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Mathematical Games

The memory game*

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Abstract

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The *memory game*, or *concentration*, as it is sometimes called, is a popular card game played by children and adults around the world. Good memory is one of the qualities required in order to succeed in it. This, however, is not enough. When it is assumed that the players have perfect memory, the memory game can be seen as a game of strategy. The game is analysed under this assumption and the optimal strategy is found. It is simple and perhaps unexpected.

In contrast to the simplicity of the optimal strategy, the analysis leading to its optimality proof is rather involved. It supplies an interesting example of concrete mathematics of the sort used in the analysis of algorithms. It is doubtful whether this analysis could have been carried out without resort to experimentation and a substantial use of automated symbolic computations.

1. The game

A pack containing n pairs of identical cards is shuffled and the cards are spread face down on a table. Each player in turn flips two cards, one after the other. If the two

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cards flipped are identical (i.e., they form a pair), they are removed from the table into the possession of the player who flipped them and he/she gets another turn. If the two cards are not identical then they are flipped back and the turn passes to the next player. The game continues until all the cards are removed from the table (or until all the players agree to end the game) and the winner is the player possessing the largest number of pairs. The gain (or loss, if negative) of a player at any stage is defined to be the number of pairs he/she holds minus the average number of pairs held by the opponents.

Any number of players can play the game but the most interesting situation occurs when there are only two of them. We will, therefore, consider this case here.

The invention of the memory game is sometimes attributed to Christopher Louis Pelman and the game is often called *Pelmanism* (refer to this entry in [4]).

A light-hearted report on some of the results obtained in this paper has recently appeared in [5].

2. Moves, positions and strategies

Each player tries to remember the position and the identity of all the cards already inspected. To focus our attention on the strategic questions involved, we will assume that the players have already reached a high level of proficiency and are able to absorb all this information (in other words, they have perfect memories).

A turn in the game is composed of two plies. The observation that triggered this work is that at each ply the player can either inspect a new card (in which case we assume that the outcome is uniformly distributed over all the as-yet-unflipped cards), or an old card whose identity is already known to both players. Inspecting an old card in the first ply, or a nonmatching old card in the second ply, are in a sense idle plies. Idle plies are not always possible. In the beginning of the game, for example, the first player has to flip two new cards.

There are at most three reasonable moves from each position.[1] The first is to pick no new cards at all. Such a move will be called a *0-move* and it is possible only if there are at least two inspected cards on the table. The two other moves, termed *1-move* and *2-move*, both begin by flipping a new card. If the new card matches a previously inspected card then in both cases the matching card is flipped, a pair is formed and the player gets another turn. If, however, the first card flipped does not match a previously inspected card then an idle ply is used in a 1-move while a new card is inspected in a 2-move. It can easily be seen that making an idle ply first and then flipping a new card is always inferior to the other moves.

While playing the game the players can have two different objectives. They could try to maximise the probability of winning the game or, alternatively, they could try to maximise their expected gain. The two objectives lead to somewhat different optimal

[1] See, however, the note at the end of the paper.

strategies. We will investigate here the strategy that maximises the expected gain. The optimal strategy for the other case could presumably be obtained using similar methods and a more involved analysis.

If a 0-move maximises the expected gain for the next player then, after this 0-move is played, the situation remains exactly the same and the second player would also like to play a 0-move. Since this can go on forever, we stop the game in such a case.

A position in the game is characterised by the number n of pairs still on the table and the number k of cards on the table which have already been inspected. We can assume that all the inspected cards are different. In the case where the last player played a 2-move and the second card flipped matches not the first card flipped but one of the previously inspected cards, the resulting pair would be immediately removed by the other player, and we may account for this as part of the present turn.

A *strategy* is a rule which determines which one of the three plausible moves should be used in each position (n, k), where $0 \leqslant k \leqslant n$ are integers. An *optimal strategy* is a strategy which maximises the expected gain assuming that both players play optimally. The *value* of a position is the expected future gain of the player who is first to play from that position assuming that both players use an optimal strategy. We shall see in the next section that the position values and an optimal strategy can be defined mutually recursively. It is easy to see that if a player is playing according to an optimal strategy then the expected gain from some position is at least the value of that position, no matter what strategy the opponent may choose.

3. The optimal strategy

We define recursively the values $e_{n,k}$ of the different positions. The only initial condition that we need is that $e_{0,0}=0$, that is, that no one gains from a null game. Assume that we have already defined $e_{n',k'}$ for $n'<n$ and $e_{n,k'}$ for $k'>k$. We will first define $e^1_{n,k}$ and $e^2_{n,k}$ which will be the expected gain from position (n, k) when beginning with a 1- or a 2-move respectively, and subsequently playing using an optimal strategy. Referring to Fig. 1, it is relatively easy to verify that

$$e^1_{n,k}=\frac{k}{2n-k}(1+e_{n-1,k-1})-\frac{2(n-k)}{2n-k}e_{n,k+1},$$

$$e^2_{n,k}=\frac{k}{2n-k}(1+e_{n-1,k-1})$$

$$-\frac{2(n-k)}{2n-k}\left(\frac{k-1}{2n-k-1}(1+e_{n-1,k})+\frac{2(n-k-1)}{2n-k-1}e_{n,k+2}\right).$$

We will explain the first relation as an example. When flipping the first card, there are k inspected cards on the table, all of them different, and $2n-k$ uninspected cards. In a 1-move an uninspected card is flipped. With probability $k/(2n-k)$, it will be a card

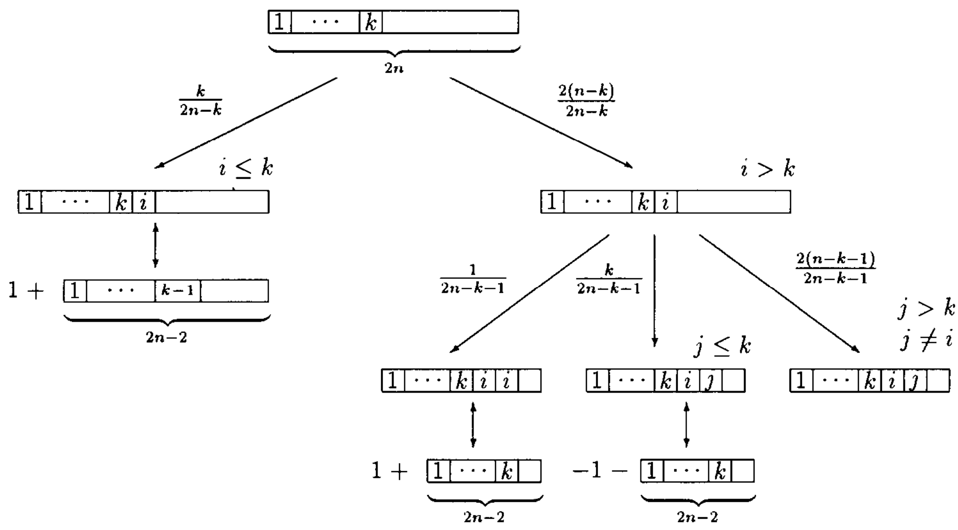

Fig. 1. Possible outcomes of moves from position (n, k).

which matches one of the previously inspected cards, in which case the player will gain a pair and will be entitled to play again from position $(n-1, k-1)$. With the complementary probability $2(n-k)/(2n-k)$, the first card flipped will not match any previously inspected card, an idle ply will follow and the opponent will play from position $(n, k+1)$. Since the gain of one player is the other's loss, the expected gain of a player from a position $(n, k+1)$ when the opponent is about to play is $-e_{n,k+1}$. This accounts for the two terms appearing in the first relation. The second relation is obtained in a similar way. (A reference to Fig. 1 may again be useful).

The value $e_{n,k}$ of the position (n, k) with $n>0$ is now defined as

$$e_{n,0} = e^2_{n,0},$$
$$e_{n,1} = \max\{e^1_{n,1}, e^2_{n,1}\},$$
$$e_{n,k} = \max\{0, e^1_{n,k}, e^2_{n,k}\} \quad \text{for } 2 \leqslant k \leqslant n.$$

These definitions are explained by the following observations. A 2-move is the only legal move from position $(n, 0)$. A 1-move and a 2-move are the only two moves allowed from position $(n, 1)$. In positions of the form (n, k) where $k \geqslant 2$, a 0-move could be used. If $e^1_{n,k}, e^2_{n,k} < 0$ then it is advantageous to use a 0-move and the game will stop with value 0.

We say that an i-move is *optimal* from position (n, k) if $e_{n,k} = e^i_{n,k}$ (where $e^0_{n,k} = 0$). It is possible that more than one move will be optimal from a certain position.

Using these recursive definitions, we can compute the values and find the optimal moves. Table 1 gives the values of positions with $n \leqslant 7$, while Table 2 gives the optimal moves for $n \leqslant 15$. For $(n, k) = (4, 3)$, it turns out that both the 0-move and the 2-move are optimal but only the 2-move is listed in the table. Similarly, for any n,

Table 1
The expected values of the simplest positions

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=0$	0							
$n=1$	1	1						
$n=2$	$-\frac{2}{3}$	$\frac{2}{3}$	2					
$n=3$	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{3}$	3				
$n=4$	$-\frac{4}{35}$	$\frac{4}{35}$	$\frac{4}{15}$	0	4			
$n=5$	$-\frac{1}{35}$	$-\frac{1}{35}$	$\frac{1}{7}$	$\frac{19}{35}$	0	5		
$n=6$	$\frac{2}{1155}$	$\frac{2}{1155}$	$\frac{2}{21}$	$\frac{13}{105}$	$\frac{27}{35}$	0	6	
$n=7$	$\frac{61}{1155}$	$\frac{61}{1155}$	$\frac{13}{495}$	$\frac{53}{231}$	$\frac{2}{21}$	$\frac{62}{63}$	0	7

Table 2
The optimal moves for $n \leqslant 15$

$n=1$	2 1
$n=2$	2 2 1
$n=3$	2 1 2 1
$n=4$	2 2 1 2 1
$n=5$	2 1 2 1 0 1
$n=6$	2 **1** 1 2 1 0 1
$n=7$	2 1 2 1 2 1 0 1
$n=8$	2 2 1 2 1 2 1 0 1
$n=9$	2 1 2 1 2 1 2 1 0 1
$n=10$	2 2 1 2 1 2 1 2 1 0 1
$n=11$	2 1 2 1 2 1 2 1 0 1 0 1
$n=12$	2 2 1 2 1 2 1 2 1 0 1 0 1
$n=13$	2 1 2 1 2 1 2 1 2 1 0 1 0 1
$n=14$	2 2 1 2 1 2 1 2 1 2 1 0 1 0 1
$n=15$	2 1 2 1 2 1 2 1 2 1 2 1 0 1 0 1

$e^1_{n,n} = e^2_{n,n} = n$; so, both the 1-move and the 2-move are optimal in this case. In fact, they are identical in this case since the first card flipped will always match a previously inspected card.

