{G}_f$ has all zeros inside the unit circle and D_f has full column rank, we get from Theorem 12

$$(\boldsymbol{V}^{-1}\boldsymbol{N}_f)^+_{opt} = \left[\begin{array}{ccc|cc} -0.2555 & -1.4924 & -2.1666 & 0.1900 & -0.1400 \\ 1.3059 & 3.0358 & 4.5169 & 0.2000 & 0.0500 \\ -0.5723 & -1.6360 & -2.4182 & 0.1000 & -0.0700 \\ \hline -0.0591 & -1.0962 & -1.5782 & 0.2000 & -0.1400 \\ 0.3650 & 2.1320 & 3.0951 & 0 & 0.2000 \end{array}\right]$$

and the optimal filter

$$\begin{aligned} \boldsymbol{Q}_{opt} &= \frac{\gamma}{\left\|(\boldsymbol{V}^{-1}\boldsymbol{N}_f)^+_{opt}\right\|_2}(\boldsymbol{V}^{-1}\boldsymbol{N}_f)^+_{opt}\boldsymbol{V}^{-1} \\ &= \left[\begin{array}{ccc|cc} -0.0740 & 0.3126 & -0.8467 & -1.2003 & 0.3892 \\ -2.8761 & 1.3720 & -5.9237 & 5.7439 & -0.2946 \\ -0.4110 & 0.2304 & -0.9359 & 0.8321 & -0.0352 \\ \hline 0.7188 & -0.7509 & 2.5122 & 0.7430 & -0.5201 \\ -1.7340 & 1.3456 & -4.8682 & 0 & 0.7430 \end{array}\right]. \end{aligned}$$

Let $\gamma = 1$ the optimal

$$\beta = \frac{\gamma}{\left\|(\boldsymbol{V}^{-1}\boldsymbol{N}_f)^+_{opt}\right\|_2} = 0.7430.$$

The singular value plots of $\boldsymbol{G}_{rd}$ and $\boldsymbol{G}_{rf}$ are shown in Figure 5.18 and Figure 5.19, respectively.

Example 10 *We consider Problem 10 for a system:*

$$A = \begin{bmatrix} -0.1964 & -0.3962 & -0.5884 \\ 1.0000 & 2.0000 & 3.0000 \\ -0.5428 & -1.0879 & -1.6291 \end{bmatrix}, B_d = \begin{bmatrix} 0.01 \\ 0.01 \\ 0 \end{bmatrix}, B_f = \begin{bmatrix} 1.0 \\ -0.5 \\ 0.5 \end{bmatrix}$$

$$C = \begin{bmatrix} -0.1964 & -0.3962 & -0.5884 \end{bmatrix}, D_d = 0.02, D_f = 0.$$

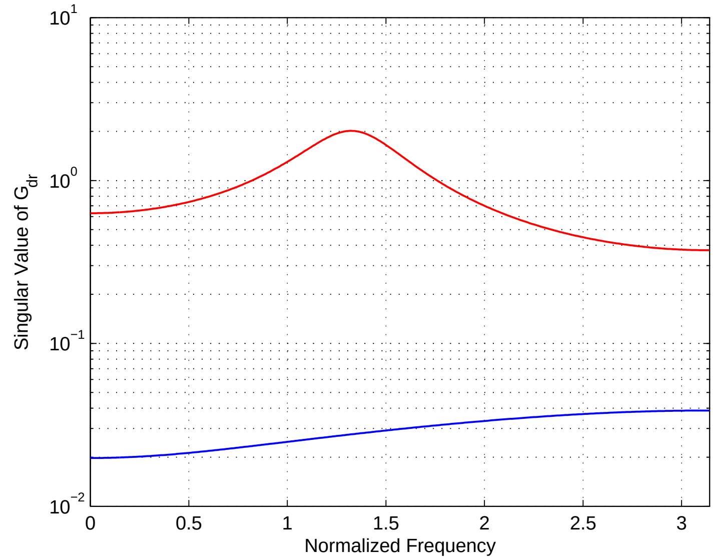

Figure 5.18: Example 9, singular value plot of $\boldsymbol{G}_{rd}$, $\|\boldsymbol{G}_{rd}\|_2 = 1$

