JOURNAL OF ECONOMIC THEORY 63, 19–41 (1994)

Indeterminacy and Increasing Returns*

JESS BENHABIB

Department of Economics, New York University,
269 Mercer Street, 7th Floor, New York, New York, 10003

AND

ROGER E. A. FARMER

Department of Economics,
2263 Bunche Hall, Los Angeles, California 90024-1477

Received May 27, 1993; revised August 31, 1993

We investigate properties of the one-sector growth model with increasing returns under two organizational structures capable of reconciling the existence of aggregate increasing returns with competitive behavior by firms. The first involves input externalities; the second involves monopolistic competition. We show, for parameters in close accord with recent literature on real business cycles, that the model displays an indeterminate steady state that can be exploited to generate a model of business fluctuations driven by self-fulfilling beliefs. In our first class of models, growth is generated by exogenous increases in factor productivity. In the second class the marginal product of capital is large enough for endogenous growth. *Journal of Economic Literature* Classification Numbers: E00, E3, O40.

1. INTRODUCTION

A number of authors have studied macroeconomic models in which the social technology may differ from the technology faced by "the representative agent" because of external effects in the production process that are not mediated by markets. If the spillover effects of knowledge acquisition are great enough they may lead to a description of the economy in which the social technology is linear in capital. Models in this class have the property that although each individual faces diminishing returns to the acquisition of knowledge, society as a whole may grow without bound. This idea has been exploited in works by Lucas [21], Romer [25, 26], and

* We thank the C.V. Starr Center at NYU and the Risk Project of the Department of Applied Economics at Cambridge for supporting this research.

0022-0531/94 $6.00

others to explain growth as an endogenous process of invention and innovation.

More recently, attention has been focused on models with significant non-competitive elements as possible explanations of a puzzle that arises from Solow's [29] concept of growth accounting. The idea behind growth accounting is to subtract a divisia index of growth in inputs from the logarithmic rate of growth of output of the aggregate economy. The residual from this exercise, under the assumptions of perfect competition and constant returns-to-scale, represents a measure of exogenous productivity growth. Hall [12] has pointed out that the "Solow residual" is predicted to be uncorrelated with any variable that is uncorrelated with the rate of growth of true productivity. In post-war U.S. data this prediction fails badly. Hall lists a number of potential reasons for this failure, two of which are the possibilities that externalities may have important effects at business cycle frequencies and that monopolistic competition may play an important role in the aggregate economy.

The focus of our paper is to study the effects of introducing either externalities or monopolistic competition into the Ramsey model of optimal growth. More precisely, we use externalities and monopolistic competition as two alternative ways of combining a social technology that displays increasing returns-to-scale with competitive behaviour by individual producers. We find, for parameter values in close accord with recent estimates of the degree of externalities in the U.S. economy, that the model displays a unique steady state that is locally stable. The implication of this finding is that there exists a continuum of equilibrium paths for any initial stock of capital, each of which is consistent with convergence to the unique steady state; that is, the equilibrium of the economy is indeterminate.

The possibility of indeterminate equilibria in the presence of increasing returns has been explored by a number of authors.[1] The difference of our work from other papers in the field involves our allowance for a variable labor supply. We find that externalities in production may cause excessive growth in the capital stock to be dampened by reductions in the equilibrium supply of labor. The result is a model in which there are many possible values of consumption, each of which is consistent with an equilibrium path of interest rates that converges back to the unique steady state.

[1] Howitt and McAfee [15, 16], Kehoe *et al.* [17], Murphy *et al.* [23], and Spear [30] as well as the early work of Shell [27] are notable examples. Hammour [13] in a recent survey points out that increasing returns in dynamic models is often destabilizing and must be coupled with congestion effects of one form or another to keep the equilibrium path of the economy within reasonable bounds, although congestion effects do not seem to be necessary in two sector models as is apparent from a recent paper by Boldrin and Rustichini [4]. See also Chamley [7] and Mulligan and Sala-i-Martin [22].

Previous work on indeterminate equilibria[2] has shown that stable steady states are associated with rational expectations models in which there exists a continuum of self-fulfilling belief-driven equilibria each of which is *stationary*.[3] We calibrate our model using estimates of the increasing returns parameter drawn from cross-section studies by Caballero and Lyons [5] and from a time series estimate of the production function by Baxter and King [2]. These studies suggest an elasticity of social output with respect to labor input of the order of 1.05 or a value for the magnitude of externalities in the social technology of around 1.5. Since our calibrated economy is consistent with the existence of equilibria that are driven by random shocks in agents' beliefs we conclude that a business cycle research program that incorporates increasing returns must face up to the possibility that "animal spirits" may be an important contributing factor to business fluctuations.

In the first part of our paper we investigate a model without growth which is easily adjusted to introduce exogenous growth. Since our work is driven by externalities of the same nature that have recently motivated the literature on *endogenous* growth, in the latter part of the paper we allow for externalities that are large enough for the social technology to be linear in capital. In this section we show that, in the presence of labor externalities, our model displays indeterminate endogenous growth in the sense that there are many sets of self-fulfilling beliefs each of which is consistent with a dynamic equilibrium that converges to the same balanced growth path but not to the same level of consumption, capital, and employment. We present an example of a model with this property which is calibrated to fit the parameter values that are typically found for the U.S. economy.

2. The Benchmark Model

To study the dynamics of capital accumulation with increasing returns to scale, we need a theory of income distribution that reconciles increasing returns at the aggregate level with competitive behavior by individuals and firms. We describe two theories, each of which is consistent with the same aggregate dynamics. The first theory is borrowed from the recent literature on endogenous growth in which one typically assumes that there are important external effects in the production technology that are not mediated by markets. Our second approach is drawn from work on

[2] See the paper by Farmer and Woodford [10] or the excellent survey by Chiappori and Guesnerie [8].

[3] These stationary rational expectations equilibria are dynamic examples of what Shell [28] and Cass and Shell [6] have labeled "sunspot equilibria." Azariadis [1] gave the first example of stationary sunspot equilibria in a macroeconomic context.

monopolistic competition by Dixit and Stiglitz [9] that has been explored in a macroeconomic model similar to ours by Kiyotaki [18] and Blanchard and Kiyotaki [3].

