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Wide-sense nonblocking for multirate 3-stage Clos networks

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Abstract

The 3-stage Clos network $C(n,m,r)$ in the multirate environment has recently been studied for strictly nonblocking and rearrangeably nonblocking, but not much is known for wide-sense nonblocking. This is not really surprising since very little is known about wide-sense nonblocking even for the classical circuit switching environment. In this paper, we propose a class of "quota" algorithms and show that by using such an algorithm the number m of center switches required is always less than that for strictly nonblocking. In particular, when no bound is set for the rate (except it is greater than zero and not exceeding the link capacity), then m required for strictly nonblocking is unbounded, while $5.75n$ suffice for our algorithm. Better results for the 2-rate and 3-rate environments are also obtained.

1. Introduction

The *3-stage Clos network* $C(n,m,r)$ is generally considered the most basic multistage interconnection network (*MIN*). A result obtained for $C(n,m,r)$ is often extendible to MIN with more than three stages. $C(n,m,r)$ is symmetric with respect to the center stage. The first stage, or the *input stage* (hence the third stage or the *output stage*), has r $n \times m$ (crossbar) switches; the center stage has m $r \times r$ (crossbar) switches. The n inlets (outlets) on each input (output) switch are the *inputs* (*outputs*) of the network. There exists exactly one link between every center switch and every input (output) switch. We will refer to the inputs and outputs as *external links* and the network links as *internal links*.

In classical circuit switching, three types of nonblocking properties have been extensively studied [1]. A *call* between an idle pair (input, output) is *routable* if there

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exists a path connecting them such that no link on the path is used by any other connection paths. A network is *strictly nonblocking* if regardless of the routing of existing connections in the network, a new call is always routable. A network is *wide-sense nonblocking* (WSNB) if a new call is always routable as long as all previous requests were routed according to a given routing algorithm. A network is *rearrangeably nonblocking,* or simply *rearrangeable,* if a new call is always routable given that we can reroute existing connections. Clearly, strictly nonblocking implies WSNB implies rearrangeable.

In the multirate environment, a call is a triple (u,v,w) where u is an inlet, v an outlet and w a weight which can be thought of as the bandwidth requirement (rate) of that call. We normalize the weights such that $1 \geqslant w > 0$. In the weakly *uniform capacity model,* each internal link has a capacity one; namely, it can carry any number of calls as along as the sum of weights of these calls does not exceed one. We also require that a call (u,v,w) can be generated only if the sum of weights of calls (u,z,w) over all z, and the sum of weights of all calls (y,v,w) over all y, currently carried in the network are both at most $\beta - w$. This is equivalent to setting the capacity of an external link to be β. For the special case $\beta = 1$, the weakly uniform capacity model becomes simply the *uniform capacity model.* When a 3-stage Clos network is expanded to 5-stage (which can be further expanded to $(2s+1)$-stage) by replacing each $r \times r$ crossbar switch in the center stage with a $C(n',m',r/n')$, then external links of $C(n',m',r/n')$ become internal links of the 5-stage network and the uniform capacity model is preserved.

Some important results have been given [3, 5–7], for the strictly nonblocking and rearrangeable multirate 3-stage Clos network, but almost nothing on WSNB except Melen and Turner [6] showed that $C(n,8n,r)$ is multirate WSNB. This is not surprising since there are very few WSNB results even for the classical circuit switching environment [1, 2, 4, 8]. The purpose of this paper is to fill such a void. We show that in the multirate environment, only $5.75n$ center switches are required for WSNB.

2. Some preliminary remarks

Since strictly nonblocking implies WSNB, we first review what is known for strictly nonblocking multirate 3-stage Clos network as a starting point for WSNB networks. Let B denote an upper bound of the weight and b a lower bound. Melen and Turner [6] proved

Theorem 2.1. *$C(n,m,r)$ is multirate strictly nonblocking if $w \in [b,1]$ and $m \geqslant 2\lfloor (n-1)/b \rfloor + 3$.*

Chung and Ross [3] improved to

Theorem 2.2. *$C(n,m,r)$ is multirate strictly nonblocking if $w \in [b,1]$ and $m \geqslant 2\lfloor 1/b \rfloor (n-1) + 1$.*

They also showed

Theorem 2.3. *$C(n,m,r)$ is multirate strictly nonblocking if $w \in (0,B]$ and*

$$m \geqslant \lim_{\varepsilon \downarrow 0} 2\left\lfloor \frac{n-B}{1-B+\varepsilon} \right\rfloor + 1.$$

Niestegge [7] gave the following result for finite number of weights.