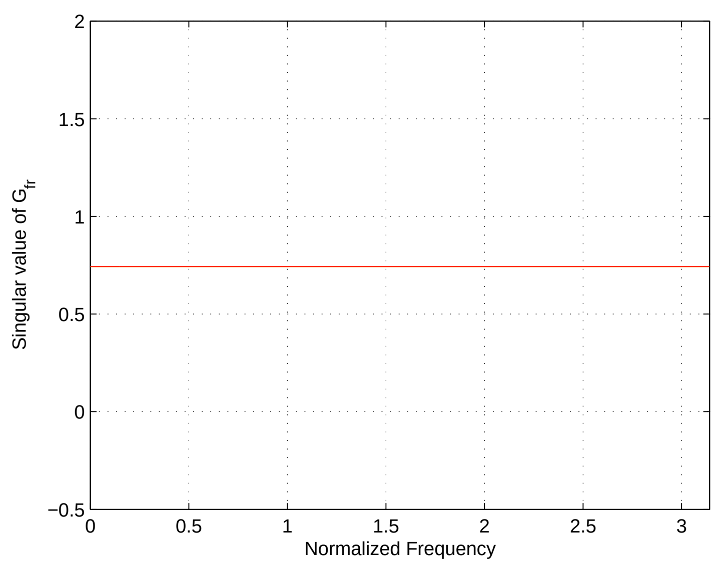

Figure 5.19: Example 9, singular value plot of $\boldsymbol{G}_{rf}$, $\|\boldsymbol{G}_{rf}\|_- = 0.7430$

where $\theta_1 = 0$ and $\theta_2 = \pi/2$.

As discussed in Chapter 4 we use optimization method to search for a good solution. Let's denote the maximum of $\|\boldsymbol{G}_{rf}\|_{-}^{[\theta_1,\theta_2]}$ as β when $\|\boldsymbol{G}_{rd}\|_2 \leq 1$. In Table 5.3 the results obtained using our optimization algorithm with different pre-defined Ψ orders are given. It is clear that the results improve with the increasing order of Ψ.

Table 5.3: Example 10, results for different Ψ's order

Ψ's order	First	Second	Third
β	14.2661	22.5345	22.8182

In particular, a third order Ψ design achieving $\beta = 22.8182$ is given by

$$\Psi = \left[\begin{array}{ccc|c} 0.1187 & -0.0019 & 0.4270 & -0.1424 \\ -0.8445 & 0.4220 & 0.3270 & 0.2363 \\ -0.3536 & -0.9012 & 0.2408 & 0.8865 \\ \hline 0.3844 & 0.0988 & 0.8079 & 0.3714 \end{array}\right].$$

The singular value plots of $\boldsymbol{G}_{rd}$ and $\boldsymbol{G}_{rf}$ are shown in Figure 5.20 and Figure 5.21 for this third order Ψ.

Figure 5.22 demonstrates how the smallest singular value of $\boldsymbol{G}_{rf}$ changes in the frequency range of $[0, \pi/2]$ with the order of Ψ. It is seen that the improvement on the performance with any Ψ of higher order than 3 is insignificant.

It is interesting to note that the Ψ is trying to invert $\tilde{\boldsymbol{N}}_f$ in the frequency interval $[0, \pi/2]$.

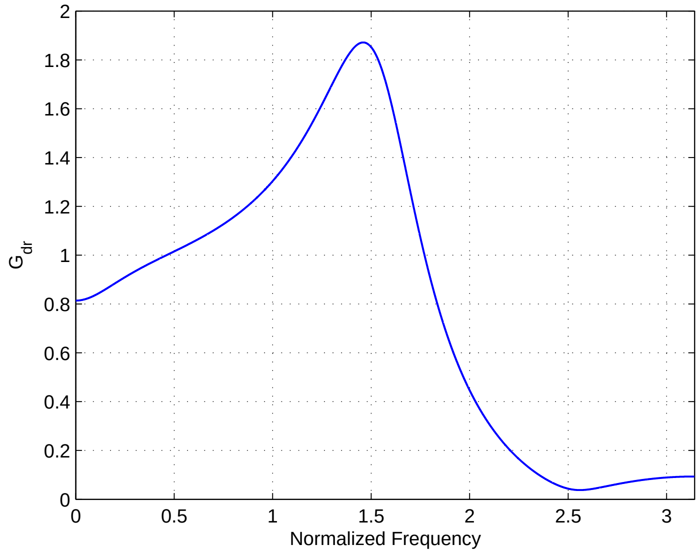

Figure 5.20: Example 10, singular value plot of $\boldsymbol{G}_{rd}$ with a third order Ψ, $\|\boldsymbol{G}_{rd}\|_2 = 1$

