Journal of Economic Theory **100**, 191–219 (2001)
doi:10.1006/jeth.2000.2785, available online at http://www.idealibrary.com on IDEAL®

Markov Perfect Equilibrium

I. Observable Actions

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Received August 2, 1999; final version received December 12, 2000;

We define Markov strategy and Markov perfect equilibrium (MPE) for games with observable actions. Informally, a Markov strategy depends only on payoff-relevant past events. More precisely, it is measurable with respect to the coarsest partition of histories for which, if all other players use measurable strategies, each player's decision-problem is also measurable. For many games, this definition is equivalent to a simple affine invariance condition. We also show that an MPE is generically robust: if payoffs of a generic game are perturbed, there exists an almost Markovian equilibrium in the perturbed game near the initial MPE. *Journal of Economic Literature* Classification Numbers: C72, C73. © 2001 Academic Press

Key Words: dynamic games; simple strategies; Markov perfect equilibrium; payoff-relevant histories; robustness.

1. INTRODUCTION

Strategic phenomena studied in economics and the other social sciences are often intrinsically intertemporal in nature, and therefore require a dynamic formulation. In their study of dynamic interactions, many researchers have focused on the class of strategies variously called "Markovian," "state-contingent," "payoff-relevant," or (for stationary games) "stationary." Such strategies make behavior in any period dependent on only a relatively small set of variables rather than on the entire history of play.

For example, consider a dynamic game in which, in every period t, player i's payoff π_t^i depends only on the vector of players' actions, $\mathbf{a}_t$, that period, and on the current (payoff-relevant) "state of the system" $\theta_t \in \Theta_t$. That is, $\pi_t^i = g_t^i(\mathbf{a}_t, \theta_t)$. Suppose, furthermore, that player i's possible

0022-0531/01 $35.00

actions A_t^i depend only on θ_t: $A_t^i = A_t^i(\theta_t)$ and that θ_t is determined (possibly randomly) by the previous period's actions $\mathbf{a}_{t-1}$ and state θ_{t-1}. Finally, assume that each player maximizes (the expectation of) a discounted sum of per period payoffs: $E(\sum_t \delta^{t-1}\pi_t^i)$. In period t, the history of the game, h_t, is the sequence of previous actions and states $h_t = ((\mathbf{a}_1, \theta_2), ..., (\mathbf{a}_{t-1}, \theta_t))$. But the only aspect of history that directly affects player i's payoffs and action sets starting in period t is the state θ_t. Hence, a Markov strategy in this model should make player i's period t action dependent only on the state θ_t rather than on the whole history h_t.

In this example, it is quite clear how a Markov strategy should be defined, primarily because the set of period t payoff-relevant states Θ_t is given *exogenously*. In an arbitrary dynamic game, by contrast, we must first *derive* the set of states in order to discuss Markov strategies. In the Markov literature, this has been done in largely *ad hoc* fashion: the question of which variables to include in Θ_t has been normally decided on a case-by-case basis.

We propose a general treatment. A major task of this paper and its companion [14] is to show that the payoff-relevant states and therefore the concept of an equilibrium in Markov strategies (Markov perfect equilibrium or MPE) can be defined naturally and consistently in a large class of dynamic games. In this paper we concentrate on games with observable actions,[1] in which case, the period t history h_t is known to all players before they choose their period t actions.

As we have noted the concept of MPE—a refinement of Nash equilibrium—figures prominently in applied game theory. In our view, this fact already justifies giving it greater theoretical attention. The concept's popularity stems in part from several practical considerations. First, MPE is often quite successful in eliminating or reducing a large multiplicity of equilibria in dynamic games, and thus in enhancing the predictive power of the model. Relatedly, MPE, by preventing non-payoff-relevant variables from affecting strategic behavior, has allowed researchers to identify the impact of state variables on outcomes; it for example has permitted researchers to obtain a clean, unobstructed analysis of strategic positioning in industrial organization. A second pragmatic reason for focusing on MPE is that Markov strategies substantially reduce the number of parameters to be estimated in dynamic econometric models. Such models can therefore be more easily estimated and the Markov restriction subjected to a specification test. The validity of the Markov restriction can also be assessed in experiments as in [15]. Finally, and relatedly, Markov models can readily

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[1] Such games are also called games of almost perfect information or games of perfect monitoring. They include games of perfect information, in which players move sequentially.

be simulated. Indeed Pakes, Gowrisankaran and McGuire developed a Gauss program capable of computing Markov perfect equilibria.[2]

MPE embodies three philosophical considerations besides its practical virtues. First, Markov strategies prescribe the *simplest form of behavior that is consistent with rationality*. Strategies depend on as few variables as possible; they involve no complex "bootstrapping" in which each player conditions on a particular variable only because others do the same. This is not to imply that we find bootstrap behavior uninteresting but only to suggest that its polar opposite—Markov behavior—is equally worthy of study. Indeed, a large literature on bounded rationality has developed arguing that simplicity of strategies is an important ingredient in good modeling.[3] Second, the Markov restriction captures the notion that "*bygones are bygones*" more completely than does the concept of subgame-perfect equilibrium. Markov perfection implies that outcomes in a subgame depend only on the relevant strategic elements of that subgame. That is, if two subgames are isomorphic in the sense that the corresponding preferences and action spaces are equivalent, then they should be played in the same

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[2] The program can be accessed by "ftping" to econ.yale.edu, using "anonymous" as the login, name and your user identification as the password, and retrieving all files from the directory "pub/mrkv-eqm." The authors ask those who access the program to send their name, institutional affiliation and e-mail address to "mrkv-eqm@econ.yale.edu." in order to keep track of the usefulness of their experiment and to send improvements.

[3] Ariel Rubinstein ([17], p. 912) offers a different perspective on the notion of simplicity: "Consider, for example, the sequential bargaining literature in which the authors *assume* (rather than conclude) that strategies are stationary. That is to say, a player is confined by *hypothesis* to the use of offers and response patterns (response to offers made by the other player) that are independent of the history of the game. This literature presents this stationarity assumption as an assumption of simplicity of behavior. Consider, for example, player 1's strategy: "Demand 50% of the surplus and reject any offer which gives you less than 50%, independent of what has happened in the past." Simplicity of behavior implies that player 1 plans to make the same offer and make the same responses independently of how player 2 has reacted in the past. However, this strategy also implies that player 2 believes that player 1 would demand 50% of the surplus even if player 1 demanded 60% of the surplus in the first, let us say, 17 periods of bargaining. Thus, stationarity implies not only the simplicity of player 1's behavior but also the passivity of player 2's beliefs. This is unusual, especially if we assume simplicity of behavior. If player 2 believes that player 1 is constrained to choose a stationary plan of action, then 2 should believe (after 17 repetitions of the demand of 60%) that player 1 will continue to demand 60%. Thus, assuming passivity of beliefs eliminates a great deal of what sequential games are intended to model: namely, the changing pattern in players' behavior and beliefs, as they accumulate experience." We are sympathetic to this point of view, but do not consider it to be a criticism of Markov strategies. Rubinstein's description suggests a game in which players are incompletely informed about each other's preferences. In this case, the appropriate version of the Markov concept is Markov perfect Bayesian equilibrium and not Markov perfect equilibrium. But in a Markov perfect Bayesian equilibrium of a game with incomplete information, beliefs are *not* "passive": beliefs about a player's type are updated on the basis of his or her behavior.

way.[4] Third, it embodies the principle that "*minor causes should have minor effects*;" that is, only those aspects of the past that are "significant" should have an appreciable influence on behavior.

We proceed as follows. In Section 2 we lay out the model and define the concept of Markov strategy. This definition requires players to make their strategies measurable with respect to a certain partition of possible histories. More specifically, a vector of partitions, one for each player, is *consistent* if, at each point of time, a player's preferences over his continuation strategies are the same for any history in a given element of his partition provided that the other players use strategies that are measurable with respect to their own partitions. Section 2 provides weak conditions under which, for any consistent vector of partitions, all players have the same partition. Hence we may refer to a consistent partition, rather than a consistent vector. We show that there is a unique, maximally coarse consistent partition. Strategies that are measurable with respect to this partition are called *Markovian*, and a subgame perfect equilibrium in Markov strategies is called a Markov perfect equilibrium (MPE). For multiperiod games in which the action spaces are finite in any period an MPE exists if the number of periods is finite or (with suitable continuity at infinity) infinite.

In Section 3 we show that, for a broad class of games there is a simple criterion by which we can check whether or not two date t histories h_t and h'_t lie in the same element of the Markov partition (and, therefore, whether or not a given strategy is Markovian): for each player i there must exist scalar $\alpha > 0$ and function $\beta(\cdot)$ such that for all vectors of current actions $\mathbf{a}_t = (a_t^i, \mathbf{a}_t^{-i})$, the player's utilities following histories h_t and h'_t are linked by the following von Neumann–Morgenstern (VNM) transformation,

$$u^i(h_t, f_t) = \alpha u^i(h'_t, f_t) + \beta(\mathbf{a}_t^{-i}) \qquad \text{for all } f_t,$$

where f_t corresponds to actions taken in period t and subsequently.

As mentioned above, the Markov restriction embodies the principle that "*minor causes should have minor effects*;" that is, only those aspects of the past that are "significant" should have an appreciable influence on behavior. Actually, the Markov restriction reflects this idea in a rather discontinuous way: payoff-relevant history affects behavior, payoff-irrelevant history does not, and there is nothing in between. Indeed, this discontinuity

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[4] Hellwig–Leininger ([10], p. 1) call this the *subgame-consistency principle*, i.e., "the behaviour principle according to which a player's behaviour in strategically equivalent subgames should be the same, regardless of the different paths by which these subgames might be reached." Similarly, Harsanyi–Selten ([9], p. 73) argue that "invariance with respect to isomorphisms" is "an indispensable requirement for any rational theory of equilibrium point selection that is based on strategic considerations exclusively." We compare our concept with that of Harsanyi and Selten later.

gives rise to the following concern. The restriction to Markov strategies has force only in a game in which there exists at least one pair of distinct histories that differ only according to payoff-irrelevant variables. But by perturbing the payoffs of such a game slightly, one can ensure that the histories differ payoff-relevantly. Formally, therefore, one might criticize the Markov assumption as being "generically" unimportant, since, in the perturbed game, Markov- and subgame-perfection are the same. But if one accepts the minor cause/minor effect principle (MCMEP), it is natural to focus only on those equilibria that are close to MPEs of the original game. By doing so, one guarantees that those aspects of history that had zero effect on future payoff functions in the original game—and, therefore, have only small effect on future payoffs in the perturbed game—have only a small influence on future equilibrium behavior in the perturbed game, which is very much in the spirit of MCMEP. Thus the principle helps ensure the wide applicability of Markov restrictions. Of course, to invoke MCMEP, there had better *exist* equilibria in the perturbed game near the original equilibrium. That is, a form of lower hemi-continuity had better hold. In Section 4, we confirm that this indeed is the case, at least generically.

2. MARKOV STRATEGIES AND MARKOV PERFECT EQUILIBRIUM

Let G be a game with n players (indexed by $i=1, ..., n$) and T periods (indexed by $t=1, ..., T$), where T can be finite or infinite. In every period t, each player i chooses an action a_t^i in his or her finite action space, where this space may depend on actions chosen in earlier periods. Note that although in this formulation players choose their actions simultaneously in period t, we can readily accommodate sequential games (i.e., games of perfect information) by endowing all but one player (the player who *really* moves in period t) with degenerate action spaces in period t. For convenience, we will restrict attention to games in which the timing of moves (who is active at date t) is independent of previous actions. Observe too that by treating "nature" as one of the players, we can incorporate stochastic payoffs or action sets; randomness corresponds simply to a mixed strategy by nature.

