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Optimal comparison strategies in Ulam's searching game with two errors

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Abstract

Suppose x is an n-bit integer. By a *comparison* question we mean a question of the form "does x satisfy either condition $a \leqslant x \leqslant b$ or $c \leqslant x \leqslant d$?". We describe strategies to find x using the smallest possible number $q(n)$ of comparison questions, and allowing up to two of the answers to be erroneous. As proved in this self-contained paper, with the exception of $n=2$, $q(n)$ is the smallest number q satisfying Berlekamp's inequality

$$2^q \geqslant 2^n \left(\binom{q}{2} + q + 1\right).$$

This result would disappear if we only allowed questions of the form "does x satisfy the condition $a \leqslant x \leqslant b$?". Since no strategy can find the unknown $x \in \{0, 1, \ldots, 2^n - 1\}$ with less than $q(n)$ questions, our result provides extremely simple optimal searching strategies for Ulam's game with two lies – the game of Twenty Questions where up to two of the answers may be erroneous.

1. Introduction

Ulam's game [7, p. 281] with l lies has two players, called Questioner and Responder. The players first fix a search space $S_n = \{0, 1, \ldots, 2^n - 1\}$. The Responder thinks of a "target" number $x \in S_n$ and the Questioner attempts to find x by asking the smallest possible number of questions. Each question can only be answered "yes" or "no", and the Responder is permitted to lie – or to be inaccurate in his answers – up to l times. There are several papers in the literature dealing with Ulam's game (see, for instance, [2, 4–6] and references therein).

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Although answers are propositions, they do not obey the rules of classical logic: for, two opposite answers to the same repeated question need not lead to contradiction, and two equal answers may be more informative than a single answer: indeed, as explained in [4], Ulam's game with l lies provides a natural interpretation of the $(l+2)$-valued calculus of Łukasiewicz.

An equivalent description of Ulam game arises if we assume that the Responder does not know when he is lying, but, as a result of distortion, the bit $b \in \{yes, no\} = \{1, 0\}$ coding his answer may occasionally be received as $1-b$. In this way, Ulam game becomes a chapter of the theory of communication with feedback, originating with Berlekamp [1]. If all questions are asked at the very beginning of the game, independently of the answers, then we have a *nonadaptive* variant of Ulam's game. In this case, finding an optimal searching strategy amounts to finding an optimal l-error-correcting code – a very difficult task already for small $l \geqslant 2$ [3]. Passing to the *adaptive* case, where the tth question is asked keeping into account the information given by the previous $t-1$ answers, if questions are allowed to range over arbitrary subsets of S_n, optimal searching strategies are rather easily described, for all l – at least for sufficiently large n. It turns out that the number of questions needed by such strategies is, up to finitely many exceptions, the smallest positive integer satisfying Berlekamp's inequality (see [2, 6]).

Since, however, questions in these strategies are quite complicated, it is of interest to investigate optimal strategies involving the simplest possible questions. In this paper we shall concentrate on *comparison* questions of the form "does x satisfy either condition $a \leqslant x \leqslant b$ or $c \leqslant x \leqslant d$?" We shall prove that, despite such limitations in the expressive power of the Questioner, there exist optimal searching strategies having precisely the same number of comparison questions as in the general, unrestricted, case.

Our main theorem is as follows: *For all $n \neq 2$, if up to two errors are allowed in the answers, an unknown n-bit number can always be found by asking $q(n)$* comparison *questions, with $q(n)$ the smallest integer q satisfying Berlekamp's inequality $2^q \geqslant 2^n(\binom{q}{2} + q + 1)$.*

This strengthens the main result of [2], and reduces to a minimum the time and space resources needed to implement optimal searching strategies in Ulam's game with two errors.

2. Questions, answers, states, strategies

Unless otherwise specified, throughout this paper we shall deal with Ulam's game with $l = 2$ errors/lies. We fix an integer $n \geqslant 1$, and we let the *search space* S_n be defined by $S_n = \{0, \dots, 2^n - 1\}$. By a *question* we mean a subset of S_n: thus for instance, the question "is the unknown number x odd?" is identified with the set of odd numbers in S_n. For any question $Q \subseteq S_n$, the *opposite* question $\neg Q$ is defined by $\neg Q = S_n \setminus Q$, the complement of Q in S_n. Suppose t questions $Q_1, \dots, Q_t$ have been asked, and answers $b_1, \dots, b_t$ have been obtained, where $b_i \in \{yes, no\}$.

If there were no errors in the answers, the Questioner could partition S_n into two sets: the set H of numbers satisfying all answers, and the set $S_n \setminus H$ of numbers falsifying at least one answer – whence they can no longer be admissible candidates for x. The characteristic function σ of H fully represents the state of knowledge of the Questioner, by assigning the truth-value 1 to all numbers in H, and 0 to the others; for each $i = 1, \ldots, t$, let $\sigma_i =$ characteristic function of Q_i iff $b_i = yes$, and $\sigma_i =$ characteristic function of $\neg Q_i$ iff $b_i = no$. Then, naturally, σ is the Boolean conjunction of the states of knowledge σ_i separately arising from the answers.

In our present case when $l = 2$, a number $y \in S_n$ might happen to falsify one or two answers, while still being an admissible candidate. To record this state of affairs, the Questioner can assign to each $y \in S_n$ the number j of answers falsified by y; any $j \geqslant 3$ may be safely collapsed to 3. Equivalently, and more conveniently for our purposes, y can be assigned the truth-value

$$\sigma(y) = 1 - \frac{j}{3} \in \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}.$$

In this way we have a natural generalization of the 0-lie setting, as follows:

Definition 2.1. For any question Q, the *positive answer* to Q is the function $\pi_Q : S \to \{\frac{2}{3}, 1\}$, given by

$$\pi_Q(y) = \begin{cases} 1 & \text{iff } y \in Q, \\ \frac{2}{3} & \text{iff } y \notin Q. \end{cases}$$

In the first case we say that y *satisfies* the answer to Q, while in the second case y *falsifies* the answer.

The *negative answer* ν_Q to Q is identified with the positive answer to the opposite question $\neg Q$, in symbols, $\nu_Q = \pi_{S_n \setminus Q}$.