In both approaches we assume that the aggregate production function is Cobb–Douglas, given by

$$Y = K^{\alpha} N^{\beta}, \qquad \alpha, \beta > 0, \ \alpha + \beta > 1. \tag{2.1}$$

K represents the aggregate stock of capital, and N is aggregate labor input. Note that the assumption $\alpha + \beta > 1$ implies that the technology exhibits increasing returns to scale.

We assume that the factor market is competitive and we use the symbols ω for the relative price of labor in terms of the unique consumption commodity and r for the capital rental rate. Both of the organizational structures that we look at imply that factors of production receive fixed shares of national income:

$$\omega N = bY, \tag{2.2}$$

$$rK = aY. \tag{2.3}$$

Note that marginal products will differ from factor shares unless $\alpha = a$ and $\beta = b$. Both of our models imply $0 < a < \alpha$ and $0 < b < \beta$.

2.1. *The Model with Externalities*

The model with externalities is the simpler and more familiar of the two. To derive (2.1), (2.2), and (2.3) from a theory of competitive behavior we redefine the aggregate technology as

$$Y = K^{a} N^{b} (\bar{K}^{a\theta_1} \bar{N}^{b\theta_2}), \tag{2.4}$$

where $\bar{K}$ and $\bar{N}$ represent the average economy-wide levels of capital and labor. The economy is assumed to consist of a large number of identical firms and from the point of view of the representative firm these terms are exogenous. They represent external effects which are not traded in markets. Equation (2.1), which describes the aggregate technology, can be derived from (2.4) by recognizing that in equilibrium $K = \bar{K}$ and $N = \bar{N}$ and by making the parameter substitutions

$$\alpha = a(1 + \theta_1) > 0$$

and

$$\beta = b(1 + \theta_2) > 0.$$

We assume that from the perspective of each firm the technology exhibits constant returns-to-scale, that is

$$a + b = 1. \tag{2.5}$$

2.2. *The Model with Monopolistic Competition*

In this version of the organizational structure we assume that the individual firm uses a technology similar to that described by Dixit and Stiglitz [9]. There is a continuum of intermediate goods $Y(i)$ where $i \in [0, 1]$. Final output is given by

$$Y = \left(\int_0^1 Y(i)^{\lambda} \, di \right)^{1/\lambda}, \tag{2.6}$$

where $\lambda \in (0, 1)$. Note that (2.6) displays constant returns to scale.

The final goods sector is competitive. If $P(i)$ is the relative price of the ith intermediate good in terms of the final good, the profits of a final goods producer are given by

$$\Pi = Y - \int_0^1 P(i) \, Y(i) \, di. \tag{2.7}$$

First order conditions for profit maximization lead to the following demand functions for intermediate goods:

$$Y(i) = P(i)^{1/(\lambda - 1)} \, Y. \tag{2.8}$$

We assume that the technology for producing an intermediate commodity is given by

$$Y(i) = K(i)^{\alpha} \, N(i)^{\beta}, \tag{2.9}$$

and to keep things simple we assume symmetry; that is, every intermediate commodity is produced with the same technology. We introduce increasing returns to scale in the intermediate goods sector with the assumption

$$\alpha + \beta > 1.$$

The profit function of the ith intermediate good producer can be expressed as follows by solving (2.8) for $P(i)$:

$$\Pi(i) = \left(\frac{Y(i)}{Y} \right)^{\lambda - 1} Y(i) - \omega N(i) - rK(i). \tag{2.10}$$

We assume that the intermediate goods producers are monopolistic competitors and we capture the degree of monopoly power of each producer by the parameter λ. When $\lambda = 1$ the intermediate goods are perfect substitutes in the production of the final good and in this case the intermediate producers face perfectly elastic demand curves. Substituting the production function into (2.10) we obtain

$$\Pi(i) = Y^{1-\lambda} N(i)^{\beta\lambda} K(i)^{\alpha\lambda} - \omega N(i) - rK(i). \tag{2.11}$$

The profit function will be concave in $N(i)$ and $K(i)$ as long as $\lambda(\alpha+\beta) \leqslant 1$.[4] Maximization of (2.11) by each monopolistic competitor leads to the following first order conditions:

$$\frac{\lambda\alpha Y(i)\, P(i)}{K(i)} = r, \tag{2.12}$$

$$\frac{\lambda\beta Y(i)\, P(i)}{N(i)} = \omega. \tag{2.13}$$

To derive the aggregate technology (2.1) from the model with monopolistic competition, we set

$$a = \lambda\alpha$$

and

$$b = \lambda\beta.$$

Since we have assumed symmetry we seek a solution in which

$$N(i) = N, \qquad K(i) = K, \qquad \text{and} \quad P(i) = \bar{P}.$$

The assumption that the final goods sector is competitive implies that

$$\Pi = Y - \int_0^1 \bar{P} Y(i)\, di = 0. \tag{2.14}$$

Using the intermediate demand functions (2.8) in the final goods zero-profit condition (2.14) we obtain the condition

$$P(i) = \bar{P} = 1, \tag{2.15}$$

[4] It is in this sense that the model with monopolistic competition is consistent with a degree of increasing returns in the technology. The closer λ is to zero, the larger α and β can be and still permit the existence of an interior solution to the profit maximization problem of each producer.

and using the symmetry assumption, $N(i)=N$, $K(i)=K$, in the production function (2.6) we can express aggregate final output as

$$Y=K^{\alpha}N^{\beta}. \tag{2.16}$$

Note that (2.16) is identical to (2.1). We can also show that factor payments plus profits in the intermediate goods sector add up to total output in the final goods sector since (2.11) implies that

$$\int_0^1 [\Pi(i)+\omega N(i)+rK(i)]\,di = Y^{1-\lambda}\int_0^1 K^{\alpha\lambda}N^{\beta\lambda}\,di = Y. \tag{2.17}$$

2.3. *The Consumer's Problem and Market Equilibrium*

In the previous section of the paper we described the structure of two alternative technologies and the associated maximization problems faced by profit maximizing firms. In this section we describe the intertemporal optimization problem faced by a representative consumer. We base our analysis in continuous time since the stability analysis of a steady state is cleaner in the continuous time framework. However, all of our results have analogs in the discrete time model and in Section 4 we point out some of the implications of our analysis for the research agenda that incorporates externalities into the real business cycle model.