The pattern emerging from Table 2 is clear. A 2-move should be used when $k=0$, since this is the only allowed move. A 1-move should be used whenever $k>0$ and $n+k$ is even. Either a 2-move or a 0-move should be used when $n+k$ is odd ($(n, k)=(6, 1)$ being the only exception). Inspecting a few more rows in the table immediately suggests that a 0-move should be used when, in addition to the requirement that $n+k$ is odd, we also have $k \geqslant 2(n+1)/3$.

We, thus, claim the following.

Theorem 3.1.

$$e_{n,k}=\begin{cases} 0 & \text{if } \left[k\geqslant\dfrac{2(n+1)}{3} \text{ and } n+k \text{ odd}\right],\\ e^1_{n,k} & \text{if } [k\geqslant 1 \text{ and } n+k \text{ even}] \text{ or } [(n,k)=(6,1)],\\ e^2_{n,k} & \text{otherwise.}\end{cases}$$

Another interesting issue is the behaviour of the values $e_{n,k}$ themselves. The following approximation gives their asymptotical behaviour.

Theorem 3.2.

$$e_{n,k}=\begin{cases} \dfrac{k}{2(n-k)+1}+O(n^{-1}) & \text{if } n+k \text{ even},\\ \dfrac{(2n-3k)(2n-k)}{16(n-k)^3}+O(n^{-2}) & \text{if } n+k \text{ odd and } k\leqslant\dfrac{2n+1}{3},\\ 0 & \text{if } n+k \text{ odd and } k\geqslant\dfrac{2(n+1)}{3}.\end{cases}$$

If we let $\lambda=k/n$ then we see that, for $\lambda<1$, $e_{n,k}=e^1_{n,k}\sim\lambda/2(1-\lambda)$ if $n+k$ is even, and $e_{n,k}=e^2_{n,k}\sim[(2-3\lambda)(2-\lambda)/16(1-\lambda)^3]\cdot 1/n$ if $n+k$ is odd and $k\leqslant(2n+1)/3$. Similarly, we can get that $e^2_{n,k}\sim-[\lambda/2(1-\lambda)]\cdot(4-12\lambda+7\lambda^2)/(2-\lambda)^2$ if $n+k$ is even and that $e^1_{n,k}=-[(4-8\lambda+5\lambda^2)/16(1-\lambda)^3]\cdot 1/n$ if $n+k$ is odd and $k\leqslant(2n+1)/3$.

A graph of $e_{n,k}=e^1_{n,k}$ and $e^2_{n,k}$ for even values of $n+k$ is given in Fig. 2. It can be seen that, unless λ is very small, $e^1_{n,k}$ is both positive and markedly superior to $e^2_{n,k}$. Similarly, a graph contrasting $e_{n,k}=e^2_{n,k}$ with $e^1_{n,k}$ for odd values of $n+k$ is presented in Fig. 3. Again, there is a sharp difference between these two options.

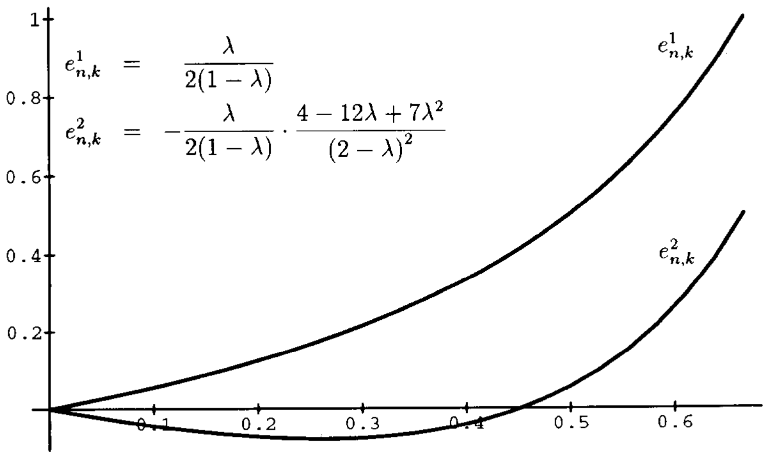

Fig. 2. The behaviour of $e_{n,k}=e^1_{n,k}$ and $e^2_{n,k}$ for $n+k$ even.

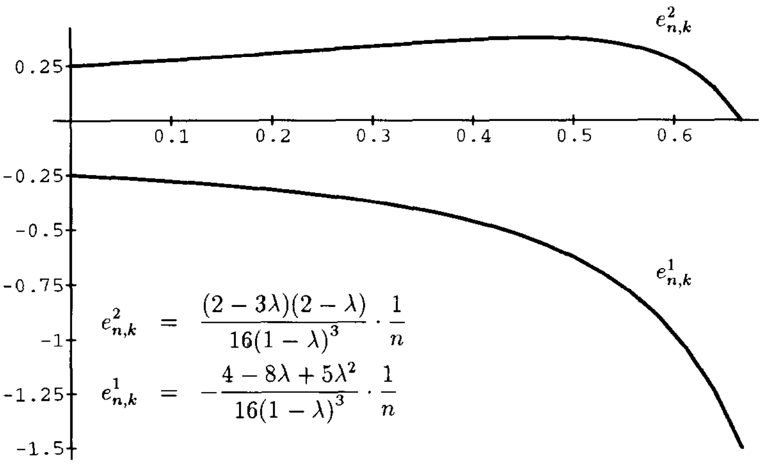

Fig. 3. The behaviour of $e_{n,k}=e^2_{n,k}$ and $e^1_{n,k}$ for $n+k$ odd.

We need better approximations, however, to show that $e_{n,k}=e^1_{n,k}$ when $n+k$ is even and $k=\mathrm{o}(n)$, and that $e_{n,k}=e^2_{n,k}$ when $n+k$ is odd and $k=(2n/3)-\mathrm{o}(n)$. These are obtained in the next section.

Another interesting question is the sign of the values of the different positions. By definition $e_{n,k}\geqslant 0$ whenever $k\geqslant 2$, but what happens when $k=1$ or $k=0$? In particular, when is it advantageous to take the first turn? It turns out that $e_{n,1}>0$ for $n\geqslant 6$, that $e_{n,0}>0$ when $n\geqslant 7$ and n is odd, and that $e_{n,0}<0$ when $n\geqslant 8$ and n is even. Thus, it is advantageous to take the first move in the game if and only if n is either 1, 6 or an odd number greater or equal to 7.

Finally, what is the expected gain or loss from a game played with n pairs of cards? It turns out that, for large n, the gain or loss is roughly $1/4n$. More precisely, we have the following result.

Theorem 3.3.

$$e_{n,0}=\begin{cases} \dfrac{1}{4n+2}+\mathrm{O}\left(\dfrac{1}{n^3}\right) & \text{if } n \text{ odd,} \\[2ex] -\dfrac{1}{4n-2}+\mathrm{O}\left(\dfrac{1}{n^3}\right) & \text{if } n \text{ even.} \end{cases}$$

The proofs of the theorems stated in this section are given in the next section.

4. Analysis

Our strategy for proving the results claimed in the previous section is the following. We first investigate the expected gains from each position when the two players play according to the alleged optimal strategy. Once we have tight estimations of these values, it will be easy to prove, by induction, that these values do in fact correspond to the optimal strategy.

4.1. Preliminary manipulations

Let $e_{n,k}$ be the expected values of the different positions when both players play according to the conjectured optimal strategy. As a "warm-up", we prove the following lemma.

Lemma 4.1. (i) $e_{n,0}=e_{e,1}$ *for odd* $n\geqslant 1$ *and* (ii) $e_{n,0}=-e_{n,1}$ *for even* $n\neq 6$.

Proof. For odd n, we have $e_{n,0}=e^2_{n,0}$ and $e_{n,1}=e^1_{n,1}$. Using the definitions of $e^2_{n,k}$ and $e^1_{n,k}$ from Section 3, we see that both $e^2_{n,0}$ and $e^1_{n,1}$ expand to the same expression

$$e^2_{n,0}=e^1_{n,1}=\frac{1}{2n-1}(1+e_{n-1,0})-\frac{2(n-1)}{2n-1}e_{n,2}.$$

This proves the first part of the lemma. For even $n\neq 6$, we have $e_{n,0}=e^2_{n,0}$ and $e_{n,1}=e^2_{n,1}$ and, therefore,

$$\begin{aligned} e_{n,0}+e_{n,1} &= \left[\frac{1}{2n-1}(1+e_{n-1,0})-\frac{2(n-1)}{2n-1}e_{n,2}\right] \\ &\quad+\left[\frac{1}{2n-1}(1+e_{n-1,0})-\frac{2(n-1)}{2n-1}\cdot\frac{2(n-2)}{2n-2}e_{n,2}\right] \\ &= \frac{2}{2n-1}(1+e_{n-1,0})-\frac{2(n-1)}{2n-1}e_{n,2}-\frac{2(n-2)}{2n-1}e_{n,3}. \end{aligned}$$

For even $n\geqslant 2$, we have $e_{n,2}=e^1_{n,2}$ and, thus,

$$\begin{aligned} e_{n,0}+e_{n,1} &= \frac{2}{2n-1}(1+e_{n-1,0})-\frac{2(n-1)}{2n-1}\left[\frac{2}{2n-2}(1+e_{n-1,1})\right. \\ &\qquad\qquad \left.-\frac{2(n-2)}{2n-2}e_{n,3}\right]-\frac{2(n-2)}{2n-1}e_{n,3} \\ &= \frac{2}{2n-1}[e_{n-1,0}-e_{n-1,1}]=0, \end{aligned}$$

where the last equality follows from the first part of the lemma. □

As an easy corollary, we get the following result.

