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Game Theoretic Analysis of a Bankruptcy Problem from the Talmud*

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DEDICATED TO THE MEMORY OF SHLOMO AUMANN, TALMUDIC SCHOLAR AND MAN OF THE WORLD, KILLED IN ACTION NEAR KHUSH-E-DNEIBA, LEBANON, ON THE EVE OF THE NINETEENTH OF SIVAN, 5742 (JUNE 9, 1982).

For three different bankruptcy problems, the 2000-year old Babylonian Talmud prescribes solutions that equal precisely the nucleoli of the corresponding coalitional games. A rationale for these solutions that is independent of game theory is given in terms of the Talmudic principle of equal division of the contested amount; this rationale leads to a unique solution for all bankruptcy problems, which always coincides with the nucleolus. Two other rationales for the same rule are suggested, in terms of other Talmudic principles. (Needless to say, the rule in question is not proportional division). *Journal of Economic Literature Classification Numbers*: 022, 026, 031, 043, 213.

1. INTRODUCTION

A man dies, leaving debts $d_1,..., d_n$ totalling more than his estate E. How should the estate be divided among the creditors?

A frequent solution in modern law is proportional division. The rationale is that each dollar of debt should be treated in the same way; one looks at dollars rather than people. Yet it is by no means obvious that this is the only equitable or reasonable system. For example, if the estate does not exceed the smallest debt, equal division among the creditors makes good sense. Any amount of debt to one person that goes beyond the entire estate might well be considered irrelevant; you cannot get more than there is.

A fascinating discussion of bankruptcy occurs in the Babylonian

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TABLE I

Estate	Debt	100	200	300
	100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
	200	50	75	75
	300	50	100	150

Talmud[1] (Kethubot 93a). There are three creditors; the debts are 100, 200, and 300. Three cases are considered, corresponding to estates of 100, 200, and 300. The *Mishna*[2] stipulates the divisions shown in Table I.

The reader is invited to study Table I. When $E = 100$, the estate equals the smallest debt; as pointed out above, equal division then makes good sense. The case $E = 300$ appears based on the different—and inconsistent—principle of proportional division. The figures for $E = 200$ look mysterious; but whatever they may mean, they do not fit any obvious extension of either equal *or* proportional division. A common rationale for all three cases is not apparent.

Over two millenia, this Mishna[3] has spawned a large literature. Many authorities disagree with it outright. Others attribute the figures to special circumstances, not made explicit in the Mishna. A few have attempted direct rationalizations of the figures as such, mostly with little success. One modern scholar, exasperated by his inability to make sense of the text, suggested errors in transcription.[4] In brief, the passage is notoriously difficult.

This paper presents a game-theoretic analysis of the general bankruptcy problem, for arbitrary $d_1, ..., d_n$ and E. We obtain (Sect. 6) an explicit characterization of the nucleolus of the coalitional game that is naturally associated with this problem. For the three cases considered in the Talmud,

[1] A 2,000-year old document that forms the basis for Jewish civil, criminal, and religious law.

[2] The basic text that forms the starting point for the discussions recorded in the Talmud.

[3] The word "Mishna" is used both for the entire text on which the Talmud is based, and for specific portions of it dealing with particular issues. Similar ambiguities occur in many languages. One may say "My son studied law" as well as "Yesterday Congress passed a law."

[4] Lewy [7, p. 106], near the end of the long footnote.

the nucleolus prescribes precisely the numbers of Table I—those in the Mishna.

Of course, it is unlikely that the sages of the Mishna were familiar with the general notion of a coalitional game, to say nothing of the nucleolus. In Sections 3, 4, and 5, we present three different justifications of the solution to the bankruptcy problem that the nucleolus prescribes, in terms that are independent of each other and of game theory, and that were well within the reach of the sages of the Mishna. The justifications also fit in well with other Talmudic principles, a consideration that is no less significant than innate reasonableness in explaining the text.

To emphasize the independence from game theory, we start with the non-game theoretic analysis. In the research process, however, the order was reversed. Only after realizing that the numbers in the Mishna correspond to the nucleolus did we find independent rationales. Without the game theory, it is unlikely that we would have hit on the analysis presented in Sections 2 through 5.

This paper addresses two related but distinct questions, "what" and "why." The Mishna explicitly gives only three numerical examples; the first, most basic question is, *what* general rule did it have in mind, what awards would it actually assign to the creditors in an arbitrary bankruptcy problem? The second question is *why* did it choose the rule that it chose, what reasoning guided Rabbi Nathan (the author of this particular Mishna)? We have three different answers to the "why" question, all of them leading to the same answer to the "what" question. And while we cannot be sure that any particular one of our answers to the "why" question really represents Rabbi Nathan's thinking, all of them together leave little doubt that our answer to the "what" question is the correct one.

It is hoped that the research reported here will be of interest in two spheres—in the study of the Talmud, and in game theory. In this paper we concentrate on the mathematical, game-theoretic side. For motivation we will, when appropriate, present and explain underlying Talmudic principles; but there will be no careful textual analysis of Talmudic passages, no lengthy citation of authorities.[5] An analysis of the latter kind—concentrating on the validity of our explanation from the source viewpoint, but skimping on the mathematics—is being planned by one of us for publication elsewhere.