The instantaneous utility of the representative consumer in our model is given by

$$U=\log C(t)-\frac{N^{1-\chi}}{1-\chi}, \tag{2.18}$$

where C is consumption, N is labor supply, and $\chi \leqslant 0$. It is well known that if one combines separability between consumption and leisure with a Cobb–Douglas production function, the use of a logarithmic utility function over consumption is the only formulation of preferences that is consistent with stationary labor supply in a growing economy.[5] The representative consumer maximizes

$$\int_0^{\infty}\left[\log C(t)-\frac{N(t)^{1-\chi}}{1-\chi}\right]e^{-\rho t}\,dt, \tag{2.19}$$

subject to

$$\dot{K}(t)=(r(t)-\delta)\,K(t)+\omega(t)\,N(t)+\Pi_{\mathrm{T}}(t)-C(t) \tag{2.20}$$

[5] We suspect that our analysis could be extended to allow for the kind of non-separable function that is exploited by Kydland and Prescott [19] in their "Time to Build..." paper, but we have not explored this generalization. The utility of labor is modeled as a power function over labor supply because this formulation allows us to derive a simple constant elasticity form for the labor supply function.

and

$$K(0) = K_0.$$

The parameter ρ represents the discount rate, δ is the depreciation rate, and $\Pi_T(t)$ is the total profits earned by the corporate sector. In the model with externalities factor payments exhaust total output and profits are zero because, by our assumption (2.5), $a + b = 1$. In the model with monopolistic competition the final goods producers make zero profits but there are positive profits in the intermediate goods sector that arise from the monopoly power of each of the intermediate producers. In this model the aggregate profits from the intermediate sector plus factor payments add up to the total output of the final sector, as shown in (2.17). For both models

$$\Pi_T(t) + r(t)\, K(t) + \omega(t)\, N(t) = Y(t). \tag{2.21}$$

The first-order conditions for the consumer's optimization problem are given by the equations

$$\frac{C(t)}{N(t)^{\chi}} = \omega_t, \tag{2.22}$$

$$\frac{\dot{C}}{C(t)} = r(t) - \rho - \delta. \tag{2.23}$$

In both of the organizational structures that we look at the real wage and the rental rate on capital will be proportional to the average products of labor and capital. Using these facts we can eliminate ω and r from the first-order conditions for the consumer to arrive at the following two expressions:

$$C(t) = b\,Y(t)\, N(t)^{\chi - 1} \tag{2.24}$$

$$\frac{\dot{C}}{C(t)} = a\,\frac{Y(t)}{K(t)} - \rho - \delta. \tag{2.25}$$

Equations (2.20), (2.24), and (2.25) describe the equilibrium growth dynamics of the economy. One also requires that the transversality condition

$$\lim_{t \to \infty} e^{-\rho t}\, \frac{K(t)}{C(t)} = 0 \tag{2.26}$$

should hold. In the next section of the paper we analyze the equilibrium dynamics of the model.

3. Analysis of the Dynamics

To simplify our analysis it is convenient to make the following logarithmic transformation of the variables. Let $y = \log(Y)$, $k = \log(K)$, $n = \log(N)$, and $c = \log(C)$. After dividing (2.20) by K one can express the two dynamic equations (2.20) and (2.25) in the form

$$\dot{k} = e^{y-k} - \delta - e^{c-k}, \tag{3.1}$$

$$\dot{c} = ae^{y-k} - \rho - \delta. \tag{3.2}$$

For (3.1) and (3.2) to be an autonomous pair of differential equations one must express $y - k$ in terms of k and c. From the production function, expressed in logarithmic form, it follows that

$$y = \alpha k + \beta n, \tag{3.3}$$

and from the first-order condition for optimal labor supply (2.24)

$$c = \log(b) + y + (\chi - 1)\, n. \tag{3.4}$$

Eliminating n from (3.3) and (3.4) allows us to obtain the expression for $y - k$,

$$y - k = \lambda_0 + \lambda_1 k + \lambda_2 c, \tag{3.5}$$

where

$$\lambda_0 = \frac{-\beta \log(b)}{\beta + \chi - 1},$$

$$\lambda_1 = \frac{(\chi - 1)(\alpha - 1) - \beta}{\beta + \chi - 1},$$

$$\lambda_2 = \frac{\beta}{\beta + \chi - 1}.$$

Using these definitions one may write the required pair of autonomous differential equations as

$$\dot{k} = e^{\lambda_0 + \lambda_1 k + \lambda_2 c} - \delta - e^{c-k}, \tag{3.6}$$

$$\dot{c} = ae^{\lambda_0 + \lambda_1 k + \lambda_2 c} - \rho - \delta. \tag{3.7}$$

Any trajectory $\{k(t), c(t)\}$ that solves (3.6) and (3.7) subject to the initial condition $k(0) = k_0$ and to the transversality condition (2.26) is an equilibrium path. We now turn to an analysis of the behavior of this pair of differential equations.

3.1. *Dynamics around a Steady State*

Under the assumptions of our model the system (3.6) and (3.7) has a unique interior steady state $\{k^*, c^*\}$. Some simple algebra yields

$$e^{(\lambda_1+\lambda_2)k^*} = \left(\frac{\rho+\delta(1-a)}{a}\right)^{-\lambda_2}\left(\frac{\rho+\delta}{a}\right)e^{-\lambda_0} > 0, \tag{3.8}$$

$$e^{c^*} = \left(\frac{\rho+\delta(1-a)}{a}\right)e^{k^*} > 0. \tag{3.9}$$

Note that

$$\left(\frac{\rho+\delta(1-a)}{a}\right) > 0,$$

since $a \leqslant a+b \leqslant 1$. This holds in both versions of our model since a and b represent factor share parameters. The (unique) solutions for k^* and c^* implied by these equations can be written as

$$k^* = \frac{1}{\lambda_1+\lambda_2}\left[\log\frac{\rho+\delta}{a} - \lambda_2\log\frac{\rho+\delta(1-a)}{a} - \lambda_0\right] \tag{3.10}$$

and

$$c^* = \log\frac{\rho+\delta(1-a)}{a} + k^*, \tag{3.11}$$

where we note that k^* and c^* are natural logarithms which may be positive or negative.

