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The ethics of intergenerational risk

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Abstract

This paper addresses the evaluation of intergenerational allocations in an uncertain world. It axiomatically characterizes a class of criteria, named *reference-dependent utilitarian*, that assess allocations relative to a stochastic reference. The characterized criteria combine social concerns for ex-ante equity—capturing the idea that generations should be treated equitably before risk is resolved—and for ex-post fairness—capturing the idea that generations should be treated equitably after risk is resolved. Social discounting is endogenous and is governed by two opposite forces: extinction risk pushes society to reduce the weight on future generations, while (uninsurable) technological risk pushes society to increase the weight on future generations.

JEL classification: D63; D81; H43

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1. Introduction

Complex ethical choices emerge in an intergenerational setting where risk resolves gradually over time. The gradual resolution of risk enriches the standard intergenerational trade-off between present and future generations with a fundamental asymmetry: today's policies have immediate consequences on the present generations, but delayed and uncertain consequences on future generations. The goal of this paper is twofold: first, to propose principles of intergenerational justice for intergenerational settings with gradual resolution of risk; and second, to show

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the implications of these principles for the measurement of social welfare, for discounting, and for policy evaluation.

To illustrate, consider first a risk-less two-period economy. In period 0, a unit of income can be partially allocated for the consumption of generation 0, say x_0, and the remaining part can be saved for future use. In period 1, production transforms savings into output, which can only be allocated for the consumption of generation 1, say x_1. In the tradition of consequentialist distributive justice, "monotonicity" of social preferences captures the social concern for efficiency: the larger the consumption of generations, the greater social welfare. Instead, the "Pigou-Dalton principle" captures the social concern for equity: a transfer of consumption from a poorer generation to a wealthier one increases inequality and, therefore, cannot increase social welfare. Consider an additively separable—"utilitarian"—social welfare function, which measures social welfare at an allocation $x \equiv (x_t)_{t \in T}$ by[1]:

$$W(x) = \sum_t u(x_t) . \tag{1}$$

Monotonicity forces the function u to be increasing. The Pigou-Dalton principle forces u to be concave. An egalitarian distribution of consumption is one where $x_0 = x_1$. Under standard assumptions, the largest egalitarian allocation is also efficient. However, society might still want to deviate from it, if doing so allows assigning sufficiently more consumption to individuals. For the utilitarian criterion (1), this trade-off between equity and efficiency is governed by the concavity of u: the more concave u is, the less society is prone to deviate from equal consumption; in the limit where u is infinitely concave, the largest equal-consumption distribution is the most desirable allocation.

Let us now introduce a gradual resolution of risk. The technology at period 1 may be characterized by either high or low productivity with known probabilities; uncertainty is resolved at period 1. An allocation now assigns a consumption x_0 to generation 0 (before risk is resolved) and a consumption prospect $x_1 = \left(\overline{x}_1, \underline{x}_1\right)$ to generation 1 (after risk is resolved), where $\bar{x}_1$ and $\underline{x}_1$ are, respectively, the consumption in the states of high and low productivity. Monotonicity of social preferences remains a compelling principle: assigning more consumption to generations increases social welfare.[2] The Pigou-Dalton principle, instead, cannot be directly applied. First, with risk, the consumption prospect of later generations is a vector (rather than a scalar). Second, with a gradual resolution of risk, the vector of contingent consumption of later generations is a higher-dimensional vector than that of earlier ones. This reflects the fact that later distributional choices can depend on a larger information set; this also implies that there is an inescapable asymmetry across generations. Third, (dis)equalizing transfers—in the spirit of the Pigou-Dalton principle—require singling out when generations are treated equally, but here many egalitarian allocations emerge. Intuitively, when $x_0 = \overline{x}_1 = \underline{x}_1$ generations are treated equally, but such an allocation does not assign all resources and is, thus, inefficient. More generally, an egalitarian treatment of generations can be achieved when a smaller consumption in the

[1] I use the term "utilitarian" for all welfare criteria numerically represented by the sum of individuals' utility functions. This is a broader interpretation than usual. According to standard utilitarianism, the utility functions express the subjective well-being of individuals. Here, instead, the utility functions are chosen by the evaluator to compare allocations.

[2] This axiom differs from "monotonicity" in Bommier et al. (2017), also named "ordinal dominance" in Chew and Epstein (1990). Here, I only impose that social welfare is strictly increasing in consumption, a requirement related to efficiency. In contrast, "ordinal dominance" is a consistency requirement between the ranking of deterministic allocations and the ranking of each randomization over these deterministic allocations.

low-productivity state, i.e., $\underline{x}_1 < x_0$, can be compensated by a sufficiently large consumption in the high-productivity state, i.e., $\overline{x}_1 > x_0$.

To address these difficulties, I extend the logic of the Pigou-Dalton principle by introducing two types of (dis)equalizing transfers, each dealing with a different type of inequality. Let an *equitable allocation* be such that: *(i)* the consumption prospects of present and future generations are equal in terms of their certainty equivalent; *(ii)* such equality of certainty equivalents continues to hold as risk resolves over time; and *(iii)* all resources are assigned.[3] Then, an *ex-ante inequality* emerges if: *(i)* a generation is assigned a consumption prospect that is, in each state, smaller than that at the equitable allocation, while *(ii)* another generation is assigned a consumption prospect that is, in each state, larger than that at the equitable allocation. Then, *ex-ante equity* postulates that a transfer of consumption (constant across states) from the former to the latter increases ex-ante inequality and, therefore, cannot increase social welfare. An *ex-post inequality* emerges if, compared to the equitable allocation, it is more likely that some inequality across generations will emerge (an ex-post inequality is obtained by a mean-preserving spread, when the consumption prospect of a generation has more weight on the tails than at the equitable allocation). Then, *ex-post equity* postulates that a further increase in the likelihood of intergenerational inequalities cannot increase social welfare.

The main result of the paper studies the normative implications of these principles and establishes a novel family of welfare criteria for intergenerational justice, named *reference-dependent utilitarian*. The criterion measures social welfare at x by:

$$W(x;r) = E\left[\sum_{t\in T} \pi_t r_t v\left(\frac{x_t}{r_t}\right)\right], \tag{2}$$

where π_t is the probability of the existence of generation t, r_t is the consumption prospect of generation t at the equitable allocation, and v is a concave iso-elastic function. The concavity of v measures social aversion to ex-ante and ex-post inequalities.[4] When v is linear, society is indifferent to inequalities: the equitable allocation r becomes irrelevant and society discounts the future uniquely based on the probability of extinction. The more concave the function v is, the more weight society places on inequalities. In the limit where v is infinitely concave, society is infinitely inequality-averse and the equitable allocation is the most desirable allocation.

Expected utilitarianism—with iso-elastic utilities—emerges as a special case of the reference-dependent utilitarian criterion (2) when the reference is constant; discounting is then uniquely determined by the probability of extinction. Expected discounted utilitarian social welfare at allocation x is represented by:

$$W(x) = E\left[\sum_{t} \beta_t u(x_t)\right], \tag{3}$$

where $\beta_t \in (0, 1)$ is the discount factor for period t and u is a concave iso-elastic function. As before, the concavity of u measures the willingness of society to deviate from "perfect equality," i.e., allocations for which $x_0 = \overline{x}_1 = \underline{x}_1$. Crucially, perfect equality is here inefficient, since the productivity differs across states. Thus, in the limit where u is infinitely concave, society cannot

[3] Asheim and Brekke (2002) suggest a similar recursive formula to define the sustainable management of a risky resource. Note that the function defining the certainty equivalent remains an ethical choice of the planner.

[4] In Section 4, I show that relaxing the additive separability assumption permits society to have a different degree of aversion to ex-ante and ex-post inequalities. However, this leads to a time inconsistent welfare criterion.

choose such inefficient allocation of perfect equality. In contrast, the criterion selects an allocation where generation 0 cannot consume more than generation 1 in the low-productivity state, i.e., $x_0 = \underline{x}_1 < \overline{x}_1$. This result holds independently of the likelihood of the low-productivity state and independently of the consumption that generation 1 achieves in the high-productivity state. This allocation is clearly not egalitarian. As a result, one might wonder why the concavity of u does not actually "control" how equally the optimal allocation treats generations. The answer is in the tension between the efficiency and equity principles underlying the expected utilitarian criterion in a setting with gradual resolution of risk. Monotonicity forces society to assign all resources and accept that some generations are assigned a larger consumption, when more turns out to be available. The underlying equity principle disregards the timing of resolution of risk and the asymmetries between generations. It pushes society to avoid all types of inequalities, even those forced by monotonicity. Thus, equity and efficiency pull social evaluation in opposite directions and contribute to explaining the broad dissatisfaction with the policy implications of the expected utilitarian criterion.[5]

The standard approach to the evaluation of intergenerational risks relies on Harsanyi's (1955) characterization of expected utilitarianism, where generations replace individuals. Arguably, however, Harsanyi's setting is not the most appropriate to address long-term intergenerational risk. Risk resolves gradually over time and not in "one shot." This implies that generations face different risks and cannot be treated anonymously. Moreover, extinction is a fundamental aspect of intergenerational discounting, but has no place in Harsanyi's framework. To address these issues, I adopt a model of gradual resolution of risk similar to that of Kreps and Porteus (1978), but with two main differences. First, I allow for the possibility of extinction. Second, I assume that each generation t is born after the risk at t is resolved. All risk is then entirely borne by society, as in Asheim and Brekke (2002) and, more recently, Asheim and Zuber (2016).[6]

The reference-dependent utilitarian criterion includes as a special case the expected utilitarian criterion. In contrast to expected utilitarianism, however, the reference-dependent utilitarian criterion can accommodate both ex-ante and ex-post distributional concerns. In some situations, society might value ex-ante egalitarianism: it might prefer giving equal chances to two individuals rather than assigning a prize to one individual only, as advocated by Diamond (1967) and Epstein and Segal (1992) (see also Grant et al. (2010)). In other situations, society might instead value ex-post fairness: it might prefer assigning a prize to both individuals with 50% probability, rather than assigning the prize for sure, as advocated by Broome (1984), Fleurbaey (2010), and Grant et al. (2012). Despite the different setting, the ex-ante and ex-post distributional concerns are also desirable properties in an intergenerational context. However, Fleurbaey (2010) and Mongin and Pivato (2016) show that there is a strong tension between these concerns, forcing "standard" welfare criteria to choose between the two. This tension does not emerge here. The *reference-dependent utilitarian* criterion uses the equitable allocation to identify ex-ante or ex-post inequalities. This allows society to oppose only these inequalities and provides enough flexibility to avoid the above-mentioned difficulties. Importantly, both the ex-ante and ex-post welfare criteria suggested in the literature are time inconsistent (see Fleurbaey and Zuber, 2015).

