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Dedekind multisets and function shells

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Abstract

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A multiset is a set containing repeated elements. We define the unique multiset determined by a function based on an idea formulated by Dedekind in 1888. The class of all functions which determine the same multiset is called a function shell. The formal theory of multisets is revised to allow for infinite repetitions of elements. The revised theory of multisets is then interpreted as a theory of function shells. With this interpretation, a new algebra of functions is defined, and interesting properties of function shells are investigated.

1. Introduction

A multiset is a set containing repeated elements. We define the unique multiset determined by a function based on an idea formulated by Dedekind in 1888. The class of all functions which determine the same multiset is called a function shell. The formal theory of multisets is revised to allow for infinite repetitions of elements. The revised theory of multisets is then interpreted as a theory of function shells. With this interpretation, a new algebra of functions is defined, and interesting properties of function shells are investigated.

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2. Dedekind multisets

In 1888, Richard Dedekind published his well-known paper "Was sind und was sollen die Zahlen?" [4, pp. 335–391] – in English translation "The nature and meaning of numbers" [3, pp. 31–115]. In his final remark (paragraph 172), Dedekind introduces the notion of a *multiset* – a set in which elements may belong more than once. His purpose is to show that the word "number" can be used in different senses. He considers the following situation (we use Dedekind's notation): let Ψ be a function from the set Σ (with n elements) onto the set $\Psi(\Sigma)$ (with m elements). If Ψ is not injective, then $m<n$. Dedekind argues that there is a sense in which one can say that the "number" of elements in $\Psi(\Sigma)$ is n. For example, if x is an element of Σ and k is the number of elements of Σ with the same image $\Psi(x)$, then $\Psi(x)$ as an element of $\Psi(\Sigma)$ can be regarded as representative of k elements "... which at least from their derivation may be considered as different from one another ..." and, therefore, $\Psi(x)$ can be counted as a k-fold element of $\Psi(\Sigma)$. Dedekind remarks, "In this way we reach the notion, very useful in many cases, of systems in which every element is endowed with a certain frequency-number which indicates how often it is to be reckoned as element of the system." He concludes "... we would say that n is the number of the elements of $\Psi(\Sigma)$ counted in this sense, while the number m ... [is the number of] ... actually different elements of this system ...".

In summary, Dedekind's idea is that an element in the range of a function can be thought of as a k-fold element of the range, where k is the number of elements in the domain that are mapped to that element. In other words, the frequency-number of an image is the number of its preimages.

After a brief introduction to multiset theory, we investigate Dedekind's concept of multiset in more detail in Section 4.

3. Multiset theory

We summarize below only those parts of multiset theory that we need in Section 4 of this paper. For further details, the interested reader is referred to [2].

A *multiset* is a collection of objects (called *elements*) in which elements may occur more than once. The number of times an element occurs in a multiset is called its *multiplicity* in the multiset. A *set* is a multiset in which every element has multiplicity one. The *root* of a multiset is the set containing the distinct elements of the multiset. The root set of a multiset M is denoted by M^*. A multiset is *simple* if its root set is a singleton set. The *empty multiset* (or set) is denoted by $\emptyset$. A multiset M is an *msubset* of a multiset N, denoted by $M \subseteq N$, if whenever x is an element of M with multiplicity α, then x is an element of N with some multiplicity $\beta \geqslant \alpha$. If x is an element of M with multiplicity α, we write $x \in^{\alpha} M$. Therefore, $M \subseteq N$ stands for $\forall x \forall \alpha (x \in^{\alpha} M \rightarrow \exists \beta (x \in^{\beta} N \wedge \beta \geqslant \alpha))$.

A multiset M is *regular* if all elements belong to M with the same multiplicity, that is,

$$\forall x\,\forall y\,\forall\alpha\,\forall\beta((x\in^{\alpha} M \wedge y\in^{\beta} M)\rightarrow\alpha=\beta) \text{ holds.}$$

For any multiset M, we write $x\in M$ to stand for $\exists\alpha(\alpha>0 \wedge x\in^{\alpha} M)$. The elements of the *union* of M and N, denoted $M\cup N$, are all elements that belong to either M or N. The multiplicity of an element in $M\cup N$ is the *maximum* of its multiplicities in M and N, where nonmembership is taken to mean multiplicity zero. The elements of the *intersection* of M and N, denoted by $M\cap N$, are all elements that belong to both M and N. The multiplicity of an element in $M\cap N$ is the *minimum* of its multiplicities in both M and N.

The cardinality of a multiset is intended to measure the total number of elements (counting both distinct elements and repeated elements) in the multiset. The *cardinality* of a multiset is, therefore, the sum of the multiplicities of its elements. We denote the cardinality of a classical set X by $|X|$ and the cardinality of a multiset M by $C(M)$. If X is a set, then $|X|=C(X)$.

A first-order formal theory of multisets (for multisets in which the multiplicities of elements are classical cardinal numbers) is developed in Section 5.

We denote the *set* containing elements $x, y, z, \ldots$ by $\{x, y, z, \ldots\}$ and we denote the multiset containing element x with multiplicity α, element y with multiplicity β, element z with multiplicity γ, ... by $[x, y, z, \ldots]_{\alpha,\beta,\gamma,\ldots}$.

A multiset M is called *finite* if $C(M)<\aleph_0$ and *infinite* if $C(M)\geqslant\aleph_0$. One can show that if $M\subseteq N$, then $C(M)\leqslant C(N)$. Since $M^*\subseteq M$, $C(M^*)\leqslant C(M)$ for all multisets M. It follows, therefore, that if M is finite, then M^* is finite. However, the converse is false. Since the multiplicity of an element may be any cardinal number, it is quite possible that M^* is finite but M itself is infinite. For example, if $M=[x]_{\lambda}$, where $\lambda\geqslant\aleph_0$, then $M^*=\{x\}$ is finite, but M is infinite since $C(M)=\lambda$. Therefore, for infinite multisets M for which M^* is finite, we say that M contains a finite number of *distinct* elements and an infinite number of *repeated* elements. For infinite multisets M for which M^* is also infinite, we say that M contains infinitely many *distinct* elements. However, any such distinct element x in M may repeat finitely or infinitely many times in M (depending upon whether its multiplicity λ in M is $\lambda<\aleph_0$ or $\lambda\geqslant\aleph_0$, respectively).

In classical mathematics, one represents multisets as sequences (n-tuples, vectors), families or functions. All these representations are equivalent to Dedekind's notion of a multiset. Consider the representation of a multiset as a *sequence* in which the multiplicity of elements equals the number of times the element occurs in the sequence. A sequence is just a function from the set $\mathbb{N}$ to the set of distinct elements which occur in the sequence. For any element in the range of this function, its multiplicity (in the multiset being represented) is exactly the cardinality of its inverse image set (the number of numbers in $\mathbb{N}$ that are mapped to it). Therefore, the notion of multiplicity in sequences is exactly Dedekind's "frequency-number" in the range of a function.

The problem with representing multisets as sequences is that an order is implied between the elements where none is intended: the multisets $[x, y, x, x, y, z]$ and $[y, x, x, x, y, z]$ are equal, but $\langle x, y, x, x, y, z\rangle$ and $\langle y, x, x, x, y, z\rangle$ are different sequences. Thus, one finds multisets defined as "unordered sequences" (see, for example, [8]). What exactly is an "unordered sequence"? Let f be a sequence from $N \subseteq \mathbb{N}$ to the set S. For any permutation π of N, define the sequence $\pi f: N \to S$ by $\pi f(n) = f(\pi(n))$ for all $n \in N$. The "unordered sequence" f is the set $\{\pi f \mid \pi \text{ is a permutation of } N\}$ of all such sequences πf. To abstract away the order of the elements, one must take all permutations of the domain of f.

One may also represent a multiset as a *family* of sets like $\mathscr{F} = \{\mathscr{F}_i\}_{i \in I}$, where the indices i range over some index set I, and where $F_i = F_j$, if $i \neq j$, represents a repeated element. But this is just a generalization of the idea of a sequence. The family $\mathscr{F}$ is a function from the index set I to the *set* of distinct $\mathscr{F}_i$'s; that is, $\mathscr{F}: I \to \{\mathscr{F}_i \mid i \in I\}$, where $\mathscr{F}(i) = \mathscr{F}_i$ for all $i \in I$. The multiplicity of some $\mathscr{F}_i$ is exactly the cardinality of its inverse image set $\mathscr{F}^{-1}(\mathscr{F}_i)$; that is, the number of distinct indices mapped to it. Again, this is exactly Dedekind's "frequency-number". If $I = \mathbb{N}$, then $\mathscr{F}$ is a sequence.