Using the definitions of λ_0, λ_1, and λ_2 and the expression for $y-k$, Eq. (3.5), we can compute the Jacobian of (3.6) and (3.7) evaluated at the steady state. The trace and the determinant of this Jacobian are given by the expressions

$$\text{Trace} = (\lambda_1 + a\lambda_2)\left(\frac{\rho+\delta}{a}\right) + \frac{\rho+\delta(1-a)}{a}, \tag{3.12}$$

$$\text{Det} = (\rho+\delta[1-a])\left(\frac{\rho+\delta}{a}\right)(\lambda_1+\lambda_2). \tag{3.13}$$

The variable k is predetermined since k_0 is given by the initial conditions of the economy while c_0 is free to be determined by the behavior of the agents in the economy. Suppose that the steady state $\{k^*, c^*\}$ is completely stable in the sense that all trajectories satisfying (3.6) and (3.7) which begin in the neighborhood of $\{k^*, c^*\}$ converge back to the steady state. In this case there will be a continuum of equilibrium paths

$\{k(t), c(t)\}$, indexed by c_0, since any path that converges to $\{k^*, c^*\}$ necessarily satisfies the transversality condition (2.26). Completely stable steady states giving rise to a continuum of equilibria will be termed "indeterminate" and in this case we say that the *stable manifold* has dimension 2.

Alternatively, if there is a one-dimensional manifold in $\{k, c\}$ space with the property that trajectories that begin on this manifold converge to the steady state but all other trajectories diverge then the equilibrium will be locally unique in the neighborhood of the steady state. In this case for every k_0 in the neighborhood of k^* there will exist a unique c_0 in the neighborhood of c^* that generates a trajectory converging to $\{k^*, c^*\}$. This c_0 is the one that places the economy on the *stable branch* of the saddle point $\{k^*, c^*\}$.

Since the trace of the Jacobian of (3.6)–(3.7) measures the sum of the roots and the determinant measures the product we can use information on the sign of the trace and the determinant to check the dimension of the stable manifold of the steady state $\{k^*, c^*\}$. When there is no capital externality (in the case of the monopolistically competitive model when the intermediate commodities become perfect substitutes), $a = \alpha$. In this case the trace evaluated at the steady state reduces to ρ, the discount rate. If the determinant, which has the sign of

$$\frac{(\alpha-1)(\chi-1)}{\beta+\chi-1},$$

is also negative then the roots must be of opposite sign and the steady state is a saddle point. This is the familiar case of "saddle path stability" in which there is a locally unique consumption rate that is consistent with convergence to the steady state for any given initial capital stock in the neighborhood of k^*.

If the determinant is positive there may be either two positive roots or two negative roots. When there are no capital externalities the case of two negative roots cannot occur since the sum of the roots is equal to ρ which is positive by the assumption of positive discounting. A positive determinant must therefore be associated with a stable manifold of dimension zero (an unstable steady state.) The eventual fate of trajectories that diverge from the steady state cannot be determined from the properties of the Jacobian evaluated at the steady state. They may eventually violate non-negativity constraints or they may settle down to a limit cycle or to some more complicated attracting set.[6]

[6] We cannot rule out the case where the Jacobian changes sign away from the steady state and we cannot therefore invoke the negative Bendixon criterion (see Guckenheimer and Holmes [11, p. 44, Theorem 1.8.2]) to rule out limit cycles.

The case of interest for our purposes is that of an indeterminate steady state, that is, the case when there is a negative trace and a positive determinant. This can occur for a relatively mild capital and labor externality when the other parameters of the model are calibrated at empirically plausible values. For example, set capital's share, a, at 1/3, labor's share, b, at 2/3, the continuously compounded discount rate at 0.02, the depreciation rate at 0.07, and the parameter χ at -0.25.[7] Then if $\alpha = 0.83$ and $\beta = 1.66$ (alternatively for $\lambda = 0.4$ in the monopolistically competitive model so that $\lambda\alpha = a = 0.33$ and $\lambda\beta = b = 0.67$,) one has a negative trace and positive determinant and consequently there exists an indeterminate steady state with a continuum of equilibrium trajectories indexed by c_0, each of which converges to the steady state. The indeterminacy of the steady state is robust to perturbations of parameters and in fact holds for a large open set of values around the ones given above.

3.2. *A Necessary Condition for Indeterminacy*

In order for there to exist a stable steady state, the determinant of the Jacobian evaluated at the steady state must be positive. But we have already shown that this condition implies that

$$\frac{(\alpha-1)(\chi-1)}{\beta+\chi-1} > 0,$$

and since χ is negative and we are generally considering economies in which $\alpha < 1$ it follows that this necessary condition will be satisfied when

$$\beta + \chi - 1 > 0.$$

Although it is extremely difficult to give intuitive explanations for the behavior of two-dimensional dynamical systems, we find the following reasoning helpful (although incomplete). In Fig. 1 we have depicted a labor demand curve parameterized by the stock of capital and a labor supply curve parameterized by the consumer's chosen rate of consumption. These equations come from breaking down the first-order condition for the optimal choice of labor (2.24) into a labor supply and a labor demand schedule as

$$\frac{C}{N^{\chi}} = \omega = b\,\frac{K^{\alpha}N^{\beta}}{N}, \tag{3.14}$$

[7] It is often assumed that preferences are logarithmic over *leisure*; see for example the work of King *et al.* [20]. Logarithmic preferences over leisure lead to a labor supply elasticity equal to $N^*/(1-N^*)$, where N^* is the steady state supply of labor. Since U.S. data suggests that the average worker allocates about one quarter of his time to productive activities it has become common to set this parameter to 0.25. The assumption that $\chi = -0.25$, using a preference function that is isoelastic over labor supply, is consistent with this tradition.