Theorem 2.4. *$C(n,m,r)$ is multirate strictly nonblocking if $w \in [b,B]$, b divides all weights and 1, and $m \geqslant 2\lfloor (n-B)/(1-B+b) \rfloor + 1$.*

A multirate environment is called a *k-rate* environment if there are only k different rates.

Corollary 2.5. *$C(n,2n-1,r)$ is 1-rate strictly nonblocking if the rate divides 1.*

Note that when $B \to 1$ and $b \to 0$, the number of center switches required is unbounded in all the above theorems. Niestegge was the first to notice that WSNB may help. He gave an example for $n = 4$ and w is either 1 or $\frac{1}{4}$. From Theorem 2.4, $m \geqslant 25$ is required. But if all calls with weight 1 are routed through one group of center switches, and all calls with weight $\frac{1}{4}$ are routed through another group, then seven center switches suffices for each group by using Corollary 2.5. Hence the necessary m is reduced from 25 to 14.

We now generalize Corollary 2.5. We first introduce some terminology. A call (u,v,w) will also be referred to as a (U,V,w) call if u is in the input switch U, and v in the output switch V. The *U-load* (resp., *V-load*) of a center switch s is the sum of weights of all calls from U (resp., to V) carried by s. The *(U,V)-load* is the sum of the U-load and the V-load.

Lemma 2.6. *Suppose that $\beta/p \geqslant B \geqslant b > \beta/(p+1)$ for some positive integers p. Then $C(n,2n-1,r)$ is strictly nonblocking.*

Proof. Suppose the call (U,V,w) is blocked. Then this call cannot be routed through a center switch s if and only if s carries p calls from U. At most $\lfloor (pn-1)/p \rfloor$ centers switches can carry p calls from U. Similarly, at most $\lfloor (pn-1)/p \rfloor$ center switches can carry p calls to V; hence $2\lfloor (pn-1)/p \rfloor + 1 = 2n-1$ center switches suffice. □

We can now generalize Theorem 2.4.

Theorem 2.7. *Suppose that the rates can be partitioned into k classes such that all rates in class i satisfy $\beta/p_i \geqslant w > \beta/(p_i+1)$ for some integer p_i. Then $C(n,k(2n-1),r)$ is WSNB.*

Proof. Use $2n-1$ center switches for each class of calls. □

Corollary 2.8. *$C(n, k(2n-1), r)$ is k-rate WSNB.*

Again, the number of center switches required is unbounded if the number of weight-classes is unbounded.

In this paper we propose a new type of routing algorithm using the "quota" scheme. Weights (or calls) are classified into *large* and *small*. $P(x,y)$ denotes the algorithm that x center switches are designated as *restricted* switches each is allowed to carry no more than y small calls, but can carry as many large calls as capacity allows. Thus, $P(x,0)$ is a reservation algorithm where x switches are reserved only for large calls, and $P(0,0)$ is an algorithm where every call can be routed through any switch with capacity. The quota scheme can also be extended to more than two types of calls, or used recursively. We show that using the quota schemes, $C(n, 5.75n, r)$ is WSNB for any set of rates. We also give better results when w can be bounded and when the environment is 2-rate or 3-rate.

3. The general multirate case

Define $p = \lfloor 1/B \rfloor$. Label a call *large* if $w > 1/(p+1)$, and *small* otherwise. For easier presentation, we ignore the integrality of m and we use $2n$, instead of the correct $2(n-w)$, as the maximum (U,V)-load before a call (U,V,w) is to be routed. We call this the *ideal* assumption.

Theorem 3.1. *$C(n,m,r)$ is WSNB under $P(x,0)$ where*

$$x = \begin{cases} \dfrac{2\beta(p+1)(Bp+B-1)\beta}{p^2} & \text{for } B < \frac{23}{32} = 0.71875, \\[2ex] 2\beta & \text{for } B \geqslant \frac{23}{32}, \end{cases}$$

if $w \in (0,B]$ and $m \geqslant m^* \equiv \min\{5.75\beta n, 2\beta(p+1)(Bp+B+p-1)n/p^2\}$.