One often represents multisets as numeric-valued functions (sometimes, cardinal-valued functions). For example, the function $f: S \to \mathbb{N}$ is interpreted as a multiset where the multiplicity of the element $x \in S$ is the number $f(x) \in \mathbb{N}$. Thus, f is a set of ordered pairs $\langle x, n\rangle$, where n represents the multiplicity of x. Dedekind multisets are (as we shall see) just cardinal-valued functions which associate with each element (in the range of the defining function) a cardinal number (the cardinality of its inverse image set under the defining function).

The identification of sets with their characteristic functions (as in $\mathbb{P}(X)$ and 2^X) suggests the representation of multisets as "generalized characteristic functions" which take other values in addition to 0 and 1. But again, this is exactly the numeric-valued function representation of a multiset.

When one first encounters Dedekind's concept of multiset (in which there are said to be as many images as preimages), one feels that it is slightly perverse or unnatural. In fact, as we have just shown, Dedekind's approach is equivalent to the most common formalizations of multisets in classical mathematics.

4. Function shells

Dedekind considered only functions between finite sets. In this case, every inverse image set is a finite set and every element of a multiset has finite multiplicity. We consider the general case in which the cardinality of an inverse image set (or the multiplicity of an element in a multiset) can be any cardinal number. Most examples, however, will involve finite multiplicities only. We now recast Dedekind's idea in a more general setting.

Let $f: X \to Y$ be an arbitrary but fixed function with domain X and co-domain Y. We denote the domain of f by $\text{dom} f$. We denote the range of f by $f(X) \subseteq Y$. For each

$y \in f(X)$, the inverse image set is $f^{-1}(y)=\{x \in X \mid f(x)=y\}$. We define *the Dedekind multiset determined by f*, denoted by M_f, as follows:

(i) $M_f^* = f(X) \subseteq Y$, and

(ii) for all $y \in M_f^*$, $y \in^{\alpha} M_f$ iff $|f^{-1}(y)| = \alpha$.

In other words, the elements of M_f are exactly the elements of $f(X)$, and the multiplicity of an element y in M_f is the unique cardinal number $|f^{-1}(y)|$. In Dedekind's words, "the frequency-number" of an image element y in $f(X)$ equals the number of its preimages in X. Since the inverse image sets $f^{-1}(y)$ partition the domain X, $\sum_{y \in f(X)} |f^{-1}(y)| = |X|$, where $\sum$ denotes the cardinal sum. The total number of preimages equals the cardinality of the domain X. Since the cardinality of a multiset is the sum of the multiplicities of its elements,

$$C(M_f) = |\mathrm{dom}\, f|.$$

For example, let $\mathbb{N}$, $\mathbb{N}^+$ and $\mathbb{Q}$, respectively, be the set of natural numbers, the set of positive natural numbers and the set of nonnegative rational numbers. Define a function $f: \mathbb{N} \times \mathbb{N}^+ \to \mathbb{Q}$ by

$$\langle n, m \rangle \mapsto [n/m],$$

where $m \neq 0$ and $[n/m]$ is the equivalence class of all nonnegative rationals equal to n/m. So, for example, $f^{-1}([1/2]) = \{\langle 1,2 \rangle, \langle 2,4 \rangle, \langle 3,6 \rangle, \dots\}$. Let $q_0, q_1, q_2, \dots$ be some countable enumeration of the nonnegative rationals $\mathbb{Q}$ (the *set* of equivalence classes – if $i \neq j$, then $q_i \neq q_j$). The Dedekind multiset M_f determined by f is the regular multiset

$$M_f = [q_0, q_1, q_2, \dots]_{\lambda, \lambda, \lambda, \dots}, \quad \lambda = \aleph_0.$$

In other words, the root set $M_f^* = \mathbb{Q} = \mathrm{range}(f)$ and, for each $q \in \mathbb{Q}$, $|f^{-1}(q)| = \aleph_0$. The multiset M_f contains a countably infinite number of distinct elements $q_0, q_1, q_2, \dots$, each with multiplicity $\aleph_0$. Therefore, $C(M_f) = \aleph_0 + \aleph_0 + \aleph_0 + \cdots = \aleph_0 \cdot \aleph_0 = \aleph_0$.

Clearly, different functions may determine the same multiset. Let $f: X \to Y$ and $g: Z \to Y$ be two Y-valued functions. (We say "Y-valued" in exactly the same sense as we say integer-valued, rational-valued, real-valued, complex-valued, ... By "Y-valued" we simply mean $f(X) \subseteq Y$ and $g(Z) \subseteq Y$.) A necessary and sufficient condition that f and g determine the same multiset is given by the following theorem.

Theorem. $M_f = M_g$ *iff*

(i) $f(X) = g(Z)$ *and*

(ii) $|f^{-1}(y)| = |g^{-1}(y)|$ *for every* y *in* $f(X) = g(Z)$.

This is exactly the definition of equality of multisets: multisets are equal iff they contain exactly the same elements with exactly the same multiplicities. Therefore, (i) and (ii) are equivalent to

$$\forall y \forall \alpha (y \in^{\alpha} M_f \Leftrightarrow y \in^{\alpha} M_g).$$

There are many examples of functions that are very different, but that, nevertheless, determine the same multiset. Let p_i be the ith prime (so p_1 is 2, p_2 is 3, p_3 is 5, ...) and let $P=\{p_1,p_2,p_3,\dots\}$. For each $p\in P$, let $u_p=\{x\in\mathbb{C}\mid x^p=1 \text{ and } x\neq 1\}$, which contains $p-1$ elements. We define a function f with domain $P\cup\bigcup_{p\in P}u_p$ and range P by

$$x\mapsto f(x)=\begin{cases} p & \text{if } x\in u_p,\\ x & \text{if } x\in P.\end{cases}$$

Clearly, for each $p\in P$, $|f^{-1}(p)|=p$. Define a function $g:\mathbb{N}\to P$ as follows: the first p_1 natural numbers are mapped to p_1, the next p_2 natural numbers are mapped to $p_2,\dots$ Clearly, for each $p\in P$, $|g^{-1}(p)|=p$. Therefore, the two functions $f:P\cup\bigcup_{p\in P}u_p\to P$ and $g:\mathbb{N}\to P$ determine the same multiset, namely, $[2,3,5,\dots]_{2,3,5,\dots}$.

The Möbius function and the character modulo three function determine the same multiset. Both the functions have domain $\mathbb{N}$ and range $\{-1,0,1\}$. The Möbius function f is defined as follows:

$$n\mapsto f(n)=\begin{cases} 1 & \text{if } n=1,\\ (-1)^k & \text{if } n \text{ is the product of } k \textit{ distinct} \text{ primes},\\ 0 & \text{if some } p_i^2 \text{ divides } n.\end{cases}$$

For example, $f(55)=f(5\cdot 11)=(-1)^2=1$, $f(42)=f(2\cdot 3\cdot 7)=(-1)^3=-1$ and $f(120)=f(2^3\cdot 3\cdot 5)=0$ since 2^2 divides 120. For each $y\in\{-1,0,1\}$, $|f^{-1}(y)|=\aleph_0$. The character modulo three function g is defined as follows:

$$n\mapsto g(n)=\begin{cases} 0 & \text{if } n\equiv 0 \pmod 3,\\ 1 & \text{if } n\equiv 1 \pmod 3,\\ -1 & \text{if } n\equiv 2 \pmod 3.\end{cases}$$

Clearly, for each $y\in\{-1,0,1\}$, $|g^{-1}(y)|=\aleph_0$. Therefore, f and g determine the same multiset $[-1,0,1]_{\lambda,\lambda,\lambda}$, where $\lambda=\aleph_0$.

Many examples of real- or complex-valued functions can be constructed with similar properties. The number of such examples is limited only by ingenuity.

The actual domains of functions that determine the same multiset are unimportant. If one function maps α elements to y, then the other function must also map α elements to y. The nature of these α elements is irrelevant. We want to define a "function shell" to be a collection of functions such that every function in the collection determines the same multiset. A "function shell" is a class (and not a set) if we define it to be *the* class of *all* functions which determine a single multiset. A "function shell", therefore, represents a domain-independent function in which the nature of domain elements is ignored, but in which the "shell" (the skeleton, the form) of the function is preserved; that is, the cardinal number partition of the domain (induced by the inverse image sets) is preserved. One removes the meat and is left with the shell.

Let $\mathscr{M}$ be an arbitrary but fixed multiset. We could define the "function shell" which determines $\mathscr{M}$ to be the class of functions $\{f \mid M_f = \mathscr{M}\}$. However, a large class of functions is rather unwieldy. Luckily, there is a much simpler approach to "function shells". We note that the "function shell" above is completely characterized by the root set $\mathscr{M}^*$ (the common range of the functions) together with the unique cardinal number associated with each element in $\mathscr{M}^*$. The multiset $\mathscr{M}$ itself completely characterizes the "function shell", and $\mathscr{M}$ can be thought of as a cardinal-valued function with domain $\mathscr{M}^*$. We, therefore, define "function shells" classically to be certain cardinal-valued functions.