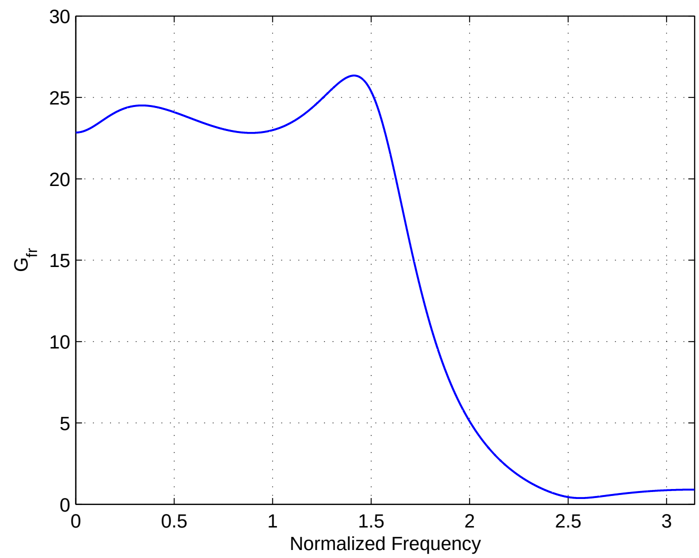

Figure 5.21: Example 10, singular value plot of $\boldsymbol{G}_{rf}$ with a third order Ψ, $\|\boldsymbol{G}_{rf}\|_{-}^{[0,\pi/2]} = 22.8182$

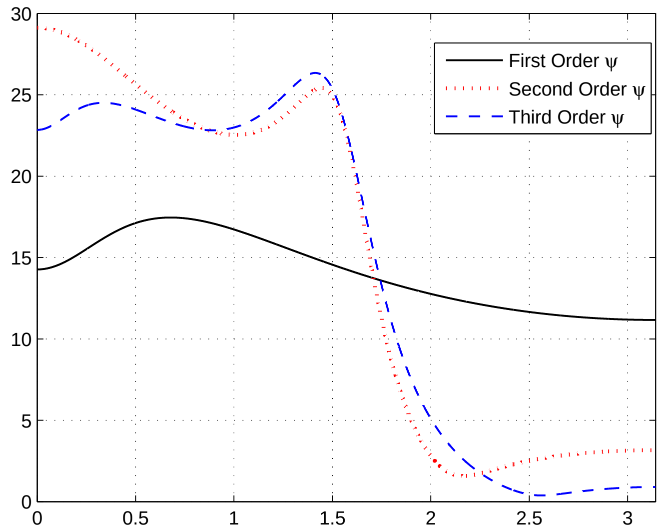

Figure 5.22: Example 10, singular value plot of $\boldsymbol{G}_{rf}$ for different order of Ψ: first order (solid line), second order (dotted line), and third order (dashed line)

5.2.4 Example for Discrete $\mathscr{H}_2/\mathscr{H}_2$ Problem

Example 11 *We study Problem 9 for the same system in Example 7.*

Let $\|\boldsymbol{G}_{rd}\|_2 = \gamma$ *and* $\|\boldsymbol{G}_{rf}\|_2 = \beta$. *Let* $\gamma = 1$,

$$\left\|\tilde{\boldsymbol{N}}_f\right\|_\infty = \max_{\theta\in[0,2\pi]}\{\bar{\sigma}(\tilde{\boldsymbol{N}}_f(e^{j\theta}))\} = 11.4598$$

and the maximum is achieved at $\theta_0 = \pi$, *so the maximum for discrete* $\mathscr{H}_2/\mathscr{H}_2$ *problem is* $11.4598\gamma = 11.4598$. *Following the design procedure after Theorem 13, first we compute*

$$u_1(e^{j\theta_0}) = \begin{bmatrix} 0.2328 + j0 \\ 0.9725 + j0 \end{bmatrix}$$