Definition 2.2. A *state* (of knowledge) is a function $\sigma : S_n \to \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. Let $b_1, \ldots, b_t$ be the list of the answers to questions $Q_1, \ldots, Q_t$, where $b_i \in \{yes, no\}$. Then the *state* determined by $Q_1, \ldots, Q_t$ and $b_1, \ldots, b_t$ is the function $\sigma : S_n \to \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ where for each $y \in S_n$,

$$\sigma(y) = \begin{cases} 1 & \text{iff } y \text{ satisfies all the answers,} \\ \frac{2}{3} & \text{iff } y \text{ falsifies precisely one answer,} \\ \frac{1}{3} & \text{iff } y \text{ falsifies precisely two answers,} \\ 0 & \text{iff } y \text{ falsifies at least three answers.} \end{cases}$$

The *support* Σ of σ is the set of all $y \in S_n$ such that $\sigma(y) > 0$. A state is *final* iff its support has at most one element. The *initial* state α_n is the constant function 1 over S_n.

For any x and y in the real unit interval $[0,1]$, the *Łukasiewicz conjunction* $\odot$ of x and y is defined by

$$x \odot y = \max(0, x + y - 1).$$

Given states β and γ, by pointwise application of Łukasiewicz conjunction we obtain the state $\delta = \beta \odot \gamma$, where $\delta(y) = \beta(y) \odot \gamma(y)$ for all $y \in S_n$.

The following result, which is an immediate consequence of the definitions, essentially states that Boolean conjunction stands to the game of Twenty Questions as Łukasiewicz conjunction stands to Ulam's game.

Lemma 2.3. *Adopt the above notation. Let σ_t be the state determined by questions $Q_1, \ldots, Q_t$ and answers $b_1, \ldots, b_t$. Let $\beta_t \in \{\pi_{Q_t}, \nu_{Q_t}\}$ be defined by $\beta_t = \pi_{Q_t}$ or $\beta_t = \nu_{Q_t}$ according as $b_t = yes$ or $b_t = no$, respectively. Then $\sigma_t = \beta_1 \odot \cdots \odot \beta_t$.*

Following tradition, for any set X, we let $\#X$ denote the number of elements of X. The set of all subsets of X is denoted by powerset(X).

For each $q = 1, 2, 3, \ldots$ we denote by $\mathcal{T}_q$ the complete binary tree of depth q. Further, $\mathcal{N}_q$ and $\mathcal{L}_q$ will, respectively, denote the set of nodes and the set of leaves of $\mathcal{T}_q$. Thus, $\#\mathcal{L}_q = 2^q$ and $\#\mathcal{N}_q = 2^q - 1$. Suppose the function $\Psi : \mathcal{N}_q \to \text{powerset}(S_n)$ assigns to each node $N \in \mathcal{N}_q$ a question $\Psi(N) = Q_N \subseteq S_n$. We then say that Ψ is a *strategy with q questions*.

Let N_{sw} and N_{se} be the two children nodes of N. Let A_{sw} and A_{se} be the edges joining N with N_{sw} and N_{se}, respectively. We now label A_{sw} with the positive answer π_{Q_N} to Q_N, and we label A_{se} with the negative answer ν_{Q_N}. For each leaf $L \in \mathcal{L}_q$, let $A_1, \ldots, A_q$ be the path of edges leading from the top node to L. Let $\beta_1, \ldots, \beta_q$ be the corresponding sequence of answers along these edges, as given by Lemma 2.3. Let $\beta(L) = \beta_1 \odot \cdots \odot \beta_q$ be their Łukasiewicz conjunction.

We then say that strategy Ψ is *winning* iff for each leaf $L \in \mathcal{L}_q$, $\beta(L)$ is a final state. More generally, given a state σ, if each conjunction $\sigma \odot \beta(L)$ is a final state we say that Ψ is a *winning strategy with q questions for state σ*.

We shall need Berlekamp's lower bound for the number q of questions of any winning strategy for a state σ; following tradition, for any function $f : X \to Y$ and element $y \in Y$ we let $f^{-1}(y) = \{x \in X \mid f(x) = y\}$.

Definition 2.4. (i) Let σ be a state. Let $0 \leqslant a, b, c$ be integers such that $\#\sigma^{-1}(1) = a$, $\#\sigma^{-1}(\frac{2}{3}) = b$, $\#\sigma^{-1}(\frac{1}{3}) = c$. Then the triplet (a, b, c) is called the *type* of σ. By abuse of notation we shall freely write

$$\#\sigma = (a, b, c).$$

(ii) Let σ be a state with $\#\sigma = (a, b, c)$. Let $q \geqslant 0$. Then the *Berlekamp weight of σ before q questions* is given by

$$w_q(\sigma) = a\left(\binom{q}{2} + q + 1\right) + b(q+1) + c = \tfrac{1}{2}a(q^2 + q + 2) + b(q+1) + c. \qquad (1)$$

(iii) The *character* $ch(\sigma)$ of a state σ is the smallest integer $t \geqslant 0$ such that $w_t(\sigma) \leqslant 2^t$.

Note that a state $ch(\sigma)=0$ iff σ is a final state.
We refer to the basic reference [1] for the proof of the following:

Lemma 2.5. (i) *Given a state σ and a question Q, let $\sigma_{yes} = \sigma \odot \pi_Q$ and $\sigma_{no} = \sigma \odot \nu_Q$. Then for any integer $q \geqslant 1$ we have the following* conservation law:

$$w_{q-1}(\sigma_{yes}) + w_{q-1}(\sigma_{no}) = w_q(\sigma). \tag{2}$$

(ii) (Berlekamp's lower bound) *Suppose the state σ has a winning strategy with q questions. Then*

$$q \geqslant ch(\sigma). \tag{3}$$

3. Comparison questions and well-shaped states

By an *interval* in S_n we either mean the empty set $\emptyset$, or a set of the form $[a,b] = \{x \in S_n \mid a \leqslant x \leqslant b\}$, for suitable integers $a \leqslant b \in S_n$.

Definition 3.1. A *comparison* question is a subset Q of S_n of the form $Q = I \cup J$ for two intervals I and J in S_n.