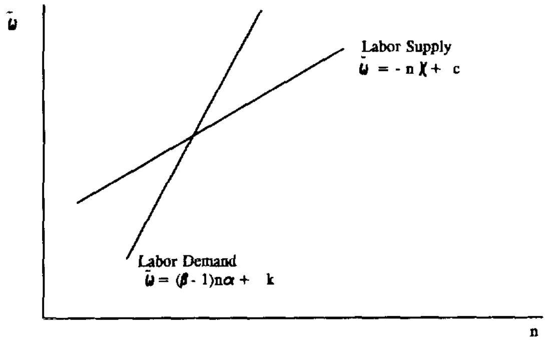

FIGURE 1

or, taking logarithms and letting $\tilde{\omega}$ represent the logarithm of the real wage,

$$c - \chi n = \tilde{\omega} = \log(b) + \alpha k + (\beta - 1)\, n. \qquad (3.15)$$

Note that the slope of the labor demand curve is given by $\beta - 1$ and the slope of the labor supply curve is equal to $-\chi$. In standard models the labor demand curve slopes down (as a function of employment) but in an economy in which increasing returns are important it may slope up. Our necessary condition for the existence of an indeterminate steady state is exactly equivalent to the requirement that the slope of the labor demand curve should exceed the slope of the labor supply curve. In this case expansions in the stock of capital which shift up the labor demand curve will tend to *lower* real wages and *reduce* employment. On an equilibrium trajectory, however, consumption will also grow with capital, and the growth in consumption will tend to *increase* real wages and *increase* employment. In Section 5 below, we show how the growth in consumption, capital, and the consumption-to-capital ratio is accompanied by the growth of employment on the transition to the stable balanced growth path.

4. Calibrated Economies and Indeterminacy

Since a number of authors[8] have begun to explore the implications of externalities in the standard real business cycle framework we think that it is important to understand the implications of this agenda for a theory of business fluctuations. The most notable of these is the possibility that a very slight departure from the standard paradigm can lead to the implication that business cycles may be generated by the self-fulfilling beliefs of

[8] Notably Baxter and King [2].

642/63/1-3

agents in addition to the effects of technological disturbances. Previous models that display this property have often been criticized for being "artificial" or for being "implausible."

Although we conduct our analysis of the growth model in continuous time the same effects hold in the class of discrete models that have become the main tool of analysis in real business cycle theory. Linearized versions of simple discrete models possess indeterminate steady states for the same magnitudes of time preference and other calibrated parameters that we use in the continuous time model. We choose to work in continous time because it simplifies the analysis and the presentation.

The original motivation for introducing externalities into the real business cycle model was to understand how demand disturbances may contribute to the cycle. In a *demand driven model* it is difficult to understand how labor productivity can be pro-cyclical, since employment fluctuations result from movements *along* a concave production function, as opposed to productivity driven shifts of the function itself. One solution to this problem is to make the technology convex by introducing increasing returns to the social production function of the same kind that we discussed in the first section of this paper. Baxter and King [2] cite studies by Caballero and Lyons [5] who find evidence of important external effects in panel data and they also present their own estimates of the marginal product of labor that are generated by using simultaneous equation estimators on aggregate data. Their findings are consistent with the view that the value of β in the technology is about one and a half times labor's share or that our parameter λ which represents the degree of monopolistic competitiveness in the economy is around 0.66.

In Tables I–III we report the theoretical values of the two eigenvalues of a perfect foresight model of the economy for different values of the parameters. Each of the three tables reports on the results of varying one of three critical parameters from an initial benchmark case. For our benchmark model we chose ρ (the rate of time preference) to be equal to 6.5% per year, a (capital's share) to be 0.42, δ (the rate of depreciation) to be 10% per year, the parameter χ (the inverse of the labor supply elasticity) to be -0.25, and we set the ratio of β to b equal to the ratio of α to a equal to 1.5. This case corresponds exactly to the benchmark increasing returns technology calibrated by Baxter and King [2]. In Table I we allow the inverse of the labor supply elasticity (the parameter χ) to vary between 0 and -0.3. This exercise is important since an influential variant of the standard model, originally introduced by Hansen [14], argues that the aggregate economy will behave as if labor were infinitely elastic.[9]

[9] The argument relies on indivisible labor at the individual level. See Hansen [14] and also Rogerson [24].

TABLE I

Varies Labor Elasticity

χ	Root 1	Root 2
0.00	0.86	−0.16
−0.05	0.70	−0.14
−0.10	0.60	−0.13
−0.15	0.54	−0.13
−0.20	0.50	−0.13
−0.25	0.46	−0.12
−0.30	0.43	−0.11

Capital's share, $a = 0.42$; externality, $\mu = 1.5$.

Note that for all values of χ in this range the model possesses one positive and one negative root. In Table II we hold labor elasticity at 4.0 ($\chi = -0.25$) and vary the externality parameter, which we define as $\mu = 1/\lambda$, from 1.0 to 1.9. In Table III we conduct a similar analysis allowing the share parameter a to fluctuate. In all of these experiments the roots of the economy split around zero; that is, the equilibrium is a saddle point.

In Table IV we allow two parameters to differ from the Baxter–King model by setting labor's share at 0.7 and letting the inverse labor elasticity fluctuate from 0.0 to −0.09. Note that for highly elastic labor supply the economy now has two stable roots and a continuum of equilibrium paths. In Fig. 2 we graph the real parts of the eigenvalues as a function of χ. For highly elastic labor supply there exists a pair of complex roots with

TABLE II

Varies Externalities

μ	Root 1	Root 2
1.00	0.26	−0.20
1.10	0.28	−0.19
1.20	0.31	−0.17
1.30	0.35	−0.16
1.40	0.40	−0.14
1.50	0.46	−0.13
1.60	0.56	−0.11
1.70	0.70	−0.09
1.80	0.94	−0.08
1.90	1.40	−0.06

Capital's share, $a = 0.42$; elasticity of labor supply, $\chi = -0.25$.