[5] In particular, this article should on no account be used as a source for Talmudic law. In citing Talmudic dicta, we mention only those of their aspects that are directly relevant to the matter at hand; additional conditions and circumstances are omitted. For more information the reader should refer to the sources, which we have been at pains to cite explicitly.

2. THE CONTESTED GARMENT

A famous Mishna (Baba Metzia 2a) states: "Two hold a garment; one claims it all, the other claims half. Then the one is awarded $\frac{3}{4}$, the other $\frac{1}{4}$."

The principle is clear. The lesser claimant concedes half the garment to the greater one. It is only the remaining half that is at issue; this remaining half is therefore divided equally.[6] Note that this is quite different from proportional division.

Let us transpose this principle to the 2-creditor bankruptcy problem with estate E and claims d_1, d_2. The amount that each claimant i concedes to the other claimant j is $(E-d_i)_+$, where

$$\theta_+ = \max(\theta, 0).$$

The amount at issue is therefore

$$E-(E-d_1)_+ - (E-d_2)_+;$$

it is shared equally between the two claimants, and, in addition, each claimant receives the amount conceded to her by the other one. Thus the total amount awarded to i is

$$x_i = \frac{E-(E-d_1)_+ - (E-d_2)_+}{2} + (E-d_j)_+. \tag{2.1}$$

We will say that this division (of E for claims d_1, d_2) is prescribed by the CG (contested garment) principle.[7,8]

If one views the solution as a function of E, one obtains the following process. Let $d_1 \leqslant d_2$. When E is small, it is divided equally. This continues until each claimant has received $d_1/2$. Each additional dollar goes to the greater claimant, until each claimant has received all but $d_1/2$ of her claim. Beyond that, each additional dollar is again divided equally. Note that the principle is *monotonic*, in the sense that for fixed claims d_1 and d_2, each of the two awards is a monotonic function of the estate E.

[6] This explanation is explicit in the eleventh century commentary of Rabbi Shlomo Yitzhaki (Rashi). Alternatively, one could say that the claims total $1\frac{1}{2}$, whereas the worth of the garment is only 1; the loss is shared equally.

[7] An additional instance of this rule may be found in the Tosefta to the first chapter of Baba Metzia, where $E=d_1=1$, $d_2=\frac{1}{3}$. (The Tosefta is a secondary source that is contemporaneous with the Mishna.)

[8] Alternatively, one may argue that neither claimant i can ask for more than $\min(E, d_i)$. If each claimant is awarded this amount, the total payment may exceed the estate; the excess is deducted in equal shares from the claimants' awards. This procedure leads to the same payoff as (2.1).

The legal circumstances of the contested garment are somewhat different from those of bankruptcy. In the garment case, there is uncertainty about the validity of the claims; they cannot both be justified. In the bankruptcy problem, all claims are definitely valid; there simply is not enough money to go around. Therefore, some authorities have held that the Mishna in Baba Metzia (about the garment) is not relevant to the bankruptcy problem.

While this certainly constitutes an important difference between the cases, it is not clear why it would make the principle of equal division of the contested amount inapplicable. Indeed, the early medieval authority Rabbi Hai Gaon (10th century) did express the opinion[9] that the Mishna in Kethubot (about bankruptcy) should be explained on the basis of that in Baba Metzia. He did not, however, make an explicit connection, and in subsequent years, this line of attack was abandoned.

3. Consistency

A *bankruptcy problem* is defined as a pair $(E; d)$, where $d = (d_1, ..., d_n)$, $0 \leqslant d_1 \leqslant \cdots \leqslant d_n$ and $0 \leqslant E \leqslant d_1 + \cdots + d_n$. A *solution* to such a problem is an n-tuple $x = (x_1, ..., x_n)$ of real numbers with

$$x_1 + \cdots + x_n = E$$

(x_i is the amount assigned to claimant i). A solution is called *CG-consistent*, or simply *consistent*, if for all $i \neq j$, the division of $x_i + x_j$ prescribed by the contested garment principle for claims d_i, d_j is (x_i, x_j).

Intuitively, a solution is consistent if any two claimants i, j use the contested garment principle to divide between them the total amount $x_i + x_j$ awarded to them by the solution. It may be verified that the solutions in Table I are consistent.

Theorem A. *Each bankruptcy problem has a unique consistent solution.*

Proof. First we prove that there is at most one consistent solution. If there were more, we could find consistent solutions x and y, and creditors i and j, with $y_i > x_i$, $y_j < x_j$, and $y_i + y_j \geqslant x_i + x_j$. Consistency implies that if just i and j are involved, the CG principle awards y_j to j when the total estate is $y_i + y_j$, and x_j when it is $x_i + x_j$. Since $y_i + y_j \geqslant x_i + x_j$, the

[9] Quoted by Rabbi Isaac Alfasi (1013–1103) in his commentary on our Mishna in Kethubot.

monotonicity of the CG principle then implies $y_j \geqslant x_j$, contradicting[10] $y_j < x_j$.