so that

$$u_1'(e^{j\theta_0}) = [0.2328e^{j0} \quad 0.9725e^{j0}].$$

Then

$$\hat{\mathbf{u}}(z) = \begin{bmatrix} 0.2328 & 0.9725 \end{bmatrix}.$$

A second order Butterworth filter is designed as

$$\boldsymbol{F}(z) = \frac{0.015466(z-1)(z+1)}{z^2 + 1.967z + 0.9691}$$

which has a pass band around θ_0*. Therefore*

$$\phi(z) = \frac{\gamma \boldsymbol{F}(z)}{\|\boldsymbol{F}(z)\|_2} = \frac{0.1244(z-1)(z+1)}{z^2 + 1.967z + 0.9691}$$

and

$$\Psi(z) = \phi(z) e_1 \hat{\mathbf{u}}(z) = \begin{bmatrix} \frac{0.02895(z-1)(z+1)}{z^2+1.967z+0.9691} & \frac{0.12094(z-1)(z+1)}{z^2+1.967z+0.9691} \\ 0 & 0 \end{bmatrix}.$$

Further computation shows that this design has $\beta = 11.3994$*, which is close to the maximum. The singular value plots of* $\boldsymbol{G}_{rd}$ *and* $\boldsymbol{G}_{rf}$ *are shown in Figure 5.23 and Figure 5.24.*

It is clear from the frequency response plot (Figure 5.24) that this fault detection filter can only detect faulty signals near the frequency π*. Hence it may not have much value in practical applications.*

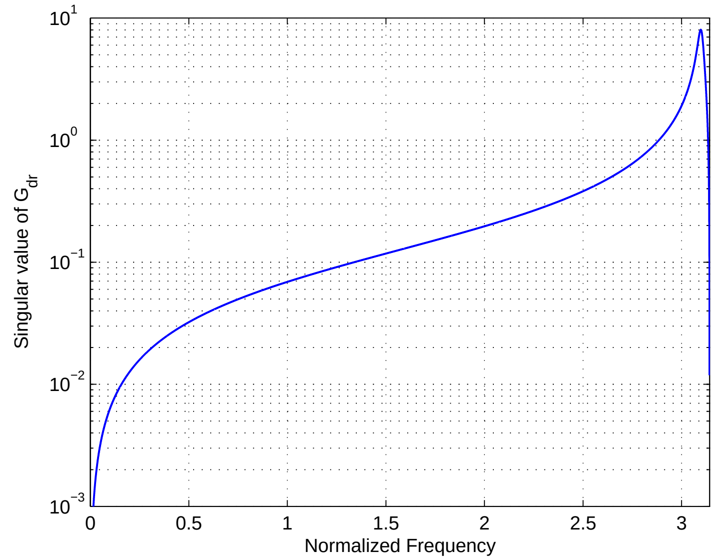

Figure 5.23: Example 11, singular value plot of $\boldsymbol{G}_{rd}$, $\|\boldsymbol{G}_{rd}\|_2 = 1$

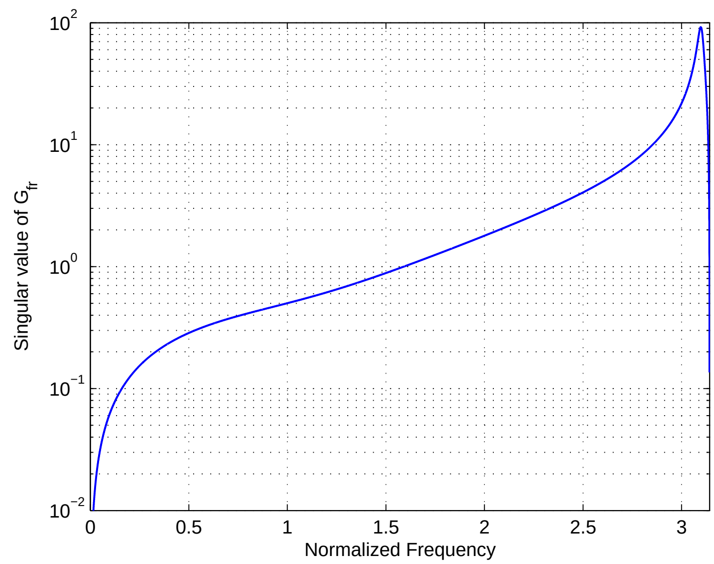

Figure 5.24: Example 11, singular value plot of $\boldsymbol{G}_{rf}$, $\|\boldsymbol{G}_{rf}\|_2 = 11.3994$

Chapter 6

Conclusion

In this dissertation, we have derived analytic and optimal solutions to various robust fault detection problems for both continuous time and discrete time cases.