TABLE III

Varies Capital's Share

a	Root 1	Root 2
0.20	2.85	−0.73
0.25	1.32	−0.43
0.30	0.87	−0.29
0.35	0.65	−0.21
0.40	0.50	−0.15
0.45	0.41	−0.10
0.50	0.34	−0.07
0.55	0.29	−0.04
0.60	0.25	−0.02
0.65	0.21	−0.01

Externality, $\mu = 1.5$; elasticity of labor supply, $\chi = -0.25$.

negative real parts which again implies that the economy possesses an indeterminate steady state. As χ moves below -0.015 the roots both become real but remain negative until at (approximately) $\chi = -0.05$ one root passes through minus infinity and reemerges as a positive real root. To explore the sensitivity of these results to fluctuations in the externality parameter, in Fig. 3 we hold the value of χ at 0.0 (the Gary Hansen model) and we let the externality parameter $\mu = 1/\lambda$ vary from 1 (no externalities) to 2. Again we find the emergence of a pair of negative real roots at a value

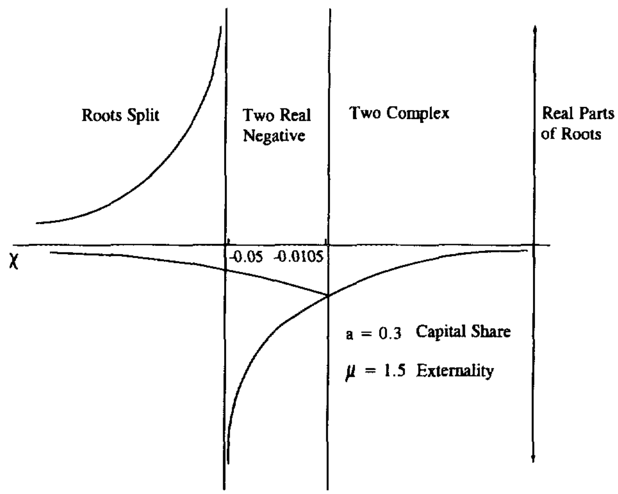

FIG. 2. Figure is not drawn to scale.

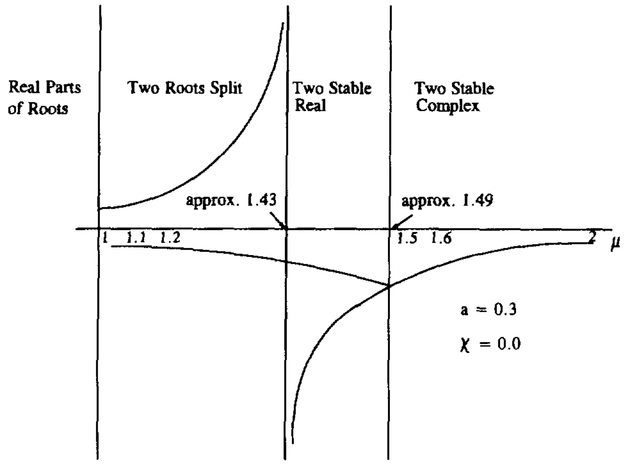

FIG. 3. Figure is not drawn to scale.

of about 1.43 for the externality parameter. At $\mu = 1.49$ these roots become complex but retain a negative real part.

We conclude from this exercise that the possibility of an indeterminate steady state is more than a pure theoretical curiosity since it occurs well within regions of the parameter space that are consistent with simple stylized facts about the business cycle.

TABLE IV

Varies Labor Elasticity

χ	Root 1	Root 2
0.00	$-0.8+0.4i$	$-0.8-0.4i$
-0.01	$-1.0+0.1i$	$-1.0-0.1i$
-0.02	-2.00	-0.70
-0.03	-3.60	-0.60
-0.04	-8.00	-0.50
-0.05	$+\infty$	-1.16
-0.06	9.30	-0.50
-0.07	5.00	-0.44
-0.08	3.40	-0.42
-0.09	2.70	-0.40

Externality, $\mu = 1.5$; capital's share, $a = 0.30$.

5. The Dynamics of Endogenous Growth

The model that we have described above will not display growth unless there is exogenous technical progress. More recently, a literature has emerged in which one tries to explain growth *endogenously*. Since a good part of this literature relies on externalities of the same kind that we have exploited in this paper we spend this section examining the connection between endogenous growth, increasing returns, and indeterminacy.

One may define a balanced growth path for the system (3.6), (3.7) as a trajectory $\{\bar{k}(t), \bar{c}(t)\}$ such that

$$\dot{k} = \dot{c} = g,$$

where g is a constant. As in the steady state case we say that a balanced growth path is *locally indeterminate* if, given a k_0 in the neighborhood of $\bar{k}(0)$, for any c_0 in the neighborhood of $\bar{c}(0)$, the equilibrium trajectory $\{k(t), c(t)\}$ converges to $\{\bar{k}, \bar{c}\}$. This definition ensures that the transversality condition holds,

$$\lim_{t \to \infty} e^{-\rho t} \frac{K(t)}{C(t)} = 0,$$

since the fact that $C(t)$ and $K(t)$ are growing at the same rate implies that $K(t)/C(t)$ is a constant.

To facilitate the analysis of balanced growth we transform the variables and set

$$q = c - k = \log(C/K). \tag{5.1}$$

Then using (3.5) and the definitions of the constants λ_0, λ_1, and λ_2, the system can be expressed as

$$\dot{k} = e^{(\lambda_1 + \lambda_2) k + \lambda_2 q + \lambda_0} - \delta - e^q, \tag{5.2}$$

$$\dot{q} = (a - 1)\, e^{(\lambda_1 + \lambda_2) k + \lambda_2 q + \lambda_0} - \rho + e^q. \tag{5.3}$$

A balanced growth path requires that $\dot{k} = g$ and $\dot{q} = \dot{c} - \dot{k} = 0$. Since $a < 1$, it is clear from (5.3) that if $\dot{k} = g \neq 0$ (if k is changing through time) then $\dot{q} = 0$ (q will be constant) only if $\lambda_1 + \lambda_2 = 0$. Therefore a necessary condition for the existence of an endogenous balanced growth path is that

$$\lambda_1 + \lambda_2 = 0.$$

From the definitions of λ_1 and λ_2,

$$\lambda_1 + \lambda_2 = \frac{(\alpha - 1)(\chi - 1)}{\beta + \chi - 1},$$

and since $\chi < 0$ this definition implies that the necessary condition for balanced growth is that $\alpha = 1$, or that social output is linear in the capital stock.