[5] In Appendix A, I present a three-period example to discuss similar drawbacks of the ex-ante and ex-post welfare criteria—often indicated as compelling alternatives to expected utilitarianism—and compare them to the *reference-dependent utilitarian* criteria.

[6] Asheim and Zuber (2016) build on recent advances in the utility-streams literature on intergenerational justice, and in particular on the rank-discounted utilitarian criterion (Zuber and Asheim, 2012). They study how to rank social situations in which each potential individual is characterized by a utility level and a probability of existence.

In contrast, time inconsistency does not arise here unless society wants to disentangle ex-ante from ex-post inequality aversion, as discussed in Section 4.

This explicit relationship between the ranking of allocations and the resource distribution problem is new to the literature on intergenerational risk. The standard approach in the literature is to define "universal" criteria. These criteria evaluate allocations independently of the problem and are, thus, simple and analytically tractable. Yet, they seem unable to provide consensual policy recommendations. A different approach is to select "optimal allocations" for each problem, as in the fair allocation theory literature (see the recent survey by Thomson 2011).[7] This approach is flexible and provides policy recommendations tailored to the specific problems faced by society. The drawback is that "optimal" allocations are often of little help in second best situations, where fine-grained welfare criteria are more appropriate. In this paper, I integrate the two approaches. This *hybrid* approach combines the simplicity and analytical tractability of fine-grained welfare criteria with the flexibility of choosing optimal allocations.[8]

The proposed criterion characterizes discounting based on the risks faced by society. Two forces govern discounting here. The first one is due to the probability of extinction (as suggested by Stern, 2007). When the extinction probability is positive, postponing consumption is more costly: with positive probability, future generations might not benefit from earlier savings. The second force is due to the gradual resolution of risk. Later generations generally face more risk than earlier ones; thus, a risk-averse society would assign them a larger expected consumption at the equitable allocation. As society measures inequalities with respect to this allocation, it attributes larger weights to future generations when the uncertainty about the future is greater. Depending on which force prevails, the discount factor can be above or below 1.

Reference-dependent utilitarianism is related to two strands of behavioral economics. The first deals with individual preferences in intertemporal models with gradual resolution of risk and includes, among others, Kreps and Porteus (1978), Epstein and Zin (1989), and Weil (1990). As in Kreps and Porteus (1978), I highlight that the timing of resolution of risk matters for the ranking of allocations. As in Epstein and Zin (1989) and Weil (1990), I restrict the focus and obtain (more tractable) iso-elastic functional forms. The second strand studies the relationship between individual preferences and the specific context in which decisions are taken. As for reference-dependent utilitarianism, Koszegi and Rabin (2006) suggest that "reference-dependent preferences" depend not only on the assigned alternative, but also on the reference point adopted for the evaluation. In behavioral economic models, the reference is usually set to the status quo. In the present setting, the reference is given a normative justification and is singled out as the unique allocation that is both efficient and equitable, in the sense of Asheim and Brekke (2002).

Finally, the present approach is also related to Dhillon and Mertens' (1999) alternative to expected utilitarianism. Dhillon and Mertens suggest additively aggregating normalized von Neumann-Morgenstern utility functions, where each individual's utility is set to have infimum 0 and supremum 1 on a set of admissible prospects. They suggest the admissible prospects to be "limited only by feasibility and justice" (1999, p. 476), but do not specify how. A distinguishing feature of the present contribution is to formalize axiomatically how the welfare criteria should depend on the problem faced by society.

[7] Applications of fair allocation theory to intergenerational justice include, among others, Asheim, 1991, Asheim and Brekke, 2002, Asheim et al., 2010, Isaac and Piacquadio, 2015, Piacquadio, 2014.

[8] For a similar approach dealing with the aggregation of individuals' preferences, see Fleurbaey and Maniquet (2011); Piacquadio (2017) and, for the intergenerational setting, see Fleurbaey and Zuber (2017).

The structure of the paper is as follows. In the next section, I discuss the class of *reference-dependent utilitarian* welfare criteria. In Section 3, I formalize the model and the axioms and derive the main results. In Section 4, I discuss a number of extensions. The proofs are presented in the appendix.

2. A two-period example

To illustrate the *reference-dependent utilitarian* criteria, I consider the following class of two-period risky intergenerational problems (similar to Selden, 1978). In period 0, an amount $\omega > 0$ of a resource is available. This can be partly allocated for consumption of generation 0, say x_0, and the remaining part can be invested in capital, say k_1. When taking decisions at 0, society has probabilistic knowledge about the future. First, extinction can arise with positive probability: let $\pi \in (0, 1]$ be the likelihood that generation 1 exists. Second, the output available in period 1, i.e., Ak_1, depends on the realization of the productivity shock A, which is a positive random variable with a finite mean.[9] In period 1, a specific level of productivity is realized, say a. The output available is then ak_1. Since period 1 is the last, output can only be allocated for the consumption of generation 1, say x_1^a. I denote the contingent consumption of generation 1 by x_1: it is a mapping that associates a consumption x_1^a to each possible realization a of the productivity shock A. A risky intergenerational problem is then identified by the endowment ω, by the survival probability π, and by the distribution of the productivity shock A. An allocation (x_0, x_1) assigns a consumption level x_0 to generation 0 and a contingent consumption x_1 to generation 1.

A key feature of this class of problems is that intergenerational risk unfolds over time. Some decisions—such as the consumption and investment choices at period 0—are taken without knowing their exact effect on future generations. Other decisions—such as the consumption choices at period 1—are "more informed" as they can depend on the realization of the technology shocks. This has two main implications. First, generations are subject to different risks. Second, unless society is willing to waste resources when more turns out to be available, this uninsurable risk makes some intergenerational inequalities unavoidable.

2.1. The equitable allocation and its role

An equitable allocation answers the following question: How would an egalitarian society distribute resources across generations? The answer is here given by an allocation that satisfies the following two principles:

- *Efficiency.* An allocation cannot be the reference if there exists another feasible allocation that assigns at least as much consumption to each generation at each state of nature and more to some.
- *Recursive equity.* The consumption assigned to each generation at each state of nature is the certainty equivalent of the consumption lottery assigned to any later generations at states of nature that can still occur.

[9] The characterization result is developed in a setting where the number of states of nature and the number of periods is finite. The presence of infinite states of nature, as in this example, is without loss of generality. The results do not rely on the linearity (or even the convexity) of technology.

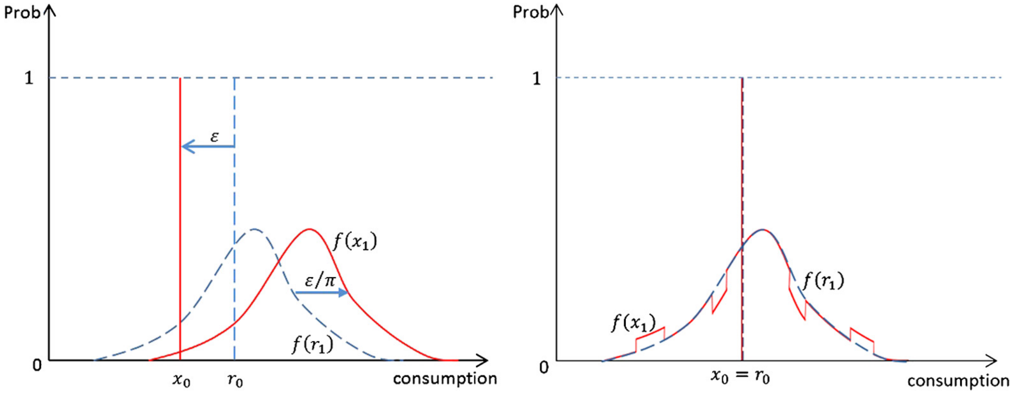

Fig. 1. Left: *Ex-ante equity* implies that (x_0, x_1) is socially less desirable than (r_0, r_1). Right: *Ex-post equity* implies that (x_0, x_1) is socially less desirable than (r_0, r_1).

Given *efficiency*, the equitable allocation is (r_0, r_1) such that $r_1 = A(\omega - r_0)$. Given *recursive equity*, the consumption level r_0 is the certainty equivalent of the contingent consumption r_1. Formally, society has to choose a real-valued function μ such that $r_0 = \mu^{-1}(E[\mu(r_1)])$. It is natural to select μ to be concave. The concavity of μ ensures that the consumption risk faced by later generations is compensated by a larger mean; the larger the concavity, the larger this compensation. When μ is linear, $r_0 = E[r_1]$ and no compensation for risk is imposed. In contrast, when μ is infinitely concave, no risk is acceptable and r_0 cannot exceed the smallest realization of r_1.