To show that there is at least one consistent solution, we exhibit it as a function of the estate E (for fixed debts $d_1,..., d_n$). Let us think of the estate as gradually growing. When it is small, all n claimants divide it equally. This continues until 1 has received $d_1/2$; for the time being she[11] then stops receiving payments, and each additional dollar is divided equally between the remaining $n-1$ claimants. This, in turn, continues until 2 has received $d_2/2$, at which point *she* stops receiving payments for the time being, and each additional dollar is divided equally between the remaining $n-2$ claimants. The process continues until each claimant has received half her claim. This happens when $E = D/2$, where

$$D := d_1 + \cdots + d_n = \text{the total debt.}$$

When $E \geqslant D/2$, the process is the mirror image of the above. Instead of thinking in terms of i's award x_i, one thinks in terms of her loss $d_i - x_i$, the amount by which her award falls short of her claim. When the total loss $D - E$ is small, it is shared equally between all creditors, so that creditor i receives her claim d_i less $(D-E)/n$. The creditors continue sharing each additional dollar of total loss equally, until 1 has lost $d_1/2$ (which is the same as receiving $d_1/2$). For the time being she then stops losing, and each additional dollar of total loss is divided equally between the remaining $n-1$ claimants. This, in turn, continues until 2 has lost $d_2/2$ (= received $d_2/2$), at which point *she* stops losing for the time being, and each additional dollar is divided equally between the remaining $n-2$ claimants. The process continues until each claimant has lost half her claim, which happens when $E = D/2$. This is precisely to where we got in describing the first part of the procedure; we have dug the tunnel form both its ends, and have met in the middle.

It will be useful to give an alternative description of the procedure, in terms of increasing award. Recall that when E was slightly less than $D/2$, claimant n was receiving all of each additional dollar of the estate. She continues to do so as E passes $D/2$, until she has received a total of $d_n - (d_{n-1}/2)$, i.e., all but $d_{n-1}/2$ of her claim. At this point, $n-1$ reenters the picture, and each additional dollar is shared equally between n and

[10] This part of the proof was generated at a seminar presentation at the IMA in Minneapolis; it replaces a more devious proof that we previously had. Mainly responsible are Y. Kannai and D. Kleitman.

[11] While we are sympathetic with the feminist movement, the reader should not conclude from our use of "she" that we write half our papers in the feminine gender. Much of the Talmud is couched in terms of case law; and while the passage under discussion does form the basis of bankruptcy law in general, the creditors in this particular case were women.

$n-1$. This continues until n and $n-1$ have each received all but $d_{n-2}/2$ of their claims (which happens at the same instant). At this point $n-2$ reenters the picture, and so on. Creditor 1 reenters the picture when *all* creditors have received all but $d_1/2$ of their claims; each additional dollar of estate is shared equally between all creditors.

Consider now two claimants i and j, where $d_i \leqslant d_j$. When E is small, they receive equal amounts. This continues until i has received $d_i/2$. Beyond that, i leaves the picture for the time being, and only j may receive any part of each additional dollar; that is, each additional dollar received by both together goes to j. This continues until j has received all but $d_i/2$ of her claim; beyond that, i and j again receive equal shares of each additional dollar. But this is precisely the verbal description of the CG solution given in the previous section. This shows that the solution we have exhibited is indeed consistent, and completes the proof of Theorem A.

Define a *rule* as a function that assigns a solution to each bankruptcy problem. The *CG-consistent* (or simply *consistent*) rule is the one that assigns the CG-consistent solution to each bankruptcy problem. A rule f is called *self-consistent* if

$$f(E; d) = x \qquad \text{implies} \quad f(x(S); d|S) = x|S$$

for each set S of creditors, where $x|S$ means "x restricted to S," and $x(S)$ is short for $\sum_{i \in S} x_i$. In words, *any* subset S of the set of all creditors (not only a 2-person subset) uses the rule f to divide among its members the total amount $x(S)$ that it gets when the rule f is applied to the original bankruptcy problem.

COROLLARY 3.1. *The CG-consistent rule is self-consistent.*

Proof. Let $(E; d)$ be a bankruptcy problem, x its CG-consistent solution, g the function that assigns to each 2-person bankruptcy problem its CG solution, and S a set of creditors. For *any* i, j, the CG-consistency of x yields $(x_i, x_j) = g(x_i + x_j;\ d_i, d_j)$; in particular, this is so for $i, j \in S$. But that means that $x|S$ is the CG-consistent solution of $(x(S); d|S)$. Q.E.D.

Self-consistency and CG-consistency are totally different kinds of concepts. Self-consistency applies to *rules*, CG-consistency to *individual solutions*. If three creditors come to a judge and ask him to divide an estate between them, and he does so in some specific way, then they cannot complain that he is not self-consistent; to do that they would have to know what he would have decided in other situations. But CG-consistency can be checked directly for each proposed solution of each case separately. If one thinks of the principle of CG-consistency as "just," then one can complain about the injustice of one particular decision; the corresponding statement cannot be made for self-consistency.

The CG-consistent rule is by no means the only self-consistent one. Others include division in proportion to the claims, and the constrained equal division solutions in the next section; these rules also play an important role in the Talmudic discussion of bankruptcy.

Various consistency conditions that are similar in spirit to those discussed here play an important role in game theory and bargaining theory. For examples, see Sections 6 and 7 below.

4. Self Duality and Constrained Equal Division

Define the *dual* f^* of a rule f by

$$f^*(E; d) := d - f(D - E; d);$$

f^* assigns awards in the same way that f assigns losses. A *self-dual* rule is one with $f^* = f$; such a rule treats losses and awards in the same way.