Let $\mathbf{a}_t \equiv (a_t^1, ..., a_t^n)$ and $\mathbf{a} \equiv (\mathbf{a}_1, ..., \mathbf{a}_T)$. The *history* in period t is the sequence of actions chosen before period t: $h_t \equiv (\mathbf{a}_1, ..., \mathbf{a}_{t-1})$. Let H_t be the *set of all possible period t histories*. We shall assume throughout that G is a game with *observable actions*: history h_t is common knowledge in period t. (See our companion piece, [14], for the case of unobservable actions.) The *future* in period t is the sequence of current and future actions:

$f_t \equiv (\mathbf{a}_t, ..., \mathbf{a}_T)$. We assume that players have von Neumann–Morgenstern preferences over action sequences. Thus, player i's preferences are representable by a utility function $u^i(\mathbf{a}) = u^i(h_t, f_t)$.

A (*behavior*) *strategy* s^i for player i is a function that, for all t and each history $h_t \in H_t$, assigns a probability distribution to the action space $A_t^i(h_t)$, i.e., $s^i(h_t) \in \Delta(A_t^i(h_t))$, where $\Delta(X)$ denotes the set of probability distributions on set X. Let S^i be the *set of strategies* for player i. Given t and history $h_t \in H_t$, let $S_t^i(h_t)$ denote the *set of continuation strategies*, i.e., the set of strategies in the $T-t+1$ period subgame starting after h_t. We will denote the vector of strategies by players other than i by $\mathbf{s}^{-i}$ and the vector of all strategies by $\mathbf{s} \equiv (s^i, \mathbf{s}^{-i})$.

Given strategy vector $\mathbf{s}$, player i's *expected utility* is $v^i(\mathbf{s}) \equiv E_{\mathbf{a}}(u^i(\mathbf{a}) \mid \mathbf{s})$. His or her *expected utility conditional on history* h_t is denoted by

$$v^i(\mathbf{s} \mid h_t) \equiv E_{f_t}(u^i(h_t, f_t) \mid \mathbf{s}).$$

That is, $v^i(\mathbf{s} \mid h_t)$ is player i's expected payoff if, after history h_t, players behave according to $\mathbf{s}$.

A *subgame-perfect equilibrium* ([18]) is a strategy vector $\mathbf{s}$ that forms a Nash equilibrium after any history; i.e., for all t, $h_t \in H_t$, and i,

$$v^i(s^i, \mathbf{s}^{-i} \mid h_t) \geqslant v^i(\hat{s}^i, \mathbf{s}^{-i} \mid h_t),$$

for any alternative strategy $\hat{s}^i$. Subgame-perfect equilibrium (SPE) refines Nash equilibrium by ruling out empty or incredible threats.

For all t, let $H_t(\cdot)$ denote a *partition* of H_t, where $H_t(h_t)$ ($\subseteq H_t$) denotes the set of period t histories that are in the same element of the partition as h_t. Let $H.(\cdot)$ denote a *collection of partitions* $\{H_t(\cdot)\}_{t=1}^T$. We shall call collection $H'.(\cdot)$ *weakly coarser* (*weakly finer*) than collection $H.(\cdot)$, if, for all t, either $H'_t(\cdot)$ is coarser (finer) than $H_t(\cdot)$ or $H_t(\cdot) = H'_t(\cdot)$. [$H'_t(\cdot)$ is coarser than $H_t(\cdot)$ if every element of the latter is contained in some element of the former and $H'_t(\cdot) \neq H_t(\cdot)$; in that case, we also say that $H_t(\cdot)$ is finer than $H'.(\cdot)$.] We shall call $H'.(\cdot)$ *strictly coarser* (*strictly finer*) than $H.(\cdot)$ if it is weakly coarser (weakly finer) and, for some t, $H'_t(\cdot)$ is coarser (finer) than $H_t(\cdot)$.

For all i, let $\bar{H}.(\cdot)$ be the collection of players' *action-space-invariant partitions*. That is, for all t and all h_t, $h'_t \in H_t$, $h'_t \in \bar{H}_t(h_t)$ if and only if $S_t^i(h_t) = S_t^i(h'_t)$ for all i.[5] If the collection $H^i.(\cdot)$ is weakly finer than $\bar{H}.(\cdot)$, then strategy s^i is *measurable* with respect to $H^i.(\cdot)$ if, for all t and for all

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[5] Or, equivalently,

$$\forall \tau \geqslant 0, \qquad \forall (\mathbf{a}_t, ..., \mathbf{a}_{t+\tau-1}),$$
$$A_{t+\tau}^i(h_t, \mathbf{a}_t, ..., \mathbf{a}_{t+\tau-1}) = A_{t+\tau}^i(h'_t, \mathbf{a}_t, ..., \mathbf{a}_{t+\tau-1}) \qquad \text{for all } i.$$

h_t, $h'_t \in H_t$, $s^i(h'_t) = s^i(h_t)$ whenever $h'_t \in H^i_t(h_t)$ (we require $H^i_\cdot(\cdot)$ to be weakly finer than $\bar{H}_\cdot(\cdot)$ in the definition of measurability because otherwise setting $s^i(h'_t) = s^i(h_t)$ may not even be feasible). Let $S^i(H^i_\cdot(\cdot))$ be the set of all strategies for player i that are measurable with respect to $H^i_\cdot(\cdot)$.

For any finite set B and two real-valued functions f, f': $B \to \mathbb{R}$, we shall write $f \sim f'$ if one function is a positive affine transformation of the other, i.e., there exist $\alpha > 0$ and β such that $f(\cdot) = \alpha f'(\cdot) + \beta$.

We shall call the vector of collections $(H^1_\cdot(\cdot), ..., H^n_\cdot(\cdot))$ *consistent* if, for all i, (a) $H^i_\cdot(\cdot)$ is weakly finer than $\bar{H}_\cdot(\cdot)$, and (b) if all other players j use strategies that are measurable with respect to their collections $H^j_\cdot(\cdot)$, then, after any two period t histories lying in the same element of $H^i_t(\cdot)$, player i's preferences over his continuation strategies are the same, i.e.,

$$\begin{gathered}\text{for all}\quad \mathbf{s}^{-i} \in \bigtimes_{k \neq i} S^k(H^k_\cdot(\cdot)), \qquad \text{for all } t,\\ \text{for all}\quad h_t, h'_t \in H_t \qquad \text{such that} \quad h'_t \in H^i_t(h_t), \\ v^i(\cdot, \mathbf{s}^{-i} \mid h_t) \sim v^i(\cdot, \mathbf{s}^{-i} \mid h'_t).\end{gathered} \tag{1}$$

It is in general possible that not all players share the same collection of partitions in a consistent vector. For instance, consider the following three-player game.

EXAMPLE 1. Player 1 moves first and chooses from $\{A, B, C\}$. Then players 2 and 3 move simultaneously and choose from $\{T, B\}$ and $\{L, R\}$ respectively. The payoffs are summarized in Fig. 1.

Notice that the pair $(H^2_2(\cdot), H^3_2(\cdot)) = (\{\{A, B\}, \{C\}\}, \{\{A\}, \{B, C\}\})$ constitutes a consistent vector (we can omit player 1 from the vector, because no history has yet occurred when he moves): Player 2's decision problem is the same whether A or B has occurred but it is different if C has occurred; player 3's decision problem is the same whether B or C has occurred but differs when A has occurred. Note, in particular, that $H^2_2(\cdot) \neq H^3_2(\cdot)$.

		Player 3	
		L	R
Player 2	T	0,1,1	0,1,0
	B	1,0,1	1,0,0

Player 1
chooses A

		Player 3	
		L	R
Player 2	T	0,1,0	0,1,1
	B	1,0,0	1,0,1

Player 1
chooses B

		Player 3	
		L	R
Player 2	T	1,0,0	1,0,1
	B	0,1,0	0,1,1

Player 1
chooses C

FIGURE 1

But example 1 is special in the sense that players 2 and 3 do not "interact": player 2's actions do not affect player 3's preferences between L and R, nor vice versa. In the "nondegenerate" case in which all pairs of players moving in the same period *do* interact—the case that subsumes most economic games of interest—it turns out (as shown in Theorem 2.1) that, in any consistent vector, all players have the *same* collection.

For any i and j, $h_t \in H_t$, strategy $\mathbf{s} \in \times S^k(\bar{H}.(\cdot))$, $a_t^i \in A_t^i(h_t)$, and $a_t^j \in A_t^j(h_t)$, let

$$w^i(a_t^i; h_t, \mathbf{s}, a_t^j) \equiv v^i(\mathbf{s} \mid h_t, a_t^i, a_t^j),$$

i.e., $w^i(\cdot\,; h_t, \mathbf{s}, a_t^j)$ represents player i's preferences over period t actions a_t^i, given that the history is h_t, player j's period t action is a_t^j, and all other players' period t actions and all players' future actions starting in period $t+1$ are determined by $\mathbf{s}$.

Call a game G *simultaneous-nondegenerate* if for all i, for all t in which player i moves (i.e., in which his or her action space is not a singleton),[6] for all h_t, $h_t' \in H_t$ such that $h_t' \in \bar{H}_t(h_t)$, and for all $j \neq i$ such that player j moves in period t, there exist strategy vector $\mathbf{s} \in \times S^k(\bar{H}.(\cdot))$ and actions a_t^j, $a_t^{j\prime} \in A_t^j(h_t) = A_t^j(h_t')$ such that

$$w^i(\cdot\,; h_t, \mathbf{s}, a_t^j) \not\sim w^i(\cdot\,; h_t', \mathbf{s}, a_t^{j\prime}). \tag{2}$$

In words, the game is simultaneous-nondegenerate if, in any period and given any two histories h_t and h_t' and any active player i, any other active player j moving simultaneously can ensure that player i's decision problem after h_t differs from that after h_t', holding some future sequence of random actions fixed. [It may not actually be possible to hold all future actions fixed, because what actions are feasible may depend on whether (h_t, a_t^j) or $(h_t', a_t^{j\prime})$ has occurred. Hence we do the next best thing by making $\mathbf{s}$ —which determines future actions—measurable with respect to the coarsest possible partition consistent with feasibility.]

THEOREM 2.1. *Let G be simultaneous-nondegenerate. If $(H_\cdot^1(\cdot), \ldots, H_\cdot^n(\cdot))$ is a consistent vector of collections, then, for all t, if players i and j both move in period t,*

$$H_t^i(\cdot) = H_t^j(\cdot).$$

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[6] Earlier we mentioned that one way to treat randomness is to introduce an additional player, "nature". If this device is invoked, however, we must exclude this artificial player from the nondegeneracy condition. The same applies to the two other nondegeneracy conditions that follow below.

Proof. Suppose that players i and j move in period t. Choose $h_t, h'_t \in H_t$ such that $h'_t \in H^i_t(h_t)$. We must show that $h'_t \in H^j_t(h_t)$. From nondegeneracy, there exist a^j_t, $a^{j\prime}_t$ and vector $\mathbf{s} \in \times S^k(\bar{H}_\cdot(\cdot))$ satisfying (2). If $h'_t \notin H^j_t(h_t)$, then player j's strategy of playing a^j_t after any history in $H^j_t(h_t)$, and $a^{j\prime}_t$ after any history in $H^j_t(h'_t)$ and otherwise playing according to s^j is measurable with respect to $H^j_\cdot(\cdot)$ (since $\bar{H}_\cdot(\cdot)$ is at least as coarse as $H^j_\cdot(\cdot)$). Hence, (2) implies a violation of (1). We conclude that $h'_t \in H^j_t(h_t)$, as claimed. ∎

Except for the second half of Theorem 2.3, we will henceforth restrict attention to simultaneous-nondegenerate games and therefore, in view of Theorem 2.1, need deal with only a (single) *consistent collection* $H^\circ_\cdot(\cdot)$ rather than with a consistent vector of collections.

Intuitively, we would expect that if players' strategies are measurable with respect to a consistent collection $H^\circ_\cdot(\cdot)$ and they do not distinguish between histories h_t and h'_t in period t (i.e., $h'_t \in H^\circ_t(h_t)$), then these strategies should not distinguish between h_t and h'_t in subsequent periods either. That this "successive coarsening" property need not hold, however, is illustrated by the following game.