If a society is egalitarian, the optimal policy is identified by the equitable allocation (r_0, r_1). The egalitarian view might be too severe as the egalitarian allocation rules out possibly attractive growth opportunities. For example, a small reduction of the consumption of generation 0 might be sufficient to generate a large increase of the consumption in period 1. In these cases, society might want to relax the normative grip of the reference standard.[10]

Let generation 0 be assigned a consumption that is smaller than that at the equitable allocation, i.e., $x_0 < r_0$. Let generation 1 be assigned a larger consumption than that at the equitable allocation (in the sense of state-wise dominance, i.e., larger in each state of nature), denoted as $x_1 \gg r_1$. Then, society should consider generation 0 worse off than generation 1: there is an *ex-ante inequality* in the consumption of generations 0 and 1. Further, assume that the expected consumption assigned to the generations is unchanged, i.e., there exists $\varepsilon > 0$ such that $x_0 = r_0 - \varepsilon$ and $x_1 = r_1 + \frac{\varepsilon}{\pi}$ (the division by the probability of existence π accounts for the fact that generation 1 might not exist). These allocations are represented in Fig. 1 (left), where $f(r_1)$ and $f(x_1)$ are the probability density functions of r_1 and x_1 respectively, while r_0 and x_0 are certain and have a probability mass of 1. Then, (x_0, x_1) can be thought of as obtained from (r_0, r_1) by transferring consumption from generation 0 to generation 1. A society that is averse to ex-ante inequalities—or satisfies *ex-ante equity*—cannot prefer (x_0, x_1) to the equitable allocation (r_0, r_1).

The reference also guides society in the evaluation of *ex-post inequalities*. Let generation 0 be assigned the consumption $x_0 = r_0$. Generation 1 is instead assigned a consumption x_1, which happens to be a mean-preserving spread of the consumption r_1. Thus, in some states

10 I thank a referee for suggesting this interpretation.

of nature, generation 1 is assigned more than the consumption at the equitable allocation; in others, it is assigned less than the consumption at the equitable allocation. It follows that, independently of which state of nature arises, the allocation (x_0, x_1) implies a larger probability of intergenerational inequality than at the equitable allocation. A society that is averse to ex-post inequalities—or satisfies *ex-post equity*—cannot prefer (x_0, x_1) to the reference (r_0, r_1). This construction is represented in Fig. 1 (right).

2.2. The reference-dependent utilitarian criterion

In Section 3, I combine *ex-ante* and *ex-post equity* with other axioms and characterize the *reference-dependent utilitarian* criterion. These other axioms are common in the literature: "monotonicity" requires social welfare to increase with consumption; "continuity" says that small changes of the allocation should not cause large jumps in the level of social welfare; "separability" gives an additive structure to the representation; and "ratio-scale invariance" requires the ranking to be invariant to rescaling allocations. In the remaining part of this section, I illustrate and discuss the *reference-dependent utilitarian* criterion and contrast it with the expected discounted utilitarian criterion. A comparison with ex-ante and ex-post welfare criteria for a three-period example is discussed in Appendix A.

Given the equitable allocation $r \equiv (r_0, r_1)$, the *reference-dependent utilitarian* welfare at allocation $x \equiv (x_0, x_1)$ can be formulated as:

$$W(x; r) = \underbrace{v\left(\frac{x_0}{r_0}\right)}_{\text{welfare of gen. 0}} + \underbrace{\pi \frac{E[r_1]}{r_0}}_{\substack{\text{risk-adjusted} \\ \text{discount factor}}} \cdot \underbrace{\frac{E\left[r_1 v\left(\frac{x_1}{r_1}\right)\right]}{E[r_1]}}_{\text{welfare of gen. 1}}, \tag{4}$$

where $v(z) = (1-\rho)^{-1} z^{1-\rho}$ with $\rho \geq 0$ (with the standard log-form when $\rho = 1$).

Society evaluates the consumption assigned to each generation in proportion to what this generation would be assigned at the equitable allocation. To understand its implications, consider how social welfare changes around the equitable allocation.

Starting from r, consider a transfer of consumption ε across generations. The new allocation x is such that $x_0 = r_0 - \epsilon$ and $x_1 = r_1 + \frac{\varepsilon}{\pi}$. Note that the transfer is weighted by the probability of extinction to ensure that, in expected terms, the total consumption is unaffected. First, in the limit where $\varepsilon \to 0$, social welfare is unchanged and, consequently, the marginal rate of substitution between the expected consumption of the two generations is 1. This ensures that there is no discrimination among generations. Second, any non-marginal transfer reduces social welfare. This is consistent with the idea that, starting from an equitable distribution of resources, the only way to increase welfare is to assign more resources.

The **risk-adjusted discount factor** defined in (4) is central to achieving this result. Two forces govern discounting. The first one is due to the gradual resolution of risk. If later generations face more consumption risk than earlier ones, their assignment at the equitable allocation consists of a larger expected consumption $E[r_1] \geq r_0$ (by the concavity of the function μ). As a result, the larger the uncertainty about the future, the larger the weights that are attributed to future generations. The second force—moving in the opposite direction—is due to the probability of extinction. When the extinction probability is positive ($\pi < 1$), saving resources is more costly as it is not certain that future generations will benefit from it. This leads to attributing smaller weights to future generations. Depending on which force prevails, the discount factor can be above or below 1.

Starting from r, consider a mean-preserving spread in the assignment of generation 1. The new allocation x is such that $x_0 = r_0$ and $x_1 = r_1 + \varepsilon z$, where z is a zero-mean noise term. First, in the limit where $\varepsilon \to 0$, social welfare is unchanged. As a result, the marginal rate of substitution between the probability-weighted consumption assigned to generation 1 at any two different states of nature is 1. This ensures that no state of nature is more important than any other. Second, when ε is non-marginal, social welfare decreases. Again, this implies that the only way to increase social welfare is to assign more resources.

The sensitivity of social welfare to dis-equalizing transfers is measured by the **inequality-aversion parameter** ρ. When $\rho = 0$, society is indifferent to inequalities and any such dis-equalizing transfers leave social welfare unchanged. The larger the value of ρ, the more society is reluctant to redistribute resources across generations away from the equitable allocation. In the limit where $\rho \to \infty$, deviations from the equitable allocation are socially unacceptable.

Importantly, the role of the concavity of μ and the role of the inequality aversion parameter ρ are very different. The first establishes how much compensation is required for future generations due to the risk they face and, thus, identifies the equitable allocation. The second determines how society trades off efficiency and equity; i.e., it determines how society prioritizes between assigning a greater number of total resources and being close to the equitable allocation. This also implies that the ethical choices of μ and ρ are independent.

2.3. *The optimal distribution of resources*

I next discuss the implications of the *reference-dependent utilitarian* criterion for optimality by comparing its first order condition with that of the expected discounted utilitarian criterion. A society that maximizes (4) would optimally select an allocation $x^* \equiv \left(x_0^*, x_1^*\right)$ such that $x_1^* = A\left(\omega - x_0^*\right)$ and:

$$v'\left(\frac{x_0^*}{r_0}\right) = E\left[A\right]\pi v'\left(\frac{E\left[x_1^*\right]}{E\left[r_1\right]}\right). \tag{5}$$

In contrast, an expected utilitarian planner would optimally select an allocation $\hat{x} \equiv \left(\hat{x}_0, \hat{x}_1\right)$ such that $\hat{x}_1 = A\left(\omega - \hat{x}_0\right)$ and:

$$u'\left(\hat{x}_0\right) = \beta E\left[Au'\left(\hat{x}_1\right)\right], \tag{6}$$

where $\beta \in (0, 1)$ is the discount factor and u the utility function of the expected discounted utilitarian criterion.

The risk-less case. With probability 1, the productivity parameter is $a > 0$. Then, the reference r assigns the same consumption $r_0 = r_1 = a\left(\omega - r_0\right)$ to both generations. The first order condition (5) can be simplified to:

$$v'\left(x_0^*\right) = a\pi v'\left(x_1^*\right), \tag{7}$$

which is equivalent to (6) when $\beta = \pi$ and $u\left(z\right) = v\left(z\right)$. The first order condition (7) is a natural requirement for the optimal allocation. Society is willing to give more (less) consumption to generation 1 with respect to generation 0 if, for each additional unit of consumption saved in period 0, more (less) than one unit can be expected to become available to generation 1. Thus, $a\pi > 1$ implies that $x_1^* > x_0^*$ and, similarly, $a\pi < 1$ implies that $x_1^* < x_0^*$.

The risky case. With gradual resolution of risk, the two first order conditions diverge. To simplify the comparison, I assume that the expected discounted utilitarian discount factor corresponds to the survival probability, i.e., $\beta = \pi$, and that the expected discounted utilitarian evaluation function is $u(z) = (1-\eta)^{-1} z^{1-\eta}$. Then, (5) can be written as:

$$\left(\frac{x_0^*}{r_0}\right)^{-\rho} = E[A]\pi\left(\frac{\omega - x_0^*}{\omega - r_0}\right)^{-\rho}, \tag{8}$$

whereas (6) can be written as:

$$\left(\hat{x}_0\right)^{-\eta} = E\left[A^{1-\eta}\right]\pi\left(\omega - \hat{x}_0\right)^{-\eta}. \tag{9}$$

As before, the optimality condition for the *reference-dependent utilitarian* criterion (8) requires assigning a smaller (larger) consumption to generation 0 than at the equitable allocation if, for each unit of consumption saved in period 0, more (less) than one unit is expected to become available in period 1. Thus, $E[A]\pi > 1$ implies that $x_0^* < r_0$ and $x_1^* \gg r_1$. Symmetrically, $E[A]\pi < 1$ implies that $x_0^* > r_0$ and $x_1^* \ll r_1$. Intuitively, the simple structure of the problem limits the choice of society; if the optimal allocation is non-wasteful, society can only redistribute resources over time and not over states of nature. Thus, it seems natural to redistribute resources away from the equitable allocation to the period in which resources provide the largest expected consumption.