For continuous fault detection problems, we have shown that an optimal filter for both $\mathscr{H}_-/\mathscr{H}_\infty$ and $\mathscr{H}_2/\mathscr{H}_\infty$ can be obtained by solving one Riccati equation. We also presented solutions to different cases of $\mathscr{H}_-/\mathscr{H}_2$ problem and $\mathscr{H}_2/\mathscr{H}_2$ problem.

In parallel with our continuous time results, we have presented analytic and optimal solutions to various robust fault detection problems for linear discrete time systems. We have shown that an optimal filter for both discrete $\mathscr{H}_-/\mathscr{H}_\infty$ and $\mathscr{H}_2/\mathscr{H}_\infty$ problems can be obtained by solving one discrete Riccati equation. It is also interesting to note that we are able to give analytic solution to a discrete $\mathscr{H}_-/\mathscr{H}_2$ problem defined on the entire frequency range $[0, 2\pi]$ when D_f has full column rank. In contrast, the corresponding continuous time problem does not make any sense [23]. The critical reason for this difference is because the entire frequency range in discrete time is finite ($\leq 2\pi$) while the entire frequency range in continuous time is infinite.

For both continuous and discrete cases, we have also shown that many design criteria considered in the literature do not give desirable fault detection designs. This result presented an in-depth understanding towards the essence of fault detection problems and provided a guideline for choosing appropriate design criteria.

Based on the results above, we believe several possible directions for future research in this area could be:

- The study of robust fault detection problems for linear time-varying systems.
- The removal of Assumption 2, i.e., to consider the case that D_d does not have full row rank.
- The study of robust fault detection problems when model uncertainties are considered.
- The study of finding the complete solution set for robust fault detection problems.

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Appendix A: The Removal of Assumption 3

It is noted that all results presented in this dissertation are derived based on three assumptions, Assumptions 1-3. While Assumptions 1 and 2 are standard, Assumption 3 may pose certain limitations. We shall now present a technique to remove Assumption 3.

We shall start with a $m \times m$ square system

$$\boldsymbol{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Lemma 18 *Let* $\boldsymbol{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *be a* $m \times m$ *square transfer matrix and assume that it has full normal rank, i.e.,* $\det \boldsymbol{G}(s) \neq 0$ *for some* $s \in \mathscr{C}$*. Suppose* $j\omega_0$ *is a transmission zero of* $\boldsymbol{G}(s)$*, i.e., there exist* $\xi \in \mathscr{C}^n$ *and* $0 \neq \eta \in \mathscr{C}^m$ *such that* $\begin{bmatrix} A - j\omega_0 I & B \\ C & D \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0.$

Let η *be normalized such that* $\eta'\eta = 1$*. Define*

$$B_\epsilon = B - \epsilon\xi\eta', \quad \boldsymbol{G}_a(s) = I - \frac{\epsilon}{s + \epsilon - j\omega_0}\eta\eta', \quad \boldsymbol{G}_m = \left[\begin{array}{c|c} A & B_\epsilon \\ \hline C & D \end{array}\right].$$

Then

$$\boldsymbol{G}(s) = \boldsymbol{G}_m(s)\,\boldsymbol{G}_a(s)$$

and

$$\|\,\boldsymbol{G}_a(s)\|_\infty = 1.$$

Proof

$$
\begin{aligned}
\boldsymbol{G}_m(s)\,\boldsymbol{G}_a(s) &= \left[\begin{array}{c|c} A & B_\epsilon \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} -\epsilon + j\omega_0 & -\epsilon\eta' \\ \hline \eta & I \end{array}\right] = \left[\begin{array}{cc|c} A & B_\epsilon\eta & B_\epsilon \\ 0 & -\epsilon + j\omega_0 & -\epsilon\eta' \\ \hline C & D\eta & D \end{array}\right] \\
&= \left[\begin{array}{cc|c} A & (A - j\omega_0 I)\xi + B\eta & B_\epsilon + \epsilon\xi\eta' \\ 0 & -\epsilon + j\omega_0 & -\epsilon\eta' \\ \hline C & C\xi + D\eta & D \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & -\epsilon + j\omega_0 & -\epsilon\eta' \\ \hline C & 0 & D \end{array}\right] \\
&= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \boldsymbol{G}(s).
\end{aligned}
$$