Let F be an arbitrary but fixed cardinal-valued function such that dom F is a set. In general, we use $F, G, H, \ldots$ to denote cardinal-valued functions whose domains are sets. The function F "represents" the *function shell* $\mathscr{F}_F$ which consists of all functions f such that

(i) the range of f equals the set dom F, and

(ii) for every element $x \in \text{dom } F$ (= range of f), f maps $F(x)$ preimages (in dom f) to x, that is, $|f^{-1}(x)| = F(x)$.

This is equivalent to defining an equivalence relation $\sim$ on functions such that $f \sim g$ iff $M_f = M_g$. The function shells are then the equivalence classes generated by $\sim$.

The function F also "represents" the *multiset* $\mathscr{M}_F$, which is defined as follows:

(i) the root set $\mathscr{M}_F^* = \text{dom } F$, and

(ii) for every element $x \in \text{dom } F$,

$$x \in^{\alpha} \mathscr{M}_F \text{ iff } \alpha = F(x).$$

Clearly, the multiset $\mathscr{M}_F$ is the multiset determined by $\mathscr{F}_F$; that is, $\mathscr{M}_F = M_f$ for every function $f \in \mathscr{F}_F$. The multiset $\mathscr{M}_F$ and the class $\mathscr{F}_F$ are completely determined by the function F. We identify the function F, the function shell $\mathscr{F}_F$ and the multiset $\mathscr{M}_F$. We can represent this single concept by Fig. 1. Since the multiset $\mathscr{M}_F$ is a nonclassical entity and since the function shell $\mathscr{F}_F$ is a rather large class of functions, we will most often work with the cardinal-valued function F (simply a *set* of ordered pairs). F is the classical anchor for both the exotic $\mathscr{M}_F$ and the unwieldy $\mathscr{F}_F$.

Consider the multiset $[a, b, c]_{5,1,3}$. We can think of this multiset in a variety of ways:

(i) It is simply a collection of three distinct elements, a, b and c, in which a repeats five times, b "repeats" once and c repeats three times, and it has cardinality $5 + 1 + 3 = 9$.

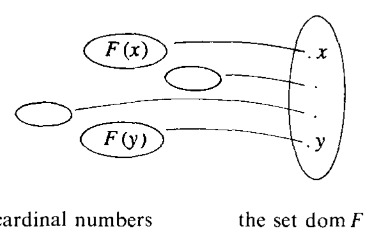


Fig. 1.

(ii) It is the unique multiset $\mathcal{M}_F$ defined by the cardinal-valued function $F=\{\langle a,5\rangle,\langle b,1\rangle,\langle c,3\rangle\}$ with $\operatorname{dom} F=\{a,b,c\}$.

(iii) It represents the large class $\mathcal{F}_F$ of functions, all of which have range $\{a,b,c\}$ and map five domain elements to a, one domain element to b, and three domain elements to c.

A candidate for the "canonical representative" of functions in $\mathcal{F}_F$ is the function $f:9\to\{a,b,c\}$ defined by

$$f(n)=\begin{cases} a & \text{if } 0\leqslant n\leqslant 4\\ b & \text{if } 5\leqslant n\leqslant 5\\ c & \text{if } 6\leqslant n\leqslant 8\end{cases}$$

where $\operatorname{dom} f=\{0,1,2,\ldots,7,8\}=9\in On$ and $|\operatorname{dom} f|=C(M_f)=9$.

The above example suggests a general definition for the "canonical representative" of the functions f in the function shell $\mathcal{F}_F$. Using the axiom of choice, the set $\operatorname{dom} F$ (the common range of the functions f in $\mathcal{F}_F$) can be well-ordered. In other words, there exists an ordinal $\beta\in On$ such that $\operatorname{dom} F\approx\beta$. We label the elements $y\in\operatorname{dom} F$ such that $\operatorname{dom} F=\{y_\alpha\mid\alpha\in\beta\}$. The *canonical representative* $\hat{f}$ of the functions f in $\mathcal{F}_F$ will be such that $\operatorname{dom}\hat{f}\in On$ and $\hat{f}$ maps the first $F(y_0)$ ordinal numbers to y_0, the next $F(y_1)$ ordinal numbers to $y_1,\ldots,$ the next $F(y_\alpha)$ ordinal numbers to $y_\alpha,\ldots$ Therefore, $\operatorname{dom}\hat{f}=\lambda\in On$, where $\lambda=\sum_{\alpha\in\beta}F(y_\alpha)$, where $\sum$ denotes the ordinal sum of the $F(y_\alpha)$'s. As defined above, $\hat{f}:\lambda\to\operatorname{dom} F$ is in $\mathcal{F}_F$.

The reader is warned not to confuse the notation for the multisets M_f and $\mathcal{M}_F$. The multiset M_f is defined for arbitrary functions f, whereas the multiset $\mathcal{M}_F$ is defined only for cardinal-valued functions F such that $\operatorname{dom} F$ is a set. The multiset M_f is the Dedekind multiset determined by the function f (that is, $M_f^*=\operatorname{range}(f)$ and $y\in^\alpha M_f$ iff $|f^{-1}(y)|=\alpha$), whereas the multiset $\mathcal{M}_F$ is the multiset defined by the cardinal-number function F (that is, $\mathcal{M}_F^*=\operatorname{dom} F$ and $y\in^\alpha\mathcal{M}_F$ iff $F(y)=\alpha$). For example, if f is the Möbius function defined earlier, then $M_f=[-1,0,1]_{\lambda,\lambda,\lambda}$, where $\lambda=\mathcal{N}_0$. If, on the other hand, F is the cardinal-valued function $F:\{-1,0,1\}\to\{\mathcal{N}_0\}$, then $\mathcal{M}_F=M_f$. In general, if $f\in\mathcal{F}_F$, then $M_f=\mathcal{M}_F$. However, the function F (simply by virtue of being a function) also determines the Dedekind multiset $M_F=[\mathcal{N}_0]_3$.

Although the function F is straightforward enough, the interesting results will arise when we think in terms of the multiset $\mathcal{M}_F$, or the functions in $\mathcal{F}_F$.

We now ask two questions:

(1) What are the properties of functions in a given function shell?

(2) What different types of multisets arise from different types of function shells?

We take the first question first. Let $\mathcal{F}_F$ be an arbitrary but fixed function shell. There are many different functions in the class $\mathcal{F}_F$. They are all different and yet they are all "alike". The function $\hat{f}\in\mathcal{F}_F$ defined above can be taken as representative of all the functions in $\mathcal{F}_F$. They all have the same range (the set $\operatorname{dom} F$). Their domains are totally arbitrary except that the cardinality of the domains is equal to the cardinal number $C(\mathcal{M}_F)$, which equals the *cardinal sum* $\sum_{y\in\operatorname{dom} F}F(y)$. In the case of $\hat{f}$,

$|\mathrm{dom}\,\hat{f}|=|\lambda|\leqslant\lambda$. Therefore, every function f in $\mathscr{F}_F$ has the same cardinality since $|f|=|\mathrm{dom}\,f|$. Moreover, for every $y\in\mathrm{dom}\,F$, every function in $\mathscr{F}_F$ maps exactly $F(y)$ preimages to y. Although the domains are arbitrary (of a fixed cardinality), the partitions of the domains induced by the inverse image sets are "isomorphic" in the sense that between any two such domains there exists a partition-preserving bijection that is the disjoint union of component bijections. For example, let $f:X\to Y$ and $g:Z\to Y$ be two functions in $\mathscr{F}_F$. For each $y\in\mathrm{dom}\,F=f(X)=g(Z)\subseteq Y$, there is a bijection $H_y:f^{-1}(y)\to g^{-1}(y)$ since $|f^{-1}(y)|=|g^{-1}(y)|$. Then $H=\bigcup_{y\in\mathrm{dom}\,F}H_y$ is the required bijection from X to Z (Fig. 2). H is a "partition-preserving" bijection in the sense that if x is mapped to y by f, then $H(x)=H_y(x)$ is also mapped to y by g. In other words, $x\in f^{-1}(y)$ iff $H(x)\in g^{-1}(y)$, or simply $f=g\circ H$.

Let f be any function in $\mathscr{F}_F$. We think of f as a set of ordered pairs $\langle x,y\rangle$. Let π be any permutation of $\mathrm{dom}\,f$. Define the function πf by $\langle\pi(x),y\rangle\in\pi f$ iff $\langle x,y\rangle\in f$. The functions f and πf have the same domain and the same range. For any element y in the common range, $|f^{-1}(y)|=|(\pi f)^{-1}(y)|$ since the *number* of ordered pairs with y as the second component is unaffected by the permutation π. Therefore, if $f\in\mathscr{F}_F$ and π is a permutation of $\mathrm{dom}\,f$, then $\pi f\in\mathscr{F}_F$. The function shell $\mathscr{F}_F$ is, therefore, closed with respect to permutations of domains. In other words, permutations of domains of functions do not alter the "shell properties" of functions.