Lemma 4.2. $e^1_{n,0}=e^2_{n,0}$ *for even* $n\neq 6$.

Proof. By definition we have $e^2_{n,0}=e_{n,0}$ and $e^1_{n,0}=-e_{n,1}$ and, thus, the result follows from the second part of the previous lemma. □

Note that 1-moves are currently not allowed from positions of the form $(n, 0)$. The previous lemma says, however, that it would not matter if we were to allow them from these positions with even $n\neq 6$. Furthermore, the 1-moves would be co-optimal in these positions and we could, therefore, use the relation $e_{n,0}=e^1_{n,0}$ as the defining relation for even $n\neq 6$. This removes the anomaly of the column $k=0$ seen in Table 1. The two remaining exceptions are $e_{6,0}=e^2_{6,0}$ and $e_{6,1}=e^1_{6,1}$.

Since the parity of $n+k$ plays a major role in the following analysis, it will be convenient to denote $e_{n,k}$ by $a_{n,k}$ when $n+k$ is even, and by $b_{n,k}$ when $n+k$ is odd. It is also convenient to write the recurrence relations defining $a_{n,k}$ and $b_{n,k}$ with the help of an auxiliary sequence $c_{n,k}$ as follows:

$$a_{n,k}=p_{n,k}(1+a_{n-1,k-1})-q_{n,k}b_{n,k+1},$$

$$b_{n,k}=[p_{n,k}(1+b_{n-1,k-1})-q_{n,k}c_{n,k+1}]\times I_{n,k},$$

$$c_{n,k}=p'_{n,k}(1+a_{n-1,k-1})+q_{n,k}b_{n,k+1},$$

where

$$p_{n,k}=\frac{k}{2n-k},\quad p'_{n,k}=\frac{k-2}{2n-k},\quad q_{n,k}=\frac{2(n-k)}{2n-k}$$

and

$$I_{n,k}=\begin{cases}1 & \text{if } k\leqslant\dfrac{2n+1}{3},\\ 0 & \text{otherwise.}\end{cases}$$

These relations hold for any (n, k) with the exception of $(6, 0)$ and $(6, 1)$. The only initial condition required is that $a_{0,0}=0$.

Note that $c_{n,k+1}$ corresponds to the expected loss from position (n, k) if one new card had already been flipped and did not match any of the previously inspected cards.

4.2. Operator notation

The following analysis is facilitated by introducing operator notation. Define an operator Φ by

$$\Phi\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}a'\\ b'\\ c'\end{pmatrix},$$

where

$$a'_{n,k} = p_{n,k} a_{n-1,k-1} - q_{n,k} b_{n,k+1},$$
$$b'_{n,k} = p_{n,k} b_{n-1,k-1} - q_{n,k} c_{n,k+1},$$
$$c'_{n,k} = p'_{n,k} a_{n-1,k-1} + q_{n,k} b_{n,k+1},$$

and an operator Z by

$$Z\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b' \\ c \end{pmatrix},$$

where

$$b'_{n,k} = \begin{cases} b_{n,k} & \text{if } k \leqslant \frac{2n+1}{3}, \\ 0 & \text{otherwise.} \end{cases}$$

We again assume that Φ has the anomalous behaviour

$$a'_{6,0} = b'_{6,1} = \frac{1}{11}(1 + b_{5,0}) - \frac{10}{11} a_{6,2}.$$

If we let $e = (a, b, c)^{\mathrm{T}}$ and $h = (p, p, p')^{\mathrm{T}}$ then it is easy to see that e satisfies the following equation:

$$e = Z(\Phi e + h). \tag{1}$$

Our task is to solve this operator equation.

4.3. Bootstrapping

We start by trying to solve the equation obtained by ignoring the presence of the operator Z in Eq. (1), i.e.,

$$e = \Phi e + h. \tag{2}$$

The solution of this equation will not only give us some useful information about the solution of Eq. (1), it also has some interest in its own right. It corresponds to the analysis of the variant of the game in which 1-moves and 2-moves are the only moves allowed.

Solving Eq. (2) amounts to inverting the operator $(I - \Phi)$, which does not seem to be an easy task. We approach this by approximating Φ by an operator $\hat{\Phi}$ for which inverting $(I - \hat{\Phi})$ is much easier. Using a method that bears some resemblance to the "bootstrapping" method described in [2, 3], we define a sequence of refining terms $E^0, E^1, \ldots$, whose sum $E^0 + E^1 + \cdots$ converges, we hope, to the required solution. This sequence is obtained in the following way:

$$E^i = (I - \hat{\Phi})^{-1} h^i, \quad i \geqslant 0,$$
$$h^{i+1} = (\Phi - \hat{\Phi}) E^i, \quad i \geqslant 0,$$

where $h^0 = h$. Let $e^0 = e$, and $e^i = e^{i-1} - E^{i-1}$ for $i \geqslant 1$ be the error of the ith approximation. It is easy to verify that

$$e^i = \Phi e^i + h^i, \quad i \geqslant 0.$$

We define $\hat{\Phi}$ as follows:

$$\hat{\Phi}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix},$$

where

$$a'_{n,k} = p_{n,k} a_{n,k} - q_{n,k} b_{n,k},$$

$$b'_{n,k} = p_{n,k} b_{n,k} - q_{n,k} c_{n,k},$$

$$c'_{n,k} = p_{n,k} a_{n,k} + q_{n,k} b_{n,k},$$

or, equivalently,

$$\hat{\Phi} = \begin{pmatrix} p & -q & 0 \\ 0 & p & -q \\ p & q & 0 \end{pmatrix}.$$

Thus,

$$(I - \hat{\Phi}) = \begin{pmatrix} 1-p & q & 0 \\ 0 & 1-p & q \\ -p & -q & 1 \end{pmatrix}, \qquad (I - \hat{\Phi})^{-1} = \frac{1}{2q^2}\begin{pmatrix} 1+q & -1 & q \\ -p & 1 & -q \\ p & q-p & q \end{pmatrix}$$

and

$$(\Phi - \hat{\Phi})\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix},$$

where

$$a'_{n,k} = p_{n,k}(a_{n-1,k-1} - a_{n,k}) - q_{n,k}(b_{n,k+1} - b_{n,k}),$$

$$b'_{n,k} = p_{n,k}(b_{n-1,k-1} - b_{n,k}) - q_{n,k}(c_{n,k+1} - c_{n,k}),$$

$$c'_{n,k} = (p'_{n,k} a_{n-1,k-1} - p_{n,k} a_{n,k}) + q_{n,k}(b_{n,k+1} - b_{n,k}).$$

The terms E^i obtained in this way become horrendously complicated even for very small values of i and it seems almost impossible to handle them manually. We used *Mathematica* to do these computations.

We now note that, for $k < \lambda n$, for some $\lambda < 1$, we have $h^0 = O(1)$ and, thus, it can easily be seen that $E^0 = O(1)$. The operator $\Phi - \hat{\Phi}$ has the characteristics of a discrete difference operator. Since each component of $E^0=(A^0, B^0, C^0)^{\mathrm{T}}$ is a rational function in n and k and, thus, continuous, in the sense that $A^0_{n-1,k-1}-A^0_{n,k}=\mathrm{O}(n^{-1})$ and so on, it is easy to see that $h^1=\mathrm{O}(n^{-1})$. By induction, we can prove in this way that as long as $\lambda=k/n$ is bounded away from 1, we have $E^i=\mathrm{O}(n^{-i})$. Therefore, each additional term E^i that we compute allows us to obtain an additional term in the asymptotic expansions of a, b and c. These computations can again be done using *Mathematica* and the expansions obtained are

$$a_{n,k}=\frac{\lambda}{2(1-\lambda)}+\frac{\lambda^2+4\lambda-4}{16(1-\lambda)^3}\cdot\frac{1}{n}+\frac{2\lambda^3-4\lambda^2+5\lambda-2}{16(1-\lambda)^5}\cdot\frac{1}{n^2}$$
$$+\frac{13\lambda^4-62\lambda^3+112\lambda^2-64\lambda+8}{64(1-\lambda)^7}\cdot\frac{1}{n^3}+\cdots,$$

$$b_{n,k}=\frac{(2-3\lambda)(2-\lambda)}{16(1-\lambda)^3}\cdot\frac{1}{n}+\frac{4\lambda^3-14\lambda^2+11\lambda-2}{16(1-\lambda)^5}\cdot\frac{1}{n^2}$$
$$+\frac{9\lambda^4-12\lambda^3-60\lambda^2+80\lambda-24}{64(1-\lambda)^7}\cdot\frac{1}{n^3}+\cdots$$

The expansion of $c_{n,k}$ is easily obtained from these two.

We claim that, by truncating these expansions after the $\mathrm{O}(n^{-i})$ terms, we get an approximation to the solution $e=(a,b,c)^{\mathrm{T}}$ of (2) with errors of $\mathrm{O}(n^{-(i+1)})$. In particular, if we let

$$A_{n,k}=\frac{\lambda}{2(1-\lambda)}+\frac{\lambda^2+4\lambda-4}{16(1-\lambda)^3}\cdot\frac{1}{n}+\frac{2\lambda^3-4\lambda^2+5\lambda-2}{16(1-\lambda)^5}\cdot\frac{1}{n^2},$$

$$B_{n,k}=\frac{(2-3\lambda)(2-\lambda)}{16(1-\lambda)^3}\cdot\frac{1}{n}+\frac{4\lambda^3-14\lambda^2+11\lambda-2}{16(1-\lambda)^5}\cdot\frac{1}{n^2},$$

$$C_{n,k}=\frac{\lambda}{2(1-\lambda)}-\frac{3\lambda^2-16\lambda+12}{16(1-\lambda)^3}\cdot\frac{1}{n}+\frac{14-41\lambda+36\lambda^2-8\lambda^3}{16(1-\lambda)^5}\cdot\frac{1}{n^2}, \tag{3}$$

where as usual $\lambda=k/n$, we claim that, for any $\lambda<c<1$, where c is a constant, we have $a_{n,k}=A_{n,k}+\mathrm{O}(n^{-3})$, $b_{n,k}=B_{n,k}+\mathrm{O}(n^{-3})$ and $c_{n,k}=C_{n,k}+\mathrm{O}(n^{-3})$.

