2009

Fault detection filter design for linear systems

Xiaobo Li
Louisiana State University and Agricultural and Mechanical College, xli12@tigers.lsu.edu

FAULT DETECTION FILTER DESIGN FOR LINEAR SYSTEMS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Electrical and Computer Engineering

by
Xiaobo Li
B.S., Central South University, China, 2003
M.S., Louisiana State University, USA, 2008
August, 2009

To My Family

Acknowledgments

I would like to gratefully and sincerely thank my advisor and committee chair, Dr. Kemin Zhou for his guidance, understanding and patience. It was him who introduced me into fault detection and robust control. He was always there to meet and discuss my progress, to proofread my papers, and to guide me to new ideas. He had confidence in me when I doubted myself and brought out the good ideas in me. Without his encouragement and constant guidance, I couldn't have finished this dissertation.

My heartful thanks also go to Dr. Guoxiang Gu and Dr. Jorge Aravena, from whom I have benefited greatly for their breadth of knowledge and inspiring thoughts. Dr. Gu's courses were a good resource of my research ideas, from where I had learned the fundamental and advanced methods and techniques in robust control, system identification, etc. I also benefited a lot from Dr. Aravena's course, and with his encouragement and help, I finished my course project successfully.

I am also very grateful to Dr. Peter Wolenski and Dr. Bijaya B. Karki for being my graduate committee members and their kind encouragements. Specifically, Dr. Wolenski's course brought my interest on nonsmooth analysis and optimization in control.

Finally, I would like to give my thanks to all my friends in the Department of Electrical and Computer Engineering, Zhongshan Wu, Nike Liu, Dimeng Wang, Lili Kong, Xi Chen, Louis Algerge, Laurentiu Marinovici and Rachinayani Kumar Phalguna.

Xiaobo Li

August, 2009

Table of Contents

List of Figures

Notation and Symbols

$A_{m \times n}$: m-row and n-column matrix

A^{-1}: Inverse of matrix A

A^{+}: Pseudo inverse of matrix A

A': Transpose of matrix A

$A^{\sim}$: Complex conjugate transpose of matrix A

I_n: Identity matrix of size $n \times n$

$0_{m \times n}$: Zero matrix of size $m \times n$

$\underline{\sigma}(A)$: Smallest singular value of matrix A

$\bar{\sigma}(A)$: Largest singular value of matrix A

$Tr\{A\}$: Trace of matrix A

$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$: Abbreviation of transfer matrix $D + C(sI - A)^{-1}B$ or $D + C(zI - A)^{-1}B$

$\|G\|_\infty$: $\mathscr{H}_\infty$ norm of system G

$\|G\|_2$: $\mathscr{H}_2$ norm of system G

$\|G\|_-$: $\mathscr{H}_-$ index of system G

$\|G\|_\infty^S$: $\mathscr{H}_\infty$ norm of system G over subspace S

$\|G\|_2^S$: $\mathscr{H}_2$ norm of system G over subspace S

$\|G\|_-^S$: $\mathscr{H}_-$ index of system G over subspace S

$\mathscr{H}_\infty$: Analytic function set in $Re(s) > 0$

$G^{\sim}$: Adjoint of system G

$\mathscr{R}$: Set of real numbers

$\mathscr{R}_+$: Set of real positive numbers

$< A, B >$: Inner product of systems A and B

$ker\{G\} : \{u : Gu = 0, u \in \mathscr{L}_2^n\}$

$image\{G\} : \{y \in \mathscr{L}_2^m : y = Gu, u \in \mathscr{L}_2^n\}$

$\mathscr{E}$: Expectation operator

List of Acronyms

ARE	algebraic Riccati equation
ARMA	autoregressive moving average model
DRE	differential Riccati equation
DDRE	difference Riccati equation
FD	fault detection
FDI	fault detection and isolation
GRE	generalized Riccati equation
LCF	left coprime factorization
LCTI	linear continuous time invariant
LCTV	linear continuous time-varying
LDTI	linear discrete time invariant
LDTV	linear discrete time-varying
LMI	linear matrix inequality
LTIS	linear time invariant systems
LTVS	linear time-varying systems
MIMO	multi-input and multi-output
MOO	multiple-objective optimization

Abstract

This dissertation considers residual generation for robust fault detection of linear systems with control inputs, unknown disturbances and possible faults.

First, multi-objective fault detection problems such as $\mathscr{H}_-/\mathscr{H}_\infty$, $\mathscr{H}_2/\mathscr{H}_\infty$ and $\mathscr{H}_\infty/\mathscr{H}_\infty$ have been formulated for linear continuous time-varying systems (LCTVS) in time domain for finite horizon and infinite horizon case, respectively. It is shown that under mild assumptions, the optimal solution is an observer determined by solving a standard differential Riccati equation (DRE). The solution is also extended to the case when the initial state for the system is unknown.

Second, the parallel problems are also solved for linear discrete time-varying systems in time domain. The solution is again an observer whose gain is determined by solving a standard recursive difference Riccati equation (DDRE). The solution is also extended to the case when the initial state for the system is unknown.

Third, for the general case in which G_d (the transfer matrix from disturbance to output) may be a tall or square transfer matrix, and D_d may not have full column rank for linear discrete time invariant systems (LDTIS), the common $\mathscr{H}_-/\mathscr{H}_\infty$, $\mathscr{H}_2/\mathscr{H}_\infty$ and $\mathscr{H}_\infty/\mathscr{H}_\infty$ frameworks are not applicable. Based on several novel definitions of norms over a certain subspace, we propose a new problem formulation with both disturbance decoupling and fault sensitivity optimization. It is shown that the solution is an observer determined by a generalized Riccati equation (or Riccati system, alternatively). To be more specific, with this

filter, some faults in certain subspace can be completely decoupled from the residual signal, while the others are optimized in terms of fault sensitivity. Furthermore, the completely non-decoupling and decoupling conditions are given. Disturbance rejection based on the solution is discussed. A direct procedure for deriving the fault detection filter in transfer matrix form is also proposed.

Finally, some potential further research problems are outlined.

Chapter 1

Introduction

1.1 Overview of Fault Detection

Fault diagnosis has received much attention for complex modern automatic systems such as car, aircraft, rockets, etc. since 1970s [8,23,58,63]. The high complexities of modern systems are vulnerable to almost unavoidable faults such as sudden breakdown or malfunction of a sensor or an actuator. Such unpredictable non-safety can cause significant performance deteriorations of control systems, and possibly damages or destructions of the whole systems. Therefore, in order to improve the system reliability and operational safety, reducing the possibility of those failures or predicting its happening before its occurrence is imperative. One direct way is to employ backup sensors or actuators for some important parts, such that the system is able to automatically replace the faulty parts once certain faults are identified. However, such a strategy may not always be feasible physically or economically, which is particularly true for some practical lower-cost systems. An alternative way, which was first introduced by R.V Beard and H.L.Jones [2, 45], is to use the analytical redundancy implied in the systems to detect and identify the faults. Thus the system is able to shut down itself or employ some procedures to tolerate the faults. The later one is the motivation of fault diagnosis [21].

There is no standard and formal definition on fault diagnosis in the current literature.

The widely accepted concept is that a fault is an unexpected change of system function although it may not represent physical failure or breakdown [8]. A monitor system which is used to detect faults and diagnosis their location and significance in a system is called a fault diagnosis system. Generally speaking, fault diagnosis is normally in charge of the following tasks [8]:

- Fault detection (FD): to make a decision whether there is something wrong or not.
- Fault isolation: to locate the fault, e.g., which sensor, actuator or system components has become faulty.
- Fault identification: to identify the size and type or nature of the fault.

As the first task of fault diagnosis, fault detection is the basis of the last two tasks and it is an absolute must in fault diagnosis. This dissertation will focus on fault detection. The reader interested in fault isolation and fault identification is referred to [8] and references therein.

The task of constructing a fault detection system [8, 23] is normally as follows:

- to design a residual generator that eliminates the effects of process input signals and if possible, also the effects of disturbances and model uncertainties on the residual;
- to design a residual evaluator by selecting a suitable evaluation function $\|\cdot\|_e$ and determining the threshold J_{th}.
- if a full elimination of the effects of disturbances and model uncertainties on the residual is not possible, to optimize the residual generator and evaluator to achieve the maximum set of detectable faults.

It can be seen that residual generation as the first step in fault detection is very critical in the fault detection design, since it plays the role of detecting the fault signal or fault information, and the poor performance in the residual generation will directly affect the next two steps significantly. Therefore, residual generation has received much attention among researchers recently [8].

Depending on the priori knowledge available to engineers, fault detection methods can be classified into two categories [77–79]: model-based fault detection and data-driven fault detection. Model-based fault detection must have a mathematical system model for the plant such as ARMA model, state-space model, or transfer function, which are often constructed from the physics of process. Data-driven fault detection is based on a great amount of historical data available, and it is suitable to be applied to large-scale industrial systems. In this dissertation, we will focus on model-based fault detection, where the model is of state-space form. Furthermore, if the model is linear time invariant, we also take it in a transfer matrix form. Reader interested in data-driven method is referred to [7] and references therein.

There are also other classifications for the fault detection. For instance, according to the system it monitors, the fault detection approaches can be classified as follows [8]: sampled-data systems FDI [41, 49, 87, 89], stochastic systems FD [10], nonlinear system FD [63], etc. Recently, active fault detection has also received great interest from researchers, in which input signal is designed for stimulating the system characteristics so that fault detection becomes easier. The fundamental issue in this direction concentrates on how to design appropriate input signal such that the system characteristics is well stimulated [4,5]. On the contrary, the other design methods are called passive fault detection [60,61]. This dissertation will focus on passive fault detection.

As the most important part of fault detection, the common residual generation techniques include [23]: eigenstructure approach [11, 57, 62, 83], observer-based approach including Leunberger observer in a deterministic framework [2] and Kalman filter in a stochastic framework [99], parity space based approach [25, 64], parameter estimation approach [38] and factorization approach [80]. This dissertation will discuss factorization approach, since all other methods can be regarded as special cases.

1.2 Model-Based Fault Detection

The first step in the model-based approach is to build a mathematical model for the monitored system as mentioned before. State space model is most widely used to represent MIMO systems. In addition, most complicated systems with possible faults can be described easily in state space form. In this dissertation the state space model of the following linear form is taken,

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u_R(t) \\ y_R(t) &= C(t)x(t) + D(t)u_R(t). \end{aligned}$$

Furthermore, if system parameters depend on time, this system will be called linear time-varying system (LTVS). Otherwise, it will be called linear time-invariant system (LTIS).

With the above model, the next stage is to characterize the fault. Generally, there are three fault sources: component fault, sensor fault and actuator fault, which can be modelled in a general framework in fault detection. Specifically, the following general form is widely accepted in fault detection community to describe plants with possible faults:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_f f(t) \\ y(t) &= Cx(t) + Du(t) + D_f f(t) \end{aligned}$$

where B_f and D_f describe how the fault f occurred in the systems. Since the system is linear time-invariant (LTI), it can also be written as the following operator form:

$$y = G_u u + G_f f$$

where G_u represents the system from input $u(t)$ to output $y(t)$ and G_f represents the system from fault $f(t)$ to output $y(t)$. Here, the model uncertainty and disturbance are not considered in this model.

Figure 1.1 shows the general framework for residual generation, where G is the plant, F is the residual generator and $r(t)$ is the residual signal. Since the faults cannot be measured directly, $f(t)$ cannot be taken as the input of F. With the control input $u(t)$ and the output $y(t)$, the residual generator must be able to produce a residual signal $r(t)$ such that it can show or predict the existence of faults $f(t)$, i.e.

$$\begin{cases} r(t) = 0, & \text{when } f(t) = 0; \\ r(t) \neq 0, & \text{when } f(t) \neq 0. \end{cases}$$

Obviously, the residual generator can be taken as a filter with the following form:

$$r = \begin{bmatrix} H_u & H_y \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = [H_u + H_y G_u]u + H_y G_f f.$$

In order to exclude the effect of $u(t)$ on the residual $r(t)$, the following fault detection decoupling condition must be satisfied:

$$\begin{cases} H_u + H_y G_u = 0 \\ H_y G_f \neq 0. \end{cases}$$

We will see later that the factorization method [80] is able to completely decouple the control input $u(t)$ from the residual signal.

Apparently, model uncertainties and disturbances will cause unexpected effects on the residual signal $r(t)$. To ensure reliable fault detection, the next stage that follows residual generation is decision-making [63]. That is, a decision rule that consists of a threshold is

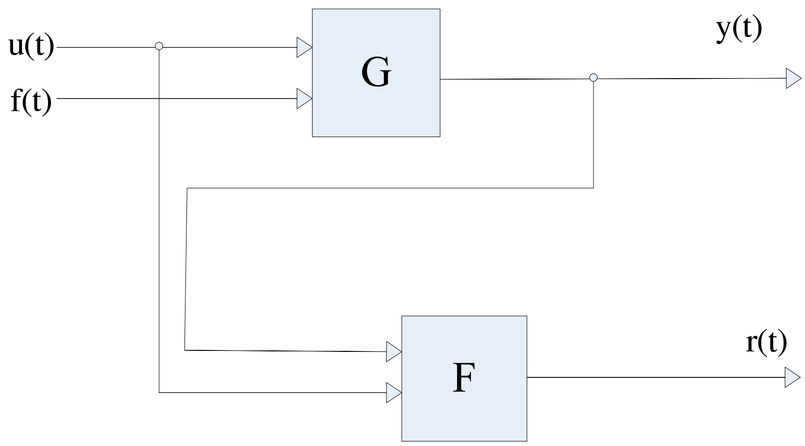

Figure 1.1: Fault Detection Filter Structure without Disturbance

applied to determine if any fault is present. Mathematically, it is to construct a residual evaluation function $H(r(t))$ and a threshold $T(t)$ so that

$$\begin{cases} H(r(t)) \leq T(t), & \text{when } f(t) = 0; \\ H(r(t)) > T(t), & \text{when } f(t) \neq 0 \end{cases}$$

where whether the function $H(r(t))$ is larger than the threshold $T(t)$ corresponds to the existence of faults. The appropriate choice of an evaluation function and a threshold is strongly related to the extend to which the system is able to detect the real faults. In the literature, there is a large amount of research on choosing appropriate residual threshold [20, 23], such as adaptive threshold [66], dynamic threshold [44] and optimal threshold selection [73].

1.3 Optimization and Robustness in Model-Based Fault Detection

The last section gives a fault detection condition characterizing how to decouple the input effect from the residual. However, there remains a question: how can we judge that one fault detection filter is better than the others? To my knowledge, a fault detection that is regarded as 'good' must have the following properties in some sense,

- fast response to faults. Or even predict faults before they occur.
- high sensitivity to faults, even under strong disturbance or/and model uncertainty.
- fewer chances of both missing alarm and false alarm: i.e. less possibility of taking disturbance as fault and vise versa.

This dissertation will only discuss the second property– fault sensitivity. The reader interested in property 1 and property 3 is referred to [8]. In order to quantify fault sensitivity, an appropriate mathematical criterion is necessary, and thus the fault detection filter design for improving the fault sensitivity turns out to be an optimization problem. In the current literature, several different criteria are available from various directions, i.e. in form of norms such as $\mathscr{H}_\infty$ and $\mathscr{H}_2$ [39, 40, 46, 47, 70]. Specifically, $\mathscr{H}_-$ index that describes the smallest detectability of a system has received great attention from researchers since it was first introduced by [15]. Several results related to $\mathscr{H}_-$ index have been obtained in [15–17,22,49,53,67] and references therein.

Due to the inevitability of modelling error, disturbance and noise in practice, robustness issue should definitely be considered in fault detection filter design, as it was in controller design [63, 97, 98]. Unfortunately, the involvements of both disturbance and model uncertainty impose significant difficulty on the fault detection filter design, since it is very difficult to distinguish the effects of disturbance and model uncertainties from faults. Specifically, if a system takes disturbance as faults, there is a false alarm. On the other hand, if the fault detection system takes the faults as disturbance, there is a missing alarm. Therefore, novel techniques for dealing with both fault detection and robustness issue are imperative.

Specifically, for illustrating our problem, consider the following plant,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + B_d(t)d(t) + B_f(t)f(t)$$

$$y(t) \;=\; C(t)x(t) + D(t)u(t) + D_d(t)d(t) + D_f(t)f(t).$$

The fault detection filter F generally can be designed as shown in Figure 1.2, in which the residual $r(t)$ should not be sensitive to control $u(t)$ (it can be guaranteed by factorization technique), less sensitive to the disturbance $d(t)$ and very sensitive to the fault $f(t)$.

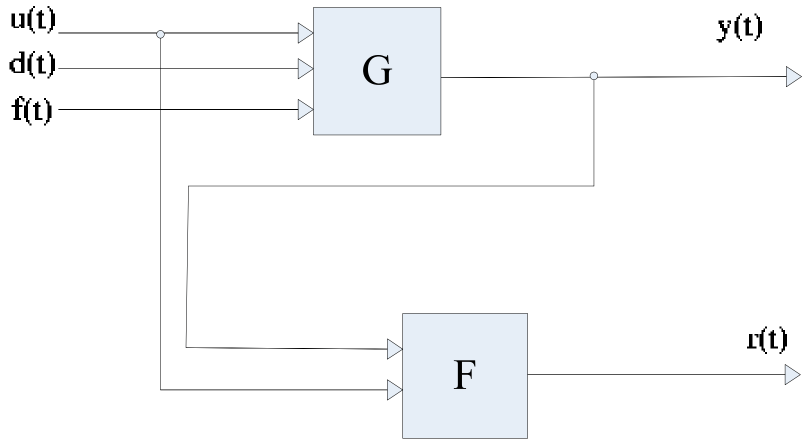

Figure 1.2: Fault Detection Filter Structure with Disturbance

The early work on this issue concentrated on how to completely decouple the disturbance from the residual, and the decoupling condition has been obtained in terms of rank condition and system zeros ([9,21,22,64]). In [9], Chen et al. proposed a necessary and sufficient condition for an unknown input observer that can decouple the disturbance from the residual. In [64], Patton and Chen presented the robust fault detection filter based on the eigenstructure assignment in the discrete-time domain for decoupling the disturbance from the residual signal. When the full decoupling cannot be realized, some research went to minimize the disturbance sensitivity, but ignored the fault sensitivity. For instance, in [47], Khosrowjerdi et al. designed a fault detection filter by minimizing disturbance for $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norm. In [12] Chung and Speyer minimized the disturbance effects via differential dynamic game.

Later on, researchers realized that when the full decoupling cannot be realized completely, fault sensitivity should also be considered in the problem in terms of fault detectability, besides disturbance sensitivity minimization problem. The reason is that the mere disturbance minimization could minimize the fault sensitivity simultaneously, thus reducing the fault detectability of the whole system. Now the problem becomes a multiple-objective optimization (MOO) problem: the residual generator should produce a residual signal that is sensitive to the fault and insensitive to the disturbance. In order to make this tradeoff in robust fault detection, several different optimization criteria have been proposed in the literature. In [18], Ding et al. presented a framework that maximizes the criterion: $\|\cdot\|_{-}/\|\cdot\|_{\infty}$, where the solution is obtained by solving one Riccati equation. The parallel results for linear discrete time periodic and sampled-data systems were given in [86]. Furthermore, the matrix inequality condition is given in [82] for the multiple optimization fault detection problem that minimizes $\gamma^2 - \beta^2$ with the constraints that disturbance sensitivity is less than γ and fault sensitivity is greater than β. In [49], Li et al. presented a multiple-objective optimization (MOO) problem in which the fault detection filter minimizes the sensitivity of the residual signal to disturbances while maintaining a minimum level of sensitivity to faults. However, one assumption in [49] is that the fault sensitivity matrix has full column rank at all frequencies, which is almost impossible in practice. In [74] Tao and Zhao dealt with the objective that minimizes the ratio of disturbance sensitivity and fault sensitivity by employing the system inverse, but it is not applicable when the inverse of the system fails to exist. Varga in [76] formulated fault detection filter design as a model matching problem and solved it by using an $\mathscr{H}_2$ or $\mathscr{H}_\infty$-norm optimization approach, but this method is difficult to be extended to solve $\mathscr{H}_-$ index problem. In [24] a reference model was assigned for the residual signal and thus the fault detection problem is transformed to minimizing the error between the reference model and the generation system. In [6] a robust deconvolution scheme for fault detection was discussed with a reference model for the residual.

Recently, the novel frameworks called $\mathscr{H}_2/\mathscr{H}_\infty$, $\mathscr{H}_\infty/\mathscr{H}_\infty$ and $\mathscr{H}_-/\mathscr{H}_\infty$ problems were proposed by Liu and Zhou in [55, 56] in which the residual generator maximizes the fault sensitivity with a bound on the disturbance sensitivity. One important fact in this research is that no assumption is made on the rank condition of the transfer matrix from fault to output. Thus, it is applicable to more general systems.

1.4 Contribution of the Dissertation

The $\mathscr{H}_2/\mathscr{H}_\infty$, $\mathscr{H}_\infty/\mathscr{H}_\infty$ and $\mathscr{H}_-/\mathscr{H}_\infty$ frameworks for linear time-invariant systems have been extensively discussed in [55, 56] under mild assumptions. The optimal filter in [55, 56] is exactly an observer for both continuous time invariant case and discrete time-invariant case. The gain of this filter is determined by solving a standard algebraic Riccati equation.

Time-varying systems describe a much more general class of systems than time invariant system in practice. In addition, most real plants in the industries can be represented or well approximated by time-varying systems [35]. However, the formulations, solutions and most concepts in the current literature are given in the frequency domain and it is hard to extend them to time domain, especially $\mathscr{H}_-$ index. In fact, to my knowledge, there are few available results on fault detection of linear time-varying systems in the literature. Authors in [59] employed dynamic game to attenuate the effect of disturbance on the residual, but the fault sensitivity was ignored. In [3], Brinsmead et al. discussed fault detection by minimizing the response to noises whilst keeping the fault response consistence via a quadratic optimization approach, but the model is too special. The first part of the dissertation will discuss fault detection of linear time-varying systems in a novel way and analytic results are obtained for linear continuous time-varying systems and linear discrete time-varying systems, respectively.

Real time algorithm is an obvious requirement of fault detection, while it has not received enough consideration from the current available methods which concentrate on the frequency domain. In [85], Zhang and Ding discussed the model-free fault detection approach from time domain data, but the method they used was based on frequency domain. In [99], Zolghadri presented an algorithm of real-time failure detection based on Kalman filter, but the disturbance is constrained to be white noise. In this dissertation, based on novel definitions in the time domain, the finite-horizon filter and infinite-horizon filter are given for $\mathscr{H}_2/\mathscr{H}_\infty$, $\mathscr{H}_\infty/\mathscr{H}_\infty$ and $\mathscr{H}_-/\mathscr{H}_\infty$ frameworks, where the finite-horizon filter compensates for this drawback in the form that it can be executed recursively, similar to a Kalman filter.

The effect of initial condition on the fault detection, cannot be taken care of by the frequency domain methods [55, 56], thus possibly causing certain performance deterioration of residual generation. In this dissertation, the effect of initial condition is appropriately considered in a new time-domain framework with a little modification of the previous framework.

Lastly, we find that the common $\mathscr{H}_2/\mathscr{H}_\infty$, $\mathscr{H}_\infty/\mathscr{H}_\infty$ and $\mathscr{H}_-/\mathscr{H}_\infty$ frameworks are not applicable to the general case in which G_d (the transfer matrix from disturbance to output) is tall and of full column rank. By the introduction of $\mathscr{H}_\infty$ norm, $\mathscr{H}_2$ norm and $\mathscr{H}_-$ index over a certain subspace, we reformulate this framework appropriately by considering both optimization and decoupling. It is shown that the optimal filter still exists in such a way that the filter not only decouples some faults from disturbances (the fault sensitivity for those faults are arbitrarily sensitive), but also maximizes the fault sensitivity in its complemental subspace. The filter is still an observer whose gain is determined by a generalized Riccati equation (or Riccati system, alternatively). In addition, we give the completely non-decoupling and decoupling conditions. Furthermore, disturbance rejection is also discussed. Finally, a procedure for fault detection filter design using polynomial matrix method is proposed.

1.5 Overview of the Dissertation

The purpose of this dissertation is to provide a systematic, self-contained and rigorous presentation for the fault detection problem of linear systems.

The dissertation is organized as follows: after the introduction part in Chapter 1, Chapter 2 provides some notations and preliminary lemmas that will be frequently used in the later chapters, followed by some current available results on fault detection design for linear time-invariant case. Chapter 3 discusses the fault detection design of continuous linear time-varying systems. Similarly, Chapter 4 discusses the fault detection design of discrete linear time-varying systems. In order to relax the constraint in the LTI case, Chapter 5 discusses the general case of fault detection design of discrete linear time-invariant systems. It is shown that the fault detection filter not only optimizes some faults in certain space, but also decouples some faults in the complement space. A novel design procedure for deriving the fault detection filter in transfer matrix form is proposed in Chapter 6. The future work based on this work is discussed in Chapter 7, followed by the conclusion in Chapter 8.

Chapter 2

Notations and Preliminary Results

In this chapter some important notations and preliminary results are given. After some notations for matrices in Section 2.1, some notations for linear time-invariant systems are given in Section 2.2. After some important notations and definitions for linear continuous time-varying systems in Section 2.3, the corresponding notations and definitions for linear discrete time-varying systems are discussed in Section 2.4. Section 2.5 gives a few important lemmas that will be frequently used in the subsequent chapters. For the completeness of the dissertation, several preliminary results on LTI case of fault detection design are given in Section 2.6.

2.1 Notations for Matrices

The set of m by n real (complex) matrices is denoted as $\mathscr{R}^{m\times n}$ ($\mathscr{C}^{m\times n}$). For a matrix $A \in \mathscr{C}^{m\times n}$ we use A^+ to denote its pseudo inverse, and A' for its complex conjugate transpose. For a Hermitian matrix $A = A' \in \mathscr{C}^{n\times n}$, $\bar{\lambda}(A)$ represents the largest eigenvalue of A and $\underline{\lambda}(A)$ represents the smallest eigenvalue of A. For any $A \in \mathscr{C}^{m\times n}$, $\bar{\sigma}(A) = \sqrt{\bar{\lambda}(AA')} = \sqrt{\bar{\lambda}(A'A)}$ denotes the largest singular value of A and $\underline{\sigma}(A) = \sqrt{\underline{\lambda}(A'A)}$ denotes the smallest singular value of A if $m \geq n$. For a matrix $A \in \mathscr{C}^{m\times m}$, $Tr\{A\}$ is the trace of A and A^{-1} is the inverse of A if it exists. A Hermitian matrix A is said to be positive semi-definite, i.e., $A \geq 0$. For

$A \geq 0$, $A^{\frac{1}{2}}$ is a matrix such that $A^{\frac{1}{2}} \times A^{\frac{1}{2}} = A$. $[A]^i$ denotes the ith row of matrix A. The $n \times n$ identity matrix is denoted as I_n and the $m \times n$ zero matrix is denoted as $0_{m\times n}$, with the subscripts dropped if they can be inferred from context.

2.2 Time-Invariant Case

2.2.1 Continuous Time-Invariant Case

We use $\mathscr{RL}_\infty^{\mathrm{m\times n}}$ to denote the set of all $m \times n$ real rational proper transfer matrices with no poles on the imaginary axis. The superscripts for dimensions will usually be dropped when they are either not important or clear from context. $\mathscr{RH}_\infty$ is a subset of $\mathscr{RL}_\infty$ with all stable transfer matrices. Similarly $\mathscr{RH}_2$ is the set of all real rational strictly proper stable transfer matrices. A state space realization of a transfer matrix $G(s)$ is denoted as

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

such that $G(s) = D + C(sI - A)^{-1}B$. Let $G^{\sim}(s) := G(-s)^T$ be the para-Hermitian complex conjugate transpose of G and $G^{-1}(s)$ be the inverse of G if $G(s)$ is square and invertible. Now suppose $G(s)$ is square and D is nonsingular, then we have from [98]

$$G^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right].$$

For $G \in \mathscr{RH}_2$ we define the $\mathscr{H}_2$ norm of G as ([98])

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Trace}\{\mathrm{G}^{\sim}(\mathrm{j}\omega)\mathrm{G}(\mathrm{j}\omega)\}\mathrm{d}\omega}.$$

For $G \in \mathscr{RH}_\infty$ we define the $\mathscr{H}_\infty$ norm of G as ([98])

$$\|G\|_\infty = \sup_{\omega\in\mathscr{R}} \bar{\sigma}(G(j\omega)).$$

Similarly, the $\mathscr{H}_-$ index of G over all frequencies is defined as ([68])

$$\|G\|_- = \inf_{\omega\in\mathscr{R}} \underline{\sigma}(G(j\omega)).$$

It should be noted that $\mathscr{H}_-$ index is sometimes called $\mathscr{H}_-$ norm in the literature although it does not satisfy the property of a norm. $\mathscr{H}_-$ index can be thought as a measurement of a system's ability of enlarging an input signal at all frequencies.

2.2.2 Discrete Time-Invariant Case

We use $\mathscr{RL}_2^{m\times n}$ to denote the set of all $m \times n$ proper real rational transfer function matrices with no poles on the unit circle. $\mathscr{RH}_\infty^{m\times n}$ $(= \mathscr{RH}_2^{m\times n})$ is the set of all $m \times n$ stable proper real rational transfer function matrices. $\mathscr{L}_2^m$ is the set of all real square summable sequences with m dimensions. For a sequence $x = (x_0, x_1, \cdots)$ in $\mathscr{L}_2^m$, $\|x\|_2 := \sqrt{\sum_{i=0}^{\infty}\sum_{j=1}^{m}|x_{ij}|^2}$ is its 2-norm, where x_{ij} is the jth element of x_i. The superscripts for dimensions will usually be dropped when they are either unimportant or clear from context. Let $G^\sim(z) := G^T(1/z)$ be the para-Hermitian complex conjugate transpose of G. A wide transfer matrix G is called co-inner if it is stable and $G(z)G^\sim(z) = I$. A tall transfer matrix G is called inner if it is stable and $G^\sim(z)G(z) = I$. A tall transfer matrix is called co-outer if it is both stable and minimum phase. We write $G(z)$ as G, when it is clear from context. For $G \in \mathscr{RL}_\infty^{m\times n}$, $ker\{G\} := \{u : Gu = 0, u \in \mathscr{L}_2^n\}$ and $image\{G\} := \{y \in \mathscr{L}_2^m : y = Gu, u \in \mathscr{L}_2^n\}$.

A state space realization of transfer matrix G is denoted as

$$G = C(zI - A)^{-1}B + D =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

We denote G^{-1} as the inverse of G if G is square and D is nonsingular. Specifically, from [98] we have

$$G^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right].$$

The following definitions for $\mathscr{H}_2$ norm, $\mathscr{H}_\infty$ norm and $\mathscr{H}_-$ index, respectively, are from [56].

For $G \in \mathscr{RH}_2$, its $\mathscr{H}_2$ norm is defined as

$$\|G\|_2 := \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} Tr\{G^{\sim}(e^{j\theta})G(e^{j\theta})\}d\theta}.$$

For $G \in \mathscr{RH}_\infty$, its $\mathscr{H}_\infty$ norm is defined as

$$\|G\|_\infty := \sup_{\theta\in[0,2\pi]} \bar{\sigma}(G(e^{j\theta})).$$

For $G \in \mathscr{RH}_\infty$, its $\mathscr{H}_-$ index is defined as

$$\|G\|_- := \inf_{\theta\in[0,2\pi]} \underline{\sigma}(G(e^{j\theta})).$$

In this dissertation, we use $\|G\|$ to represent any one of them if no confusion exists.

In order to describe system gain when input signal is in a subspace, we introduce the following definitions.

Definition 1 *For a system $G \in \mathscr{RH}_\infty^{m\times p}$, its $\mathscr{H}_-$ index over a subspace S is defined as*

$$\|G\|_-^S := \inf\left\{\frac{\|Gu\|_2}{\|u\|_2} : u \neq 0, u \in S \subseteq \mathscr{L}_2^p\right\}.$$

Definition 2 *Let a subspace $S \subseteq \mathscr{L}_2^p$ be the set of all outputs generated by an inner W (i.e., $S = image\{W\}$). For a system $G \in \mathscr{RH}_2^{m\times p}$ with input u and output y, its $\mathscr{H}_2$ index over S is defined as*

$$\|G\|_2^S := \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} Tr\{W^{\sim}(e^{j\theta})G^{\sim}(e^{j\theta})G(e^{j\theta})W(e^{j\theta})\}d\theta}.$$

Definition 3 *For a system $G \in \mathscr{RH}_\infty^{m\times p}$, its $\mathscr{H}_\infty$ index over a subspace S is defined as*

$$\|G\|_\infty^S := \sup\left\{\frac{\|Gu\|_2}{\|u\|_2} : u \neq 0, u \in S \subseteq \mathscr{L}_2^p\right\}.$$

In this paper, we use $\|G\|^S$ to represent any one of them if no confusion exists.

Remark 1 *Those definitions above are more general than the definition over $\mathscr{R}^p$. It will be frequently used in Chapter 5 to emphasize the fault sensitivity for the faults that cannot be decoupled from residual.*

Since $\mathscr{H}_-$ index is not a norm, it is difficult to compute it in both frequency domain and state space. The natural way to do it is to sample the system at frequencies among the real axis and compute the singular value for the system at each frequency separately. Therefore, the smallest one can be taken as the $\mathscr{H}_-$ index of the system. However, this method is time consuming and small sampling rate on the frequency will result in a significant error.

2.3 Continuous Time-Varying Case

For any $T \in R_+$, let $\mathscr{L}_2[0,T]$ denote the usual Hilbert space of square integrable functions endowed with usual inner product and norm (denoted $< \cdot, \cdot >_{2,[0,T]}$ and $\| \cdot \|_{2,[0,T]}$, respectively). Throughout this chapter, we compress the notation and write $\mathscr{L}_2[0,\infty)$ or $\mathscr{L}_2[0,T]$ as $\mathscr{L}_2$ and $\| \cdot \|_{2,[0,T]}$ as $\| \cdot \|_2$ whenever there is no possibility of confusion.

We use $G : S_1 \mapsto S_2$ to denote a system from input space S_1 to output space S_2. If we use w and y to represent input signal and output signal, respectively, the system can be denoted as $G : w \mapsto y = Gw$.

We also use I to denote identity system in which inputs are equivalent to outputs, and $G_{m\times p}$ to denote the linear system with p inputs and m outputs.

The state space representation of a linear system $G_{m\times p} : w \mapsto y = Gw$ can be written as

$$\dot{x}(t) = A(t)x(t) + B(t)w(t), \quad (2.1)$$

$$y(t) = C(t)x(t) + D(t)w(t) \quad (2.2)$$

where $x(t) \in \mathscr{R}^n$ is state vector, $w(t) \in \mathscr{R}^p$ is input vector and $y(t) \in \mathscr{R}^m$ is output vector. Here, $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are piecewise continuous bounded functions of t with compatible dimensions. If the initial state $x(0)$ is known in advance, it is assumed that the initial state $x(0) = 0$ without loss of generality. Specifically, if $x(0) \neq 0$ but is known in advance, we can subtract its contribution from the initial condition, then it becomes a problem with zero initial state by constructing a new output vector $\overline{y}(t) = y(t) - C(t)\Phi(t,0)x(0)$, where $\Phi(\cdot,\cdot)$ is the state transition matrix associated with $A(t)$.

Definition 4 *A system G is exponentially stable if there exist $c_1, c_2 > 0$ such that $\|\Phi_G(t,\tau)\| \leq c_1 e^{-c_2(t-\tau)}$, $\forall t \geq \tau$, where τ is a positive number and Φ_G is the state transition matrix of the homogeneous part of state space realization.*

Definition 5 *For a system G, if there exists a system P with compatible dimension such that*

$$< PGw, y >=< w, y >$$

for all w, y, that is, $PG = I$, P is called a left inverse of system G. In this chapter, we use G^{-1} to denote the left inverse of system G. The existence condition for left inverse is that the D term of state-space realization of system G always has full column rank for all $t \geq 0$.