To solve for the balanced growth path we define[10]

$$X = e^q = \frac{C}{K},$$

and write (5.3) as

$$\dot{q} = \frac{\dot{X}}{X} = (a-1)\, e^{\lambda_0} X^{\lambda_2} - \rho + X \equiv f(X). \tag{5.4}$$

The balanced growth paths of the system are given by the solutions to the equation

$$\dot{X} = Xf(X) = 0, \tag{5.5}$$

and the interior balanced growth paths are the solutions to

$$f(X) = 0. \tag{5.6}$$

The derivative of f is given by the expression

$$f'(X) = \lambda_2(a-1)\, e^{\lambda_0} X^{\lambda_2 - 1} + 1 > 0. \tag{5.7}$$

There are two cases to consider, $\lambda_2 < 0$ and $\lambda_2 \geqslant 1$.[11] First note that if $\lambda_2 < 0$, that is if $b + \chi - 1 < 0$, then there exists a unique interior balanced growth path. This follows from the facts that $f(0) = -\infty$, $f(+\infty) = +\infty$, f is continuous, and $f' > 0$ for $\lambda_2 < 0$.

The fact that $f'(\bar{X}) > 0$ implies that this balanced growth path is unstable in the sense that any initial value $X_0 \neq \bar{X}$ generates a trajectory that monotonically diverges from $\bar{X}$. For any initial value of K_0, a choice of $\bar{C}$ such that $C_0/K_0 = \bar{X}$ generates a unique equilibrium trajectory along which

[10] Recall that we are using uppercase letters to refer to levels and lowercase letters to refer to logarithms.

[11] If λ_2, where

$$\lambda_2 = \frac{\beta}{\beta + \chi - 1},$$

is greater than zero, then

$$\beta + \chi - 1 > 0,$$

and since $\chi < 0$ from the convexity of preferences in N it must also be true that

$$\lambda_2 - 1 = \frac{1 - \chi}{\beta + \chi - 1} > 0.$$

the consumption capital ratio is constant and from (5.2) and (5.3) it follows that the growth rate is given by:

$$g(X)|_{X=\bar{X}} = e^{\lambda_0}\bar{X}^{\lambda_2} - \delta - \bar{X} = \frac{a}{1-a}\bar{X} - \frac{\rho + \delta(1-a)}{1-a}. \tag{5.8}$$

When the parameter λ_2 is greater than one, these results must be modified since the function $f(X)$ need no longer be monotonic. In this case it is possible to establish that either an interior balanced growth path will not exist or two such paths will exist. This follows from the fact that if $\lambda_2 > 1$ then $f(X)$ is a concave function since

$$\frac{\partial^2 \dot{X}}{\partial X^2} = \lambda_2(\lambda_2 - 1)(a-1)\, e^{\lambda_0} X^{\lambda_2 - 2} < 0. \tag{5.9}$$

But if f is concave then it can cross zero at most two times.

Suppose that there are two balanced growth paths $\bar{X}_1$ and $\bar{X}_2$ with $\bar{X}_1 < \bar{X}_2$; it follows immediately that $\bar{X}_1$ is unstable and $\bar{X}_2$ is stable since $f'(\bar{X}_1)$ must be positive and $f'(\bar{X}_2)$ negative from the concavity of f. In this case, there exists a continuum of equilibrium trajectories. Given a K_0, any initial C_0 such that $C_0/K_0 \geqslant \bar{X}_1$ gives rise to an equilibrium trajectory. Furthermore, for all C_0 such that $C_0/K_0 > \bar{X}_1$, $C(t)/K(t)$ converges to $\bar{X}_2$. These trajectories are associated with higher consumption–capital ratios at all times than the balanced consumption–capital ratio $\bar{X}_1$ but, paradoxically, they result in *higher* asymptotic growth rates as is easily seen from (5.8). This apparent paradox occurs since a higher consumption–capital ratio is also associated with a higher level of employment and with a higher average product of capital.

In Fig. 4 we have depicted the functions $f(X)$ and $g(X)$ for an economy that is calibrated to match U.S. data. We set $\alpha = 1$ to permit endogenous growth and $\beta = 1.2$ to capture the labor externality, but we have allowed α and β to differ from $a = 0.2$ and $b = 0.66$.[12] We chose the inverse of the labor elasticity $\chi = 0$, the discount rate $\rho = 0.03$, and the depreciation rate $\delta = 0.1$. This calibrated economy has two steady states, one of which is determinate (unstable) and the other of which is indeterminate with a continuum of convergent paths. The indeterminate growth path is associated with a growth rate of about 2% and a ratio of consumption to income of around 0.84.

It is worth noting the striking differences between the equilibria that can arise in the endogenous growth model when there is indeterminacy. These

[12] Note that we have not chosen a and b to sum to one since we are allowing for positive profits in the intermediate goods sector which will appear as a residual return to capital in the accounting data.

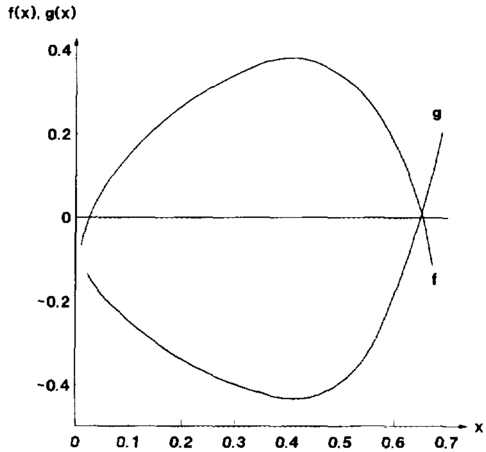

FIGURE 4

differences are particularly dramatic when we contrast an economy on the lower, determinate balanced growth path with another economy that has the same initial capital stock but which places itself on the higher indeterminate balanced growth path by choosing higher levels of initial consumption and employment. In the first case the economy contracts. In the second it also may contract initially, but the ratio of consumption to capital grows. The growing ratio of consumption to capital is associated with an increasing level of employment, which eventually raises the growth rate and the economy converges to a high balanced growth path with a positive growth rate. Observe from (5.4) that a path which begins with a higher consumption-to-capital ratio will always have a higher consumption-to-capital ratio, since solution trajectories cannot intersect. When there is an indeterminate growth path, it also follows that the optimal employment level is increasing in the consumption-to-capital ratio.[13] The overall welfare level in the high consumption, high employment economy can be dramatically higher, despite the higher level of employment. To see this consider two economies which both start out with the same capital stock, say, without loss of generality, at $K(0)=1$. The welfare levels on the two balanced growth paths can then be computed, given their initial consumption levels and their respective growth rates. For the parameter values given above, the indeterminate balanced growth path, which grows at a rate of about 2%, generates a welfare level of 6.46, while the determinate balanced growth path, which actually contracts at a rate of 13%, has a welfare level of about -261.33.