We now consider the second question. We first consider different types of multisets $\mathscr{M}_F$ and the function shells $\mathscr{F}_F$ which give rise to them. We also look at various types of function shells $\mathscr{F}_F$ and determine their corresponding multisets $\mathscr{M}_F$. If $\mathscr{M}_F$ is the empty multiset $\emptyset$, then $\mathrm{dom}\,F=\emptyset$ and $\mathscr{F}_F=\{\emptyset\}$. If $\mathscr{M}_F$ is a singleton set $\{x\}$, then $F:\{x\}\to\{1\}$ is the cardinal-valued function $\{\langle x,1\rangle\}$. The functions $f:\mathrm{dom}\,f\to\{x\}$ are all of the form $\{\langle z,x\rangle\}$, where z is arbitrary ($\mathrm{dom}\,f=\{z\}$). Therefore, if $\mathscr{M}_F$ is a singleton $\{x\}$, then F is a singleton $\{\langle x,1\rangle\}$ and all f in $\mathscr{F}_F$ are singletons $\{\langle z,x\rangle\}$ with arbitrary singleton domains $\{z\}$, and the singleton range $\{x\}$. If $\mathscr{M}_F$ is a simple multiset of the form $[x]_\lambda$, then F is the singleton function $\{\langle x,\lambda\rangle\}$. All functions in $\mathscr{F}_F$ have range $\{x\}$ and arbitrary domains of cardinality λ. Therefore, $\mathscr{F}_F$ is exactly the class of all constant functions (single-valued) with range $\{x\}$ and with domains of cardinality λ. The canonical representative of functions in $\mathscr{F}_F$ is the function $\hat{f}:\lambda\to\{x\}$ since $|\lambda|=\lambda$.

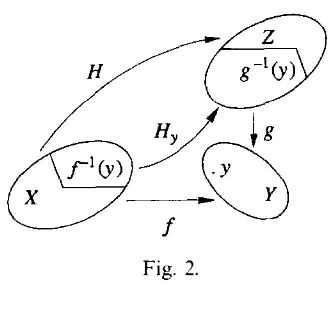


Fig. 2.

If $\mathcal{M}_F$ is a set, then F is a constant $\{1\}$-valued function with $\operatorname{dom} F=\mathcal{M}_F^*=\mathcal{M}_F$ and every function f in $\mathscr{F}_F$ is an injection (that is, every inverse image set is a singleton) with range $\mathcal{M}_F^*=\mathcal{M}_F$. In this case, $|\operatorname{dom} f|=|\operatorname{dom} F|=C(\mathcal{M}_F)$. In general, functions in $\mathscr{F}_F$ are injections iff $\mathcal{M}_F$ is a set. By contraposition, functions in $\mathscr{F}_F$ are many–one (noninjective) iff $\mathcal{M}_F$ is a nonset (that is, contains a repeated element).

If $C(\mathcal{M}_F^*)\geqslant\mathcal{N}_0$ (that is, if $\mathcal{M}_F$ contains infinitely many *distinct* elements), then $\mathcal{M}_F$ is called a *near set* if all but a finite number of distinct elements have multiplicity 1. If $\mathcal{M}_F$ is a near set, then F has an infinite domain and a finite range (which includes 1) and every function in $\mathscr{F}_F$ has range $\mathcal{M}_F^*$ and all but a finite number of its inverse image sets are singletons.

If $\mathcal{M}_F$ is a finite multiset, then F is a finite set of ordered pairs (since $\operatorname{dom} F$ is a finite set) and all functions $f\in\mathscr{F}_F$ are finite sets of ordered pairs since $|\operatorname{dom} f|=C(\mathcal{M}_F)<\mathcal{N}_0$.

If $\mathcal{M}_F$ is regular with all elements having multiplicity λ, then the function F is a constant $\{\lambda\}$-valued function with $\operatorname{dom} F=\mathcal{M}_F^*$. In this case, for every function $f\in\mathscr{F}_F$, every inverse image set is exactly the same size; that is, $|f^{-1}(x)|=|f^{-1}(y)|=\lambda$ for every x and y in $\operatorname{dom} F$. In other words, the partition of the arbitrary domains is *uniform* (every disjoint subset has cardinality λ).

Finally, if $\mathcal{M}_F$ is an infinite multiset, then $\operatorname{dom} F$ may be finite or infinite. If $\operatorname{dom} F$ is finite, then F itself is finite and the finite range of F must contain at least one cardinal number $\lambda\geqslant\mathcal{N}_0$. In this case, every $f\in\mathscr{F}_F$ has the finite-range $\operatorname{dom} F=\mathcal{M}_F^*$ and an infinite domain since at least one inverse image set must be infinite. If, on the other hand, $\operatorname{dom} F$ is infinite, then F is infinite and its range may be finite or infinite and may or may not contain an infinite cardinal number. In this case, the functions $f\in\mathscr{F}_F$ have the infinite range $\operatorname{dom} F=\mathcal{M}_F^*$ and infinite domains (since $|\operatorname{dom} f|\geqslant|\operatorname{dom} F|$) which may or may not contain an infinite inverse image subset.

We may also ask: Given a particular type of class $\mathscr{F}_F$, what do the corresponding F and $\mathcal{M}_F$ look like? Suppose, for example, that the partition of the arbitrary domains of functions $f\in\mathscr{F}_F$ is such that for every $n\in\omega^+$ (where $\omega^+=\omega-\{0\}$ is the set of *positive* natural numbers), there exists one and only one inverse image set of cardinality n. This is a generalization of a previous example for which the above condition holds for all prime numbers. In this case, the common range of the functions f must be countable and the arbitrary domains will also be countable. A canonical representative for such functions $f\in\mathscr{F}_F$ is the function $\hat{f}:\omega^+\to\omega^+$ defined by $\hat{f}$ takes 1 to 1, $\hat{f}$ takes the next 2 elements to $2,\ldots,\hat{f}$ takes the next n elements to $n,\ldots$ Therefore, $\hat{f}=\{\langle 1,1\rangle,\langle 2,2\rangle,\langle 3,2\rangle,\langle 4,3\rangle,\langle 5,3\rangle,\langle 6,3\rangle,\ldots\}$. Hence, for all $n\in\omega^+$, $|\hat{f}^{-1}(n)|=n$. The corresponding multiset $\mathcal{M}_F$ is $[1,2,3,\ldots]_{1,2,3,\ldots}$ and the function $F:\omega^+\to\omega^+$ is the identity map.

An alternative approach to such characterizations could proceed via the function F. For example, if $F=\emptyset$, then $\mathcal{M}_F=\emptyset$ and $\mathscr{F}_F=\{\emptyset\}$. If the range of F is a singleton (F is a constant cardinal-valued function), then $\mathcal{M}_F$ is a regular multiset and the domains of functions f in $\mathscr{F}_F$ are uniformly partitioned by the inverse image sets. If, in particular, the range of F is $\{1\}$, then $\mathcal{M}_F$ is a set and all functions f in $\mathscr{F}_F$ are injections.

5. Cardinal-valued multisets: the formal theory MSTC

The theory MST developed in [2] is a first-order two-sorted theory for multisets in which elements possess a unique finite multiplicity, a positive natural number. The theory MST contains an exact copy of ZFC and is shown to be relatively consistent (a model of MST is constructed in ZFC). We now give a brief description of a procedure to change the theory MST into a new theory, MSTC, which is intended to formalize multisets in which elements possess a unique (finite or infinite) cardinal number multiplicity. The theory MSTC will be interpreted as a formal theory of function shells in Section 6.

A rather lengthy discussion of the advantages and disadvantages of formalizing multisets with infinite multiplicities is given in [1, Chapter VI]. It is not our intention to get bogged down in formal details here. We avoid the formal difficulties where possible, while at the same time preserving as much of MST within MSTC as is reasonable and practical.

In [5–7, 9], multisets are defined as classical cardinal-valued functions in ZFC. Hickman, in particular, has laid down a great deal of the conceptual groundwork for the definitions and the algebra of cardinal-valued multisets. In the development of the formal theory MSTC, we have attempted, wherever possible, to remain faithful to the naive (nonaxiomatic) concepts defined in [5–7, 9].