To better deal with comparison questions, it is convenient to visualize the search space S_n as a necklace. To this purpose, by an *arc* in S_n we either mean an interval or its complement in S_n. The *immediate successor* relation $\prec$ on S_n is defined by $0 \prec 1 \prec 2 \prec \cdots \prec 2^n - 1 \prec 0$. For each $a \in S_n$ we let a' denote the immediate successor of a. For any two elements $a, b \in S_n$ we define the arc $\langle a, b \rangle$ in S_n as follows: starting from a we take immediate successors $a', a'', a''', \ldots$ until b is reached. We naturally say that $\langle a, b \rangle$ is the arc obtained by *scanning S_n with positive orientation from a to b*. In case $a = b$ we stipulate that $\langle a, b \rangle = \{a\}$, thus obtaining a *singleton arc*. In case $a \leqslant b$, the arc $\langle a, b \rangle$ coincides with the interval $[a, b]$. In case $a > b$, $\langle a, b \rangle$ is the disjoint union of the two intervals $[a, 2^n - 1]$ and $[0, b]$. Every nonempty arc in S_n has the form $\langle a, b \rangle$ for suitably chosen $a, b \in S_n$.

For any $X \subseteq S_n$, an *interval in X* is a subset of X of the form $I \cap X$, for some interval I in S_n. An *arc in X* is a subset of X of the form $A \cap X$, for some arc A in S_n. Two arcs in X are said to be *adjacent in X* iff they are disjoint and their union is an arc in X.

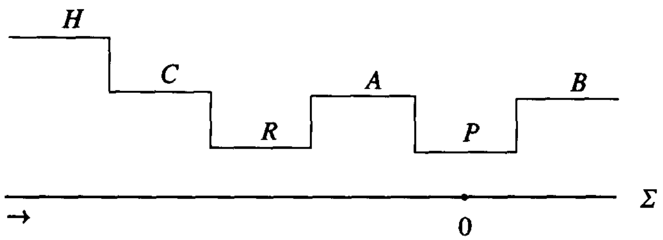
Definition 3.2. Let σ be a state with support Σ. Then σ is *well-shaped* iff it satisfies the following conditions:

$\sigma^{-1}(1)$ is an arc H in Σ;

$\sigma^{-1}(\frac{2}{3})$ is the disjoint union of three arcs A, B, C in Σ, with both B and C adjacent to H in Σ;

$\sigma^{-1}(\frac{1}{3})$ is the disjoint union of two arcs P and R in Σ, both of them being adjacent to A in Σ.

A typical well-shaped state has the form



Starting from H, and scanning S_n with positive orientation, without loss of generality we can list the above six arcs in Σ using the following self-explanatory notation:

$$\sigma|_\Sigma = H^1 C^{\frac{2}{3}} R^{\frac{1}{3}} A^{\frac{2}{3}} P^{\frac{1}{3}} B^{\frac{2}{3}}. \tag{4}$$

Theorem 3.3. *Let σ be a well-shaped state such that $\#\sigma^{-1}(i/3)$ is even, for each $i = 1, 2, 3$. Then there exists a comparison question Q such that both states $\sigma_{yes} = \sigma \odot \pi_Q$ and $\sigma_{no} = \sigma \odot \nu_Q$ are well-shaped and have the same type.*

Proof. Using the above notation (4), let us write $\sigma|_\Sigma = H^1 C^{2/3} R^{1/3} A^{2/3} P^{1/3} B^{2/3}$, for suitable arcs H, A, B, C, P, R in Σ. To avoid trivialities we assume that both H and A are nonempty, whence we can write $A = \langle a_1, a_2 \rangle \cap \Sigma$, and $H = \langle h_1, h_2 \rangle \cap \Sigma$, for suitable elements $a_1, a_2, h_1, h_2 \in \Sigma$. It follows that there is an element $h^* \in H$ such that, letting $H^* = \langle h_1, h^* \rangle \cap \Sigma$,

$$\#H^* = \tfrac{1}{2}\#H.$$

We can safely assume $\#P \geqslant \#R$ whence, to avoid trivialities, $P \neq \emptyset$. Then we can write $P = \langle p_1, p_2 \rangle \cap \Sigma$ for some $p_1, p_2 \in \Sigma$. There is an element $p^* \in P$ such that, letting $P^* = \langle p_1, p^* \rangle \cap \Sigma$,

$$\#P^* = \tfrac{1}{2}\#\sigma^{-1}\left(\frac{1}{3}\right).$$

Case 1: $\#B \geqslant \#C$.

Skipping all inessentials, let us further assume $B \neq \emptyset$, whence $B = \langle b_1, b_2 \rangle \cap \Sigma$, for some $b_1, b_2 \in \Sigma$. Note that $b_2 \in B$ and $h_1 \in H$ are adjacent elements (= adjacent singleton arcs) in Σ. The same is true for a_2 and p_1.

Subcase 1.1: $\#A \geqslant \#B + \#C$.

Then there is an element $a^* \in A$ such that, letting $A^* = \langle a^*, a_2 \rangle \cap \Sigma$,

$$\#A^* = \tfrac{1}{2}\#\sigma^{-1}\left(\frac{2}{3}\right).$$

Let V be the union of the two disjoint arcs in S_n given by $\langle a^*, p^*\rangle$ and $\langle h_1, h^*\rangle$. Depending on whether $0 \notin V$ or $0 \in V$, either V or its complement $S_n \backslash V$ is a comparison question.

Assume first V to be a comparison question. Then define $Q = V$ and let Σ^+ and Σ^- be the supports of the states $\sigma_{yes} = \sigma \odot \pi_Q$ and $\sigma_{no} = \sigma \odot \nu_Q$ obtainable from σ as a consequence of the two possible answers to Q. Then we have

$$\sigma_{yes}|_{\Sigma^+} = H^{*1}(H\backslash H^*)^{2/3}C^{1/3}(A\backslash A^*)^{1/3}A^{*2/3}P^{*1/3}B^{1/3}$$

and

$$\sigma_{no}|_{\Sigma^-} = H^{*2/3}(H\backslash H^*)^1 C^{2/3}R^{1/3}(A\backslash A^*)^{2/3}A^{*1/3}(P\backslash P^*)^{1/3}B^{2/3},$$

and both states are well-shaped and have the same type.

In case $S_n\backslash(\langle a^*, p^*\rangle \cup \langle h_1, h^*\rangle)$ is a comparison question, letting $Q = S_n\backslash(\langle a^*, p^*\rangle \cup \langle h_1, h^*\rangle)$, one proceeds in a similar way with the roles of σ_{yes} and σ_{no} reversed.

Subcase 1.2: $\#A < \#B + \#C$.