Proof.

Case (i): $B < \frac{23}{32}$. Suppose a large call (U,V,w) is blocked. Then each of the xn restricted switches must carry p calls either from U or to V, hence a (U,V)-load exceeding $p/(p+1)$. Furthermore, each of the $2\beta(p+1)n/p$ nonrestricted switches must carry a load exceeding $(1-w) \geqslant (1-B)$. Therefore, the total (U,V)-load carried exceeds

$$\frac{2\beta(p+1)(Bp+B-1)n}{p^2}\frac{p}{p+1} + \frac{2\beta(p+1)n}{p}(1-B) = 2\beta n,$$

contradicting the fact that both the U-load and the V-load are upper bounded by βn (hence the (U,V)-load upper bounded by $2\beta n$).

Table 1

B	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
x	0.022	0.096	$0.1\dot{7}$	0.3	0.75	0.8	1.6	2	2	2
m^*/n	2.222	2.496	$2.8\dot{4}$	3.3	3.75	4.85	5.6	5.75	5.75	5.75
m^0/n	$2.\dot{2}$	2.5	2.857	$3.\dot{3}$	4	5	6.6	10	20	∞

Next suppose a small call (U, V, w) is blocked. Then each nonrestricted switch must carry a (U, V)-load exceeding $(1-w) \geqslant p/(p+1)$. Thus, the total (U, V)-load carried exceeds

$$\frac{2\beta(p+1)n}{p}\frac{p}{p+1} = 2\beta n,$$

again, a contradiction.

Note that for $B = 0.5$, $m^* = 3.75\beta n$.

Case (ii): $B \geqslant \frac{23}{32}$. By Lemma 2.6, $2n-1$ center switches can carry all large calls. By the result in Case (i), $3.75\beta n$ additional center switches can carry all small calls. □

We compare m^* with $m^0 \equiv 2n/(1-B)$ which is the m-value given in Theorem 2.3 for strictly nonblocking except under the ideal assumption (β is omitted) (see Table 1). Thus, we see that $m^* < m^0$ always, and the difference increases with B and is unbounded.

In many practical applications, the environment is k-rate with small k. We show that we can do better than Corollary 2.8 and Theorem 3.1 for 2-rate and 3-rate in the next section.

4. The 2-rate and 3-rate cases for the uniform capacity model

First consider the 2-rate environment. Let B and b, $1 \geqslant B > b > 0$ be the two rates.

Theorem 4.1. *$C(n, 3n, r)$ is WSNB if $B \leqslant \frac{1}{2}$.*

Proof. If $\frac{1}{3} \geqslant B > b$, then Theorem 4.1 follows from Theorem 3.1 by setting $p = 3$, $B = \frac{1}{3}$ and noting

$$\frac{2(p+1)(Bp+B+p-1)n}{p^2} = \frac{2(4)(\frac{10}{3})n}{9} = \frac{80n}{27}.$$

If $\frac{1}{2} \geqslant B > b > \frac{1}{3}$, then Theorem 4.1 follows from Lemma 2.6. Therefore, it suffices to consider the case $\frac{1}{2} \geqslant B > \frac{1}{3} \geqslant b$. Define q_0, q_1, q_2 in

$$q_0 b \leqslant 1 < (q_0+1)b,$$

$$B + q_1 b \leqslant 1 < B + (q_1+1)b,$$

$$2B + q_2 b \leqslant 1 < 2B + (q_2+1)b.$$

Since

$$2(B+q_1b)\leqslant 2<(q_0+1)b+2B+(q_2+1)b$$

and

$$q_0b+2B+q_2b\leqslant 2<2[B+(q_1+1)b],$$

we have

$$-1\leqslant\delta\equiv q_0+q_2-2q_1\leqslant 1.$$

We also have

$$\frac{q_2}{q_0+1}<q_2b\leqslant 1-2B<\frac{1}{3}.$$

Hence,

$$3q_2\leqslant q_0=2q_1-q_2+\delta,$$

$$2q_2\leqslant q_1+\delta/2,$$

which implies (by the integrality of q_1 and q_2)

$$2q_2\leqslant q_1 \quad \text{if } \delta=0 \text{ or } 1.$$

and

$$2q_2+1\leqslant q_1 \quad \text{if } \delta=-1.$$

Suppose $P(x,q_2)$ is the algorithm, where x is to be defined later. Consider the $2n$ external links of U and V. Assuming the worst scenario, every such external link generates a maximal set of calls, i.e., it generates q_2 b-calls, or 1 B-call and q_1 b-calls, or 2 B-calls and q_2 b-calls. Let c_0n, c_1n, c_2n denote the numbers on external links of U and V generating these sets of calls, respectively.