Consider now the expected discounted utilitarian criterion. The crucial aspect of the optimality condition (9) is that technological risk is accounted for through the term $E\left[A^{1-\eta}\right]$, i.e., the $1-\eta$ raw moment of the productivity shock A.[11] When $\eta = 0$, society maximizes the total expected consumption, as for the *reference-dependent utilitarian* criterion with $\rho = 0$. As η increases, society becomes more and more concerned with small realizations of the shock. In the limit where $\eta \to \infty$, society adopts the precautionary principle and selects x_0^* to be equal to the smallest possible realization of $x_1^* = A\left(\omega - x_0^*\right)$. In contrast, in the limit where $\rho \to \infty$, society selects the equitable allocation r.

2.4. *The social discount rate*

The social discount rate is the typical measure adopted in the economic literature to describe the importance today of a unit of expected consumption tomorrow. The social discount factor expresses the trade-off between the marginal change in a future period and the marginal change at period 0 that leaves social welfare unchanged. I consider here a stochastic version of the social discount rate, where a reduction of consumption assigned at period 0, i.e., dx_0, allows investing a fraction $d\varepsilon$ in a project with stochastic return A. For computational simplicity, I assume that the return on the investment A and the growth rate at x are jointly log-normally distributed. Finally, let the status-quo allocation be a non-wasteful distribution of resources x such that $x_1 = A(\omega - x_0)$.[12]

[11] Interestingly, when $\eta > 1$, the $1-\eta$ raw moment of the productivity shock A might not exist. This happens, for example, if A^{-1} is fat tailed. In such cases, the expected marginal benefit of an additional unit of consumption for generation 1 is unbounded. This problem does not emerge for the *reference-dependent utilitarian* criterion.

[12] Since x is a non-wasteful distribution of resources, the return on the investment A and the growth rate at x have perfect positive correlation. In this case, the formula for the stochastic social discount rate is significantly simplified, both for the expected utilitarian criterion and for the *reference-dependent utilitarian* criterion.

Then, the social discount rate for the *reference-dependent utilitarian* criterion is:

$$d\,(x;r) = \delta_\pi + \alpha + \rho\,(\bar{g}_x - \bar{g}_r)\,, \tag{10}$$

where $\delta_\pi \equiv -\ln \pi$ is the rate of pure time preference, $\alpha \equiv -\ln E\,[A]$ is the expected return of the stochastic project, and $\bar{g}_x$ and $\bar{g}_r$ are, respectively, the expected consumption growth rates at the allocation x and at the equitable allocation r. The social trade-off between consumption at different periods is the sum of three terms. The first term reflects the probability of extinction; the higher the probability π that generation 1 exists, the smaller the rate of pure time preference δ. The second term reflects the expected return of the stochastic project (in the literature, it is usually set to 0 by assuming that $E\,[A] = 1$). The third term is the product between the inequality aversion parameter ρ and the difference in expected growth between the allocation x and the reference r. If the growth rate at x is larger than that at r, i.e., $(\bar{g}_x - \bar{g}_r) > 0$, generation 1 is assigned a larger consumption than that at the reference, while generation 0 is assigned a smaller consumption than that at the reference. This justifies discounting the consumption of generation 1 at a higher rate. The larger the difference between the growth rates, the larger the priority society attributes to the worse-off generation. The degree to which society reacts to such a difference is measured by the inequality aversion parameter ρ.

The social discount rate for the expected discounted utilitarian criterion is:

$$d\,(x) = \delta_\beta + \alpha + \eta \bar{g}_x,$$

where $\delta_\beta \equiv -\ln \beta$ is the rate of pure time preference. The main difference with respect to (10) is the absence of $\bar{g}_r$. The *reference-dependent utilitarian* criterion justifies future generations receiving more consumption, on average, with respect to earlier ones due to the larger risks they face. The expected discounted utilitarian criterion does not. As a consequence, a growth rate of $\bar{g}_x = 2\%$ and a small inequality aversion (say $\eta = 1$) are sufficient to generate a significant social discount factor for the expected discounted utilitarian criterion ($d\,(x) > 2\%$). In contrast, even with a larger inequality aversion ρ, the social discount factor for the *reference-dependent utilitarian* criterion can be very small or even negative. In fact, it is largely determined by the difference in growth rates $\bar{g}_x - \bar{g}_r$. If the growth rate at x exceeds the equitable one, a lower weight on generation 1 is required; if the growth rate at x is too small, a higher weight on generation 1 is required.

3. The characterization result

3.1. A risky intergenerational problem

Time is discrete and the horizon is finite: $T \equiv \left\{0, ..., \bar{t}\right\}$, with $\bar{t} \geq 2$.[13] At period 0, a stock of capital $k_0 > 0$ is available. Production takes place and transforms capital into output. Let Φ denote the set of all production functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ that are strictly increasing, continuous, and satisfy $\phi\,(0) = 0$.[14] The output can be partly assigned for the consumption of the current one-period living generation or, for the remaining part, saved for use in later periods.

[13] Vector inequalities are defined as follows: $x \geq y \Leftrightarrow \left[x_i \geq y_i \;\forall i\right]$; $x > y \Leftrightarrow [x \geq y \text{ and } x \neq y]$; and $x \gg y \Leftrightarrow \left[x_i > y_i \;\forall i\right]$.

[14] Note that the technology need not be concave. This also implies that the results extend to introducing social concerns with respect to an increasing transformation of consumption (rather than the consumption level). For example, one could take the ethical stand that what matters for social welfare are generations' cardinal and interpersonally comparable utilities; the results are unaffected, although the interpretation of the axioms varies accordingly.

Two types of risk characterize later periods. First, extinction may arise before the end-period $\bar{t}$. Second, technology is randomly selected from Φ. To clarify, at each period, the decision concerning how to share output between consumption and investment is made without knowing whether later generations will exist and, conditional on existence, without knowing what technology will be available.

To formalize risk and its resolution over time, information disclosure takes the form of an event tree. An **event tree** N is a finite collection of nodes. Each node $n \in N$ is either associated with a technology $\phi^n \in \Phi$ or with extinction. At period 0 there is a unique initial node n_0: only technology ϕ^{n_0} is known, while no future risk is yet resolved so that all final nodes $N_{\bar{t}} \subset N$ can be reached from n_0. As time flows, risk resolves. At the final period $\bar{t} \in T$, the full history of technology and extinction is known. Without loss of generality, each final node $n \in N_{\bar{t}}$ is reached with positive probability $\pi^n \in (0, 1]$, with $\sum_{n \in N_{\bar{t}}} \pi^n = 1$.

At each period $t \in T$, society knows the realization of history until t. Let $N_t \subset N$ be the subset of nodes at t. Each node $n \in N_t$ is uniquely identified by the subset of final nodes $N_{\bar{t}}(n)$ that can be reached from n.[15] Extinction is irreversible: assume $n \in N$ is associated with extinction; then, each strict successor of n, i.e., $n' \in N(n)$, is also associated with extinction. Let $N^\ell \subseteq N$ be the tree obtained from N by eliminating the nodes that are associated with extinction. For each period $t \in \{1, ..., \bar{t}\}$, the number of non-extinction nodes is larger than 2.[16] Let $\pi_t \in (0, 1]$ be the (unconditional) existence probability of generation t.

An **allocation** x specifies a consumption x^n for each node $n \in N_t^\ell$ and each generation $t \in T$. By construction, the assignment of generation t at node n can only depend on the information available at n, which consists of: *(i)* the technology realized and the consumption and investment decisions taken until n; and *(ii)* the structure, the intensity, and the time resolution of technological risk, summarized by the event tree N. The domain of allocations is $X \equiv \mathbb{R}_{++}^{|N^\ell|}$.

An allocation $x \in X$ is **feasible** if there exists a saving plan $s \equiv (s^n)_{n \in N^\ell}$ such that: *(i)* for each period $t \in T$ and each node $n \in N_t^\ell$, $\phi^n(k^n) \geq x^n + s^n$; *(ii)* for each period $t \in \{1, ..., \bar{t}\}$ and each node $n \in N_t^\ell$, $k^n = s^{n^-}$, where n^- denotes the unique predecessor of n; and *(iii)* $k^{n_0} = k_0$. Let $X^f \subset X$ be the set of feasible allocations.

A **social ranking** is a complete and transitive ordering of allocations. Let R denote such ranking; then, $x\,R\,x'$ means that allocation x is socially at least as desirable as allocation x'. The strict preference relation P and the indifference relation I are the asymmetric and symmetric counterparts of R.

3.2. Identifying the reference

Society is guided by two main objectives of distributive justice when ranking allocations. The first is related to an effective use of resources. The most appealing allocation according to this objective can be defined as follows.

Efficiency: *An allocation $x \in X^f$ satisfies efficiency if there exists no allocation $x' \in X^f$ such that $x' > x$.*

[15] As standard in the literature, this requires that later partitions of possible histories be finer. Formally, for each $t \in T$, each $n \in N_t$, and each $n' \in N_{t+1}$, either $N_{\bar{t}}(n) \supseteq N_{\bar{t}}(n')$ or $N_{\bar{t}}(n) \bigcap N_{\bar{t}}(n') = \emptyset$.