Two prominent features of the consistent rule, both of which follow from the explicit characterization in the proof of Theorem A, are its self-duality, and the qualitative change in the rule that occurs at $E = D/2$. In this section we discuss these features and show that they are strongly rooted in the Talmudic literature. We end the section with an alternative characterization (Theorem B) of the consistent rule in terms of these features, a characterization not directly related to the CG principle.

The basic idea behind duality is that there are certain types of division problem in which it is natural to think in terms of dividing the award (amount received, gain), and other problems in which it is more natural to think in terms of dividing the loss. All other things being equal, it seems appropriate to apply dual solution rules to these problems. In still other problems, it is equally natural to think in terms of losses or of gains; in such cases a self-dual rule is called for.

We start by illustrating the notion of duality with some rules other than the consistent one. A *constrained equal award* (CEA) solution of a bankruptcy problem $(E; d)$ is one of the form $(\alpha \wedge d_1, \ldots, \alpha \wedge d_n)$, where $a \wedge b := \min(a, b)$. In words, this means that all claimants get the same award α, except that those who claim less than α get their claims. Note that

$$\text{each bankruptcy problem has a unique CEA solution.} \tag{4.1}$$

Indeed, $\sum_{i=1}^{n} \alpha \wedge d_i$ is a continuous strictly increasing function of α on the interval $[0, d_n]$, and maps this interval onto $[0, D]$; hence every point in $[0, D]$ is attained precisely once, proving (4.1).

In our Mishna, the CEA rule prescribes equal awards to all creditors up to $E = 300$; it prescribes (100, 150, 150) for $E = 400$, and (100, 200, 200) for

$E = 500$. The rule, which divides each additional dollar equally between those claimants who still have an outstanding claim, seems natural enough; it has been adopted as law by most major codifiers, including Maimonides[12] (1135–1204).

Maimonides's great intellectual adversary, Rabad,[13] adopted a different rule. This provides equal awards for all creditors when $E \leqslant d_1$. When E passes d_1, 1 leaves the picture, and each additional dollar is divided equally among the remaining $n-1$ creditors. When E passes d_2, also 2 leaves the picture, and each additional dollar is divided equally among the remaining $n-2$ creditors; and so on. The rule is not defined beyond $E = d_n$. This, too, seems quite natural; when $E \leqslant d_n$ one might say that all creditors have a claim on the first d_1 dollars, only $2, ..., n$ on the next $d_2 - d_1$ dollars, and so on.[14]

We would not discuss these rules in such detail if it were not for the remarkable fact that their duals also make an explicit appearance in the Talmud (Erakhin 27b).[15] At an auction there are n bidders, who bid $b_1 < b_2 < \cdots < b_n$. If n reneges, the object is acquired by $n-1$, and the seller sustains a loss of $b_n - b_{n-1}$; this loss must be paid by n, as the price of being allowed out of his contract. Suppose now that all n bidders renege, and that for one reason or another the object cannot be sold to anyone else; then the loss to the seller is b_n, and this must somehow be shared among the bidders. How?

In this case Maimonides says that the loss is divided equally among all bidders, subject, of course, to no bidder paying more than his bid.[16] Rabad,[17] on the other hand, divides the loss into the n successive increments $b_1, b_2 - b_1, b_3 - b_2, ..., b_n - b_{n-1}$. The first increment is paid by all bidders in equal shares, the second in equal shares by all except 1, and

[12] *The Laws of Lending and Borrowing*, Chapter 20, Section 4.

[13] Acronym for Rabbi Abraham ben David (1125–1198).

[14] This rule is implicit already in the Babylonian Talmud's discussion of our Mishna, and so goes back at least to the third or fourth century. It first appears explicitly in Alfasi (op. cit.), and is also mentioned by Rashi and many other medieval commentators. Only Rabad, though, seems to have adopted it as law; see his gloss on Alfasi (op. cit.). The rule is also mentioned by the mathematician Abraham Ibn Ezra in connection with certain types of inheritance problem [3, p. 60ff.]; cf. Rabinovitch [16, p. 162] and O'Neill [13]. In modern times it has surfaced again as the solution to the airport landing problem [8]; it is closely connected with the Shapley value [18], a game-theoretic solution concept that is conceptually quite different from the nucleolus.

[15] We are grateful to Y. Aumann for bringing this reference to our attention.

[16] *The Laws of Appraisal*, Chapter 8, Section 4. Unlike for bankruptcy, Maimonides here gives no clear general rule; for the specific numerical example treated in the Talmud (and by Maimonides), his division is equal. But it is difficult to imagine that Maimonides would ever require a reneging bidder to pay more than his bid.

[17] See his gloss on Maimonides, op. cit.

so on. These rules are the exact duals of the rules adopted by these same authorities for bankruptcy.

It is apparent that these authorities had precise ideas on how to deal with division problems, and that they applied them to the award or the loss according to whether the funds were to be received (as in bankruptcy) or paid (as in the auction). But one can also approach the problem not so much from the technical viewpoint of the direction in which money flows at the specific time of the court decision, but from the more substantive viewpoint of whether the protagonists themselves consider the transaction an award or a loss. In bankruptcy, for example, the creditors will in the end receive checks as a result of the court proceedings. Nevertheless, they are worse off than before making the loan, and they may well conceive of the transaction as a loss rather than an award.