Claim. *Suppose that* $z_0+z_1+z_2=z$. *Then*

$$(z_1+2z_2)(q_1-q_2)+z_0q_0+z_1q_1+z_2q_2=z(2q_1-q_2)+z_0\delta.$$

Proof.

$$\begin{aligned}(z_1+2z_2)(q_1-q_2)&+z_0q_0+z_1q_1+z_2q_2\\&=(z-z_0+z_2)(q_1-q_2)+z_0q_0+z_1q_1+z_2q_2\\&=z(q_1-q_2)+z_0q_0-(z_0-z_1-z_2)q_1+z_0q_2\\&=z(q_1-q_2)+z_0q_0+(z-2z_0)q_1+z_0q_2\\&=z(2q_1-q_2)+z_0(q_0-2q_1+q_2)\\&=z(2q_1-q_2)+z_0\delta.\end{aligned}$$

Let yn denote the number of nonrestricted switches. We consider four cases.

Case (i): $\delta = 1$. Let

$$x = \frac{8q_1^2 - 12q_1q_2 + 4q_2^2 - 2q_2 - 2}{8q_1^2 - 14q_1q_2 + 4q_2^2 + 6q_1 - 6q_2 + 1},$$

$$y = \frac{16q_1^2 - 32q_1q_2 + 12q_2^2 + 16q_1 - 16q_2 + 4}{8q_1^2 - 14q_1q_2 + 4q_2^2 + 6q_1 - 6q_2 + 1}.$$

Suppose a call (U, V, B) is blocked. Then each restricted switch must carry 2 B-calls and each nonrestricted switch a load exceeding $1 - B$, the minimal such loads are $q_1 + 1$ b-calls, 1 B-call and $q_2 + 1$ b-calls, 2 B-calls. Let y_0, y_1 and $y_2 (y_0 + y_1 + y_2 = y)$ denote the numbers of nonrestricted switches carrying these loads, respectively. Counting the number of b-calls and B-calls generated and carried (recall one B-call is generated but not carried), we have

$$y_0(q_1 + 1) + y_1(q_2 + 1) \leqslant c_0q_0 + c_1q_1 + c_2q_2,$$

$$2x + y_1 + 2y_2 < c_1 + 2c_2.$$

Multiplying the second inequality by $(q_1 - q_2)$ and adding the first, then the left-hand side of the new inequality is

$$\begin{aligned}
&(2x + y_1 + 2y_2)(q_1 - q_2) + y_0(q_1 + 1) + y_1(q_2 + 1) \\
&\quad = 2x(q_1 - q_2) + (y_1 + 2y_2)(q_1 - q_2) + y(q_1 + 1) \\
&\qquad - (y_1 + y_2)(q_1 + 1) + y_1(q_2 + 1) \\
&\quad = 2x(q_1 - q_2) + y(q_1 + 1) + y_2(q_1 - 2q_2 - 1),
\end{aligned}$$

while the right-hand side is $2(2q_1 - q_2) + c_0$ by the claim. Therefore,

$$\begin{aligned}
2x(q_1 - q_2) + y(q_1 + 1) &< 2(2q_1 - q_2) + c_0 - y_2(q_1 - 2q_2 - 1) \\
&\leqslant 2(2q_1 - q_2) + 2 - (c_1/2 + c_2) + y_2 \\
&< 2(2q_1 - q_2) + 2 - (2x + y_1 + 2y_2)/2 + y_2 \\
&\leqslant 2(2q_1 - q_2 + 1) - x,
\end{aligned}$$

which is false by a straightforward verification (substituting in the specified x and y values).