[16] For the main result, it is sufficient that the total number of non-extinction nodes is larger than 2. Both restrictions can be avoided by introducing a consistency requirement with respect to the duplication of possible histories.

The second is related to equity. A natural requirement is the following.

Equality: *An allocation $x \in X^f$ satisfies equality if, for each $n, n' \in N^\ell$, $x^n = x^{n'}$.*

Unfortunately, these two objectives are compatible only in the absence of risk. The intuition is simple. Let $C \equiv \left\{ c \in \mathbb{R}_+ \middle| (c, ..., c) \in X^f \right\}$. Due to the assumptions made concerning technology, this set has a maximal element $\bar{c} \in C$. In the absence of risk, the allocation that assigns $\bar{c}$ to each generation satisfies *efficiency* and, by construction, also satisfies *equality*. The compatibility between *efficiency* and *equality* does not extend to the presence of risk. The example in Section 2 illustrates this difficulty. For the same resources saved in period 0, the amount of consumption that can be distributed at period 1 differs across states. Thus, any efficient distribution of resources cannot assign the same consumption to each generation at each node.

Giving priority to *efficiency*, I suggest weakening *equality*. The consumption assigned to each generation at each node should be as desirable as the lottery over consumption assigned to later generations, restricted to the states of nature that can still occur. This idea is closely related to the concept of sustainability proposed by Asheim and Brekke (2002).

Recursive equity: *An allocation $x \in X^f$ satisfies recursive equity if there exists a concave function $\mu : \mathbb{R}^+ \to \mathbb{R}$ such that for each $t, s \in T$ with $s > t$ and each $n \in N_t^\ell$, $x^n = \mu^{-1} \left(E \left[\mu \left(x_s \left(n \right) \right) \right] \right)$, where E is the expectation operator and $x_s(n)$ is the random variable that takes the value of $x^{\bar{n}}$ if $\bar{n} \in N_s^\ell(n)$ occurs.*

The following result states that efficiency is compatible with recursive equity and that it is unique up to the choice of μ in the definition of recursive equity. The proof is in the appendix.

Proposition 1. *There exist allocations satisfying efficiency and recursive equity. Moreover, each such allocation $r \in X^f$ is uniquely identified by the function μ satisfying the conditions of recursive equity.*

3.3. The social ranking: axioms

The first axiom is related to efficiency. Among two allocations, society prefers the one which assigns more consumption.

Monotonicity: *For each pair $x, \bar{x} \in X$, $x > \bar{x}$ implies that $x \, P \, \bar{x}$.*

Next, the social ranking is required to be continuous. Small changes of the allocation are associated with small changes in the level of social welfare.

Continuity: *For each $x \in X$, the sets $\{\bar{x} \in X \, | \bar{x} \, R \, x \}$ and $\{\bar{x} \in X \, | x \, R \, \bar{x} \}$ are closed.*

The next two axioms are central to the analysis of intergenerational ethics. They convey the idea that *some* inequalities, measured by contrast to the reference, are bad for society and reduce (or at least cannot increase) social welfare.

The first deals with ex-ante inequalities. Comparing the assignments of two generations, say t and t', there is an **ex-ante inequality** if, at each state of nature, t is assigned more than the consumption at the equitable allocation, while t' is assigned less than that at the equitable allocation. Generation t is then considered better-off than generation t'. The next axiom requires that society does not rank higher allocations with larger ex-ante inequality. The formalization is similar in spirit to Dalton's (1920) transfer principle.

Ex-ante (intergenerational) equity: *For each pair $x, \bar{x} \in X$, each pair $t, t' \in T$, and each $\varepsilon \in \mathbb{R}_+$, if*

(i) $x^n = \bar{x}^n - \frac{\varepsilon}{\pi_t} \geq r^n$ *for each* $n \in N_t^\ell$;
(ii) $x^n = \bar{x}^n + \frac{\varepsilon}{\pi_{t'}} \leq r^n$ *for each* $n \in N_{t'}^\ell$;
(iii) $x^n = \bar{x}^n$ *for each* $n \in N^\ell \setminus \left\{N_t^\ell \bigcup N_{t'}^\ell\right\}$;
then $x \, R \, \bar{x}$.

The axiom reads as follows. At allocation $\bar{x}$, generation t is assigned a larger consumption than that at the equitable allocation in each state (Condition *i*); generation t' is assigned a smaller consumption than that at the equitable allocation in each state (Condition *ii*). Define a transfer ε from t to t' that is weighted by the respective extinction-probabilities and uniform across states, such that the ex-ante inequality is only reduced (but not overturned) by the transfer. Then, *ceteris paribus* (Condition *iii*), the after-transfer allocation x is socially at least as desirable as allocation $\bar{x}$.

The second equity axiom deals with ex-post inequalities. Assume all generations but t are assigned the consumption corresponding to the reference. An **ex-post inequality** emerges when generation t is assigned more consumption than that at the equitable allocation at one node and less than that at the equitable allocation at another node. If the first node is reached, generation t will be better off than later generations; if the latter node is reached, generation t will be worse off than later generations. In either case, some intergenerational inequality occurs. The next axiom requires that society does not rank higher allocations with larger ex-post inequalities. The formalization is similar to a mean preserving spread (Rothschild and Stiglitz, 1970).[17] For each $t \in T$ and each $n \in N_t^\ell$, the unconditional probability that node n is reached is $\pi^n \equiv \sum_{\bar{n} \in N_{\bar{t}}(n)} \pi^{\bar{n}}$.

Ex-post (intergenerational) equity: *For each pair $x, \bar{x} \in X$, each $t \in T$, each pair $n, n' \in N_t^\ell$, and each $\varepsilon \in \mathbb{R}_+$, if*

(i) $x^n = \bar{x}^n - \frac{\varepsilon}{\pi^n} \geq r^n$;
(ii) $x^{n'} = \bar{x}^{n'} + \frac{\varepsilon}{\pi^{n'}} \leq r^{n'}$;
(iii) $x^{\tilde{n}} = \bar{x}^{\tilde{n}} = r^{\tilde{n}}$ *for each* $\tilde{n} \in N^\ell \setminus \{n, n'\}$;
then $x \, R \, \bar{x}$.

The axiom reads as follows. At allocation $\bar{x}$, the assignment of generation t at node n is larger than that at the equitable allocation (Condition *i*); the assignment of generation t at node n' is instead smaller than that at the equitable allocation (Condition *ii*). At all other nodes, generations are assigned the equitable consumption prospect both at x and $\bar{x}$ (Condition *iii*). Define a transfer ε from n to n' that is weighted by the probability that these nodes occur, such that the ex-post

[17] The mean preserving spread is obtained by transferring probability mass to the tales of the distribution, but can be equivalently expressed as a regressive transfer across states of nature, weighted by the likelihood of each. See Atkinson (1970).

inequality is only reduced (but not overturned) by the transfer. Then, the after-transfer allocation x is socially at least as desirable as the initial one $\bar{x}$.

Next, the social ranking should be invariant to scale changes in consumption. This axiom ensures that society's distributional concern is limited to the relative level (and not the absolute level) of generations' assignments.[18]

Ratio-scale invariance: *For each pair $x, \bar{x} \in X$ and each $\alpha > 0$, $x \, R \, \bar{x}$ if and only if $\alpha x \, R \, \alpha \bar{x}$.*

The axiom of *ratio-scale invariance* extends the applicability of *ex-ante* and *ex-post equity* and, thus, acts as a "lifting property." When, for example, two generations receive less consumption than that at the equitable allocation, *ex-ante equity* does not apply and one cannot establish whether a transfer between them improves social welfare. Given *ratio-scale invariance*, one can rescale the allocations (before and after the transfer) and ensure that one generation is assigned more and one less consumption than that at the equitable allocation. *Ex-ante equity* then applies to the rescaled allocation and the effect of the transfer on social welfare can be established.

The last axiom introduces additive separability to the evaluation. This requires the ranking of allocations to be independent of the consumption assignment at an unaffected node. Note that this axiom also implies that the ranking is time consistent.[19] I show the implications of weaker forms of separability in Section 4.

Separability: *For each $x, \bar{x}, \tilde{x}, \hat{x} \in X$ and each $t \in T$ and each $n \in N_t^\ell$ such that:*
(i) *$x^n = \bar{x}^n$ and $\tilde{x}^n = \hat{x}^n$;*
(ii) *$x^{n'} = \tilde{x}^{n'}$ and $\bar{x}^{n'} = \hat{x}^{n'}$ for each $n' \in N^\ell \setminus \{n\}$;*
then $x \, R \, \bar{x}$ if and only if $\tilde{x} \, R \, \hat{x}$.

Ratio-scale invariance and separability are demanding requirements. Yet, they have valuable implications and are, thus, common in the literature. First, they provide informational parsimony; the comparison of allocations only requires information about the relative consumption of the generations affected by the choice. Second, they significantly simplify the social welfare function and, thus, can be applied more easily for the computation of optimization problems. Finally and most importantly, they ensure tractability of the representation result and help to elucidate the effects of the equitable allocation on the social evaluation.

3.4. The social ranking: representation result

Let v be a real-valued and concave iso-elastic function, i.e., for each $z \in \mathbb{R}_+$, $v(z) = z^{1-\gamma}$ if $\gamma \neq 1$ and $v(z) = \ln z$ if $\gamma = 1$. The **reference-dependent utilitarian social welfare function** W is defined by setting for each $x \in X$:

[18] As clarified by Blackorby and Donaldson (1982), this axiom "involves picking an interpersonally significant norm such as a poverty line...and the positivity restriction prevents the use of this information by assuming that everyone is above..." this poverty line. In the present setting, this poverty line corresponds to a consumption level equal to 0 and can be interpreted as the consumption associated with a life barely worth living.