It is easy to see $\|\boldsymbol{G}_a\|_\infty = 1$. □

Note that the above factorization can be applied recursively to factorize out all imaginary axis transmission zeros so that the remaining part of the system does not contain any imaginary axis transmission zeros. It is easy to illustrate this process for a scalar transfer function. For example, suppose a transfer function has a pair of imaginary axis zeros at $j\omega_0$ and $-j\omega_0$. Then it can be written as

$$\boldsymbol{G} = \frac{n_1(s)(s^2 + \omega_0^2)}{d(s)}$$

where $n_1(s)$ and $d(s)$ are polynomials with no imaginary axis roots. Then for any $\epsilon > 0$, we can factorize $\boldsymbol{G}(s)$ as

$$\boldsymbol{G}(s) = \frac{n_1(s)((s+\epsilon)^2 + \omega_0^2)}{d(s)} \; \frac{s^2 + \omega_0^2}{(s+\epsilon)^2 + \omega_0^2}$$

and

$$\frac{s^2 + \omega_0^2}{(s+\epsilon)^2 + \omega_0^2} \approx 1$$

for a sufficiently small $\epsilon > 0$. Since the transfer matrix $\boldsymbol{G}_d$ in this paper is not square, we need first augment it to a square transfer matrix. Let $\tilde{D} \in \mathscr{R}^{(n_d - n_y)\times n_d}$ be such that

$$\begin{bmatrix} D_d \\ \tilde{D} \end{bmatrix}$$

is nonsingular. Let

$$\boldsymbol{G}_{daug} = \begin{bmatrix} \boldsymbol{G}_d \\ \tilde{D} \end{bmatrix} = \left[\begin{array}{c|c} A & B_d \\ \hline C & D_d \\ 0 & \tilde{D} \end{array}\right]$$

and apply Lemma 18 to $\boldsymbol{G}_{daug}$ to get

$$\boldsymbol{G}_{daug} = \begin{bmatrix} \boldsymbol{G}_m \\ \tilde{D} \end{bmatrix} \boldsymbol{G}_a.$$

Hence we can always factorize $\boldsymbol{G}$ as

$$\boldsymbol{G}_d = \boldsymbol{G}_m \boldsymbol{G}_a$$

so that $\boldsymbol{G}_m$ has no zero on the imaginary axis. Note that

$$\sigma_i\left(\boldsymbol{G}_a(j\omega)\right) = 1,\ i = 1, 2, \ldots, n_d - 1$$

and

$$\sigma_{n_d}\left(\boldsymbol{G}_a(j\omega)\right) = \left|\frac{j\omega - j\omega_0}{j\omega + \epsilon - j\omega_0}\right|$$

for any frequency ω. If $\epsilon > 0$ is sufficiently small, then we have for every singular value

$$\sigma_i\left(\boldsymbol{G}_a(j\omega)\right) \approx 1$$

for every frequency ω. Hence Assumption 3 can be removed without sacrificing much of the performance since

$$\|\boldsymbol{Q}\boldsymbol{G}_d\|_\infty \approx \|\boldsymbol{Q}\boldsymbol{G}_m\|_\infty, \quad \|\boldsymbol{Q}\boldsymbol{G}_d\|_2 \approx \|\boldsymbol{Q}\boldsymbol{G}_m\|_2$$

for a sufficiently small $\epsilon > 0$.