The multiset variable symbols $x, y, z, \ldots$ of MST (which denote multisets and elements of multisets) remain unchanged in MSTC. However, the numeric variable symbols of MST (whch denote multiplicities of elements in multisets) must be changed in MSTC. We replace the numeric variable symbols $k, l, m, n, \ldots$ of MST (intended to range over positive integers) by the numeric variable symbols $\alpha, \beta, \gamma, \ldots$ (intended to range over cardinal numbers). We do not concern ourselves with the specific collection of axioms that are necessary to characterize the arithmetic of cardinal numbers needed in MSTC. We simply assume that the numeric variable symbols $\alpha, \beta, \gamma, \ldots$ of MSTC denote classical cardinal numbers in ZFC. Since we are not concerned with specific cardinal arithmetic axioms, we do not specify all nonlogical symbols of the language L of MSTC, since these do depend upon the specific axioms chosen. However, L will most certainly contain the ternary predicated symbol $\in$ (where the intended interpretation of the atomic formula $x \in^{\alpha} y$ is "x is an element of y with multiplicity α"), the unary function symbols $\hat{}$ (where $\hat{\alpha}$ is the hereditary set of MSTC that corresponds to the cardinal number α) and $\sum$ (where Σx is the cardinal sum of "cardinals" $\hat{\alpha}$ in x), the binary function symbols $+$ and $\cdot$ (binary cardinal addition and multiplication) and the numeric constant symbol 0 (which denotes the cardinal number zero).

In addition to expressions of the form $x \in^{\alpha} y$, the only other atomic formulae of L are expressions of the forms $x = y$ and $\alpha = \beta$. The wffs of L are defined in the usual way from the atomic formulae using the logical connectives of the first-order predicate calculus. With respect to quantifiers, since L is a two-sorted language, if Φ is a wff of

L and x and α are arbitrary (multiset and numeric) variable symbols, then $\exists x\Phi$, $\forall x\Phi$, $\exists\alpha\Phi$ and $\forall\alpha\Phi$ are wffs of L.

With this minimal amount of formal preparation, we are now in a position to rewrite and revise the axioms and definitions of MST to obtain the theory MSTC. The exact multiplicity axiom I, the axiom of extensionality II, the empty multiset axiom III and the elementary multisets axioms IV require only a rewrite using the new variable symbols $\alpha, \beta, \gamma, \ldots$ We repeat them here for the convenience of the reader:

(I) $\forall x \forall y \forall\alpha \forall\beta((x \in^{\alpha} y \wedge x \in^{\beta} y) \rightarrow \alpha=\beta)$.

(II) $\forall x \forall y(\forall z \forall\alpha(z \in^{\alpha} x \Leftrightarrow z \in^{\alpha} y) \rightarrow x=y)$.

(III) $\exists y \forall x \forall\alpha \sim x \in^{\alpha} y$. Denote y by $\emptyset$.

(IV(i)) $\forall x \forall\alpha \exists y(x \in^{\alpha} y \wedge \forall z(z \in y \Leftrightarrow z=x))$, where $z \in y$ stands for $\exists\beta(\beta \neq 0 \wedge z \in^{\beta} y)$. Denote y by $[x]_{\alpha}$ and $[x]_1$ by $\{x\}$.

(IV(ii)) $\forall x \forall y(x \neq y \rightarrow \forall\alpha \forall\beta \exists z(x \in^{\alpha} z \wedge y \in^{\beta} z \wedge \forall z'(z' \in z \Leftrightarrow (z'=x \vee z'=y))))$. Denote z by $[x, y]_{\alpha,\beta}$ and $[x, y]_{1,1}$ by $\{x, y\}$. Axioms V (the powerset axiom) and VI (the axiom of foundation) remain unchanged (although the definitions in Axiom V must be rewritten):

(V) $\forall x \exists y\,(\mathrm{Set}(y) \wedge \forall z(z \in y \Leftrightarrow z \subseteq x))$, where $\mathrm{Set}(y)$ stands for $y=\emptyset \vee \forall z \forall\alpha(z \in^{\alpha} y \rightarrow \alpha=1)$, $z \subseteq x$ stands for $\forall z' \forall\alpha(z' \in^{\alpha} z \rightarrow \exists\beta(\alpha \leqslant \beta \wedge z' \in^{\beta} x))$ and $\alpha \leqslant \beta$ stands for $\exists\gamma(\beta=\alpha+\gamma)$. Denote y by $\mathbb{P}(x)$.

(VI) $\forall y(y \neq \emptyset \rightarrow \exists x(x \in y \wedge \forall z(z \in x \rightarrow z \notin y)))$. Since the least upper bound of cardinal numbers is a cardinal number, the union axiom of MST is simplified in MSTC:

(VII) $\forall x \exists z' \forall z \forall\alpha(z \in^{\alpha} z' \Leftrightarrow [\exists y(z \in y \wedge y \in x) \wedge \forall y \forall\beta((z \in^{\beta} y \wedge y \in x) \rightarrow \beta \leqslant \alpha) \wedge \forall\alpha'(\forall y \forall\beta((z \in^{\beta} y \wedge y \in x) \rightarrow \beta \leqslant \alpha') \rightarrow \alpha \leqslant \alpha'))])$. Denote z' by $\bigcup x$. Let $x \cup y$ stand for $\bigcup\{x, y\}$ if $x \neq y$, and $\bigcup\{x\}$ otherwise.

(VIII) The additive union axiom (the additive union of x is denoted by $\uplus x$) is, in fact, proveable in MST. It is not necessary, therefore, to include it as an axiom of MSTC (it can be shown to be proveable in MSTC from the other axioms of MSTC). The multiset $\uplus x$ is described below. The separation and replacement schemes must be rewritten using the new variable symbols $\alpha, \beta, \gamma, \ldots$ The *separation scheme* of MSTC reads as follows: for every wff $\Phi(x, \alpha)$ of L with free variables including x and α but excluding z and α', the universal closure of

(IX_{Φ}) $\forall x \forall\alpha \forall\alpha'((\Phi(x, \alpha) \wedge \Phi(x, \alpha')) \rightarrow \alpha=\alpha') \rightarrow \forall y \exists z \forall x \forall\alpha(x \in^{\alpha} z \Leftrightarrow ([x]_{\alpha} \subseteq y \wedge \Phi(x, \alpha)))$ is an axiom of MSTC. We say that "the msubset $z \subseteq y$ is defined by separation on y using the wff $\Phi(x, \alpha)$". The *replacement scheme* of MSTC reads as follows: for every wff $\Phi(x, y)$ of L with free variables including x and y but excluding y' and z', the universal closure of

(X_{Φ}) $\forall x \forall y \forall y'((\Phi(x, y) \wedge \Phi(x, y')) \rightarrow y=y') \rightarrow \forall z \exists z' \forall y \forall\alpha(y \in^{\alpha} z' \Leftrightarrow [\exists x(x \in^{\alpha} z \wedge \Phi(x, y)) \wedge \forall x \forall\beta((x \in^{\beta} z \wedge \Phi(x, y)) \rightarrow \alpha \leqslant \beta)])$ is an axiom of MSTC. We say that "the multiset z' is defined by replacement on z using the wff $\Phi(x, y)$". Axiom XI (the axiom of infinity) is exactly the same as in MST, whereas axiom XII (the choice multiset axiom) requires a rewrite:

(XI) $\exists y(\emptyset \in y \wedge \forall x(x \in y \rightarrow x \cup \{x\} \in y)$

(XII) $\forall y[[y \neq \emptyset \wedge \forall x(x \in y \rightarrow x \neq \emptyset) \wedge \forall x \forall z((x \in y \wedge z \in y \wedge x \neq z) \rightarrow x \cap z = \emptyset)] \rightarrow \exists y'(\forall x \forall \alpha(x \in^{\alpha} y \rightarrow \exists x'(x' \in^{\alpha} y' \wedge x' \in x \wedge \forall x''((x'' \in x \wedge x'' \in y') \rightarrow x'' = x'))) \wedge \forall x' \forall \alpha (x' \in^{\alpha} y' \rightarrow \exists x(x \in^{\alpha} y \wedge x' \in x)))]$, where $x \cap z$ stands for $\bigcap \{x, z\}$. The unary operation $\bigcap$ is defined below. Any multiset y' that satisfies XII is called a *choice multiset* for y.

By using separation on the multiset $\bigcup x$ we can define a unique msubset $\bigcap x \subseteq \bigcup x$ such that

$$\forall z \forall \alpha(z \in^{\alpha} \bigcap x \Leftrightarrow [\forall y(y \in x \rightarrow z \in y) \wedge \forall y \forall \beta((z \in^{\beta} y \wedge y \in x) \rightarrow \alpha \leqslant \beta)$$

$$\wedge \exists y(z \in^{\alpha} y \wedge y \in x)]) \text{ holds.}$$

Let $x \cap y$ stand for $\bigcap \{x, y\}$ if $x \neq y$, and $\bigcap \{x\}$ otherwise.