Then, by our standing hypothesis of Case 1, there exists an element $b^* \in B$ such that, letting $B^* = \langle b^*, b_2\rangle \cap \Sigma$,

$$\#A + \#B^* = \tfrac{1}{2}\#\sigma^{-1}\left(\frac{2}{3}\right).$$

Let $U = \langle a_1, p^*\rangle \cup \langle b^*, h^*\rangle$. Then either U or its complement $S_n\backslash U$ is a comparison question. In the first case, let us define $Q = U$. Then letting Σ^+ and Σ^- be the supports of $\sigma_{yes} = \sigma \odot \pi_Q$ and $\sigma_{no} = \sigma \odot \nu_Q$, we have

$$\sigma_{yes}|_{\Sigma^+} = H^{*1}(H\backslash H^*)^{2/3}C^{1/3}A^{2/3}P^{*1/3}(B\backslash B^*)^{1/3}B^{*2/3}$$

and

$$\sigma_{no}|_{\Sigma^-} = H^{*2/3}(H\backslash H^*)^1 C^{2/3}R^{1/3}A^{1/3}(P\backslash P^*)^{1/3}(B\backslash B^*)^{2/3}B^{*1/3},$$

whence both states σ_{yes} and σ_{no} are well-shaped and have the same type.

If, on the other hand, $S_n\backslash U$ is a comparison question, let $Q = S_n\backslash U$. Then the desired conclusion follows by reversing the roles of σ_{yes} and σ_{no}.

Case 2: $\#B < \#C$

This case is handled as Case 1, by reversing the orientation of S_n. □

4. Critical indexes

By Definition 2.4, the character and the weight of any state σ only depend on the type (a, b, c) of σ. Accordingly, by abuse of notation we shall write

$$ch(a,b,c) \quad \text{and} \quad w_q(a,b,c)$$

to denote the character, resp., the weight before q questions, of *any* state of type (a, b, c).

We are interested in the values of $ch(1,n,\binom{n}{2})$ for $n\geqslant 3$. A direct computation yields $ch(1,3,3)=ch(1,4,6)=6$, $ch(1,5,10)=\cdots=ch(1,8,28)=7$, $ch(1,9,36)=\cdots=ch(1,14,91)=8$.

Definition 4.1. Let $\chi\geqslant 4$ be an arbitrary integer. The *first critical index* n_χ is the largest integer $n\geqslant 0$ such that $ch(1,n,\binom{n}{2})=\chi$. Thus,

$$ch\left(1,n_\chi,\binom{n_\chi}{2}\right)=\chi \quad\text{and}\quad ch\left(1,n_\chi+1,\binom{n_\chi+1}{2}\right)>\chi. \tag{5}$$

The *second critical index* p_χ is defined by

$$p_\chi=2^\chi-\tfrac{1}{2}(\chi^2+\chi+2)-n_\chi(\chi+1)=2^\chi-w_\chi(1,n_\chi,0). \tag{6}$$

The two functions $\chi\mapsto n_\chi$ and $\chi\mapsto p_\chi$ are well defined for all $\chi\geqslant 4$. For instance we have

$$n_4=1,\ n_5=2,\ n_6=4,\ n_7=8,\ n_8=14,\ n_9=22,\ n_{10}=34,\ldots \tag{7}$$

$$p_4=0,\ p_5=4,\ p_6=14,\ p_7=35,\ p_8=93,\ p_9=246,\ p_{10}=594,\ldots \tag{8}$$

As usual, for every real number ρ we denote by $\lfloor\rho\rfloor$ the largest integer $k\leqslant\rho$.

Lemma 4.2. *Let $\chi\geqslant 5$ be an arbitrary integer.*

(i) *If χ is odd then $n_\chi=2^{(\chi+1)/2}-\chi-1$.*

(ii) *If χ is even then, letting $n^*=\lfloor 2^{(\chi+1)/2}\rfloor-\chi-1$, we either have $n_\chi=n^*$, or $n_\chi=n^*+1$.*

Proof. (i) Let $m=2^{(\chi+1)/2}-\chi-1$. We shall prove the following two identities:

$$ch\left(1,m,\binom{m}{2}\right)=\chi \quad\text{and}\quad ch\left(1,m+1,\binom{m+1}{2}\right)=\chi+1. \tag{9}$$

The first identity is equivalent to the following two inequalities:

$$2^\chi-w_\chi\left(1,m,\binom{m}{2}\right)\geqslant 0 \tag{10}$$

and

$$w_{\chi-1}\left(1,m,\binom{m}{2}\right)-2^{\chi-1}>0. \tag{11}$$

The second identity is equivalent to the following two inequalities:

$$2^{\chi+1}-w_{\chi+1}\left(1,m+1,\binom{m+1}{2}\right)\geqslant 0 \tag{12}$$

and

$$w_\chi\left(1,m+1,\binom{m+1}{2}\right)-2^\chi>0. \tag{13}$$

A tedious but straightforward simplification of the left-hand sides of the above four inequalities yields the following equivalent reformulation of (10)–(13):

$$-1+2^{(\chi-1)/2}\geqslant 0, \tag{14}$$

$$2+2^{\chi-1}-3(2^{(\chi-1)/2})>0, \tag{15}$$

$$-2+2^{\chi}-3(2^{(\chi-1)/2})\geqslant 0, \tag{16}$$

$$1+2^{(\chi-1)/2}>0. \tag{17}$$

A direct inspection now establishes these four inequalities, for all odd integers $\chi\geqslant 5$. Thus (9) holds true, and the proof of (i) is complete.

(ii) We first observe that inequalities (14)–(17) also hold for all even integers $\chi\geqslant 6$. Thus, if in formula (1), we extend to arbitrary real numbers $a,b,c\geqslant 0$ the domain of definition of the function $w_q(a,b,c)=a(q^2+q+2)/2+b(q+1)+c$, and use the monotonicity properties of this function over the extended domain, then the same computations yielding (9), are now to the effect that

$$ch\left(1,k,\binom{k}{2}\right)\leqslant\chi \quad \text{for all integers } k\leqslant 2^{(\chi+1)/2}-\chi-1, \tag{18}$$

and

$$ch\left(1,h,\binom{h}{2}\right)\geqslant\chi+1 \quad \text{for all integers } h\geqslant 2^{(\chi+1)/2}-\chi. \tag{19}$$