Appendix B: Matlab Code

Matlab Code for Computing $\mathscr{H}_-$ Index

This is a Matlab function programmed by the author. The function calculates the $\mathscr{H}_-$ index of a given system matrix.

```
hminnorm.m

% out = hminnorm(sys,tol)
% Compute the H minus index of a given SYSTEM matrix.
% This function takes two inputs: the system matrix and the tolerance.
% The default value for the tolerance is 0.0001.
% The output of this function is the H minus norm.
% Max iteration

function out = hminnorm(sys,tol)
 if nargin == 0
   disp('usage: out = hminnorm(sys,ttol)')
   return
 end
 if nargin == 1
   tol = 0.0001;
 end

 zerodef = 1e-09;

 [type,rows,cols,num] = minfo(sys);
 if type == 'vary'
   disp('A VARYING matrix has not H minus definition')
 elseif type == 'cons'
   if nargout == 1
     out = norm(sys);
   else
     disp(['System is a CONSTANT matrix and
     the H minus norm is ' num2str(norm(sys))])
   end
```

```
elseif type == 'syst'
  [A,B,C,D] = unpck(sys);
  if max(real(eig(A))) >= 0
    disp('System is not stable')
    if nargout == 1
      out = nan;
    end
    return
  end

  lbound = 0;
  temp = frsp(sys, 1e3);
  ubound = abs(temp(1,1));  %initial guess of upper bound

  if (ubound-lbound <= tol)  %if the initial guess is good enough
    if nargout == 1
      out = (ubound-lbound)/2;
    else
      disp(num2str((ubound-lbound)/2));
    end
    return
  end

  maxiter = 100;
  i = 1;

  while (ubound-lbound > tol)&&(i <= maxiter)  %bisection
    mid = (ubound+lbound)/2;
    R = mid*mid*eye(cols)-D'*D;
    S = inv(R);
    [m n] = size(D*S*D');
    %Computation of Hamitonian matrix
    H = [           A+B*S*D'*C               B*S*B'
         -C'*(eye(m)+D*S*D')*C       -(A+B*S*D'*C)'];
    try
      E = eig(H,'nobalance');
    catch
      disp('The Hamitonian matrix is ill-conditioned')
      rethrow(lasterror);
    end
    m = length(E);
    NoImagEig = 1;
    for i = 1:m
      eig_real = abs(real(E(i)));
```

```
      eig_imag = abs(imag(E(i)));
      if (eig_real <= zerodef)    %altercating form of bounded real lemma
        NoImagEig = 0;
      end
    end
    if NoImagEig == 1
      lbound=mid;
    else
      ubound = mid;
    end %end if
    i = i+1;
  end %end while
  if i <= 100       %Iterations less than maxiter
    if nargout == 1
      out = mid;
    else
      disp(num2str(mid))
    end
  else   %iteration more than maxiter
    disp('Max iterations limit reached, algorithm fails to converge.')
    disp(['Current lower bound: ' num2str(lbound)])
    disp(['Current upper bound: ' num2str(ubound)])
  end
 else
  disp('A SYSTEM matrix input is required')
 end
}
```

Matlab Code for Two Stage Optimization

This is the example code for two stage optimization for Problem 5 (Example 5). The objective function was set for fourth order Ψ.

```
hmintwo.m (The main routine)

global r1 r2 Nf1 Q Gfr

A=[-10 0     5      0
   0 -5    0      2.5
   0     0     -2.5      0
   0     5       0   -3.75];
Bd=[0.8  0.04
     -2.4  0.08
     1.6  0.08
```

```
     0.8  0.08];
C=[1  0  0  1
   1  0  1  1];
Dd=[0.2 0.04
    0.4 0.06];
Bf=[4 4 8 -8]';
Df=[2 -1]';
%E=alpha*[-1 -2 -3 -4];
%L=place(A',-C',E);
L=-lqr(A',C',eye(4),eye(2));
L=L';
M=ss(A+L*C,L,C,eye(2));
Nf=ss(A+L*C,Bf+L*Df,C,Df);
Nd=ss(A+L*C,Bd+L*Dd,C,Dd);
%Nf=ssbal(Nf);
%Nd=ssbal(Nd);
[A1,B1,C1,D1]=ssdata(Nd);
R1=D1*D1';
Y=are(A1-B1*D1'*inv(R1)*C1, C1'*inv(R1)*C1, B1*(eye(2)-D1'*inv(R1)*D1)*B1');
V=ss(A1,(B1*D1'+Y*C1')*R1^(-0.5),C1,R1^0.5);
%V=ssbal(V);
V1=inv(V);
%V1=ssbal(V1);
Nf1=V1*Nf;
%Gfr=0.5435*V1*Nf;
%Gdr=minrealm(Gdr);
Nf1=minreal(Nf1);

options=gaoptimset('PopInitRange',[-1;1],'PopulationSize',1000,
'Generation',1000,'StallTimeLimit',500,'Display','iter');
[x0,fval,reason,output,population,scores]=ga(@fun_mintwo4,22,options);
options=optimset('LargeScale','off','Display','final');
[x,fval,exitflag,output]=fminsearch(@fun_mintwo4,x0,options);

w=logspace(log10(0.001),log10(1000),10000);
Gs=pck(Gfr.a,Gfr.b,Gfr.c,Gfr.d);
Gf=frsp(Gs,w);
[u,s,d]=vsvd(Gf);
vplot('liv,lm',s), grid;
%title('Gfr');

%[r1,w0]=norm(Gdr,inf,0.00001);
%r2=v;
r1
```

```
r2
Q
```