Example 2. Player 1 moves first and chooses U or D. Then player 2 moves and chooses L or R. Finally player 3 moves and chooses L or R. The payoffs are as in Fig. 2. Notice that $H^\circ_2(\cdot) = \{U, D\}$ and $H^\circ_3(\cdot) = \{\{UL, UR\}, \{DL, DR\}\}$ constitute a consistent collection but violate the successive coarsening property: player 2's decision problem does not depend on which of U or D occurred, but player 3's decision problem *does* depend on the choice of U or D.

Still, example 2 is degenerate in a way that is similar to example 1: player 3 has no effect on player 2's decision problem. Accordingly, call a game *backward sequential-nondegenerate* if, for all i, for all t in which player i moves (his or her action set is not a singleton), for all $h_t, h'_t \in H_t$ such that $h'_t \in \bar{H}_t(h_t)$, for all $\mathbf{a}_t \in A_t(h_t)$, and for all $j \neq i$ such that j moves in period

		Player 2	
		L	R
Player 1	U	0, 1, 0	1, 0, 0
	D	1, 1, 1	0, 0, 1

Player 3 chooses L

		Player 2	
		L	R
Player 1	U	0, 1, 1	1, 0, 1
	D	1, 1, 0	0, 0, 0

Player 3 chooses R

FIGURE 2

$t+1$ there exist a^j_{t+1}, $a^j_{t+1}{}'$ and $\mathbf{s} \in \times S^k(\bar{H}.(\cdot))$ such that $a^j_{t+1}, a^j_{t+1}{}' \in A^j_{t+1}(h_t, \mathbf{a}_t) = A^j_{t+1}(h'_t, \mathbf{a}_t)$ and

$$w^i(\cdot\,; h_t, \mathbf{a}_t^{-i}, a^j_{t+1}, \mathbf{s}) \not\sim w^i(\cdot\,; h'_t, \mathbf{a}_t^{-i}, a^j_{t+1}{}', \mathbf{s}), \tag{3}$$

where, analogous to before, $w^i(\cdot\,; h_t, \mathbf{a}_t^{-i}, a^j_{t+1}, \mathbf{s})$ denotes player i's preferences over a^i_t, given that the history is h_t, other players' period t actions are $\mathbf{a}_t^{-i}$, player j's period $t+1$ action is a^j_{t+1}, and the non-j players' period $t+1$ actions and all players' actions starting in period $t+2$ are determined by $\mathbf{s}$. In words, the game is backward sequential-nondegenerate if in any period t, any active player i's period t decision problem can be affected by the action of any other active player j moving in period $t+1$, for some fixed choice of other actions (here we make the same qualification about fixing other actions as after our definition of simultaneous nondegeneracy.) It is readily verified that most economic applications of Markov equilibrium that have been considered in the literature are backward sequential-nondegenerate.

We now show that the successive coarsening property holds in games satisfying the two nondegeneracy conditions defined thus far.

THEOREM 2.2. *Let $H^\circ_\cdot(\cdot)$ be a consistent collection in a simultaneous- and backward sequential-nondegenerate game. If, for some t and $h_t, h'_t \in H_t$, we have $h'_t \in H^\circ_t(h_t)$, then, for any $\mathbf{a}_t \in A_t(h_t)$ $(=A_t(h'_t))$,*

$$(h'_t, \mathbf{a}_t) \in H^\circ_{t+1}(h_t, \mathbf{a}_t). \tag{4}$$

Proof. Consider h_t, h'_t and $\mathbf{a}_t$ as in the statement of the theorem. Choose player i who moves in period t and j who moves in period $t+1$. Backward sequential-nondegeneracy implies that there exist a^j_{t+1}, $a^j_{t+1}{}'$ and $\mathbf{s}$ satisfying (3). Now the strategy by each player $k \neq i, j$ to play a^k_t in period t and subsequently according to s^k is obviously measurable with respect to $H^\circ_\cdot(\cdot)$. If $(h'_t, \mathbf{a}_t) \notin H^\circ_{t+1}(h_t, \mathbf{a}_t)$, then the strategy for player j of playing a^j_{t+1} in period $t+1$ (followed by s^j) after any history $h_{t+1} \in H^\circ_{t+1}(h_t, \mathbf{a}_t)$ and otherwise playing $a^j_{t+1}{}'$ (followed by s^j) is also measurable with respect to $H^\circ_{t+1}(\cdot)$. But then (3) violates $h'_t \in H^\circ_t(h_t)$. We conclude that (4) holds after all.[7] ∎

Define a consistent collection $H^\circ_\cdot(\cdot)$ to be *maximally coarse* if there exists no other consistent collection $H^{\circ\circ}_\cdot(\cdot)$ that is strictly coarser than $H^\circ_\cdot(\cdot)$. From this definition, it may appear as though there could exist a multiplicity of maximally coarse consistent collections (since the relation "coarser

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[7] In the proof of this theorem, the only use of simultaneous nondegeneracy is in its implication, given consistency, that all players partition history the same way. Were we to drop the hypothesis, the theorem would still hold for a consistent *vector* of collections.

than" is only a partial ordering), but in fact there can be only one. The intuition for this uniqueness result is that the coarsening operation exhibits a form of complementarity across players: If all other players switch to a coarser partition, the remaining player can use a partition that is weakly coarser than before (i.e., at worst the same as before and possibly strictly coarser), as he or she "needs to remember less." Hence, if $H_t^\circ(\cdot)$ and $H_t^{\circ\circ}(\cdot)$ are both maximally coarse consistent partitions, and all players but i make their strategies measurable with respect to the finest common coarsening $H_t^\circ(\cdot) \wedge H_t^{\circ\circ}(\cdot)$, player i can do the same. Hence $H_t^\circ(\cdot)$ and $H_t^{\circ\circ}(\cdot)$ could not have been maximally coarse after all.

THEOREM 2.3. (i) *If a game is simultaneous-nondegenerate, then there exists a unique maximally coarse consistent collection* $H_\cdot^*(\cdot)$. $H_t^*(h_t)$ *constitutes the state of the system or the payoff-relevant history.*

(ii) *If we do not impose simultaneous-nondegeneracy, there is a unique maximally coarse consistent vector of collections* $H_\cdot^{i*}(\cdot)$, $i = 1, ..., n$.

Proof. (i) Let Σ be the set of all consistent collections. Notice that Σ is nonempty because it includes the collection in which each period t history h_t is in its own separate partition element. Define $H_\cdot^*(\cdot)$ so that, for all t,

$$H_t^*(\cdot) = \bigwedge_{H_\cdot^\circ(\cdot) \in \Sigma} H_t^\circ(\cdot).$$

That is, $H_t^*(\cdot)$ is the finest common coarsening (i.e., the *meet*) of all partitions $H_t^\circ(\cdot)$ for which the corresponding collection $H_\cdot^\circ(\cdot)$ is consistent. Note that because H_t is finite, $H_t^*(\cdot)$ is the meet of only finitely many distinct partitions (even though there may be infinitely many consistent collections if T is infinite). We claim that $H_\cdot^*(\cdot)$ is consistent. To see this, choose i, t, $\mathbf{s}^{-i} \in \times_{j \neq i} S^j(H_\cdot^*(\cdot))$, h_t, $h'_t \in H_t$ such that $h'_t \in H_t^*(h_t)$. By definition of $H_\cdot^*(\cdot)$ there exists a sequence $\{h_t(1), ..., h_t(m)\}$ of period t histories such that $h_t(1) = h_t$, $h_t(m) = h'_t$, and, for all $k = 1, ..., m-1$, there exists a consistent collection ${}^kH_\cdot^\circ(\cdot)$ such that

$$h_t(k+1) \in {}^kH_t^\circ(h_t(k)). \tag{5}$$

Because $\mathbf{s}^{-i}$ is measurable with respect to ${}^kH_\cdot^\circ(\cdot)$, (1) and (5) imply that, for all k,

$$v^i(\cdot, \mathbf{s}^{-i} \mid h_t(k)) \sim v^i(\cdot, \mathbf{s}^{-i} \mid h_t(k+1)). \tag{6}$$

Hence (6) continues to hold when $h_t(k)$ is replaced by h_t and $h_t(k+1)$ by h'_t. Therefore $H_\cdot^*(\cdot)$ is consistent, as claimed.

Now, by construction, $H_t^*(\cdot)$ is at least as coarse as any other consistent collection $H_t^\circ(\cdot)$. Therefore, $H_t^*(\cdot)$ is uniquely maximally coarse.

(ii) The proof when we drop simultaneous-nondegeneracy is completely analogous. ▌

We shall define a strategy to be *Markovian* if it is measurable with respect to $H_t^*(\cdot)$. That is, $\mathbf{s}$ is Markovian if, for all t, $\mathbf{s}(h_t) = \mathbf{s}(h'_t)$ whenever $h'_t \in H_t^*(h_t)$ (in which case we say that h_t and h'_t are *Markov-equivalent*). We call $H_t^*(\cdot)$ the *Markov collection* of partitions. Hence, Markov strategies are the simplest strategies (i.e., the strategies measurable with respect to the coarsest partition and hence dependent on the fewest variables) that are consistent with rationality in the sense that, if the other players make their strategies measurable with respect to some coarser partition $\hat{H}_t(\cdot)$, it would *not* always be optimal for a player to make his or her choice between any two given continuation strategies measurable with respect to $\hat{H}_t(\cdot)$. Note that, if one's measure of a strategy's complexity is the number of states an automaton requires to execute it, then the Markovian strategies are also the least complex strategies consistent with rationality.

Recall the simple example at the beginning of this paper. In that example, a Markov strategy was measurable with respect to the state space Θ_t. Notice that this is precisely the same conclusion that we would draw using the general concept of Markov strategy just defined. We shall define a *Markov Perfect Equilibrium* (MPE) to be a subgame perfect equilibrium in which all players use Markov strategies.

Continuous action spaces. We can readily extend the above analysis to games with continuous action spaces. Indeed there is no change in any of the arguments except that it may no longer be clear that maximally coarse consistent partitions exist. To establish existence, consider a sequence of consistent partitions $H_t^1(\cdot)$, $H_t^2(\cdot)$, ... such that, for all m, $H_t^{m+1}(\cdot)$ is coarser than $H_t^m(\cdot)$. Then for each element $e_t^1 \in H_t^1(\cdot)$ we can find elements $e_t^m \in H_t^m(\cdot)$, $m = 2, 3, \ldots$ such that $e_t^1 \subseteq e_t^2 \subseteq \cdots$. Define $H_t^\infty(\cdot)$ so that for each $e_t^1 \in H_t^1(\cdot)$, the corresponding element of $H_t^\infty(\cdot)$ is $\bigcup_{m=1}^\infty e_t^m$. $H_t^\infty(\cdot)$ is evidently a partition (by construction its elements are collectively exhaustive; to see that they are mutually exclusive, note that if $h_t \in (\bigcup_{m=1}^\infty e_t^m) \cap (\bigcup_{m=1}^\infty \hat{e}_t^m)$ then there exists m such that $h_t \in e_t^m \cap \hat{e}_t^m$ which implies that $e_t^m = \hat{e}_t^m$ and so $\bigcup_{m=1}^\infty e_t^m = \bigcup_{m=1}^\infty \hat{e}_t^m$) and consistent. Moreover it is coarser than any partition in the sequence. Hence, Zorn's Lemma implies that maximally coarse consistent partitions exist.