Since $\chi-1$ is odd, by (i) and by definition of first critical index we can write

$$ch\left(1,n_{\chi-1}+1,\binom{n_{\chi-1}+1}{2}\right)=\chi,$$

and since $n^*\geqslant n_{\chi-1}+1$, by monotonicity we get from (18)

$$ch\left(1,n^*,\binom{n^*}{2}\right)=\chi. \tag{20}$$

In a similar way, from (5), (19), and (i) we obtain

$$ch\left(1,n^*+2,\binom{n^*+2}{2}\right)=\chi+1. \tag{21}$$

Arguing now by cases, if $ch(1,n^*+1,\binom{n^*+1}{2})=\chi+1$, then by (20), $n_\chi=n^*$. Otherwise, if $ch(1,n^*+1,\binom{n^*+1}{2})=\chi$, then by (21), $n_\chi=n^*+1$. This completes the proof of (ii). □

Corollary 4.3. *For all integers $\chi\geqslant 6$ we have*

$$\frac{n_{\chi+1}}{2}\leqslant n_\chi\leqslant n_{\chi+1}, \tag{22}$$

$$w_\chi(1,n_\chi,p_\chi)=2^\chi, \tag{23}$$

$$p_\chi \geqslant \binom{n_\chi}{2}, \tag{24}$$

$$p_\chi \geqslant n_{\chi+1} - n_\chi, \tag{25}$$

$$p_\chi \leqslant p_{\chi+1}. \tag{26}$$

Proof. Eq. (22) is proved by direct inspection, using (7) and Lemma 4.2. Eqs. (23) and (24) are immediate consequences of definitions (1) and (6). To prove Eq. (25), by (7) together with (22) and (24) we have

$$n_{\chi+1} - n_\chi \leqslant \frac{n_{\chi+1}}{2} \leqslant \frac{1}{2}\left[\frac{n_{\chi+1}}{2}\left(\frac{n_{\chi+1}}{2} - 1\right)\right] = \binom{n_{\chi+1}/2}{2} \leqslant \binom{n_\chi}{2} \leqslant p_\chi.$$

To prove Eq. (26), first of all, by (8) we have the inequalities $p_6 < p_7 < p_8$. Using now Eqs. (22) and (24), after a routine computation we get

$$\begin{aligned} p_{\chi+1} - p_\chi &= 2^\chi - w_{\chi+1}(1, n_{\chi+1}, 0) + w_\chi(1, n_\chi, 0) \\ &\geqslant 2^\chi - w_{\chi+1}(1, n_{\chi+1}, 0) + w_\chi(1, n_{\chi+1}/2, 0) \\ &= 2^\chi - w_{\chi+1}(0, n_{\chi+1}/2, 0) - w_{\chi+1}(1, n_{\chi+1}/2, 0) + w_\chi(1, n_{\chi+1}/2, 0) \\ &= 2^\chi - \chi - 1 - \tfrac{1}{2}(\chi + 3)n_{\chi+1}. \end{aligned}$$

Further, by Lemma 4.2 we have

$$n_{\chi+1} \leqslant \lfloor 2^{(\chi+2)/2} \rfloor - \chi - 1 \leqslant 2^{(\chi+2)/2} - \chi - 1,$$

whence a fortiori,

$$\begin{aligned} p_{\chi+1} - p_\chi &\geqslant 2^\chi - \chi - 1 - \tfrac{1}{2}(\chi + 3)n_{\chi+1} \\ &\geqslant 2^\chi - \chi - 1 - \tfrac{1}{2}(\chi + 3)(2^{(\chi+2)/2} - \chi - 1) \geqslant 2^\chi - (\chi + 3)2^{\chi/2}. \end{aligned}$$

Since the inequality $2^\chi \geqslant (\chi + 3)2^{\chi/2}$ holds for all $\chi \geqslant 8$, the proof is complete. □

Definition 4.4. Let $0 \leqslant b, c$. A state of type $(0, b, c)$ is said to be *simple* iff $c \geqslant b - 1$.

Definition 4.5. Let $0 \leqslant i, j, k$ be integers. A question Q is said to be $[i, j, k]$*-like* for a state σ iff $\#(Q \cap \sigma^{-1}(1)) = i$, $\#(Q \cap \sigma^{-1}(2/3)) = j$, and $\#(Q \cap \sigma^{-1}(1/3)) = k$.

Theorem 4.6. *Let $\chi \geqslant 6$ be an integer, and σ be a well-shaped state of type $(1, n_{\chi+1}, p_{\chi+1})$. Then there is a* comparison *question Q such that the two states $\sigma_{yes} = \sigma \odot \pi_Q$ and $\sigma_{no} = \sigma \odot \nu_Q$ satisfy the following conditions*:

(i) *both σ_{yes} and σ_{no} are well-shaped and have the same character χ*;

(ii) *one of σ_{yes} and σ_{no} is of type $(1, n_\chi, p_\chi)$, and the other is simple and of type $(0, 1 + n_{\chi+1} - n_\chi, p_{\chi+1} - p_\chi + n_{\chi+1})$.*

Proof. Let Σ be the support of σ. Following (4) we can write

$$\sigma|_\Sigma = H^1 C^{2/3} R^{1/3} A^{2/3} P^{1/3} B^{2/3},$$

for suitable arcs H,A,B,C,P,R in Σ. To avoid trivialities, let us further assume that none of these arcs is empty, whence for suitable elements a_1,a_2, b_1,b_2, c_1,c_2, p_1,p_2, $r_1,r_2 \in \Sigma$ we can write

$$A=\langle a_1,a_2\rangle \cap \Sigma, \qquad B=\langle b_1,b_2\rangle \cap \Sigma, \qquad C=\langle c_1,c_2\rangle \cap \Sigma,$$

$$P=\langle p_1,p_2\rangle \cap \Sigma, \qquad R=\langle r_1,r_2\rangle \cap \Sigma,$$

with $\#H=1, \#A+\#B+\#C=n_{\chi+1}, \#P+\#R=p_{\chi+1}$. It is no loss of generality to assume

$$\#C \geqslant \#B. \tag{27}$$

Case 1: $n_\chi \geqslant \min(\#A+\#B, \#A+\#C, \#B+\#C)$.

Then, by (27), there are only two cases to consider:

Subcase 1.1: $n_\chi \geqslant \#A+\#B$.