Obviously, for the system G with state space realization given by (2.3) and (2.4) (here, $D(t)$ has full column rank for all $t \geq 0$), a state space realization of a left inverse system G^{-1} is:

$$\begin{aligned} \dot{p}(t) &= \left[A(t) - B(t)D^{+}(t)C(t)\right]p(t) - B(t)D^{+}(t)y(t) \\ r(t) &= D^{+}(t)C(t)p(t) + D^{+}(t)y(t). \end{aligned}$$

We have $(GH)^{-1} = H^{-1}G^{-1}$ and $(G^{-1})^{-1} = G$, for any systems G and H with compatible dimensions.

Definition 6 *We use $G^\sim$ to denote the adjoint of system G. Suppose $G : S_1 \mapsto S_2$ is a linear system and S_1 and S_2 are Hilbert spaces such as $\mathscr{L}_2[0,T]$ or $\mathscr{L}_2[0,\infty)$. The adjoint system is the linear system $G^\sim : S_2 \mapsto S_1$ that has the property $< Gw, y >=< w, G^\sim y >$ for all $w \in S_1$ and all $y \in S_2$.*

For system G with state space realization given by (2.3) and (2.4), one realization of the adjoint system $G^\sim$ is:

$$\begin{aligned} \dot{p}(t) &= -A'(t)p(t) - C'(t)y(t) \\ u(t) &= B'(t)p(t) + D'(t)y(t), \end{aligned}$$

where $p(T) = 0$ for finite-horizon case, and $p(\infty) = 0$ for infinite-horizon case. Obviously, $(GH)^\sim = H^\sim G^\sim$, and $(G^\sim)^\sim = G$ for any linear systems G and H with compatible dimensions.

Definition 7 *[26] $(C(t), A(t))$ is observable if the pair $(y(t), w(t))$, $t \in [0,T]$, uniquely determines $x(0)$. This is equivalent to $X(t) > 0$, in which $X(t)$ is the observability gramian satisfying*

$$-\dot{X}(t) = X(t)A(t) + A'(t)X(t) + C'(t)C(t), \quad X(T) = 0.$$

$(A(t), B(t))$ is controllable if and only if, for any $x(0) \in \mathscr{R}^n$, there exists a $w(t)$, $t \in [0,T]$, such that $x(T) = x_T$. This is equivalent to $Y(t) > 0$, in which $Y(t)$ is the controllability gramian satisfying

$$\dot{Y}(t) = Y(t)A'(t) + A(t)Y(t) + B(t)B'(t), \quad Y(0) = 0.$$

It can be shown that $(A(t), B(t))$ is controllable if and only if $(B'(t), A'(t))$ is observable.

Hence, the controllability of system G is equivalent to the observability of its adjoint system $G^\sim$.

Definition 8 *[59] The system G given by (2.3) and (2.4) is said to be stabilizable (respectively, detectable) if there exists a bounded matrix function $K(t)$ (respectively, $L(t)$) such that the system $\dot{x}(t) = [A(t) - B(t)K(t)]\,x(t)$ (respectively, $\dot{x}(t) = [A(t) - L(t)C(t)]\,x(t)$) is exponentially stable.*

Definition 9 *For a linear system $G : w \mapsto y$, its $\mathscr{H}_\infty$ norm is defined as*

$$\|G\|_\infty = \sup_{w \in \mathscr{L}_2} \frac{\|y\|_2}{\|w\|_2} = \sup_{w \in \mathscr{L}_2} \frac{\|Gw\|_2}{\|w\|_2}$$

For finite-horizon case, $\mathscr{H}_\infty$ norm of system G is defined as

$$\|G\|_{\infty,[0,T]} = \sup_{w \in L_2[0,T]} \frac{\|Gw\|_{2,[0,T]}}{\|w\|_{2,[0,T]}}$$

Definition 10 *[26] Suppose S_1 and S_2 are normed signal spaces such as $\mathscr{L}_2[0,T]$ or $\mathscr{L}_2[0,\infty)$, then G is co-isometric if*

$$\|G^{\sim} w\|_{S_2} = \|w\|_{S_1}$$

for all $w \in S_1$. Here, $\|\cdot\|_S$ means the 2-norm of signal defined in the space S.

Consequently, G is a co-isometric system between two Hilbert spaces if and only if $GG^{\sim} = I$.

Definition 11 *For a linear system $G : w \mapsto y$, its $\mathscr{H}_2$ norm is the expected root-mean square value of the output when the input is a realization of a unit variance white noise process. That is, $w(t)$ is a unit variance white noise process when $t \in [0,T]$, otherwise, $w(t) = 0$, and $y = Gw$, the finite-horizon 2-norm of G is defined by*

$$\|G\|_{2,[0,T]} = \sqrt{\mathscr{E}\{\frac{1}{T}\int_0^T y'(t)y(t)dt\}},$$

in which E is the expectation operator. When $T \to \infty$, we obtain the infinite-horizon 2-norm of G

$$\|G\|_2 = \lim_{T\to\infty} \sqrt{\mathscr{E}\{\frac{1}{T}\int_0^T y'(t)y(t)dt\}}.$$

Similar to $\mathscr{H}_\infty$ norm, we can extend $\mathscr{H}_-$ index into time domain, avoiding the involvement of some concepts of frequency domain.

Definition 12 *For a linear system $G : w \mapsto y$, its $\mathscr{H}_-$ index is defined as*

$$\|G\|_- = \inf_{w \in \mathscr{L}_2} \frac{\|Gw\|_2}{\|w\|_2}.$$

For finite-horizon case,

$$\|G\|_{-,[0,T]} = \inf_{w \in \mathscr{L}_2[0,T]} \frac{\|Gw\|_{2,[0,T]}}{\|w\|_{2,[0,T]}}.$$

We also define coprime factorization for linear time-varying system, which is similar to linear time-invariant system.

Definition 13 *[69] Let G be a finite dimensional linear time-varying system. We say that G admits an exponentially stable proper left-coprime factorization if there exist exponentially stable finite dimensional linear time-varying systems M, N, X and Y such that $G = M^{-1}N$ and $XN + YM = I$. Here, (N, M) is called the coprime pair of system G.*

Similar to spectral factorizations for the linear time invariant systems in frequency domain, we present the corresponding ones for time varying systems in time domain.

Definition 14 *For a linear system $G_{m\times p}$ with $m \le p$, if there exists a linear system $W_{m\times m}$ such that*

$$WW^\sim = GG^\sim$$

that is,

$$< W^\sim u, W^\sim y >=< G^\sim u, G^\sim y >$$

and W^{-1} exists for all u and y, we say W is one of spectral factorization of system G. Here, W must be exponentially stable for infinite-horizon case. Since W^{-1} exists, $W^{-1}G$ is a co-isometric system.

2.4 Discrete Time-Varying Case

For any $T \in \mathscr{R}_+$, let $l_2[0,T]$ denote the usual Hilbert space of square summable sequences endowed with usual inner product and norm (denoted $< \cdot, \cdot >_{2,[0,T]}$ and $\| \cdot \|_{2,[0,T]}$, respectively). Throughout the dissertation, we compress the notation and write $l_2[0,\infty)$ or $l_2[0,T]$ as l_2 and $\| \cdot \|_{2,[0,T]}$ as $\| \cdot \|_2$ whenever there is no possibility of confusion.

We use $G : S_1 \mapsto S_2$ to denote a system from input space S_1 to output space S_2. If we use w and y to represent input signal and output signal, respectively, the system is denoted as $G : w \mapsto y = Gw$.

We also use I to denote identity system that inputs are equivalent to outputs, and $G_{m\times p}$ to denote the linear system with p inputs and m outputs.

A state space representation of linear discrete time system $G_{m\times p} : w \mapsto y = Gw$ can be written as

$$x(t+1) = A(t)x(t) + B(t)w(t) \tag{2.3}$$

$$y(t) = C(t)x(t) + D(t)w(t) \tag{2.4}$$

where $x(t) \in \mathscr{R}^n$ is state vector, $w(t) \in \mathscr{R}^p$ is input vector and $y(t) \in \mathscr{R}^m$ is output vector. $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are bounded matrices of t with compatible dimensions. Here, we assume the initial state $x(0) = 0$. We argue that it is without losing of generality to make this assumption, if the initial state $x(0)$ is known in advance. Specifically, if $x(0) \neq 0$ but is known in advance, we can subtract its contribution from the output vector $y(t)$, then it becomes a problem with zero initial state by constructing a new output vector $\overline{y}(t) = y(t) - C(t)\Phi(t,0)x(0)$, where $\Phi(\cdot,\cdot)$ is the state transition matrix associated with $A(t)$.

For system G, if there exists a system P with compatible dimensions such that

$$< PGw, y >=< w, y >$$

for all w, y, that is, $PG = I$, P is called a left inverse of system G. In this chapter, we use G^{-1} to denote the left inverse of system G. The existence condition for the left inverse is that the D term of state-space realization of the system G always has full column rank for all $t \geq 0$.

One state space realization of G^{-1} is

$$\begin{aligned} x(t+1) &= [A(t) - B(t)D^{-1}(t)C(t)]x(t) - B(t)D^{-1}(t)y(t) \\ w(t) &= D^{-1}(t)C(t)x(t) + D^{-1}(t)y(t). \end{aligned}$$

We have $(GH)^{-1} = H^{-1}G^{-1}$ and $(G^{-1})^{-1} = G$, for any systems G and H with compatible dimensions.

We use $G^{\sim}$ to denote the adjoint of the system G. Suppose $G : S_1 \mapsto S_2$ is a linear system and S_1 and S_2 are Hilbert spaces such as $l_2[0, T]$ or $l_2[0, \infty)$. The adjoint system is the linear system $G^{\sim} : S_2 \mapsto S_1$ that has the property $< Gw, y >=< w, G^{\sim}y >$ for all $w \in S_1$ and all $y \in S_2$. Its state space realization is as follows

$$\begin{aligned} p(t-1) &= A'(t)p(t) + C'(t)y(t) \\ w(t) &= B'(t)p(t) + D'(t)y(t), \quad p(T) = 0. \end{aligned}$$

Note that $p(\infty) = 0$ for the infinite-horizon case.

Obviously, $(GH)^{\sim} = H^{\sim}G^{\sim}$, and $(G^{\sim})^{\sim} = G$ for any linear systems G and H with compatible dimensions.

Definition 15 *Suppose S_1 and S_2 are normed signal spaces such as $l_2[0, T]$ or $l_2[0, \infty)$, then*

G is co-isometric if

$$\|G^{\sim}w\|_{S_2} = \|w\|_{S_1}$$

for all $w \in S_1$. Here, $\|\cdot\|_S$ means the 2-norm of signal defined in the space S. Obviously, $GG^{\sim} = I$.

Definition 16 *A system is said to be observable on $[t_0, t_f]$ if any initial state $x(t_0)$ is uniquely determined by the corresponding zero-input output $y(t)$ for $t = t_0, \cdots, t_{f-1}$.*

Definition 17 *A system is said to be controllable (it is called reachable in some literature) on $[t_0, t_f]$ if for any initial time t_0 and any initial state $x(t_0)$, there exists a time l and a corresponding input $u(t)$ such that the system state goes to zero under this input.*

It can be shown that a system is controllable if and only if its adjoint system is observable.

Definition 18 *[35] A system is said to be exponentially stable on $[t_0, \infty)$ if*

$$|\Phi(t,s)| \leq c\alpha^{t-s}, \quad \forall t_0 \leq s \leq t < \infty$$

for some constants $c > 0$ and $0 < \alpha < 1$ independent of s and t. $\Phi(t,s)$ is the state transition matrix of $A(t)$.

Definition 19 *The pair $(A(t), B(t))$ is said to be stabilizable on $[t_0, \infty)$ if there exists a bounded matrix $K(t)$ such that $A(t) + B(t)K(t)$ is exponentially stable on $[t_0, \infty)$. The pair $(C(t), A(t))$ is said to be detectable on $[t_0, \infty)$ if there exists a bounded matrix $L(t)$ such that $A(t) + L(t)C(t)$ is exponentially stable on $[t_0, \infty)$.*

Definition 20 *Let G be a finite dimensional linear time-varying system. We say that G admits an exponentially stable proper left-coprime factorization if there exist exponentially stable finite dimensional linear time-varying systems M, N, X and Y such that $G = M^{-1}N$ and $XN + YM = I$. Here, (N, M) is called the coprime pair of system G.*

Definition 21 *For a linear operator $G_{m\times p}$ with $m \leq p$, if there exists a linear operator $W_{m\times m}$ such that*

$$WW^{\sim} = GG^{\sim}.$$

that is,

$$< W^{\sim}u, W^{\sim}y >=< G^{\sim}u, G^{\sim}y >$$

and W^{-1} exists for all u and y, we say W is a spectral factorization of operator G. Here, W must be exponentially stable for the infinite-horizon case.

2.5 Preliminary Lemmas

2.5.1 Two Important Matrix Inequalities

It is easy to show that we have the following results by the definition of matrix singular value [32].

Lemma 1 *Let $A \in \mathscr{C}^{m\times n}$ and $B \in \mathscr{C}^{n\times p}$ be two matrices with appropriate dimensions, then $\underline{\sigma}(AB) \leq \bar{\sigma}(A)\underline{\sigma}(B)$.*

Lemma 2 *([98]) For $A \in \mathscr{C}^{m\times n}$ and $B \in \mathscr{C}^{l\times n}$ with $l \geq n$, the following inequality holds $\underline{\sigma}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) \geq \underline{\sigma}(B)$*

2.5.2 Linear Time-Invariant Case

Lemma 3 *(Left Coprime Factorization) Let $P(s)$ or $P(z)$ be a proper real rational transfer matrix. A left coprime factorization (LCF) of P is a factorization*

$$P = M^{-1}N$$

where N and M are left-coprime over $\mathscr{RH}_\infty$. Let

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

be a detectable state-space realization of P and L be a matrix with appropriate dimensions such that $A + LC$ is stable, then a left coprime factorization of P is given by

$$\begin{bmatrix} M & N \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & L & B + LD \\ \hline C & I & D \end{array}\right].$$

Lemma 4 *(Spectral Factorization: Continuous Time Case) Let $G(s)$ be a proper real rational transfer matrix and*

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

be a detectable realization of G. Suppose D has full row rank and $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ has full row rank for all $\omega \in \mathscr{R}$. Let $R := DD' > 0$ and let $Y \geq 0$ be the stabilizing solution to the following algebraic Riccati equation

$$(A - BD'R^{-1}C)X + X(A - BD'R^{-1}C)' - XC'R^{-1}CX + B(I - D'R^{-1}D)B' = 0$$

such that $A - BD'R^{-1}C - XC'R^{-1}C$ is stable. Then the following spectral factorization holds

$$WW^{\sim} = GG^{\sim}$$

where $W^{-1} \in \mathscr{RH}_\infty$ and

$$W = \left[\begin{array}{c|c} A & (BD' + XC')R^{-1/2} \\ \hline C & R^{1/2} \end{array}\right].$$

Lemma 5 *(Spectral Factorization: Discrete Time Case) Let $G(z)$ be a proper real rational transfer matrix and*

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

be a detectable realization of G. *Suppose* D *has full row rank and* $\begin{bmatrix} A - e^{j\theta}I & B \\ C & D \end{bmatrix}$ *has full row rank for all* $\theta \in [0, 2\pi]$. *Let* $P \geq 0$ *be the stabilizing solution to the following algebraic Riccati equation*

$$APA' - P - (APC' + BD')(DD' + CPC')^{-1}(DB' + CPA') + BB' = 0$$

such that $A - (APC' + BD')(DD' + CPC')^{-1}C$ *is stable and let* $R := DD' + CPC'$. *Then the following spectral factorization holds*

$$WW^{\sim} = GG^{\sim}$$

where $W^{-1} \in \mathscr{RH}_{\infty}$ *and*

$$W = \left[\begin{array}{c|c} A & (APC' + BD')R^{-1/2} \\ \hline C & R^{1/2} \end{array}\right].$$

We also give a co-inner-outer factorization factorization for tall transfer matrices, where co-inner is square and co-outer is a tall matrix.

Lemma 6 *(Co-Inner-Outer and Spectral Factorization: Discrete Time Case [28]) Let a transfer matrix* $H(z) \in \mathscr{RH}_{\infty}^{p\times m}$ $(p \geq m)$ *be*

$$H(z) = C(zI - A)^{-1}B + D =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where $A \in \mathscr{R}^{n\times n}$, $B \in \mathscr{R}^{n\times m}$, $C \in \mathscr{R}^{p\times n}$ *and* $D \in \mathscr{R}^{p\times m}$.

Assume that (A, C) *is detectable,* $D \neq 0$ *and*

$$rank\left\{\begin{bmatrix} A - e^{j\theta}I & B \\ C & D \end{bmatrix}\right\} = n + m \quad \forall\theta \in [0, 2\pi].$$

Then we have the following conclusions,

1. *Among all positive-semidefinite solutions* ($Y = Y' \geq 0$) *of the following algebraic Riccati equation (ARE)*

$$\begin{aligned} Y &= AYA' - S_Y(DD' + CYC')^+ S_Y' + BB' \\ &= (A + LC)Y(A + LC)' + (B + LD)(B + LD)' \end{aligned}$$

where $L = -(AYC' + BD')(DD' + CYC')^+$ *and* $S_Y = AYC' + BD'$, *there exists* Y_{max} *such that*

$$Y_{max} \geq Y \geq 0$$

and $(A+L_{max}C)$ *is a stability matrix with* $L_{max} = -(AY_{max}C'+BD')(DD'+CY_{max}C')^+$.

2. $\Pi = DD' + CYC'$ *has rank* m.

3. *If* A *is stable, there holds the co-inner-outer factorization* $H(z) = H_0(z)H_i(z)$ *with*

$$\begin{aligned} H_i(z) &= \Omega_m^+ \left[\begin{array}{c|c} A + L_{max}C & B + L_{max}D \\ \hline C & D \end{array} \right] \\ H_0(z) &= \left[\begin{array}{c|c} A & -L_{max} \\ \hline C & I \end{array} \right] \Omega_m. \end{aligned}$$

where Ω_m *is of full column rank and* $\Omega_m \Omega_m' = DD' + CY_{max}C'$.

The algorithm for obtaining Y_{max} *is as follows*

1. *Let* Y_0 *be the solution of the Lyapunov equation* $Y_0 = AY_0A' + BB'$.

2. *Do the following procedures iteratively until* $\|Y_N - Y_{N+1}\|$ *is smaller than some tolerance bound.*

$$\begin{aligned} L_k &= -(AY_kC' + BD')(DD' + CY_kC')^+ \\ Y_{k+1} &= (A + L_kC)Y_k(A + L_kC)' + (B + L_kD)(B + L_kD)' \end{aligned}$$

3. Let $Y_{max} = Y_{N+1}$.

Alternatively, the L_{max} and Y_{max} can be obtained by solving the following Riccati system [36]:

$$\begin{bmatrix} I_{n\times n} & L_{max} \end{bmatrix} \begin{bmatrix} AY_{max}A' - Y_{max} + BB' & AY_{max}C' + BD' \\ (AY_{max}C' + BD')' & DD' + CY_{max}C' \end{bmatrix} = 0.$$

2.5.3 Continuous Time-Varying Case

The following lemma aims to compute $\mathscr{H}_\infty$ norm.

Lemma 7 *[26] For the system G given by (2.3) and (2.4) with $D = 0$, $\|G\|_{\infty,[0,T]} < \beta$ if and only if the differential Riccati equation*

$$-\dot{P}(t) = P(t)A(t) + A'(t)P(t) + \beta^{-2}P(t)B(t)B'(t)P(t) + C'(t)C(t), \quad P(T) = 0$$

has a solution $P(t) \geq 0$ on $[0, T]$.

For infinite-horizon case, $\|G\|_\infty < \beta$ if and only if there exists a bounded symmetric matrix function $P(t) \geq 0$ for $t \in [0, \infty)$ that is absolutely continuous and differentiable and satisfies the differential Riccati equation

$$-\dot{P}(t) = P(t)A(t) + A'(t)P(t) + \beta^{-2}P(t)B(t)B'(t)P(t) + C'(t)C(t), \quad P(\infty) = 0$$

and $A(t) + \beta^{-2}B(t)B'(t)P(t)$ is such that the following linear time-varying system

$$\dot{p}(t) = \left[A(t) + \beta^{-2}B(t)B'(t)P(t)\right] p(t)$$

is exponentially stable.

Remark 2 *If the coefficients of linear time-varying system are restricted to constants, that is, system is time invariant, the definitions above are reduced to their corresponding ones in frequency domain.*

Lemma 8 *Let $A, B : S_1 \mapsto S_2$ are two systems with appropriate dimensions, where $S_1, S_2 = \mathscr{L}_2[0,\infty)$, then*

$$\|AB\|_\infty \leq \|A\|_\infty \|B\|_\infty \tag{2.5}$$

$$\|AB\|_2 \leq \|A\|_\infty \|B\|_2 \tag{2.6}$$

$$\|AB\|_- \leq \|A\|_\infty \|B\|_- \tag{2.7}$$

If $S_1, S_2 = \mathscr{L}_2[0, T]$, then

$$\|AB\|_{\infty,[0,T]} \leq \|A\|_{\infty,[0,T]} \|B\|_{\infty,[0,T]} \tag{2.8}$$

$$\|AB\|_{2,[0,T]} \leq \|A\|_{\infty,[0,T]} \|B\|_{2,[0,T]} \tag{2.9}$$

$$\|AB\|_{-,[0,T]} \leq \|A\|_{\infty,[0,T]} \|B\|_{-,[0,T]} \tag{2.10}$$

Proof Inequality (2.5) and inequality (2.8) are obvious since the submultiplicative property of norm.

For inequality (2.6), let $y = Bw$, we have

$$\begin{aligned}
\|AB\|_2^2 &= \lim_{T\to\infty} \mathscr{E}\left\{\frac{1}{T}\int_0^T z'z dt\right\} \\
&= \lim_{T\to\infty} \mathscr{E}\left\{\frac{1}{T}\int_0^T (Ay)'(Ay) dt\right\} \\
&= \lim_{T\to\infty} \mathscr{E}\left\{\frac{1}{T}\int_0^T y'A'Ay dt\right\} \\
&\leq \|A\|_\infty^2 \lim_{T\to\infty} \mathscr{E}\left\{\frac{1}{T}\int_0^T y'y dt\right\} \\
&= \|A\|_\infty^2 \lim_{T\to\infty} \mathscr{E}\left\{\frac{1}{T}\int_0^T (Bw)'(Bw) dt\right\} \\
&= \|A\|_\infty^2 \|B\|_2^2.
\end{aligned}$$

For inequality (2.7), let $y = Bw$, we have

$$\|AB\|_- = \inf_{w\in\mathscr{L}_2} \frac{\|ABw\|_2}{\|w\|_2}$$

$$
\begin{aligned}
&= \inf_{w\in\mathscr{L}_2}\left\{\frac{\|ABw\|_2}{\|Bw\|_2}\cdot\frac{\|Bw\|_2}{\|w\|_2}\right\}\\
&\le \sup_{y\in\mathscr{L}_2}\frac{\|Ay\|_2}{\|y\|_2}\cdot\inf_{w\in L_2}\frac{\|Bw\|_2^2}{\|w\|_2}\\
&= \|A\|_\infty\|B\|_-.
\end{aligned}
$$

The proofs for inequalities (2.9) and (2.10) are similar to those of inequalities (2.6) and (2.7), respectively. □

Lemma 9 *Let G be a linear time-varying system with a realization given by (2.3) and (2.4), and assume that $(C(t), A(t))$ is detectable (for infinite-horizon $t \in [0,\infty)$). If $L(t)$ is a matrix function with appropriate dimensions such that system $\dot{x}(t) = [A(t) + L(t)C(t)]\,x(t)$ is exponentially stable, then G admits a left coprime factorization pair (N, M) with a realization for system N*

$$
\begin{aligned}
\dot{x}(t) &= [A(t) + L(t)C(t)]\,x(t) + [B(t) + L(t)D(t)]\,w(t)\\
y(t) &= C(t)x(t) + D(t)w(t)
\end{aligned}
$$

and a realization for system M

$$
\begin{aligned}
\dot{x}(t) &= [A(t) + L(t)C(t)]\,x(t) + L(t)w(t)\\
y(t) &= C(t)x(t) + w(t)
\end{aligned}
$$

Proof The result can be easily verified by the multiplication of systems. □

Remark 3 *For finite-horizon case, since M and N are not necessarily exponentially stable, $L(t)$ can be any bounded and piecewise continuous matrix function with appropriate dimension.*

Lemma 10 *[69] Assume that $(A(t), B(t))$ is stabilizable and $(C(t), A(t))$ is detectable, then there exists a unique, bounded solution $Q(t) \geq 0$ to the filter Riccati equation*

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A'(t) - Q(t)C'(t)C(t)Q(t) + B(t)B'(t), \quad Q(0) \geq 0.$$

furthermore, the system $\dot{x}(t) = [A(t) - Q(t)C'(t)C(t)]\, x(t)$ is exponentially stable.

2.5.4 Discrete Time-Varying Case

Lemma 11 *Let $A : y \mapsto z$ and $B : w \mapsto y$ be two systems with appropriate dimensions, where x, y and z are signals in $l_2[0, \infty)$, then*

$$\begin{aligned} \|AB\|_\infty &\leq \|A\|_\infty \|B\|_\infty && (2.11) \\ \|AB\|_2 &\leq \|A\|_\infty \|B\|_2 && (2.12) \\ \|AB\|_- &\leq \|A\|_\infty \|B\|_- && (2.13) \end{aligned}$$

If $S_1, S_2 = l_2[0, T]$,then

$$\begin{aligned} \|AB\|_{\infty,[0,T]} &\leq \|A\|_{\infty,[0,T]} \|B\|_{\infty,[0,T]} && (2.14) \\ \|AB\|_{2,[0,T]} &\leq \|A\|_{\infty,[0,T]} \|B\|_{2,[0,T]} && (2.15) \\ \|AB\|_{-,[0,T]} &\leq \|A\|_{\infty,[0,T]} \|B\|_{-,[0,T]} && (2.16) \end{aligned}$$

Proof Inequality (2.11) and inequality (2.14) are obvious since the submultiplicative property of norm.

For inequality (2.12), we have

$$\begin{aligned} \|AB\|_2^2 &= \lim_{T\to\infty} \mathscr{E} \left\{ \frac{1}{T} \sum_{t=0}^{T} z'z \right\} \\ &= \lim_{T\to\infty} \mathscr{E} \left\{ \frac{1}{T} \sum_{t=0}^{T} (Ay)'(Ay) \right\} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{T\to\infty} \mathscr{E}\left\{\frac{1}{T}\sum_{t=0}^{T} y'A'Ay\right\} \\
&\le \|A\|_\infty^2 \lim_{T\to\infty} \mathscr{E}\left\{\frac{1}{T}\sum_{t=0}^{T} y'y\right\} \\
&= \|A\|_\infty^2 \lim_{T\to\infty} \mathscr{E}\left\{\frac{1}{T}\sum_{t=0}^{T} (Bw)'(Bw)\right\} \\
&= \|A\|_\infty^2 \|B\|_2^2.
\end{aligned}
$$

For inequality (2.13), we have

$$
\begin{aligned}
\|AB\|_- &= \inf_{w\in l_2} \frac{\|ABw\|_2}{\|w\|_2} \\
&= \inf_{w\in l_2} \left\{\frac{\|ABw\|_2}{\|Bw\|_2}\cdot\frac{\|Bw\|_2}{\|w\|_2}\right\} \\
&\le \sup_{y\in l_2} \frac{\|Ay\|_2}{\|y\|_2}\cdot \inf_{w\in l_2}\frac{\|Bw\|_2}{\|w\|_2} \\
&= \|A\|_\infty \|B\|_-.
\end{aligned}
$$

The proofs for inequalities (2.15) and (2.16) are similar to those of inequalities (2.12) and (2.13), respectively. □

Lemma 12 *[35] Suppose that $(C(t), A(t))$ is detectable. The system G given by (2.3) and (2.4) is exponentially stable if and only if there exists a bounded nonnegative solution to*

$$A'(t)X(t+1)A(t) + C'(t)C(t) = X(t).$$

Suppose that $(A(t), B(t))$ is stabilizable on $[t_0, \infty)$. The system G is exponentially stable on $[t_0, \infty)$ if and only if there exists a bounded nonnegative solution to

$$A(t)Y(t)A'(t) + B(t)B'(t) = Y(t+1), \quad Y(t_0) = 0.$$

Lemma 13 *Suppose G is a system given by equations (2.3) and (2.4) with $x(0) = 0$. If* *there exists a bounded and symmetric matrix $X(t)$ satisfying*

$$\begin{aligned} A(t)X(t)A'(t)+B(t)B'(t) &= X(t{+}1) \quad (2.17)\\ B(t)D'(t)+A(t)X(t)C'(t) &= 0 \quad (2.18)\\ D(t)D'(t)+C(t)X(t)C'(t) &= I \quad (2.19)\end{aligned}$$

with $X(0)=0$ for all $t\in[0,T]$, then G is co-isometric on $l_2[0,T]$. If the system G is observable, these conditions are also necessary.

The result is also true for infinite-horizon case ($T\to\infty$).

Proof Controllability gramian for system G is $X(t)$ such that

$$A(t)X(t)A'(t)+B(t)B'(t)=X(t{+}1)$$

with the initial condition $X(0)=0$.

Its adjoint system is

$$\begin{aligned} p(t{-}1) &= A'(t)p(t)+C'(t)y(t)\\ w(t) &= B'(t)p(t)+D'(t)y(t), \quad p(T)=0.\end{aligned}$$

When $X(0)=0$,

$$\begin{aligned} \|w(t)\|^2_{2,[0,T]} &= \sum_{t=0}^{T} w'(t)w(t)\\ &= \sum_{t=0}^{T}\{B'(t)p(t)+D'(t)y(t)\}'\{B'(t)p(t)+D'(t)y(t)\}\\ &= \sum_{t=0}^{T}\{p'(t)B(t)B'(t)p(t)+2p'(t)B(t)D'(t)y(t)+y'(t)D(t)D'(t)y(t)\}\\ &= \sum_{t=0}^{T}\{p'(t)B(t)B'(t)p(t)+2p'(t)B(t)D'(t)y(t)+y'(t)D(t)D'(t)y(t)\}\\ &\quad -\sum_{t=0}^{T}\{p'(t)X(t{+}1)p(t)-p'(t{-}1)X(t)p(t{-}1)\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=0}^{T}\{p'(t)B(t)B'(t)p(t) + 2p'(t)B(t)D'(t)y(t) + y'(t)D(t)D'(t)y(t)\} \\
&\quad - \sum_{t=0}^{T}\{p'(t)X(t+1)p(t) - [A'(t)p(t) + C'(t)y(t)]'X(t)[A'(t)p(t) + C'(t)y(t)]\} \\
&= \sum_{t=0}^{T}\{p'(t)[B(t)B'(t) + A(t)X(t)A'(t) - X(t+1)]p(t) \\
&\quad +2p'(t)[B(t)D'(t) + A(t)X(t)C'(t)]y(t) + y'(t)[D(t)D'(t) + C(t)X(t)C'(t)]y(t)\}
\end{aligned}
$$

When $B(t)B'(t) + A(t)X(t)A(t) - X(t+1) = 0$, $B(t)D'(t) + A(t)X(t)C'(t) = 0$ and $D(t)D'(t) + C(t)X(t)C'(t) = I$, we have

$$
\|y(t)\|_{2,[0,T]}^2 = \|w(t)\|_{2,[0,T]}^2.
$$

Conversely, let $X(t)$ be the controllability gramian, we have

$$
\begin{aligned}
& \|w(t)\|_{2,[0,T]}^2 - \|y(t)\|_{2,[0,T]}^2 \\
&= \|G^{\sim}y(t)\|_{2,[0,T]}^2 - \|y(t)\|_{2,[0,T]}^2 \\
&= \sum_{t=0}^{T}\{p'(t)[B(t)B'(t) + A(t)X(t)A'(t) - X(t+1)]p(t) \\
&\quad +2p'(t)[B(t)D'(t) + A(t)X(t)C'(t)]y(t) + y'(t)[D(t)D'(t) + C(t)X(t)C'(t) - I]y(t)\} \\
&= 0.
\end{aligned}
$$

When the system is observable, the adjoint system is controllable. Consider $y_t(t) = P_T y(t)$, where P_T is the truncation operator. Since controllability ensures that $p(t)$ spans $\mathscr{R}^n$ as y ranges over $l_2[0,T]$, we have $B(t)D'(t) + A(t)X(t)C'(t) = 0$ and $D(t)D'(t) + C(t)X(t)C'(t) - I = 0$.

The proof for the infinite-horizon case is similar. □

Lemma 14 *Let G be a linear time varying system with a realization given by (2.3) and (2.4), and assume that $(C(t), A(t))$ is detectable (for the infinite-horizon case $t \in [0,\infty)$). If $L(t)$*

is a matrix function with appropriate dimension such that $(A(t)+L(t)C(t))$ *is exponentially stable, then* G *admits a left coprime factorization pair* (N, M) *with the realization for system* M

$$\begin{aligned} x(t+1) &= [A(t)+L(t)C(t)]x(t)+L(t)u(t) \\ y(t) &= C(t)x(t)+u(t) \end{aligned}$$

and system N

$$\begin{aligned} x(t+1) &= [A(t)+L(t)C(t)]x(t)+[B(t)+L(t)D(t)]u(t) \\ y(t) &= C(t)x(t)+D(t)u(t). \end{aligned}$$

Lemma 15 *For the system* G *given by equations (2.3) and (2.4), one of its spectral factorization is a system* W *with the following realization:*

$$\begin{aligned} x(t+1) &= A(t)x(t)-L_0(t)R^{1/2}(t)u(t) \\ y(t) &= C(t)x(t)+R^{1/2}(t)u(t) \end{aligned}$$

where $R(t) = D(t)D'(t)+C(t)P(t)C'(t)$, $L_0(t) = -[A(t)P(t)C'(t)+B(t)D'(t)]R^{-1}(t)$ *and* $P(t)$ *is the solution of the following difference Riccati equation*

$$A(t)P(t)A'(t)-L_0(t)R(t)L_0'(t)+B(t)B'(t) = P(t+1)$$

with $P(0) = 0$.

Proof Since one state space realization of the system W^{-1} is

$$\begin{aligned} x(t+1) &= [A(t)+L_0(t)C(t)]x(t)+L_0(t)u(t) \\ y(t) &= R^{-1/2}(t)C(t)x(t)+R^{-1/2}(t)u(t), \end{aligned}$$

the system $W^{-1}G$ can be realized as:

$$\begin{aligned} x(t+1) &= [A(t) + L_0(t)C(t)]x(t) + [B(t) + L_0(t)D(t)]u(t) \\ y(t) &= R^{-1/2}(t)C(t)x(t) + R^{-1/2}(t)D(t)u(t). \end{aligned}$$

To show that $W^{-1}G$ to be co-isometric, it is necessary to check three conditions:

$$\begin{aligned} [A(t) + L_0(t)C(t)]P(t)[A(t) + L_0(t)C(t)]' + [B(t) + L_0(t)D(t)][B(t) + L_0(t)D(t)]' &= P(t+1) \\ [B(t) + L_0(t)D(t)]\,(R^{-1/2}(t)D(t))' + [A(t) + L_0(t)C(t)]P(t)[R^{-1/2}(t)C(t)]' &= 0 \\ \left[R^{-1/2}(t)D(t)\right][R^{-1/2}(t)D(t)]' + [R^{-1/2}(t)(t)C(t)]P(t)[R^{-1/2}C(t)]' &= I. \end{aligned}$$

The first equation can be simplified as the difference Riccati equation (2.20) with initial condition $P(0) = 0$. The second one is satisfied naturally and the third one turns out to be $R(t) = C(t)P(t)C'(t) + D(t)D'(t)$. □

In order to prove the exponential stability in the next section, we give a lemma about difference Riccati equation with a small change of [35].