Furthermore, we prove in this section that these expansions are also valid for the solution $e=(a,b,c)^{\mathrm{T}}$ of (1), corresponding to the full version of the game, provided that λ is less than and bounded away from 2/3. We thus see that in this region there is hardly any difference between the two variants of the game.

4.4. Boundary layer influence

In this section we return to the study of Eq. (1) that corresponds to the full version of the game.

Let Φ^* be the operator defined as follows:

$$\Phi^* \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix},$$

where

$$a'_{n,k} = \begin{cases} p_{n,k} a_{n-1,k-1} - q_{n,k} b_{n,k+1} & \text{if } k \leqslant \dfrac{2n-2}{3}, \\ p_{n,k} a_{n-1,k-1} & \text{if } k \geqslant \dfrac{2n-1}{3}, \end{cases}$$

$$b'_{n,k} = \begin{cases} p_{n,k} b_{n-1,k-1} - q_{n,k} c_{n,k+1} & \text{if } k \leqslant \dfrac{2n+1}{3}, \\ 0 & \text{if } k \geqslant \dfrac{2n+2}{3}, \end{cases}$$

$$c'_{n,k} = \begin{cases} p'_{n,k} a_{n-1,k-1} + q_{n,k} b_{n,k+1} & \text{if } k \leqslant \dfrac{2n-2}{3}, \\ p'_{n,k} a_{n-1,k-1} & \text{if } k \geqslant \dfrac{2n-1}{3}. \end{cases}$$

It is easy to verify that $Z\Phi^* = \Phi^*$ and that if $e = Ze$ then $\Phi e = \Phi^* e$. If we let $h' = Zh$, we, therefore, get that Eq. (1) is equivalent to

$$e = \Phi^* e + h'. \tag{4}$$

Examining this equation, we see that the values of $a_{n,k}$ for $k \leqslant (2n+3)/3$, the values of $b_{n,k}$ for $k \leqslant (2n+1)/3$, and the values of $c_{n,k}$ for $k \leqslant (2n+4)/3$ do not depend on values outside these regions. We denote this "closed" region by Ω and consider the behaviour of e on it first.

The values of $a_{n,k}$ for $(2n-1)/3 \leqslant k \leqslant (2n+3)/3$, of $b_{n,k}$ for $(2n-4)/3 \leqslant k \leqslant (2n+1)/3$ and of $c_{n,k}$ for $(2n-1)/3 \leqslant k \leqslant (2n+4)/3$ are the values in Ω affected most directly by the vanishing of the $b_{n,k}$ terms for $k \geqslant (2n+2)/3$. We call the narrow region of Ω containing these values the *boundary layer* of Ω and denote it by $\partial\Omega$. It is convenient to think of the differences between the actual values $a_{n,k}$, $b_{n,k}$ and $c_{n,k}$ in Ω and those predicted by the asymptotic expansion of the previous subsection as being caused by this boundary layer. The shapes of the region Ω and the boundary layer $\partial\Omega$ are depicted in Fig. 4.

Note that on $\Omega - \partial\Omega$ the operators Φ and Φ^* agree, while on $\partial\Omega$ the operator Φ^* has missing $\pm q_{n,k} b_{n,k+1}$ terms. Since in the boundary layer $b_{n,k+1} = \mathrm{O}(n^{-2})$ (or, more precisely, $B_{n,k+1} = \mathrm{O}(n^{-2})$), we expect the boundary layer to have only an

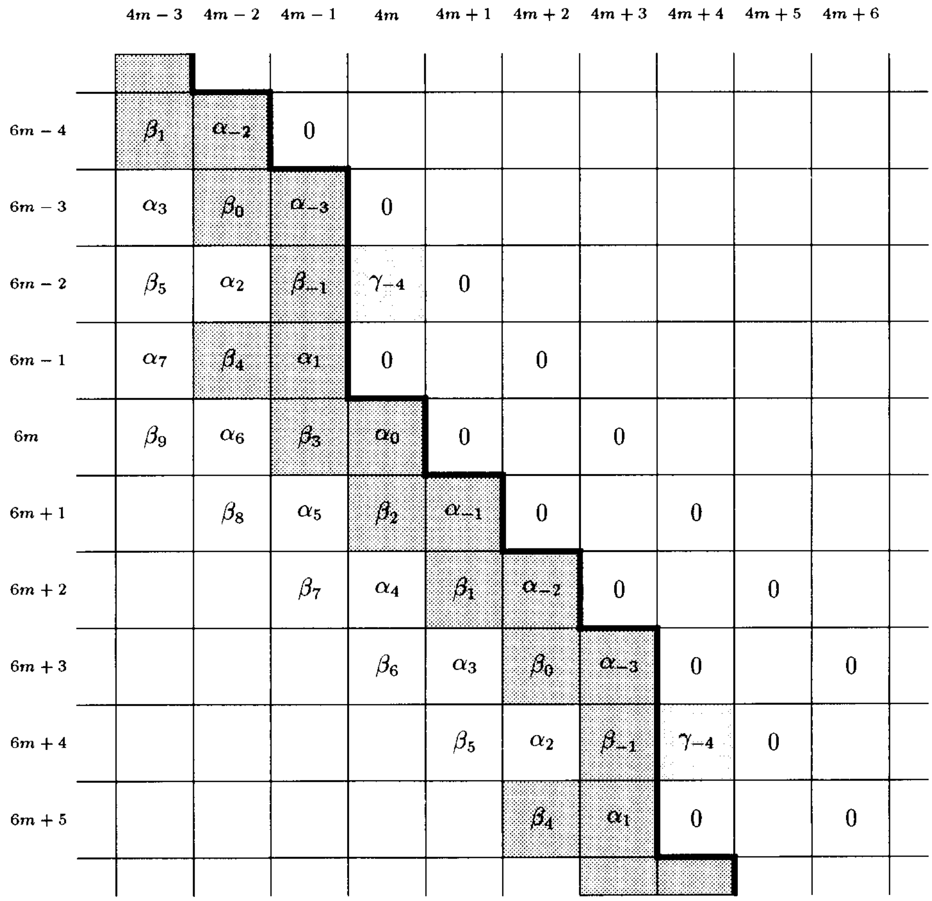

Fig. 4. The region Ω and the boudary layer $\partial\Omega$.

$O(n^{-2})$ influence on values close to the boundary layer. We shall further see that this influence fades very quickly as we move away from the boundary area.

We will now try to find an approximation $\mathscr{E}$ with $O(n^{-3})$ error for the solution of (4), valid for the whole of Ω. This approximation will enable us to establish in Section 4.7 the validity of the alleged optimal strategy. As implied by the previous paragraph, this approximation must include not only the first terms obtained by the bootstrapping process but also terms corresponding to the boundary layer influence.

If $e=\Phi^*e+h'$ and $\varepsilon=e-\mathscr{E}=O(n^{-3})$ in Ω, then we must also have $\mathscr{H}=(I-\Phi^*)(e-\mathscr{E})=h'-(I-\Phi^*)\mathscr{E}=O(n^{-3})$ in Ω. Note that ε satisfies the equation $\varepsilon=\Phi^*\varepsilon+\mathscr{H}$. In the next subsection we will see that, under certain conditions, the last implication can be reversed. More precisely, if $\mathscr{H}=O(n^{-3})$ and if it satisfies a certain additional condition then $\varepsilon=e-\mathscr{E}=O(n^{-3})$.

Let us first look at $H=(R,S,T)^{\mathrm{T}}=h'-(I-\Phi^*)E$, where $E=(A,B,C)^{\mathrm{T}}$, with A,B,C defined in (3). Easy manipulations show that $R_{n,k}, S_{n,k}, T_{n,k}=\mathrm{O}(n^{-3})$ in $\Omega-\partial\Omega$ (this is ensured by the bootstrapping process) but that in $\partial\Omega$

$$R_{n,k}=\frac{9(2n-3k-1)}{8}\cdot\frac{1}{n^2}+\mathrm{O}(n^{-3})$$

and

$$T_{n,k}=-\frac{9(2n-3k-1)}{8}\cdot\frac{1}{n^2}+\mathrm{O}(n^{-3}).$$

The quantity $2n-3k$ measures the horizontal distance of position (n,k) from the boundary layer $\partial\Omega$. This suggests that one could try to work with an approximation $\mathscr{E}=(\mathscr{A},\mathscr{B},\mathscr{C})^{\mathrm{T}}$ of the form

$$\mathscr{A}_{n,k}=A_{n,k}+\frac{\alpha_{2n-3k}}{n^2},\qquad \mathscr{B}_{n,k}=B_{n,k}+\frac{\beta_{2n-3k}}{n^2},$$

$$\mathscr{C}_{n,k}=C_{n,k}+\frac{\gamma_{2n-3k}}{n^2}, \tag{5}$$

where the $A_{n,k}, B_{n,k}, C_{n,k}$ are again those of (3), and, thus, represent the global behaviour in Ω, while the sequences $\{\alpha_l\}$, $\{\beta_l\}$ and $\{\gamma_l\}$ represent the effect of the boundary layer $\partial\Omega$. We expect the sequences $\{\alpha_l\}$, $\{\beta_l\}$ and $\{\gamma_l\}$ to be quickly, in fact exponentially, diminishing, so that their contribution far from the boundary layer will indeed be negligible.

The sequences $\{\alpha_l\}$, $\{\beta_l\}$ and $\{\gamma_l\}$ should be chosen in a way that ensures that $\mathscr{H}=\mathrm{O}(n^{-3})$ in the whole of Ω. To that end, as we shall see shortly in the proof of Theorem 4.3, the sequences $\{\alpha_l\}$, $\{\beta_l\}$ and $\{\gamma_l\}$ should satisfy the following linear recurrence relations:

$$\begin{aligned}
&\alpha_l-\frac{1}{2}\alpha_{l+1}-\frac{9(l-1)}{8}=0, && -3\leqslant l\leqslant 1,\\
&\alpha_l-\frac{1}{2}\alpha_{l+1}+\frac{1}{2}\beta_{l-3}=0, && 1<l,\\
&\beta_l-\frac{1}{2}\beta_{l+1}+\frac{1}{2}\gamma_{l-3}=0, && -1\leqslant l,\\
&\gamma_l-\frac{1}{2}\alpha_{l+1}-\frac{9(l-1)}{8}=0, && -4\leqslant l\leqslant 1,\\
&\gamma_l-\frac{1}{2}\alpha_{l+1}-\frac{1}{2}\beta_{l-3}=0, && 1<l,
\end{aligned} \tag{6}$$

together with the additional requirement that $\alpha_l, \beta_l, \gamma_l\to 0$ as $l\to\infty$.