Now suppose a call (U, V, b) is blocked. Let $y_0, y_1, y_2, y_0 + y_1 + y_2 = y$ be the numbers of nonrestricted switches carrying q_0 b-calls, 1 B-call and q_1 b-calls, 2 B-calls and q_2 b-calls, respectively. Then we have

$$y_0q_0 + y_1q_1 + y_2q_2 + xq_2 < c_0q_0 + c_1q_1 + c_2q_2,$$

$$y_1 + 2y_2 \leqslant c_1 + 2c_2.$$

Again, multiplying the second inequality by $(q_1 - q_2)$ and adding the first, we obtain (by the claim)

$$y(2q_1 - q_2) + y_0 + q_2 x < 2(2q_1 - q_2) + c_0$$

or

$$y(2q_1 - q_2) + xq_2 < 2(2q_1 - q_2) + 2 - c_1 - c_2 - y_0$$

$$< 2(2q_1 - q_2 + 1) - (y_1 + 2y_2)/2 - y_0 \leqslant 2(2q_1 - q_2 + 1) - y/2,$$

which is also false by a straightforward verification.

The analyses of the other three cases are similar to Case (i) except slightly different conditions induce different values for x and y. We will merely list the implied inequalities which can be verified to be false.

Case (ii): $\delta = 0$, $q_1 \geqslant 2q_2 + 1$. Let

$$x = \frac{4q_1^2 - 6q_1q_2 + 2q_2^2 - 4q_1 + 2q_2}{4q_1^2 - 7q_1q_2 + 4q_2^2 - q_2}, \qquad y = \frac{8q_1^2 - 16q_1q_2 + 6q_2^2}{4q_1^2 - 7q_1q_2 + 4q_2^2 - q_2}.$$

Then

$$2x(q_1 - q_2) + y(q_1 + 1) < 2(2q_1 - q_2) - y_2(q_1 - 2q_2 - 1) \leqslant 2(2q_1 - q_2) \quad \text{for } B\text{-call},$$

$$y(2q_1 - q_2) + xq_2 < 2(2q_1 - q_2) \quad \text{for } b\text{-call}.$$

Case (iii): $\delta = 0$, $q_1 = 2q_2$. Let

$$x = \frac{4q_1^2 - 6q_1q_2 + 2q_2^2}{4q_1^2 - 7q_1q_2 + 2q_2^2 + 2q_1 - 2q_2}, \qquad y = \frac{8q_1^2 - 16q_1q_2 + 6q_2^2 + 4q_1 - 4q_2}{4q_1^2 - 7q_1q_2 + 2q_2^2 + 2q_1 - 2q_2}.$$

Then

$$2(q_1 - q_2) + y(q_1 + 1)$$

$$< 2(2q_1 - q_2) + y_2 < 2(2q_1 - q_2) + (c_1 + 2c_2 - 2x - y_1)/2$$

$$\leqslant 2(2q_1 - q_2) + (4 - 2x)/2 = 2(2q_1 - q_2) + 2 - x \quad \text{for } B\text{-call},$$

$$y(2q_1 - q_2) + xq_2 < 2(2q_1 - q_2) \quad \text{for } b\text{-call}.$$

Case (iv): $\delta = -1$, which implies $q_1 \geqslant 2q_2 + 1$. Let

$$x = \frac{4q_0q_1^2 - 6q_0q_2q_2 + 2q_0q_2^2 - 4q_0q_1 + 2q_0q_2 - 2q_1^2 - 2q_1}{4q_0q_1^2 - 7q_0q_1q_2 + 2q_0q_2^2 - q_0q_2 - q_1q_2 - q_2},$$

$$y = \frac{8q_0q_1^2 - 16q_0q_1q_2 + 6q_0q_2^2 + 4q_1^2 - 8q_1q_2 + 2q_2^2}{4q_0q_1^2 - 7q_0q_1q_2 + 2q_0q_2^2 - q_0q_2 - q_1q_2 - q_2}.$$

Then

$$2x(q_1 - q_2) + y(q_1 + 1) < 2(2q_1 - q_2) - c_0 - y_2(q_1 - 2q_2 - 1)$$
$$\leqslant 2(2q_1 - q_2) \quad \text{for } B\text{-call;}$$
$$y(2q_1 - q_2) + xq_2 < 2(2q_1 - q_2) + y_0 - c_0 < 2(2q_1 - q_2)$$
$$+ (c_1q_1 + c_2q_2 - y_1q_1 - y_2q_2 - xq_2)/q_0$$
$$\leqslant 2(2q_1 - q_2) + (2q_1 - xq_2)/q_0 \quad \text{for } b\text{-call.} \qquad \square$$

Remark. In each of the four cases considered in the proof, we actually gave the (x, y) pair which minimizes $x + y$.