This suggests a rule in which (i) awards and losses are treated dually, and (ii) it makes no difference whether we think of the outcome as an award or a loss.[18] Together, (i) and (ii) call for a self-dual rule.

Self-duality was just one of the two prominent features of the consistent rule mentioned at the beginning of this section. The other was the qualitative change in the rule that occurs at the "halfway point," $E = D/2$. This, too, is strongly rooted in the Talmud; there are dozens—perhaps hundreds—of discussions hinging on the principle that "more than half[19] is like the whole" (Hulin 27a). For example, kosher slaughter of an animal calls for cutting through the windpipe and the foodpipe; but as long as more than half of each pipe is cut, the meat is still kosher (op. cit.).

Another example is based on the Talmudic principle that in general, a lender automatically has a lien on the borrower's real property. But when his entire property is worth less than half the loan, the borrower may in certain cases dispose of it "free and clear" (Erakhin 23b). Rashi explains that when the property is worth more than half the loan, the lien is of considerable importance, and the lender relies on it as a guarantee. But when it is worth less than half the loan, the lien will not help very much anyway; the loan was presumably made "on trust," and we are not justified in repossessing the property from the bona fide recipient.

Again, the principle involved here is "more than half is like the whole;" property amounting to more than half the loan is conceptually close to covering it all, and cannot be ignored. Less than half is like nothing; property covering less than half the loan is inconsiderable, need not be taken into account. This is not merely a legal convention, but is explicitly based on a psychological presumption. In the "less-than-half" case, the len-

[18] Specifically, whether we think of the outcome to Creditor i as an award of x_i or a loss of $d_i - x_i$.

[19] "Rov."

der is presumed not to rely on the lien, to lend "on trust." Psychological presumptions of this kind often play an important role in Talmudic law.[20]

In the bankruptcy problem, too, the half-way point is a psychological watershed. If you get more than half your claim, your mind focusses on the full debt, and your concern is with the size of your loss. If you get less than half, your mind writes off the debt entirely, and is "happy" with whatever it can get; your concern is with your award. Moreover, it is socially unjust for different creditors to be on opposite sides of this watershed; for one creditor to get most of his claim, while another one loses most of his. Subject to this constraint, therefore, the losses are divided equally when $E \geqslant D/2$, the awards when $E \leqslant D/2$. In brief, we have

THEOREM B. *The consistent rule is the unique self-dual rule that, when* $E \leqslant D/2$, *assigns to* $(E; d)$ *the constrained equal award solution of* $(E; d/2)$.

Mathematically, this theorem is simply a concise expression of the explicit construction in the proof of Theorem A. Conceptually, though, it provides an additional, independent characterization of the consistent rule.

We end this section by mentioning two properties of the consistent rule that will be useful in the sequel. Call a rule f *monotonic* if $f_i(E; d)$ is a non-decreasing function of E when i and d are held fixed; that is, no claimant loses from an increase in the estate. Call a solution x to a bankruptcy problem $(E; d)$ *order-preserving* if

$$0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \quad \text{and} \quad 0 \leqslant d_1 - x_1 \leqslant \cdots \leqslant d_n - x_n.$$

That is, a person with a higher claim than another gets an award that is no smaller and suffers a loss that is no smaller;[21] ordinally, one might say, both the award and the loss are scaled to the claim. As we have said, the consistent rule is monotonic and yields order-preserving solutions. So are the other rules we have considered here: Division in proportion to the claims, the CEA rule, its dual, and the solution adopted by Rabad (insofar as it is defined).

[20] E.g., the finder's right to found property depends on whether the loser can be presumed to have "despaired" of regaining it (Baba Metzia 21a ff.).

[21] Also, awards are non-negative and do not exceed the claims; but this statement follows from the other if we include outsiders, whose claim is 0.

5. Coalition Formation and the Jerusalem Talmud

In discussing our Mishna, the Jerusalem Talmud[22] says as follows: "Samuel says, the Mishna takes it that the creditors empower each other; specifically, that the third empowers the second to deal with the first. She may say to her, 'Your claim is 100, right? Take 50 and go.'"

Samuel is obviously referring to the cases $E = 200$ and $E = 300$. The second and third creditors (whose claims are 200 and 300, respectively) form a coalition "against" the first (whose claim is 100). This leaves two effective protagonists, with claims of 500 and 100, respectively; applying the contested garment rule yields the first creditor 50, with the remainder going to the coalition. If the coalition again applies the contested garment rule to divide its award among its members, the numbers in the Mishna result.

If applied to the case $E = 100$, this procedure would lead to the payoff vector (50, 25, 25), which is not order-preserving: 1's award is larger than that of 2 or of 3. It begins yielding order-preserving results at $E = 150$; as E rises, it continues to do so until $E = 450$. Beyond that, order preservation again fails, this time on the loss side. At $E = 500$, for example, we obtain (50, 175, 275), so that 1's loss is larger than that of 2 or of 3.

We may proceed in the same way in any problem with three creditors. First 2 and 3 pool their claims and act as a single agent vis-a-vis 1. The CG solution of the resulting problem yields awards to 1, and to the coalition of 2 and 3; to divide its award among its members, the coalition again applies the CG principle. The result is order preserving if and only if $3d_1/2 \leqslant E \leqslant D - (3d_1/2)$. If one divides the awards equally when $E \leqslant 3d_1/2$, and the losses equally when $E \geqslant D - (3d_1/2)$, one obtains precisely the consistent solution over the entire range $0 \leqslant E \leqslant D$.