For every multiset x, there exists a unique multiset $\uplus x$, called the *additive union* of x, such that $\bigcap x \subseteq \bigcup x \subseteq \uplus x$ and $\forall z \ (z \in \uplus x \Leftrightarrow z \in \bigcup x)$ hold. For every $z \in (\uplus x)^* = (\bigcup x)^*$, $z \in^{\lambda} \uplus x \Leftrightarrow \lambda = \sum \alpha \cdot \beta$, where the (possibly infinite) cardinal sum $\sum$ is taken over *all* multisets y such that $z \in^{\alpha} y \wedge y \in^{\beta} x$ holds (λ is a sum of products of cardinal numbers). Let $x \uplus y$ stand for $\uplus \{x, y\}$ if $x \neq y$, and $\uplus [x]_2$ otherwise.

Since the multisets $\bigcup x$ and $\uplus x$ are defined differently in MSTC, the algebra of multisets in MSTC will differ slightly from that in MST. These differences, however, do not affect the axioms, definitions and theorems of MSTC used in Section 6. The vast majority of theorems which are proveable in MST are also proveable (when rewritten) in MSTC.

In MST, in order to determine the cardinality of a multiset y, it is necessary (for each element $x \in y$) to convert the n copies of x in y (where $x \in^{n} y$ holds) into n distinct ordered pairs in the hereditary set $H(y)$ – a multiset z is a *hereditary set* if every element of $TC(\{z\})$ is a set. The hereditary sets of MST (and of MSTC) are the exact analogs of classical sets. The cardinality of y, denoted by $C(y)$, is then defined to be the cardinality of $H(y)$ which is defined classically [the least ordinal equinumerous to $H(y)$]. The motivation for this definition is that the multiplicity of each element should contribute to the cardinality of the multiset as a whole. In MSTC, we simply define the cardinality of a multiset to be the cardinal sum of the multiplicities of its elements; that is, $C(y) = \beta$, where $\hat{\beta} = \sum \{\hat{\alpha} \mid x \in y^* \wedge x \in^{\alpha} y\}$.

In MST, Cantor's theorem ($\forall y \ y < \mathbb{P}(y)$) fails (as it does also in MSTC) but $\forall y(C(y) < C(\mathbb{P}(y))$ is proveable in MST. The limiting cases in MST are simple multisets of the form $[x]_n$, where $C([x]_n) = \hat{n}$ and $C(\mathbb{P}([x]_n)) = C(\{\emptyset, \{x\}, [x]_2, \ldots, [x]_n\}) = n \hat{+} 1$. In MSTC, however, this fails. For example, let $y = [x]_{\lambda}$, where $\lambda = \mathcal{N}_0$. We have that

$$C(\mathbb{P}(y)) = C(\{\emptyset, \{x\}, [x]_2, \ldots, [x]_{\lambda}\}) = \mathcal{N}_0 = C(y).$$

In MSTC, a multiset y is *finite* (or *infinite*) if its cardinality $C(y)$ is finite (or infinite). The theorem of MST "If y^* is finite, then y is finite" and its contraposition "If y is infinite, then y^* is infinite" are not theorems of MSTC. Consider again $y = [x]_{\lambda}$, where $\lambda = \mathcal{N}_0$. The root set $y^* = \{x\}$ is finite, but y itself is infinite. The reason is clear: in

MST, an element can contribute at most a finite multiplicity to the cardinality, whereas in MSTC, an element may contribute any cardinal number to the cardinality. If $\forall x\,\forall\alpha(x\in^{\alpha} y \rightarrow \alpha<\mathcal{N}_0)$ holds, then y is said to have *at most finite repetitions* (y itself may be finite or infinite). Otherwise, y is said to have *infinite repetitions* (and y itself is infinite). If y is infinite but y^* is finite, then y has infinite repetitions. If y^* is finite, then y is said to contain *finitely many distinct elements.* If y^* is infinite, then y is said to contain *infinitely many distinct elements.* A multiset is finite iff it has at most finite repetitions *and* finitely many distinct elements. Contrapositively, a multiset is infinite iff it has infinite repetitions *or* infinitely many distinct elements.

Our purpose has been to provide enough of an outline of the theory MSTC so that we may meaningfully interpret it in Section 6. A detailed investigation of the properties of multisets in MSTC (and how they compare to those in MST) will be undertaken elsewhere. Clearly, MSTC can be shown to contain an exact copy ZFC′ of ZFC and a model of MSTC can be constructed in ZFC (a hierarchy of positive cardinal-valued functions) using methods similar to those described in [2] for MST. We, therefore, assume (without proof) that MSTC contains a copy of ZFC and is relatively consistent.

6. Interpreting MSTC as a theory of function shells

The obvious model of MSTC is a hierarchy of positive cardinal-valued functions of ZFC similar to the model of MST defined in Section III of [2]. The numeric variable symbols $\alpha, \beta, \gamma, \ldots$ would range over the class of positive cardinal numbers, denoted by Card^+. The multiset variable symbols $x, y, z, \ldots$ would range over the class $\mathbb{F}$ (a hierarchy of Card^+-valued functions) defined as follows:

$$\mathbb{F}_0=\emptyset,$$

$$\mathbb{F}_{\alpha+1}=\{F:\mathrm{dom}\,F\rightarrow\mathrm{Card}^+ \mid \mathrm{dom}\,F\subseteq\mathbb{F}_\alpha\},$$

$$\mathbb{F}_\lambda=\bigcup_{\alpha<\lambda}\mathbb{F}_\alpha \quad \text{if } \lambda \text{ is a limit ordinal, and}$$

$$\mathbb{F}=\bigcup_{\alpha\in On}\mathbb{F}_\alpha.$$

The range of each function $F\in\mathbb{F}$ is some subset of Card^+ and the elements of $\mathrm{dom}\,F$ are other functions in $\mathbb{F}$ of lesser rank than F.

The proof that $\langle\mathrm{Card}^+,\mathbb{F}\rangle$ is a model of MSTC proceeds exactly as in Section III of [2]. We do not repeat it here. With this model, we interpret the atomic formula $x\in^{\alpha} y$ of L as "$y(x)=\alpha$" in the model (or, equivalently, "$\langle x,\alpha\rangle\in y$"). However, we want to interpret multisets of MSTC, not as certain positive cardinal-valued functions F of ZFC, but rather as the corresponding function shells $\mathscr{F}_F$ (the function shell $\mathscr{F}_F$ characterized by F that determines the multiset $\mathscr{M}_F$). Therefore, the atomic formula $x\in^{\alpha} y$

of L is interpreted as "x is the image of α domain elements under every function in $\mathscr{F}_y$, the function shell characterized by y". Under such an interpretation, the axioms, definitions and theorems of MSTC will define properties of function shells, and properties of functions that are the elements of function shells.

The interpretation of axiom I of MSTC is simply that the cardinality of inverse image sets is well-defined. If x has α preimages in $\mathscr{F}_y$ and x has β preimages in $\mathscr{F}_y$, then $\alpha=\beta$. The interpretation of axiom II states that "if, for every image z and every α, z is the image of α elements in $\mathscr{F}_x$ iff z is the image of α elements in $\mathscr{F}_y$, then $\mathscr{F}_x$ and $\mathscr{F}_y$ are the same function shell". This is exactly the equality of function shells $\mathscr{F}_F=\mathscr{F}_G$ iff $F=G$. (Equality of two classes of functions in terms of the equality of two functions.)

We note that if F and G in $\mathbb{F}$ are the interpretations of multisets x and y in MSTC, then $x=y$ iff $F=G$ [that is, $\operatorname{dom} F=\operatorname{dom} G$ and $\forall x\in\operatorname{dom} F$, $F(x)=G(x)$] iff $\mathscr{M}_F=\mathscr{M}_G$ (they contain exactly the same elements with exactly the same multiplicities) iff $\mathscr{F}_F=\mathscr{F}_G$ (equality of classes in ZFC $-f\in\mathscr{F}_F$ iff $f\in\mathscr{F}_G$).

The interpretation of axiom III of MSTC asserts the existence of the empty cardinal-valued function $F\in\mathbb{F}_1$ since $\operatorname{dom} F=\emptyset\subseteq\mathbb{F}_0$. In this case, $\mathscr{F}_F=\{\emptyset\}$ is a set containing the single empty function $\emptyset$ with empty range equal to $\operatorname{dom} F$.

Axioms IV(i) and IV(ii) assert the existence of Card^+-valued functions $F=\{\langle x,\alpha\rangle\}$ and $G=\{\langle x,\alpha\rangle,\langle y,\beta\rangle\}$, respectively, where $x\neq y$. The functions F and G determine the multisets $\mathscr{M}_F=[x]_\alpha$ and $\mathscr{M}_G=[x,y]_{\alpha,\beta}$. The function shell $\mathscr{F}_F$ contains all constant x-valued functions with domains of cardinality α (for example, $\alpha\to\{x\}$ is in $\mathscr{F}_F$). The function shell $\mathscr{F}_G$ contains all two-valued functions g (with range $\{x,y\}$) with $|\operatorname{dom} g|=\alpha+\beta$ such that $|g^{-1}(x)|=\alpha$ and $|g^{-1}(y)|=\beta$ (for example, $\hat{g}:\gamma\to\{x,y\}$, where γ is the ordinal sum of α and β, is in $\mathscr{F}_G$).