[19] Time consistency is the requirement that the ranking of allocations does not change as the perspective of the decision maker changes from the initial period $t = 0$ to some later period. It thus implies that a decision maker will not have any incentive to deviate from her original plan after some risk is resolved.

$$W(x;r) = E\left[\sum_{t\in T} \pi_t r_t v\left(\frac{x_t}{r_t}\right)\right]. \tag{11}$$

A social ranking is **reference-dependent utilitarian** if it can be numerically represented by a reference-dependent utilitarian social welfare function W as defined in (11). That is, for each pair of allocations $x, \bar{x} \in X$:

$$x\, R\, \bar{x} \Leftrightarrow W(x;r) \geq W(\bar{x};r).$$

The main result establishes the equivalence between the axioms introduced above and the *reference-dependent utilitarian* criterion.[20] The proof is in the appendix.

Theorem 1. *A social ranking satisfies monotonicity, continuity, ex-ante equity, ex-post equity, ratio-scale invariance, and separability if and only if it is reference-dependent utilitarian.*

4. Extensions

4.1. Ex-ante and ex-post inequality disentanglement

I next show that introducing two weaker separability axioms allows the welfare criteria to disentangle ex-ante and ex-post inequality aversion.

The first separability axiom is over time. Assume that the assignment of generation t is the same at two allocations x and $\bar{x}$. Then, the social ranking of these allocations should not depend on this generation's assignment. This requirement is closely related to the "independence of the utility of the dead" (Blackorby et al., 2005).

Intergenerational separability: *For each $x, \bar{x}, \tilde{x}, \hat{x} \in X$ and each $t \in T$ such that:*

(i) $x^n = \bar{x}^n$ *and* $\tilde{x}^n = \hat{x}^n$ *for each* $n \in N_t^\ell$;
(ii) $x^n = \tilde{x}^n$ *and* $\bar{x}^n = \hat{x}^n$ *for each* $n \in N^\ell \backslash N_t^\ell$;
then $x\, R\, \bar{x}$ *if and only if* $\tilde{x}\, R\, \hat{x}$.

The second separability condition is across nodes, but within a period of time. Consider two allocations x and $\bar{x}$ that assign the same consumption to each generation apart from generation t. Furthermore, the assignment of generation t at a node n is unaffected by the choice. Then, the social ranking of these allocations should not depend on this generations' assignment at node n.

Intragenerational separability: *For each $x, \bar{x}, \tilde{x}, \hat{x} \in X$, each $t \in N$, and each $n \in N_t^\ell$ such that:*

(i) $x^n = \bar{x}^n$ *and* $\tilde{x}^n = \hat{x}^n$;
(ii) $x^{n'} = \tilde{x}^{n'}$ *and* $\bar{x}^{n'} = \hat{x}^{n'}$ *for each* $n' \in N_t^\ell \backslash \{n\}$;
(iii) $x^{n''} = \bar{x}^{n''} = \tilde{x}^{n''} = \hat{x}^{n''}$ *for each* $n'' \in N^\ell \backslash N_t^\ell$;
then $x\, R\, \bar{x}$ *if and only if* $\tilde{x}\, R\, \hat{x}$.

[20] This result is independent of the choice of the reference. More precisely, the *reference-dependent utilitarian* criterion does not rely on the axioms of *efficiency* and *recursive equity* for the selection of the reference. For instance, combining *equality* with a weakening of *efficiency*, the reference would assign the same consumption at each node. The corresponding *reference-dependent utilitarian* criterion would then be the iso-elastic additive criterion defined in (3).

Let v and w be two real-valued and concave iso-elastic functions. Let the social welfare function W be defined by setting for each $x \in X$:

$$W(x;r) = \sum_{t \in T} \pi_t w\left(v^{-1}\left(E\left[r_t v\left(\frac{x_t}{r_t}\right)\right]\right)\right). \tag{12}$$

A social ranking is **reference-dependent utilitarian with disentanglement** if it can be numerically represented by a social welfare function (12). Clearly, when $w = v$, the *reference-dependent utilitarian* criterion emerges as a special case. The next result characterizes this criterion.

Theorem 2. *A social ranking satisfies monotonicity, continuity, ex-ante equity, ex-post equity, ratio-scale invariance, intergenerational separability, and intragenerational separability if and only if it is reference-dependent utilitarian with disentanglement.*

While allowing for more flexibility in the treatment of generations, this criterion fails to be time consistent. Time consistency postulates that when a criterion is applied at different times and different (partial) realizations of risk, the ranking of allocations should be unchanged. Here, the failure to satisfy time consistency is similar to that discussed in Epstein and Segal (1992): their criterion is a special case of (12) when the equitable allocation is constant and $w \circ v^{-1}$ is quadratic.

4.2. Variable population

The model assumes that population is constant (net of extinction). I next discuss how to adapt the transfer principles to account for the number of individuals at each node. For each node $n \in N_t^\ell$ and each generation $t \in T$, let $L^n > 0$ denote the mass of identical individuals alive at node n, let $\bar{L}_t \equiv \sum_{n \in N_t^\ell} \pi^n L^n$ denote the expected number of individuals of generation t, and let x^n denote per-capita consumption. Writing the allocation $x \in X$ in per-capita terms ensures that the definition of the equitable allocation extends to this setting. In particular, society considers two generations equally well off when their per-capita consumption prospects are attached the same certainty equivalent.

I can thus proceed with modifying the axioms. The main difference is that to ensure that the allocations "before" and "after" the transfer allocate the same amount of total resources, the transfer needs to be weighted by the population size.

Ex-ante population-weighted equity: *For each pair $x, \bar{x} \in X$, each pair $t, t' \in T$, and each $\varepsilon \in \mathbb{R}_+$, if*

(i) $x^n = \bar{x}^n - \frac{\varepsilon}{\pi_t L_t} \geq r^n$ *for each* $n \in N_t^\ell$;
(ii) $x^n = \bar{x}^n + \frac{\varepsilon}{\pi_{t'} L_{t'}} \leq r^n$ *for each* $n \in N_{t'}^\ell$;
(iii) $x^n = \bar{x}^n$ *for each* $n \in N^\ell \setminus \{N_t^\ell \bigcup N_{t'}^\ell\}$;
then $x \, R \, \bar{x}$.

Similar, the axiom of *ex-post equity* becomes:

Ex-post population-weighted equity: *For each pair $x, \bar{x} \in X$, each $t \in T$, each pair $n, n' \in N_t^\ell$, and each $\varepsilon \in \mathbb{R}_+$, if*

(i) $x^n = \bar{x}^n - \frac{\varepsilon}{\pi^n L^n} \geq r^n$;

(ii) $x^{n'} = \bar{x}^{n'} + \frac{\varepsilon}{\pi^{n'} L^{n'}} \leq r^{n'}$;
(iii) $x^{\tilde{n}} = \bar{x}^{\tilde{n}} = r^{\tilde{n}}$ *for each* $\tilde{n} \in N^{\ell} \setminus \{n, n'\}$;
then $x \, R \, \bar{x}$.

Let the **reference-dependent utilitarian criterion for exogenous population dynamics** be a social ranking that can be represented by:

$$W(x; r) = E\left[\sum_{t \in T} \bar{L}_t R_t v\left(\frac{x_t}{r_t}\right)\right], \tag{13}$$

where R_t is a random variable that, for each node $n \in N_t^{\ell}$, specifies the total consumption $L^n r^n$ at the equitable allocation r. The next result extends the characterization of reference-dependent utilitarianism to exogenous population dynamics. Since the population size enters the model in the same way as the existence probability, the proof of Theorem 1 can be easily extended.[21]

Theorem 3. *A social ranking satisfies monotonicity, continuity, ex-ante population-weighted equity, ex-post population-weighted equity, ratio-scale invariance, and separability if and only if it is reference-dependent utilitarian for exogenous population dynamics.*

4.3. Infinite horizon

The above model has a finite time horizon. This is not without loss of generality. However, difficulties with an infinite time horizon are well known and emerge already in a risk-less setting. In the following, I briefly discuss how to extend the present analysis to an infinite-horizon setting.

The first challenge pertains the definition of the equitable allocation. Even without risk, equal treatment of generations and efficiency are incompatible. The intuition is immediate. Assume that a non-renewable resource has to be shared among infinite generations. Any strictly positive consumption assigned equally to all generations is unfeasible. In contrast, assigning zero consumption to all generations is wasteful. Therefore, generations cannot be treated equally at efficient allocations. This tension disappears if technology is sufficiently productive, as assumed in Asheim et al. (2001). Thus, to obviate this difficulty, one possibility is to restrict the technology to ensure that, eventually, the return to savings is sufficiently high and the existence of equitable and efficient allocations is guaranteed. An alternative is to weaken the axiom of recursive equity to allow for some discrimination if the technology cannot sustain equality.

Independent of the choice of the reference, the second challenge is the definition of a continuous social ranking that combines equity and efficiency. The impossibility of combining these desiderata is known since the seminal work of Diamond (1965) (see also Asheim, 2010).[22] The solutions proposed in the literature include relaxing: *(i)* the completeness of the social ranking;

[21] Note that the reference-dependent utilitarian criterion with constant population (11) can be rewritten as $W(x;r) = \sum_{t \in T} \pi_t \sum_{n \in N_t^{\ell}} \pi^n r^n v\left(\frac{x^n}{r^n}\right)$, while with exogenous population dynamics (13) can be rewritten as $W(x;r) = \sum_{t \in T} \bar{L}_t \sum_{n \in N_t^{\ell}} \pi^n L^n r^n v\left(\frac{x^n}{r^n}\right)$. Since $\pi_t = \sum_{n \in N_t^{\ell}} \pi^n$ and $\bar{L}_t = \sum_{n \in N_t^{\ell}} \pi^n L^n$, the population size enters the formula in the same way as the existence probability. This directly follows from the definitions of *ex-ante* and *ex-post population-weighted equity*.