By using induction, one may generalize this in a natural way to arbitrary n. Suppose we already know the solution for $(n-1)$-person problems. Depending on the values of E and d, we treat a given n-person problem in one of the following three ways:

(i) Divide E between $\{1\}$ and $\{2, \ldots, n\}$ in accordance with the CG solution of the 2-person problem $(E; d_1, d_2 + \cdots + d_n)$, and then use the $(n-1)$-person rule, which we know by induction, to divide the amount assigned to the coalition $\{2, \ldots, n\}$ between its members.

(ii) Assign equal awards to all creditors.

(iii) Assign equal losses to all creditors.

[22] The Jerusalem Talmud is based on the same source (the Mishna) as the Babylonian Talmud, and is contemporaneous with it. Although considered less authoritative, it is valued as an independent parallel source, which often sheds light on obscure passages in the Babylonian Talmud. We are very grateful to Yehonatan Aumann for calling to our attention the remarkable passage that forms the basis for this section.

Specifically, (i) is applied whenever it yields an order-preserving result, which is precisely when $nd_1/2 \leqslant E \leqslant D-(nd_1/2)$. We apply (ii) when $E \leqslant nd_1/2$, and (iii) when $E \geqslant D-(nd_1/2)$. This is called the *coalitional procedure*.

For example, let $n=5$, $d_i=100\,i$, $E=510$. At the first step of the induction, the coalition {2. 3, 4, 5} forms; its joint claim is 1400, while 1's claim is 100. Applying the CG rule yields 50 to 1, and 460 to the coalition. To divide the 460 between 2, 3, 4, and 5, one splits {2, 3, 4, 5} into 2 and {3, 4, 5}, and again applies the CG rule. This yields 100 to 2, and 360 to {3, 4, 5}. If one were again to split {3, 4, 5} into 3 and {4, 5}, then 3 would be awarded 150, leaving 4 and 5 only 210 to divide between them. At least one of them would therefore receive $\leqslant$ 105, so that the result would not be order-preserving. At this point, therefore, the 360 are split equally between 3, 4, and 5. The final result is (50, 100, 120, 120, 120), which *is* order-preserving.

Note that this is the consistent solution of the above problem. More generally, it may be verified that

THEOREM C. *The coalitional procedure yields the consistent solution for all bankruptcy problems.*

Theorem A and C both use the CG principle to characterize the same rule; but the two characterizations are conceptually totally different. Theorem A applies the CG principle to pairs of *individuals* only, and it is applied to all $\binom{n}{2}$ such pairs. The $\binom{n}{2}$ resulting conditions are desiderata of a solution, but they do not tell us directly how we should arrive at one; it is not a priori clear that they have any simultaneous solution at all, or that they do not have more than one. Theorem C applies the CG principle to pairs of *coalitions*; and it describes an orderly step-by-step process, which by its very definition must lead to a unique result. But it uses only certain carefully selected pairs of coalitions, not all such pairs.[23]

6. THE NUCLEOLUS AND THE KERNEL

It will be recalled that a (*coalitional*) *game* is a function v that associates a real number $v(S)$ with each subset S of a finite set N. The members of N

[23] As described—with the coalitions {1} and {2,..., n}—the coalitional procedure yields a monotonic rule and order-preserving solutions. Moreover, it appears that they are the *only* such coalitions, though we have no satisfactory formulation and proof of such a result. Also, it appears that if the creditors may form coalitions as they wish, then for $E \leqslant D/2$, the incentives lead to the coalitions suggested by the coalitional procedure. These matters call for further study.

are called *players*, the sets S *coalitions*. Intuitively, $v(S)$ represents the total amount of payoff that the coalition S can get by itself, without the help of other players; it is called the *worth* of S. By convention, $v(\emptyset)=0$. A *payoff vector* is a vector x with components indexed by the players; x_i represents the payoff to i.

Solution concepts associate payoff vectors with games; each such concept represents a specific notion of stability, expected outcome, or the like. In many cases a solution concept associates several payoff vectors with a game, or none at all. Only two of the better known solution concepts associate a unique payoff vector with each game; they are the value [18] and the nucleolus [17].

As it stands, the bankruptcy problem considered here is not a game; coalitions do not appear explicitly in its formulation. A natural way to associate a game with a bankruptcy problem $(E; d)$ is to take the worth of a coalition S to be what it can get without going to court; i.e., by accepting either nothing, or what is left of the estate E after each member i of the complementary coalition $N\backslash S$ is paid his complete claim d_i. Thus we define the (*bankruptcy*) *game* $v_{E;d}$ *corresponding* to the bankruptcy problem $(E; d)$ by

$$v_{E;d}(S) := (E-d(N\backslash S))_+ \tag{6.1}$$

THEOREM D. *The consistent solution of a bankruptcy problem is the nucleolus of the corresponding game.*