Lemma 16 *Suppose that $D(t)D'(t) > 0$, $(A(t), B(t))$ is stabilizable and $(C(t), A(t))$ is detectable. There exists a nonnegative bounded solution to the following difference Riccati equation*

$$A(t)Y(t)A'(t) + B(t)B'(t) - [A(t)Y(t)C'(t)]R^{-1}(t)[A(t)Y(t)C'(t)]' = Y(t+1)$$

with $Y(0) = 0$ such that $A(t) - A(t)Y(t)C'(t)R^{-1}(t)C(t)$ is exponentially stable, where $R(t) = D(t)D'(t) + C(t)Y(t)C'(t)$.

Proof The difference Riccati equation can be returned to the Lemma in [35] by defining

$$\widetilde{C}(t) := (D(t)D'(t))^{1/2}C(t).$$

Since $D(t)D'(t)$ is of full rank, that $((D(t)D'(t))^{1/2}C(t), A(t))$ is detectable is equivalent to that $(C(t), A(t))$ is detectable. Furthermore, we have that $A(t) - A(t)Y(t)C'(t)R^{-1}(t)C(t)$ is exponentially stable. □

2.6 Fault Detection of Linear Time-Invariant Systems

For the completeness of the dissertation, we summarize the fault detection result of linear time-invariant systems [55, 56], which can be thought as a special case of the results in Chapter 5.

2.6.1 Continuous Time-Invariant Case

Consider a linear continuous time invariant system (LTI) with disturbance and possible faults as:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d d(t) + B_f f(t) \tag{2.20}$$

$$y(t) = Cx(t) + Du(t) + D_d d(t) + D_f f(t) \tag{2.21}$$

where $x(t) \in \mathscr{R}^n$ is the state vector, $y(t) \in \mathscr{R}^{n_y}$ is the output measurement, $d(t) \in \mathscr{R}^{n_d}$ represents the unknown/uncertain disturbance and measurement noise, and $f(t) \in \mathscr{R}^{n_f}$ denotes the process, sensor or actuator fault vector. $f(t)$ and $d(t)$ can be modeled as different types of signals, depending on specific situations under consideration.

Assumption 1 (C, A) *is detectable.*

This is a standard assumption for all fault detection problems.

Assumption 2 D_d *has full row rank.*

Assumption 3 $\begin{bmatrix} A - j\omega I & B_d \\ C & D_d \end{bmatrix}$ *has full row rank for all* $\omega \in \mathscr{R}$. *Or, equivalently, the transfer function matrix* $G_d := \left[\begin{array}{c|c} A & B_d \\ \hline C & D_d \end{array}\right]$ *has no transmission zero on the imaginary axis.*

Assumption 4 $n_y \geq n_f$.

By taking Laplace transform of equations (2.20) and (2.21) we have the system input/output equation

$$y = G_u u + G_d d + G_f f \tag{2.22}$$

where G_u, G_d, and G_f are $n_y \times n_u$, $n_y \times n_d$ and $n_y \times n_f$ transfer matrices respectively and their state-space realizations are

$$\begin{bmatrix} G_u & G_d & G_f \end{bmatrix} = \left[\begin{array}{c|ccc} A & B & B_d & B_f \\ \hline C & D & D_d & D_f \end{array}\right]. \tag{2.23}$$

Since the state-space realization of G_u, G_d and G_f share the same A and C matrices, applying Lemma 3 we can find a LCF for the system (2.23)

$$\begin{bmatrix} G_u & G_d & G_f \end{bmatrix} = M^{-1} \begin{bmatrix} N_u & N_d & N_f \end{bmatrix} \tag{2.24}$$

where

$$\begin{bmatrix} M & N_u & N_d & N_f \end{bmatrix} = \left[\begin{array}{c|cccc} A + LC & L & B + LD & B_d + LD_d & B_f + LD_f \\ \hline C & I & D & D_d & D_f \end{array}\right] \tag{2.25}$$

and L is a matrix such that $A + LC$ is stable.

It has been shown in [23] that, without loss of generality, the fault detection filter can take the following general form

$$r = Q(My - N_u u) = Q \begin{bmatrix} M & -N_u \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \tag{2.26}$$

where r is the residual vector for detection, $Q \in \mathscr{RH}_\infty^{n_y \times n_y}$ is a free stable transfer matrix to be designed. The filter structure is shown in Figure 2.1.

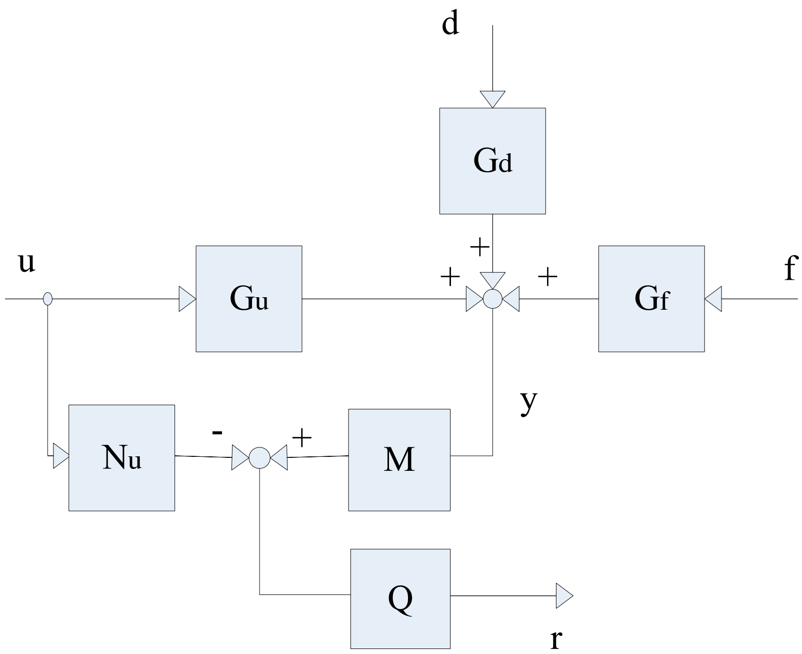

Figure 2.1: Fault Detection Filter Structure–LTI Case

Problem 1 *($\mathscr{H}_{-}/\mathscr{H}_{\infty}$ Problem) Let an uncertain system be described by equations (2.20)-(2.23) and let $\gamma > 0$ be a given disturbance rejection level. Find a stable transfer matrix $Q \in \mathscr{R}\mathscr{H}_{\infty}^{\mathrm{n_y}\times\mathrm{n_y}}$ such that $\|G_{rd}\|_{\infty} \leq \gamma$ and $\|G_{rf}\|_{-}$is maximized, i.e.*

$$\max_{Q\in\mathscr{R}\mathscr{H}_{\infty}^{\mathrm{ny}\times\mathrm{ny}}}\{\|QN_f\|_{-} : \|QN_d\|_{\infty} \leq \gamma\}$$

Problem 2 *($\mathscr{H}_{2}/\mathscr{H}_{\infty}$ Problem) Let an uncertain system be described by equations (2.20)-(2.23) and let $\gamma > 0$ be a given disturbance rejection level. Find a stable transfer matrix $Q \in \mathscr{R}\mathscr{H}_{\infty}^{\mathrm{n_y}\times\mathrm{n_y}}$ such that $\|G_{rd}\|_{\infty} \leq \gamma$ and $\|G_{rf}\|_{2}$ is maximized, i.e.*

$$\max_{Q\in\mathscr{R}\mathscr{H}_{\infty}^{\mathrm{ny}\times\mathrm{ny}}}\{\|QN_f\|_{2} : \|QN_d\|_{\infty} \leq \gamma\}$$

Problem 3 *($\mathscr{H}_{\infty}/\mathscr{H}_{\infty}$ Problem) Let an uncertain system be described by equations (2.20)-(2.23) and let $\gamma > 0$ be a given disturbance rejection level. Find a stable transfer matrix $Q \in \mathscr{R}\mathscr{H}_{\infty}^{\mathrm{n_y}\times\mathrm{n_y}}$ such that $\|G_{rd}\|_{\infty} \leq \gamma$ and $\|G_{rf}\|_{\infty}$ is maximized, i.e.*

$$\max_{Q\in\mathscr{R}\mathscr{H}_{\infty}^{\mathrm{ny}\times\mathrm{ny}}}\{\|QN_f\|_{\infty} : \|QN_d\|_{\infty} \leq \gamma\}$$

Theorem 1 *Suppose Assumptions 1-3 are satisfied. Let* $R_d := D_d D_d' > 0$ *and let* $Y \geq 0$ *be the stabilizing solution to the Riccati equation*

$$(A - B_d D_d' R_d^{-1} C)Y + Y(A - B_d D_d' R_d^{-1} C)' - YC' R_d^{-1} CY + B_d(I - D_d' R_d^{-1} D_d)B_d' = 0$$

such that $A - B_d D_d' R_d^{-1} C - YC' R_d^{-1} C$ *is stable. Define*

$$L_0 = -(B_d D_d' + YC')R_d^{-1}.$$

Then an optimal fault detection filter for the three problems above has the following form

$$r = Q_{opt} \begin{bmatrix} M & -N_u \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}$$

where

$$Q_{opt} \begin{bmatrix} M & -N_u \end{bmatrix} = \gamma \left[\begin{array}{c|cc} A + L_0 C & -L_0 & B + L_0 D \\ \hline -R_d^{-1/2} C & R_d^{-1/2} & -R_d^{-1/2} D \end{array} \right]$$

In other words, the optimal fault detection filter is the following observer:

$$\begin{aligned} \dot{\hat{x}}(t) &= (A + L_0 C)\hat{x}(t) - L_0 y(t) + (B + L_0 D)u(t) \\ r(t) &= \gamma R_d^{-1/2} \left(y(t) - C\hat{x}(t) - Du(t) \right). \end{aligned}$$

2.6.2 Discrete Time-Invariant Case

Consider a discrete time invariant system with disturbance and possible faults as:

$$x(t+1) = Ax(t) + Bu(t) + B_d d(t) + B_f f(t) \tag{2.27}$$

$$y(t) = Cx(t) + Du(t) + D_d d(t) + D_f f(t) \tag{2.28}$$

where $x(t) \in \mathscr{R}^n$ is the state vector, $y(t) \in \mathscr{R}^{n_y}$ is the output measurement, $d(t) \in \mathscr{R}^{n_d}$ represents the unknown/uncertain disturbance and measurement noise, and $f(t) \in \mathscr{R}^{n_f}$ denotes the process, sensor or actuator fault vector. $f(t)$ and $d(t)$ can be modelled as different type of signals, depending on specific situations under consideration.

Assumption 5 *(C, A) is detectable.*

Assumption 6 *D_d has full row rank.*

Assumption 7 $\begin{bmatrix} A - e^{j\theta} I & B_d \\ C & D_d \end{bmatrix}$ *has full row rank for all $\theta \in [0, 2\pi]$. Or, equivalently, the transfer function matrix* $G_d := \left[\begin{array}{c|c} A & B_d \\ \hline C & D_d \end{array}\right]$ *has no transmission zero on the unit circle.*

Similarly, we can formulate the three optimization problems as before.

Theorem 2 *Suppose Assumptions 5-7 are satisfied. Let $P \geq 0$ be the stabilizing solution to the Riccati equation*

$$APA' - P - (APC' + B_dD_d')(D_dD_d' + CPC')^{-1}(D_dB_d' + CPA') + B_dB_d' = 0 \qquad (2.29)$$

such that $A - (APC' + B_dD_d')(D_dD_d' + CPC')^{-1}C$ is stable and let $R_d = D_dD_d' + CPC'$. Define

$$L_0 = -(APC' + B_dD_d')R_d^{-1}.$$

Then an optimal fault detection filter for the three problems has the following state space representation

$$r = Q_{opt} \begin{bmatrix} M & -N_u \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}$$

where

$$Q_{opt} \begin{bmatrix} M & -N_u \end{bmatrix} = \gamma \left[\begin{array}{c|cc} A + L_0C & -L_0 & B + L_0D \\ \hline -R_d^{-1/2}C & R_d^{-1/2} & -R_d^{-1/2}D \end{array}\right]$$

In other words, the optimal fault detection filter is the following observer:

$$\hat{x}(k+1) = (A + L_0C)\hat{x}(k) - L_0y(t) + (B + L_0D)u(k) \qquad (2.30)$$

$$r(k) = \gamma R_d^{-1/2} \left(y(k) - C\hat{x}(k) - Du(k)\right). \qquad (2.31)$$

Chapter 3

Fault Detection Filter Design for Continuous Time-Varying Systems*

This chapter is dedicated to fault detection for linear continuous time-varying systems (LCTVS). Problem formulation is given in Section 3.1. Our fault detection design is presented in Section 3.2. Section 3.3 extends the result to the plant with non-zero initial states. One example is shown in Section 3.4.

3.1 Problem Formulation

Consider a linear time-varying system G with disturbance and possible faults of the following state-space realization

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + B_d(t)d(t) + B_f(t)f(t) \quad (3.1)$$

$$y(t) = C(t)x(t) + D(t)u(t) + D_d(t)d(t) + D_f(t)f(t) \quad (3.2)$$

where $t \in [0, T]$ ($t \in [0, \infty)$ for the infinite-horizon case). T is a positive scalar. $x(t) \in \mathscr{R}^n$ is the state vector, $y(t) \in \mathscr{R}^{n_y}$ is the output measurement, $d(t) \in \mathscr{R}^{n_d}$ represents the unknown/uncertain disturbance and measurement noise, and $f(t) \in \mathscr{R}^{n_f}$ denotes the process, sensor or actuator fault vector. $A(t)$, $B(t)$, $C(t)$, $D(t)$, $B_d(t)$, $D_d(t)$, $B_f(t)$ and $D_f(t)$ are piecewise continuous bounded functions of t with compatible dimensions. $f(t)$

*Reprinted by permission of Elsevier Science and $d(t)$ can be modeled as different type of signals, depending on specific situations under consideration. Different assumptions on $d(t)$ and $f(t)$ will lead to different fault detection problem formulations and the solutions for all these problems will be discussed in this chapter.

For all coefficient matrices in equations (3.1) and (3.2) the following three assumptions are made.

Assumption 8 *$(C(t), A(t))$ is detectable;*

Assumption 9 *$D_d(t)$ has full row rank for all $t \geq 0$, that is, $R_d(t) = D_d(t)D_d'(t) > 0$;*

Define

$$A_R(t) := A(t) - B_d(t)D_d'(t)R_d^{-1}(t)C(t)$$
$$B_R(t) := B_d(t)\left[I - D_d'(t)R_d^{-1}(t)D_d(t)\right].$$

Assumption 10 *$(A_R(t), B_R(t))$ is stabilizable.*

Remark 4 *Assumption 1 is a standard assumption for all fault detection problems for infinite-horizon case. This assumption guarantees the existence of a $L(t)$ such that*

$$\dot{x}(t) = [A(t) + L(t)C(t)]\, x(t)$$

is exponentially stable.

Remark 5 *Assumption 2 means that $n_y \leq n_d$ and every measurement of the output signals is either affected by some disturbance or corrupted with some measurement noise. We argue that this assumption can be made without loss of any generality since it is impossible to take perfect measurement in any practical system and furthermore it is reasonable to assume that the measurement noise is independent of each other. So it is reasonable to assume that the* *measurement noise is independent of each other and that $D_d(t)$ has full row rank for all $t \geq 0$ (see [55] for detailed description).*

In the case of some simplified model where $D_d(t)$ does not have full row rank, we can simply add some columns to make it full row rank. For example, suppose $D_d(t)$ is not full row rank, then let

$$\tilde{d} = \begin{bmatrix} d \\ d_\epsilon \end{bmatrix}, \ \tilde{B}_d = \begin{bmatrix} B_d & 0_{n \times n_y} \end{bmatrix}, \ \tilde{D}_d = \begin{bmatrix} D_d & \epsilon I_{n_y} \end{bmatrix}$$

for a small $\epsilon > 0$. Then $\tilde{D}_d(t)$ has full row rank for all $t \geq 0$. Because ϵ can be made as small as possible, we argue that the performance degradation caused by the fictitious disturbances should not be large.

This assumption might be restrictive in some applications when the external disturbances and measurement noise are different class of signals so that it is impossible to combine them together in our framework. Actually, when D_d is not wide or square, in other words, the number of outputs is greater than the number of disturbances, the decoupling of some disturbances from residual without sacrificing the fault detection ability is possible. Therefore, our optimization framework is still useful when it is possible to remove some disturbances that can be decoupled from the residual([18]). This will be discussed in detail in Chapter 5.

Remark 6 *Assumption 3 is an additional assumption for guaranteeing the existence of an unique and exponentially stable filter for infinite-horizon case. In particular, when the plant and measurement noise are independent (i.e. $B_d(t)D_d'(t) = 0$), the Assumption 3 can be simplified as $(A(t), B_d(t))$ is stabilizable. This assumption can be significantly relaxed in the linear time-invariant case (see [55]).*

The system realization given by (3.1) and (3.2) can be written as:

$$y = \begin{bmatrix} G_u & G_d & G_f \end{bmatrix} \begin{bmatrix} u \\ d \\ f \end{bmatrix}$$

where G_u, G_d, G_f are $n_y \times n_u$, $n_y \times n_d$ and $n_y \times n_f$ systems with the following state space realizations:

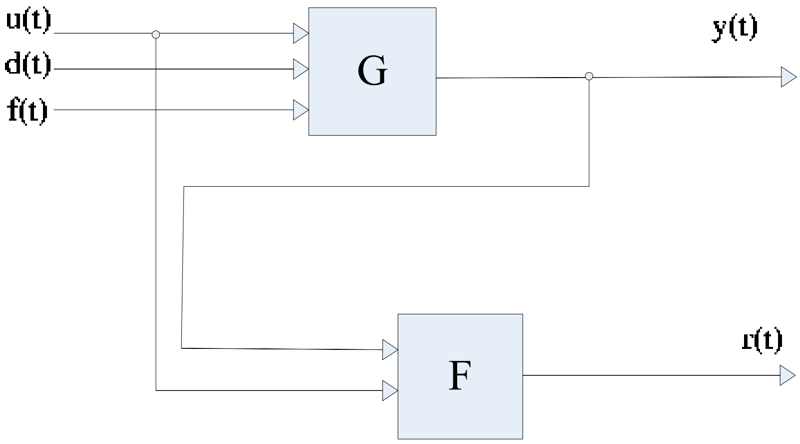

Figure 3.1: Fault Detection Filter Structure–LCTV Case

System G_u:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y_u(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

System G_d:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_d(t)d(t) \\ y_d(t) &= C(t)x(t) + D_d(t)d(t) \end{aligned}$$

System G_f:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_f(t)f(t) \\ y_f(t) &= C(t)x(t) + D_f(t)f(t) \end{aligned}$$

The filter F to be designed next is an linear bounded system from $u(t)$ and $y(t)$ to residual signal $r(t)$. Figure 3.1 shows the general form of fault detection system.

Since $(C(t), A(t))$ is detectable, according to Lemma 9, system G admits the following left coprime factorization

$$G = M^{-1}N = M^{-1}[N_u \; N_d \; N_f]. \tag{3.3}$$

To decouple the residual signal from input signal completely, the fault detection filter can take the following form:

$$r = Q(My - N_u u) = Q \begin{bmatrix} M & -N_u \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \tag{3.4}$$

which is shown in Figure 3.2, where N_u and M are linear systems with appropriate dimensions given in the coprime factorization (3.3) and Q is a linear bounded system to be designed. Specifically, the systems have the following state-space realizations, respectively.

System N_u:

$$\begin{aligned} \dot{x}_1(t) &= [A(t) + L(t)C(t)]\, x_1(t) + [B(t) + L(t)D(t)]\, u(t) \\ y_1(t) &= C(t)x_1(t) + D(t)u(t) \end{aligned}$$

System M:

$$\begin{aligned} \dot{x}_2(t) &= [A(t) + L(t)C(t)]\, x_2(t) + L(t)y(t) \\ y_2(t) &= C(t)x_2(t) + y(t) \end{aligned}$$

System N_d:

$$\begin{aligned} \dot{x}_3(t) &= [A(t) + L(t)C(t)]\, x_3(t) + [B_d(t) + L(t)D_d(t)]\, d(t) \\ y_3(t) &= C(t)x_3(t) + D_d(t)d(t) \end{aligned}$$

System N_f:

$$\begin{aligned} \dot{x}_4(t) &= [A(t) + L(t)C(t)]\, x_4(t) + [B_f(t) + L(t)D_f(t)]\, f(t) \\ y_4(t) &= C(t)x_4(t) + D_f(t)f(t) \end{aligned}$$

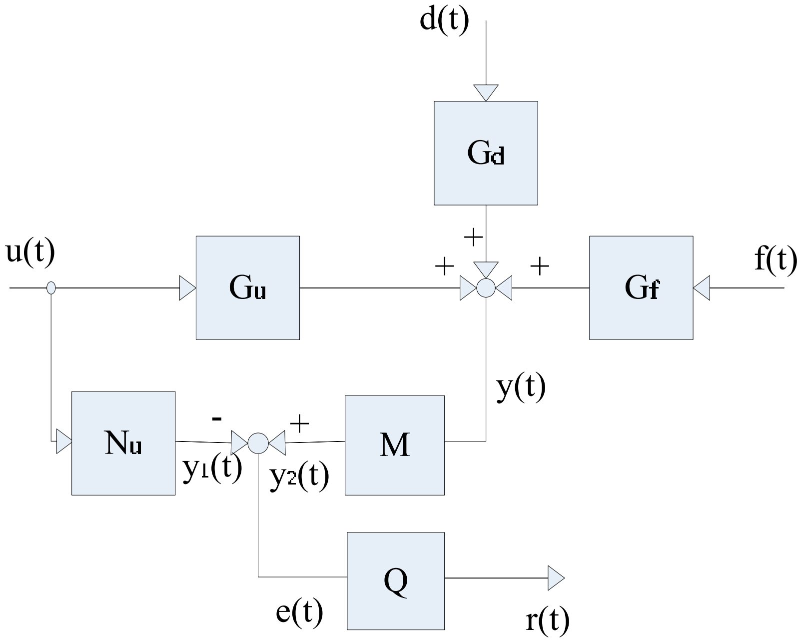

Figure 3.2: Filter Structure Decoupling Control Input–LCTV Case

By simple computation it is verified that signal $e(t) = y_2(t) - y_1(t)$ is decoupled from the input signal $u(t)$. In addition, the system from $d(t)$ to $r(t)$ is QN_d, and system from $f(t)$ to $r(t)$ is QN_f. Specifically, we have

$$r = QN_d d + QN_f f.$$

In general, a good fault detection filter must make a tradeoff between two conflicting performance objectives: robustness to disturbance rejection and sensitivity to faults. Therefore, the next step is how to design a system $Q(t)$ such that the residual signal $r(t)$ is sensitive to fault $f(t)$, but insensitive to the disturbance $d(t)$.

To achieve good robustness to disturbance, the influence of disturbance must be minimized at the output of the residual signals. On the other hand, the residual signal should be as sensitive as possible to the faults. Therefore, we need to choose certain performance criteria for measuring these two aspects so that the fault detection filter design has satisfactory fault detection sensitivity and guaranteed disturbance rejection effect.

Assume that G_{rf} is the system from fault signal to residual, and G_{rd} is the system from disturbance signal to residual. $\|G_{rf}\|_-$ is a reasonable performance criterion for measuring fault detection sensitivity if $f(t)$ is modeled as unknown energy or power bounded signals. If $d(t)$ is modeled as unknown energy or power bounded signals, then $\mathscr{H}_\infty$ is a widely accepted worst case measure and $\|G_{rd}\|_\infty$ is a good indicator of disturbance rejection performance. On the other hand, if $d(t)$ and/or $f(t)$ are white noise, the $\mathscr{H}_2$ norms of G_{rd} and/or G_{rf} seem to be more suitable criteria (see [55] for details).

Based on the definitions of norms in Chapter 2, we can formulate the following three fault detection filter design problems.

- ($\mathscr{H}_-/\mathscr{H}_\infty$ problem) Let an uncertain system be described by equations (3.1) and (3.2) and let $\beta > 0$ be a given disturbance rejection level. Find a $n_y \times n_y$ linear bounded system Q such that $\|QN_d\|_\infty \leq \beta$ and $\|QN_f\|_-$ is maximized, i.e.

$$\max_Q \{\|QN_f\|_- : \|QN_d\|_\infty \leq \beta\},$$

  where Q must be exponentially stable for infinite-horizon case.

- ($\mathscr{H}_2/\mathscr{H}_\infty$ problem) Let an uncertain system be described by equations (3.1) and (3.2) and let $\beta > 0$ be a given disturbance rejection level. Find a $n_y \times n_y$ linear bounded system Q such that $\|QN_d\|_\infty \leq \beta$ and $\|QN_f\|_2$ is maximized, i.e.

$$\max_Q \{\|QN_f\|_2 : \|QN_d\|_\infty \leq \beta\},$$

  where Q must be exponentially stable for infinite-horizon case.

- ($\mathscr{H}_\infty/\mathscr{H}_\infty$ problem) Let an uncertain system be described by equations (3.1) and (3.2) and let $\beta > 0$ be a given disturbance rejection level. Find a $n_y \times n_y$ linear bounded system Q such that $\|QN_d\|_\infty \leq \beta$ and $\|QN_f\|_\infty$ is maximized, i.e.

$$\max_Q \{\|QN_f\|_\infty : \|QN_d\|_\infty \leq \beta\},$$

where Q must be exponentially stable for infinite-horizon case.

Remark 7 *We should point out that the $\mathscr{H}_\infty/\mathscr{H}_\infty$ problem itself is not interesting. This is because making $\|QN_f\|_\infty$ large does not imply good sensitivity to the faults since the faults may not occur in the direction where $\|QN_f\|_\infty$ is large. We include this problem formulation here because it has been widely considered in the literature and it also turns out that our optimal solution to other problems is also an optimal solution to this problem. However, other optimal solutions for this particular criterion may not be a good candidate of fault detection filter, since the optimal solution to this problem is generally not unique.*

Before the main result is given, two preliminary results are needed.

Lemma 17 *Suppose G is a state space system with realization given by equations (2.3) and (2.4), with $x(0) = 0$. If there exists a bounded and symmetric matrix $X(t)$ satisfying*

$$\begin{aligned} X(t)A'(t) + A(t)X(t) + B(t)B'(t) &= \dot{X}(t) \\ D(t)B'(t) + C(t)X(t) &= 0 \\ D(t)D'(t) &= I \end{aligned}$$

with $X(0) = 0$ for all $t \in [0, T]$, then G is co-isometric on $\mathscr{L}_2[0, T]$. If the system G is observable, these conditions are also necessary.

The result is also true for infinite-horizon case ($T \to \infty$).

Proof The adjoint system $G^{\sim}$ is:

$$\begin{aligned} \dot{p}(t) &= -A'(t)p(t) - C'(t)y(t) \\ u(t) &= B'(t)p(t) + D'(t)y(t) \end{aligned}$$

with $p(T) = 0$. When $X(0) = 0$,

$$\begin{aligned} & \|u\|^2_{2,[0,T]} \\ = & \|G^{\sim}y\|^2_{2,[0,T]} \\ = & \int_0^T (B'p + D'y)'(B'p + D'y) - \frac{d}{dt}(p'Xp)dt \\ = & \int_0^T y'DD'y + 2y'(DB' + CX)p + p'(-\dot{X} + XA' + AX + BB')pdt \\ = & \|y\|^2_{2,[0,T]} \end{aligned}$$

when $DB' + CX = 0$ and $DD' = I$ and $-\dot{X} + XA' + AX + BB' = 0$.

Conversely, let $X(t)$ be the controllability gramian, we have

$$\begin{aligned} & \|u\|^2_{2,[0,T]} - \|y\|^2_{2,[0,T]} \\ = & \|G^{\sim}y\|^2_{2,[0,T]} - \|y\|^2_{2,[0,T]} \\ = & \int_0^T \{(B'p + D'y)'(B'x + D'y) - \frac{d}{dt}(p'Xx) - y'y\}dt \\ = & \int_0^T y'(DD' - I)y + 2y'(DB' + CX)p \\ & + p'(-\dot{X} + XA' + AX + BB')pdt \\ = & \int_0^T y'(DD' - I)y + 2y'(DB' + CX)pdt. \end{aligned}$$

When the system is observable, the adjoint system is controllable. Consider $y_t = P_T y$, where P_T is the truncation operator. Since controllability ensures that $p(t)$ spans $\mathscr{R}^n$ as y ranges over $\mathscr{L}_2[0, T]$. So $DB' + CX = 0$, hence, $DD' = I$.

The proof for infinite-horizon case is similar. □

The following lemma aims to find the state space realization of spectral factorization for $N_d N_d^{\sim}$, where N_d is the system in the left coprime factorization $G = M^{-1}N = M^{-1}[N_u \, N_d \, N_f]$.

Lemma 18 *Assume a state space realization of the linear system N_d is:*

$$\begin{aligned}\dot{x}_3(t) &= [A(t) + L(t)C(t)]\, x_3(t) + [B_d(t) + L(t)D_d(t)]\, d(t) \\ y_3(t) &= C(t)x_3(t) + D_d(t)d(t).\end{aligned}$$

Let V be an invertible and causal system such that $V^{-1}N_d$ is co-isometric. That is,

$$(V^{-1}N_d)(V^{-1}N_d)^{\sim} = I.$$

Then, a state space representation of V is given by

$$\begin{aligned}\dot{p}(t) &= [A(t) + L(t)C(t)]\, p(t) + (([B_d(t) + L(t)D_d(t)]\, D_d' + Y(t)C'(t))R_d^{-1/2}(t))u(t) \\ y(t) &= C(t)p(t) + R_d^{1/2}(t)u(t)\end{aligned}$$

with the state space representation for V^{-1}:

$$\begin{aligned}\dot{q}(t) &= [A(t) + L_0(t)C(t)]\, q(t) + [L_0(t) - L(t)]\, y(t) \\ u(t) &= R_d^{-1/2}(t)C(t)q(t) + R_d^{-1/2}(t)y(t)\end{aligned}$$

where $L_0(t) = -\left[B_d(t)D_d'(t) + Y(t)C'(t)\right] R_d^{-1}(t)$ and $Y(t)$ satisfies the following differential Riccati equation:

$$A_R(t)Y(t) + Y(t)A_R'(t) - Y(t)C'(t)R_d^{-1}(t)C(t)Y(t) + B_R(t)B_R'(t) = \dot{Y}(t) \quad (3.5)$$

with $Y(0) = 0$.

If $(C(t), A(t))$ is detectable and $(A_R(t), B_R(t))$ is stabilizable, there exists an unique and bounded solution $Y(t) \geq 0$ for the differential Riccati equation (3.5), and the realization V^{-1} is exponentially stable. Furthermore, $L(t)$ can be chosen so that

$$\dot{x}(t) = [A(t) + L(t)C(t)]\, x(t)$$

is exponentially stable.

Proof According to Lemma 17, to show that $V^{-1}N_d$ is a co-isometric system with the following state space realization:

$$\begin{aligned}
\dot{x}(t) &= [A(t) + L_0(t)C(t)]\, x(t) + [B_d(t) + L_0(t)D_d(t)]\, d(t) \\
r(t) &= R_d^{-1/2}(t)C(t)x(t) + R_d^{-1/2}(t)D_d(t)d(t),
\end{aligned}$$

it is sufficient to check the following three conditions:

$$\begin{aligned}
Y(t)\, [A(t) + L_0(t)C(t)]' + [A(t) + L_0(t)C(t)]\, Y(t) \qquad & \\
+ [B_d(t) + L_0(t)D_d(t)]\, [B_d(t) + L_0(t)D_d(t)]' &= \dot{Y}(t) \qquad (3.6) \\
R_d^{-1/2}(t)D_d(t)\, [B_d(t) + L_0(t)D_d(t)]' + R_d^{-1/2}(t)C(t)Y(t) &= 0 \qquad (3.7) \\
\left[R_d^{-1/2}(t)D_d(t)\right] \left[R_d^{-1/2}(t)D_d(t)\right]' &= I. \qquad (3.8)
\end{aligned}$$

Obviously, equation (3.8) is equivalent to $R_d(t) = D_d(t)D_d'(t) > 0$, since $D_d(t)$ has full row rank for all $t \geq 0$. Equation (3.7) is equivalent to $L_0(t) = -\left[B_d(t)D_d'(t) + Y(t)C'(t)\right] R_d^{-1}(t)$. Replacing L_0 by $-\left[B_d(t)D_d'(t) + Y(t)C'(t)\right] R_d^{-1}(t)$ in equation (3.6), we get the differential Riccati equation (3.5).

According to Lemma 10, when $(C(t), A(t))$ is detectable and $(A_R(t), B_R(t))$ is stabilizable, the solution for differential Riccati equation (3.5) $Y(t)$ is unique and bounded and $Y(t) \geq 0$, and system

$$\begin{aligned}
\dot{x}(t) &= \left[A_R(t) - Y(t)C'(t)R_d^{-1}(t)C(t)\right] x(t) \\
&= [A(t) + L_0(t)C(t)]\, x(t)
\end{aligned}$$

is exponentially stable. Therefore, system V^{-1} is exponentially stable. □

3.2 Main Results

We shall now present the solutions for all $\mathscr{H}_-/\mathscr{H}_\infty$, $\mathscr{H}_2/\mathscr{H}_\infty$ and $\mathscr{H}_\infty/\mathscr{H}_\infty$ fault detection problems.

Theorem 3 *For the linear time-varying system G with realization given by equations (3.1) and (3.2) for both finite-horizon case and infinite-horizon case, an optimal filter for all $\mathscr{H}_-/\mathscr{H}_\infty$, $\mathscr{H}_2/\mathscr{H}_\infty$ and $\mathscr{H}_\infty/\mathscr{H}_\infty$ problems is the following observer:*

$$\begin{aligned} \dot{\hat{x}}(t) &= [A(t) + L_0(t)C(t)]\,\hat{x}(t) + [B(t) + L_0(t)D(t)]\,u(t) - L_0(t)y(t) \\ r(t) &= \beta R_d^{-1/2}(t)(y(t) - C(t)\hat{x}(t) - D(t)u(t)). \end{aligned}$$

where

$$R_d(t) = D_d(t)D_d'(t) > 0,$$

$$L_0(t) = -\left[B_d(t)D_d'(t) + Y(t)C'(t)\right]R_d^{-1}(t)$$

and $Y(t)$ is the solution to the following differential Riccati equation:

$$A_R(t)Y(t) + Y(t)A_R'(t) - Y(t)C'(t)R_d^{-1}(t)C(t)Y(t) + B_R(t)B_R'(t) = \dot{Y}(t)$$

with $Y(0) = 0$.

For an infinite-horizon $t \in [0, \infty)$, we have $Y(t) \geq 0$ is bounded and unique, and the filter above is also exponentially stable.