[22] Note that Diamond's anonymity requirement is here implied by ex-ante equity and separability for situations without risk.

(ii) continuity; *(iii)* efficiency; or *(iv)* equity. I leave to future work the extension of the *reference-dependent utilitarian* criterion to infinite horizon settings.

5. Conclusions

In the literature, welfare issues involving intergenerational risks are generally addressed by analogy to Harsanyi's (1955) pioneering contribution to the evaluation of risky social situations. Agents are simply reinterpreted as generations and time discounting is added. I claim that such an approach disregards essential aspects of intergenerational risks. First, risk resolves gradually over time. Second, a gradual resolution of risk exposes generations to different types and quantity of risk. Third, intergenerational risk is largely uninsurable. Consequently, some inequalities across generations are inevitable.

The principles of justice introduced here are inspired by the Pigou-Dalton principle and take into account these aspects of intergenerational risk. The first step is to choose a reference allocation. This allocation is identified as the most equitable among the efficient allocations. It thus accounts for the time resolution of risk, the heterogeneous risk faced by different generations, and the unavoidable inequalities among generations. The second step is to assess allocations by contrast to this reference. More specifically, the main principles introduced here indicate that more inequalities—as measured with respect to the reference—cannot improve social welfare.

The axiomatic analysis singles out the class of *reference-dependent utilitarian* criteria. In contrast to previous proposals, these criteria avoid serious drawbacks of expected discounted utilitarianism. In particular, these criteria can jointly accommodate social concerns for ex-ante equity—capturing the idea that generations should be treated equitably before risk is resolved—and for ex-post fairness—capturing the idea that generations should be treated equitably after risk is resolved.

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Appendix A. A three-period example

Let $\omega = 3$ be the endowment at period 0. The consumption of generation 0 and investment need to satisfy $x_0 + k_1 \leq 3$. At period 1, risk resolves and one of two states is realized. With probability π^G, a high productivity state G is realized: the consumption of generation 1, x_1^G, and of generation 2, x_2^G, need to satisfy $x_1^G + x_2^G \leq 5k_1$. With probability $\pi^B = 1 - \pi^G$, a low productivity state B is realized: the consumption of generation 1, x_1^B, and of generation 2, x_2^B, need to satisfy $(1+\varepsilon)\, x_1^B + (1-\varepsilon)\, x_2^B \leq k_1$. The parameter $\varepsilon \in (0, 1)$ introduces a technological asymmetry between periods 1 and 2. This economy is characterized by two feasibility constraints:

$$\begin{cases} 5\,(3-x_0) \geq x_1^G + x_2^G \\ (3-x_0) \geq (1+\varepsilon)\,x_1^B + (1-\varepsilon)\,x_2^B, \end{cases}$$

which also express the timing of resolution of risk, since the consumption of generation 0 cannot be state-specific.

A.1. Ex-ante, ex-post, and reference-dependent criteria

In the following, I compare the policy recommendations of the *reference-dependent utilitarian* criterion with that of the ex-ante welfare criterion (Grant et al., 2010) and the ex-post welfare criterion (Fleurbaey, 2010; Grant et al., 2012), of which the expected utilitarian criterion is a special case. Due to the absence of extinction risk, I disregard discounting.

The ex-ante and the ex-post criteria capture their specific ethical concerns in the choice of two concave functions, f and u. The ex-ante welfare criterion can be defined by:

$$W^{ea}(x) = \sum_{t\in\{0,1,2\}} f \circ \underbrace{u^{-1}\left(E\left[u\left(x_t\right)\right]\right)}_{\text{certainty equivalent}}, \tag{A.1}$$

where E denotes the expectation taken over the states of nature $s \in \{G, B\}$. Ex-ante social welfare is the sum of a concave transformation of each generation's certainty equivalent. When $f = u$, the expected utilitarian criterion emerges.

The ex-post welfare criterion W^{ep} can be defined as:

$$W^{ep}(x) = E\left[u \circ \underbrace{f^{-1}\left(\sum_{t\in\{0,1,2\}} f\left(x_t\right)\right)}_{\text{equally distributed equivalent}}\right].$$

Ex-post social welfare is the expected utility associated with the equally-distributed equivalent. Again, when $f = u$, the expected utilitarian criterion emerges.

Finally, the *reference-dependent utilitarian* criterion is represented by:

$$W(x;r) = \sum_{t\in\{0,1,2\}} E\left[r_t v\left(\frac{x_t}{r_t}\right)\right],$$

where $v(z) = (1-\rho)^{-1} z^{1-\rho}$ with $\rho \geq 0$. The expected utilitarian criterion (with power functions) emerges as a special case if the reference is constant over time and states.[23]

A.2. Concavity, risk, and inequality aversion

In the following, I illustrate the role of concavity of the functions f and u for the ex-ante and ex-post criteria and contrast it to the role of concavity for the functions μ (identifying the reference) and v for the *reference-dependent utilitarian* criterion. It is well known that introducing a

[23] This endogenously emerges if technological risk is "perfectly insurable," i.e., there exist investment choices such that an allocation assigning the same consumption to each generation at each state is efficient.

concave transformation to additively aggregate components is equivalent to averting variation in such components.

For the ex-ante criterion, the concavity of f measures aversion to (purely) ex-ante intergenerational inequality, where inequality is measured with respect to the certainty equivalent at each period; the concavity of u is a measure of aversion to risk, where risk is described in terms of the variation across states of each period's prospect. When the concavity of f increases, society becomes less willing to accept (purely) ex-ante inequalities across generations. In the limit where f is infinitely concave, society is (purely) ex-ante egalitarian and all generations achieve the same certainty equivalent. However, maximizing the lowest certainty equivalent is less egalitarian than it seems. While $x_0 = u^{-1} (E [u (x_t)])$ for each t, this is achieved by letting $x_1^G > x_2^G$ and $x_1^B < x_2^B$. At period 1 in the high productivity state G, the (purely) ex-ante egalitarian society justifies assigning more to generation 1 than to generation 2 based on the counterfactual scenario B where the reverse could have happened. Yet, when choosing how to allocate resources in state G, society knows that state B cannot occur. The ex-ante criterion disregards the timing of resolution of uncertainty and is time inconsistent.

For the ex-post criterion, the concavity of u is also a measure of aversion to risk, but now risk is described in terms of the variation across states of the equally-distributed equivalent at each state. The concavity of f is a measure of aversion to (purely) ex-post intergenerational inequality, where inequality is measured with respect to the variation across time of the assignments at each state. When the concavity of f increases, society becomes less willing to accept (purely) ex-post inequalities across generations. In the limit where f is infinitely concave, society is (purely) ex-post egalitarian: in state B, all generations achieve the consumption $x_0 = x_1^B = x_2^B = 1$; in state G and since x_0 is already determined, society can only avoid inequalities between the remaining generations and sets $x_1^G = x_2^G = 5$. As a result, the (purely) ex-post egalitarian society disregards the probabilities of the states and adopts a "precautionary principle" perspective. No matter how (un-)likely state B is, the assignment of generation 0 cannot be larger than the lowest assignment in state B. Also, the ex-post criterion disregards the timing of resolution of uncertainty and is time inconsistent.

The *reference-dependent utilitarian* criterion avoids these drawbacks by allowing the evaluation to depend on the endogenous reference that is both ex-ante and ex-post egalitarian. The reference is ex-ante egalitarian as it demands that $r_0 = \mu^{-1} (E [\mu (x_t)])$ for each t. It is also ex-post egalitarian as it imposes the same restriction after any risk is resolved: for this simple example, the recursive equity requirement leads to $r_1^G = r_2^G$ and $r_1^B = r_2^B$. Moreover, by setting the concavity of μ, society determines how much to compensate later generations for facing the technological risk of states G and B. This compensation is also reflected in the risk-adjusted discount factor $(r_0)^{-1} E [r_t]$ which increases with the concavity of μ. Given this equitable allocation as a reference, the concavity of v (measured by the inequality aversion parameter ρ) controls how much society is willing to deviate from the reference. Note that this criterion takes into account the timing of resolution of uncertainty through the reference while, by being additively separable, it remains time consistent.

Appendix B. Proofs

B.1. Proposition 1

Proof. I first show existence. Let $X^{f+} \subset \mathbb{R}_+^{|N^\ell|}$ be such that $x \in X^{f+}$ if there exists $x' \in X^f$ with $x' \geq x$.

Let μ be a concave function $\mu : \mathbb{R}^+ \to \mathbb{R}$. Define $X^{RE} \subseteq X^{f+}$ as the subset of allocations satisfying the conditions of *recursive equity* for function μ. That is, for each $t, s \in T$ with $s > t$ and each $n \in N_t^\ell$, $x^n = \mu^{-1}\left(E\left[\mu\left(x_s\left(n\right)\right)\right]\right)$. Let

$$C_0 \equiv \left\{ c \in \mathbb{R}_+ \middle| x_0 = c \text{ for some } x \in X^{RE} \right\}.$$

The set C_0 is non-empty: by assumption $X^f \neq \emptyset$ and, since the production functions are strictly increasing, concave, and satisfy no free lunch, there exists $x \in X^f$ and $c > 0$ such that $x^n = c$ for each $n \in N^\ell$; thus, $x_0 = c \in C_0$. The set C_0 is bounded: this immediately follows from X^f and X^{f+} being bounded. The set C_0 is compact: this follows from the continuity of technology F, the concavity of function μ, and the compactness of X^{f+}. Let $x^* \in X^{RE}$ be such that x_0^* is the maximal element of C_0. By construction, x^* satisfies *recursive equity*. By contradiction, assume that x^* does not satisfy *efficiency*; then, there exists $x' \in X^f$ such that $x' > x$. Due to the mentioned assumptions on technology, there exists a $x'' \in X^{RE}$ such that $x'' \gg x^*$, contradicting x_0^* being a maximal element of C_0. Finally, since $x_0^* > 0$ and technology is strictly increasing and continuous, $x^* \gg 0$ and, thus, $x^* \in X^f$.