The values of α_l and β_l are easily computed using generating function techniques. The first values are $\alpha_{-3} \simeq -6.83199877$, $\alpha_{-2} \simeq -4.66399755$, $\alpha_{-1} \simeq -2.57799510$ and $\beta_{-1} \simeq -2.08745613$, $\beta_0 \simeq -1.96591166$, $\beta_1 = -1.76382209$. In general, $\alpha_l = \sum_{i=1}^{6} u_i \theta_i^{-l}$, $\beta_l = \sum_{i=1}^{6} v_i \theta_i^{-l}$ and $\gamma_l = \sum_{i=1}^{6} w_i \theta^{-l}$, where u_i, v_i, w_i are some fixed complex numbers and $\theta_1, \ldots, \theta_6$ are the six complex roots of the equation $x^8 - x^7 + 4x^2 - 4x + 1 = 0$, with modulus greater than 1. The values of the roots θ_i and of the coefficients u_i and v_i are given in Table 3.

Assuming that $\mathscr{E}$ does indeed approximate e to within an $O(n^{-3})$ error, we get (for fixed values of l) the following behaviour of $a_{n,k}$ and $b_{n,k}$ near the boundary layer:

$$a_{n,(2n-l)/3} = 1 - \frac{3(l+1)}{2} \cdot \frac{1}{n} + \left(\frac{3(2l^2+2l+3)}{4} + \alpha_l \right) \cdot \frac{1}{n^2} + O\left(\frac{1}{n^3} \right),$$

$$b_{n,(2n-l)/3} = \left(\frac{9(l+1)}{4} + \beta_l \right) \cdot \frac{1}{n^2} + O\left(\frac{1}{n^3} \right).$$

In particular,

$$b_{n,(2n+1)/3} \approx \frac{0.162544}{n^2}, \qquad b_{n,2n/3} \approx \frac{2.534088}{n^2},$$

$$b_{n,(2n-1)/3} \approx \frac{4.986178}{n^2}.$$

Having chosen the sequences α_l, β_l and γ_l in this way, we can indeed prove that $\mathscr{H} = O(n^{-3})$ in the whole of Ω. Furthermore, we show that $\mathscr{H}$ satisfies an additional "continuity" condition that, together with the condition $\mathscr{H} = O(n^{-3})$, will allow us to infer in the next subsection that $\varepsilon = O(n^{-3})$.

Theorem 4.3.

$$|\mathscr{H}_{n,k}| \leqslant \frac{100}{n^3}, \qquad |\mathscr{H}_{n,k} - q_{n,k}^{(2)} \mathscr{H}_{n,k+2}| \leqslant (1 - q_{n,k}^{(2)}) \cdot \frac{100}{n^3},$$

where $q_{n,k}^{(2)} = q_{n,k} q_{n,k+1}$ *for all positions in* Ω *with* $n \geqslant 1000$.

Proof. We first clarify the statement of the theorem. If $\mathscr{H} = (\mathscr{R}, \mathscr{S}, \mathscr{T})^{\mathrm{T}}$ then we claim that $|\mathscr{R}_{n,k}|, |\mathscr{S}_{n,k}|, |\mathscr{T}_{n,k}| \leqslant 100/n^3$ for $n \geqslant 1000$ and $k \leqslant (2n+3)/3$, $k \leqslant (2n+1)/3$,

Table 3
The values of the roots θ_i and the coefficients u_i and v_i

$\theta_{1,2} \approx -1.108812 \pm 0.625391i$	$u_{1,2} \approx 0.021830 \mp 0.048470i$	$v_{1,2} \approx 0.060084 \pm 0.035674i$
$\theta_{3,4} \approx 1.121061 \pm 0.562315i$	$u_{3,4} \approx -1.027472 \mp 0.0663188i$	$v_{3,4} \approx -0.227399 \pm 1.791415i$
$\theta_{5,6} \approx -0.018515 \pm 1.239618i$	$u_{5,6} \approx -0.122426 \pm 0.028318i$	$v_{5,6} \approx -0.015567 \pm 0.142098i$
$\theta_7 \approx 0.539036$	$u_7 = 0.000000$	$v_7 = 0.000000$
$\theta_8 \approx 0.473498$	$u_8 = 0.000000$	$v_8 = 0.000000$

$k \leqslant (2n+4)/3$, respectively, and that $|\mathscr{R}_{n,k} - q^{(2)}_{n,k}\mathscr{R}_{n,k+2}|$, $|\mathscr{S}_{n,k} - q^{(2)}_{n,k}\mathscr{S}_{n,k+2}|$, $|\mathscr{T}_{n,k} - q^{(2)}_{n,k}\mathscr{T}_{n,k+2}| \leqslant (1-q^{(2)}_{n,k})100/n^3$ for $n \geqslant 1000$ and $k \leqslant (2n-3)/3$, $k \leqslant (2n-5)/3$, $k \leqslant (2n-2)/3$, respectively.

The rigorous proof of these inequalities is rather lengthy and technical. Here we shall only "demonstrate" the validity of two of them (those involving $\mathscr{R}_{n,k}$) using high-level asymptotic analysis.

Assume first that $k \leqslant (2n-2)/3$ (the case $(2n-1)/3 \leqslant k \leqslant (2n+3)/3$ will be dealt with separately). Using the definition of $(I-\Phi^*)$, we get that

$$\begin{aligned}\mathscr{R}_{n,k} &= -\mathscr{A}_{n,k} + p_{n,k}(1+\mathscr{A}_{n-1,k-1}) - q_{n,k}\mathscr{B}_{n,k+1} \\ &= \left.\begin{matrix} -A_{n,k} + p_{n,k}(1+A_{n-1,k-1}) - q_{n,k}B_{n,k+1} \end{matrix}\right\} = R_{n,k} \\ &\quad \left.-\frac{\alpha_{2n-3k}}{n^2} + p_{n,k}\cdot\frac{\alpha_{2n-3k+1}}{(n-1)^2} - q_{n,k}\cdot\frac{\beta_{2n-3k-3}}{n^2}\right\} = \rho_{n,k}.\end{aligned}$$

The term $R_{n,k}$ is a rational expression in n and k and automatic manipulations show that

$$R_{n,k} = -\frac{8-26\lambda^2+11\lambda^3}{16(2-\lambda)(1-\lambda)^5}\cdot\frac{1}{n^3} + \mathrm{O}\left(\frac{1}{n^4}\right).$$

The coefficient of $1/n^3$ above attains its maximum absolute value in the range [0, 2/3] at $\lambda \approx 0.57$, where it evaluates to approximately -4.73. We thus see that this term does not give us any cause for concern.

If we let $l = 2n - 3k$, we get

$$\begin{aligned}\rho_{n,k} &= \frac{1}{n^2}\left[-\alpha_l + p_{n,k}\cdot\frac{n^2}{(n-1)^2}\cdot\alpha_{l+1} - q_{n,k}\cdot\beta_{l-3}\right] \\ &= \underbrace{\frac{1}{n^2}\left[-\alpha_l + \frac{1}{2}\alpha_{l+1} - \frac{1}{2}\beta_{l-3}\right]}_{0} \\ &\quad + \frac{1}{n^2}\left[\left(p_{n,k}\cdot\frac{n^2}{(n-1)^2} - \frac{1}{2}\right)\alpha_{l+1} - \left(q_{n,k} - \frac{1}{2}\right)\beta_{l-3}\right].\end{aligned}$$

The first expression in the last line disappears as it is one of the defining relations of the sequences $\{\alpha_l\}$, $\{\beta_l\}$, $\{\gamma_l\}$. We may assume now that $l = \mathrm{o}(n)$ since, otherwise, α_{l+1} and β_{l-3} are exponentially small and we have nothing to worry about. We can, therefore, use the relations

$$p_{n,k} = \frac{2n-l}{4n+l} = \frac{1}{2} - \frac{3l}{8n} + \mathrm{o}\left(\frac{1}{n}\right),$$

$$q_{n,k} = \frac{2(n+l)}{4n+l} = \frac{1}{2} + \frac{3l}{8n} + \mathrm{o}\left(\frac{1}{n}\right),$$

together with the fact that

$$\frac{n^2}{(n-1)^2}=1+\frac{2}{n}+\mathrm{O}\left(\frac{1}{n^2}\right),$$

to get that

$$\rho_{n,k}=\left[\left(1-\frac{3l}{8}\right)\alpha_{l+1}-\frac{3l}{8}\beta_{l-3}\right]\cdot\frac{1}{n^3}+\mathrm{o}\left(\frac{1}{n^3}\right).$$

The coefficient of $1/n^3$ here is maximised when $l=5$, where we get $\rho_{n,(2n-4)/3}\simeq -2.80/n^3$. Hence, for large enough n, and $k\leqslant(2n-2)/3$ we even expect to have $|\mathscr{R}_{n,k}|\leqslant 10/n^3$.

Assume now that $(2n-1)/3\leqslant k\leqslant(2n+3)/3$. Proceeding in a similar way, we get that

$$\begin{aligned}\mathscr{R}_{n,k}&=-\mathscr{A}_{n,k}+p_{n,k}(1+\mathscr{A}_{n-1,k-1})\\&=\left.-A_{n,k}+p_{n,k}(1+A_{n-1,k-1})-\frac{9(2n-3k-1)}{8n^2}\right\}=R'_{n,k}\\&\left.-\frac{\alpha_{2n-3k}}{n^2}+p_{n,k}\cdot\frac{\alpha_{2n-3k+1}}{(n-1)^2}+\frac{9(2n-3k-1)}{8n^2}\right\}=\rho'_{n,k}.\end{aligned}$$

Again, if $l=2n-3k$ then

$$R'_{n,k}\sim\frac{-135+198l-36l^2}{16}\cdot\frac{1}{n^3}$$

and the maximum of this expression in the range $-3\leqslant l\leqslant 1$ is attained when $l=-3$ and $R'_{n,(2n+3)/2}\sim -65.8125/n^3$. As for $\rho'_{n,k}$, we get

$$\begin{aligned}\rho'_{n,k}&=\frac{1}{n^2}\left[-\alpha_l+p_{n,k}\cdot\frac{n^2}{(n-1)^2}\cdot\alpha_{l+1}+\frac{9(2n-3k-1)}{8}\right]\\&=\underbrace{\frac{1}{n^2}\left[-\alpha_l+\frac{1}{2}\alpha_{l+1}+\frac{9(l-1)}{8}\right]}_{0}\\&\quad+\frac{1}{n^2}\left[\left(p_{n,k}\cdot\frac{n^2}{(n-1)^2}-\frac{1}{2}\right)\alpha_{l+1}\right]\\&\sim\left(1-\frac{3l}{8}\right)\alpha_{l+1}\cdot\frac{1}{n^3}.\end{aligned}$$

The maximum absolute value is again attained when $l=-3$, where $\rho'_{n,(2n+3)/2}\simeq -9.91/n^3$. So, for large enough n, and any $k\leqslant(2n+3)/2$, we expect to have $|\mathscr{R}_{n,k}|\leqslant 80/n^3$. The slackness that we have introduced by requiring only that $|\mathscr{R}_{n,k}|\leqslant 100/n^3$ allows us to prove this inequality for every $n\geqslant 1000$.