Stationary strategies. Many economic models entail games that are *stationary* in the sense that they "look the same" starting in any period, i.e., they do not depend on calender time. (Clearly, such games must have infinite horizons). For these games it is natural to make Markov strategies independent of calendar time as well. To capture this independence, we shall say that two histories h_t and $h'_{t'}$ are action-space equivalent provided that $\forall \tau \geqslant 1$, $\forall(\mathbf{a}_1, ..., \mathbf{a}_{\tau-1})$, $\forall i$, $A^i_{t+\tau}(h_t, \ \mathbf{a}_1, ..., \ \mathbf{a}_{\tau-1}) = A^i_{t'+\tau}(h'_{t'}, \ \mathbf{a}_1, ..., \ \mathbf{a}_{\tau-1})$ (we are slightly abusing notation here). One then generalizes the notion of partition by allowing two histories, h_t and $h'_{t'}$, at two different dates to belong to the same element of a partition. The vector of partitions $(H^1(\cdot), ..., H^n(\cdot))$ is *consistent* if given that all other players j play strategies that are measurable with respect to their partitions $H^j(\cdot)$ then player i's preferences over his or her continuation strategies are the same for all t, t', h_t, and $h'_{t'}$ such that $h_t \in H^i(h'_{t'})$. That is, condition (1) remains the same except that elements of a partition can encompass histories at different dates. One then proceeds as previously to define the *stationary* partition as the maximally coarse consistent partition. A strategy is *stationary* provided that it is measurable with respect to the stationary equivalence classes, i.e.,

$$s^i(h_t) = s^i(h'_{t'}) \qquad \text{if } h_t \text{ and } h'_{t'} \text{ are stationary-equivalent.}$$

Note that in "cyclical" games—in which the payoff structure in period t is the same as that in period $t + km$ ($k = 1, 2, 3, ...$), where m is the length of the cycle—stationary strategies will, in general, depend on "cyclical time" (i.e., the time from the beginning of the last cycle), rather than on calendar time.[8]

Establishing the existence of stationary perfect equilibria in stationary games that are continuous at infinity and have finite action spaces each period is not a difficult matter: Constrain each player to choose the same behavioral strategy each time he or she moves. Require also that this strategy be measurable with respect to the Markov partition. If we introduce some (Markov-measurable) trembles, then the limit of the fixed points as the trembles go to zero will be a stationary perfect equilibrium. This proof is actually borrowed from the stochastic games literature, which aims (among other things) at demonstrating existence of equilibria in "state space strategies", where the "state" is defined in some arbitrary manner. The proofs apply in particular to the situation in which the state space is our Markovian state space.

Existence. Proving existence of MPE for finite action spaces follows the standard lines (see our discussion paper [13] for more detail.) Assume

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[8] Cyclical games are not quite stationary according to our informal definition of "looking the same starting in any period." However, they become stationary if we adopt the perspective that an entire cycle corresponds to a period. Other "modifications at the margin" of the Markov concept can be envisioned as well. For example, one might require that two action sets that are identical up to a relabelling be treated as the same action set.

that, for all i, t, and h_t, $A_t^i(h_t)$ is finite. In the case of an infinite horizon game, assume further that the game is continuous at infinity (this condition is satisfied if, for example, players discount future payoffs at a constant rate $r > 0$). Then there exists a Markov perfect equilibrium.

To prove existence for a finite horizon game, one can, as usual, work backwards, and select the same Nash equilibrium for all histories in an equivalence class so as to obtain Markov measurability. An instructive method of proof in the infinite horizon case consists of taking the limit of (finite horizon) MPEs of truncated games. By standard arguments (see [6]), this limit is a perfect equilibrium. That this limit equilibrium is also Markovian results roughly speaking from the fact that at a fixed date t players "should remember more" if the horizon is longer. That is, the partition of histories into equivalence classes in the limiting game is *finer* than that in any game along the convergent subsequence (i.e., if h_t and h'_t are equivalent in the limiting game, they are equivalent for any game in the subsequence). In Section 4 we shall exhibit an example of a different sort of sequence of games in which the limiting partition is *coarser* than those in the sequence (i.e., if h_t and h'_t are equivalent for some game in the sequence, the same is true in the limiting game.) As we shall see, this coarseness implies that the limit of MPEs need not be Markovian.

3. A SIMPLE CRITERION FOR MARKOV STRATEGIES

We have defined Markov strategies as those that are measurable with respect to the maximally coarse consistent partition (the Markov partition). Although, we would argue, this is the right definition *conceptually*, it is a bit cumbersome *practically*. How does one go about finding this partition? We claim, however, that for a broad class of games there is a pair of readily checked conditions that enable us to determine whether or not two histories belong to the same element of the Markov partition.

Let $H_\cdot^{**}(\cdot)$ be the collection defined so that, for all t and for all h_t, $h'_t \in H_t$, $h'_t \in H_t^{**}(h_t)$ if and only if

(i) $\bar{H}_t(h_t) = \bar{H}_t(h'_t)$

and

(ii) for all i there exist scalar $\alpha > 0$ and function β: $A_t^{-i}(h_t) \to \mathbb{R}$ such that

$$u^i(h'_t, f_t) = \alpha u^i(h_t, f_t) + \beta(\mathbf{a}_t^{-i}) \qquad \text{for all } f_t. \tag{7}$$

Note that (i) requires us only to verify that action spaces following h_t and h'_t are the same, whereas (ii) simply involves checking that continuation utility functions are appropriate affine transformations of one another. Indeed, condition (ii) may be viewed as a multiplayer extension of the familiar von Neumann–Morgenstern invariance condition for one-person decision-problems. Note also that since the goal of this section is to find a criterion for determining whether or not strategies are Markovian that does not require actually computing the Markov partition, the criterion essentially has to be expressed in terms of "constant" (i.e., uncontingent) strategies, since only those strategies are assured of being Markovian regardless of what the Markov partition turns out to be.

We first show that, in general, $H_t^{*}(\cdot)$ is at least as coarse as $H_t^{**}(\cdot)$ (Theorem 3.1). To illuminate when the two partitions are, in fact, equal, we then introduce an auxiliary partition $H_t^{***}(\cdot)$ such that $h'_t \in H_t^{***}(h_t)$ if and only if

(i) $\bar{H}_t(h_t) = \bar{H}_t(h'_t)$, and

(ii) for all i there exist scalar $\alpha > 0$ and function $\gamma\colon \times_{\tau = t, \ldots, T} A_\tau^{-i}(h_\tau) \rightarrow \mathbb{R}$ such that

$$u^i(h'_t, f_t) = \alpha u^i(h_t, f_t) + \gamma(f_t^{-i}) \qquad \text{for all } f_t. \tag{8}$$

It can be shown, (see [11]), that if $h'_t \in H_t^{*}(h_t)$, then (i) and (ii) are satisfied (Theorem 3.2). Hence, $H_t^{***}(\cdot)$ is always at least as coarse as $H_t^{*}(\cdot)$. (Example 3 below illustrates that $H_t^{***}(\cdot)$ can be strictly coarser). However, provided that the decision problem of any player moving in period $t+1$ can be affected by any player moving in period t (i.e., forward sequential nondegeneracy holds) and a fairly mild indifference condition holds, we can show, using $H_t^{***}(\cdot)$, that $H_t^{*}(\cdot) = H_t^{**}(\cdot)$.

We first show that for simultaneous-nondegenerate games, the "coarser than" relation always holds in one direction: $H_{\cdot}^{*}(\cdot)$ is at least as coarse as $H_{\cdot}^{**}(\cdot)$. The proof of this theorem as well as that of Theorem 3.3 are in the Appendix.[9]

THEOREM 3.1. *Suppose that a game is simultaneous-nondegenerate. For all t, the Markov partition* $H_t^{*}(\cdot)$ *is at least as coarse as* $H_t^{**}(\cdot)$.

Consider now the following coarsening of $H_{\cdot}^{**}(\cdot)$. Let $H_{\cdot}^{***}(\cdot)$ be the collection defined so that, for all t and for all $h_t, h'_t \in H_t$, $h'_t \in H_t^{***}(h_t)$ if and only if (i) and (ii) above hold. Notice that (8) is the same as (7) except

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[9] The reader will note that the proofs in this section often make reference to paths of actions. That is, we will refer to preferences over strategies s_t^i inducing constant action paths f_t^i. It so happens that we can prove the theorems appealing only to paths of actions.

		Player 2's move in period $t+1$	
		L	R
Player 1's move in period t	T	2, 1	1, 0
	B	3,0	0, 1

(a) Payoffs conditioned on h_t

		Player 2's move in period $t+1$	
		L	R
Player 1's move in period t	T	0,1	2,0
	B	1, 0	1,1

(b) Payoffs conditioned on h'_t

FIGURE 3

that $\gamma(\cdot)$ may depend on other players' *entire* future, whereas $\beta(\cdot)$depends only on other players' actions in period t.

THEOREM 3.2. *$H^{***}(\cdot)$ is at least as coarse as $H^{*}(\cdot)$.*

Proof. For i, t, h_t and h'_t such that $h'_t \in H^{*}_t(h_t)$, and any random actions $\tilde{f}_t^{-i}$, the consistency of $H^{*}(\cdot)$ implies that player i's preferences over f^i_t given $\tilde{f}_t^{-i}$ are independent of whether the history is h_t or h'_t.

We can think of $\tilde{f}_t^{-i}$ as a random variable representing the state of nature. We are, therefore, in the realm of state-dependent preferences, and can thus appeal to Karni, Schmeidler, and Vind [11] to conclude that there exist $\alpha > 0$ and $\gamma(\cdot)$ such that (8) holds.[10] Hence $h'_t \in H^{***}_t(h_t)$. ▮

In general, $H^{***}(\cdot)$ is actually strictly coarser then $H^{*}(\cdot)$. To see this, consider the following two-person game:

EXAMPLE 3. Player 1 moves in period t and player 2 in period $t+1$ and conditional on history h_t payoffs are as in Fig. 3a, whereas conditional on history h'_t payoffs are as in Fig. 3b.

Notice that, whether h_t or h'_t has occurred, it is optimal for player 2 to play L if player 1 has played T and to play R if player 1 has played B. However, $h'_t \notin H^{*}_t(h_t)$ because, given that player 2 behaves in this way, player 1 prefers T to B after h_t but B to T after history h'_t. Nevertheless $h'_t \in H^{***}_t(h_t)$ since if

$$\gamma(a^2_{t+1}) = \begin{cases} -2, & \text{if } \ a^2_{t+1} = L \\ 1, & \text{if } \ a^2_{t+1} = R \end{cases}$$

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[10] There is no difficulty in associating a state with $\tilde{f}_t^{-i}$: for each act s^i, the objective uncertainty corresponding to $\tilde{f}_t^{-i}$ is just the lottery over payoffs (prizes) $v^i(s^i, \tilde{f}_t^{-i} \mid h_t)$. With the appropriate change of notation, Theorem 3.2 is nothing more than the Karni–Schmeidler–Vind representation theorem.

we have

$$u^1(h_t', a_t^1, a_{t+1}^2) = u^1(h_t, a_t^1, a_{t+1}^2) + \gamma(a_{t+1}^2)$$

for all a_t^1, a_{t+1}^2.

This example illustrates the idea that if we permit γ to depend on f_{t+1}^{-i} then, indirectly, it can depend on a_t^i (in which case there is no reason to expect $h_t' \in H_t^*(h_t)$) since players other than i may condition their behavior in period $t+1$ (or after) on a_t^i. This suggests that, in a sufficiently "rich" game (such as the example), we will have to restrict γ to depend only on $\mathbf{a}_t^{-i}$ (as $H_t^{**}(\cdot)$ requires) in order to obtain consistency.

To capture the idea of richness, call a game *forward sequentially-nondegenerate* if for all j, all $t+1$ in which j moves, all $h_t \in H_t$, all $\mathbf{s} \in \mathbf{S}(\bar{H}.(\cdot))$ and all $i \neq j$ such that i moves in period t, there exist a_t^i, $a_t^{i\prime} \in A_t^i(h_t)$ such that

$$w^j(\cdot\,; h_t, a_t^i, \mathbf{s}) \not\sim w^j(\cdot\,; h_t, a_t^{i\prime}, \mathbf{s}), \tag{9}$$

where the argument of w^j is player j's date $t+1$ action.

In words, the game is forward sequentially-nondegenerate if in any period $t+1$, any active player j's period $t+1$ decision problem can be affected by the action of any player i moving in period t, for any choice of others' actions. To see that this property is satisfied in Example 3, observe that player 1 does indeed affect player 2's ranking between L and R.