Proof Since N_d admits the following spectral factorization

$$N_d N_d^{\sim} = VV^{\sim},$$

we have

$$\begin{aligned}(QN_d)(QN_d)^{\sim} &= QN_dN_d^{\sim}Q^{\sim} \\ &= QVV^{\sim}Q^{\sim} \\ &= (QV)(QV)^{\sim}.\end{aligned}$$

Hence

$$\|QV\|_\infty = \|QN_d\|_\infty.$$

Using the above equality with $\|QN_d\|_\infty \leq \beta$ and Lemma 11, we have

$$\begin{aligned}\|QN_f\|_- &= \|QVV^{-1}N_f\|_- \\ &\leq \|QV\|_\infty\|V^{-1}N_f\|_- \\ &= \|QN_d\|_\infty\|V^{-1}N_f\|_- \\ &\leq \beta\|V^{-1}N_f\|_-.\end{aligned}$$

Obviously, the inequity is also true for $\mathscr{H}_2$ norm and $\mathscr{H}_\infty$ norm according to Lemma 11. That is

$$\begin{aligned}\|QN_f\|_2 \leq & \ \beta\|V^{-1}N_f\|_2, \\ \|QN_f\|_\infty \leq & \ \beta\|V^{-1}N_f\|_\infty.\end{aligned}$$

When $QV = \beta I$, that is, $Q = \beta V^{-1}$, the filter is optimal since the equality can be obtained.

Thus, according to Lemma 18, one state space realization for Q is

$$\begin{aligned}\dot{\hat{x}}(t) &= [A(t) + L_0(t)C(t)]\,\hat{x}(t) + [L_0(t) - L(t)]\,e(t) \\ r(t) &= \beta R_d^{-1/2}(t)C(t)\hat{x}(t) + \beta R_d^{-1/2}(t)e(t).\end{aligned}$$

In addition, the corresponding state space realization for the filter F in Figure 2 is,

$$\begin{aligned}\dot{\hat{x}}(t) &= [A(t) + L_0(t)C(t)]\,\hat{x}(t) - L_0(t)y(t) + + [B(t) + L_0(t)D(t)]\,u(t) \\ r(t) &= \beta R_d^{-1/2}(t)(y(t) - C(t)\hat{x}(t) - D(t)u(t)).\end{aligned}$$

According to Lemma 18, for infinite-horizon case, M, N_u, and V^{-1} are exponentially stable. Hence, filter F is exponentially stable. □

Remark 8 *For infinite-horizon case, when system G is linear time invariant, the filter converges to a time invariant one that is the same as the one given in [56] (see Section 2.5.2).*

Remark 9 *The solution is optimal to all $\mathscr{H}_-/\mathscr{H}_\infty$, $\mathscr{H}_2/\mathscr{H}_\infty$ and $\mathscr{H}_\infty/\mathscr{H}_\infty$ problems.*

Remark 10 *The solution is independent of the choice of $L(t)$ matrix function, that is, independent of coprime factorization.*

Remark 11 *If $D_d(t)$ is a nonsingular square matrix, the differential Riccati equation can be simplified as*

$$A_R(t)Y(t) + Y(t)A'_R(t) - Y(t)C'(t)R_d^{-1}(t)C(t)Y(t) = \dot{Y}(t), \tag{3.9}$$

since $D'_d(t)R_d^{-1}(t)D_d(t) = I$. According to Lemma 10, if $(A_R(t), 0)$ is stabilizable and $(D_d^{-1}(t)C(t), A_R(t))$ is detectable, in other words, $\dot{x}(t) = A_R(t)x(t)$ is exponentially stable, the solution for the equation (3.9) is unique and bounded, that is $Y(t) = 0$. Hence, $L_0(t) = -B_d(t)D_d^{-1}(t)$ and the filter becomes

$$\begin{aligned}\dot{\hat{x}}(t) &= A_R(t)\hat{x}(t) + \left[B(t) - B_d(t)D_d^{-1}(t)D(t)\right]u(t) + B_d(t)D_d^{-1}(t)y(t) \\ r(t) &= \beta D_d^{-1}(t)\left[y(t) - C(t)\hat{x}(t) - D(t)u(t)\right].\end{aligned}$$

3.3 Extension to Unknown Initial State

In the previous sections, we have assumed that the original system given by (3.1) and (3.2) has the initial state $x(0) = 0$. If $x(0) \neq 0$ but is known in advance, we can subtract its contribution from the output and transform the original problem into a problem with zero initial state that we have solved in the previous section.

In this section, we formulate a new optimization problem with extra terms considering the effect of uncertain initial condition in the fault detection filter design.

The same procedures in the previous sections can be carried out to completely decouple the residual signal from the input signal $u(t)$ so that

$$r = QN_d d + QN_f f$$

where Q is a bounded linear time-varying system to be designed. Here, the system N_d and N_f are the same as those in the previous sections but with an important difference that N_d has an unknown initial state $x(0)$ and N_f has a zero initial state.

The standard system we consider is $G_{m\times n}$:

$$\dot{x}(t) = A(t)x(t) + B(t)w(t), \tag{3.10}$$

$$y(t) = C(t)x(t) + D(t)w(t),\ x(0) = x_0 \text{ unknown} \tag{3.11}$$

It can also be written in a simple way,

$$G : (x_0, w) \rightarrow y$$

In order to consider the effect of initial condition, we define the following inner product for the system given by equations (3.10) and (3.11).

Definition 22 *The inner product in* $\mathscr{R}^n \times \mathscr{L}_2[0,T]$ *is defined as:*

$$< (x_1, w_1), (x_2, w_2) >_{2,[0,T]} = x_1' R x_2 + < w_1, w_2 >_{2,[0,T]}$$

where $R = R' > 0$, x_1, x_2, w_1 and w_2 are initial states and input signals, respectively. The positive matrix R with compatible dimensions can be thought as penalty that reflects the knowledge we know about the initial state. The more we know the state, the bigger R should be.

Based on the definition of the inner product, we give a new definition of adjoint system with initial condition.

Definition 23 ([59]) *Adjoint system in $\mathscr{L}_{2,[0,T]}$ is a map from output signal to initial state and input signal*

$$G^{\sim} : y \to (p(0), w)$$

It includes two parts: initial condition $p(0)$ and a dynamic system:

$$G^{\sim} = \begin{bmatrix} R^{-1}\int_0^T \Phi'(s,0)C'(s)y(s)ds \\ \int_t^T B'(t)\Phi'(s,t)C'(s)y(s)ds + D'(t)y(t) \end{bmatrix}$$

A state space realization of $G^{\sim}$ can be obtained as:

- *Initial condition:*

$$x_0 = R^{-1}p(0)$$

- *Dynamic model:*

$$\begin{aligned} \dot{p}(t) &= -A'(t)p(t) - C'(t)y(t) \quad p(T) = 0 \\ w(t) &= B'(t)p(t) + D'(t)y(t), \end{aligned}$$

Note that it is sufficient to let $T \to \infty$ when considering systems in $\mathscr{L}_2$.

In this section, we will use this adjoint system to derive our fault detection filter.

Definition 24 *Similar to co-isometric system defined before, the co-isometric system with uncertain initial state is as follows:*

$$GG^{\sim} = I$$

More specifically, system defined by equations (3.10) and (3.11) is co-isometric if and only if

$$\|w\|_2^2 + x_0' R x_0 = \|y\|_2^2$$

where the 2 norms of w and y can be defined in either $\mathscr{L}_2$ or $\mathscr{L}_{2,[0,T]}$.

Next, we give a revised version of Lemma 5 that considers the effect of unknown initial state.

Lemma 19 *Suppose that G is a state space system with realization given by equations (3.10) and (3.11) with unknown $x(0)$. If there exists a bounded and symmetric matrix $X(t)$ satisfying*

$$\begin{aligned} X(t)A'(t) + A(t)X(t) + B(t)B'(t) &= \dot{X}(t) \\ D(t)B'(t) + C(t)X(t) &= 0 \\ D(t)D'(t) &= I \end{aligned}$$

with $X(0) = R^{-1}$ for all $t \in [0, T]$, then G is co-isometric on $\mathscr{L}_2[0, T]$. If the system G is observable, these conditions are also necessary.

If $T \to \infty$, the result is also true.

Proof The adjoint system $G^{\sim}$ is:

initial condition: $x_0 = R^{-1} p(0)$

dynamic model:

$$\begin{aligned}\dot{p}(t) &= -A'(t)p(t) - C'(t)y(t)\\ u(t) &= B'(t)p(t) + D'(t)y(t) \ with \ p(T) = 0.\end{aligned}$$

$$\begin{aligned}&\|u\|^2_{2,[0,T]} + x_0'Rx_0\\ =&\|G^{\sim}y\|^2_{2,[0,T]} + p'(0)R^{-1}p(0)\\ =&\int_0^T (B'p + D'y)'(B'p + D'y) - \frac{d}{dt}(p'Xp)dt + p'Xp|_0^T + p'(0)R^{-1}p(0)\\ =&\int_0^T y'DD'y + 2y'(DB' + CX)p\\ &+ p'(-\dot{X} + XA' + AX + BB')pdt + p'(0)(R^{-1} - X(0))p(0)\\ =&\|y\|^2_{2,[0,T]}\end{aligned}$$

when $DB' + CX = 0$ and $DD' = I$ and $-\dot{X} + XA' + AX + BB' = 0$ and $X(0) = R^{-1}$.

Conversely, let $X(t)$ be the controllability gramian, we have

$$\begin{aligned}&\|u\|^2_{2,[0,T]} + x_0'Rx_0 - \|y\|^2_{2,[0,T]}\\ =&\|G^{\sim}y\|^2_{2,[0,T]} + p'(0)R^{-1}p(0) - \|y\|^2_{2,[0,T]}\\ =&\int_0^T \left\{(B'p + D'y)'(B'p + D'y) - \frac{d}{dt}(p'Xp) - y'y\right\} dt\\ &+ p'Xp|_0^T + p'(0)R^{-1}p(0) + p'(0)(R^{-1} - X(0))p(0)\\ =&\int_0^T y'(DD' - I)y + 2y'(DB' + CX)p\\ &+ p'(-\dot{X} + XA' + AX + BB')pdt + p'(0)(R^{-1} - X(0))p(0)\\ =&\int_0^T y'(DD' - I)y + 2y'(DB' + CX)pdt + p'(0)(R^{-1} - X(0))p(0).\end{aligned}$$

When the system is observable, the adjoint system is controllable. Consider $y_t = P_T y$, where P_T is the truncation operator. Since controllability ensures that $p(t)$ spans $\mathscr{R}^n$ as y ranges over $\mathscr{L}_2[0,T]$, we have $DB' + CX = 0$ and $Y(0) = R^{-1}$. Hence, $DD' = I$.

If $T \to \infty$, the result is obvious. $\square$

Now, the three problems mentioned before can be revised as follows,

Problem 4 *($\mathscr{H}_-/\mathscr{H}_\infty$ problem)*

$$\max_Q \left\{ \|QN_f\|_- : \sup_{d(t),x_0} \sqrt{\frac{\|r(t)\|_2^2}{x_0' R x_0 + \|d(t)\|_2^2}} \le \beta \right\}$$

Problem 5 *($\mathscr{H}_\infty/\mathscr{H}_\infty$ problem)*

$$\max_Q \left\{ \|QN_f\|_\infty : \sup_{d(t),x_0} \sqrt{\frac{\|r(t)\|_2^2}{x_0' R x_0 + \|d(t)\|_2^2}} \le \beta \right\}$$

Problem 6 *($\mathscr{H}_2/\mathscr{H}_\infty$ problem)*

$$\max_Q \left\{ \|QN_f\|_2 : \sup_{d(t),x_0} \sqrt{\frac{\|r(t)\|_2^2}{x_0' R x_0 + \|d(t)\|_2^2}} \le \beta \right\}$$

Note that the effects of noise and uncertain initial state are considered together in these three problems and the initial state for the system QN_f is zero. More specifically, the initial states of system N_d is also x_0 that is unknown, while the initial states for the other systems (i.e. N_u, M and Q) are all zeros. Therefore, the results and definitions in the previous sections can be used directly except that the initial condition of differential Riccati equation in Lemma 18 becomes $Y(0) = R^{-1}$ according to Lemma 22.

Theorem 4 *For the system given by equations (3.1) and (3.2) with unknown initial state $x(0) = x_0$, an optimal fault detection filter for the new problems mentioned above is*

$$\begin{aligned}
\dot{\hat{x}}(t) &= [A(t) + L_0(t)C(t)]\,\hat{x}(t) + [B(t) + L_0(t)D(t)]\,u(t) - L_0(t)y(t) \\
r(t) &= \beta R_d^{-1/2}(t)\,[y(t) - C(t)\hat{x}(t) - D(t)u(t)]\,.
\end{aligned}$$

where $R_d(t) = D_d(t)D_d'(t) > 0$,

$$L_0(t) = -\left[B_d(t)D_d'(t) + Y(t)C'(t)\right] R_d^{-1}(t)$$

and $Y(t)$ is the solution to the differential Riccati equation:

$$A_R(t)Y(t) + Y(t)A'_R(t) - Y(t)C'(t)R_d^{-1}(t)C(t)Y(t) + B_R(t)B'_R(t) = \dot{Y}(t)$$

with $Y(0) = R^{-1}$. For infinite-horizon $t \in [0, \infty)$, we have $Y(t) \geq 0$ is bounded and unique, and the filter above is also optimal and exponentially stable.

Proof The proof is the same as that in the previous section and thus omitted. □

Remark 12 *The filter designed is the same as that for the known initial state, except that the initial condition for the differential Riccati equation becomes $Y(0) = R^{-1}$.*

3.4 Example

We shall illustrate our fault detection filter design with a simple linear continuous time-varying system.

Example 1. Consider the following linear time-varying system with

$$A(t) = \begin{bmatrix} -0.1 & 10(1 - e^{-t/50}) \\ 0 & -0.2 \end{bmatrix}, \quad B_d(t) = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix},$$

$$B(t) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \ B_f(t) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \ C(t) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix},$$

$$D(t) = 0, \quad D_d(t) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, D_f(t) = 0.$$

Assume that the input $u(t)$, the disturbance $d(t)$, and the fault $f(t)$ are the following forms, respectively (see Figure 3.3),

$$\begin{aligned} u(t) &= 0 \\ d(t) &= \begin{bmatrix} 0.2sin(0.5t) & 0.2cos(0.5t) \end{bmatrix}' \\ f(t) &= \begin{cases} 0.1, \ 20s \leq t < 30s \\ 0, \ \text{elsewhere.} \end{cases} \end{aligned}$$

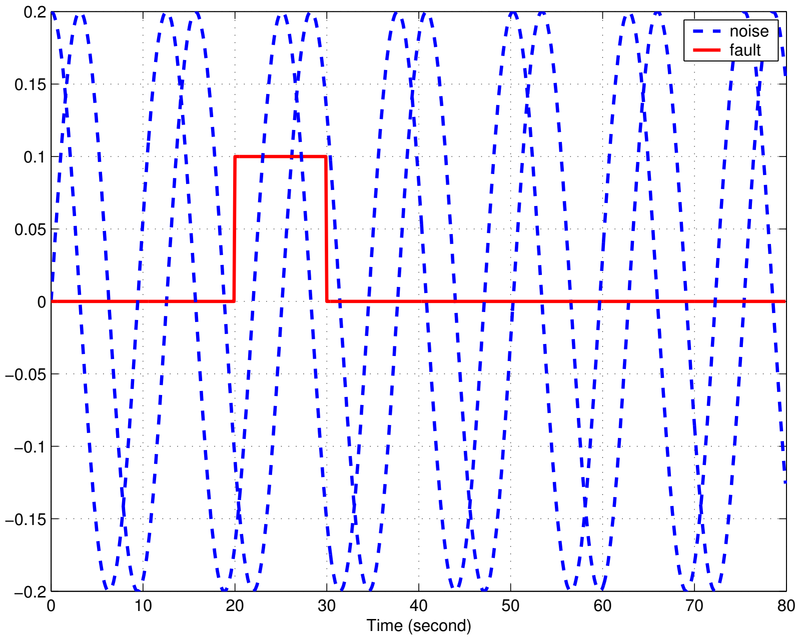

Figure 3.3: Disturbance $d(t)$ and Fault $f(t)$–LCTV Case

For $\mathscr{H}_-/\mathscr{H}_\infty$, $\mathscr{H}_2/\mathscr{H}_\infty$ and $\mathscr{H}_\infty/\mathscr{H}_\infty$ problems mentioned above with $\beta = 1$, we now compare three different filter design methods. The first is the optimal filter given by Theorem 1, where $L_0(t)$ is computed by solving the differential Riccati equation (3.5). The second is the frozen time filter designed by considering time-varying system as time invariant system at each instant of time, where $L_0(t)$ is computed by solving the algebra Riccati equation

$$A_R(t)Y(t) + Y(t)A'_R(t) - Y(t)C'(t)R_d^{-1}(t)C(t)Y(t) + B_R(t)B'_R(t) = 0$$

at each instant of time t (see [55] and Section 2.6). The third is to replace the time-varying terms by their steady states and thus design a linear time invariant filter by [55]. In this example, we let

$$A = \begin{bmatrix} -0.1 & 10 \\ 0 & -0.2 \end{bmatrix}.$$

Figure 3.4 shows different $L_0(t)$ for the three filters. Since the time-varying term $e^{-t/50}$ in the $A(t)$ matrix approaches 0 slowly, the optimal filter gain $L_0(t)$ and the $L_0(t)$ in the

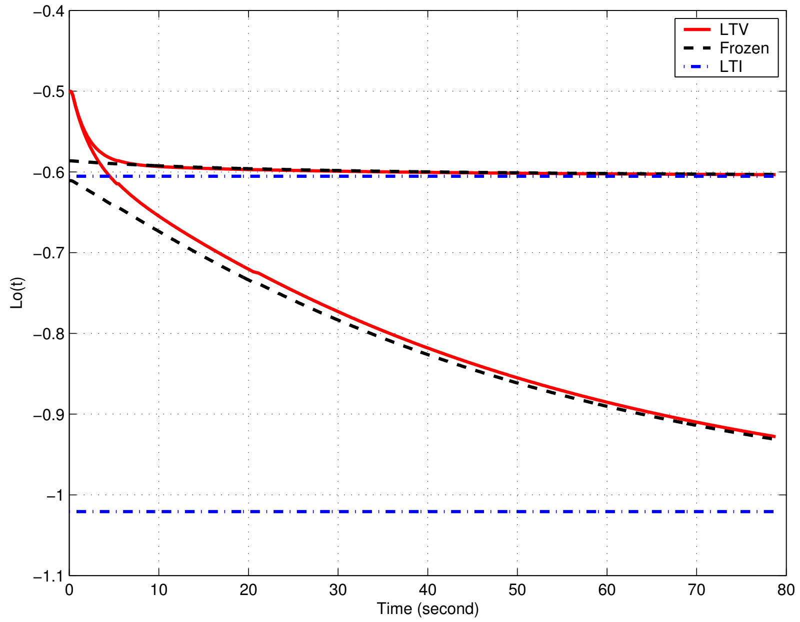

Figure 3.4: Fault Detection Filter Gain $L_0(t)$–LCTV Case

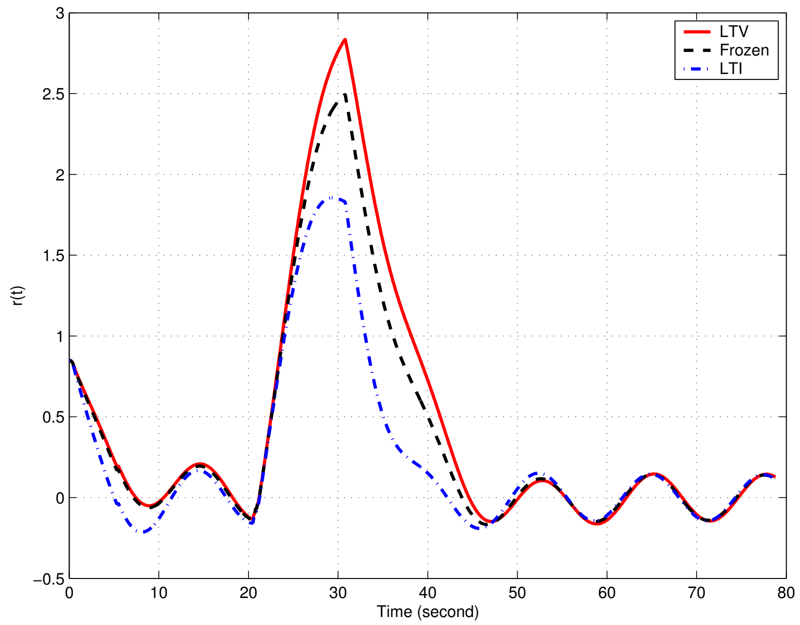

Figure 3.5: Residuals $r(t)$ vs Time t –LCTV Case

frozen time filter converge to that of the linear time invariant filter. Figure 3.5 shows different residual signals generated by the three methods. It is clear that the residual signal generated by the time-varying filter is much more sensitive to faults than those of the other two filters. Furthermore, it can also be shown that the residual signal generated by linear time-invariant filter strongly depends on the input signal $u(t)$, since input signal is not decoupled from residual signal at all time instants.

Chapter 4

Fault Detection Filter Design for Discrete Time-Varying Systems

This chapter is dedicated to fault detection for linear discrete time-varying systems (LDTVS). Problem formulation is given in Section 4.1. Our fault detection design is presented in Section 4.2. Section 4.3 extends the result to the plant with non-zero initial state. One example is given in Section 4.4 for illustration.

4.1 Problem Formulation

Consider a linear discrete time-varying system G with disturbance and possible faults in the following state-space realization

$$x(t+1) = A(t)x(t) + B(t)u(t) + B_d(t)d(t) + B_f(t)f(t) \tag{4.1}$$

$$y(t) = C(t)x(t) + D(t)u(t) + D_d(t)d(t) + D_f(t)f(t) \tag{4.2}$$

where $t = 0, 1, \ldots, T$ ($t = 0, 1, \ldots$ for the infinite-horizon case). T is a positive integer. $x(t) \in \mathscr{R}^n$ is the state vector, $u(t) \in \mathscr{R}^{n_u}$ is the input vector, $y(t) \in \mathscr{R}^{n_y}$ is the output measurement, $d(t) \in \mathscr{R}^{n_d}$ represents the unknown/uncertain disturbance and measurement noise, and $f(t) \in \mathscr{R}^{n_f}$ denotes the process, sensor or actuator fault vector. $A(t)$, $B(t)$, $C(t)$, $D(t)$, $B_d(t)$, $D_d(t)$, $B_f(t)$ and $D_f(t)$ are bounded sequences of t with compatible dimensions. $f(t)$ and $d(t)$ can be modelled as different type of signals, depending on specific situations under consideration. Different assumptions on $d(t)$ and $f(t)$ will lead to different fault detection problem formulations and the solutions for three problems will be discussed in this chapter.

Define

$$R_t(t) := D_d'(t)(D_d(t)D_d'(t))^{-1}.$$

The following assumptions are made:

Assumption 11 *$(C(t), A(t))$ is detectable;*

Assumption 12 *$D_d(t)$ has full row rank for all $t \geq 0$. In other words, $D_d(t)D_d'(t) > 0$;*

Assumption 13 *$\left(A(t) - B_d(t)R_t(t)C(t), B_d(t)[I - R_t(t)D_d(t)]\right)$ is stabilizable.*

Remark 13 *Assumption 1 is a standard assumption for all fault detection problems for the infinite-horizon case ([50] [55] [56]). This assumption guarantees the existence of a $L(t)$ such that*

$$x(t+1) = [A(t) + L(t)C(t)]\, x(t)$$

is exponentially stable.

Remark 14 *Assumption 2 means that $n_y \leq n_d$ and every measurement of the output signals is either affected by some disturbance or corrupted with some measurement noise. We argue that this assumption can be made without loss of any generality since it is impossible to take perfect measurement in any practical system and furthermore it is reasonable to assume that the measurement noise is independent of each other. So it is reasonable to assume that the measurement noise is independent of each other and that $D_d(t)$ has full row rank for all $t \geq 0$ (see [55] for detailed description).*

In the case of some simplified model where $D_d(t)$ does not have full row rank, we can simply add some columns to make it full row rank. For example, suppose $D_d(t)$ is not full row rank, then let

$$\tilde{d} = \begin{bmatrix} d \\ d_\epsilon \end{bmatrix}, \ \tilde{B}_d = \begin{bmatrix} B_d & 0_{n\times n_y} \end{bmatrix}, \ \tilde{D}_d = \begin{bmatrix} D_d & \epsilon I_{n_y} \end{bmatrix}$$

for a small $\epsilon > 0$. Then $\tilde{D}_d(t)$ has full row rank for all $t \geq 0$. Because ϵ can be made as small as possible, we argue that the performance degradation caused by the fictitious disturbances should not be large.

This assumption might be restrictive in some applications when the external disturbances and measurement noise are different classes of signals so that it is impossible to combine them together in our framework. Actually, when D_d is not wide or square, in other words, the number of outputs is greater that the number of disturbances, the decoupling of some disturbances from residual without sacrificing the fault detection ability is possible. Therefore, our optimization framework is still useful when it is possible to remove some disturbances that can be decoupled from the residual([18]). This will be discussed in details in Chapter 5.

Remark 15 *The assumption 3 is an additional assumption for guaranteeing the existence of a unique and exponentially stable filter for the infinite-horizon case. In particular, when the plant and measurement noise are independent (i.e., $B_d(t)D_d'(t) = 0$), this assumption is equivalent to that $(A(t), B_d(t))$ is stabilizable. This assumption can be significantly relaxed in the linear time-invariant case [55].*

The system realization given by (4.1) and (4.2) can be written as:

$$y = \begin{bmatrix} G_u & G_d & G_f \end{bmatrix} \begin{bmatrix} u \\ d \\ f \end{bmatrix}$$

where G_u, G_d, G_f are $n_y \times n_u$, $n_y \times n_d$ and $n_y \times n_f$ systems with the following state space realizations:

System G_u:

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) \\ y_u(t) &= C(t)x(t) + D(t)u(t). \end{aligned}$$

System G_d:

$$\begin{aligned} x(t+1) &= A(t)x(t) + B_d(t)d(t) \\ y_d(t) &= C(t)x(t) + D_d(t)d(t). \end{aligned}$$

System G_f:

$$\begin{aligned} x(t+1) &= A(t)x(t) + B_f(t)f(t) \\ y_f(t) &= C(t)x(t) + D_f(t)f(t). \end{aligned}$$

Since $(C(t), A(t))$ is detectable, according to Lemma 14, system G admits the following left coprime factorization

$$G = M^{-1}N = M^{-1}[N_u \ N_d \ N_f].$$

In order to decouple the residual signal from the input signal completely, the fault detection can take the following form:

$$r = Q(My - N_u u) = Q \begin{bmatrix} M & -N_u \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}$$

which is shown in Figure 4.1, where both N and M are linear systems with appropriate dimensions and Q is a bounded system to be designed. In addition, the systems in the Figure 4.1 have the following state-space realizations.

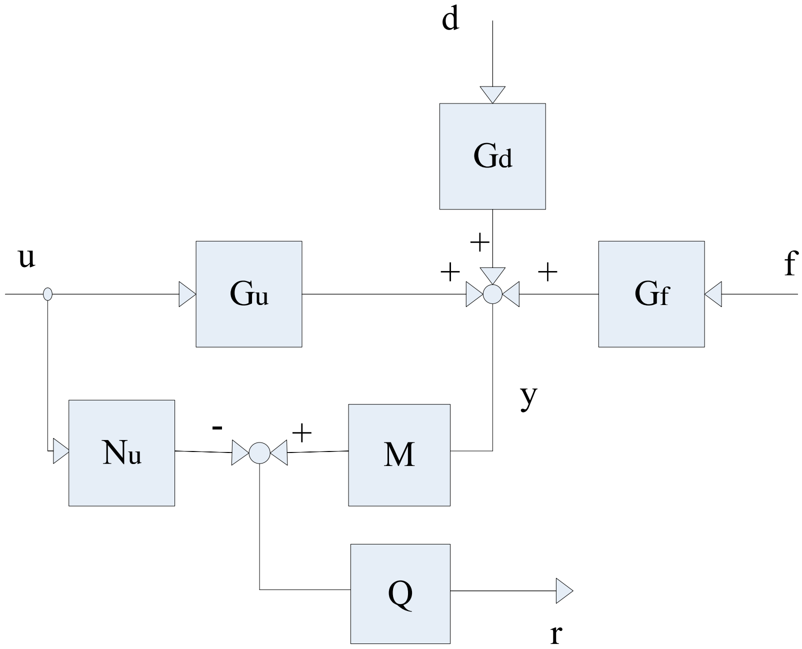

Figure 4.1: Filter Structure Decoupling Control Input–LDTV Case

System N_u:

$$
\begin{aligned}
x_1(t+1) &= [A(t) + L(t)C(t)]x_1(t) + [B(t) + L(t)D(t)]u(t) \\
y_1(t) &= C(t)x_1(t) + D(t)u(t).
\end{aligned}
$$

System M:

$$
\begin{aligned}
x_2(t+1) &= [A(t) + L(t)C(t)]x_2(t) + L(t)y(t) \\
y_2(t) &= C(t)x_2(t) + y(t).
\end{aligned}
$$

System N_d:

$$x_3(t+1) = [A(t) + L(t)C(t)]x_3(t) + [B_d(t) + L(t)D_d(t)]d(t) \tag{4.3}$$

$$y_3(t) = C(t)x_3(t) + D_d(t)d(t). \tag{4.4}$$

System N_f:

$$
\begin{aligned}
x_4(t+1) &= [A(t) + L(t)C(t)]x_4(t) + [B_f(t) + L(t)D_f(t)]f(t) \\
y_4(t) &= C(t)x_4(t) + D_f(t)f(t).
\end{aligned}
$$

By simple computation, it is easy to verify that the signal $e(t) = y_2(t) - y_1(t)$ is decoupled from the input signal $u(t)$. In addition, the system from $d(t)$ to $r(t)$ is QN_d and the system from $f(t)$ to $r(t)$ is QN_f.

Similar to [55], we formulate the following problems:

- ($\mathscr{H}_-/\mathscr{H}_\infty$ problem) Let an uncertain system be described by equations (4.1) and (4.2) and let $\beta > 0$ be a given disturbance rejection level. Find a $n_y \times n_y$ linear system Q such that $\|QN_d\|_\infty \leq \beta$ and $\|QN_f\|_-$ is maximized, i.e.

$$\max_Q \{\|QN_f\|_- : \|QN_d\|_\infty \leq \beta\},$$

  where Q must be exponentially stable for the infinite-horizon case.

- ($\mathscr{H}_2/\mathscr{H}_\infty$ problem) Let an uncertain system be described by equations (4.1) and (4.2) and let $\beta > 0$ be a given disturbance rejection level. Find a $n_y \times n_y$ linear system Q such that $\|QN_d\|_\infty \leq \beta$ and $\|QN_f\|_2$ is maximized, i.e.

$$\max_Q \{\|QN_f\|_2 : \|QN_d\|_\infty \leq \beta\},$$

  where Q must be exponentially stable for the infinite-horizon case.

- ($\mathscr{H}_\infty/\mathscr{H}_\infty$ problem) Let an uncertain system be described by equations (4.1) and (4.2) and let $\beta > 0$ be a given disturbance rejection level. Find a $n_y \times n_y$ linear system Q such that $\|QN_d\|_\infty \leq \beta$ and $\|QN_f\|_\infty$ is maximized, i.e.

$$\max_Q \{\|QN_f\|_\infty : \|QN_d\|_\infty \leq \beta\},$$

  where Q must be exponentially stable for the infinite-horizon case.

Before proving the main result, we introduce the following lemma to provide a spectral factorization of the system N_d.

Lemma 20 *For the system N_d given by equations (4.3) and (4.4), a spectral factorization is system V:*

$$\begin{aligned} x(t+1) &= [A(t) + L(t)C(t)]x(t) + [L(t) - L_1(t)]R_d^{1/2}(t)u(t) \\ y(t) &= C(t)x(t) + R_d^{1/2}(t)u(t) \end{aligned}$$

with V^{-1}:

$$\begin{aligned} x(t+1) &= [A(t) + L_1(t)C(t)]x(t) + [L_1(t) - L(t)]u(t) \\ y(t) &= R_d^{-1/2}(t)C(t)x(t) + R_d^{-1/2}(t)u(t) \end{aligned}$$

where $R_d(t) = D_d(t)D_d'(t) + C(t)P(t)C'(t)$, $L_1(t) = -[B_d(t)D_d'(t) + A(t)P(t)C'(t)]R_d^{-1}(t)$ and $P(t)$ is the solution of the following difference Riccati equation

$$A(t)P(t)A'(t) - L_1(t)R_d(t)L_1'(t) + B_d(t)B_d'(t) = P(t+1) \tag{4.5}$$

with $P(0) = 0$.

Proof Since Assumptions are satisfied, Lemma 15 can be applied to N_d to get $VV^\sim = N_d N_d^\sim$ where $P(t)$ satisfies the following difference Riccati equation

$$A_{LC}(t)P(t)A_{LC}'(t) - T(t)R_d(t)T'(t) + B_{LD}(t)B_{LD}'(t) = P(t+1)$$

where

$$\begin{aligned} A_{LC}(t) &= A(t) + L(t)C(t), \\ B_{LD}(t) &= B_d(t) + L(t)D_d(t), \\ T(t) &= [A_{LC}(t)P(t)C'(t) + B_{LD}(t)D_d'(t)]R_d^{-1}(t). \end{aligned}$$

It is easy to show that the above difference Riccati equation can be simplified to the difference Riccati equation (4.5). The rest of the proof follows from some simple algebraic manipulations. □

The following lemma is to show that under the three assumptions mentioned above, there is always a bounded solution for the difference Riccati equation and the system V^{-1} is exponentially stable.

Lemma 21 *Suppose* $\big(A(t)-B_d(t)R_t(t)C(t), B_d(t)[I-R_t(t)D_d(t)]\big)$ *is stabilizable and* $(C(t), A(t))$ *is detectable. There exists a nonnegative bounded solution* $P(t) \geq 0$ *to the filter Riccati equation (4.5), and the system* V^{-1} *is exponentially stable.*

Proof In order to simplify the problem, let's consider the special case $B_d(t)D_d'(t) = 0$.

Now, the assumptions turn out to be that $D_d(t)D_d'(t) > 0$, $(A(t), B_d(t))$ is stabilizable and $(C(t), A(t))$ is detectable.

According to Lemma 16, the following difference Riccati equation

$$A(t)P(t)A'(t) - [A(t)P(t)C'(t)]R_d^{-1}(t)[A(t)P(t)C'(t)]' + B_d(t)B_d'(t) = P(t+1)$$

has a nonnegative bounded solution, and $A(t) - A(t)P(t)C'(t)R_d^{-1}(t)C(t)$ is exponential stable.

Now we try to relax the constraint that $B_d(t)D_d'(t) = 0$.

Note that the first state equation of the system N_d can be written as

$$\begin{aligned} x_3(t+1) &= [A(t) - B_d(t)R_t(t)C(t) + L(t)C(t)]x_3(t) + B_d(t)R_t(t)y_3(t) \\ &\quad +[B_d(t) - B_d(t)R_t(t)D_d(t) + L(t)D_d(t)]d(t). \end{aligned}$$

Its state $x_3(t)$ can be decomposed as

$$x_3(t) = \widetilde{x}(t) + \widehat{x}(t),$$

where $\widetilde{x}(t)$ depends on $x(0)$ and $[B_d(t) - B_d(t)R_t(t)D_d(t) + L(t)D_d(t)]d(t)$, while $\widehat{x}(t)$ is the contribution due to $B_d(t)R_t(t)y_3(t)$ term, and it is known exactly. Therefore, we only need to consider $\widetilde{x}(t)$. So the state equations become

$$\begin{aligned}\widetilde{x}(t+1) &= [\widetilde{A}(t) + L(t)C(t)]\widetilde{x}(t) + [\widetilde{B}_d(t) + L(t)D_d(t)]d(t) \\ y_3(t) &= C(t)\widetilde{x}(t) + D_d(t)d(t)\end{aligned}$$

where $\widetilde{A}(t) = A(t) - B_d(t)R_t(t)C(t)$ and $\widetilde{B}_d(t) = B_d(t)[I - R_t(t)D_d(t)]$. Obviously, the new state equations satisfy $\widetilde{B}_d(t)D'_d(t) = 0$.