Turning our attention to the second inequality involving $\mathscr{R}_{n,k}$, we note that, for $k \leqslant (2n-8)/3$, we have

$$\frac{\mathscr{R}_{n,k} - q^{(2)}_{n,k}\mathscr{R}_{n,k+2}}{1-q^{(2)}_{n,k}} = \frac{R_{n,k} - q^{(2)}_{n,k} R_{n,k+2}}{1-q^{(2)}_{n,k}} + \frac{\rho_{n,k} - q^{(2)}_{n,k}\rho_{n,k+2}}{1-q^{(2)}_{n,k}}.$$

A simple manipulation yields

$$\frac{R_{n,k} - q^{(2)}_{n,k} R_{n,k+2}}{1-q^{(2)}_{n,k}} = R_{n,k} + \frac{q^{(2)}_{n,k}}{1-q^{(2)}_{n,k}}(R_{n,k} - R_{n,k+2}).$$

Note now that $R_{n,k} - R_{n,k+2} = \mathrm{O}(n^{-4})$ or, more precisely,

$$R_{n,k} - R_{n,k+2} = \frac{-88+152\lambda+64\lambda^2-126\lambda^3+33\lambda^4}{16(2-\lambda)^2(1-\lambda)^6}\cdot\frac{1}{n^4} + \mathrm{O}\left(\frac{1}{n^5}\right).$$

The coefficient of $1/n^4$ here is, of course, twice the derivative of the coefficient of $1/n^3$ in the corresponding expansion of $R_{n,k}$. It can be easily checked that $|R_{n,k} - R_{n,k+2}| \leqslant 3/n^4$ for say $\lambda \leqslant 1/10$. Now $q^{(2)}_{n,k}/(1-q^{(2)}_{n,k}) < 2n$ for every $k \geqslant 0$ and, furthermore, $q^{(2)}_{n,k}/(1-q^{(2)}_{n,k}) = \mathrm{O}(1)$ whenever λ is bounded away from 0.

The term $(\rho_{n,k} - q^{(2)}_{n,k}\rho_{n,k+2})/(1-q^{(2)}_{n,k})$ attains a maximum of about $-2.33/n^3$ for $l=12$ and, thus, we can again obtain the desired inequality.

Combining these facts, we get the desired bound for $k \leqslant (2n-8)/3$. The case $(2n-7)/3 \leqslant k \leqslant (2n-3)/3$ should again be treated separately. We omit the details.

The inequalities involving $\mathscr{S}_{n,k}$ and $\mathscr{T}_{n,k}$ can be "verified" in a similar manner. □

4.5. *Bounding the errors*

We saw in the previous subsection that ε, the error of our estimation, satisfies the equation $\varepsilon = \Phi^*\varepsilon + \mathscr{H}$, where $\mathscr{H}$ satisfies the conditions of Theorem 4.3. We now show that this implies $\varepsilon = \mathrm{O}(n^{-3})$.

Theorem 4.4. *If $e = \Phi^* e + h$, where $e = (a, b, c)^{\mathrm{T}}$, $h = (r, s, t)^{\mathrm{T}}$ and*

$$|h_{n,k}| \leqslant \frac{H}{n^3}, \quad |h_{n,k} - q^{(2)}_{n,k} h_{n,k+2}| \leqslant (1-q^{(2)}_{n,k})\cdot\frac{H}{n^3}$$

for all positions in Ω with $n \geqslant n_0 \geqslant 1000$, and

$$|a_{n,k}|, |c_{n,k}| \leqslant \frac{15H}{n^3}, \quad |b_{n,k}| \leqslant \frac{10H}{n^3}$$

for all positions in Ω with $n = n_0, n_0+1$, then the same bounds on $a_{n,k}$, $b_{n,k}$ and $c_{n,k}$ hold for all positions in Ω with $n \geqslant n_0$.

Proof. What conditions should two constants A and B satisfy if we are to succeed in proving, by induction, that $|a_{n,k}| \leqslant A/n^3$ and that $|b_{n,k}| \leqslant B/n^3$? Assuming the basis of the induction to be already established, we check what conditions on A and B enable us to derive the induction step.

Using the induction hypothesis and the conditions on $h_{n,k}$, we can bound $a_{n,k}$ as follows:

$$|a_{n,k}| \leqslant p_{n,k} \cdot |a_{n-1,k-1}| + q_{n,k} \cdot |b_{n,k+1}| + |r_{n,k}|$$
$$\leqslant \left[\frac{p_{n,k}}{(n-1)^3}\right] \times A + \left[\frac{q_{n,k}}{n^3}\right] \times B + \left[\frac{1}{n^3}\right] \times H \quad .$$

If $A > B + 2H$ then the last expression is less than A/n^3 for any sufficiently large n and $k \leqslant (2n+3)/2$. This is because in Ω we have $p_{n,k} \leqslant \frac{1}{2} + o(1)$.

In particular, if we choose $A = 15H$, $B = 10H$, we must verify that

$$\left[\frac{p_{n,k}}{(n-1)^3}\right] \times A + \left[\frac{q_{n,k}}{n^3}\right] \times B + \left[\frac{1}{n^3}\right] \times H \leqslant \frac{A}{n^3}$$

for any $n \geqslant 1000$ and $0 \leqslant k \leqslant 0.67n$. This inequality involves only quantities like $p_{n,k}$ and $q_{n,k}$ that were explicitly defined. Expanding these definitions, we find that the claim to be verified is equivalent to the claim that

$$-2(4+3\lambda)(1-3n+3n^2) + (8-9\lambda)n^3 \geqslant 0$$

for any $n \geqslant 1000$ and $0 \leqslant \lambda = k/n \leqslant 0.67$. This inequality *is* easily verified.

The choice that $A > B$ has so far been to our advantage. It will, however, make our lives much more difficult in the sequel.

By expanding the recursive definitions of $a_{n,k}$, $b_{n,k}$ and $c_{n,k}$ in the way depicted in Fig. 5, we get that

$$\begin{aligned} b_{n,k} = {} & [p_{n,k}] \cdot b_{n-1,k-1} + [q_{n,k}^{(4)}] \cdot b_{n,k+4} \\ & - [q_{n,k}(q_{n,k+1} p_{n,k+2} - p'_{n,k+1} q_{n-1,k})] \cdot b_{n-1,k+1} \\ & - [q_{n,k} p_{n,k+1}^{(2)}] \cdot a_{n-2,k-1} + [q_{n,k}^{(3)} p'_{n,k+3}] \cdot a_{n-1,k+2} \\ & - [q_{n,k} p_{n,k+1}^{(2)}] \cdot r_{n-1,k} + (s_{n,k} - [q_{n,k}^{(2)}] \cdot s_{n,k+2}) \\ & - [q_{n,k}] \cdot (t_{n,k+1} - [q_{n,k+1}^{(2)}] \cdot t_{n,k+3}), \end{aligned}$$

where, as before, $q_{n,k}^{(2)} = q_{n,k} q_{n,k+1}$, $q_{n,k}^{(3)} = q_{n,k}^{(2)} q_{n,k+2}$, $q_{n,k}^{(4)} = q_{n,k}^{(3)} q_{n,k+3}$ and $p_{n,k}^{(2)} = p_{n,k} p_{n-1,k-1}$. We assume here that $k \leqslant (2n-11)/3$ so that all the terms given are indeed present. The case $(2n-10)/3 \leqslant k \leqslant (2n+1)/3$ must be dealt with separately and the details are omitted.

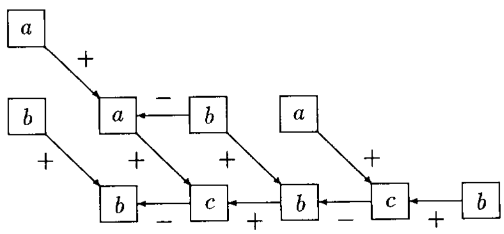

Fig. 5. Expanding the definition of $b_{n,k}$.

The important point to note here is the fact that $b_{n-1,k+1}$ contributes to $b_{n,k}$ along two different paths, once with a positive sign and once with a negative sign. Since $q_{n,k+1}p_{n,k+2} \approx p'_{n,k+1}q_{n-1,k}$, these two contributions almost cancel each other out. Thus, when we add up (the absolute values of) the coefficients of all the $a_{n',k'}$ and $b_{n',k'}$ appearing in this expansion for $b_{n,k}$, we get a quantity $\sigma_{n,k}$ which for $0<\lambda$ is significantly less than 1. In fact, it is easy to check that $\sigma_{n,k} \sim \sigma(\lambda) = (8-20\lambda+22\lambda^2-9\lambda^3)/(2-\lambda)^3$. The function $\sigma(\lambda)$ attains the value $\frac{3}{4}$ at $\lambda=\frac{2}{3}$, the minimal value of $19/27(\simeq 0.704)$ at $\lambda=\frac{1}{2}$ and the maximal value of 1 at $\lambda=1$. We see therefore that a choice $A, B \gg H$ should enable us to prove the induction step when λ is bounded away from 0 and n is large enough. We might expect trouble when $\lambda \approx 0$, but this is exactly the place where the additional condition of Theorem 4.3 comes to our rescue. We have gone far enough in the expansion shown in Fig. 5 to obtain a configuration in which the driving terms tend to cancel each other in pairs.