Since by Corollary 4.3, Eq. (22), $n_\chi \leqslant n_{\chi+1}$, there must be an element $c^* \in C$ such that, letting $C^*=\langle c_1,c^*\rangle \cap \Sigma$,

$$\#A+\#B+\#C^*=n_\chi.$$

By (25) and (26), there exist elements $p^* \in P$ and $r^* \in R$ such that, letting $P^*=\langle p_1,p^*\rangle \cap \Sigma$ and $R^*=\langle r^*,r_2\rangle \cap \Sigma$,

$$\#P^*+\#R^*=p_\chi-n_{\chi+1}+n_\chi.$$

Defining now $W=\langle b_1,c^*\rangle \cup \langle r^*,p^*\rangle$, by direct verification we see that $\langle r^*,p^*\rangle \supseteq A$ and $\langle b_1,c^*\rangle \supseteq B\cup H$. Since W is the union of two arcs in S_n, either W or $S_n\setminus W$ is a comparison question. In the first case we take $Q=W$, and let Σ^+ and Σ^- be the supports of $\sigma_{yes}=\sigma \odot \pi_Q$ and $\sigma_{no}=\sigma\odot \nu_Q$. By direct inspection, Q is $[1,n_\chi,p_\chi-n_{\chi+1}+n_\chi]$-like for σ. Further,

$$\sigma_{yes}|_{\Sigma^+}=H^1{C^*}^{2/3}(C\setminus C^*)^{1/3}{R^*}^{1/3}A^{2/3}{P^*}^{1/3}B^{2/3}$$

and

$$\sigma_{no}|_{\Sigma^-}=H^{2/3}{C^*}^{1/3}(C\setminus C^*)^{2/3}(R\setminus R^*)^{1/3}A^{1/3}(P\setminus P^*)^{1/3}B^{1/3},$$

whence both states σ_{yes} and σ_{no} are well-shaped. By (23) we can write

$$w_\chi(1,n_\chi,p_\chi)=2^\chi \quad \text{and} \quad w_{\chi+1}(1,n_{\chi+1},p_{\chi+1})=2^{\chi+1},$$

and by Definition 4.1, $ch(1,n_\chi,p_\chi)=\chi$. From Lemma 2.5(i) it follows that $w_\chi(\sigma_{no})=2^\chi$ and $ch(\sigma_{no})=\chi$. Further,

$$\begin{aligned}\#\sigma_{no}&=(0,\ 1+n_{\chi+1}-n_\chi, p_{\chi+1}+n_\chi-(p_\chi-(n_{\chi+1}-n_\chi)))\\&=(0,\ 1+n_{\chi+1}-n_\chi, p_{\chi+1}-p_\chi+n_{\chi+1}),\end{aligned}$$

whence by (26), σ_{no} is a simple state, and both conditions (i) and (ii) are satisfied. In case $S_n\setminus W$ is a comparison question, taking $Q=S_n\setminus W$ we obtain the desired conclusion by reversing the roles of σ_{yes} and σ_{no}.

Subcase 1.2: $n_\chi \geqslant \#C + \#B$.

Then by (22) there is an element $a^\dagger \in A$ such that, letting $A^\dagger = \langle a_1, a^\dagger \rangle \cap \Sigma$, we have $\#A^\dagger + \#B + \#C = n_\chi$. Moreover, in the light of (25) and (26), there are elements $p^\dagger \in P$ and $r^\dagger \in R$ such that, upon defining

$$P^\dagger = \langle p^\dagger, p_2 \rangle \cap \Sigma \quad \text{and} \quad R^\dagger = \langle r_1, r^\dagger \rangle \cap \Sigma,$$

we have $\#P^\dagger + \#R^\dagger = p_\chi - n_{\chi+1} + n_\chi$.

Let $V = \langle p^\dagger, r^\dagger \rangle \cup \langle a_1, a^\dagger \rangle$. Note that $\langle p^\dagger, r^\dagger \rangle \supseteq B \cup H \cup C$. Then either V or $S_n \backslash V$ is a comparison question. In the first case, define $Q = V$ and let Σ^+ and Σ^- be the supports of $\sigma_{yes} = \sigma \odot \pi_Q$ and $\sigma_{no} = \sigma \odot \nu_Q$. Again note that Q is $[1, n_\chi, \ p_\chi - n_{\chi+1} + n_\chi]$-like for σ. We have

$$\sigma_{yes}|_{\Sigma^+} = H^1 C^{2/3} {R^\dagger}^{1/3} {A^\dagger}^{2/3} (A \backslash A^\dagger)^{1/3} {P^\dagger}^{1/3} B^{2/3}$$

and

$$\sigma_{no}|_{\Sigma^-} = H^{2/3} C^{1/3} (R \backslash R^\dagger)^{1/3} {A^\dagger}^{1/3} (A \backslash A^\dagger)^{2/3} (P \backslash P^\dagger)^{1/3} B^{1/3},$$

whence both σ_{yes} and σ_{no} satisfy conditions (i) and (ii). In case $S_n \backslash V$ is a comparison question, one gets a similar conclusion by defining $Q = S_n \backslash V$ and reversing the roles of σ_{no} and σ_{yes}.

Case 2: $n_\chi < \min(\#A + \#B, \ \#A + \#C, \ \#B + \#C)$.

Then by (27) and (22) we can write $\#B \leqslant n_{\chi+1}/2 \leqslant n_\chi$. Furthermore, from (25) we get

$$0 \leqslant p_\chi - n_{\chi+1} + n_\chi \leqslant \#P + \#R = p_{\chi+1}.$$

We can now choose elements $p^\ddagger \in P$ and $r^\ddagger \in R$ in such a way that, upon defining

$$P^\ddagger = \Sigma \cap \langle p^\ddagger, p_2 \rangle \quad \text{and} \quad R^\ddagger = \Sigma \cap \langle r^\ddagger, r_2 \rangle,$$

we have $\#P^\ddagger + \#R^\ddagger = p_\chi - n_{\chi+1} + n_\chi$. Further, by our standing hypothesis in the present case, there is an element $a^\ddagger \in A$ such that, letting

$$A^\ddagger = \Sigma \cap \langle a_1, a^\ddagger \rangle,$$

we have $n_\chi = \#A^\ddagger + \#B$. We now define $Z = \langle r^\ddagger, a^\ddagger \rangle \cup \langle p^\ddagger, h \rangle$, where h is the only element of H. Note that $\langle p^\ddagger, h \rangle \supseteq B$. Again, either Z or $S_n \backslash Z$ is a comparison question. In the first case, we take $Q = Z$ and let Σ^+ and Σ^- be the supports of $\sigma_{yes} = \sigma \odot \pi_Q$ and $\sigma_{no} = \sigma \odot \nu_Q$. Note that Q is $[1, n_\chi, \ p_\chi - n_{\chi+1} + n_\chi]$-like for σ. Further,

$$\sigma_{yes}|_{\Sigma^+} = H^1 C^{1/3} {R^\ddagger}^{1/3} {A^\ddagger}^{2/3} (A \backslash A^\ddagger)^{1/3} {P^\ddagger}^{1/3} B^{2/3}$$

and

$$\sigma_{no}|_{\Sigma^-} = H^{2/3} C^{2/3} (R \backslash R^\ddagger)^{1/3} {A^\ddagger}^{1/3} (A \backslash A^\ddagger)^{2/3} (P \backslash P^\ddagger)^{1/3} B^{1/3},$$

whence both σ_{yes} and σ_{no} satisfy conditions (i) and (ii). In the other case, taking $Q = S_n \backslash Z$, we get a similar conclusion by reversing the roles of σ_{no} and σ_{yes}. □