In other words, the constraint $B_d(t)D'_d(t) = 0$ is relaxed with the following new matrices

$$\begin{aligned}\widetilde{A}(t) &\leftarrow A(t) - B_d(t)D'_d(t)(D_d(t)D'_d(t))^{-1}C(t) \\ \widetilde{B_d}(t) &\leftarrow B_d(t)[I - D'_d(t)(D_d(t)D'_d(t))^{-1}D_d(t)].\end{aligned}$$

Now, the difference Riccati equation becomes

$$\widetilde{A}(t)P(t)\widetilde{A}'(t) - (\widetilde{A}(t)P(t)C'(t))\widetilde{R}_d^{-1}(t)(\widetilde{A}(t)P(t)C'(t))' + \widetilde{B_d}(t)\widetilde{B_d}'(t) = P(t+1)$$

where $\widetilde{L}_1(t) = -(\widetilde{B}(t)D'_d(t) + A(t)P(t)C'(t))\widetilde{R}_d^{-1}(t)$ and $\widetilde{R_d}(t) = D_d(t)D'_d(t) + C(t)P(t)C'(t)$.

It can be simplified as the difference Riccati equation (4.5) by some simple algebraic manipulations, and furthermore, $\widetilde{A}(t) + \widetilde{A}(t)P(t)C'(t)\widetilde{R}_d^{-1}(t)C(t)$ is exponentially stable. That is, $A(t) + L_1(t)C(t)$ is exponentially stable. Hence, V^{-1} is exponentially stable. □

4.2 Main Results

We shall now present the solutions for all $\mathscr{H}_-/\mathscr{H}_\infty$, $\mathscr{H}_2/\mathscr{H}_\infty$ and $\mathscr{H}_\infty/\mathscr{H}_\infty$ fault detection problems.

Theorem 5 *For the system given by equations (4.1) and (4.2) under three above assumptions (Assumption 3 only for infinite-horizon case) for both infinite and finite case, an optimal fault detection filter for all $\mathscr{H}_-/\mathscr{H}_\infty$, $\mathscr{H}_2/\mathscr{H}_\infty$ and $\mathscr{H}_\infty/\mathscr{H}_\infty$ cases, has the following* *state space representation*

$$\hat{x}(t+1) = [A(t) + L_1(t)C(t)]\hat{x}(t) - L_1(t)y(t) + [B(t) + L_1(t)D(t)]u(t) \quad (4.6)$$

$$r(t) = \beta R_d^{-1/2}(t)[y(t) - C(t)\hat{x}(t) - D(t)u(t)] \quad (4.7)$$

where $L_1(t) = -(A(t)P(t)C'(t) + B_d(t)D_d'(t))R_d^{-1}(t),\ R_d(t) = D_d(t)D_d'(t) + C(t)P(t)C'(t)$ *and* $P(t)$ *satisfies the following difference Riccati equation:*

$$A(t)P(t)A'(t) - L_1(t)R_d(t)L_1'(t) + B_d(t)B_d'(t) = P(t+1)$$

with initial condition $P(0) = 0$.

For the infinite-horizon case, this fault detection filter is also exponentially stable.

Proof Since N_d admits the following spectral factorization

$$N_d N_d^{\sim} = VV^{\sim},$$

we have

$$\begin{aligned}(QN_d)(QN_d)^{\sim} &= QN_d N_d^{\sim} Q^{\sim}\\ &= QVV^{\sim}Q^{\sim}\\ &= (QV)(QV)^{\sim}.\end{aligned}$$

Hence

$$\|QV\|_\infty = \|QN_d\|_\infty.$$

Using the above equality with $\|QN_d\| \leq \beta$ and Lemma 11, we have

$$\begin{aligned}\|QN_f\|_- &= \|QVV^{-1}N_f\|_-\\ &\leq \|QV\|_\infty \|V^{-1}N_f\|_-\\ &= \|QN_d\|_\infty \|V^{-1}N_f\|_-\\ &\leq \beta\|V^{-1}N_f\|_-.\end{aligned}$$

Obviously, the inequity is also true for $\mathscr{H}_2$ norm and $\mathscr{H}_\infty$ norm according to Lemma 11. That is

$$\begin{aligned} \|QN_f\|_2 \leq & \quad \beta\|V^{-1}N_f\|_2, \\ \|QN_f\|_\infty \leq & \quad \beta\|V^{-1}N_f\|_\infty. \end{aligned}$$

When $QV = \beta I$, that is, $Q = \beta V^{-1}$, the filter is optimal since the equality can be obtained.

Thus, according to Lemma 20, one state space realization for the system Q is

$$\begin{aligned} \hat{x}(t+1) &= [A(t) + L_1(t)C(t)]\hat{x}(t) + [L_1(t) - L(t)]e(t) \\ r(t) &= \beta R_d^{-1/2}(t)C(t)\hat{x}(t) + \beta R_d^{-1/2}(t)e(t). \end{aligned}$$

Further, the corresponding state space realization for the filter F is given by equations (4.6) and (4.7).

According to Lemma 21, for the infinite-horizon case, N_d and V^{-1} are exponentially stable. Hence, the filter F is exponentially stable. □

Remark 16 *Assume that all coefficients of equations (4.1) and (4.2) are θ-periodic, that is, the system G is periodic. Since periodic system is a special case of time-varying system, the same filter is derived, while all coefficients of the filter are θ-periodic. This result is the same as the solution in [90].*

4.3 Extension to Unknown Initial State

In the previous sections, we have assumed that the original system given by (4.1) and (4.2) has initial state $x(0) = 0$. If $x(0) \neq 0$ but is known in advance, we can subtract its contribution from output and transform the original problem into a problem with zero initial state that we have solved in the previous section.

In this section, we formulate a new optimization problem with extra terms considering the effect of uncertain initial condition.

The same procedures in the previous sections can be carried out to completely decouple the residual signal from the input signal $u(t)$ so that

$$r = QN_d d + QN_f f$$

where Q is a bounded linear time-varying system to be designed. Here, the system N_d and N_f are the same as those in the previous sections but with an important difference that N_d has an unknown initial state $x(0)$ and N_f has a zero initial state.

The standard system we consider is $G_{m\times n}$:

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)w(t) && (4.8) \\ y(t) &= C(t)x(t) + D(t)w(t),\ x(0) = x_0 \textit{ unknown}. && (4.9) \end{aligned}$$

It can also be written in a simple way,

$$G : (x_0, w) \to y.$$

In order to consider the effect of initial condition, we define the following inner product for the system given by equations (4.8) and (4.9).

Definition 25 *The inner product in $\mathscr{R}^n \times l_2[0,T]$ is defined as:*

$$< (x_1, w_1), (x_2, w_2) >_{2,[0,T]} = x_1' R x_2 + < w_1, w_2 >_{2,[0,T]}$$

where $R = R' > 0$, x_1, x_2, w_1 and w_2 are initial states and input signals, respectively. The positive matrix R with compatible dimensions can be thought as penalty that reflects the *knowledge we know about the initial state. The more we know the state, the bigger R should be.*

Based on the definition of the inner product, we give a new definition of the adjoint system with an initial state.

Definition 26 *Adjoint system in $l_2[0,T]$ is a map from output signal to initial state and input signal*

$$G^{\sim} : y \rightarrow (x_0, w).$$

It includes two parts: initial condition $p(-1)$ and a dynamic system:

$$G^{\sim} = \begin{bmatrix} R^{-1}\sum_{t=0}^{T}\Phi'(t,0)C'(t)y(t) \\ \sum_{s=t}^{T}[B'(t)\Phi'(s,t)C'(s)y(s)] + D'(t)y(t) \end{bmatrix}$$

A state space realization of $G^{\sim}$ can be obtained as:

- *Initial condition:*

$$x_0 = R^{-1}p(-1);$$

- *Dynamic model:*

$$\begin{aligned} p(t-1) &= A'(t)p(t) + C'(t)y(t) \\ w(t) &= B'(t)p(t) + D'(t)y(t), \quad p(T) = 0. \end{aligned}$$

Note that it is sufficient to assume $T \rightarrow \infty$ when considering systems in $l_2[0,\infty)$.

In this section, we will use this adjoint system to derive the fault detection filter.

Definition 27 *Similar to co-isometric system defined before, the co-isometric system with uncertain initial state is as follows*

$$GG^{\sim} = I.$$

More specifically, the system defined by equations (4.8) and (4.9) is co-isometric if and only if

$$\|w\|^2 + x_0' R x_0 = \|y\|^2.$$

where the 2 norms of w and y can be defined in either $l_2[0,\infty)$ or $l_2[0,T]$.

Next, we give a revised version of Lemma 5 that considers the effect of unknown initial state.

Lemma 22 *Suppose G is a state space system with realization given by equations (4.8) and (4.9) with unknown initial state $x(0) = x_0$. If there exists a bounded and symmetric matrix $X(t)$ satisfying*

$$\begin{aligned} A(t)X(t)A'(t) + B(t)B'(t) &= X(t+1) \\ B(t)D'(t) + A(t)X(t)C'(t) &= 0 \\ D(t)D'(t) + C(t)X(t)C'(t) &= I \end{aligned}$$

with $X(0) = R^{-1}$ for all $t \in [0,T]$, then G is co-isometric on $l_2[0,T]$. If the system G is observable, these conditions are also necessary.

If $T \to \infty$, the result is also true.

Proof The adjoint system $G^{\sim}$ is:

initial condition: $x_0 = R^{-1}p(-1)$

dynamic model:

$$\begin{aligned} p(t-1) &= A'(t)p(t) + C'(t)y(t) \\ w(t) &= B'(t)p(t) + D'(t)y(t), \qquad p(T) = 0. \end{aligned}$$

$$
\begin{aligned}
&\|w(t)\|^2_{2,[0,T]} + x_0' R x_0 \\
=&\|G^{\sim} y(t)\|^2_{2,[0,T]} + p'(-1)R^{-1}p(-1) \\
=&\sum_{t=0}^{T}[B'(t)p(t) + D'(t)y(t)]'[B'(t)p(t) + D'(t)y(t)] - \sum_{t=0}^{T}\{p'(t)X(t{+}1)p(t) - p'(t{-}1)X(t)p(t{-}1)\} \\
&+ p'(t)X(t+1)p(t)|_{t=-1}^{T} + p'(-1)R^{-1}p(-1) \\
=&\sum_{t=0}^{T}\{y'(t)[D(t)D'(t) + C(t)X(t)C'(t)]y(t) + 2p'(t)[B(t)D'(t) + A(t)X(t)C'(t)]y(t) \\
&+ p'(t)[-X(t{+}1) + A(t)X(t)A'(t) + B(t)B'(t)]p(t)\} \\
&+ p'(-1)[R^{-1} - X(0)]p(-1) \\
=&\|y(t)\|^2_{2,[0,T]}
\end{aligned}
$$

when $B(t)D'(t)+A(t)X(t)C'(t) = 0$, $-X(t{+}1)+A(t)X(t)A'(t)+B(t)B'(t) = 0$, $D(t)D'(t)+C(t)X(t)C'(t) = I$ and $X(0) = R^{-1}$.

Conversely, let $X(t)$ be the controllability gramian, we have

$$
\begin{aligned}
&\|w\|^2_{2,[0,T]} + x_0' R x_0 - \|y\|^2_{2,[0,T]} \\
=&\|G^{\sim} y(t)\|^2_{2,[0,T]} + p'(-1)R^{-1}p(-1) - \|y\|^2_{2,[0,T]} \\
=&\sum_{t=0}^{T}\{[B'(t)p(t) + D'(t)y(t)]'[B'(t)p(t) + D'(t)y(t)] - y'(t)y(t)\} \\
&- \sum_{t=0}^{T}\{p'(t)X(t{+}1)p(t) - p'(t{-}1)X(t)p(t{-}1)\} \\
&+ p'(t)X(t+1)p(t)|_{t=-1}^{T} + p'(-1)R^{-1}p(-1) \\
=&\sum_{t=0}^{T}\{y'(t)[D(t)D'(t) + C(t)X(t)C'(t) - I]y(t) + 2p'(t)[B(t)D'(t) + A(t)X(t)C'(t)]y(t) \\
&+ p'(t)[-X(t{+}1) + A(t)X(t)A'(t) + B(t)B'(t)]p(t)\} \\
&+ p'(-1)[R^{-1} - X(0)]p(-1) \\
=&\sum_{t=0}^{T}\{y'(t)[D(t)D'(t) + C(t)X(t)C'(t) - I]y(t) + 2p(t)[B(t)D'(t) + A(t)X(t)C'(t)]y(t)\} \\
&+ p'(-1)[R^{-1} - X(0)]p(-1).
\end{aligned}
$$

When the system is observable, the adjoint system is controllable. Consider $y_t(t) = P_T y(t)$, where P_T is the truncation operator. Since controllability ensures that $p(t)$ spans R^n as y ranges over $l_2[0,T]$, we have $B(t)D'(t) + A(t)X(t)C'(t) = 0$ and $X(0) = R^{-1}$. Hence, $D(t)D'(t) + C(t)X(t)C'(t) - I = 0$.

If $T \to \infty$, the result is obvious. □

Now, the three problems mentioned before can be revised as follows,

Problem 7 *($\mathscr{H}_-/\mathscr{H}_\infty$ problem)*

$$\max_Q \left\{ \|QN_f\|_- : \sup_{d(t),x_0} \sqrt{\frac{\|r(t)\|^2}{x_0' R x_0 + \|d(t)\|^2}} \leq \beta \right\}$$

Problem 8 *($\mathscr{H}_\infty/\mathscr{H}_\infty$ problem)*

$$\max_Q \left\{ \|QN_f\|_\infty : \sup_{d(t),x_0} \sqrt{\frac{\|r(t)\|^2}{x_0' R x_0 + \|d(t)\|^2}} \leq \beta \right\}$$

Problem 9 *($\mathscr{H}_2/\mathscr{H}_\infty$ problem)*

$$\max_Q \left\{ \|QN_f\|_2 : \sup_{d(t),x_0} \sqrt{\frac{\|r(t)\|^2}{x_0' R x_0 + \|d(t)\|^2}} \leq \beta \right\}$$

Note that the effects of noise and uncertain initial condition are considered together in these three problems and the initial condition for the system QN_f is zero. More specifically, the initial condition of system N_d is also x_0 that is unknown, while the initial conditions for the other systems (i.e. N_u, M and Q) are all zeros. Therefore, the results and definitions in the previous sections can be used directly except that the initial condition of difference Riccati equation in Lemma 13 becomes $X(0) = R^{-1}$ according to Lemma 22.

Theorem 6 *For the system given by equations (4.1) and (4.2) with unknown initial state $x(0) = x_0$, an optimal fault detection filter for the new problems formulated above is*

$$\begin{aligned} \hat{x}(t+1) &= [A(t) + L_1(t)C(t)]\hat{x}(t) - L_1(t)y(t) + [B(t) + L_1(t)D(t)]u(t) \\ r(t) &= \beta R_d^{-1/2}(t)[y(t) - C(t)\hat{x}(t) - D(t)u(t)], \quad \textit{with } \hat{x}(0) = 0 \end{aligned}$$

where $R_d(t) = D_d(t)D_d'(t)+C(t)P(t)C'(t) > 0$, $L_1(t) = -(B_d(t)D_d'(t)+A(t)P(t)C'(t))R_d^{-1}(t)$ *and* $P(t)$ *is the solution to the difference Riccati equation:*

$$A(t)P(t)A'(t) - L_1(t)R_d(t)L_1'(t) + B_d(t)B_d'(t) = P(t+1)$$

with $P(0) = R^{-1}$. *For the infinite-horizon case* $t \in [0, \infty)$, $P(t) \geq 0$ *is bounded, and the filter is also asymptotically stable.*

Proof The derivation of the filter formula and optimal property of the fault detection filter is the same as that in the previous section and thus omitted.

Regarding to the stability of the fault detection filter for the infinite-horizon case, instead of proving it directly, we find it can be transformed into a Kalman filter problem. Specifically, for the system

$$\begin{aligned} x(t+1) &= A(t)x(t) + B_d(t)w(t), \\ y(t) &= C(t)x(t) + D_d(t)w(t), \end{aligned}$$

with assumption $B_d(t)D_d'(t) = 0$, under the standard assumptions for the Kalman filter, its Kalman filter form is given by

$$\begin{aligned} \hat{x}(t+1|t) &= [A(t) + L_1(t)C(t)]\hat{x}(t|t-1) - L_1(t)y(t), \quad \hat{x}(0|-1) = \overline{x}_0 \\ L_1(t) &= -A(t)P(t)C'(t)[D(t)D'(t) + C(t)P(t)C'(t)]^{-1}, \quad P_0 = R^{-1} \\ P(t+1) &= A(t)P(t)A'(t) - A(t)P(t)C'(t)[D(t)D'(t) + C(t)P(t)C'(t)]^{-1}C(t)P(t)A'(t). \end{aligned}$$

From [27], we know that, under three assumptions above, the Kalman filter is asymptotically stable, so is the fault detection filter.

The assumption $B_dD_d' = 0$ can be relaxed by the methods in Lemma 21. □

Remark 17 *The filter designed is the same as that for the known initial state, except that the initial condition for the difference Riccati equation becomes $P(0) = R^{-1}$.*

4.4 Example

Consider a linear discrete time-varying system with the following coefficients

$$A(t) = \begin{bmatrix} -0.1 & 1+10\times 0.9^t \\ 0 & -0.2-0.1^t \end{bmatrix}, \quad B_d(t) = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix},$$

$$B(t) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \ B_f(t) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \ C(t) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix},$$

$$D(t) = 0, \ \ D_d(t) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \ D_f(t) = 0.$$

where $t = 0, 1, 2, \cdots$. Assume that the input $u(t)$, the noise $d(t)$, and the fault $f(t)$ are the following forms, respectively,

$$\begin{aligned} u(t) &= \begin{cases} 0, & t < 20s \\ 20, & \text{elsewhere;} \end{cases} \\ d(t) &= \begin{bmatrix} 0.2sin(0.5t) & 0.2cos(0.5t) \end{bmatrix}'; \\ f(t) &= \begin{cases} 0.1, & 5s \le t < 10s \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

For $\mathscr{H}_-/\mathscr{H}_\infty$, $\mathscr{H}_2/\mathscr{H}_\infty$ and $\mathscr{H}_\infty/\mathscr{H}_\infty$ problems mentioned above with $\beta = 1$, we now compare three different filter design methods. The first is the optimal filter given by Theorem 1, where $L_1(t)$ is computed by solving the difference Riccati equation (4.5). The second is the frozen time filter designed by considering time-varying system as time invariant system at each instant of time, where $L_1(t)$ is computed by solving the Algebra Riccati equation

$$A(t)P(t)A'(t) - L_1(t)R_d(t)L_1'(t) + B_d(t)B_d'(t) = 0$$

at each instant of time t (see [56] and Section 2.6). The third is to replace the time-varying terms by their steady states and thus design a linear time invariant filter by [56]. In this

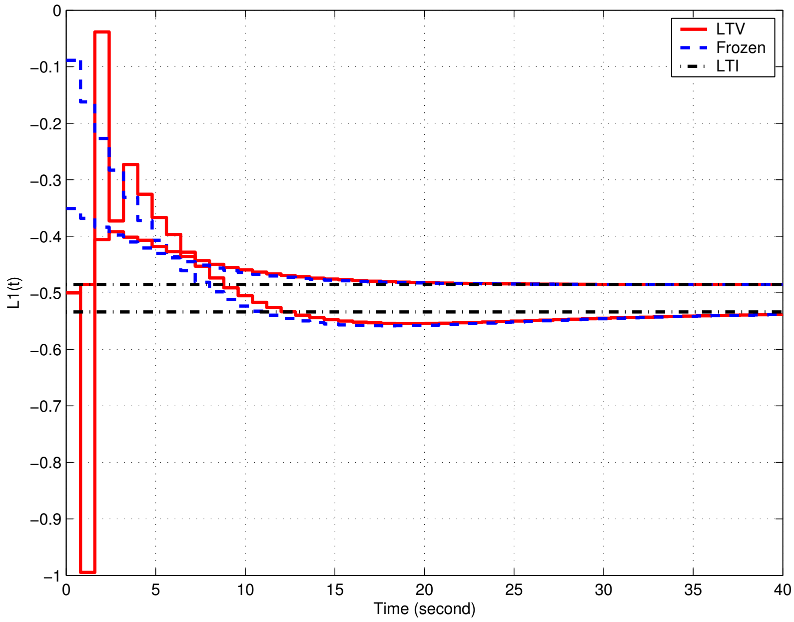

Figure 4.2: $L_1(t)$ vs Time t–LDTV Case

example, we let

$$A = \begin{bmatrix} -0.1 & 1 \\ 0 & -0.2 \end{bmatrix}.$$

Figure 4.2 shows different $L_1(t)$ for the three filters. Since the time-varying term 0.9^t in the $A(t)$ matrix approaches 0 gradually, the optimal filter gain $L_1(t)$ and the $L_1(t)$ in the frozen time filter converge to that of the linear time-invariant filter. Figure 4.3 shows different residual signals generated by the three methods. It is clear that the residual signals generated by these three filters are sensitive to faults. However, it can also be shown that the residual signal generated by the linear time-invariant filter strongly depend on the input signal $u(t)$, since input signal is not decoupled from residual signal at all time instants. Specially, when $t > 20s$ that is the step instant of the input signal, the residual produces a strong fluctuation even that no fault exists at this instant.

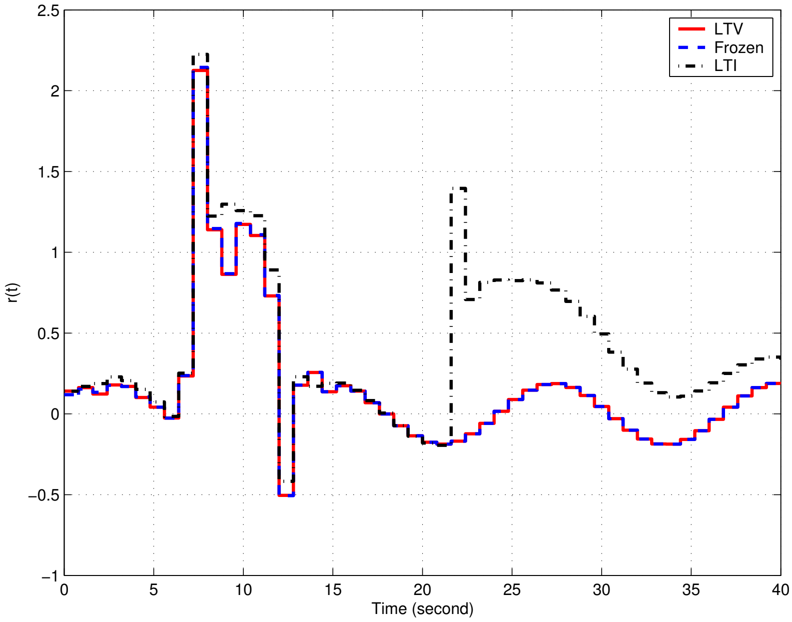

Figure 4.3: Residual $r(t)$ vs Time t–LDTV Case

Chapter 5

Fault Detection Filter Design with Partial Disturbance Decoupling and Optimization

In this chapter, we consider the multiple-objective optimization (MOO) criteria such as $\mathscr{H}_-/\mathscr{H}_\infty$, $\mathscr{H}_2/\mathscr{H}_\infty$ and $\mathscr{H}_\infty/\mathscr{H}_\infty$ in Section 2.6 for fault detection to a more general case when G_d is a tall transfer matrix and D_d may not have full column rank for linear discrete time-invariant plants. It is shown that the sensitivity of residual to some faults could be unbounded because of the free parameter in the filter. Actually, faults in a certain subspace can have bounded sensitivity, while the others that can be decoupled from the residual could have unbounded sensitivity. We also find that this subspace is strongly related to the image spaces of G_d and G_f. Furthermore, we also derive a method to decouple some disturbances, without changing the fault sensitivity. Section 5.1 is our problem formulation and motivations. The main result is given in Section 5.2. Section 5.3 discusses the non-decoupling and decoupling conditions. Section 5.4 discusses disturbance rejection. Several examples are given in Section 5.5 to illustrate our results.

5.1 Problem Formulation and Motivations

Consider the following linear discrete time-invariant system with disturbance and possible faults

$$x(t+1) = Ax(t) + Bu(t) + B_d d(t) + B_f f(t) \quad (5.1)$$

$$y(t) = Cx(t) + Du(t) + D_d d(t) + D_f f(t) \quad (5.2)$$

where $x(t) \in \mathscr{R}^n$ is the state vector, $u(t) \in \mathscr{R}^{n_u}$, $f(t) \in \mathscr{R}^{n_f}$, $d(t) \in \mathscr{R}^{n_d}$ and $y(t) \in \mathscr{R}^{n_y}$ are control input, fault, disturbance and output, respectively. With respect to different situations, d and f can be modeled as different type of signals, which results in different problem formulations. All coefficient matrices are assumed to be constant and known.

By taking z-transformations of (5.1) and (5.2), we have the system input-output equation:

$$y = G_u u + G_d d + G_f f$$

where G_u, G_d and G_f are the following transfer matrices, respectively,

$$G = \begin{bmatrix} G_u & G_d & G_f \end{bmatrix} = \left[\begin{array}{c|ccc} A & B & B_d & B_f \\ \hline C & D & D_d & D_f \end{array}\right].$$

Furthermore, the following assumptions are made:

1. $rank\left\{\begin{bmatrix} A - e^{i\theta}I & B_d \\ C & D_d \end{bmatrix}\right\} = n + n_d, \ \forall \theta \in [0, 2\pi].$
2. (C, A) is detectable.
3. $D_d \neq 0$.
4. $n_y \geq n_f$.
5. $n_y \geq n_d$.

Remark 18 *The assumption 1 means that G_d has no transmission zeros on the unit circle. It can be removed by the method discussed in Chapters 3 and 4.*

Remark 19 *The assumption 2 is a standard assumption for filter design.*

Remark 20 *The assumption 3 is made without loss of generality since any causal transfer matrix G_d can be written as $G_d = z^{-k}\widetilde{G}_d$ such that the constant term of $\widetilde{G}_d$ is nonzero. Thus we can design a filter F for $\widetilde{G}_d$, and take F as the filter for G_d.*

Remark 21 *The assumption 4 is necessary for guaranteeing that every faults can be identified [63]. The assumption 5 is the case we shall solve in this chapter. The opposite case that $n_y \leq n_d$ has been solved in [56], which is also summarized in Chapter 2.*

Since (C, A) is detectable, Lemma 3 guarantees that there exists a left coprime factorization for G

$$G = M^{-1}N$$

and

$$\begin{bmatrix} M & N \end{bmatrix} = \begin{bmatrix} M & N_u & N_d & N_f \end{bmatrix} = \left[\begin{array}{c|cccc} A+LC & L & B+LD & B_d+LD_d & B_f+LD_f \\ \hline C & I & D & D_d & D_f \end{array}\right] \tag{5.3}$$

where L is a matrix such that $A + LC$ is stable.

It can be shown in [55] that, without loss of generality, the fault detection filter can take the following general form

$$r = Q(My - N_u u) = Q \begin{bmatrix} M & -N_u \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} =: F \begin{bmatrix} y \\ u \end{bmatrix}$$

where r is the residual vector for detection, $Q \in \mathscr{RH}_\infty^{n_y \times n_y}$ is a free stable transfer matrix to be designed. The framework is shown in Figure 5.1. In general, Q can be any system in

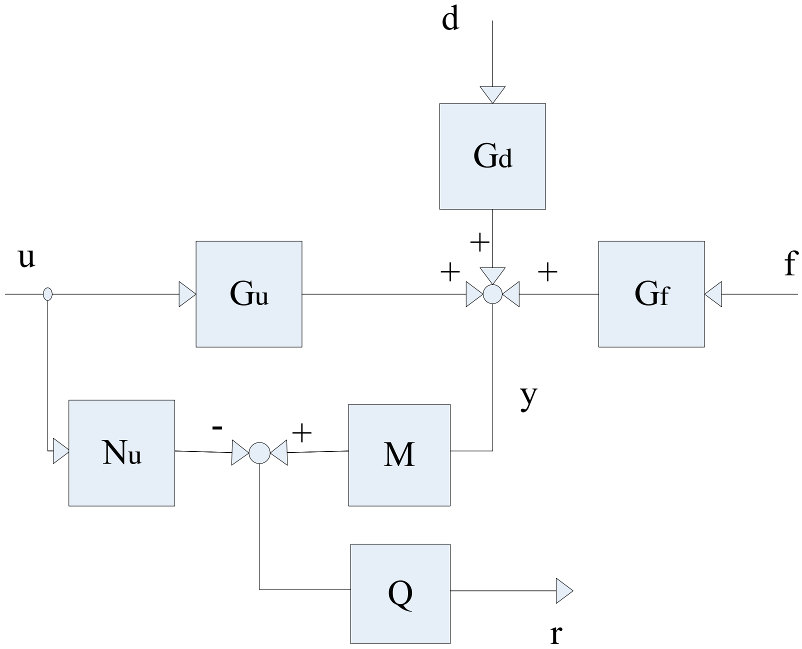

Figure 5.1: Fault Detection Filter Structure–General Case

$\mathscr{RH}_\infty^{p\times n_y}$ with $p \geq n_y$, while it is without loss of generality to assume $Q \in \mathscr{RH}_\infty^{n_y\times n_y}$ ([55]). Hence, we have

$$r = Q \begin{bmatrix} N_d & N_f \end{bmatrix} \begin{bmatrix} d \\ f \end{bmatrix} = QN_d d + QN_f f$$

where $N_f \in \mathscr{RH}_\infty^{n_y\times n_f}$ and $N_d \in \mathscr{RH}_\infty^{n_y\times n_d}$. Here, the effect of input u has been completely decoupled from the residual. QN_d measures the extend to which noise/disturbance affects residual signal and QN_f measures the extend to which faults affect the residual.

In general, an ideal fault detection filter must be able to generate residual signal r such that r is identically zero when no fault shows up, that is, it rejects completely the noise/disturbance d , while r must be as sensitive as possible for any fault f.

Therefore, the ideal fault detection problem can be formulated as follows,

Problem 10 *Design a system $Q \in \mathscr{RH}_\infty^{n_y\times n_y}$ to generate a residual r such that*

$$\begin{cases} r = 0, & \text{if } f = 0; \\ r \neq 0, & \text{if } f \neq 0. \end{cases}$$

Further, it should be possible to identify the faults.

In our framework, it is equivalent to the following problem.

Problem 11 *Design a system* $Q \in \mathscr{RH}_\infty^{n_y \times n_y}$ *such that*

$$\begin{cases} QN_d = 0, \\ QN_f f \neq 0, \quad \forall f \neq 0. \end{cases}$$

Further, it should be possible to identify the faults.

Remark 22 *The condition that* $QN_f f \neq 0, \forall f \neq 0$ *is stronger than* $QN_f \neq 0$*. For instance, consider a simple case with*

$$QN_f = \begin{bmatrix} 0 & 0 \\ 0 & \frac{z+0.3}{z+0.7} \end{bmatrix} \neq 0.$$

We have $QN_f f = 0$ *for the fault signal* $f = \begin{bmatrix} f_1 & 0 \end{bmatrix}^T$*, where* f_1 *is any nonzero signal. Similarly, this condition is also stronger than* $\|QN_f\|_\infty \neq 0$ *and* $\|QN_f\|_2 \neq 0$*.*

Remark 23 *The filter design method that only looks for a stable transfer matrix* Q *such that the noise/disturbance effect is completely rejected (i.e.* $QN_d = 0$*) is not always reasonable since it may also simultaneously reject some faults in a certain subspace. For instance,* $QN_f f$ *could be zero for* f *in certain subspace if* $QN_d = 0$*, which implies that the fault in some subspace cannot be detected.*

We argue that there may exist no solution to Problem 11 for some faulty systems. It is possible that for any Q such that $QN_d = 0$, there always exists f such that $QN_f f = 0$.

For instance, consider a simple case with

$$N_d = \begin{bmatrix} \frac{z+0.3}{z+0.7} \\ 0 \end{bmatrix}, \quad N_f = \begin{bmatrix} \frac{2z+0.6}{z+0.7} \\ 0 \end{bmatrix}.$$

For any $Q \in \mathscr{RH}_\infty^{2\times 2}$ such that $QN_d = 0$ (i.e., $Q = \begin{bmatrix} 0 & Q_{12} \\ 0 & Q_{22} \end{bmatrix}$ with $Q_{12} \in \mathscr{RH}_\infty^{1\times 1}$ and $Q_{22} \in \mathscr{RH}_\infty^{1\times 1}$), it follows that $QN_f = 0$, which implies that any fault f cannot be detected.

Therefore, as long as no solution exists for Problem 11, new criteria are required to make a tradeoff between two objectives: robustness to disturbance rejection and sensitivity to faults.

In Section 2.6, when G_d is a wide or square matrix, the problem that maximizes the fault detection sensitivity and simultaneously constrains the disturbance rejection level is formulated as a multi-objective optimization problem

$$\max_{Q \in \mathscr{R}\mathscr{H}_\infty^{ny \times ny}} \{\|QN_f\| : \|QN_d\|_\infty \leq \gamma, \gamma > 0\} \tag{5.4}$$

where $\|\cdot\|$ can be anyone of $\mathscr{H}_\infty$ norm, $\mathscr{H}_2$ norm and $\mathscr{H}_-$ index. Here the scalar γ represents the disturbance rejection level, and fault sensitivity is maximized for all possible faults.

However, when G_d is a tall transfer matrix, the situation becomes much more complicated. To start with, since some faults could be decoupled from the disturbances, their fault sensitivities can be arbitrarily assigned. Fault sensitivity in terms of $\|QN_f\|$ could be arbitrarily large and thus $\|QN_f\|$ might not be appropriate to characterize the fault sensitivity. In contrast, some faults could have bounded sensitivity. Therefore, it is appropriate to differentiate those faults, that is, to find the subspace of f (denoted as S) in which any fault can have bounded sensitivity, while the fault out of the subspace S could have unbounded sensitivity (i.e. decoupled). The fault detection filter should be able to maximize the fault sensitivities for the faults in the subspace S, that is, $\max \|QN_f\|^S$. Furthermore, it may be possible to decouple some disturbances from the residual, without changing the fault sensitivity. That is, it is desired to find a subspace of d (denoted as E) in which the disturbance is completely rejected by the filter.

Therefore, we formulate the following problem.

Problem 12 *Let $\gamma \geq 0$ be a scalar that represents the disturbance rejection level. The fault detection filter design is:*

1. *Find a proper transfer matrix $Q \in \mathscr{RH}_\infty^{n_y \times n_y}$ and a subspace S of faults f such that the following criteria are satisfied*

$$\begin{cases} \|QN_d\|_\infty \leq \gamma \\ \max_Q \in \mathscr{RH}_\infty^{n_y \times n_y} \|QN_f\|^S \\ \|QN_f\|^{\bar{S}} \textit{ can be arbitrarily assigned} \end{cases}$$

   where $\bar{S} := \{f : f \in \mathscr{L}_2^{n_f}, f \notin S\}$ and $\|QN_f\|^S$ represents any one of $\|QN_f\|_-^S$, $\|QN_f\|_2^S$ and $\|QN_f\|_\infty^S$.

2. *If possible, find a subspace $E \subseteq \mathscr{L}_2^{n_d}$ and a transfer matrix $\hat{Q} \in \mathscr{RH}_\infty^{n_y \times n_y}$ such that*

$$\hat{Q} N_d d = 0, \quad \forall d \in E \subseteq \mathscr{L}_2^{n_d}$$

   and the fault sensitivity in Step 1 is unchanged.