Relying on the induction hypothesis and the conditions on $h_{n,k}$, we get that

$$|b_{n,k}| \leqslant \left[\frac{q_{n,k}p^{(2)}_{n,k+1}}{(n-2)^3} + \frac{q^{(3)}_{n,k}p'_{n,k+3}}{(n-1)^3}\right] \times A$$
$$+ \left[\frac{p_{n,k}}{(n-1)^3} + \frac{q_{n,k}(q_{n,k+1}p_{n,k+2} - p'_{n,k+1}q_{n-1,k})}{(n-1)^3} + \frac{q^{(4)}_{n,k}}{n^3}\right] \times B$$
$$+ \left[\frac{q_{n,k}p^{(2)}_{n,k+1}}{n^3} + \frac{(1-q^{(2)}_{n,k})(1+q_{n,k})}{n^3}\right] \times H,$$

where $q_{n,k+1}p_{n,k+2} - p'_{n,k+1}q_{n-1,k} = 6(n-k-1)/(2n-k-1)(2n-k-2)$ is indeed positive for all relevant values of n and k.

We want to find values for A and B for which the last expression is less than or equal to B/n^3 for all large enough n and k in the appropriate range. Since we are not interested in finding the optimal constants A and B, we just point out that again the choice $A=15H$ and $B=10H$ suffices, i.e., the bound in the last inequality is less than B/n^3 for any $n \geqslant 1000$ and $k \leqslant 0.67n$. Expanding again the definitions of $p_{n,k}$, $q_{n,k}$ and of $p^{(2)}_{n,k}$, $q^{(2)}_{n,k}$, $q^{(3)}_{n,k}$, $q^{(4)}_{n,k}$, we get that the condition that we have to verify is that the expression

$$\begin{aligned}
&48(272-422\lambda+141\lambda^2)\\
&\quad +8(-9904+18558\lambda-9863\lambda^2+1551\lambda^3)n\\
&\quad +4(51824-115326\lambda+81991\lambda^2-22071\lambda^3+1692\lambda^4)n^2\\
&\quad +2(-147888+391146\lambda-350617\lambda^2+130959\lambda^3-18428\lambda^4+564\lambda^5)n^3\\
&\quad +(247232-782580\lambda+861216\lambda^2-415735\lambda^3+84636\lambda^4-5076\lambda^5)n^4\\
&\quad +(-122384+473052\lambda-630812\lambda^2+377241\lambda^3-100814\lambda^4+9306\lambda^5)n^5\\
&\quad +(34392-170768\lambda+275520\lambda^2-198141\lambda^3+65080\lambda^4-7857\lambda^5)n^6\\
&\quad +(-4808+34920\lambda-69032\lambda^2+58361\lambda^3-22308\lambda^4+3159\lambda^5)n^7\\
&\quad +(2-\lambda)(112-1756\lambda+3652\lambda^2-2590\lambda^3+567\lambda^4)n^8\\
&\quad +\lambda(2-\lambda)(80-220\lambda+178\lambda^2-39\lambda^3)n^9
\end{aligned}$$

is nonnegative for $n \geqslant 1000$ and $0 \leqslant \lambda \leqslant 0.67$. To show that this is plausible, we note that the function $\lambda(2-\lambda)(80-220\lambda+178\lambda^2-39\lambda^3)$, which is the coefficient of n^9 in the above expression, is positive for $0<\lambda \leqslant 0.67$. For values of λ close to 0 we also have to consider the coefficient of n^8, which is approximately 224 when $\lambda \approx 0$. With slightly more technical work, the positivity of this expression can be established rigorously.

Finally, for $c_{n,k}$, we get

$$|c_{n,k}| \leqslant p'_{n,k}|a_{n-1,k-1}|+q_{n,k}|b_{n,k+1}| \leqslant \frac{(15p'_{n,k}+10q_{n,k})H}{n^3} \leqslant \frac{12.5H}{n^3}. \qquad \square$$

Theorem 4.5. *If* $e=(a,b,c)^{\mathrm{T}}$ *is the solution of Eq.* (1) (*or* (4)) *and* $\mathscr{E}=(\mathscr{A},\mathscr{B},\mathscr{C})^{\mathrm{T}}$ *is defined by* (3), (5) *and* (6) *then*

$$|a_{n,k}-\mathscr{A}_{n,k}| \leqslant \frac{1500}{n^3} \quad \text{for } n \geqslant 1000 \text{ and } 0 \leqslant k \leqslant \frac{2n+3}{2},$$

$$|b_{n,k}-\mathscr{B}_{n,k}| \leqslant \frac{1000}{n^3} \quad \text{for } n \geqslant 1000 \text{ and } 0 \leqslant k \leqslant \frac{2n+1}{2},$$

$$|c_{n,k}-\mathscr{C}_{n,k}| \leqslant \frac{1500}{n^3} \quad \text{for } n \geqslant 1000 \text{ and } 0 \leqslant k \leqslant \frac{2n+4}{2}.$$

Proof. It can be verified directly that these inequalities hold for $n=1000$, 1001 and all admissible values of k. The theorem then follows by combining Theorems 4.3 and 4.4. $\square$

4.6. Beyond the boundary layer

We have to consider the values of $a_{n,k}$ only for $k \geqslant (2n+4)/3$. The values of $b_{n,k}$ for $k \geqslant (2n+2)/3$ are identically zero, by definition, and the values of $c_{n,k}$ for $k \geqslant (2n+5)/3$ are of no interest since they are never used. For $a_{n,k}$ in this region, we have the simple relation

$$a_{n,k}=p_{n,k}(1+a_{n-1,k-1}) \quad \text{for } k \geqslant \frac{2n-1}{3}.$$

By induction, we can prove that, for $k \geqslant (2n-1)/3$, we have

$$a_{n,k}=\frac{k}{2(n-k)+1}-\frac{k!}{(2n-k)!}\cdot\frac{[4(n-k)-2]!}{[2(n-k-1)]!} \cdot\left(\frac{2(n-k-1)}{2(n-k)+1}-a_{3(n-k)-2,\,2(n-k-1)}\right). \qquad (7)$$

Note that the value $a_{3(n-k)-2,\,2(n-k-1)}$ lies in Ω, just outside the boundary layer as it is of the form $a_{n',(2n'-2)/3}$.

Using Stirling's formula, we get, for $k=\lambda n$ with $\frac{2}{3}<\lambda<1$, that

$$\frac{k!}{(2n-k)!}\cdot\frac{[4(n-k)-2]!}{[2(n-k-1)]!}\approx\sqrt{\frac{\lambda}{2-\lambda}}\cdot e^{-L(\lambda)n},$$

where

$$L(\lambda)=\ln\left[\frac{(2-\lambda)^{(2-\lambda)}}{\lambda^{\lambda}[8(1-\lambda)]^{2(1-\lambda)}}\right].$$

It can be checked that $L(2/3)=L(1)=0$ while $L(\lambda)>0$ for $\lambda\in(2/3,1)$. Thus, for any k with $\lambda=k/n$ bounded away from both 2/3 and 1, we get that $a_{n,k}\simeq k/[2(n-k)+1]$ with an exponentially small error! The most accurate approximation is obtained for $k=\lambda n$ where $\lambda=1-1/\sqrt{65}\simeq 0.875965$, for which $L(\lambda)\approx 0.249353$.

For $a_{n,n-l}$, equation (7) becomes

$$a_{n,n-l}=\frac{n-l}{2l+1}-\frac{(n-l)!}{(n+l)!}\cdot\frac{[2(2l-1)]!}{[2(l-1)]!}\cdot\left(\frac{2(l-1)}{2l+1}-a_{3l-2,2(l-1)}\right).$$

We can thus get explicit formulae for $a_{n,n-l}$, where l is a constant. All we have to know for this purpose is the single value of $a_{3l-2,2(l-1)}$. In particular, we get

$$a_{n,n}=n,$$

$$a_{n,n-2}=\frac{n-2}{5}-\frac{48}{(n+2)(n+1)n(n-1)},$$

$$a_{n,n-4}=\frac{n-4}{9}-\frac{2983680}{(n+4)(n+3)\cdot\ldots\cdot(n-3)},$$

and, in general,

$$a_{n,n-l}=\frac{n-l}{2l+1}+\mathrm{O}(n^{-2l}).$$

Hence, the diagonals in the $e_{n,k}$ table behave essentially as linear progressions.

4.7. *Verifying the optimal strategy*

For $n\leqslant 1000$, the validity of the optimal strategy can be verified directly.

We now prove the validity of the optimal strategy for $n>1000$ by induction. Suppose that we have already verified the claimed optimal strategy for all positions (n', k'), with either $n'<n$ or $n'=n$ and $k'>k$. This means that, so far, the values of the positions agree with those obtained from Eq. (1), and, thus, all the estimations of the previous subsections are valid. If $n+k$ is even and $k\neq 0, n$, we use these estimates to show that $e^1_{n,k}>0, e^2_{n,k}$ (if $k=0$ or n, we already know that $e^1_{n,k}=e^2_{n,k}$). If $n+k$ is odd and $k\leqslant(2n+1)/3$, we use these estimates to show that $e^2_{n,k}>0, e^1_{n,k}$, and if $n+k$ is odd and $k>(2n+1)/3$, we use them to show that $e^1_{n,k}, e^2_{n,k}<0$. This will prove, by induction, the validity of the optimal strategy for every position.

As can be seen from Figs. 2 and 3, the only inequalities for which we really need the $O(1/n^2)$ terms in our approximations are those that claim that $e^2_{n,k} = b_{n,k} > 0$ when $n+k$ is odd and $k \leqslant (2n+1)/3$, and that $e^2_{n,k} < 0$ when $n+k$ is odd and $k \geqslant (2n+2)/3$. Even here, the $O(1/n^2)$ terms are needed only when $\lambda \approx \frac{2}{3}$.

5. Variants of the game

As the reader has probably realised by now, there is no point in flipping back the cards after inspection if both players will remember them anyway. This convention also allows the game to be played as a game of strategy by players with imperfect memories. A 0-move now simply corresponds to a decision to end the game, while a 1-move will mean literally the inspection of one new card, without the useless ritual of inspecting an old one too. With these new conventions, it seems natural to allow 0-moves and 1-moves from all positions (even those of the form $(n, 0)$ and $(n, 1)$) and we shall do so throughout this section.