fun_mintwo4.m (The optimization objective function)

```
function f=fun_mintwo4(x)
global Nf1 Q r1 r2 Gfr
gamma=1;
Bq=[x(1) x(2); x(3) x(4); x(5) x(6);x(7) x(8)];
Cq=[x(9) x(10) x(11) x(12) ; x(13) x(14) x(15) x(16)];
Dq=zeros(2,2);
Aqk=[0 x(17) x(18) x(19); -x(17) 0 x(20) x(21); -x(18) -x(20) 0 x(22);
-x(19) -x(21) -x(22) 0];
if trace(Bq'*Bq)>gamma
  Pe=1000;
  f=Pe;
else
  Pe=20;
  Aqs=-0.5*Cq'*Cq;
  Aq=Aqk+Aqs;
  Q=ss(Aq,Bq,Cq,Dq);
  norm1=norm(Q,2);
  %disp(['The H_inf norm of Q is ',num2str(norm(1))]);
  Gfr=Q*Nf1;
  Gfr=minrealm(Gfr);

  A2=Gfr.a;
  B2=Gfr.b;
  C2=Gfr.c;
  D2=Gfr.d;
  G=ss(A2,B2,C2,D2);
  w1=0;
  w2=10;
  v1=0;
  Gf=bode(G,w2);
  v2=min(svd(Gf));
  %v2=1000;

  while((v2-v1)>0.00001)
  v=(v1+v2)/2;
  R2=v*v*eye(1)-D2'*D2;
  S2=inv(R2);
  H=[          A2+B2*S2*D2'*C2               B2*S2*B2'
     -C2'*(eye(2)+D2*S2*D2')*C2       -(A2+B2*S2*D2'*C2)'];
```

```
  E=eig(H,'nobalance');
  m=length(E);
  NoImagEig=1;
  for i=1:m
    eig_real=abs(real(E(i)));
    eig_imag=abs(imag(E(i)));
    if (eig_real<=0.00001)&&(eig_imag>=w1)&&(eig_imag<=w2)
      NoImagEig=0;
    end
  end
  if NoImagEig==1
    v1=v;
  else
    v2=v;
  end %end if
  end %end while

  r1=norm1(1);
  r2=v;
  if (r1>gamma)
    Pe=1000;
  end
  f=Pe-r2;
end
```

Appendix C: Authorization Letter

Copyright Authorization
Rasha Magdy <rasha.magdy@hindawi.com> Tue, Mar 18, 2008 at 2:52 AM
To: nick liu <nickliu79@gmail.com>

Dear Mr. liu,

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Please let me know if I can be of any further help.

Best regards,

Rasha Magdy

Rasha Magdy
Editorial Staff
Hindawi Publishing Corp.
http://www.hindawi.com

Vita

Nike Liu was born in Wuhan, China, on May 1st, 1979. He received his bachelor's and master's of engineering degrees in automatic control from Huazhong University of Science and Technology in 2000 and 2002, respectively. In fall 2003, he entered the graduate program in the Department of Electrical and Computer Engineering at Louisiana State University. He received his master of science degree in electrical engineering in December 2005. Now he is a candidate for the degree of Doctor of Philosophy in electrical and computer engineering.