The proof of Theorem D makes use of several concepts and results of cooperative game theory. Let v be a game, S a coalition, x a payoff vector. The *reduced game* $v^{S,x}$ is defined [1, 19, 9, 14] on the player space S as follows:

$$\begin{aligned} v^{S,x}(T) &= x(T) && \text{if} \quad T=S \text{ or } T=\emptyset, \\ &= \max\{v(Q\cup T)-x(Q)\colon Q\subset N\backslash S\} && \text{if} \quad \emptyset \subsetneqq T \subsetneqq S. \end{aligned}$$

In the reduced game, the players of S consider how to divide the total amount assigned to them by x under the assumption that players i outside S get exactly x_i. Together, all the players of S get $x(S)$; as always, the empty set gets nothing. If a non-empty proper subcoalition T of S chooses a set Q of "partners" outside S, it will have total worth $v(Q\cup T)$; but to keep the partners satisfied, it will have to pay them the total $x(Q)$ assigned to them by x. Thus T will choose its partners Q to maximize the amount $v(Q\cup T)-x(Q)$ left for it after paying off the partners.

LEMMA 6.2. *Let x be a solution of the bankruptcy problem $(E; d)$, such that $0\leqslant x_i\leqslant d_i$ for all i. Then for any coalition S,*

$$v_{E;d}^{S,x}=v_{x(S);d|S}. \tag{6.2}$$

(In words, the reduced bankruptcy game is the game corresponding to the "reduced bankruptcy problem.")

Proof. Set $v := v_{E;d}$ and $v^S := v^{S,x}_{E;d}$. First let $\emptyset \subsetneqq T \subsetneqq S$, and let the maximum in the definition of $v^S(T)$ be attained at Q. Since $x_i \geqslant 0$ and $a_+ - b_+ \leqslant (a-b)_+$ for all a and b, we have

$$\begin{aligned} v^S(T) &= v(T \cup Q) - x(Q) = (E - d(N \setminus (Q \cup T)))_+ - (x(Q))_+ \\ &\leqslant (x(N) - d(N \setminus (Q \cup T)) - x(Q))_+ \\ &= [x(S) - d(S \setminus T) - (d-x)(N \setminus (S \cup Q))]_+ \\ &\leqslant (x(S) - d(S \setminus T))_+, \end{aligned} \tag{6.3}$$

where the last inequality follows from $x_i \leqslant d_i$. On the other hand, setting $Q = N \setminus S$ yields

$$\begin{aligned} v^S(T) &\geqslant v(T \cup (N \setminus S)) - x(N \setminus S) \\ &= (E - d(N \setminus (T \cup (N \setminus S))))_+ - (x(N) - x(S)) \\ &\geqslant (E - d(S \setminus T)) - (E - x(S)) = x(S) - d(S \setminus T); \end{aligned} \tag{6.4}$$

and setting $Q = \emptyset$ yields

$$v^S(T) \geqslant v(T \cup \emptyset) - x(\emptyset) = v(T) = (E - d(N \setminus T))_+ \geqslant 0. \tag{6.5}$$

Formulas (6.4) and (6.5) together yield

$$v^S(T) \geqslant (x(S) - d(S \setminus T))_+ ;$$

together with (6.3), this yields

$$v^S(T) = (x(S) - d(S \setminus T))_+ = v_{x(S);d|S}(T). \tag{6.6}$$

When $T = \emptyset$ or $T = S$, formula (6.6) is immediate, so the proof of the lemma is complete.

We also make use of the solution concepts called kernel [1] and pre-kernel [9]. Let v be a game. For each payoff vector x and players i, j, define

$$s_{ij}(x) = \max\{v(S) - x(S) \colon S \text{ contains } i \text{ but not } j\}.$$

The *pre-kernel* of v is the set of all payoff vectors x with $x(N) = v(N)$ and $s_{ij}(x) = s_{ji}(x)$ for all i and j. The *kernel* of v is the set of all payoff vectors x with $x(N) = v(N)$, $x_i \geqslant v(i)$ for[24] all i, and for all i and j,

$$s_{ij}(x) > s_{ji}(x) \qquad \text{implies} \quad x_j = v(j).$$

[24] We do not distinguish between i and $\{i\}$.

One more definition is required. The *standard solution* of a 2-person game v with player set $\{1, 2\}$ is given by

$$x_i = \frac{v(12) - v(1) - v(2)}{2} + v(i). \tag{6.7}$$

Note that this is equivalent to $x_1 + x_2 = v(12)$, $x_1 - x_2 = v(1) - v(2)$. In words, the standard solution gives each player i the amount $v(i)$ that he can assure himself, and divides the remainder equally between the two players. The nucleolus and kernel,[25] the pre-kernel and the Shapley value of a 2-person game all coincide with its standard solution; so do most of the better-known bargaining solutions [12, 4, 11]. Indeed, the standard solution constitutes the only symmetric and efficient point-valued solution concept for 2-person games that is covariant under strategic equivalence.[26]

LEMMA 6.8. *Let x be in the pre-kernel of a game v, and let S be a coalition with exactly two players. Then $x|S$ is the standard solution of $v^{S,x}$.*