At this stage, it is clear that elementary properties of multisets in MSTC are modelled by certain properties of function shells in ZFC. Continuing this process for axioms V–XII is exactly the proof that the L-structure $\langle\mathrm{Card}^+,\mathbb{F}\rangle$ is a model of MSTC (that is, one proves that the interpretation of each axiom of MSTC holds in $\langle\mathrm{Card}^+,\mathbb{F}\rangle$). Instead, we proceed directly to the interpretation of interesting definitions and theorems of MSTC.

Let F and G in $\mathbb{F}$ be the interpretation of multisets x and y, respectively, in MSTC. The binary union $x\cup y$ is such that $z\in^\alpha x\cup y$ iff $(z\in^\alpha x\ \wedge\ z\notin y)\ \vee\ (z\in^\alpha y\ \wedge\ z\notin x)\ \vee\ (z\in^\beta x\ \wedge\ z\in^\gamma y\ \wedge\ \alpha=\max(\beta,\gamma))$ holds. Clearly, the function H in $\mathbb{F}$ that is the interpretation of the multiset $x\cup y$ is such that $\operatorname{dom} H=\operatorname{dom} F\cup\operatorname{dom} G$ (where $\cup$ here denotes binary union of sets in ZFC). If J is some function in $\operatorname{dom} H$ (the interpretation of some $z\in x\cup y$), then

$$H(J)=\begin{cases} F(J) & \text{if } J\in\operatorname{dom} F\ \wedge\ J\notin\operatorname{dom} G,\\ G(J) & \text{if } J\in\operatorname{dom} G\ \wedge\ J\notin\operatorname{dom} F,\\ \max(F(J),G(J)) & \text{if } J\in\operatorname{dom} F\ \wedge\ J\in\operatorname{dom} G.\end{cases}$$

We can identify $x\cup y$ with the multiset $\mathscr{M}_F\cup\mathscr{M}_G=\mathscr{M}_H$ (where $\cup$ here denotes binary union of multisets). What exactly is the relationship of the function shell $\mathscr{F}_H$ to the function shells $\mathscr{F}_F$ and $\mathscr{F}_G$? If $f \in \mathscr{F}_F$ and $g \in \mathscr{F}_G$ and $h \in \mathscr{F}_H$, then range$(h)=$ range$(f) \cup$ range(g) (where $\cup$ here is binary union of sets in ZFC). In words, the range of any function in $\mathscr{F}_H$ is the union of the range of any function in $\mathscr{F}_F$ and the range of any function in $\mathscr{F}_G$. Every function h: dom $h \to$ range(h) in $\mathscr{F}_H$ is such that, for every $z \in$ range(h),

$$|h^{-1}(z)| = \begin{cases} |f^{-1}(z)| & \text{if } z \in \text{range}(f) \ \wedge \ z \notin \text{range}(g), \\ |g^{-1}(z)| & \text{if } z \notin \text{range}(f) \ \wedge \ z \in \text{range}(g), \\ \max(|f^{-1}(z)|, |g^{-1}(z)|) & \text{if } z \in \text{range}(f) \cap \text{range}(g). \end{cases}$$

The cardinality $|\text{dom}\, h|$ of the domains of functions h in $\mathscr{F}_H$ is such that

$$|\text{dom}\, f|, |\text{dom}\, g| \leqslant |\text{dom}\, h| \leqslant |\text{dom}\, f| + |\text{dom}\, g|.$$

What does such a "union" of function shells mean for the functions that are elements of the shells? Let f and g be real-valued functions such that range$(f) \cap$ range$(g) = [0, 1] \subseteq \mathbb{R}$. Let M_f and M_g be the Dedekind multisets determined by f and g. Then $(M_f \cap M_g)^* = [0, 1]$. Let h be any real-valued function such that

(i) range$(h) = ($range$(f) \cup$ range$(g)) \subseteq \mathbb{R}$

and

(ii) for all images $z \in$ range(h),

$$|h^{-1}(z)| = \begin{cases} |f^{-1}(z)| & \text{if } z \in \text{range}(f) - \text{range}(g), \\ |g^{-1}(z)| & \text{if } z \in \text{range}(g) - \text{range}(f), \\ \max(|f^{-1}(z)|, |g^{-1}(z)|) & \text{if } z \in [0, 1]. \end{cases}$$

For any real-valued function h satisfying (i) and (ii), the Dedekind multiset $M_h = M_f \cup M_g$. Such a "union" of real-valued functions is independent of their domains, but depends only on their ranges and the cardinalities of their inverse image sets. We note that joining f and g together in this way does not necessarily alter the *values* of f and g, but only the *numbers* of preimages.

Let us consider an even simpler case. Let f and g be *specific* two-valued real functions (with ranges $\{x, y\} \subseteq \mathbb{R}$ and $\{y, z\} \subseteq \mathbb{R}$, respectively). Let h with range $\{x, y, z\} \subseteq \mathbb{R}$ be a "union" of f and g in the above sense. A conceptual representation of this situation is shown in Fig. 3. Given the funcions f and g, how does one construct such a real-valued function h? For simplicity, let us assume that dom $f \cap$ dom $g = \emptyset$ and $|f^{-1}(y)| \geqslant |g^{-1}(y)|$. Then define the function h as follows:

$$\text{range}(h) = \text{range}(f) \cup \text{range}(g) = \{x, y, z\},$$

$$\text{dom}\, h = \underbrace{f^{-1}(x) \cup f^{-1}(y)}_{\text{dom}\, f} \cup g^{-1}(z)$$

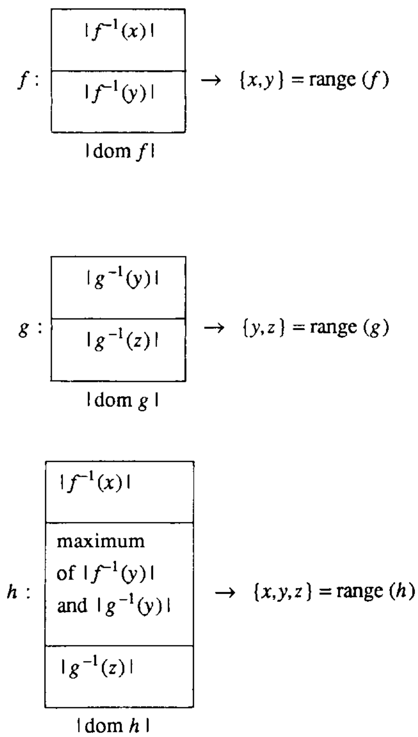


Fig. 3.

and, for all $z' \in \text{dom}\, h$, define

$$h(z') = \begin{cases} f(z') & \text{if } z' \in \text{dom}\, f, \\ g(z') & \text{if } z' \in g^{-1}(z). \end{cases}$$

In such a construction, we take the largest domain (the largest inverse image set for each common image).

For the "additive union" h of f and g with $\text{dom}\, f \cap \text{dom}\, g = \emptyset$, one would simply define

$$\text{range}(h) = \text{range}(f) \cup \text{range}(g),$$

$$\text{dom}\, h = \text{dom}\, f \cup \text{dom}\, g$$

and, for all $z' \in \text{dom}\, h$, define

$$h(z') = \begin{cases} f(z') & \text{if } z' \in \text{dom}\, f, \\ g(z') & \text{if } z' \in \text{dom}\, g. \end{cases}$$

So, we have

$$h^{-1}(y) = f^{-1}(y) \cup g^{-1}(y)$$

and

$$|h^{-1}(y)|=|f^{-1}(y)|+|g^{-1}(y)|.$$

Therefore, the multiplicity of y in M_h is the *sum* of its multiplicities in M_f and M_g.

It is important to emphasize both the size and the complexity of these interpretations of multisets as function shells. Not only is each multiset x of MSTC interpreted as a Card$^+$-valued function F of ZFC (which, in turn, represents a large class $\mathscr{F}_F$ of ZFC functions), but every element of x is interpreted as a Card$^+$-valued function (an element of dom F) which also represents a large class of ZFC functions. Each $\mathscr{F}_F$ is a large class of functions such that each element of dom F is also a large class of functions. The images themselves represent function shells. In other words, a multiset represents a large class of functions with a common range, and each image point in the common range also represents a large class of functions.