What is the effect of allowing 0- and 1-moves from positions of the form $(n, 0)$ and 0-moves from positions of the form $(n, 1)$? Since the value of every position in the new game is, by definition, nonnegative, some changes are bound to occur but, as we shall soon see, the overall effect is minimal. The values of the simplest positions under the new rules are given in Table 4. These new values will, of course, influence the values of almost all other positions, but it turns out that the changes are exponentially diminishing in n for every $k = \lambda n$ with $\lambda < c < 1$. The new optimal moves from positions (n, k) with $k \leqslant n \leqslant 15$ are given in Table 5. There are again some exceptions when $n \leqslant 5$ but, apart from that, the only difference between Table 5 and Table 2, corresponding to the original version of the game, is that a 0-move is now used from positions $(n, 0)$ with n even. This was to be expected as the values of these positions were hitherto negative.

The analysis of this version of the game is almost identical to the one carried out in the previous section. The only difference is that a second boundary layer now exists when $\lambda \approx 0$, caused by the 0-moves used from positions $(n, 0)$ with n even.

Table 4
The expected values of the simplest positions when 0- and 1-moves are allowed everywhere

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=1$	1	1						
$n=2$	0	$\frac{2}{3}$	2					
$n=3$	0	0	$\frac{1}{3}$	3				
$n=4$	0	$\frac{1}{7}$	$\frac{1}{3}$	0	4			
$n=5$	0	0	$\frac{1}{7}$	$\frac{4}{7}$	0	5		
$n=6$	0	$\frac{1}{231}$	$\frac{11}{105}$	$\frac{5}{42}$	$\frac{11}{14}$	0	6	
$n=7$	$\frac{151}{3003}$	$\frac{151}{3003}$	$\frac{20}{693}$	$\frac{272}{1155}$	$\frac{19}{210}$	$\frac{125}{126}$	0	7

Table 5
The optimal moves for $n \leqslant 15$ when 0- and 1-moves are allowed everywhere

$n=1$	2 1
$n=2$	0 2 1
$n=3$	0 0 2 1
$n=4$	0 2 1 0 1
$n=5$	0 0 2 1 0 1
$n=6$	0 2 1 2 1 0 1
$n=7$	2 1 2 1 2 1 0 1
$n=8$	0 2 1 2 1 2 1 0 1
$n=9$	2 1 2 1 2 1 2 1 0 1
$n=10$	0 2 1 2 1 2 1 2 1 0 1
$n=11$	2 1 2 1 2 1 2 1 0 1 0 1
$n=12$	0 2 1 2 1 2 1 2 1 0 1 0 1
$n=13$	2 1 2 1 2 1 2 1 2 1 0 1 0 1
$n=14$	0 2 1 2 1 2 1 2 1 2 1 0 1 0 1
$n=15$	2 1 2 1 2 1 2 1 2 1 2 1 0 1 0 1

We now turn to the study of variants of the game obtained by restricting the set of allowed moves. We have already encountered an example of this kind in Section 4.3, where we have assumed that 1-moves and 2-moves are the only moves allowed. We consider two other restricted versions.

5.1. Version 1

In this section we investigate the version of the game in which 1-moves are the only moves allowed. While there is no question of finding the optimal strategy in this case, the analysis of the expected gains from the different positions turns out to be interesting.

If we denote again by $e_{n,k}$ the expected gain from position (n, k), we get immediately the following recurrence relation:

$$e_{n,k} = p_{n,k}(1 + e_{n-1,k-1}) - q_{n,k} e_{n,k+1}, \tag{8}$$

where the only initial condition required is $e_{0,0} = 0$.

It turns out that, in this version, each diagonal $e_{n,n-r}$ for a fixed r forms an arithmetical progression,

$$e_{n,n-r} = \alpha_r n + \beta_r. \tag{9}$$

By substituting this relation into (8), we can prove (9) by induction and get the following recurrence relations for $r \geqslant 1$:

$$\alpha_r = \frac{2r}{2r-1}\left(\frac{1}{2r} - \alpha_{r-1}\right),$$

$$\beta_r = \frac{1}{2}(\alpha_r - 1) - \beta_{r-1},$$

where $\alpha_0=1$, $\beta_0=0$. By expanding the definition of α_r, we get

$$\alpha_r=(-1)^r\frac{2\cdot 4\cdots 2r}{3\cdot 5\cdots(2r+1)}\left[1-\frac{1}{2}+\frac{1\cdot 3}{2\cdot 4}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}+\cdots +(-1)^r\frac{1\cdot 3\cdots(2r-1)}{2\cdot 4\cdots 2r}\right].$$

Recalling the Wallis product

$$\frac{\pi}{2}=\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdots},$$

we have

$$\frac{2\cdot 4\cdots 2r}{3\cdot 5\cdots(2r+1)}\sim\sqrt{\frac{\pi}{2(2r+1)}}.$$

The terms inside the square bracket have decreasing absolute values and alternating signs. By Leibnitz's theorem, the limit of the sum of this series exists as $r\to\infty$, and can be shown to be $\sqrt{2}/2$.

We conclude that

$$\alpha_r\sim\left(\frac{\pi}{8}\right)^{1/2}\frac{(-1)^r}{\sqrt{r}}.$$

Similarly, we can expand the definition of β_r and get that

$$\beta_r=\tfrac{1}{2}\cdot(\alpha_r-1)-\tfrac{1}{2}\cdot(\alpha_{r-1}-1)+\tfrac{1}{2}\cdot(\alpha_{r-2}-1) -\cdots+\tfrac{1}{2}\cdot(-1)^r(\alpha_0-1),$$

or, equivalently,

$$\beta_r=\frac{1}{2}\cdot\sum_{s=0}^{r}(-1)^{r-s}\alpha_s-\begin{cases}0 & \text{if } r \text{ odd,}\\ \frac{1}{2} & \text{otherwise.}\end{cases}$$

Since $\alpha_r\sim(\pi/8)^{1/2}(-1)^r r^{-1/2}$, we immediately get that

$$\beta_r\sim\left(\frac{\pi}{8}\right)^{1/2}(-1)^r\sqrt{r}.$$

Hence, if $n-k\to\infty$ then

$$e_{n,k}=\alpha_{n-k}n+\beta_{n-k}\sim\left(\frac{\pi}{8}\right)^{1/2}(-1)^{n-k}\left[\frac{n}{\sqrt{n-k}}+\sqrt{n-k}\right],$$

and, in particular,

$$e_{n,0}\sim\left(\frac{\pi}{2}\right)^{1/2}(-1)^n\sqrt{n}.$$

The behaviour of the $e_{n,k}$ as well as the method used to find it are, therefore, quite different in this case.

5.2. Version 2

In this section we check what happens if 2-moves are the only moves allowed. The analysis in this case can serve as an introductory example to the use of the bootstrapping method of Section 4.3. We omit the details but point out that, in contrast to what we have seen so far, the parity of $n+k$ does not play a major role, unless $\lambda = k/n \approx 1$. The asymptotic expansion for $e_{n,k}$ obtained by bootstrapping is

$$e_{n,k} = \frac{\lambda^2}{4(2-\lambda)(1-\lambda)} + \frac{16-64\lambda+64\lambda^2-19\lambda^3}{16(2-\lambda)^2\,(1-\lambda)^2}\cdot\frac{1}{n}$$

$$+\frac{64-144\lambda+216\lambda^2-198\lambda^3+69\lambda^4}{64(2-\lambda)^3(1-\lambda)^3}\cdot\frac{1}{n^2}+\cdots$$

and is again valid whenever λ is bounded away from 1.

6. More possibilities

How should one play against players who only use 1-moves? The optimal strategy against such players is to play 1-moves from positions (n, k) with $n+k$ even, and 0-moves from positions (n, k) with $n+k$ odd. The expected gains are then the absolute values of the corresponding expected gains when both players are always using 1-moves. This is just version 1 of the game analysed in the previous section.

How should one play against players who only use 2-moves? The optimal strategy here is to play 1-moves from "almost" all the positions. The exact details here are more complicated and not entirely known to us.

What happens if the objective of the players is to maximise their probability of winning? A position is now characterised by a triplet (n, k, l), where l is the *lead* of the player to play next. The lead is the difference between the number of pairs held by the two players. When a player is in the lead, or at least even (i.e., $l \geqslant 0$), her optimal moves are almost identical to those of the gain-maximising strategy. If a player is trailing behind, then she has no other choice but to take her chances and play 2-moves whenever 0-moves are suggested by the gain-optimising strategy. Obtaining an exact formulation of the optimal strategy here is an interesting problem.

What happens if more players join the game? The right move to make depends in this case mainly on $(n-k)$ mod p, where p is the number of players. The optimal move from position (n, k) in the three-player game, for example, is a 0-move if $n-k \equiv 2$ and $k \geqslant 3$, a 1-move if $n-k \equiv 0$ and $k \geqslant 1$, and a 2-move otherwise. All these congruences are, of course, modulo 3.

We assumed in this paper that the players have perfect memories. What happens if the players have imperfect memories?

7. Concluding remarks

The optimal strategy for playing the memory game turns out to be very simple. The analysis and proof presented here were however extremely involved. Is there an easier way of proving the results stated in Section 3?

While the results of this work are mainly of recreational value, we hope that the methods used here will prove useful elsewhere. We would like to stress again the indispensable role played in this work by experimentation and by automated symbolic computations.

Acknowledgment

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Note added in proof. After completing this work, we heard that Sabih H. Gerez and Frits Göbel [1] had previously considered the analysis of the memory game. They had empirically found the optimal strategy of Section 3 and explained parts of it theoretically. They also considered the version of the game in which 0-moves are not allowed. They discovered that in this version a surprising move optimises the expected profit from positions of the form $(n, n-1)$ where $n \geqslant 8$. In this move, a new card is flipped in the first ply. If this does not match any known card, a second new card is flipped. But if the first card flipped does match a known card then an old card *not* matching the first card is chosen in the second ply. This *sacrifice* deliberately leaves a matching pair on the table! The next player would collect this pair but then be in a similarly awkward position in which the sacrifice move is again optimal. With this new move $e_{n,n-1} \approx -3/2 + 5/2n$.

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