Proof. Let $S = \{i, j\}$. Then

$$\begin{aligned} s_{ij}(x) &= \max_{Q \subset N \setminus S} (v(Q \cup i) - x(Q \cup i)) \\ &= \max_{Q \subset N \setminus S} (v(Q \cup i) - x(Q)) - x_i = v^{S,x}(i) - x_i; \end{aligned}$$

similarly $s_{ji}(x) = v^{S,x}(j) - x_j$. Since by the definition of pre-kernel, $s_{ij}(x) = s_{ji}(x)$, it follows that $v^{S,x}(i) - x_i = v^{S,x}(j) - x_j$. Hence

$$x_i - x_j = v^{S,x}(i) - v^{S,x}(j)$$

and

$$x_i + x_j = x(i,j) = v^{S,x}(i, j),$$

which proves the lemma.[27]

Remark. The converse of this lemma is also true; i.e., if $x(N) = v(N)$ and $x|S$ is the standard solution of $v^{S,x}$ for all 2-person coalitions S, then x is in the pre-kernel of v. From this it follows that if $|N| \geqslant 3$, then x is in the pre-kernel of v if and only if $x(N) = v(N)$ and $x|S$ is in the pre-kernel of $v^{S,x}$ for all coalitions $S \subsetneqq N$ [14].

[25] When they are non-empty, which is the case whenever there is at least one payoff vector x that is both individually rational ($x_i \geqslant v(i)$ for all i) and efficient ($x(N) = v(N)$).

[26] A similar remark is made in [14]. Rules such as proportional division or the CEA rule are not covariant in terms of the game v.

[27] This lemma is a special case of Lemma 7.1 in [1].

LEMMA 6.9. *The contested garment solution of 2-person bankruptcy problem is the standard solution of the corresponding game.*

Proof. Follows from (2.1), (6.1), and (6.7).

PROPOSITION 6.10. *The kernel of a bankruptcy game $v_{E;d}$ consists of a single point, namely the consistent solution of the problem $(E; d)$.*

Proof. Set $v = v_{E;d}$, and let x be in its kernel. By its definition (6.1), v is superadditive $(S \cap T = \varnothing \Rightarrow v(S) + v(T) \leqslant v(S \cup T))$ and hence 0-monotonic $(S \subset T \Rightarrow v(S) + \sum_{i \in T \setminus S} v(i) \leqslant v(T))$. In 0-monotonic games, the kernel coincides with the pre-kernel [10]. Hence x is in the pre-kernel of v.

Now let S be an arbitrary 2-person coalition. By Lemma 6.8, $x|S$ is the standard solution of $v^{S,x}$, and hence by Lemma 6.2, of $v_{x(S);d|S}$. Hence by Lemma 6.9, $x|S$ is the CG-solution of $(x(S); d|S)$; but that means that x is the consistent solution of $(E; d)$. Q.E.D.

Theorem D follows from Proposition 6.10, since the nucleolus is always in the kernel [17].[28]

We stated Theorem D in terms of the nucleolus because it is better known than the kernel and conceptually simpler (it is point valued). In fact, though, the idea of CG-consistency is more closely related to the kernel than it is to the nucleolus.[29] We have already noted that the contested garment solution is in fact simply the standard solution of 2-person games. Thus an appropriate generalization of the notion of a consistent solution to an arbitrary game v is a payoff vector x such that whenever S is a 2-person coalition, $x|S$ is the standard solution of the reduced game $v^{S,x}$. Lemma 6.8, together with the theorem of Peleg cited in the succeeding remark, show that with this definition, the set of all consistent solutions of an arbitrary game is precisely its pre-kernel; and as we have noted, this coincides with the kernel for 0-monotonic games.

7. HISTORICAL NOTES

In the Talmudic bankruptcy literature, the rule that is perhaps closest to ours was proposed by Piniles [15, p. 64]; it coincides with ours for $E \leqslant D/2$, but beyond that they differ. Evidently, Piniles was unaware of the connection with the contested garment, since even for the CG Mishna itself

[28] Since this is the only property of the nucleolus that we require, we will not cite its definition here. The interested reader may consult the original article [17], or any one of several equivalent characterizations [5, 6, 19].

[29] The nucleolus has more to do with self-consistent rules (Sect. 3) than with CG-consistent solutions. See [19].

(where $E = 1 > \frac{3}{4} = D/2$), his rule gives $(\frac{5}{8}, \frac{3}{8})$ rather than $(\frac{3}{4}, \frac{1}{4})$. We owe the reference to Erakhin 23b (see Sect. 4) to Piniles.

In bargaining theory, the idea of consistency first appeared in [2, p. 328], where Harsanyi characterized the product maximization solution to an n-person bargaining problem as the unique solution x at which each two players i, j use the Nash solution [12] to divide between them what remains if every other player k gets x_k.

In the context of apportionment, self-consistency is discussed in Balinski and Young's "Fair Representation" (Yale University Press, 1982, 43–45 and 141–149). Alexander Hamilton's apportionment method, vetoed by George Washington but nevertheless used in the U.S. from 1852 until 1901 (and in many countries until today), is not self-consistent. Just before Oklahoma became a state in 1907, the House of Representatives had 386 seats. Oklahoma was allocated 5, bringing the total to 391. Nothing else changed, so presumably the apportionment among the old states should have remained the same. Yet under Hamilton's method, Oklahoma's joining meant New York losing a seat to Maine!

Others who have recently used ideas related to consistency include H. Moulin and W. Thomson. A particularly striking result, as yet unpublished, is by T. Lensberg, who showed that Nash's bargaining solution [12] can be characterized by a set of axioms in which the Independence of Irrelevant Alternatives is replaced by a self-consistency axiom.

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