THEOREM 3.3. *Suppose that a game is simultaneous-nondegenerate and forward sequentially-nondegenerate. Suppose, furthermore, that for all t, all* $h_t, h_t' \in H_t$, *all players i who move in period t, all feasible* $\tilde{f}_t^{-i}$ *and* $\tilde{f}_t^{-i\prime}$, *there exist* $\tilde{f}_t^i$ *and* $\tilde{f}_t^{i\prime}$ *such that* $u^i(h_t, \tilde{f}_t^i, \tilde{f}_t^{-i}) = u^i(h_t', \tilde{f}_t^{i\prime}, \tilde{f}_t^{-i\prime})$. *Then* $H_\cdot^*(\cdot) = H_\cdot^{**}(\cdot)$.

Remark. The last hypothesis in Theorem 3.3 is the fairly mild requirement that, given histories h_t and h_t' (with fixed plays by other players corresponding to each history) player i has a way of playing after each history such that his or her overall payoff is the same in either case. (Note that this condition can be satisfied in *mixed* strategies). The condition can alternatively be formulated as follows: $\forall h_t$ and h_t', $\forall \tilde{f}_t^{-i}$ and $\tilde{f}_t^{-i\prime}$, $\{u^i \mid \text{there exists } \tilde{f}_t^i \text{ such that } u^i = u^i(h_t, \tilde{f}_t^i, \tilde{f}_t^{-i})\} \cap \{u^i \mid \text{there exists } \tilde{f}_t^{i\prime} \text{ such that } u^i = u^i(h_t', \tilde{f}_t^{i\prime}, \tilde{f}_t^{-i\prime})\}$ is a nondegenerate interval. Expressed this way, it is obviously an *open* condition (as are the other two hypotheses of Theorem 3.3), in the sense that if it is satisfied by game $\mathbf{u}$, then it is also satisfied by all games in an open neighborhood of $\mathbf{u}$.

	L	R
T	1, 1, 0	1, 0, 0
M	1, 1, 0	1, 5, 0
D	1, 3, 0	0, 4, 0

Player 1 chooses A

	L	R
T	1, 2, 0	1, 0,0
M	0, 2, 0	1, 5, 0
D	1, 4, 0	1, 4, 0

Player 1 chooses B

	L	R
T	1, 1, 0	1, 0,0
M	1, 1, 0	1, 5, 0
D	1, 3, 0	1, 4, 0

Player 1 chooses C

FIGURE 4

Theorem 3.3 amounts essentially to establishing that $\gamma(f_t^{-i})$ in (8) is independent of f_{t+1}^{-i}. The argument makes important use of forward sequential-nondegeneracy. To see what happens when this condition fails, consider the following example.

EXAMPLE 4. Player 1 moves first and chooses from $\{A, B, C\}$. Player 2 then chooses from $\{T, M, D\}$. Finally, player 3 chooses from $\{L, R\}$. The payoffs are as in Fig. 4. Notice that the game is simultaneous- and backward sequentially-nondegenerate and satisfies the indifference condition in Theorem 3.3. However, it is not forward sequentially-nondegenerate because all period 3 histories belong to the same (unique) element of $H_3^*(\cdot)$, and hence condition (9) cannot be satisfied for player 3. Indeed, even though $B \in H_2^*(A)$, we do *not* have $B \in H_2^{**}(A)$. Instead,

$$u^2(B, a_2, a_3) = u^2(A, a_2, a_3) + \gamma(a_3) \qquad \text{for all } (a_2, a_3),$$

where

$$\gamma(a_3) = \begin{cases} 1, & \text{if } \ a_3 = L \\ 0, & \text{if } \ a_3 = R. \end{cases}$$

Because γ depends on period 3 actions, it violates (7)

Remark 1. With one qualification discussed below, our Markov concept is stronger than the invariance with respect to isomorphisms proposed in Harsanyi–Selten ([9], p. 73). Consider the two-player example of Fig. 5, in which the two players move simultaneously.

	Player 2	
Player 1	1, 1	0, 0
	0, 0	1, 1

Subgame after h_t

	Player 2	
Player 1	8, -2	-3, -4
	6, 3	-1, 5

Subgame after h'_t

FIGURE 5

Harsanyi and Selten consider two subgames G and G' to be *isomorphic* if, for all players i, there exist scalars $\alpha^i > 0$ and β^i and a bijection between player i's continuation strategy spaces in the two subgames such that, for any player i payoff p^i in subgame G, the corresponding payoff (as determined by the bijections) in G' is $\alpha^i p^i + \beta^i$. By this criterion, the subgames after h_t and h'_t are *not* isomorphic. [Note that player 1's possible payoffs after h_t are $\{0, 1\}$, whereas after h'_t they are $\{8, 6, -3, -1\}$. Clearly there is no affine transformation that maps the former set into the latter.] However, it is readily verified that h_t and h'_t are Markov-equivalent (because for each i there exist $\alpha^i > 0$ and $\beta^i(a_t^{-i})$ such that each payoff p^i after h_t gets mapped into $\alpha^i p^i + \beta^i(a_t^{-i})$ after h'_t).

On the other hand, as noted above, Harsanyi and Selten [9] allow for "bijection equivalence". Their concept deems strategies played in two different subgames to be equivalent even if the action spaces in these subgames are not identical, but only related by a bijection. We have not allowed for bijection equivalence both for notational simplicity and because in most applications action spaces are in fact the same in (Harsanyi–Selten [9]) equivalent subgames, so that no extra power is obtained from imposing the stronger concept. However, we could readily incorporate "bijection equivalence" within our definition of Markov-equivalence, in which case our notion would be unambiguously stronger than that of Harsanyi and Selten [9].

Remark 2. In deeming two decision problems to be equivalent if and only if they give rise to the same preferences over actions, we are working in the tradition of Savage who demanded that *all* acts (even strictly dominated acts[11]) be ranked. Another tradition, dating back at least to Chernoff, asks only that the *best* act (or acts) be identified. In a game theoretic setting this alternative requirement would imply that two problems are equivalent if their best-reply sets are the same (see Mertens [16] who pursues this second notion of equivalence).[12]

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[11] Actually, however, we could easily strengthen our concept of Markov-equivalence by eliminating strictly dominated strategies (this would be a change that would require no other alteration in our formation).

[12] While we find our point of view attractive, it is thus not the unique "right" way of proceeding. Which philosophy one adopts is ultimately a matter of taste. Let us point out, however, that even if we went the Mertens route and used best-response sets as the basis of equivalence, nothing would change *practically*. Again, in every economic application of Markov equilibrium that we have seen, the decision-theoretic and best-response approaches give the same equilibria. This is because in applied work, researchers almost invariably use utility functions with single-crossing properties, so that if a variable changes, all of a player's strategies are affected in the same direction. That is, there is nothing to particularly distinguish the best-response set from the other strategies. And so, one does not get a coarser partition by focussing only on the best-response set.

4. CONTINUITY

The assumption that strategies are Markovian is restrictive only if there is more than one history in some element of the Markov partition. Otherwise all strategies are Markovian, and an MPE is the same thing as a subgame-perfect equilibrium. Moreover, it is easy to see that in a "generic" extensive form game in which every player has at least three available actions each time he or she moves,[13] the Markov partition consists entirely of singletons. Thus formally the Markov requirement has no bite generically. Nevertheless, as we suggested in the introduction, the principle that minor causes should have minor effects might enable us to extend the Markov "spirit" more generally.

The issue is one of lower hemi-continuity. Fix a finite extensive form of a game with T periods, but omit the payoffs. We identify this extensive form with the set $\mathscr{A}$ of possible sequences of action vectors $\mathbf{a} = (\mathbf{a}_1, ..., \mathbf{a}_T)$. Given this form the description of a game is then completed by specifying a vector of payoff functions $\mathbf{u} = (u^1, ..., u^n)$ defined on $\mathscr{A}$. (In what follows, we will keep the extensive form fixed and abuse terminology by identifying a game with its payoffs $\mathbf{u}$.) Let $H_\cdot^{\mathbf{u}}(\cdot) \equiv \{H_t^{\mathbf{u}}(h_t)\}_{t, h_t}$ denote the collection of Markov partitions for game $\mathbf{u}$. Consider a sequence $\{\mathbf{u}_m\}$ converging to $\mathbf{u}$.[14] The associated Markov partitions $H_t^{\mathbf{u}_m}(\cdot)$ may well differ from $H_t^{\mathbf{u}}(\cdot)$. We ask whether for each MPE $\mathbf{s}$ of game $\mathbf{u}$ there exists a sequence $\mathbf{s}_m$ converging to $\mathbf{s}$ such that each $\mathbf{s}_m$ is an MPE of $\mathbf{u}_m$. If so (and if this is true of any other sequence converging to $\mathbf{u}$), then the MPE correspondence satisfies *lower hemi-continuity* (lhc) at $\mathbf{s}$, and, for sufficiently high m, the strategies $\mathbf{s}_m$ put little weight on aspects of the past that are not payoff-relevant in game $\mathbf{u}$. In that respect, they are the true embodiment of the Markov spirit, even though, literally interpreted, the Markov restriction may be ineffective in game $\mathbf{u}_m$.

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[13] When there are at least three actions, a generic perturbation of payoffs changes the von Neumann–Morgenstern preferences.

[14] Given a fixed extensive form, the *distances* between two games $\mathbf{u}$ and $\hat{\mathbf{u}}$ and between two strategy vectors $\mathbf{s}$ and $\hat{\mathbf{s}}$ are defined by

$$\|\hat{\mathbf{u}} - \mathbf{u}\| \equiv \max_{\{i, \mathbf{a}\}} |\hat{u}^i(\mathbf{a}) - u^i(\mathbf{a})|$$

and

$$\|\hat{\mathbf{s}} - \mathbf{s}\| \equiv \max_{\{i, t, h_t\}} \|\hat{s}^i(h_t) - s^i(h_t)\|,$$

where $\|\hat{s}^i(h_t) - s^i(h_t)\|$ denotes the Euclidean distance between the probability vectors $\hat{s}^i(h_t)$ and $s^i(h_t)$.

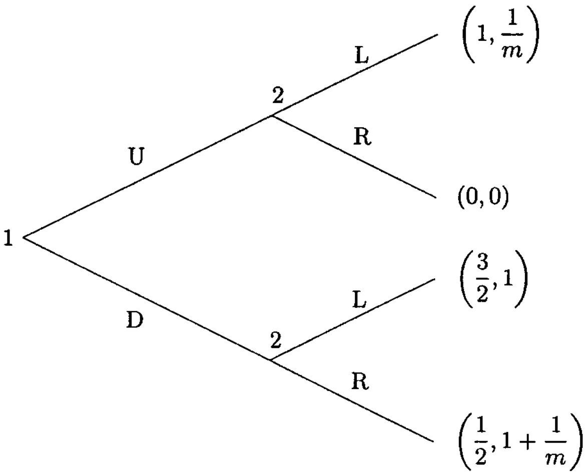

FIGURE 6

Unfortunately, lhc does not hold for all games. For a counterexample, consider the sequence $\{\mathbf{u}_m\}$ depicted in Fig. 6.[15]

In each game $\mathbf{u}_m$, player 1 chooses U or D in period 1. Player 2 observes player 1's choice and in period 2 picks L or R. Game $\mathbf{u}_m$ has a unique subgame-perfect (and therefore Markov perfect) equilibrium: Player 2 plays L following U and R following D, and player 1 plays U. Equilibrium payoffs are $(1, 1/m)$ and converge to $(1, 0)$ as m tends to infinity. In the limit game $\mathbf{u} = \lim_{m \to \infty} \mathbf{u}_m$, however, player 1's choice does not affect player 2's von Neumann–Morgenstern preferences ($u^2(D, a_2) = u^2(U, a_2) + 1$ for all $a_2 = L$ or R). Thus in any MPE of $\mathbf{u}$, player 2's strategy must be independent of a_1, which implies that player 1 plays D. We conclude that lhc fails for $\mathbf{u}$. Indeed, the set of MPE payoffs for $\mathbf{u}$ is $\{(\frac{1}{2} + x, 1)\}_{x \in [0, 1]}$, which is quite different from the limit point $(1, 0)$.