Theorem 4.2. *Consider the 2-rate environment where* $B > \frac{1}{2} \geqslant b$, $B + q_1 b \leqslant 1 < B + (q_1 + 1)b$, $q_0 b \leqslant 1 < (q_0 + 1)b$. *Then* $C(n,m,r)$ *is WSNB if*

$$m > m^* \equiv \begin{cases} 2 + \dfrac{2(q_0 - q_1)(q_0 - q_1 - 1)}{q_0^2 - q_0q_1 - q_1^2 - q_1} & \text{for } q_0 \geqslant 2q_1 + 1, \\ 2 + \dfrac{2q_1}{q_1 + 1} & \text{for } q_0 \leqslant 2q_1. \end{cases}$$

Proof. Consider the algorithm $P(x, q_1)$ where

$$x = \begin{cases} \dfrac{2q_0(q_0 - q_1 - 1)}{q_0^2 - q_0q_1 - q_1^2 - q_1} & \text{for } q_0 \geqslant 2q_1 + 1, \\ 0 & \text{for } q_0 \leqslant 2q_1. \end{cases}$$

Suppose a call from input switch U to output switch V is blocked. In the worst scenario, each external link of U and V generates a maximal set of calls. Assume that among the $2n$ external links, $c_0 n$ of them generate q_0 b-calls each and $c_1 n$ 1 B-call and q_1 b-calls each, where $c_0 + c_1 = 2$ (note that the blocked call is also counted). Define $y = m^* - x$, so yn is the number of nonrestricted switches.

(i) *The blocked call is a B-call.* Then each switch must carry a load exceeding $1 - B$, which means, at least $(q_1 + 1)b$ or B. In the worst scenario, all switches carry either $q_1 + 1$ b-calls or 1 B-call. Let $y_0 n$ and $y_1 n$ denote the numbers of nonrestricted switches carrying these two types of load respectively, where $y_0 + y_1 = y$. By comparing the numbers of b-calls and B-calls generated by (U, V) and carried by the center switches, we obtain

$$y_0(q_1 + 1) \leqslant c_0q_0 + c_1q_1$$

and

$$x + y_1 < c_1 \quad \text{(since the blocked call is not carried).}$$

The first inequality can be written as

$$y_0 \leqslant \frac{(2 - q)q_0 + c_1q_1}{q_1 + 1} = \frac{2q_0 - (c_0 - c_1)q_1}{q_1 + 1}.$$

Adding the two inequalities, we obtain

$$m^* = x + y < \frac{2q_0 - (q_0 - 2q_1 - 1)c_1}{q_1 + 1}.$$

Suppose $q_0 \geqslant 2q_1 + 1$. Then $q_0 - 2q_1 - 1 \geqslant 0$. Since $c_1 > x$,

$$m^* < \frac{2q_0 - (q_0 - 2q_1 - 1)x}{q_1 + 1} = \frac{2q_0 - (q_0 - 2q_1 - 1)(m^* - 2)q_0/(q_0 - q_1)}{q_1 + 1}$$

$$= \frac{2q_0(2q_0 - 3q_1 - 1) - q_0(q_0 - 2q_1 - 1)m^*}{(q_1 + 1)(q_0 - q_1)},$$

or

$$m^* < \frac{2q_0(2q_0 - 3q_1 - 1)}{q_0^2 - q_0q_1 - q_1^2 - q_1} \equiv m^*,$$

a contradiction.

Suppose $q_0 \leqslant 2q_1$. Then $q_0 - 2q_1 - 1 < 0$. Since $c_1 \leqslant 2$,

$$m^* = x + y < \frac{2q_0 - 2(q_0 - 2q_1 - 1)}{q_1 + 1} = \frac{2q_1 + 1}{q_1 + 1} = m^*,$$

a contradiction.