How is the msubset relation $\subseteq$ interpreted for function shells? Equivalently, how is the relationship $\mathscr{M}_F \subseteq \mathscr{M}_G$ interpreted for $\mathscr{F}_F$ and $\mathscr{F}_G$? Clearly, $\mathscr{M}_F \subseteq \mathscr{M}_G$ means that dom $F \subseteq$ dom G and for all $x \in$ dom F, $F(x) \leqslant G(x)$. Let f and g be arbitrary but fixed functions in $\mathscr{F}_F$ and $\mathscr{F}_G$, respectively. Hence, range$(f) \subseteq$ range(g) and for all images $x \in$ range(f), $|f^{-1}(x)| \leqslant |g^{-1}(x)|$. In words, the common range of functions in $\mathscr{F}_F$ is a subset of the common range of functions in $\mathscr{F}_G$ and, further, for any image common to both, the number of domain elements with that image in $\mathscr{F}_F$ is less than or equal to the number of domain elements with that image in $\mathscr{F}_G$.

Let us consider a simple example where $f: \text{dom} f \to \{x, y\}$ is in $\mathscr{F}_F$ and $g: \text{dom}\, g \to \{x, y, z\}$ is in $\mathscr{F}_G$. A conceptual representation of this situation is shown in Fig. 4. Then $\mathscr{M}_F \subseteq \mathscr{M}_G$ (or, equivalently, $M_f \subseteq M_g$) if $|f^{-1}(x)| \leqslant |g^{-1}(x)|$ and $|f^{-1}(y)| \leqslant |g^{-1}(y)|$. The cardinality $|g^{-1}(z)| = \gamma'$ is irrelevant since z is not a common image. In multiset symbols, $[x, y]_{\alpha,\beta} \subseteq [x, y, z]_{\alpha',\beta',\gamma'}$ iff $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$.

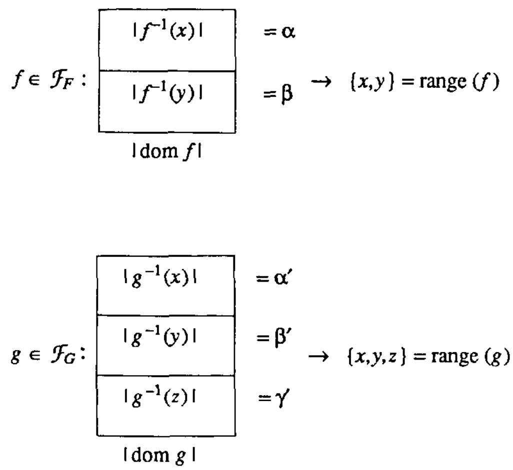


Fig. 4.

In general, the "union" of $\mathscr{F}_F$ and $\mathscr{F}_G$ is a "new" function shell $\mathscr{F}_H$. If, however, $\mathscr{F}_F$ is a "subset" of $\mathscr{F}_G$, then their "union" is $\mathscr{F}_G$.

Set $(\mathscr{M}_F)$ means that either $F=\emptyset$ (that is, $\operatorname{dom} F=\emptyset$) or $\forall x \in \operatorname{dom} F$, $F(x)=1$. In other words, every inverse image set is a singleton (every f in $\mathscr{F}_F$ is an injection).

If $\mathscr{F}_F$ is a "subset" of $\mathscr{F}_G$ and $\mathscr{F}_G$ is a "subset" of $\mathscr{F}_F$, then $\mathscr{F}_F$ and $\mathscr{F}_G$ are the same function shell. The "root" of $\mathscr{F}_F$ is the function shell of all injections with common range equal to $\operatorname{dom} F$. The "root" of $\mathscr{F}_F$ determines the root set $\mathscr{M}_F^*$.

What is the "powerset" of $\mathscr{F}_F$? Since $\forall x$ $\operatorname{Set}(\mathbb{P}(x))$ in MSTC, the "powerset" function $\mathbb{P}F$ is a $\{1\}$-valued function with

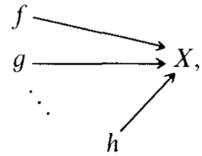
$$\operatorname{dom} \mathbb{P}F=\{G \mid G \subseteq F\}$$

$$=\{G \mid \operatorname{dom} G \subseteq \operatorname{dom} F \ \wedge \ \forall x \in \operatorname{dom} G \ G(x) \leqslant F(x)\}.$$

Therefore, the "powerset" of $\mathscr{F}_F$ is the class of all injections with range equal to $\operatorname{dom} \mathbb{P}F$.

The "intersection" of the two function shells $\mathscr{F}_F$ and $\mathscr{F}_G$ is the class of all functions with range $\operatorname{dom} F \cap \operatorname{dom} G$ and such that, for every $z \in \operatorname{dom} F \cap \operatorname{dom} G$, z has exactly $\min(F(z), G(z))$ preimages. We call this class of functions $\mathscr{F}_H$. Therefore, $h \in \mathscr{F}_H$ implies $\operatorname{range}(h)=\operatorname{dom} F \cap \operatorname{dom} G$ and $\forall z \in \operatorname{range}(h)$, $|h^{-1}(z)|=\min(F(z), G(z))$.

The copy ZFC′ of ZFC in MSTC is interpreted as a class hierarchy of function shells, each of which is a class of all injections with a common range such that each common image (element of the common range) is itslef a class of all injections with a common range. Therefore, ZFC can be thought of as a formal theory of function shells of *injections* of the form



where every element of X is also a function shell of injections of the same form.

The theorem of MSTC that states

$$\forall x \, \forall y (x \cap y=\emptyset \ \rightarrow \ x \uplus y=x \cup y)$$

becomes the fact that if $\mathscr{M}_F \cap \mathscr{M}_G=\emptyset$ (that is, $\operatorname{dom} F \cap \operatorname{dom} G=\emptyset$), then the "union" and the "additive union" of $\mathscr{F}_F$ and $\mathscr{F}_G$ are the same function shell. This is true anyway for infinite inverse image sets because $\lambda+\kappa=\max(\lambda, \kappa)$ for $\lambda, \kappa \geqslant \aleph_0$.

Certain operations on functions are only defined for compatible functions; for example, $f \circ g$ requires that $\operatorname{range}(g) \subseteq \operatorname{dom} f$. But combining f and g (that is, combining the function shells $\mathscr{F}_F$ containing f and $\mathscr{F}_G$ containing g) in $\langle \text{Card}^+, \mathbb{F} \rangle$ requires *no* compatibility conditions on f and g at all.

It is important to note that it is the "injective" function shells $\mathscr{F}_F$ that give rise to multisets $\mathscr{M}_F$ that are "sets". Injections are the "sets" of functions; that is, the function

f determines the *multiset* M_f, whereas an injection f determines the *set* $M_f = M_f^*$. The degree of "many–one character" of f (its distance from "injectiveness" measured by comparing $|\mathrm{dom}\, f|$ to $|\mathrm{range}(f)|$) gives a measure of the degree of multiplicity in the multiset M_f (its distance from "setness"). The greater the "compression" of f (the ratio $|\mathrm{dom}\, f|/|\mathrm{range}(f)| \geqslant 1$), the greater the degree of multiplicity of elements in M_f.

Every function f can be decomposed into an injection g and a *maximal many–one* function h (that is, $|h^{-1}(z)| = |f^{-1}(z)| > 1$ for all $z \in \mathrm{range}(h) \subseteq \mathrm{range}(f)$) such that $g \subseteq f$, $h \subseteq f$, $\mathrm{dom}\, g \cap \mathrm{dom}\, h = \emptyset$, $\mathrm{range}(g) \cap \mathrm{range}(h) = \emptyset$ and $f = g \cup h$. For the corresponding Dedekind multisets, the multiset M_f decomposes into the subset $M_g = M_g^*$ and the whole msubset M_h (M is a *whole* msubset of N if $\forall x\, \forall \alpha (x \in^\alpha M \rightarrow x \in^\alpha N)$ holds) such that $M_g \cap M_h = \emptyset$ and $M_f = M_g \cup M_h = M_g \uplus M_h$. If $\mathscr{M}_F = M_f$, $\mathscr{M}_G = M_g$ and $\mathscr{M}_H = M_h$, then $f \in \mathscr{F}_F$, $g \in \mathscr{F}_G$ and $h \in \mathscr{F}_H$. If this is the case, then the function shell $\mathscr{F}_F$ can be decomposed into the "subset" function shell $\mathscr{F}_G$ (of all injections with range dom $G = M_g^*$) and the "disjoint subset" function shell $\mathscr{F}_H$ (of all maximal many–one functions with range dom $H = M_h^*$) and the "union" (or, the "additive union") of $\mathscr{F}_G$ and $\mathscr{F}_H$ equals the function shell $\mathscr{F}_F$.

Section 6 may be summarized as follows: properties of multisets in MSTC become certain properties of cardinal-valued functions in the model $\langle \mathrm{Card}^+, \mathbb{F} \rangle$ of MSTC. These, in turn, determine the properties of the corresponding function shells. When the consequences of these properties are applied to the elements of function shells, a new algebra of functions is defined.

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