5. Amenable states

Definition 5.1. A state σ is said to be *amenable* iff there is a winning strategy for σ with $ch(\sigma)$ many comparison questions.

Proposition 5.2. *Every well-shaped simple state σ is amenable.*

Proof. By induction on $\chi = ch(\sigma)$. The cases $\chi = 0$ or $\chi = 1$ are trivial. For the induction step, assume $2 \leqslant \chi$. Let Σ be the support of σ. Since by hypothesis σ is well-shaped and simple, we can write (4) in the following simplified form:

$$\sigma|_\Sigma = R^{1/3}A^{2/3}P^{1/3}B^{2/3}.$$

Skipping all trivialities, and assuming that none of these arcs in Σ is empty, let us display R, A, P, B as follows:

$$R = \langle r_1, r_2 \rangle \cap \Sigma, \quad A = \langle a_1, a_2 \rangle \cap \Sigma, \quad P = \langle p_1, p_2 \rangle \cap \Sigma, \quad B = \langle b_1, b_2 \rangle \cap \Sigma,$$

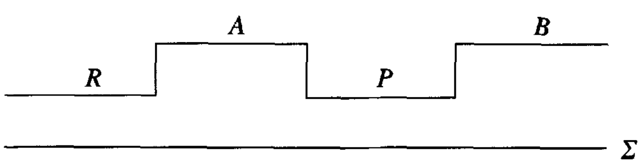
for suitable elements a_1, a_2, b_1, b_2, p_1, p_2, $r_1, r_2 \in \Sigma$. Let us agree to say that a question W is *appropriate* for σ iff both $\sigma_{yes} = \sigma \odot \pi_W$ and $\sigma_{no} = \sigma \odot \nu_W$ are simple states of character $< \chi$.

Claim. *There exists a comparison question W appropriate for σ, having the additional property that both σ_{yes} and σ_{no} are well-shaped.*

As a matter of fact, by [5, Lemma 3], the set of appropriate questions for σ is nonempty. Choose an arbitrary question W that is appropriate for σ. Then W will be $[0, j, k]$-like for σ, for suitable integers j, k with

$$0 \leqslant j \leqslant \#A + \#B \quad \text{and} \quad 0 \leqslant k \leqslant \#R + \#P.$$

Any question W' which is $[0, j, k]$-like for σ will automatically be appropriate for σ; furthermore, $\#\sigma'_{yes} = \#\sigma_{yes}$ and $\#\sigma'_{no} = \#\sigma_{no}$, whence in particular, both σ'_{yes} and σ'_{no} will be simple states of character $< \chi$. Let us display σ as follows:



We can choose elements $r^* \in R$, $a^* \in A$, $p^* \in P$, and $b^* \in B$ such that

$$\#((\langle r^*, r_2 \rangle \cup \langle p^*, p_2 \rangle) \cap \Sigma) = k \quad \text{and} \quad \#((\langle a_1, a^* \rangle \cup \langle b_1, b^* \rangle) \cap \Sigma) = j.$$

By the above discussion, the question $W'' = \langle r^*, a^* \rangle \cup \langle p^*, b^* \rangle$ is still $[0, j, k]$-like for σ. Since W'' is the union of two arcs in S_n, then either W'' or its complement $S_n \backslash W''$ is a comparison question such that the two resulting states σ''_{yes} and σ''_{no} are well-shaped. We can safely identify W and W''. Our claim is settled.

Recalling now that the character of the well-shaped simple states σ''_{yes} and σ''_{no} is $<\chi$, and applying the induction hypothesis, we obtain the desired conclusion. □

Proposition 5.3. *Every well-shaped state* σ *of type* $(1,4,14)$ *is amenable.*

Proof. Let Σ be the support of σ. Following (4) we can write $\sigma|_\Sigma = H^1C^{2/3}R^{1/3}A^{2/3}P^{1/3}B^{2/3}$. Skipping all inessentials, let us assume that none of these arcs in Σ is empty. It is no loss of generality to assume $\#P \geqslant \#R$, whence $\#P \geqslant 7$. Assume question W to be $[1,1,7]$-like for σ. We can safely assume that the seven elements of $W \cap \sigma^{-1}(1/3)$ form an arc $U \subseteq P$ in Σ and U is adjacent to B in Σ. Picking now the first element $z \in B$, and letting $T = \{z\} \cup U \cup H$, it follows that there are two arcs A_1, A_2 in S_n such that $T = (A_1 \cup A_2) \cap \Sigma$. Thus either $A_1 \cup A_2$ or its complement in S_n is a comparison question Q, and can be safely identified with W. By construction, Q is still $[1,1,7]$-like for σ and has the additional property that the two states σ_1 and τ_1 obtainable from Q are well-shaped, and have character equal to 5. Without loss of generality, $\#\sigma_1 = (1,1,10)$, and $\#\tau_1 = (0,4,8)$. By Proposition 5.2, τ_1 is amenable. Choose now a $[1,0,4]$-like question Q_1 for σ_1. Again, by suitably rearranging Q_1, we can safely assume that either Q_1 or $S_n \backslash Q_1$ is a comparison question yielding from σ_1 two well-shaped states σ_2 and τ_2, both of character equal to 4; further, $\#\sigma_2 = (1,0,5)$ and $\#\tau_2 = (0,2,6)$. Another application of Proposition 5.2 shows that τ_2 is amenable. Whenever a question Q_2 is $[1,0,1]$-like for σ_2, then Q_2 is automatically a comparison question; the two resulting states σ_3 and τ_3 are well-shaped, have character 3, and type $(1,0,1)$ and $(0,1,4)$ respectively; in addition, τ_3 is amenable. To conclude the proof, choosing for σ_3 a $[1,0,0]$-like question Q_3, it follows that Q_3 is a comparison question, giving two well-shaped states σ_4 and τ_4, of type $(1,0,0)$ and $(0,1,1)$, respectively. The former state is final, the latter has character 2 and is amenable, by Proposition 5.2. □