(ii) *The blocked call is a b-call.* Then each switch must carry a load exceeding $1 - b$, which means, either $q_0 b$ or $B + q_1 b$ in the worst scenario. By the definition of $P(x, q_1)$, each restricted switch carries q_1 b-calls. Assume that y_0 nonrestricted switches carry q_0 b-calls each and y_1 carry 1 B-call and q_1 b-calls each, where $y_0 + y_1 = y$. Again, comparing the numbers of b-calls and B-calls generated by (U, V) and carried by center switches, we obtain

$$c_0q_0 + c_1q_1 > y_0q_0 + (x + y_1)q_1$$

and

$$c_1 \geqslant y_1.$$

This implies

$$(2 - y_1)q_0 + y_1q_1 \geqslant c_0q_0 + c_1q_1 > y_0q_0 + (x + y_1)q_1$$

or

$$2q_0 > yq_0 + xq_1.$$

Suppose $q_0 \geqslant 2q_1 + 1$. Then we have

$$2q_0 > mq_0 - x(q_0 - q_1)$$

$$= 2q_0 + x(q_0 - q_1) - x(q_0 - q_1) = 2q_0,$$

a contradiction.

Suppose $q_0 \leqslant 2q_1$. Then we have

$$2q_0 > yq_0 = \left(2 + \frac{2q_1}{q_1+1}\right) q_0,$$

again, a contradiction. □

Clearly, $m^* < 4n$ for $q_0 \leqslant 2q_1$. If $q_0 > 2q_1$, then

$$q_0^2 - q_0q_1 - q_1^2 - q_1 - (q_0 - q_1)(q_0 - q_1 - 1) = (q_0 - 2q_1)(q_1 + 1) > 0.$$

Hence, $m^* < 4n$ also for $q_0 > 2q_1$.

When b divides B and 1, then we can also use Theorem 2.4. We now show that Theorem 4.2 requires a smaller m. Note that $q_0 - 2q_1 = 1/b - 2(1-B)/b = (2B-1)/b > 0$. Therefore, $q_0 \geqslant 2q_1 + 1$. We show that the m^* never exceeds $2n/(1-B+b)$, which is the m in Theorem 2.4 under the ideal assumption:

$$\begin{aligned}
&\frac{2}{1-B+b} - \left(2 + \frac{2(q_0-q_1)(q_0-q_1-1)}{q_0^2 - q_0q_1 - q_1^2 - q_1}\right) \\
&= \frac{2q_0}{q_1+1} - 2 - \frac{2(q_0-q_1)(q_0-q_1-1)}{q_0^2 - q_0q_1 - q_1^2 - q_1} \\
&= \frac{2(q_0-q_1-1)}{q_1+1} - \frac{2(q_0-q_1)(q_0-q_1-1)}{q_0^2 - q_0q_1 - q_1^2 - q_1} \\
&= \frac{2(q_0-q_1-1)[q_0^2 - q_0q_1 - q_1^2 - q_1 - (q_1+1)(q_0-q_1)]}{(q_1+1)(q_0^2 - q_0q_1 - q_1^2 - q_1)} \\
&= \frac{2(q_0-q_1-1)q_0(q_0-2q_1-1)}{(q_1+1)(q_0^2 - q_0q_1 - q_1^2 - q_1)} \geqslant 0.
\end{aligned}$$

Theorem 4.3. *Consider the 3-rate environment with three weights $B > w > b$. Then $C(n, 5n, r)$ is WSNB.*

Proof. (i) $b > \frac{1}{2}$. By Lemma 2.6, $2n$ center switches suffice for all calls.

(ii) $w > \frac{1}{2} \geqslant b$. By Lemma 2.6, $2n$ center switches suffice for all B-calls and w-calls; another $2n$ suffice for all b-calls.

(iii) $B > \frac{1}{2} \geqslant w$. By Lemma 2.6, $2n$ center switches suffice for all B-calls. By Theorem 4.1, another $3n$ suffice for all w-calls and b-calls.

(iv) $\frac{1}{2} \geqslant B$. By Theorem 3.1, $3.75n$ center switches suffice for all calls. □

5. Conclusion

We proposed a new class of algorithms using the quota scheme. We show that $C(n, 5.75n, r)$ is WSNB for any set of rates under the uniform model. The required

m-value can be reduced if the upper bound $B < \frac{23}{32}$. Furthermore, $C(n, 4n, r)$ is WSNB for any two rates, and $C(n, 3n, r)$ is 2-rate WSNB if $B \leqslant 0.5$. Finally, $C(n, 5n, r)$ is WSNB for any three rates.

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