Remark 24 *The criteria in Problem 12 is more reasonable than the problem stated in (5.4), since Problem 12 emphasizes on decoupling some disturbances and some faults from the residual, besides maximizing the fault sensitivity. Actually, Problem 12 is identical to the problem stated in (5.4) when $S = \mathscr{L}_2^{n_f}$ and $E = 0$.*

Remark 25 *When $\gamma = 0$, we have $QN_d = 0$ in the sense that disturbance can be completely decoupled/rejected from residual.*

5.2 Fault Detection Filter Design

In this section, we shall derive our fault detection filter for Problem 12 without considering disturbance rejection (i.e. $E = \{0\}$), based on the co-inner-outer factorization in Lemma 6.

Lemma 23 *$N_d(z) \in \mathscr{RH}_\infty^{n_y \times n_d}$ with $z = e^{j\theta}$ in (5.3) has rank n_d for any $\theta \in [0, 2\pi]$.*

Proof It can be easily shown by

$$\begin{bmatrix} A+LC-e^{j\theta}I & B_d+LD_d \\ C & D_d \end{bmatrix} = \begin{bmatrix} I & L \\ 0 & I \end{bmatrix} \begin{bmatrix} A-e^{j\theta}I & B_d \\ C & D_d \end{bmatrix}.$$

Then the conclusion follows from Assumption 1. □

Since $N_d(z)$ with $z = e^{j\theta}$ has full column rank for any $\theta \in [0, 2\pi]$ by Lemma 23, Lemma 6 can be employed to obtain the following co-inner-outer factorization

$$N_d = N_o N_i$$

where $N_o \in \mathscr{RH}_\infty^{n_y \times n_d}$ is a co-outer and $N_i \in \mathscr{RH}_\infty^{n_d \times n_d}$ is a co-inner. Specially, by defining $L_m = L + L_{max}$, we have

$$\begin{aligned} N_o &= \left[\begin{array}{c|c} A+LC & -L_{max} \\ \hline C & I \end{array}\right] \Omega_m \\ &= \left[\begin{array}{c|c} A+LC & L-L_m \\ \hline C & I \end{array}\right] \Omega_m, \end{aligned}$$

$$\begin{aligned} L_m &= -(AY_mC' + B_dD_d')(D_dD_d' + CY_mC')^+ \\ &\quad +L\left[I - (D_dD_d' + CY_mC')(D_dD_d' + CY_mC')^+\right], \end{aligned}$$

$$\Omega_m\Omega_m' = D_dD_d' + CY_mC'$$

where $\Omega_m \in \mathscr{R}^{n_y \times n_d}$ is of full column rank and the positive semi-definite definite matrix $Y_m \in \mathscr{R}^{n \times n}$ is the maximal solution of the following Riccati equation

$$\begin{aligned} Y_m &= AY_mA' - S_Y(D_dD_d' + CY_mC')^+S_Y' + B_dB_d' \\ &= (A + L_mC)Y_m(A + L_mC)' + (B_d + L_mD_d)(B + L_mD_d)' \end{aligned}$$

where $S_Y = (AY_mC' + BD')$.

Alternatively, Y_m and L_m can be thought as the solution of the following Riccati system:

$$\begin{bmatrix} I_{n\times n} & L_m \end{bmatrix} \begin{bmatrix} AY_mA' - Y_m + B_dB_d' & AY_mC' + B_dD_d' \\ (AY_mC' + B_dD_d')' & D_dD_d' + CY_mC' \end{bmatrix} = 0.$$

By defining

$$V := \left[\begin{array}{c|c} A + LC & L - L_m \\ \hline C & I \end{array}\right] \in \mathscr{RH}_\infty^{n_y \times n_y},$$

N_d can be written as

$$N_d = N_oN_i = V\Omega_mN_i. \tag{5.5}$$

Since $\Omega_m \in \mathscr{R}^{n_y\times n_d}$ is a tall matrix with full column rank, the following singular value decomposition exists

$$\Omega_m = U_m \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V_m$$

where $U_m \in \mathscr{R}^{n_y\times n_y}$ and $V_m \in \mathscr{R}^{n_d\times n_d}$ are unitary matrices, and $\Sigma \in \mathscr{R}^{n_d\times n_d}$ is a diagonal matrix.

Hence,

$$QN_d = QV\Omega_mN_i = QVU_m \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V_mN_i$$

Let $Q = \begin{bmatrix} Q_1\Sigma^{-1} & Q_2 \end{bmatrix} U_m'V^{-1}$ with $Q_1 \in \mathscr{RH}_\infty^{n_y\times n_d}$ and $Q_2 \in \mathscr{RH}_\infty^{n_y\times(n_y-n_d)}$. It follows that

$$QN_d = \begin{bmatrix} Q_1\Sigma^{-1} & Q_2 \end{bmatrix} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V_mN_i = Q_1V_mN_i.$$

Since V_m is a unitary matrix and N_i is a co-inner, we have

$$\|QN_d\|_\infty = \|Q_1V_mN_i\|_\infty = \|Q_1\|_\infty \le \gamma.$$

By defining

$$\widetilde{N}_f = U_m'V^{-1}N_f = \begin{bmatrix} \widetilde{N}_{1f} \\ \widetilde{N}_{2f} \end{bmatrix}$$

where $\widetilde{N}_{1f} \in \mathscr{RH}_\infty^{n_d \times n_f}$ and $\widetilde{N}_{2f} \in \mathscr{RH}_\infty^{(n_y-n_d)\times n_f}$, QN_f can be written as

$$\begin{aligned} QN_f &= \begin{bmatrix} Q_1\Sigma^{-1} & Q_2 \end{bmatrix} U_m' V^{-1} N_f \\ &= Q_1\Sigma^{-1}\widetilde{N}_{1f} + Q_2\widetilde{N}_{2f} \end{aligned}$$

where $Q_1 \in \mathscr{RH}_\infty^{n_y\times n_d}$ is bounded by $\|Q_1\| \le \gamma$ and $Q_2 \in \mathscr{RH}_\infty^{n_y\times(n_y-n_d)}$ is totally free.

Now we shall give our fault detection filter from the signal point of view.

Define

$$\begin{cases} \widetilde{r}_1 := \gamma V_m N_i d + \gamma\Sigma^{-1}\widetilde{N}_{1f} f \\ \widetilde{r}_2 := \alpha\widetilde{N}_{2f} f \end{cases}$$

where $\widetilde{r}_1 \in \mathscr{L}_2^{n_d}$ and $\widetilde{r}_2 \in \mathscr{L}_2^{n_y-n_d}$ can be thought as signals representing the information of fault f. Let

$$Q_1 = \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix}.$$

The residual r bocomes

$$\begin{aligned} r &= \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = QN_f f + QN_d d \\ &= \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} \Sigma^{-1}\widetilde{N}_{1f} \\ \widetilde{N}_{2f} \end{bmatrix} f + \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} V_m N_i d \qquad (5.6) \\ &= \begin{bmatrix} Q_{11}\Sigma^{-1}\widetilde{N}_{1f} f + Q_{12}\widetilde{N}_{2f} f + Q_{11} V_m N_i d \\ Q_{21}\Sigma^{-1}\widetilde{N}_{1f} f + Q_{22}\widetilde{N}_{2f} f + Q_{21} V_m N_i d \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\gamma}Q_{11}\widetilde{r}_1 + \frac{1}{\alpha}Q_{12}\widetilde{r}_2 \\ \frac{1}{\gamma}Q_{21}\widetilde{r}_1 + \frac{1}{\alpha}Q_{22}\widetilde{r}_2 \end{bmatrix} \qquad (5.7) \\ &= \begin{bmatrix} \frac{1}{\gamma}Q_{11} & \frac{1}{\alpha}Q_{12} \\ \frac{1}{\gamma}Q_{21} & \frac{1}{\alpha}Q_{22} \end{bmatrix} \begin{bmatrix} \widetilde{r}_1 \\ \widetilde{r}_2 \end{bmatrix}. \end{aligned}$$

By taking

$$\begin{aligned} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} &= \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \\ &= \begin{bmatrix} \gamma I_{n_d\times n_d} & 0 \\ 0 & \alpha I_{(n_y-n_d)\times(n_y-n_d)} \end{bmatrix} \qquad (5.8) \end{aligned}$$

where α is an arbitrary nonzero real number, we have

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \gamma V_m N_i d + \gamma \Sigma^{-1} \widetilde{N}_{1f} f \\ \alpha \widetilde{N}_{2f} f \end{bmatrix} = \begin{bmatrix} \widetilde{r}_1 \\ \widetilde{r}_2 \end{bmatrix},$$

in the sense that all the fault information included in $\widetilde{r}_1$ and $\widetilde{r}_2$ are transferred to the residual r, while the disturbance effect is constrained by $\|Q_{11}\| = \gamma$. Furthermore, any filter in the form of (5.7) with the constraint $\|Q_1\|_\infty \leq \gamma$ can be produced by our filter (taking Q_1 and Q_2 as that in (5.8).

Putting Q_1 and Q_2 in (5.8) into Q, the residual signal becomes

$$r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \gamma \Sigma^{-1} \widetilde{N}_{1f} \\ \alpha \widetilde{N}_{2f} \end{bmatrix} f + \begin{bmatrix} \gamma V_m N_i \\ 0 \end{bmatrix} d.$$

The fault sensitivity is

$$\|QN_f\| = \left\| \begin{bmatrix} \gamma \Sigma^{-1} \widetilde{N}_{1f} \\ \alpha \widetilde{N}_{2f} \end{bmatrix} \right\|.$$

Specifically, we have

$$\begin{aligned} Q &= \begin{bmatrix} \gamma \Sigma^{-1} & 0 \\ 0 & \alpha I_{(n_y - n_d)} \end{bmatrix} U_m' V^{-1} \\ &= \begin{bmatrix} \gamma \Sigma^{-1} & 0 \\ 0 & \alpha I_{(n_y - n_d)} \end{bmatrix} U_m' \left[\begin{array}{c|c} A + L_m C & L_m - L \\ \hline C & I \end{array} \right]. \end{aligned}$$

$$QN_f = \begin{bmatrix} \gamma \Sigma^{-1} & 0 \\ 0 & \alpha I_{(n_y - n_d)} \end{bmatrix} U_m' \left[\begin{array}{c|c} A + L_m C & B_f + L_m D_f \\ \hline C & D_f \end{array} \right].$$

$$\begin{aligned} L_m &= -(AY_m C' + B_d D_d')(D_d D_d' + C Y_m C')^+ \\ &\quad + L \left[I - (C Y_m C' + D_d D_d')(C Y_m C' + D_d D_d')^+ \right]. \end{aligned}$$

$L \in R^{n\times n}$ is any matrix such that $A + L_m C$ is stable. The fault detection filter is

$$\begin{aligned} F &= Q \begin{bmatrix} M & -N_u \end{bmatrix} \\ &= \begin{bmatrix} \gamma\Sigma^{-1} & 0 \\ 0 & \alpha I_{(n_y - n_d)} \end{bmatrix} U_m' V^{-1} \begin{bmatrix} M & -N_u \end{bmatrix} \\ &= \begin{bmatrix} \gamma\Sigma^{-1} & 0 \\ 0 & \alpha I_{(n_y - n_d)} \end{bmatrix} U_m' \left[\begin{array}{c|cc} A + L_m C & -L_m & B + L_m D \\ \hline C & I & -D \end{array}\right]. \end{aligned}$$

In summary, we have the following theorem.

Theorem 7 *For the plant given by the equations (5.1) and (5.2), under Assumptions (1-5), assume Y_m is the maximal solution of the following difference Riccati equation*

$$Y_m = (A + L_m C) Y_m (A + L_m C)' + (B_d + L_m D_d)(B + L_m D_d)'$$

where

$$\begin{aligned} L_m &= -(A Y_m C' + B_d D_d')(D_d D_d' + C Y_m C')^+ \\ &\quad + L\left[I - (C Y_m C' + D_d D_d')(C Y_m C' + D_d D_d')^+\right]. \end{aligned}$$

Alternatively, L_m and Y_m are also the solutions of the following Riccati system:

$$\begin{bmatrix} I_{n\times n} & L_m \end{bmatrix} \begin{bmatrix} A Y_m A' - Y_m + B_d B_d' & A Y_m C' + B_d D_d' \\ (A Y_m C' + B_d D_d')' & D_d D_d' + C Y_m C' \end{bmatrix} = 0.$$

Let $\Omega_m \in \mathscr{R}^{n_y \times n_d}$ be the Cholesky factorization $\Omega_m \Omega_m' = D_d D_d' + C Y_m C'$ and Ω_m has the following singular value decomposition

$$\Omega_m = U_m \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V_m.$$

Then a fault detection filter for Problem 12 is

$$F = \begin{bmatrix} \gamma\Sigma^{-1} & 0 \\ 0 & \alpha I_{(n_y - n_d)} \end{bmatrix} U_m' \left[\begin{array}{c|cc} A + L_m C & -L_m & B + L_m D \\ \hline C & I & -D \end{array}\right]$$

and

$$\|QN_f\| = \left\| \begin{bmatrix} \gamma\Sigma^{-1}\widetilde{N}_{1f} \\ \alpha\widetilde{N}_{2f} \end{bmatrix} \right\|.$$

Here, α is an arbitrary real number.

Remark 26 *Theorem 7 shows that the sensitivity $\|QN_f\|$ is related to a free parameter α that can be used to increase the sensitivity for faults in space $\overline{ker\{\widetilde{N}_{2f}\}}$, while it cannot provide more fault information, since any other filter in this form can be recovered by our filter (5.9).*

Theorem 8 *For fault $f \in ker\{\widetilde{N}_{2f}\}$, we have*

$$QN_f f = \begin{bmatrix} \gamma\Sigma^{-1}\widetilde{N}_{1f} f \\ 0 \end{bmatrix}$$

which is independent of the free parameter L. Thus,

$$\|QN_f\|^{ker(\widetilde{N}_{2f})} = \left\| \begin{bmatrix} \gamma\Sigma^{-1}\widetilde{N}_{1f} \\ 0 \end{bmatrix} \right\|^{ker(\widetilde{N}_{2f})}$$

is bounded.

Proof Let

$$L_m = L_0 + L_\delta$$

where

$$\begin{aligned} L_0 &:= -(AY_mC' + B_dD_d')(D_dD_d + CY_mC')^+ \\ L_\delta &:= L\left[I - (DD' + CYC')(DD' + CYC')^+\right] \\ &= LU_m \begin{bmatrix} 0 & 0 \\ 0 & I_{n_y - n_d} \end{bmatrix} U_m' \end{aligned}$$

where we used the fact that

$$\Omega_m = U_m \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V_m \text{ and } \Omega_m\Omega_m' = DD' + CYC'.$$

By defining

$$V_\delta^{-1} = \left[\begin{array}{c|c} A + L_m C & L_\delta \\ \hline C & I \end{array}\right]$$

and

$$V_0^{-1} = \left[\begin{array}{c|c} A + L_0 C & L_0 - L \\ \hline C & I \end{array}\right]$$

it follows that

$$\begin{aligned} V^{-1} &= V_\delta^{-1} V_0^{-1} \\ Q &= \begin{bmatrix} \gamma \Sigma^{-1} & 0 \\ 0 & \alpha I_{n_y - n_d} \end{bmatrix} U_m' V_\delta^{-1} V_0^{-1}. \end{aligned}$$

Thus, we have

$$\begin{aligned} QN_d &= \begin{bmatrix} \gamma V_m N_i \\ 0 \end{bmatrix} \\ QN_f &= \begin{bmatrix} \gamma \Sigma^{-1} & 0 \\ 0 & \alpha I_{n_y - n_d} \end{bmatrix} U_m' V_\delta^{-1} \left[\begin{array}{c|c} A + L_0 C & B_f + L_0 D_f \\ \hline C & D_f \end{array}\right] \\ &= \begin{bmatrix} \gamma \Sigma^{-1} & 0 \\ 0 & \alpha I_{n_y - n_d} \end{bmatrix} \left[\begin{array}{c|c} A + L_m C & L_\delta \\ \hline U_m' C & U_m' \end{array}\right] \left[\begin{array}{c|c} A + L_0 C & B_f + L_0 D_f \\ \hline C & D_f \end{array}\right] \\ &= \begin{bmatrix} \gamma \Sigma^{-1} & 0 \\ 0 & \alpha I_{n_y - n_d} \end{bmatrix} \left[\begin{array}{c|c} A + L_m C & LU_m \begin{bmatrix} 0 & 0 \\ 0 & I_{n_y - n_d} \end{bmatrix} \\ \hline U_m' C & I \end{array}\right] U_m' \\ &\quad \left[\begin{array}{c|c} A + L_0 C & B_f + L_0 D_f \\ \hline C & D_f \end{array}\right] \\ &= \begin{bmatrix} \gamma \Sigma^{-1} & 0 \\ 0 & \alpha I_{n_y - n_d} \end{bmatrix} \tilde{\tilde{N}}_f + \left[\begin{array}{c|c} A + L_m C & LU_m \begin{bmatrix} 0 & 0 \\ 0 & I_{n_y - n_d} \end{bmatrix} \\ \hline U_m' C & 0 \end{array}\right] U_m' \\ &\quad \left[\begin{array}{c|c} A + L_0 C & B_f + L_0 D_f \\ \hline C & D_f \end{array}\right] \\ &= \begin{bmatrix} \gamma \Sigma^{-1} & 0 \\ 0 & \alpha I_{n_y - n_d} \end{bmatrix} \tilde{\tilde{N}}_f + \left[\begin{array}{c|c} A + L_m C & LU_m \\ \hline U_m' C & 0 \end{array}\right] \begin{bmatrix} 0 & 0 \\ 0 & I_{n_y - n_d} \end{bmatrix} U_m' \end{aligned}$$

$$
\begin{array}{cl}
& \left[\begin{array}{c|c} A+L_0C & B_f+L_0D_f \\ \hline C & D_f \end{array}\right] \\
= & \begin{bmatrix} \gamma\Sigma^{-1}\tilde{\tilde{N}}_{1f} \\ \alpha\tilde{\tilde{N}}_{2f} \end{bmatrix} + \left[\begin{array}{c|c} A+L_mC & LU_m \\ \hline U_m'C & 0 \end{array}\right] \begin{bmatrix} 0 & 0 \\ 0 & I_{n_y-n_d} \end{bmatrix} \begin{bmatrix} \tilde{\tilde{N}}_{1f} \\ \tilde{\tilde{N}}_{2f} \end{bmatrix} \\
= & \begin{bmatrix} \gamma\Sigma^{-1}\tilde{\tilde{N}}_{1f} \\ \alpha\tilde{\tilde{N}}_{2f} \end{bmatrix} + \left[\begin{array}{c|c} A+L_mC & LU_m \\ \hline U_m'C & 0 \end{array}\right] \begin{bmatrix} 0 \\ \tilde{\tilde{N}}_{2f} \end{bmatrix} \\
= & \begin{bmatrix} \gamma\Sigma^{-1}\tilde{\tilde{N}}_{1f} \\ 0 \end{bmatrix} + \left[\begin{array}{c|c} A+L_mC & LU_m \\ \hline U_m'C & \alpha I_{n_y} \end{array}\right] \begin{bmatrix} 0 \\ \tilde{\tilde{N}}_{2f} \end{bmatrix}
\end{array}
$$

where

$$\tilde{\tilde{N}}_f := U_m'V_0^{-1}N_f = \begin{bmatrix} \tilde{\tilde{N}}_{1f} \\ \tilde{\tilde{N}}_{2f} \end{bmatrix}$$

which is independent of L.

Furthermore, we have

$$QN_f = \begin{bmatrix} \gamma\Sigma^{-1}\tilde{N}_{1f} \\ \alpha\tilde{N}_{2f} \end{bmatrix}.$$

Therefore, if $f \in ker\{\tilde{\tilde{N}}_{2f}\}$, we also have $f \in ker\{\tilde{N}_{2f}\}$, the n_d+1 to n_y outputs of residual are identically zero, and the first n_d outputs of residual is independent of L_2. It follows that the fault sensitivity $\|QN_f\|^{ker\{\tilde{N}_{2f}\}}$ is bounded. □

Theorem 9 *$\|QN_f\|$ achieves the maximum for faults in $ker\{\tilde{N}_{2f}\}$. $\|QN_f\|^{\overline{ker\{\tilde{N}_{2f}\}}}$ can be arbitrarily assigned.*

Proof Given $f \in ker\{\tilde{N}_{2f}\}$, we have $\tilde{N}_{2f}f = 0$, $\forall f \in \mathscr{L}_2^{n_f}$.

$$
\begin{aligned}
\|QN_ff\|_2 &= \left\|QVU_mU_m'V^{-1}N_ff\right\|_2 \\
&= \left\|QVU_m \begin{bmatrix} \tilde{N}_{1f}f \\ \tilde{N}_{2f}f \end{bmatrix}\right\|_2 \\
&= \left\|QVU_m \begin{bmatrix} \Sigma/\gamma & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \gamma\Sigma^{-1}\tilde{N}_{1f}f \\ 0 \end{bmatrix}\right\|_2
\end{aligned}
$$

$$\leq \left\| QVU_m \begin{bmatrix} \Sigma/\gamma \\ 0 \end{bmatrix} \right\|_\infty \cdot \left\| \begin{bmatrix} \gamma\Sigma^{-1}\widetilde{N}_{1f}f \\ 0 \end{bmatrix} \right\|_2 .$$

In addition, we have

$$\begin{aligned} \|QN_d\|_\infty &= \|QN_oN_i\|_\infty = \left\| QVU_m \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V_mN_i \right\|_\infty \\ &= \left\| QVU_m \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \right\|_\infty \leq \gamma. \end{aligned}$$

Therefore,

$$\|QN_f\|_2 \leq \gamma \left\| \begin{bmatrix} \Sigma^{-1}\widetilde{N}_{1f}f \\ 0 \end{bmatrix} \right\|_2 = \gamma \left\| \Sigma^{-1}\widetilde{N}_{1f}f \right\|_2$$

$$\sup_{f\in ker\widetilde{N}_{2f}} \frac{\|QN_f\|_2}{\|f\|_2} \leq \gamma \sup_{f\in ker\widetilde{N}_{2f}} \frac{\left\|\Sigma^{-1}\widetilde{N}_{1f}f\right\|_2}{\|f\|_2}$$

$$\inf_{f\in ker\widetilde{N}_{2f}} \frac{\|QN_f\|_2}{\|f\|_2} \leq \gamma \inf_{f\in ker\widetilde{N}_{2f}} \frac{\left\|\Sigma^{-1}\widetilde{N}_{1f}f\right\|_2}{\|f\|_2}.$$

It follows that

$$\|QN_f\|_\infty^{ker\widetilde{N}_{2f}} \leq \gamma \left\|\Sigma^{-1}\widetilde{N}_{1f}\right\|_\infty^{ker\widetilde{N}_{2f}}$$

and

$$\|QN_f\|_-^{ker\widetilde{N}_{2f}} \leq \gamma \left\|\Sigma^{-1}\widetilde{N}_{1f}\right\|_-^{ker\widetilde{N}_{2f}}$$

where the equalities can be achieved when $\|QN_d\|_\infty = \gamma$. For our filter (5.9), the equalities hold.

The similar proof can be derived for 2-norm based on Definition 2. □

Remark 27 *Since γ is a free parameter, we have*

$$\left\| \begin{bmatrix} \gamma\Sigma^{-1}\widetilde{N}_{1f} \\ \alpha\widetilde{N}_{2f} \end{bmatrix} \right\|_\infty \geq \alpha\|\widetilde{N}_{2f}\|_\infty$$

$$\left\| \begin{bmatrix} \gamma\Sigma^{-1}\widetilde{N}_{1f} \\ \alpha\widetilde{N}_{2f} \end{bmatrix} \right\|_2 \geq \alpha\|\widetilde{N}_{2f}\|_2$$

Hence, fault sensitivity in terms of $\mathscr{H}_\infty$ and $\mathscr{H}_2$ norm can be arbitrarily assigned.

Remark 28 *If $\widetilde{N}_{2f}$ is a tall or square matrix ($n_y - n_d \geq n_f$), the following inequality holds from (2)*

$$\left\| \begin{bmatrix} \gamma\Sigma^{-1}\widetilde{N}_{1f} \\ \alpha\widetilde{N}_{2f} \end{bmatrix} \right\|_- \geq \alpha\|\widetilde{N}_{2f}\|_-.$$

Hence, we can arbitrarily assign $\mathscr{H}_-$ sensitivity for any fault, given $\|\widetilde{N}_{2f}\|_- \neq 0$.

Similarly, if $\widetilde{N}_{1f}$ is tall or square matrix ($n_d \geq n_f$), the following inequality holds

$$\left\| \begin{bmatrix} \gamma\Sigma^{-1}\widetilde{N}_{1f} \\ \alpha\widetilde{N}_{2f} \end{bmatrix} \right\|_- \geq \gamma\|\Sigma^{-1}\widetilde{N}_{1f}\|_-.$$

Hence, we have guaranteed $\mathscr{H}_-$ sensitivity for any fault, given $\|\widetilde{N}_{1f}\|_- \neq 0$.

Remark 29 *For $n_y = n_d$, the filter can be simplified as*

$$F = \gamma\Sigma^{-1}U'_m \left[\begin{array}{c|cc} A + L_mC & -L_m & B + L_mD \\ \hline C & I & -D \end{array} \right].$$

Since pre-multiplying a unitary matrix on a system does not affect its $\mathscr{H}_\infty$ norm, V_mF is also an optimal filter for our criteria, which is identical to the solution given in Section 2.6 in which G_d is of full row rank.

5.3 Decoupling Condition

It has been shown in the last section that the faults in the space $ker\{\widetilde{N}_{2f}\}$ could have bounded sensitivity in term of $\mathscr{H}_-$ index, while the other faults in the space $\overline{ker\{\widetilde{N}_{2f}\}}$ can be arbitrarily sensitive. In this section we shall derive the condition under which the fault sensitivity for any fault $f \in \mathscr{L}_2^{n_f}$ is bounded.

Theorem 10 *(completely non-decoupling condition) If $rank\{[G_d(e^{j\theta})\ G_f(e^{j\theta})]\} = rank\{G_d(e^{j\theta})\} = n_d, \forall\theta \in [0, 2\pi]$, that is, $image\{G_f\} \subseteq image\{G_d\}$, the fault detection filter has a bounded sensitivity in $\mathscr{H}_-$ index.*

Proof Assume

$$rank\Big\{ \begin{bmatrix} G_d(e^{j\theta}) & G_f(e^{j\theta}) \end{bmatrix} \Big\} = rank\{G_d(e^{j\theta})\} = n_d, \ \forall\theta \in [0, 2\pi].$$

In other words,

$$rank\left\{ \begin{bmatrix} A - e^{j\theta}I & B_d & B_f \\ C & D_d & D_f \end{bmatrix} \right\} = rank\left\{ \begin{bmatrix} A - e^{j\theta}I & B_d \\ C & D_d \end{bmatrix} \right\} = n + n_d, \ \forall\theta \in [0, 2\pi]$$

that is, $image\{G_f\} \subseteq image\{G_d\}$. Specifically, there exists a stable (possibly improper) transfer matrix X such that $G_f = G_d X$.

Furthermore, it is easy to show that $N_f = N_d X$ is equivalent to $G_f = G_d X$ by the coprime factorization (5.3).

Note that

$$\begin{aligned} U_m' V^{-1} N_d &= U_m' V^{-1} N_o N_i \\ &= U_m' V^{-1} V \Omega_m N_i \\ &= U_m' \Omega_m N_i \\ &= \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V_m N_i \\ &= \begin{bmatrix} \Sigma V_m N_i \\ 0 \end{bmatrix}. \end{aligned}$$

$$\widetilde{N}_f = U_m' V^{-1} N_f = \begin{bmatrix} \widetilde{N}_{1f} \\ \widetilde{N}_{2f} \end{bmatrix} = \begin{bmatrix} \Sigma V_m N_i X \\ 0 \end{bmatrix}.$$

According to the derivation in Theorem 8, $\widetilde{N}_{2f} = 0$ implies $V_\delta^{-1} = I$ and thus $V^{-1} = V_0^{-1}$. By (5.5), we have

$$\begin{aligned} \|QN_f\|_- &= \|\gamma \Sigma^{-1} \Sigma V_m N_i X\|_- = \gamma \|V^{-1} N_d X\|_- \\ &= \gamma \|V^{-1} N_f\|_- = \gamma \|V_0^{-1} N_f\|_-. \end{aligned}$$

In other words, the fault detection filter has a bounded sensitivity in $\mathscr{H}_-$ index. □

Theorem 11 *The completely non-decoupling condition in Theorem 10 is necessary for $\mathscr{H}_-$ index if ker* $\left\{\widetilde{N}_{2f}\right\} = \{0\}$. *If ker* $\left\{\widetilde{N}_{2f}\right\} \neq \{0\}$, *it is also necessary for $\mathscr{H}_-$ index over the space ker*$\{\widetilde{N}_{2f}\}$.

Proof Assume that $image\{G_f\}$ is not a subset of $image\{G_d\}$, that is, $rank\{[G_d\ G_f]\} > rank\{G_d\} = n_d,\ \forall\theta \in [0, 2\pi]$.

According to the derivation in Section 5.3, our system Q can transform $[N_d\ N_f]$ into the following form:

$$Q\left[\begin{array}{cc} N_d & N_f \end{array}\right] = \left[\begin{array}{cc} \gamma V_m N_i & \gamma\Sigma^{-1}\widetilde{N}_{1f} \\ 0 & \alpha\widetilde{N}_{2f} \end{array}\right]$$

where $V_m N_i$ is of rank n_d. Since $rank\{[G_d\ G_f]\} = rank\{[N_d\ N_f]\} > n_d$, we have $\widetilde{N}_{2f} \neq 0$.

Therefore, the outputs of residual from $n_d + 1$ to n_y can be arbitrarily assigned if $ker\{\widetilde{N}_{2f}\} = \{0\}$. Thus, the fault sensitivity is unbounded. If $ker\{\widetilde{N}_{2f}\} \neq \{0\}$, the output energy from $n_d + 1$ to n_y can be arbitrary for some fault not in $ker\{\widetilde{N}_{2f}\}$. Therefore, the fault can have unbounded sensitivity that is inconsistent with the condition. □

Remark 30 *This condition is also true when G_d is a square or wide matrix ($n_y \leq n_d$). In [55, 56], when G_d is assumed to be full row rank, this condition is satisfied automatically, which implies the existence of the upper bound.*

Remark 31 *If there exists a nonzero $X \in \mathscr{R}^{n_d \times n_f}$ solving the linear equation*

$$\left[\begin{array}{c} B_f \\ D_f \end{array}\right] = \left[\begin{array}{c} B_d \\ D_d \end{array}\right] X,$$

then any disturbance d cannot be decoupled from residual signal, since the following equality holds

$$\left[\begin{array}{cc} G_d & G_f \end{array}\right] = \left[\begin{array}{cc} C(zI - A)^{-1} & I \end{array}\right] \left[\begin{array}{cc} B_d & B_f \\ D_d & D_f \end{array}\right].$$

Remark 32 *The completely decoupling condition is* $rank\{[G_d\ G_f]\} = rank\{G_d\}+rank\{G_f\}$. *Otherwise, if* $rank\{G_d\} < rank\{[G_d\ G_f]\} < rank\{G_d\} + rank\{G_f\}$, *partial decoupling can be realized. [22] provided the same condition for perfect fault isolation.*

5.4 Decoupling Some Disturbances

In the previous sections we showed that some faults in $ker(\widetilde{N}_{2f})$ can be decoupled so that their sensitivities can be arbitrarily assigned. In this section we shall discuss how to decouple (or reject) some disturbances without changing the fault sensitivity. In other words, find a space $E \subseteq \mathscr{L}_2^{n_d}$ and a transfer matrix $\hat{Q}^{n_y \times n_y} \in \mathscr{RH}_\infty^{n_y \times n_y}$ such that

$$\hat{Q} N_d d = 0, \forall d \in E \subseteq \mathscr{L}_2^{n_d}$$

and the fault sensitivity in Step 1 is unchanged.

Theorem 12 *Rejecting the disturbance in the following space* E *does not affect the fault sensitivity.*

$$E = \left\{ d \in \mathscr{L}_2^{n_d} : [QN_d]^i d \neq 0, [QN_f]^i f = 0, f \in ker\{\widetilde{N}_{2f}\} \text{ for some } i \in \{1, \cdots, n_d\} \right\}$$

where $[A]^i$ *represents the* i*th row of matrix* A.

Proof The residual can be written as

$$r = \begin{bmatrix} \gamma \Sigma^{-1} \widetilde{N}_{1f} \\ \alpha \widetilde{N}_{2f} \end{bmatrix} f + \begin{bmatrix} \gamma V_m N_i \\ 0 \end{bmatrix} d.$$

The residual at channel i is

$$[r]^i = \begin{cases} \left[\Sigma^{-1} \widetilde{N}_{1f}\right]^i f + \gamma \left[V_m N_i\right]^i d, & 1 \leq i \leq n_d; \\ \alpha \left[\widetilde{N}_{2f}\right]^i f, & n_d < i \leq n_y. \end{cases}$$

Given $\left[\widetilde{N}_{1f}\right]^i f = 0$ for any $f \in ker\{\widetilde{N}_{2f}\}$, $[r]^i$ is independent of f, and thus it is appropriate to reject the disturbance at output channel i, i.e.

$$\hat{Q}\,[V_m N_i]^i = 0.$$

□

We use S_i to denote the set of all i's corresponding to E, that is,

$$S_i = \left\{ i : [QN_d]^i d \neq 0, d \in \mathscr{L}_2^{n_d}, [QN_f]^i f = 0, f \in ker\{\widetilde{N}_{2f}\} \text{ for some } i \in \{1, \cdots, n_d\} \right\}$$

Therefore, it is appropriate to revise Q to $\hat{Q}$ such that for any disturbance $d \in E$, we have

$$[\hat{Q}N_d]^i d = 0 \text{ and } [\hat{Q}N_f]^i f = 0,\ \forall f \in ker\{\widetilde{N}_{2f}\},\ \forall i \in S_i$$

which can be done by setting $\hat{Q} = \Lambda Q$, where Λ is a diagonal matrix with some zero diagonal element.

Here we consider a special case:

$$\bar{E} \;=\; \left\{ d \in \mathscr{L}_2^{n_d} : [QN_d]^i d \neq 0, [QN_f]^i = 0, \text{ for some } i \in \{1, \cdots n_d\} \right\}.$$

Apparently, $\bar{E}$ denotes the case that some rows of $\widetilde{N}_{1f}$ are zero or even $\widetilde{N}_{1f} = 0$, which is a subspace of E. In this case, the filter can be improved by modifying (5.9) in the previous sections.

If $\widetilde{N}_{1f} = 0$, the residual expression (5.7) becomes

$$r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} Q_{21}\widetilde{N}_{2f} \\ Q_{22}\widetilde{N}_{2f} \end{bmatrix} f + \begin{bmatrix} Q_{11}V_m N_i \\ 0 \end{bmatrix} d.$$

Using the similar method in the previous section, we can set $Q_{11} = 0$ such that the disturbance has been completely decoupled from the residual. More specifically,

$$r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha\widetilde{N}_{2f} \end{bmatrix} f.$$

Thus we have the following remark.

Remark 33 *When $\widetilde{N}_{1f} = 0$, we can revise Q as*

$$\begin{aligned}\hat{Q} &= \begin{bmatrix} 0_{n_d} & 0 \\ 0 & \alpha I_{(n_y-n_d)} \end{bmatrix} U_m' V^{-1} \\ &= \begin{bmatrix} 0_{n_d} & 0 \\ 0 & \alpha I_{(n_y-n_d)} \end{bmatrix} U_m' \left[\begin{array}{c|c} A+L_mC & L_m - L \\ \hline C & I \end{array}\right]\end{aligned}$$

to completely decouple disturbance from the residual.

Similarly, when some rows of $\widetilde{N}_{1f}$ are zero, the disturbance effect on the residual from the same rows can be made zero, while it does not change the fault effect on the residual. In other words, some disturbances are decoupled from the residual.