Proposition 5.4. *Let* χ *be an integer* $\geqslant 6$. *Let* σ *be a well-shaped state of type* $(1, n_\chi, p_\chi)$. *Then* σ *is amenable.*

Proof. The case $\chi = 6$ is taken care of by Proposition 5.3. In case $\chi > 6$, Theorem 4.6 yields a comparison question Q_χ and two well-shaped states $\sigma_1 = \sigma \odot \pi_{Q_\chi}$ and $\tau_1 = \sigma \odot \nu_{Q_\chi}$, both of character $\chi - 1$, with the additional property that σ_1 is of type $(1, n_{\chi-1}, p_{\chi-1})$ and τ_1 is simple. By Proposition 5.2, τ_1 is amenable. Proceeding by induction we have a sequence of comparison questions,

$$Q_\chi, Q_{\chi-1}, \ldots, Q_{\chi-i}, \ldots$$

for each $i = 0, 1, 2, \ldots$, together with a sequence of well-shaped states

$$\sigma_1, \sigma_2, \ldots, \sigma_j, \ldots \quad \text{and} \quad \tau_1, \tau_2, \ldots, \tau_j, \ldots$$

$(j \geqslant 1)$ with $\#\sigma_j = (1, n_{\chi-j}, p_{\chi-j})$, $ch(\sigma_j) = ch(\tau_j) = \chi - j$, each τ_j being simple and amenable. Iterated applications of Theorem 4.6 finally yield a well-shaped state σ^* of type $(1,4,14)$, and character equal to 6. By Proposition 5.3, σ^* is amenable. □

6. Main result and final remarks

Theorem 6.1. *Let $n \geqslant 1$ be an integer different from 2. Then an unknown n-bit number $x \in S_n = \{0, 1, \ldots, 2^n - 1\}$ can always be found with $q(n)$ many* comparison *questions, allowing up to two lies in the answers, where $q(n)$ is the smallest integer $q \geqslant 0$ satisfying Berlekamp's inequality*

$$2^q \geqslant 2^n \left(\binom{q}{2} + q + 1 \right).$$

There cannot exist any winning strategy with less than $q(n)$ questions.

Proof. The last statement follows from Berlekamp's inequality (3) in Lemma 2.5(ii). There remains to be proved that for each $2 \neq n \geqslant 1$, the initial state α_n over S_n is amenable. The case $n = 1$ is trivial, since in $S_1 = \{0, 1\}$ every state is well-shaped, $ch(\alpha_1) = 5$, and every question is a comparison question. A proof that α_2 is not amenable easily follows from [2, p. 75]. Now assume $n \geqslant 3$. Let $\xi \geqslant 9$ be the character of the initial state α_n. Theorem 3.3 yields a comparison question Q, and two well-shaped states $\beta_0 = \alpha_n \odot \nu_Q$ and $\beta_1 = \alpha_n \odot \pi_Q$ having the same type $(2^{n-1}, 2^{n-1}, 0)$, and the same character $\xi - 1$. Another application of the theorem yields two comparison questions and four well-shaped states β_{00}, β_{01}, β_{10}, and β_{11}, of type $(2^{n-2}, 2^{n-1}, 2^{n-2})$ and character $\xi - 2$. Repeated use of Theorem 3.3 yields a strategy with n comparison questions such that each leaf L determines a well-shaped state β_L of the same type $(1, n, \binom{n}{2})$ and character $\chi = \xi - n$. We shall prove that β_L is amenable. First of all, since $ch(1, 3, \binom{3}{2}) = ch(1, 4, \binom{4}{2}) = 6$, we can restrict to the case $n \geqslant 4$. More generally, since by Definition 4.1, $n \leqslant n_\chi$, it is sufficient to prove the amenability of every well-shaped state of type $(1, n_\chi, \binom{n_\chi}{2})$, for each $\chi \geqslant 6$. Since by (24), $p_\chi \geqslant \binom{n_\chi}{2}$, it suffices to prove the amenability of every well-shaped state of type $(1, n_\chi, p_\chi)$, for $\chi \geqslant 6$. This is done in Proposition 5.4. □

Final Remarks. (1) An *interval question* has the form "does x satisfy the condition $a \leqslant x \leqslant b$?". Starting from α_n with $n \geqslant 4$ and mimicking the above proof, we easily get a strategy with 3 interval questions and 8 well-shaped states of the same form $H^1 C^{2/3} P^{1/3} B^{2/3}$, where $\#H = 2^{n-3}$, $\#C = 2^{n-2}$, $\#P = 3 \times 2^{n-3}$, $\#B = 2^{n-3}$. For any such state σ with support Σ there is *no* interval question Q such that $\sigma_{yes} = \sigma \odot \pi_Q$ and $\sigma_{no} = \sigma \odot \nu_Q$ have the same type. For otherwise, assuming without loss of generality that $Q \cap \Sigma$ coincides with the second half of H and contains an initial segment C^* of C, then C^* must have $3 \times 2^{n-4}$ elements. Since Q is an arc in S_n, Q is disjoint from P, whence $\#\sigma_{yes}^{-1}(\frac{1}{3}) \neq \#\sigma_{no}^{-1}(\frac{1}{3})$, and the counterpart of Theorem 3.3 fails for interval questions.

(2) Since comparison questions are expressible by the endpoints of two intervals, using Lemma 2.3 and Theorem 6.1 we can substantially simplify the description of states and the computation of optimal comparison strategies in Ulam's game with two errors.

(3) Already in the case with no lies, if all questions are asked at the beginning of the game, independently of the answers, then the number of *comparison* questions needed to find the unknown number $x \in S_n$ grows *exponentially* with n. Theorem 6.1 points out the role of interactiveness in reducing to Berlekamp's theoretical minimum the number of comparison questions in the game with two errors.

(4) What are the analogues of well-shaped states, comparison questions, and shape-preserving optimal strategies in Ulam's game with $l \geqslant 3$ errors?

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