[13] To see this, set $\alpha = 1$ and totally differentiate (3.15).

3. Conclusion

We have explored a variant of the Ramsey growth model in which the economy displays increasing returns to scale. We find that for realistic parameter values the model may possess multiple dynamic equilibria all converging to a unique steady state. In a version of the model which displays endogenous growth we have shown that there may exist two balanced growth paths and that one of these paths will be associated with multiple dynamic equilibria, all of which converge to the same asymptotic growth rate.

Our work suggests that slight departures from the real business cycle model are consistent with the idea that economic fluctuations may be driven not only by productivity disturbances but also by the self-fulfilling beliefs of agents. Since the welfare implications of equilibria that depend only on fundamentals are very different from those which are driven by "sunspots" or "animal spirits," our work suggests that it may be important to explore the possibility that some classes of policy interventions may be associated with higher economic welfare.

References

1. C. Azariadis, Self-fulfilling prophecies, *J. Econ. Theory* **25** (1981), 380–396.
2. M. Baxter and R. King, "Productive Externalities and Cyclical Volatility," Working Paper 245, University of Rochester, 1990.
3. O. J. Blanchard and N. Kiyotaki, Monopolistic competition and the effects of aggregate demand, *Amer. Econ. Rev.* **77** (1987), 647–666.
4. M. Boldrin and A. Rustichini, Indeterminacy of equilibria in models with infinitely lived agents and external effects, *Econometrica* **62** (1994), 323–342.
5. R. J. Caballero and R. K. Lyons, The role of external economies in U.S. manufacturing, mimeo, Columbia University, New York, 1990.
6. D. Cass and K. Shell, Do sunspots matter?, *J. Polit. Econ.* **91** (1983), 193–227.
7. C. Chamley, Externalities and dynamics in models of learning or doing, *Int. Econ. Rev.* **34** (1993), 583–610.
8. P. A. Chiappori and R. Guesnerie, "Self-Fulfilling Theories: The Sunspot Connection," Working Paper, Ecole Des Hautes Etudes en Sciences Sociales, Paris, 1983.
9. A. Dixit and J. Stiglitz, Monopolistic competition and optimum product diversity, *Amer. Econ. Rev.* **67** (1977), 297–308.
10. R. Farmer and M. Woodford, "Self-Fulfilling Prophecies and the Business Cycle," CARESS Working Paper 84-12, University of Pennsylvania, Philadelphia, 1984.
11. J. Guckenheimer and P. Holmes, "Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields," Springer-Verlag, New York, 1983.
12. R. E. Hall, Invariance properties of Solow's productivity residual, *in* "Growth–Productivity–Unemployment" (P. Diamond, Ed.), pp. 71–112, MIT Press, Cambridge, MA, 1990.

13. M. L. HAMMOUR, "Social Increasing Returns in Macro Models with External Effects," Working Paper, Columbia University, 1989.
14. G. HANSEN, Indivisible labor and the business cycle, *J. Monet. Econ.* **16** (1985), 309–325.
15. P. HOWITT AND P. R. MCAFEE, Stability of equilibria with trade externalities, *Quart. J. Econ.* **103** (1988), 261–277.
16. P. HOWITT AND P. R. MCAFEE, Animal spirits, *Amer. Econ. Rev.* **82** (1992), 493–507.
17. T. J. KEHOE, D. LEVINE, AND P. M. ROMER, Characterizing equilibria of optimization problems with externalities and taxes as solutions to optimization problems, *Econ. Theory* **2** (1992), 43–68.
18. N. KIYOTAKI, Multiple expectational equilibria under monopolistic competition, *Quart. J. Econ.* **103** (1988), 695–714.
19. F. KYDLAND AND E. C. PRESCOTT, Time to build and aggregate fluctuations, *Econometrica* **50**, No. 6 (1982), 1345–1370.
20. R. G. KING, C. I. PLOSSER, AND S. T. REBELO, Production growth and business cycles. II. New directions, *J. Monet. Econ.* **21** (1988), 309–342.
21. R. E. LUCAS, On the mechanics of economic development, *J. Monet. Econ.* **22** (1988), 1–42.
22. C. B. MULLIGAN AND X. SALA-I-MARTIN, Transitional dynamics in two capital goods models of endogenous growth, mimeo, Yale University, 1991.
23. K. MURPHY, A. SCHLEIFER, AND R. VISHNY, "Building Blocks of Market Clearing Bussiness Cycle Models," NBER Macroeconomics Annual 1989, MIT Press, Cambridge, MA, 1989.
24. R. ROGERSON, Indivisible labor, lotteries and equilibrium, *J. Monet. Econ.* **21** (1988), 3–16.
25. P. ROMER, Increasing returns and long run growth, *J. Polit. Econ.* **94** (1986), 1002–1073.
26. P. ROMER, Endogenous technological change, *J. Polit. Econ.* **98**, suppl., No. 5, part 2 (1990), s71–s102.
27. K. SHELL, A model of inventive activity and capital accumulation, *in* "Essays on the Theory of Optimal Economic Growth" (K. Shell, Ed.), pp. 67–85, MIT Press, Cambridge, MA, 1967.
28. K. SHELL, "Monnaie et Allocation Intertemporelle", mimeo., Séminaire d'Econométrie Roy–Malinvaud, Centre Nationale de la Recherche Scientifique, Paris, November 21, 1977. [title and abstract in French, text in English]
29. R. M. SOLOW, Technical change and the aggregate production function, *Rev. Econ. Statist.* **39**, No. 3 (1957), 312–320.
30. S. SPEAR, Growth, externalities, and sunspots, *J. Econ. Theory* **54** (1991), 215–223.