This game, however, is nongeneric within the class of games having the same Markov partitions as $\mathbf{u}$. This class consists of all those games for which

$$H_2(U) = H_2(D). \tag{10}$$

Game $\mathbf{u}$ is degenerate because it not only satisfies (10) but also the property that player 2 is indifferent between L and R. In view of this degeneracy, we are not especially concerned about the failure of lhc here. Indeed, one implication of Theorem 4.2 below is that lhc *is* satisfied in *well-behaved* (i.e., generic) games satisfying (10).

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[15] This example also shows that the MPE correspondence fails to be *upper* hemi-continuous (and that therefore it cannot be described by a set of continuous equations).

For a given collection of partitions $H.(\cdot)$, consider the set $\mathscr{U}(H.(\cdot))$ of payoff vectors $\mathbf{u}$ for which the Markov partitions are $H.(\cdot)$. Note that $\mathscr{U}(H.(\cdot))$ is defined by a set of linear equations, and so is isomorphic to $\mathbf{R}^{\ell}$ for some ℓ. We will say that a proposition is true for *generic* (or *almost all*) $\mathbf{u}$ in $\mathscr{U}(H.(\cdot))$ if it is true for an open and dense subset of $\mathscr{U}(H.(\cdot))$; that is, the proposition is generic in the class of payoff functions whose Markov partitions are $H.(\cdot)$. Note that if $H.(\cdot)$ consists entirely of singletons (i.e., no two histories are equivalent), then a proposition that is true for generic $\mathbf{u} \in \mathscr{U}(H.(\cdot))$ is, in fact, true for generic $\mathbf{u}$ in the class of *all* games with the same extensive form (since, as we have noted, the singleton property is itself generic). An MPE $\mathbf{s}$ of a game $\mathbf{u}$ is *robust* if, for all $\epsilon > 0$, there exists ξ such that any game $\hat{\mathbf{u}}$ satisfying $\|\hat{\mathbf{u}} - \mathbf{u}\| < \xi$ has an MPE $\hat{\mathbf{s}}$ for which $\|\hat{\mathbf{s}} - \mathbf{s}\| < \varepsilon$. That is, an MPE is robust if, for any small payoff-perturbation, one can find a nearby MPE. Note that we do not require that the nearby game $\hat{\mathbf{u}}$ have the same Markov partitions as $\mathbf{u}$ (indeed, generically it will not unless $H.(\cdot)$ consists entirely of singletons).

A game is *essential* if it has only finitely many MPEs and all of these are robust. Essentiality thus ensures lower hemi-continuity. Essential equilibria have received some attention in the literature on strategic-form games[16]. Almost all such games are essential (where the definition of essentiality is unchanged except that "MPE" is replaced by "Nash equilibrium"):

THEOREM 4.1. (a) (*Wu and Jiang* [22]) *For a generic strategic-form game having fixed and finite strategy spaces, all Nash equilibria are robust*;

(b) (*Harsanyi* [8]) *a generic strategic-form game has finitely many equilibria.*[17]

Theorem 4.1 does not answer the question of whether *extensive-form* games are generically essential because the class of games having a particular extensive form is nongeneric in the class of games having the corresponding strategic form.[18] Moreover, a Nash equilibrium need not be an MPE. Theorem 4.2 establishes, however, that for a given collection of Markov partitions $H.(\cdot)$, almost all games $\mathbf{u}$ in $\mathscr{U}(H.(\cdot))$ are essential. To prove Theorem 4.2, we will make use of the following

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[16] See van Damme [20] for a good exposition.

[17] See Wilson [21] for the first result on the generic finiteness of the Nash equilibrium set.

[18] Suppose for instance that player 1 chooses between U and D in period 1 and that player 2 reacts by picking L or R in period 2. The payoff vector following U is independent of the action that player 2 would have picked if player 1 had chosen D. The associated equality constraints therefore make the payoff vector nongeneric in the space of strategic form games with two pure strategies for player 1 and four pure strategies for player 2.

LEMMA. *If* $\mathbf{u}$ *is essential, then*

(a) *for all* $\mathbf{a}_1$, $\mathbf{u}(\mathbf{a}_1, \cdot)$ *is essential, and*

(b) *if, for all* $\mathbf{a}_1$, $\mathbf{s}_{\mathbf{a}_1}$ *is a second-period continuation MPE for* $\mathbf{u}$ *and* $\mathbf{s}_\cdot$ *is measurable with respect to* $H_2^{\mathbf{u}}(\cdot)$, (*i.e.*, $\mathbf{s}_{\hat{\mathbf{a}}'_1} = \mathbf{s}_{\hat{\mathbf{a}}_1}$ *if* $\hat{\mathbf{a}}'_1 \in H_2^{\mathbf{u}}(\hat{\mathbf{a}}_1)$), *then the "one-period game"* $\mathbf{u}(\cdot, \mathbf{s}_\cdot)$ *is essential.*

Proof. (a) Note first that for any $\mathbf{a}_1$, $\mathbf{u}(\mathbf{a}_1, \cdot)$ has only finitely many MPEs; otherwise, $\mathbf{u}$ itself would have infinitely many MPEs (since each MPE of $\mathbf{u}(\mathbf{a}_1, \cdot)$ is a continuation of an MPE of $\mathbf{u}$). Consider an MPE $\mathbf{s}_{\mathbf{a}_1}$ of $\mathbf{u}(\mathbf{a}_1, \cdot)$. We must show that it is robust. Let $\mathbf{s} = (\tilde{\mathbf{a}}_1, \mathbf{s}_\cdot)$ be a corresponding MPE for the overall game (note that $\mathbf{s}_\cdot$ must be measurable with respect to $H_2^{\mathbf{u}}(\cdot)$). Consider a $T-1$-period game $\hat{\mathbf{u}}$, near $\mathbf{u}(\mathbf{a}_1, \cdot)$. To prove the robustness of $\mathbf{s}_{\mathbf{a}_1}$, we must show that $\hat{\mathbf{u}}$ has an MPE near $\mathbf{s}_{\mathbf{a}_1}$.

Now $\hat{\mathbf{u}}$ can be extended to a T-period game $\hat{\mathbf{u}}'$ near $\mathbf{u}$ by setting

$$\hat{\mathbf{u}}'(\mathbf{a}'_1, \cdot) = \begin{cases} \hat{\mathbf{u}}(\cdot), & \text{if } \mathbf{a}'_1 = \mathbf{a}_1 \\ \mathbf{u}(\mathbf{a}'_1, \cdot), & \text{if } \mathbf{a}'_1 \neq \mathbf{a}_1. \end{cases}$$

Because $\mathbf{u}$ is essential (by hypothesis), $\hat{\mathbf{u}}'$ has an MPE $\hat{\mathbf{s}}' = (\hat{\tilde{\mathbf{a}}}'_1, \hat{\mathbf{s}}'_\cdot)$ near $\mathbf{s}$. Hence, the continuation strategy $\hat{\mathbf{s}}'_{\mathbf{a}_1}$ constitutes an MPE of $\hat{\mathbf{u}}(\cdot)$ near $\mathbf{s}_{\mathbf{a}_1}$, and so $\mathbf{s}_{\mathbf{a}_1}$ is indeed robust.

(b) Let

$$\mathbf{u}^*(\mathbf{a}_1) = \mathbf{u}(\mathbf{a}_1, \mathbf{s}_{\mathbf{a}_1}) \qquad \text{for all } \mathbf{a}_1.$$

We must show that $\mathbf{u}^*$ is essential. Any Nash equilibrium $\tilde{\mathbf{a}}_1$ of $\mathbf{u}^*$ is, by construction, part of an MPE $\mathbf{s} = (\tilde{\mathbf{a}}_1, \mathbf{s}_\cdot)$ of $\mathbf{u}$. Hence, because $\mathbf{u}$ has only finitely many MPEs, $\mathbf{u}^*$ has only finitely many Nash equilibria. To establish, therefore, that $\mathbf{u}^*$ is essential, it remains to show that each Nash equilibrium $\tilde{\mathbf{a}}_1$ is robust. Consider a one-period game $\mathbf{u}^\circ$ near $\mathbf{u}^*$. We can extend $\mathbf{u}^\circ$ to a T-period game $\mathbf{u}^{\circ\circ}$ near $\mathbf{u}$ as follows. Let

$$\varepsilon(\mathbf{a}_1) = \mathbf{u}^\circ(\mathbf{a}_1) - \mathbf{u}^*(\mathbf{a}_1)$$

and

$$\mathbf{u}^{\circ\circ}(\mathbf{a}_1, f_2) \equiv \mathbf{u}(\mathbf{a}_1, f_2) + \varepsilon(\mathbf{a}_1)$$

for all $\mathbf{a}_1$ and f_2. Because $\mathbf{u}^\circ$ is near $\mathbf{u}^*$, $\mathbf{u}^{\circ\circ}$ is close to $\mathbf{u}$. Furthermore, for any $\mathbf{a}_1$, $\mathbf{u}^{\circ\circ}$ and $\mathbf{u}$ have the same second-period continuation MPEs. Because $\mathbf{u}$ is essential, $\mathbf{s} = (\tilde{\mathbf{a}}_1, \mathbf{s}_\cdot)$ is robust, and so $\mathbf{u}^{\circ\circ}$ has an MPE

$\mathbf{s}^{\circ\circ} = (\tilde{\mathbf{a}}_1^{\circ\circ}, \mathbf{s}_{\cdot}^{\circ\circ})$ near $\mathbf{s}$. Since, moreover, $\mathbf{u}$ and $\mathbf{u}^{\circ\circ}$ have the same set of continuation MPEs and $\mathbf{u}$ has only finitely many MPEs, $\mathbf{s}_{\cdot}^{\circ\circ}$ must coincide with $\mathbf{s}_{\cdot}$ if $\mathbf{u}^{\circ\circ}$ is near enough $\mathbf{u}$. Therefore, for all $\mathbf{a}_1$,

$$\begin{aligned}\mathbf{u}^{\circ\circ}(\mathbf{a}_1, \mathbf{s}_{\mathbf{a}_1}^{\circ\circ}) &= \mathbf{u}(\mathbf{a}_1, \mathbf{s}_{\mathbf{a}_1}) + \varepsilon(\mathbf{a}_1)\\ &= \mathbf{u}(\mathbf{a}_1, \mathbf{s}_{\mathbf{a}_1}) + \mathbf{u}^{\circ}(\mathbf{a}_1) - \mathbf{u}^*(\mathbf{a}_1)\\ &= \mathbf{u}^{\circ}(\mathbf{a}_1).\end{aligned}$$

And so, $\tilde{\mathbf{a}}_1^{\circ\circ}$ is a Nash equilibrium of $\mathbf{u}^{\circ}$. Because it is near $\tilde{\mathbf{a}}_1$ this establishes the robustness of $\tilde{\mathbf{a}}_1$ and the essentiality of $\mathbf{u}^*$. ▮

THEOREM 4.2. *Fix a finite-horizon game with corresponding collection of Markov partitions* $H_{\cdot}(\cdot)$. *Then, almost all games* $\mathbf{u}$ *in* $\mathscr{U}(H_{\cdot}(\cdot))$ *are essential.*

Proof. We must establish two things: (i) if $\mathbf{u} \in \mathscr{U}(H_{\cdot}(\cdot))$ is essential, then there exists an open neighborhood N of $\mathbf{u}$ in $\mathscr{U}(H_{\cdot}(\cdot))$ such that each $\mathbf{u}' \in N$ is essential; and (ii) for any $\mathbf{u} \in \mathscr{U}(H_{\cdot}(\cdot))$ there exists a sequence $\{\mathbf{u}_m\} \subseteq \mathscr{U}(H_{\cdot}(\cdot))$ such that $\mathbf{u}_m \to \mathbf{u}$ and $\mathbf{u}_m$ is essential for all m.

To establish claim (i), we must show that if $\mathbf{u}' \in \mathscr{U}(H_{\cdot}(\cdot))$ is near enough $\mathbf{u}$, it is essential too. We shall establish this by induction on T, the number of stages. Note that for $T = 1$ the claim follows from Theorem 4.1.