Remark 34 *When some rows of $\widetilde{N}_{1f}$ are zeros, we can revise Q as $\hat{Q}$*

$$\begin{aligned}\hat{Q} &= \Lambda \begin{bmatrix} \gamma\Sigma^{-1} & 0 \\ 0 & \alpha I_{(n_y-n_d)} \end{bmatrix} U_m' V^{-1} \\ &= \Lambda \begin{bmatrix} \gamma\Sigma^{-1} & 0 \\ 0 & \alpha I_{(n_y-n_d)} \end{bmatrix} U_m' \left[\begin{array}{c|c} A+L_mC & L_m - L \\ \hline C & I \end{array}\right]\end{aligned}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n_y} \end{bmatrix} \text{ and } \lambda_i = \begin{cases} 0, & [\widetilde{N}_{1f}]^i = 0; \\ 1, & \text{otherwise.} \end{cases}$$

5.5 Examples

In this section several examples are given to illustrate our results.

Example 1. Consider the following plant with $G_u = 0$:

$$y = G_d d + G_f f = \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} d + \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 3 \end{bmatrix} f.$$

Assume $\gamma = 1$. Then $M = I_3$, $N_d = G_d$ and $N_f = G_f$.

According to Theorem 7, we can design

$$F = Q = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \quad \alpha \neq 0 \text{ is a free parameter.}$$

Now

$$QG_d = QN_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and $\|QN_d\|_\infty = 1 = \gamma$,

$$QG_f = QN_f = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 3\alpha \end{bmatrix}.$$

Now, the residual signal is

$$r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} d + \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 3\alpha \end{bmatrix} f$$

and

$$\|QG_f\|_- = \left\| \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 3\alpha \end{bmatrix} \right\|_- = \min\{0.5, 3\alpha\}.$$

$$\|QG_f\|_\infty = 3\alpha.$$

It follows that fault sensitivity in $\mathscr{H}_\infty$ norm can be arbitrarily assigned due to the free parameter α. Given $\alpha > 1/6$, $\|QG_f\|_- = 0.5$ shows that this fault detection filter gives at least 0.5 fault sensitivity for any fault.

Furthermore, for any fault in the space

$$ker\left\{\widetilde{N}_{2f}\right\} = \left\{ \begin{bmatrix} f_1 \\ 0 \end{bmatrix} : f_1 \in \mathscr{L}_2^1 \right\}$$

the fault sensitivity is 0.5.

Example 2. Consider the following plant

$$y = D_d d + D_f f = \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} d + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} f.$$

The design criterion is $\max_Q\{\|QG_f\| : \|QG_d\| \leq \gamma\}$ and $\gamma = 1$.

According to Section 5, the disturbance can be completely decoupled from the residual since $rank\{[G_d\ G_f]\} = rank\{G_d\} + rank\{G_f\}$ holds. We design fault detection filter as follows

$$F = Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \quad \alpha \neq 0 \text{ is a free parameter}$$

Thus, we have

$$QG_d = QN_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which implies that residual r is independent of disturbance d. Furthermore, we have

$$QG_f = QN_f = \begin{bmatrix} 0 \\ 0 \\ 3\alpha \end{bmatrix}$$

and $\|QG_f\|_- = 3\alpha$. It implies that the worst sensitivity of residual signal is 3α for any fault.

Example 3. Consider the following plant

$$y = D_d d + D_f f = \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} d + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} f.$$

The design criterion is $\max_Q\{\|QG_f\| : \|QG_d\| \leq \gamma\}$ and $\gamma = 1$.

According to Theorem 7, we have the following fault detection filter

$$Q = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \qquad \alpha \neq 0.$$

Thus,

$$QN_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and $\|QN_d\|_\infty = 1$,

$$QN_f = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 0 \\ \alpha & 0 & \alpha \end{bmatrix}$$

$$\|QN_f\|_- = \min\{0.5, \alpha\}$$

Given $\alpha > 0.5$, the worst fault sensitivity is 0.5, which is only for any fault in

$$ker\left\{\widetilde{N}_{2f}\right\} = \left\{\begin{bmatrix} f_1 \\ 0 \\ f_3 \end{bmatrix} : f_1 + f_3 = 0\right\}$$

while fault sensitivity may be arbitrarily assigned for faults in $\overline{ker\left\{\widetilde{N}_{2f}\right\}}$.

Furthermore, since $[QN_f]^2 = 0$, it is appropriate to revise Q as

$$\hat{Q} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix}.$$

Thus we have

$$\hat{Q}N_d = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$\|\hat{Q}N_d\|_\infty = \gamma = 1$ and

$$\hat{Q}N_f = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 0 \\ \alpha & 0 & \alpha \end{bmatrix} = QN_f.$$

It can be shown that $\hat{Q}$ is better than Q, since this fault detection filter can reject the noise/disturbance from the direction $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$, while fault sensitivity is not changed. For instance, assume that the disturbance is $d = \begin{bmatrix} 0 & sin(t) \end{bmatrix}^T$ $(t \in \{0, 1, \cdots\})$, when no fault exists, the fault detection filter Q generates a residual $r = \begin{bmatrix} 0 & 4sin(t) & 0 \end{bmatrix}^T \neq 0$, while the fault detection filter $\hat{Q}$ produces the residual $r = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, which shows clearly on the non-existence of fault.

Example 4: Consider a dynamic plant with the following coefficients

$$A = \begin{bmatrix} -0.2 & 0 & 5 & 0 \\ 0 & -2.5 & 0 & 2.5 \\ 0 & 0 & 1.65 & 0 \\ 0 & 0 & 0 & -1.85 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 0.8 & 0.4 \\ -0.4 & 1 \\ 0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix} \quad B_f = \begin{bmatrix} 1 \\ 0.5 \\ 2 \\ 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & 0 & 0 & 2 \\ 1 & 0 & 3 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0.3 \\ 0.5 \\ 0.4 \end{bmatrix}$$

$$D_d = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.6 \\ 0.3 & 0.6 \end{bmatrix} \quad D_f = \begin{bmatrix} 32 \\ -3.1 \\ 0.3 \end{bmatrix}$$

By the algorithm in Chapter 2, we obtain

$$Y_m = 0$$

$$\Omega_m = \begin{bmatrix} 0.2040 & 0 \\ 0.4040 & 0.0196 \\ 0.3040 & 0.0098 \end{bmatrix}$$

$$L_m = \begin{bmatrix} 4.5883 & -6 & 2.2745 \\ -9.0855 & 9 & -4.6097 \\ 1.5455 & -2 & -0.3637 \\ 5.1460 & -5 & 0.5693 \end{bmatrix}$$

$$U_m = \begin{bmatrix} -0.4136 & 0.3696 & -0.8321 \\ -0.6663 & -0.7456 & 0 \\ -0.6204 & 0.5544 & 0.5547 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 1.0796 & 0 \\ 0 & 0.0668 \end{bmatrix}$$

Note that we used the free parameter L to guarantee that $A+L_mC$ is stable in the algorithm.

We take $\gamma = 1$. Given $\alpha = 1$, the optimal filter is F with the following coefficients

$$\hat{A} = \begin{bmatrix} 2.529 & 1.15 & 3.428 & 0.475 \\ 0.8403 & -0.1553 & 0.6409 & -0.05557 \\ 0.6763 & -0.9238 & -4.184 & -2.451 \\ 5.945 & -0.3431 & 7.254 & 0.9723 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} -5.806 & 9.325 & -3.543 & 0.7847 \\ -1.608 & 2.763 & -1.271 & -0.1101 \\ -0.2529 & 1.271 & -1.544 & -1.974 \\ -8.316 & 7.103 & -2.328 & 0.983 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} 2.748 & -0.3346 & -5.688 & -0.4438 \\ 3.898 & 1.126 & -3.46 & 13.99 \\ 1.356 & 1.988 & 0.2368 & 2.637 \end{bmatrix}$$

$$\hat{D} = \begin{bmatrix} -0.3831 & -0.6172 & -0.5747 & 0.6534 \\ 5.534 & -11.16 & 8.301 & 0.6013 \\ -0.8321 & 0 & 0.5547 & 0.02774 \end{bmatrix}$$

Further, we have

$$QN_d = \begin{bmatrix} \frac{-0.4959z^4-0.4156z^3-0.07765z^2-0.001465z}{z^4+0.838z^3+0.1566z^2+0.002916z} & \frac{-0.8684z^4-0.7277z^3-0.136z^2-0.00257z}{z^4+0.838z^3+0.1566z^2+0.002916z} \\ \frac{-0.8684z^4-0.7277z^3-0.136z^2-0.00263z}{z^4+0.838z^3+0.1566z^2+0.002916z} & \frac{0.4959z^4+0.4156z^3+0.07761z^2+0.001251z-0.0001254}{z^4+0.838z^3+0.1566z^2+0.002916z} \\ 0 & 0 \end{bmatrix}$$

$$QN_f = \begin{bmatrix} \frac{0.9747z^4-75.74z^3-893.2z^2-2898z-1086}{z^4+0.838z^3+0.1566z^2+0.002916z} \\ \frac{48.16z^4-700.4z^3-6088z^2-1.415e004z-5013}{z^4+0.838z^3+0.1566z^2+0.002916z} \\ \frac{-1.498z^4-193.4z^3-1265z^2-2703z-943.4}{z^4+0.838z^3+0.1566z^2+0.002916z} \end{bmatrix}$$

The last row of QN_d is zero, which means the disturbance effect on the output 3 of residual is removed. The poles of QN_f are 0, -0.0209, -0.2430 and -0.5741, which are inside of the unit disk and implies the stability.

Since the filter is related to the free parameter α, we compare the filters with different α ($\alpha = 1$, 5, 10 and 50). We have $\|QN_d\| \approx 1$ for all α. It can be seen from Figure 5.2 that the frequency responses of QN_d at each channels for different α are the same and the output 3 is identical zero. That is, QN_d is independent of α. Figure 5.3 shows the three singular values of QN_d for $\theta = [0, 2\pi]$. It can be seen that the disturbance sensitivity is bounded by 1. Figure 5.4 shows the frequency responses of QN_f at each channels. Figure 5.5 shows the singular values of QN_f for $\theta = [0, 2\pi]$. It can be seen that increasing α can improve the fault sensitivity.

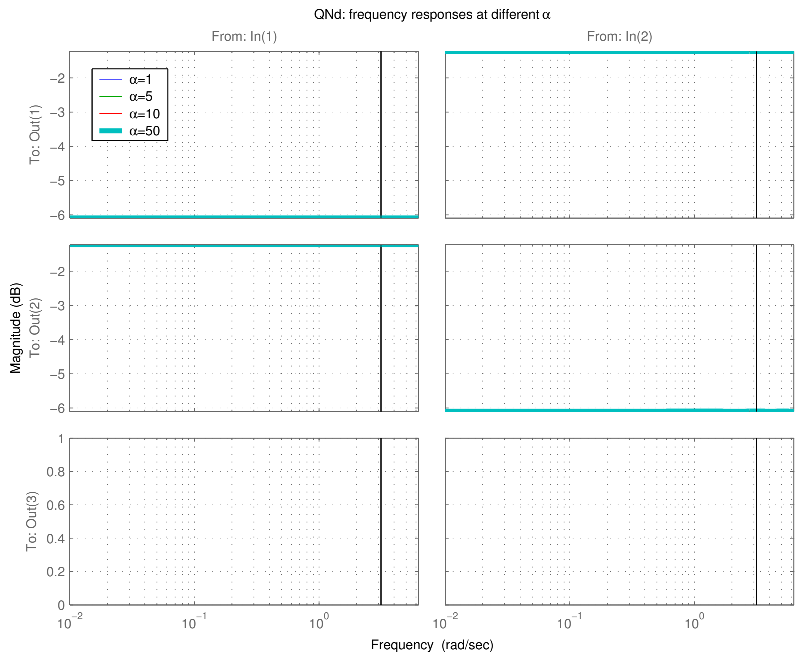

Figure 5.2: Frequency Responses of QN_d at Different α (Example 4)

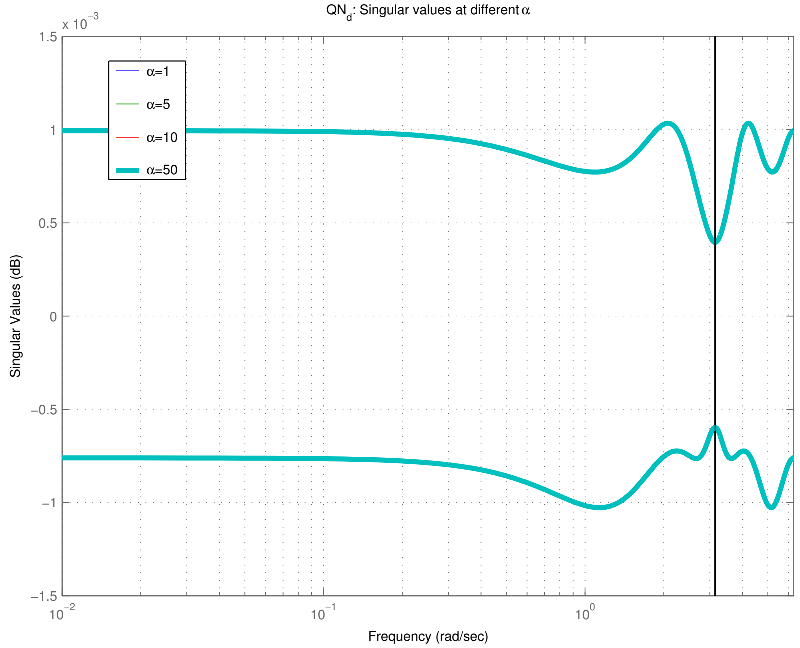

Figure 5.3: Singular Values of QN_d over θ at Different α (Example 4)

Example 5: Consider the dynamic system:

$$\left[\begin{array}{c|cc} A & B_d & B_f \\ \hline C & D_d & D_f \end{array}\right] = \begin{bmatrix} \frac{z+0.7}{z+0.5} & \frac{z+0.8}{z+0.5} \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{c|cc} -0.5 & 0.2 & 0.3 \\ \hline \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}\right].$$

Define

$$L_m = \begin{bmatrix} l_1 & l_2 \end{bmatrix}.$$

By solving the following Riccati system

$$\begin{bmatrix} 1 & l_1 & l_2 \end{bmatrix} \begin{bmatrix} 0.04 - 0.75Y_m & 0.5Y_m + 0.2 & 0 \\ 0.5Y_m + 0.2 & 1 + Y_m & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

we obtain the solution:

$$Y_m = 0, l_1 = -0.2, l_2 \ arbitary.$$

We have

$$Q = \left[\begin{array}{c|c} -0.7 & \begin{bmatrix} -0.2 & l_2 \end{bmatrix} \\ \hline \begin{bmatrix} \gamma \\ 0 \end{bmatrix} & \begin{bmatrix} \gamma & 0 \\ 0 & \alpha \end{bmatrix} \end{array}\right]$$

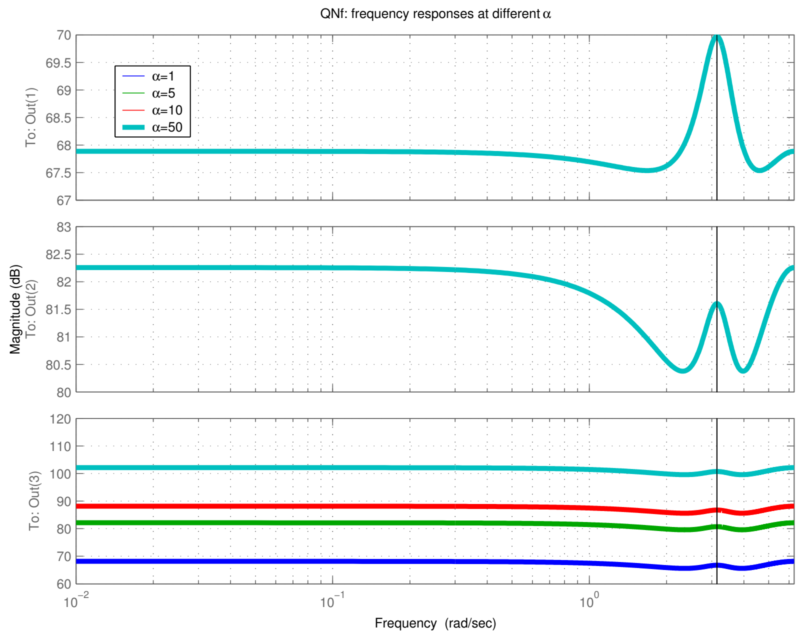

Figure 5.4: Frequency Response of QN_f at Different α (Example 4)

$$QN_d = \begin{bmatrix} \gamma \\ 0 \end{bmatrix}, \quad QN_f = \begin{bmatrix} \gamma \frac{z+0.8}{z+0.7} \\ 0 \end{bmatrix}.$$

Obviously, we have $ker\left\{\widetilde{N}_{2f}\right\} = \mathscr{L}_2^1$ which implies that all faults in $\mathscr{L}_2^1$ have bounded sensitivities. Actually, we have $\|QN_f\|_- = \gamma \left\|\frac{z+0.8}{z+0.7}\right\|_- = 0.6667$ which can be achieved at frequency $\omega = \pi$.

Example 6: Consider the following dynamic system with $G_u = 0$

$$\begin{bmatrix} G_d & G_f \end{bmatrix} = \begin{bmatrix} \frac{z+0.7}{z+0.5} & \frac{z+0.8}{z+0.5} \\ 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|cc} -0.5 & 0.2 & 0.3 \\ \hline \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}\right].$$

Define

$$L_m = \begin{bmatrix} l_1 & l_2 \end{bmatrix}$$

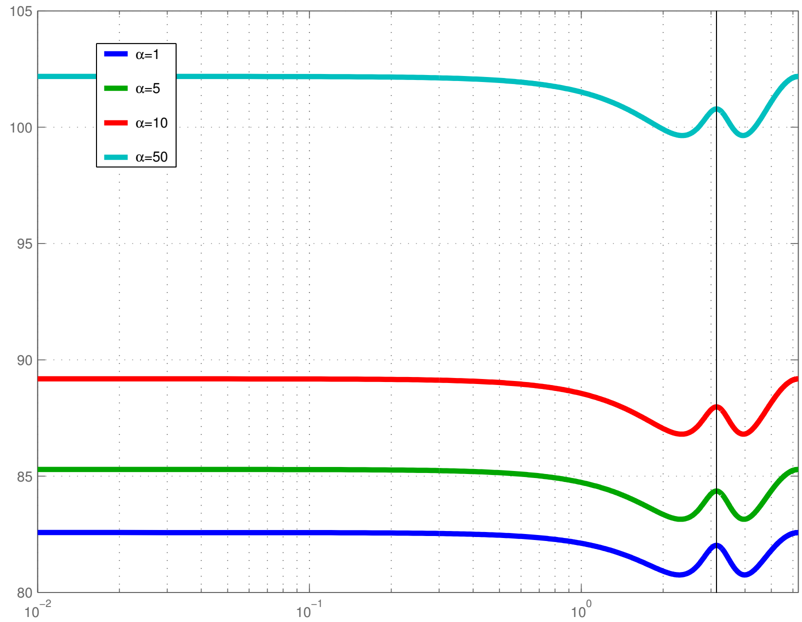

Figure 5.5: Singular Values of QN_f over θ at Different α (Example 4)

By solving the following Riccati system

$$\begin{bmatrix} 1 & l_1 & l_2 \end{bmatrix} \begin{bmatrix} 0.04 - 0.75Y_m & 0.5Y_m + 0.2 & 0 \\ 0.5Y_m + 0.2 & 1 + Y_m & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

we obtain the solution:

$$Y_m = 0, l_1 = -0.2, l_2 \ arbitary.$$

We have

$$Q = \left[\begin{array}{c|c} -0.7 & \begin{bmatrix} -0.2 & l_2 \end{bmatrix} \\ \hline \begin{bmatrix} \gamma \\ 0 \end{bmatrix} & \begin{bmatrix} \gamma & 0 \\ 0 & \alpha \end{bmatrix} \end{array} \right]$$

$$QN_d = \begin{bmatrix} \gamma \\ 0 \end{bmatrix}, \quad QN_f = \begin{bmatrix} \gamma \frac{z+0.8+l_2}{z+0.7} \\ \alpha \end{bmatrix}.$$

Obviously, $ker\left\{\widetilde{N}_{2f}\right\} = \{0\}$, which means that any fault in $\mathscr{L}_2^1$ cannot have bounded sensitivity. When α is big enough, we have

$$\|QN_f\|_- = \gamma \left\| \frac{z + 0.8 + l_2}{z + 0.7} \right\|_- .$$

Assume $\gamma = 1$. Figure 5.6 shows the singular values of QN_f for all $\theta \in [0, 2\pi]$ at different l_2. Figure 5.7 shows $\|QN_f\|_-$ at different l_2. It can be seen that l_2 can be used to improve the fault sensitivity $\|QN_f\|_-$ and the worst fault sensitivity can be arbitrarily assigned.

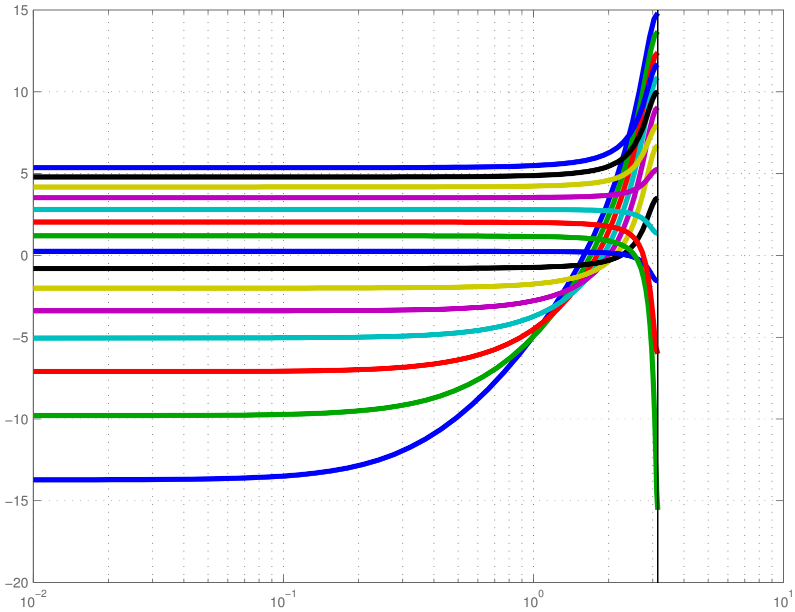

Figure 5.6: Singular Values of $\|QN_f\|_-$ vs θ at Different l_2 (Example 6)

Example 7: Consider the following dynamic system with $G_u = 0$

$$\begin{bmatrix} G_d & G_f \end{bmatrix} = \begin{bmatrix} \frac{z+0.7}{z+0.5} & \frac{z+0.8}{z+0.5} \\ 1 & 0 \end{bmatrix}$$

$$= \left[\begin{array}{c|cc} -0.5 & 0.2 & 0.3 \\ \hline \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}\right].$$

Define

$$L_m = \begin{bmatrix} l_1 & l_2 \end{bmatrix}.$$

By solving the following Riccati system

$$\begin{bmatrix} 1 & l_1 & l_2 \end{bmatrix} \begin{bmatrix} 0.04 - 0.75Y_m & 0.5Y_m + 0.2 & 0.2 \\ 0.5Y_m + 0.2 & 1 + Y_m & 1 \\ 0.2 & 1 & 1 \end{bmatrix} = 0$$

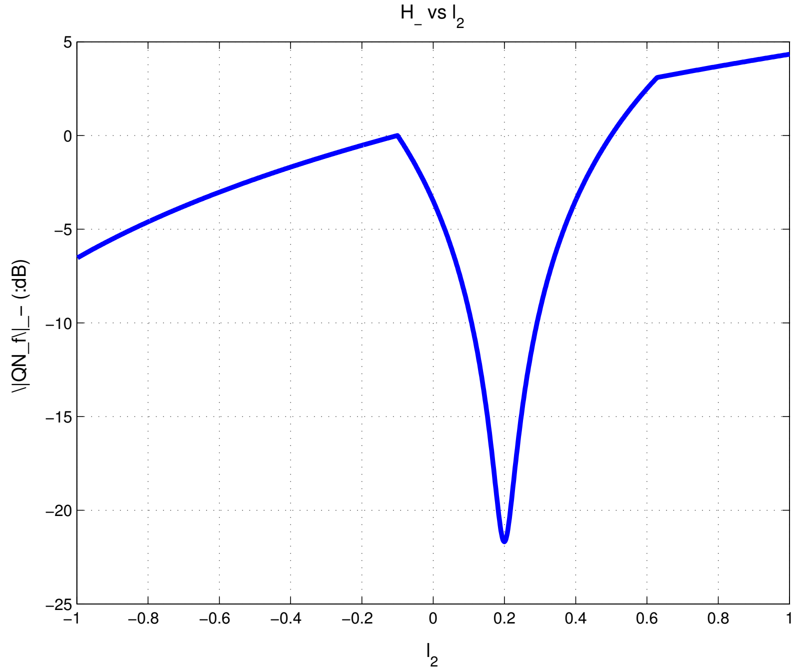

Figure 5.7: $\|QN_f\|_-$ at Different l_2 (Example 6)

we obtain the solution:

$$Y_m = 0 \text{ and } l_1 + l_2 + 0.2 = 0.$$

We have

$$Q = \left[\begin{array}{c|c} -0.5 + l_1 & \begin{bmatrix} l_1 & l_2 \end{bmatrix} \\ \hline \begin{bmatrix} -\frac{1}{2}\gamma \\ -\frac{\sqrt{2}}{2}\alpha \end{bmatrix} & \begin{bmatrix} -\frac{\sqrt{2}}{2}\gamma & -\frac{\sqrt{2}}{2}\gamma \\ -\frac{\sqrt{2}}{2}\alpha & \frac{\sqrt{2}}{2}\alpha \end{bmatrix} \end{array}\right].$$

System is stable when $|-0.5 + l_1| < 1$, that is, $-0.5 < l_1 < 1.5$.

$$QN_d = \begin{bmatrix} -\gamma \\ 0 \end{bmatrix}, \quad QN_f = \begin{bmatrix} -\frac{1}{2}\gamma\frac{z+0.8}{z+0.5-l_1} \\ -\frac{\sqrt{2}}{2}\alpha\frac{z+0.8}{z+0.5-l_1} \end{bmatrix}.$$

Thus, we have $\|QNd\| = \gamma$ and $ker\left\{\widetilde{N}_{2f}\right\} = \left\{f : \frac{z+0.8}{z+0.5-l_1}f = 0\right\} = \{0\}$, which means every fault could have arbitrary sensitivity. Actually, for any fault $f \in \mathscr{L}_2^1$ and $f \neq 0$, the residual energy at the second output could be arbitrarily large due to free parameter α. Thus the fault sensitivity in terms of $\|QN_f\|_\infty$ and $\|QN_f\|_2$ could be arbitrarily large.

As for $\|QN_f\|_-$, it is related to the free parameter l_1. Given that α is zero and $\gamma = 2$, we have $\|QN_f\|_- = \frac{1}{2}\gamma\left\|\frac{z+0.8}{z+0.5-l_1}\right\|_- = 1$ which is achieved when $l_1 = -0.3$.

Figure 5.8 shows the singular values of QN_f for all $\theta \in [0, 2\pi]$ at different l_1. Figure 5.9 shows $\|QN_f\|_-$ at different l_1. It can be seen that l_1 can be used to increase the fault sensitivity $\|QN_f\|_-$, but the maximum of the worst fault sensitivity is bounded.

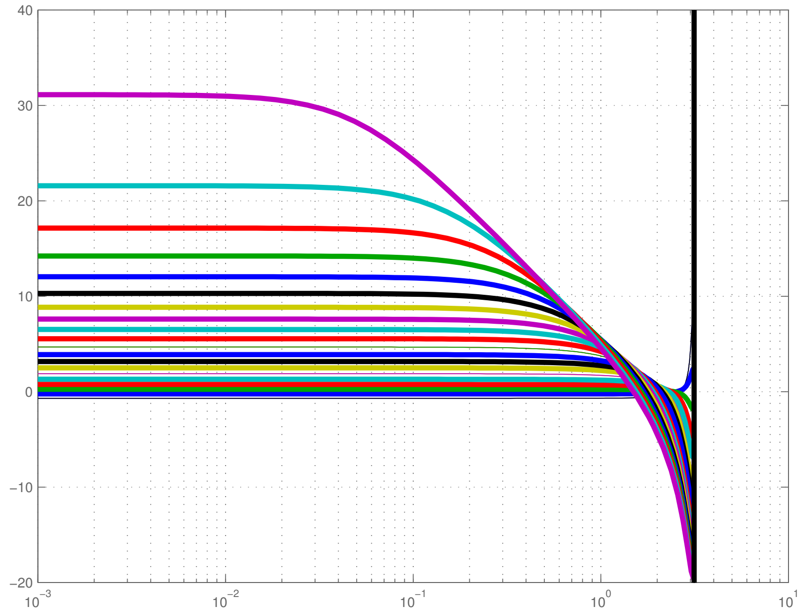

Figure 5.8: Singular Values of QN_f over θ at Different l_1 (Example 7)

Example 8:

$$\begin{bmatrix} G_d & G_f \end{bmatrix} = \left[\begin{array}{c|cc} \frac{z+0.7}{z+0.5} & \frac{z+0.8}{z+0.5} & \frac{z+0.6}{z+0.5} \\ 0 & 1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|cc} -0.5 & 0.2 & \begin{bmatrix} 0.3 & 0.1 \end{bmatrix} \\ \hline \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{array}\right].$$

Define

$$L_m = \begin{bmatrix} l_1 & l_2 \end{bmatrix}.$$

By solving the following Riccati system

$$\begin{bmatrix} 1 & l_1 & l_2 \end{bmatrix} \begin{bmatrix} 0.04 - 0.75Y_m & 0.5Y_m + 0.2 & 0 \\ 0.5Y_m + 0.2 & 1 + Y_m & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

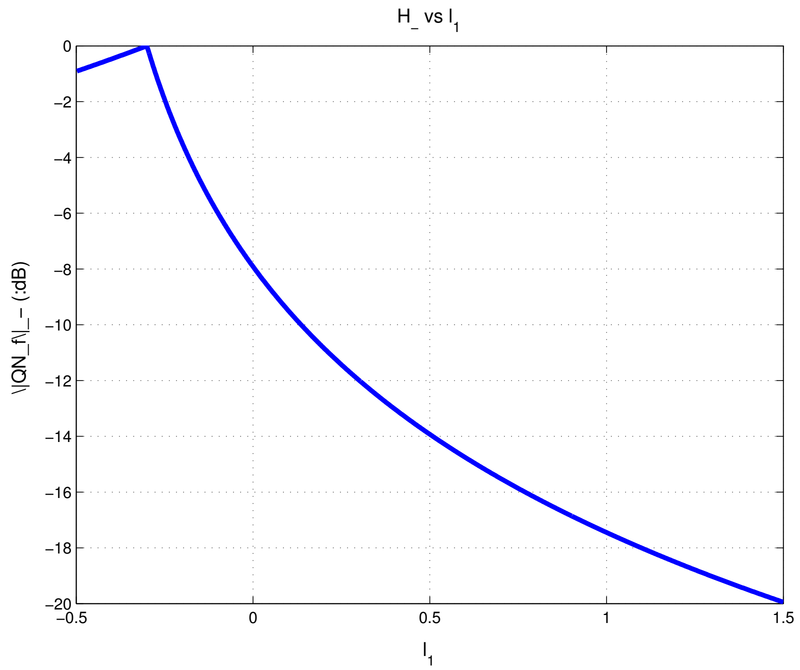

Figure 5.9: $\|QN_f\|_-$ at Different l_1 (Example 7)

we obtain the solution:

$$Y_m = 0, l_1 = -0.2, l_2 \; arbitary.$$

We have

$$Q = \left[\begin{array}{c|c} -0.7 & \begin{bmatrix} -0.2 & l_2 \end{bmatrix} \\ \hline \begin{bmatrix} \gamma \\ 0 \end{bmatrix} & \begin{bmatrix} \gamma & 0 \\ 0 & \alpha \end{bmatrix} \end{array}\right].$$

$$QN_d = \begin{bmatrix} \gamma \\ 0 \end{bmatrix} \quad QN_f = \begin{bmatrix} \gamma\frac{z+0.8+l_2}{z+0.7} & \gamma\frac{z+0.6+l_2}{z+0.7} \\ \alpha & \alpha \end{bmatrix}.$$

It follows that

$$ker\left\{\widetilde{N}_{2f}\right\} = \left\{ f : f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, f_1 + f_2 = 0, f \in \mathscr{L}_2^{n_f} \right\}.$$

Further,

$$\|QN_f\|_-^{ker\{\widetilde{N}_{2f}\}} = \frac{\sqrt{2}}{2} \cdot \gamma \left\| \frac{z+0.2}{z+0.7} \right\|_- = 0.0832\gamma$$

which is independent of l_2.

From the expression of QN_f, it can be seen that for the fault not in $ker\{\widetilde{N}_{2f}\}$ $(f_1+f_2 \neq 0)$, the fault sensitivity can be arbitrary due to the free parameter α.

Example 9: Consider the following dynamic plant with $G_u = 0$

$$\left[\begin{array}{cc} G_d & G_f \end{array}\right] = \left[\begin{array}{c|cc} \frac{z+0.7}{z+0.5} & \frac{z+0.8}{z+0.5} & \frac{z+0.6}{z+0.5} \\ 1 & 1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|cc} -0.5 & 0.2 & \left[\begin{array}{cc} 0.3 & 0.1 \end{array}\right] \\ \hline \left[\begin{array}{c} 1 \\ 0 \end{array}\right] & \left[\begin{array}{c} 1 \\ 1 \end{array}\right] & \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right] \end{array}\right]$$

Define

$$L_m = \left[\begin{array}{cc} l_1 & l_2 \end{array}\right]$$

By solving the following Riccati system

$$\left[\begin{array}{ccc} 1 & l_1 & l_2 \end{array}\right] \left[\begin{array}{ccc} 0.04 - 0.75Y_m & 0.5Y_m + 0.2 & 0.2 \\ 0.5Y_m + 0.2 & 1 + Y_m & 1 \\ 0.2 & 1 & 1 \end{array}\right] = 0$$

we obtain the solution:

$$Y_m = 0 \quad l_1 + l_2 + 0.2 = 0.$$

We have

$$Q = \left[\begin{array}{c|c} -0.5 + l_1 & \left[\begin{array}{cc} l_1 & l_2 \end{array}\right] \\ \hline \left[\begin{array}{c} -\frac{1}{2}\gamma \\ -\frac{\sqrt{2}}{2}\alpha \end{array}\right] & \left[\begin{array}{cc} -\frac{\sqrt{2}}{2}\gamma & -\frac{\sqrt{2}}{2}\gamma \\ -\frac{\sqrt{2}}{2}\alpha & \frac{\sqrt{2}}{2}\alpha \end{array}\right] \end{array}\right]$$

System is stable when $|-0.5 + l_1| < 1$, that is, $-0.5 < l_1 < 1.5$.

$$QN_d = \left[\begin{array}{c} -\gamma \\ 0 \end{array}\right]$$

$$QN_f = \left[\begin{array}{cc} -\frac{1}{2}\gamma\frac{z+0.8}{z+0.5-l_1} & -\frac{1}{2}\gamma\frac{z+0.6}{z+0.5-l_1} \\ -\frac{\sqrt{2}}{2}\alpha\frac{z+0.8}{z+0.5-l_1} & -\frac{\sqrt{2}}{2}\alpha\frac{z+0.6}{z+0.5-l_1} \end{array}\right]$$

Further, we have

$$ker\left\{\widetilde{N}_{2f}\right\} = \left\{\left[\begin{array}{c} f_1 \\ f_2 \end{array}\right] : f_1 = -\frac{z+0.8}{z+0.6}f_2\right\}$$

and

$$\|QN_f\|_{-}^{ker\left\{\widetilde{N}_{2f}\right\}} = 0$$

which is independent of l_2.