We first note that if $\mathbf{u}'$ is near enough $\mathbf{u}$ in $\mathscr{U}(H_{\cdot}(\cdot))$, then all its MPEs are near MPEs of $\mathbf{u}$. If not, then for some $\varepsilon > 0$ there exists a sequence $\mathbf{u}_m \to \mathbf{u}$ such that, for all m, $\mathbf{u}_m \in \mathscr{U}(H_{\cdot}(\cdot))$ and $\mathbf{u}_m$ has an MPE $\mathbf{s}_m$ for which $\|\mathbf{s} - \mathbf{s}_m\| > \varepsilon$ for each MPE $\mathbf{s}$ of $\mathbf{u}$. But any convergent subsequence of $\{\mathbf{s}_m\}$ converges to an MPE of $\mathbf{u}$ (since all the games in the subsequence share $H_{\cdot}(\cdot)$ as their Markov partitions), a contradiction.

Next, we must show that if $\mathbf{u}'$ is near enough $\mathbf{u}$ in $\mathscr{U}(H_{\cdot}(\cdot))$, it has only finitely many MPEs. Suppose to the contrary that it has infinitely many MPEs. Then either (a) for some vector of first-period actions $\mathbf{a}_1$ the continuation game $\mathbf{u}'(\mathbf{a}_1, \cdot)$ has infinitely many MPEs, or else (b) for some collection $\{\mathbf{s}'_{\mathbf{a}'_1}\}_{\mathbf{a}'_1 \in A_1}$, where for all $\mathbf{a}'_1$, $\mathbf{s}'_{\mathbf{a}'_1}$ is a continuation MPE following $\mathbf{a}'_1$ and where $\mathbf{s}'_{\mathbf{a}'_1} = \mathbf{s}'_{\hat{\mathbf{a}}'_1}$ if $\hat{\mathbf{a}}'_1 \in H_2(\mathbf{a}'_1)$, the first-period game $\mathbf{u}'(\cdot, \mathbf{s}'_{\cdot})$ has infinitely many Nash equilibria. Now, from part (a) of the lemma, $\mathbf{u}(\mathbf{a}_1, \cdot)$ is essential. Hence, since we are assuming that claim 1 holds for $T-1$ stage games, $\mathbf{u}'(\mathbf{a}_1, \cdot)$ is also essential, ruling out case (a). In case (b), because the MPEs of $\mathbf{u}'$ are close to those of $\mathbf{u}$ (from the previous paragraph), $\mathbf{s}'_{\mathbf{a}_1}$ is close to a continuation MPE $\mathbf{s}_{\mathbf{a}_1}$ for each $\mathbf{a}_1$. Therefore, the one-period game $\mathbf{u}'(\cdot, \mathbf{s}'_{\cdot})$ is near $\mathbf{u}(\cdot, \mathbf{s}_{\cdot})$. Now, from part (b) of the lemma, $\mathbf{u}(\cdot, \mathbf{s}_{\cdot})$ is essential. Hence, by Theorem 4.1, $\mathbf{u}'(\cdot, \mathbf{s}'_{\cdot})$ is essential too, and so has only finitely many equilibria, a contradiction.

Finally, we must show that if $\mathbf{u}'$ is near $\mathbf{u}$ in $\mathscr{U}(H_{\cdot}(\cdot))$, any MPE $\mathbf{s}' = (\tilde{\mathbf{a}}'_1, \mathbf{s}'_{\cdot})$ of $\mathbf{u}'$ is robust. For any $\mathbf{a}_1$, we noted above that the inductive hypothesis implies that $\mathbf{u}'(\mathbf{a}_1, \cdot)$ is essential. Hence $\mathbf{s}'_{\mathbf{a}_1}$ is robust and so, if $\hat{\mathbf{u}}'$ is near $\mathbf{u}'$ in the space of all games, $\hat{\mathbf{u}}'(\mathbf{a}_1, \cdot)$ has an MPE $\hat{\mathbf{s}}'_{\mathbf{a}_1}$ near $\mathbf{s}'_{\mathbf{a}_1}$. Now if $\hat{\mathbf{u}}'$ is near enough $\mathbf{u}'$, $H_2^{\hat{\mathbf{u}}'}(\cdot)$ either equals or is a refinement of $H_2^{\mathbf{u}'}(\cdot)$ $(= H_2(\cdot))$. To see this, note that if $h'_2 \notin H_2^{\mathbf{u}'}(h_2)$ then if we rescale $u'^i(h'_2, \cdot)$ and $u'^i(h_2, \cdot)$ so that their minimum values are 0 and their maximum values are 1, we obtain

$$\|u'^i(h_2, \cdot) - u'^i(h'_2, \cdot)\|_{sup} \equiv \max_{f_2} |u'^i(h_2, f_2) - u'^i(h'_2, f_2)| > 0. \qquad (11)$$

Thus if $\hat{\mathbf{u}}'$ is near $\mathbf{u}'$ (and rescaled so that its minimum value is 0 and its maximum 1), (11) also holds for $\hat{\mathbf{u}}'$, and so $h'_2 \notin H_2^{\hat{\mathbf{u}}'}(h_2)$ (if $h'_2 \in H_2^{\hat{\mathbf{u}}'}(h_2)$, then $\|\hat{u}'^i(h_2, \cdot) - \hat{u}'^i(h'_2, \cdot)\| = 0$). So indeed $H_2^{\hat{\mathbf{u}}'}(\cdot)$ either equals or is a refinement of $H_2^{\mathbf{u}'}(\cdot)$. Hence if $\mathbf{a}'_1 \in H_2^{\hat{\mathbf{u}}'}(\mathbf{a}_1)$, then $\mathbf{a}'_1 \in H_2^{\mathbf{u}'}(\mathbf{a}_1)$ and so $\mathbf{s}'_{\mathbf{a}_1} = \mathbf{s}'_{\mathbf{a}'_1}$ (since $\mathbf{s}'$ is an MPE for $\mathbf{u}'$). This in turn means that we can assume that $\hat{\mathbf{s}}'_{\mathbf{a}_1} = \hat{\mathbf{s}}'_{\mathbf{a}'_1}$, i.e., $\hat{\mathbf{s}}'_{\cdot}$ is measurable with respect to $H_2^{\hat{\mathbf{u}}'}(\cdot)$. Now, from part (b) of the lemma, the one-period game $\mathbf{u}'(\cdot, \mathbf{s}'_{\cdot})$ is essential. Thus since $\hat{\mathbf{s}}'_{\cdot}$ is near $\mathbf{s}'_{\cdot}$, $\hat{\mathbf{u}}'(\cdot, \hat{\mathbf{s}}'_{\cdot})$ is near $\mathbf{u}'(\cdot, \mathbf{s}'_{\cdot})$ and so (from Theorem 4.1) has a Nash equilibrium $\hat{\tilde{\mathbf{a}}}'_1$ near $\tilde{\mathbf{a}}'_1$. Thus $\hat{\mathbf{s}}' \equiv (\hat{\tilde{\mathbf{a}}}'_1, \hat{\mathbf{s}}'_{\cdot})$ constitutes an MPE of $\hat{\mathbf{u}}'$ near $\mathbf{s}'$, establishing that $\mathbf{s}'$ is robust and thus claim (i).

To establish claim (ii), we will again proceed by induction on T. The case $T = 1$ is handled by Theorem 4.1. Now, from each element of $H_2^{\mathbf{u}}(\cdot)$ choose a first-period action vector $\mathbf{a}_1$ and consider $\mathbf{u}(\mathbf{a}_1, \cdot)$. From inductive hypothesis, there exists a sequence $\mathbf{u}_m(\mathbf{a}_1, \cdot) \to \mathbf{u}(\mathbf{a}_1, \cdot)$ such that each $\mathbf{u}_m(\mathbf{a}_1, \cdot)$ is essential and $\mathbf{u}_m(\mathbf{a}_1, \cdot) \in \mathscr{U}(H_2^{\mathbf{u}(\mathbf{a}_1, \cdot)})$. Now, from Theorem 3.1, if $\mathbf{a}'_1 \in H_2^{\mathbf{u}}(\mathbf{a}_1)$, then for all i there exist $\alpha^i > 0$ and $\beta^i(\cdot)$ such that

$$u^i(\mathbf{a}'_1, f_2) = \alpha^i u^i(\mathbf{a}_1, f_2) + \beta^i(\mathbf{a}_2^{-i}) \qquad \text{for all } f_2.$$

Hence, because $\mathbf{u}_m(\mathbf{a}_1, \cdot)$ is essential and converges to $\mathbf{u}(\mathbf{a}_1, \cdot)$, if, for all i, we define

$$u_m^i(\mathbf{a}'_1, f_2) = \alpha^i u_m^i(\mathbf{a}_1, f_2) + \beta^i(\mathbf{a}_2^{-i}) \qquad \text{for all } f_2,$$

then $\mathbf{u}_m(\mathbf{a}'_1, \cdot)$ is essential and converges to $\mathbf{u}(\mathbf{a}'_1, \cdot)$. In other words, when we piece together all the games $\{\mathbf{u}_m(\mathbf{a}'_1, \cdot)\}_{\mathbf{a}'_1 \in A_1}$, we obtain a T-period game $\mathbf{u}_m \in \mathscr{U}(H_{\cdot}^{\mathbf{u}}(\cdot))$ such that $\mathbf{u}_m(\mathbf{a}'_1, \cdot)$ is essential for all $\mathbf{a}'_1$. Now, $\mathbf{u}_m$ may not be essential. But we will show that, for each m, there exists a game in $\mathscr{U}(H_{\cdot}^{\mathbf{u}})$ near $\mathbf{u}_m$ that *is* essential, and this sequence of essential games will establish claim (ii).

To see this, for all m, list all the functions $\mathbf{s}_{\cdot 1}^m, ..., \mathbf{s}_{\cdot k_m}^m$ that map first-period actions to continuation MPEs of $\mathbf{u}_m$. Because $\mathbf{u}_m(\mathbf{a}'_1, \cdot)$ is essential for all $\mathbf{a}'_1$, there are only finitely many of these (i.e., k_m is finite). Define

$$\hat{\mathbf{u}}_{m,1}(\cdot) \equiv \mathbf{u}_m(\cdot, \mathbf{s}_{\cdot 1}^m).$$

From Theorem 4.1, there exists an essential one-period game $\mathring{\mathbf{u}}_{m,1}$ near $\hat{\mathbf{u}}_{m,1}$.

Let

$$\varepsilon_{m,1}(\mathbf{a}'_1) \equiv \mathring{\mathbf{u}}_{m,1}(\mathbf{a}'_1) - \hat{\mathbf{u}}_{m,1}(\mathbf{a}'_1)$$

and

$$\mathbf{u}_{m,1}(\mathbf{a}'_1, f_2) \equiv \mathbf{u}_m(\mathbf{a}'_1, f_2) + \varepsilon_{m,1}(\mathbf{a}'_1) \qquad \text{for all } \mathbf{a}'_1, f_2.$$

Because $\mathbf{u}_m \in \mathscr{U}(H^{\mathbf{u}}_{\cdot}(\cdot))$ and $\mathbf{u}_m(\mathbf{a}'_1, \cdot)$ is essential for all $\mathbf{a}'_1$, the corresponding properties are true of $\mathbf{u}_{m,1}$ and $\mathbf{u}_{m,1}(\mathbf{a}'_1, \cdot)(\mathbf{u}_{m,1}(\mathbf{a}'_1, \cdot)$ is just $\mathbf{u}_m(\mathbf{a}'_1, \cdot)$ plus a constant). Since $\mathring{\mathbf{u}}_{m,1}$ is essential, $\mathbf{u}_{m,1}$ has only finitely many MPEs that entail $\mathbf{s}^m_{\cdot 1}$ (this follows because $\mathbf{u}_{m,1}(\cdot, \mathbf{s}^m_{\cdot 1}) \equiv \mathring{\mathbf{u}}_{m,1}$) and all of these are robust. (To see the robustness, let $(\tilde{\mathbf{a}}^m_{1,1}, \mathbf{s}^m_{\cdot 1})$ be an MPE of $\mathbf{u}_{m,1}$. If $\mathbf{u}'_{m,1}$ is near $\mathbf{u}_{m,1}$, then by inductive hypothesis $\mathbf{u}'_{m,1}(\mathbf{a}'_1, \cdot)$ is essential for all $\mathbf{a}'_1$, and so there exists $\hat{\mathbf{s}}^m_{\cdot 1}$ near $\mathbf{s}^m_{\cdot 1}$ such that, for $\mathbf{a}'_1$, $\hat{\mathbf{s}}^m_{\mathbf{a}'_1,1}$ is an MPE for $\mathbf{u}'_{m,1}(\mathbf{a}'_1, \cdot)$. Moreover, by the same argument we used with $\hat{\mathbf{s}}'_{\cdot}$ in the proof of claim (i), $\hat{\mathbf{s}}^m_{\cdot 1}$ can be chosen to be measurable with respect to $H_2^{\hat{\mathbf{u}}'_{m,1}}(\cdot)$. Now $\mathbf{u}'_{m,1}(\cdot, \hat{\mathbf{s}}^m_{\cdot 1})$ is near $\mathring{\mathbf{u}}_{m,1}$, and so, by the essentiality of the latter, the former has an equilibrium $\hat{\tilde{\mathbf{a}}}^m_{1,1}$ near $\tilde{\mathbf{a}}^m_{1,1}$, which is an equilibrium of $\mathring{\mathbf{u}}_{m,1}$. Hence, $(\hat{\tilde{\mathbf{a}}}^m_{1,1}, \hat{\mathbf{s}}^m_{\cdot 1})$ is an MPE for $\mathbf{u}'_{m,1}$ near $(\tilde{\mathbf{a}}^m_{1,1}, \mathbf{s}^m_{\cdot 1})$, establishing the robustness of the latter.)

The fact that the MPEs of $\mathbf{u}_{m,1}$ in which the continuation MPE is $\mathbf{s}^m_{\cdot 1}$ are finite in number and robust does not establish that $\mathbf{u}_{m,1}$ is essential because it says nothing about, say, MPEs of the form $(\tilde{\mathbf{a}}^m_1, \mathbf{s}^m_{\cdot 2})$. We next show, however, that we can find $\mathbf{u}_{m,2}$ near $\mathbf{u}_{m,1}$ all of whose continuation MPEs are in the set $\{\mathbf{s}^m_{\cdot 1}, ..., \mathbf{s}^m_{\cdot k_m}\}$ and whose MPEs with continuation $\mathbf{s}^m_{\cdot 1}$ or $\mathbf{s}^m_{\cdot 2}$ are robust and finite in number.

Specifically, define the one-period game $\hat{\mathbf{u}}_{m,2}(\cdot) = \mathbf{u}_{m,1}(\cdot, \mathbf{s}^m_2)$. From Theorem 4.1, there exists an essential one-period game $\mathring{\mathbf{u}}_{m,2}$ near $\hat{\mathbf{u}}_{m,2}$. Let

$$\varepsilon_{m,2}(\mathbf{a}'_1) \equiv \mathring{\mathbf{u}}_{m,2}(\mathbf{a}'_1) - \hat{\mathbf{u}}_{m,2}(\mathbf{a}'_1)$$

and

$$\mathbf{u}_{m,2}(\mathbf{a}'_1, f_2) \equiv \mathbf{u}_{m,1}(\mathbf{a}'_1, f_2) + \varepsilon_{m,2}(\mathbf{a}'_1).$$

for all $(\mathbf{a}'_1, f_2)$. Then $\mathbf{u}_{m,2} \in \mathscr{U}(H^{\mathbf{u}}_{\cdot}(\cdot))$ (since $\mathbf{u}_{m,1} \in \mathscr{U}(H^{\mathbf{u}}_{\cdot}(\cdot))$). Moreover, by argument analogous to that for $\mathbf{u}_{m,1}$, $\mathbf{u}_{m,2}$ has only finitely many MPEs that entail $\mathbf{s}^m_{\cdot 2}$, and all of these are robust. Finally, because $\mathbf{u}_{m,1}(\cdot, \mathbf{s}^m_{\cdot 1})$ is essential, inductive hypothesis implies that for $\varepsilon_{m,2}(\cdot)$ small enough, $\mathbf{u}_{m,2}(\cdot, \mathbf{s}^m_{\cdot 1})$ is essential, and so $\mathbf{u}_{m,2}$ has only finitely many MPEs that entail $\mathbf{s}^m_{\cdot 1}$ and all of these are robust.

Continuing iteratively, we can define $\mathbf{u}_{m,3}, \ldots, \mathbf{u}_{m,k_m}$, where $\mathbf{u}_{m,k_m}$ is near $\mathbf{u}_m$ and has only finitely many MPEs that entail $\mathbf{s}^m_{\cdot 1}, \ldots,$ or $\mathbf{s}^m_{\cdot k_m}$, all of which are robust. But, by construction, $\mathbf{u}_{m,k_m}$ has the same second-period continuation MPEs as $\mathbf{u}_m$, and so these MPEs entailing $\mathbf{s}^m_{\cdot 1}, \ldots,$ or $\mathbf{s}^m_{\cdot k_m}$ are the only ones that $\mathbf{u}_{m,k_m}$ has. Hence $\mathbf{u}_{m,k_m}$ is essential. Moreover since $\mathbf{u}_m \to \mathbf{u}$, and $\mathbf{u}_{m,k_m}$ is near $\mathbf{u}_m$, we have $\mathbf{u}_{m,k_m} \to \mathbf{u}$. ∎

APPENDIX

Proof of Theorem 3.1

Because $H^*_\cdot(\cdot)$ is maximally coarse among consistent collections, it suffices to show that $H^{**}_\cdot(\cdot)$ is consistent. Given i, consider t, h_t, $h'_t \in H_t$ such that $h'_t \in H^{**}_t(h_t)$. There exist $\alpha > 0$ and β: $A^{-i}_t(h_t) \to \mathbb{R}$ such that for all $\tau \geqslant 0$, $\mathbf{a}_t, \mathbf{a}_{t+1}, \ldots, \mathbf{a}_{t+\tau}$, and $f_{t+\tau+1}$,

$$\begin{aligned}\mathbf{u}^i(h'_t, \mathbf{a}_t, \ldots, \mathbf{a}_{t+\tau}, f_{t+\tau+1})\\ = \alpha \mathbf{u}^i(h_t, \mathbf{a}_t, \ldots, \mathbf{a}_{t+\tau}, f_{t+\tau+1}) + \beta(\mathbf{a}^{-i}_t).\end{aligned} \tag{A.1}$$

And so because $\beta(\cdot)$ in (A1) does not depend on $f_{t+\tau+1}$

$$(h'_t, \mathbf{a}_t, \ldots, \mathbf{a}_{t+\tau}) \in H^{**}_{t+\tau+1}(h_t, \mathbf{a}_t, \ldots, \mathbf{a}_{t+\tau}). \tag{A.2}$$

Consider strategies $\mathbf{s}^{-i}$ that are measurable with respect to $H^{**}_\cdot(\cdot)$. Then (A.2) implies that

$$\mathbf{s}^{-i}(h'_t, \mathbf{a}_t, \ldots, \mathbf{a}_{t+\tau}) = \mathbf{s}^{-i}(h_t, \mathbf{a}_t, \ldots, \mathbf{a}_{t+\tau}),$$

and so

$$v^i(s^i, \mathbf{s}^{-i} \mid h'_t) = \alpha v^i(s^i, \mathbf{s}^{-i} \mid h_t) + \beta(\mathbf{a}^{-i}_t), \qquad \text{for all } s^i.$$

Thus $H^{**}_\cdot(\cdot)$ is consistent, as claimed:

$$v^i(\cdot, \mathbf{s}^{-i} \mid h_t) \sim v^i(\cdot, \mathbf{s}^{-i} \mid h'_t). \quad ∎$$

Proof of Theorem 3.3

It suffices to show that $H^{**}_\cdot(\cdot)$ is at least as coarse as $H^*_\cdot(\cdot)$. Suppose therefore that for some t, h_t and h'_t, $h'_t \in H^*_t(h_t)$. We must show that $h'_t \in H^{**}_t(h_t)$. Now, from Theorem 3.2, $h'_t \in H^{***}_t(h_t)$. Hence there exist $\alpha > 0$ and $\gamma(\cdot)$ such that

$$u^i(h'_t, f_t) = \alpha u^i(h_t, f_t) + \gamma(f^{-i}_t) \qquad \text{for all } f_t \tag{A.3}$$

Write $f_t^{-i} = (\mathbf{a}_t^{-i}, f_{t+1}^{-i})$. We must show that

$$\gamma(\mathbf{a}_t^{-i}, f_{t+1}^{-i}) = \gamma(\mathbf{a}_t^{-i}, f_{t+1}^{-i\,\prime}) \qquad \text{for all } f_{t+1}^{-i\,\prime}.$$

Suppose, to the contrary that there exists $f_{t+1}^{-i\,\prime}$ for which

$$\gamma(\mathbf{a}_t^{-i}, f_{t+1}^{-i}) \neq \gamma(\mathbf{a}_t^{-i}, f_{t+1}^{-i\,\prime}). \tag{A.4}$$

Fix $\mathbf{s} \in \mathbf{S}(\bar{H}_{\cdot}(\cdot))$. Consider a player j who moves in period $t+1$. From forward sequential-nondegeneracy, there exist $a_t^i, a_t^{i\,\prime} \in A_t^i(h_t)$ such that

$$w^j(\,\cdot\,; h_t, a_t^i, \mathbf{a}_t^{-i}, \mathbf{s}) \not\sim w^j(\,\cdot\,; h_t, a_t^{i\,\prime}, \mathbf{a}_t^{-i}, \mathbf{s}),$$

where the argument of w^j is a_{t+1}^j. Hence, from simultaneous nondegeneracy and Theorem 2.1,

$$H_{t+1}^*(h_t, a_t^i, \mathbf{a}_t^{-i}) \neq H_{t+1}^*(h_t, a_t^{i\,\prime}, \mathbf{a}_t^{-i}). \tag{A.5}$$

Now, from the last hypothesis of the Theorem, there exist $\tilde{f}_{t+1}^i$ and $\tilde{f}_{t+1}^{i\,\prime}$ such that

$$u^i(h_t, a_t^i, \mathbf{a}_t^{-i}, \tilde{f}_{t+1}^i, f_{t+1}^{-i}) = u^i(h_t, a_t^{i\,\prime}, \mathbf{a}_t^{-i}, \tilde{f}_{t+1}^{i\,\prime}, f_{t+1}^{-i\,\prime}). \tag{A.6}$$

Now, from (A.5), the strategies of players $-i$ are measurable with respect to $H_{\cdot}^*(\cdot)$ if they play f_{t+1}^{-i} after $(h_t, a_t^i, \mathbf{a}_t^{-i})$ and $f_{t+1}^{-i\,\prime}$ after $(h_t, a_t^{i\,\prime}, \mathbf{a}_t^{-i})$. Hence because $h_t' \in H_t^*(h_t)$, player i's ranking of $(a_t^i, \tilde{f}_{t+1}^i)$ and $(a_t^{i\,\prime}, \tilde{f}_{t+1}^{i\,\prime})$ should not depend on whether h_t or h_t' occurred. But from (A6), player i is indifferent between these choices after h_t, yet, from (A3) and (A4), player i is not indifferent between them after h_t', a contradiction. We conclude that (A4) cannot hold after all.

ACKNOWLEDGMENT

Research support from the National Science Foundation and the Taussig Visiting Professorship at Harvard University and able research assistance by Daniel Chen, Thomas Mariotti, E. Somanathan, and Steve Tadelis are gratefully acknowledged. Joint work and discussions with Drew Fudenberg helped give shape to the ideas developed here. The technical advice of Andreu Mas-Colell was invaluable. We also thank David Easley, Martin Hellwig, Edi Karni, Sylvain Sorin, and Geoffrey Tate for helpful comments.

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