Chapter 6

Polynomial Matrix Approach to Fault Detection Filter Design

In contrast to the previous chapter on the state-space model, we shall derive the fault detection filter directly from transfer matrix. Section 6.1 presents the filter design procedure, while Section 6.2 gives an example to illustrate the result.

6.1 Design Procedure

From the derivations in the previous chapter, the optimal filter design for linear discrete time-invariant systems can be formulated as a problem of finding an appropriate Q such that some measure of QN_f is optimized while QN_d is constrained in some sense. In general, under the optimal condition, we have $\|QN_d\|_\infty = \gamma$. From matrix theory, QN_d and QN_f can be thought as transformations on the N_d and N_f respectively, which can be done by row elementary operations. Therefore, we can try to do row elementary operations for N_d till we find an appropriate Q. Specifically, if we are able to do elementary operations for $\begin{bmatrix} N_d & N_f & I \end{bmatrix}$ till N_d becomes diagonal, it is easy to find its inverse. In addition, the information of the elementary operations is stored into the third block (the previous identity matrix I). Based on its inverse and the third block, the filter Q can be constructed. By following this idea, we have the following procedure for fault detection filter design.

1. Find a left coprime factorization for $G = \begin{bmatrix} G_u & G_d & G_f \end{bmatrix}$ as follows

$$G = M^{-1}N = M^{-1}\begin{bmatrix} N_u & N_d & N_f \end{bmatrix}.$$

2. Define $q := z^{-1}$. Construct a combined transfer matrix $\bar{N}$ as follows

$$\bar{N} = \left[\begin{array}{c|c|c} N_d(q) & N_f(q) & I_{n_y} \end{array}\right].$$

Note: | in $\left[\begin{array}{c|c|c} N_d(q) & N_f(q) & I_{n_y} \end{array}\right]$ is used to separate three blocks.

3. Pull out the denominator part for $N_d(q)$ and then do row elementary operations iteratively for $\bar{N}$ till $\bar{N}$ becomes the following form

$$\bar{N} \rightarrow \widetilde{N} = \begin{bmatrix} \widetilde{N}_d & \widetilde{N}_f & \widetilde{Q} \end{bmatrix}$$

where

$$\widetilde{N}_d = \begin{bmatrix} R_1 & & 0 \\ & \ddots & \\ 0 & & R_{n_d} \\ 0 & \cdots & 0 \end{bmatrix}.$$

This transformation is possible since $rank\{N_d\} = n_d$, $\forall\theta \in [0, 2\pi]$.

4. Design Q as

$$Q = \Psi\widetilde{Q}(q)$$

where

$$\Psi = \begin{bmatrix} \gamma T_1^{-1}(q) & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \gamma T_{n_d}^{-1}(q) & 0 \\ 0 & \cdots & 0 & \alpha I_{(n_y-n_d)\times n_d} \end{bmatrix},$$

α is a free scalar and T_i $(i = 1, \ldots n_d)$ are the spectral factors of R_i. In other words, T_i can be obtained by reflecting the unstable zeros of R_i into the unit circle.

Note that since G_d has no transmission zero on the unit circle, T_i^{-1} $(i = 1, \cdots, r)$ are stable.

5. Given some rows of $\widetilde{N}_{1f}$ are zero, it is appropriate to revise Q as

$$\hat{Q} = \Lambda Q = \Lambda \Psi \widetilde{Q}(q)$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n_y} \end{bmatrix}$$

and

$$\lambda_i = \begin{cases} 0, & [\widetilde{N}_{1f}]^i = 0; \\ 1, & \text{otherwise.} \end{cases}$$

6. The fault detection filter is

$$F = \hat{Q} \begin{bmatrix} M & -N_u \end{bmatrix}.$$

Remark 35 *The row elementary operations in Step 3 include three operations: exchanging rows, adding one row on another row and multiplying a polynomial in terms of operator q on one row.*

6.2 Example

We give the following example to illustrate our procedure in Section 6.1.

Assume that we have already obtained the following matrix after left coprime factorization

$$\left[\begin{array}{c|c|c} N_d & N_f & I \end{array}\right] = \left[\begin{array}{c|c|cc} \frac{z+4}{z+0.5} & \frac{z+0.8}{z+0.5} & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c|cc} \frac{4q+1}{0.5q+1} & \frac{0.8q+1}{0.5q+1} & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right].$$

Step 1: By pulling out the common denominators, we have

$$\left[\begin{array}{c|c|c} N_d(q) & N_f(q) & I \end{array}\right] = \frac{1}{0.5q+1} \left[\begin{array}{c|c|cc} 4q+1 & 0.8q+1 & 0.5q+1 & 0 \\ 0.5q+1 & 0 & 0 & 0.5q+1 \end{array}\right].$$

Step 2: Row elementary operations:

1. After the first row $\times(0.5q+1)$ and the second row $\times(4q+1)$, we have

$$\frac{1}{0.5q+1}\left[\begin{array}{c|c|cc}(4q+1)(0.5q+1) & (0.5q+1)(0.8q+1) & (0.5q+1)^2 & 0\\(4q+1)(0.5q+1) & 0 & 0 & (4q+1)(0.5q+1)\end{array}\right].$$

2. After row 2 minus row 1, we have

$$\frac{1}{0.5q+1}\left[\begin{array}{c|c|cc}(4q+1)(0.5q+1) & (0.5q+1)(0.8q+1) & (0.5q+1)^2 & 0\\0 & -(0.5q+1)(0.8q+1) & -(0.5q+1)^2 & (4q+1)(0.5q+1)\end{array}\right].$$

3. To simplify it, we have

$$\left[\begin{array}{c|c|cc}(4q+1) & 0.8q+1 & 0.5q+1 & 0\\0 & -(0.8q+1) & -(0.5q+1) & 4q+1\end{array}\right].$$

4. Thus, we can have the following result:

$$Q=\begin{bmatrix}\gamma\frac{1}{q+4} & 0\\0 & \alpha\end{bmatrix}\begin{bmatrix}0.5q+1 & 0\\-(0.5q+1) & (4q+1)\end{bmatrix}=\begin{bmatrix}\gamma\frac{0.5q+1}{q+4} & 0\\-\alpha(0.5q+1) & \alpha(4q+1)\end{bmatrix}.$$

In addition, we can easily verify

$$QN_d=\begin{bmatrix}\gamma\frac{0.5q+1}{q+4} & 0\\-\alpha(0.5q+1) & \alpha(4q+1)\end{bmatrix}\begin{bmatrix}\frac{4q+1}{0.5q+1}\\1\end{bmatrix}=\begin{bmatrix}\gamma\frac{4q+1}{q+4}\\0\end{bmatrix}=\begin{bmatrix}\gamma\frac{z+4}{4z+1}\\0\end{bmatrix},$$

$\|QN_d\|_\infty=\gamma$ and

$$QN_f=\begin{bmatrix}\gamma\frac{0.5q+1}{q+4} & 0\\-\alpha(0.5q+1) & \alpha(4q+1)\end{bmatrix}\begin{bmatrix}\frac{0.8q+1}{0.5q+1}\\0\end{bmatrix}=\begin{bmatrix}\gamma\frac{0.8q+1}{q+4}\\-\alpha(0.8q+1)\end{bmatrix}=\begin{bmatrix}\gamma\frac{z+0.8}{4z+1}\\-\alpha\frac{z+0.8}{z}\end{bmatrix}.$$

From the expression QN_d, it can be seen that α is able to improve the fault sensitivity at the second output of residual signal, but not the first one. Thus, the fault sensitivity at the first output is bounded. In addition, we can do more row elementary operations in Step 2, which can result in different Q. Therefore, Q is not unique.

Chapter 7

Future Work

Robust fault detection filter design is an exciting area for both theoretical research and practical application. As we saw in the previous chapters, our framework is suitable for robust fault detection of LTI and LTV cases. In order to sufficiently explore this framework on fault detection, the following future work could be a significant complement.

First, it is not immediate to extend the result in Chapter 5 to linear continuous time systems. In Chapter 5 we have obtained the solution for the case in which G_d may not be square or wide and D_d may not have full column rank for linear discrete time invariant systems. Parallel results may be developed for linear continuous time-invariant systems by employing similar techniques. However, it may be a little harder to relax the condition $D_d = 0$.

The relaxation of the condition that $\left[\begin{array}{c|c} A - j\omega I & B_d \\ \hline C & D_d \end{array}\right]$ has full rank may be much harder. This condition means there exists no transmission zero on the imaginary axis or the unit circle for G_d. This condition is not necessary for the practical systems. It is highly possible that without this condition there may be no rational solution in our framework. In this situation, there are two issues: one is to explore more general solution set, i.e. nonlinear filter; the other is to find approximated rational solution, but approximation error should be evaluated in terms of a bound.

In addition, further work should be addressed on the complete comparison and relation with other criteria. Actually, there are some other criteria in robust fault detection with the same model such as maximizing the ratio of fault sensitivity and disturbance sensitivity [18]. An obvious drawback of this criterion is that the disturbance sensitivity cannot be zero, otherwise, the objective function could be infinity. Thus, it cannot handle with the decoupling problem simultaneously. Therefore, our decoupling and optimization mechanism could be developed for this criterion.

since the general case contains both decoupling and optimization simultaneously and it is related to a general Riccati equation (GRE), it is necessary to explore an efficient algorithm for this equation. The current available methods for GRE can be classified into two categories: one is the so called deficient subspace method, which is an extension of invariant space method, but it is too complicated to compute [37]; the other is an optimization-based method that aims to look for a matrix sequence that converges to the optimal solution, but its convergence speed is not guaranteed for the general Riccati equation, especially when the discrete time system's poles close to the unit circle [28].

One assumption in the fault detection of linear time-varying systems is that $D_d(t)$ is full row rank for all time $t > 0$. However, this restriction could be a little strong for some time varying systems that rank deficiency happens at only a few 'singular' points. For instance, for a system with time-varying term

$$D_d(t) = \begin{bmatrix} 2-t & 1 \\ 1 & 1 \end{bmatrix}$$

$D_d(t)$ is of full rank for all time t except $t = 1$. The relaxation of our assumption to include the case that $D_d(t)$ is rank deficient at some discrete points makes sense. Mathematically, the rank deficient set is of countable points.

Another issue is the extension to sampled-data systems. This work will make much sense because, as we all know, most modern systems that have computer involved are sampled-data systems, a kind of hybrid systems that have continuous input and output for the plant, but have discrete-time input and output for the controller. Actually, there are quite a few of works on fault detection of sampled-data systems [39]. In [39] Izadi et al. the authors considered maximizing the ratio of fault sensitivity and disturbance sensitivity in terms of $\mathscr{H}_2$ or $\mathscr{H}_\infty$ norm, while the popular $\mathscr{H}_-$ index was not considered. By employing the similar lifting technique in frequency domain or time domain, it may be possible to extend our framework to sampled-data systems.

Our framework could be extended to infinite-dimensional systems as well. Infinite-dimensional systems are systems in which system coefficients may be represented by operators in Hilbert space, but not finite dimensional matrices. The extension of the result to infinite-dimensional systems could be rather hard and the Riccati equation involved in our fault detection framework could be an operator Riccati equation for infinite-dimensional systems.

Optimal robust fault detection with model uncertainties could be a challenging work. In this case, the plant in the framework is not exactly known, i.e., it may be represented by a nominal plant G and model error G_Δ. The optimal fault detection filter design will be much difficult since the residual generator should attenuate the negative effect of the uncertain model error G_Δ on the residual signal.

Nonlinear robust fault detection is another challenging work. One way is to look for a broader solution set, i.e. nonlinear filter. The other is to linearize the nonlinear plant to obtain a time varying (possibly time invariant) nominal model with plant error. This linearization method turns out to be the robust fault detection problem with model uncertainty.

Chapter 8

Conclusion

Several multi-objective fault detection problems such as $\mathscr{H}_-/\mathscr{H}_\infty$, $\mathscr{H}_2/\mathscr{H}_\infty$ and $\mathscr{H}_\infty/\mathscr{H}_\infty$ have been given for linear continuous time-varying systems (LCTVS) in time domain for finite horizon and infinite horizon case, respectively. It has been proven that the optimal solution is an observer determined by solving a standard differential Riccati equation. The solution has also been extended to the case when the initial state for the system is unknown. An example has also been given to illustrate the results.

The parallel problems have also been solved for linear discrete time-varying systems in time domain. The solution is also an observer whose gain is determined by solving a standard recursive difference Riccati equation (DDRE). The solution is also extended to the case when the initial state for the system is unknown. An example has also been given to illustrate the results.

We have also extended the framework to the more general case in which G_d may be a tall or square transfer matrix, and D_d may not have full column rank for linear discrete time-invariant systems (LDTIS). In this situation, the common $\mathscr{H}_-/\mathscr{H}_\infty$, $\mathscr{H}_2/\mathscr{H}_\infty$ and $\mathscr{H}_\infty/\mathscr{H}_\infty$ frameworks are not applicable. Based on several novel definitions of norms over a certain subspace, we have proposed some new frameworks for both decoupling and optimization and the solution has been given in state space form. Here, the solution is related to a generalized

Riccati equation (GRE). To be more specific, with this filter, some faults in certain subspace can be completely decoupled from the residual signal, while the others are optimized in terms of fault sensitivity. Furthermore, disturbance rejection based on the solution has been discussed. In addition, we have provided a procedure for computing the fault detection filter in transfer matrix form. Several examples have been given to illustrate the results.

Finally, some further explorations based on this framework have been also discussed.

Bibliography

[1] B. D. Appleby and J. R. Dowdle, Robust estimator design using μ synthesis, *Proceedings of the 30th Conference on Decision and Control*, Brighton, England, December 1991.

[2] R.V. Beard, *Failure Accomodation in Linear Systems Through Self Reorganization*, Ph.D dissertation, Massachusetts Institute of Technology, Cambridge, MA, 1971.

[3] T. S. Brinsmead, J. S. Gibson, G. C. Goodwin, G. H. Lee and D. L. Mingori, Fault detection: a quadratic optimisation approach, *Proceedings of the 36th IEEE Conference on Decison and Control*, San Diego, CA, USA, pp. 3970-3975, Dec. 10-12, 1997.

[4] S. L. Campbell, K. G. Horton and R. Nikoukhah, Auxiliary signal design for rapid multi-model identification using optimization, *Automatica*, Vol. 38, 2002, pp. 1313-1325.

[5] S. L. Campbell and R. Nikoukhah, *Auxiliary Signal Design for Failure Detection*, Princeton, NJ: Princeton Univ. Press, 2004.

[6] A. Casavola, D. Famularo and G. Franze, A robust deconvolution scheme for fault detection and isolation of uncertain linear systems: an LMI approach, *Automatica*, 41, 2005, pp. 1463-1472.

[7] L. H. Chiang, E. L. Russell and R. D. Braatz, *Fault Detection and Diagnosis in Industrical Systems*, Springer, 2001.

[8] J. Chen and R. J. Patton, *Robust Model-Based Fault Diagnosis for Dynamic Systems*, Kluwer Academic Publishers, Boston, 1999.

[9] J. Chen, R. J. Patton and H. Y. Zhang, Design of unknown input observers and robust fault detection filters, *International Journal of Control*, Vol. 63, Issue 1, January 1996, pp. 85-105.

[10] R. H. Chen and D. Lewis Mingori, Jason L. Speyer, Optimal stochastic fault detection filter, *Automatica*, 39, 2003, pp. 377-390.

[11] R. H. Chen and Jason L. Speyer, Detection filter analysis and design using eigenstructure assignment, *Proceedings of the 2006 American Control Conference*, Minneapolis, Minnesota, USA, June 14-16, 2006.

[12] W. H. Chung and J. L. Speyer, A game theoretic fault detection filter, *IEEE Transactions on Automatic Control*, Vol. 43, No. 2, pp. 143-161, 1998.

[13] B. Darkhovski and M. Staroswiecki, A game-theoretic approach to decision in FDI, *IEEE Transactions on Automatic Control*, vol. 48, No. 5, May 2003.

[14] S. X. Ding, P.M. Frank, E.L. Ding and T. Jeinsch, Fault detection system design based on a new trade-off strategy, *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia, December, 2000.

[15] X. Ding, P.M. Frank and L. Guo. An approach to residual generator and evaluator design and synthesis, *Proceedings of IFAC World Congress*, Sydney, Australia, 1993.

[16] X. Ding, L. Guo and P.M. Frank. A frequency domain approach to fault detection of uncertain dynamic systems. *Proceedings of IEEE Conference on Decesion and Control*, pp. 1722-1727, San Antonio, TX, 1993. IEEE Press, New York.

[17] X. Ding, L. Guo and T. Jeinsch, A characterization of parity space and its application to robust fault detection, *IEEE Transactions on Automatic Control*, 44(2), pp. 337-343, 1999.

[18] S. X. Ding, J. Jeinsch, P. M. Frank and E. L. Ding, A unified approach to the optimization of fault detection systems, *International Journal of Adaptive Control and Signal Processing*, Vol. 14, pp. 725-745, 2000.

[19] A. M. Edelmayer and J. Bokor, Scaled $\mathscr{H}_\infty$ filtering for senstivity optimization of detection filters, *SAFEPROCESS 2000*, Vol. 1, pp. 324-330, Budapest, Hungary, June 2000.

[20] A. Emami-Naeini, M. M. Akhter and S. M. Rock, Effect of model uncertainty on failure detection: the threshold selector, *IEEE Transactions On Automatic Control*, Vol. 33, No. 12, pp. 1106-1115, 1988.

[21] P. M. Frank, Fault diagnosis in dynamic systems using analytical and knowledge-based redundancy–a survey and some new results, *Automatica*, Vol. 26, No.3, pp. 459-474, 1990.

[22] P. M. Frank and S. X. Ding, Frequency domain approach to optimally robust residual generation and evaluation for model-based fault diagnosis, *Automatica*, 30(5), pp. 789-804, 1994.

[23] P. M. Frank and X. Ding, Survey of robust residual generation and evaluation methods in observer-based fault detection systems, *Journal of Process Control*, Vol. 7, No. 6, pp. 403-424, 1997.

[24] E. Frisk and L. Nielsen, Robust residual generation for diagnosis including a reference model for residual behavior, *Automatica*, 42, 2006, pp. 437-445.

[25] J. Gertler and D. Singer, A new structure framework for parity equations based failure detection and isolation, *Automatica*, Vol.26, pp. 381-388, 1990.

[26] M. Green and D. J. N. Limebeer, *Linear Robust Control*, Prentice Hall, Englewood Cliffs, New Jersey, 1995.

[27] G. Gu, *Theory and Design of Linear Systems in Discrete-time Feedback Control and Data Communications*, Class Notes, Louisiana State University, 2005.

[28] G. Gu, X. Cao and H. Badr, Generalized LQR control and Kalman filtering with relations to computations of inner-outer and spectral factorizations, *IEEE Transactions on Automatic Control*, Vol. 51, No.4, April, 2007.

[29] D. Henry and A. Zolghadri, Design and analysis of robust residual generators for systems under feedback control, *Automatica*, 41-2005, pp. 251-264.

[30] D. Henry and A. Zolghadri, Design of fault diagnosis filters: a multi-objective approach, *Journal of the Franklin Institute*, Vol. 342, pp. 421-446, 2005.

[31] D. Henry and A. Zolghadri, Norm-based design of robust FDI schemes for uncertain systems under feedback control: Comparison of two approaches, *Control Engineering Practice*, Vol. 14, pp. 1081-1097, 2006.

[32] R. C. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[33] M. Hou and R. J. Patton, An LMI approach to $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observers, *Proceedings of UKACC International Conference on Control*, pp. 305-310, 1996.

[34] T. Hsiao and M. Tomizuka, Threshold selection for timely fault detection in feedback control systems, *American Control Conference*, June 8-10, 2005, Portland, OR, USA.

[35] A. Ichikawa and H. Katayama, *Linear Time Varying Systems and Sampled-data Systems*, Lecture Notes in Control and Information Sciences 265, Springer, London, 2001.

[36] V. Ionescu and C. Oară, "Spectral and inner-outer factorization for discrete-time systems," *IEEE transactions on Automatic Control*, vol. 41, pp. 1840–1845, December 1996.

[37] V. Ionescu, C. Oară and M. Wiess, *Generalized Riccati Theory and Robust Control–a Popov Function Approach*, John Wiley and Sons Ltd, England, 1999.

[38] R. Isermann, Process fault detection based on modelling and estimation methods–a survey, *Automatica*, Vol. 20, pp. 387-404, 1984.

[39] I. Izadi, T. W. Chen and Q. Zhao, Norm invariant discretization for sampled-data fault detection, *Automatica*, Vol. 41, pp. 1633-1637, 2005.

[40] I. Izadi, T. W. Chen and Q. Zhao, An $\mathscr{H}_\infty$ approach to fast rate fault detection for multirate sampled-data system, *Journal of Process Control*, Vol. 16, pp. 651-658, 2006.

[41] I. Izadi, Q. Zhao and T. Chen, Analysis of performance criteria in sampled-data fault detection, *Systems and Control Letters*, Vol. 56, No. 4, pp. 320-325, 2006.

[42] I. M. Jaimoukha, Z. Li and V. Papakos, A matrix factorization solution to the $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection problem, *Automatica*, Vol. 42, No. 11, pp. 1907-1912, 2006.

[43] B. Jiang, M. Staroswiecki and V. Cocquempot, $\mathscr{H}_\infty$ fault detection filter design for linear discrete time systems with multiple time delays, *International Journal of Systems Science*, Vol. 34, No. 5, pp. 365-373, 2003.

[44] A. Johansson, M. Bask and T. Norlander, Dynamic thresthod generators for robust fault detection in linear systems with parameter uncertainty, *Automatica*, Vol. 42, 2006, pp. 1095-1106.

[45] H. L. Jones, *Failure Detection in Linear Systems*, Ph.D dissertation, Massachusetts Institute of Technology, Cambridge, MA, 1973.

[46] M. J. Khosrowjerdi, R. Nikoukhah and N. Safari-Shad, A mixed $\mathscr{H}_2/\mathscr{H}_\infty$ approach to simutaneous fault detection and control, *Automatica*, Vol. 40, No. 2, pp. 261-267, 2004.

[47] M. J. Khosrowjerdi, R. Nikoukhah and N. Safari-Shad, Fault detection in a mixed $\mathscr{H}_2/\mathscr{H}_\infty$ setting, *IEEE Transactions on Automatic Control*, Vol. 50, No. 7, July 2005.

[48] L. Li and G. Gu, Design of optimal zero-forcing precoders for MIMO channels via optimal full information control, *IEEE Transactionss on Signal Processing*, Vol. 53, No. 8, pp. 3238-3246, 2005.

[49] Z. Li, E. Mazars and I. Jaimoukha, State space solution to the H_-/H_∞ fault detection problem, *Proceedings of the 45th IEEE Conference on Decision and Control*, San Diego, CA, USA, Dec. 13-15, 2006.

[50] X. Li and K. Zhou, A time domain approach to robust fault detection of linear time-varying systems, *Automatica*, Vol. 45, Issue 1, pp. 94-102 January, 2009.

[51] X. Li and K. Zhou, Fault detection for linear discrete time-invariant systems with decoupling and optimization, submitted to the 49th IEEE Conference on Decision and Control, March 2009.

[52] X. Li and K. Zhou, Robust fault detection of linear discrete time-varying systems, Accepted by the seventh Asian Control Conference, May 2009.

[53] J. Liu, J. L. Wang and G. H. Yang, An LMI approach to worst case analysis for fault detection observers, *Proceedings of the American Control Conference*, Denver, Colorado, USA, June 2003, pp. 2985-2990.

[54] J. Liu, J. L. Wang and G. H. Yang, An LMI approach to minimum sensitivity analysis with application to fault detection, *Automatica*, Vol. 41, No. 11, pp. 1995-2004, 2005.

[55] N. Liu and K. Zhou, Optimal solutions to multi-objective robust fault detection problems, *Procedings of the 46th IEEE Conference on Decision and Control*, pp. 981-988, New Orleans, LA, USA, December 12-14, 2007.

[56] N. Liu and K. Zhou, Optimal robust fault detection for linear discrete time systems, *Journal of Control Science and Engineering*, Vol. 2008.

[57] M. A. Massoumnia, A geometric approach to the synthesis of failure detection filters, *IEEE Transactions on Automatic control*, Vol. 31, No.9, pp. 839-846, September 1986.

[58] N. Meskin and K. Khorasani, Fault detection and isolation of actuator faults in spacesraft formation flight, *Proceedings of the 45th IEEE conference on Decision and Control*, San Diego, CA, USA, Dec. 13-15, 2006.

[59] K. M. Nagpal and P. P. Khargonekar, Filtering and smoothing in an $\mathscr{H}_\infty$ setting, *IEEE Transactions on Automatic Control*, Vol. 36, No. 2, pp. 152-166, Feb. 1991.

[60] H. Niemann, Fault tolerant control based on active fault diagnosis, *Proceedings of the 2005 merican Control Conference*, June 8-10, 2005, pp. 2224-2229, Portland, OR, USA.

[61] H. Niemann, A setup for active fault diagosis, *IEEE Transactions on Automatic control*, Vol. 51, No. 9, pp. 1572-1578, Sep. 2006.

[62] J. Park and G. Rizzoni, An eigenstructure assignment alogorithm for the design of fault detection filters, *IEEE Transactions on Automatic Control*, Vol. 39, No. 7, July 1994.

[63] R. J. Patton, Robustness in model-based fault diagnosis: the 1995 situation, *IFAC Annual Reviews*, No. 21, pp. 103-123, 1997.

[64] R. J. Patton and J. Chen, A review of parity space approaches to fault diagnosis, *Procedings of IFAC/IMACS Symp. SAFEPROCESS'91*, Baden, Germany, 1991, Vol. 1, pp. 239-255.

[65] R. J. Patton, J. Chen and J. H. P. Millar, A robust disturbance decoupling approach to fault detection in process system, *Proceedings of the 30th Conference on Decision and Control*, pp. 1543-1548, Brighton, England, 1991.

[66] R. J. Patton,, P. M. Frank and R. N. Clark (eds), *Fault Diagnosis in Dynamic Systems, Theory and Application*, Prentice Hall, 1989.

[67] F. Rambeaux, F. Hamelin and D. Sauter, Robust residual generation via LMI, *Proceedings of the 14th World Congress of IFAC*, Beijing, PRC, pp. 240-246, July 1999.

[68] M. L. Rank and H. Niemann, Norm based design of fault detectors, *International Journal of Control*, Vol. 72, No. 9, pp. 773-783, 1999.

[69] R. Ravi, A. M. Pascoal and P. P. Khargonekar, Normalized coprime factorizations and the graph metric for linear time-varying systems, *Proceedings of the 29th Conference on Desicion and Control*, Honolulu, Hawail, December 1990.

[70] M. A. Sadrnia, J. Chen and R. J. Patton, Robust fault diagnosis observer design using $\mathscr{H}_\infty$ optimization and μ synthesis, *IEEE Colloquium on Modelling and Signal Processing for Fault Diagnosis*, 9/1-196, Leicester, UK, September 1996.

[71] D. Sauter and F. Hamelin, Frequency-domain optimization for robust fault detection and isolation in dynamic systems, *IEEE Transactions on Automatic Control*, Vol. 44, No. 4, April 1999.

[72] J. Stoustrup, M. Grimble and H. Niemann, Design of integrated systems for the control and detection of actuator/sensor faults, *Sensor Review*, Vol. 17, No. 2, pp. 138-149, 1997.

[73] J. Stoustrup, H. Niemann and A. J. Cour-Harbo, Optimal threshold functions for fault detection and isolation, *Proceedings of the 2003 American Control Conference*, Vol. 2 June 4-6, 2003, pp. 1782-1787.

[74] F. Tao and Q. Zhao, Fault detection observer design with unknown inputs, *Proceeding of the 2005 IEEE Conference on Control Applications*, pp. 1275-1280, Toronto, Canada, August 2005.

[75] A. Varga, Computational issues in fault detection filter design, *Proceeding of the 41th IEEE conference on Decision and control*, Las Vegas, Nevada USA, Decemember 2002.

[76] A. Varga, Numericall reliable methods for optimal design of fault detection filters, *Proceeding of the 44th IEEE Conference on Decision and Control*, Seville, Spain, Decemember 12-15, 2005.

[77] V. Venkatasubramanian, R. Rengaswamy, K. Yin, and S. N. Kavuri, A review of process fault detection and diagnosis part i: Quantitative modelbased methods. Computers and Chem. Eng. 27, pp. 293-311, 2003, 170.

[78] V. Venkatasubramanian, R. Rengaswamy, K. Yin, and S. N. Kavuri, A review of process fault detection and diagnosis part II: Qualitative models and search strategies. Computers and Chem. Eng. 27, pp. 313-326, 2003, 171.

[79] V. Venkatasubramanian, R. Rengaswamy, K. Yin, and S. N. Kavuri, A review of process fault detection and diagnosis part III: Process history based methods. Computers and Chem. Eng. 27, pp. 327-346, 2003.

[80] N. Viswanadham, J. H. Taylor and E. C. Luce, A frequency domain approach to failure detection and isolation with application, *Control Theory and Advanced Technology*, 3, 1987, pp. 45-72.

[81] H. B. Wang, L. Lam, S. X. Ding and M. Y. Zhong, Iterative linear matrix inequality algorithms for fault detection with unknown inputs, *Journal of Systems and Control Engineering*, Vol. 219, No. 2, pp. 161-172, 2005.

[82] J. L. Wang, G. H. Yang and J. Liu, An LMI approach to $\mathscr{H}_-$ index and mixed $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design, *Automatica*, Vol. 43, No. 9, pp. 1656-1665, 2007.

[83] J. E. White and J. L. Speyer, Detection filter design: spectral theory and algorithms, *IEEE Transactions on Automatic control*, Vol. 32, No. 7, pp. 593-603, July 1987.

[84] H. Ye, S. X. Ding and G. Wang, Integrated design of fault detection systems in time-frequency domain, *IEEE Transactions on Automatic Control*, Vol. 47, No. 2, pp. 384-390, 2002.

[85] P. Zhang and S. X. Ding, A model-free fault detection approach of continuous-time systems from time domain data, *Institute of Automation, Chinese Academy of Sciences*, pp. 189-194, Vol. 4, No. 2, April, 2007

[86] P. Zhang and S. X. Ding, On the fault detection in linear discrete-time periodic and sampled-data systems, *Journal of Control Science and Engineering*, 2008.

[87] P. Zhang, S. X. Ding, G. Z. Wang and D. H. Zhou, An FDI approach for sampled-data systems, *Proceedings of the 2001 American Control Conference*, Vol. 4, pp. 2702-2707, Arlington, VA, USA, June 25-27, 2001.

[88] P. Zhang, S. X. Ding, G. Z. Wang and D. H. Zhou, Fault detection for uncertain sampled data system, *Proceedings of the 4th World Congress on Intelligent Control and Automation*, pp. 2728-2732, Shanghai, China, June 2002.

[89] P. Zhang, S. X. Ding, G. Z. Wang and D. H. Zhou, A frequency domain approach to fault detection in sampled-data systems. *Automatica*, Vol. 39, No. 7, pp. 1303-1307, 2003.

[90] P. Zhang, S. X. Ding, G. Z. Wang and D. H. Zhou, Fault detection of linear discrete time periodic systems, *IEEE Transactionss on Automatic Control*, Vol. 50, No. 2, pp. 239-244, February 2005.

[91] P. Zhang, S. X. Ding, G. Z. Wang, D. H. Zhou and E. L. Ding, An $\mathscr{H}_\infty$ approach to fault detection for sampled-data, *Proceedings of the American Control Conference*, pp. 2196-2201, Anchorage, USA, May 2002.

[92] Y. M. Zhang and J. Jiang, Design of integrated fault detection, diagnosis and reconfigurable control sytems, *Procedings of 38th IEEE Conference on Decision and Control*, Vol. 4, pp. 3587-3592, Phoenix, USA, December 1999.

[93] Y. M. Zhang and J. Jiang, Issues on integration of fault diagnosis and reconfigurable control in active fault-tolerant control systems, *SAFEPROCESS 2006*, pp. 1513-1524, Beijing, China, August 2006.

[94] P. Zhang, H. Ye, S. X. Ding, G. Z. Wang and D. H. Zhou, On the relationship between parity space and $\mathscr{H}_2$ approaches to fault detection, *System and Control Letters*, Vol. 55, pp. 94-100, 2006.

[95] M. Zhong, S. X. Ding, J. Lam and H. Wang, An LMI approach to design robust fault detection filter for uncertain LTI systems, *Automatica*, Vol. 39, No. 3, pp. 543-550, 2003.

[96] M. Zhong, S. X. Ding, B. Tang, P. Zhang and T. Jeinsch, A LMI approach to robust fault detection filter design for discrete-time systems with model uncertainty, *Proceedings of the 40th Conference on Desision and Control*, pp. 3613-3618, Orlando, USA, December 2001.

[97] K. Zhou and J. C. Doyle, *Essentials of Robust Control*, Prentice Hall, New Jersey, 1998.

[98] K. Zhou, J. C. Doyle and K. Clover, *Robust and Optimal Control*, Prentice Hall, New Jersey, 1996.

[99] A. Zolghadri, An algorithm for real-time failure-detection in Kalman filters, *IEEE Transactions on Automatic control*, Vol. 41, pp. 1537-1539, 1996.

Appendix: Authorization

The authorization for reprinting the article [50] is from the website: *http://www.elsevier.com/wps/find/supportfaq.cws_home/rightsasanauthor*.

As an author, you retain rights for a large number of author uses, including use by your employing institute or company. These rights are retained and permitted without the need to obtain specific permission from Elsevier. These include:

- the right to make copies of the article for your own personal use, including for your own classroom teaching use;
- the right to make copies and distribute copies (including through e-mail) of the article to research colleagues, for the personal use by such colleagues (but not commercially or systematically, e.g. via an e-mail list or list serve);
- the right to post a pre-print version of the article on Internet web sites including electronic pre-print servers, and to retain indefinitely such version on such servers or sites (see also our information on electronic preprints for a more detailed discussion on these points.);
- the right to post a revised personal version of the text of the final article (to reflect changes made in the peer review process) on the author's personal or institutional web site or server, with a link to the journal home page (on External link elsevier.com);
- the right to present the article at a meeting or conference and to distribute copies of such paper or article to the delegates attending the meeting;
- for the author's employer, if the article is a 'work for hire', made within the scope of the author's employment, the right to use all or part of the information in (any version of) the article for other intra-company use (e.g. training);
- patent and trademark rights and rights to any process or procedure described in the article;
- the right to include the article in full or in part in a thesis or dissertation (provided that this is not to be published commercially);
- the right to use the article or any part thereof in a printed compilation of works of the author, such as collected writings or lecture notes (subsequent to publication of the article in the journal); and
- the right to prepare other derivative works, to extend the article into book-length form, or to otherwise re-use portions or excerpts in other works, with full acknowledgement of its original publication in the journal.

Other uses by authors should be authorized by Elsevier through the Global Rights Department (for addresses see Obtaining Permissions), and authors are encouraged to let Elsevier know of any particular needs or requirements.

Vita

Xiaobo Li was born in Shimen, Hunan, China, in September, 1981. He received his Bachelor in Electrical Engineering degree from Central South University in 2003. From fall 2003 to spring 2005, he was with the Department of Control Theory and Engineering at Zhejiang University, China. In fall 2005, he entered the graduate program in the Department of Electrical and Computer Engineering at Louisiana State University. He received his Master of Science in Electrical Engineering degree in May 2008. Now he is a candidate for the degree of Doctor of Philosophy from Electrical and Computer Engineering at Louisiana State University.