I next show that μ identifies a unique reference. By contradiction, assume there exists a pair $x, \bar{x} \in X^f$ with $x \neq \bar{x}$ that satisfies *efficiency* and *recursive equity* for the same function μ. Let $t \in T$ be the first period for which $x^n \neq \bar{x}^n$ for some $n \in N_t^\ell$. If $t = 0$, $x_0 \gtrless \bar{x}_0$ and the same argument as above leads to a contradiction of *efficiency* for one of the two allocations. Assume $t > 0$ and define:

$$X^{RE}(n) \equiv \left\{ \hat{x} \in X^{RE} \middle| \hat{x}^{n'} = x^{n'} = \bar{x}^{n'} \text{ for each } n' \in N_{t'}^\ell \text{ with } t' < t \right\}, \text{ and}$$

$$C(n) \equiv \left\{ c \in \mathbb{R}_+ \middle| c = \hat{x}^n \text{ for some } \hat{x} \in X^{RE}(n) \right\}.$$

Again, the same reasoning as that applied for C_0 allows concluding that either $x^n = \bar{x}^n$ or one of the two allocations does not satisfy *efficiency*. This constitutes a contradiction. □

B.2. Theorems 1 and 2

Part 1. *If a social ranking satisfies the axioms, then it is reference-dependent utilitarian (with disentanglement).*

Note that *separability* implies *intergenerational* and *intragenerational separability*. I thus first show the implications of the axioms of Theorem 2 and, at the end, impose *separability* to prove Theorem 1.

The argument is divided into several steps. Assume that the social ranking satisfies *monotonicity, continuity, ex-ante equity, ex-post equity, ratio-scale invariance, intergenerational separability,* and *intragenerational separability.*

The first step shows that the social ranking R admits a specific functional representation, which is continuous, increasing, and additive over time and, for each period, additive across nodes. This is an implication of *monotonicity, continuity, intergenerational separability,* and *intragenerational separability.*

Step 1. *For each $t \in T$ and each $n \in N_t^\ell$, there exist continuous and strictly increasing functions q_t and $\bar{v}^n$ such that R is represented by:*

$$V(x; r) = \sum_{t \in T} q_t \left(\sum_{n \in N_t^\ell} \bar{v}^n \left(x^n\right) \right). \tag{B.1}$$

Proof. By assumption, there are at least three periods and, for each period, there are at least three nodes of non-extinction. *Monotonicity* implies the axiom of “strict essentiality”.[24] Thus, Gorman’s (1968) theorem on overlapping separable sets applies: “strict essentiality”, *continuity*, and *intergenerational separability* imply that there exist continuous functions $\tilde{q}_t$ (one for each $t \in T$) such that R can be represented by $V(x;r) = \sum_{t\in T} \tilde{q}_t \left(\{x^n\}_{n\in N_t^\ell} ; r \right)$. Given “strict essentiality”, *continuity*, and *intragenerational separability*, for each $t \in T$, there exist a continuous function q_t and continuous functions $\bar{v}^n$ (one for each $n \in N_t^\ell$) such that $\tilde{q}_t \left(\{x^n\}_{n\in N_t^\ell} ; r \right) = q_t \left(\sum_{n\in N_t^\ell} \bar{v}^n (x^n) \right)$. Substituting leads to R admitting the representation (B.1). Given *monotonicity*, for each $t \in T$ and each $n \in N_t^\ell$, either q_t and $\bar{v}^n$ are all strictly increasing or these are all strictly decreasing. Either choices lead to ordinally equivalent representations of R. □

The next step highlights that, given *ex-post equity* and *ratio-scale invariance*, the function $\bar{v}^n$ is a concave transformation of the “relative consumption” x^n/r^n and is equal across nodes belonging to the same period (up to an additive constant).

Step 2. *For each $t \in T$, there exists a strictly increasing and concave function $v_t : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each $x^n \in \mathbb{R}_+$, each $n \in N_t^\ell$, and some $\chi^n \in \mathbb{R}$:*

$$\bar{v}^n \left(x^n \right) = \pi^n r^n v_t \left(\frac{x^n}{r^n} \right) + \chi^n .$$

Proof. For each $t \in T$, each $n \in N_t^\ell$, and each $x^n \in \mathbb{R}_+$ define:

$$v^n \left(\frac{x^n}{r^n} \right) \equiv \frac{\bar{v}^n (x^n)}{\pi^n r^n} .$$

Since $\bar{v}^n$ is strictly increasing (by Step 1), v^n is as well. Let a pair $x, \bar{x} \in X$ be such that for some $t \in T$, a pair $n, n' \in N_t^\ell$, and a $\varepsilon \in \mathbb{R}_+$ the following conditions hold: *(i)* $x^n = \bar{x}^n - \frac{\varepsilon}{\pi^n} \geq r^n$; *(ii)* $x^{n'} = \bar{x}^{n'} + \frac{\varepsilon}{\pi^{n'}} \leq r^{n'}$; and *(iii)* $x^{\tilde{n}} = \bar{x}^{\tilde{n}}$ for each $\tilde{n} \in N^\ell \setminus \left\{ n, n' \right\}$. Given *ex-post equity*, $x \, R \, \bar{x}$. By Step 1, this implies that $V(x;r) - V(\bar{x};r) \geq 0$ or, using *(iii)*, that:

$$\bar{v}^n \left(x^n \right) - \bar{v}^n \left(x^n + \frac{\varepsilon}{\pi^n} \right) + \bar{v}^{n'} \left(x^{n'} \right) - \bar{v}^{n'} \left(x^{n'} - \frac{\varepsilon}{\pi^{n'}} \right) \geq 0 \tag{B.2}$$

Substituting the functions v^n and $v^{n'}$ in (B.2) gives:

$$\pi^n r^n \left[v^n \left(\frac{x^n}{r^n} \right) - v^n \left(\frac{x^n}{r^n} + \frac{\varepsilon}{\pi^n r^n} \right) \right] + \pi^{n'} r^{n'} \left[v^{n'} \left(\frac{x^{n'}}{r^{n'}} \right) - v^{n'} \left(\frac{x^{n'}}{r^{n'}} - \frac{\varepsilon}{\pi^{n'} r^{n'}} \right) \right] \geq 0.$$

If v^n and $v^{n'}$ are differentiable at $\left(\frac{x^n}{r^n} \right)$ and $\left(\frac{x^{n'}}{r^{n'}} \right)$, respectively, dividing by ε and taking the limit for $\varepsilon \to 0$, yields:

24 “Strict essentiality” states that each individual’s assignment matters for the social ranking; see also Blackorby and Donaldson (1982).

$$\left.\frac{\partial v^n(a)}{\partial a}\right|_{a=\frac{x^n}{r^n}} \leq \left.\frac{\partial v^{n'}(a)}{\partial a}\right|_{a=\frac{x^{n'}}{r^{n'}}}. \tag{B.3}$$

Since v^n and $v^{n'}$ are strictly increasing, these are differentiable almost everywhere. Thus, (B.3) holds for almost all $\left(\frac{x^n}{r^n}\right) \geq 1 \geq \left(\frac{x^{n'}}{r^{n'}}\right)$ and, symmetrically, the reverse inequality holds for almost all $\left(\frac{x^n}{r^n}\right) \leq 1 \leq \left(\frac{x^{n'}}{r^{n'}}\right)$. Thus, if the functions are differentiable at 1, $\left.\frac{\partial v^n(a)}{\partial a}\right|_{a=1} = \left.\frac{\partial v^{n'}(a)}{\partial a}\right|_{a=1}$.

Given *proportionality* and by Step 1, $V(x;r) \geq V(\bar{x};r)$ if and only if $V(bx;r) \geq V(b\bar{x};r)$ for each $b > 0$. Thus, equation (B.3) holds almost everywhere for each $\left(\frac{x^n}{r^n}\right) \geq b \geq \left(\frac{x^{n'}}{r^{n'}}\right)$ and each $b > 0$. Moreover, $\left.\frac{\partial v^n(a)}{\partial a}\right|_{a=b} = \left.\frac{\partial v^{n'}(a)}{\partial a}\right|_{a=b}$ almost everywhere for each $b > 0$. This implies that for each $t \in T$, there exists a strictly increasing and concave function $v_t : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each $x^n > 0$, each $n \in N_t^\ell$, and some constant $\chi^n \in \mathbb{R}$, $v_t(x^n) = v^n(x^n) - \frac{\chi^n}{\pi^n r^n}$. Substituting the definition of v^n yields the result. □

By imposing *ratio-scale invariance*, the next step proves that v_t is a power function.

Step 3. *For each $t \in T$, there exist constants $\eta_t \in \mathbb{R}_+$ and $\gamma_t, \bar{\eta}_t \in \mathbb{R}$ such that for each $a \in \mathbb{R}_+$:*

$$v_t(a) = \frac{\eta_t}{1-\gamma_t} a^{1-\gamma_t} + \bar{\eta}_t \text{ if } \gamma_t \neq 1 \text{ and}$$
$$v_t(a) = \eta_t \ln a + \bar{\eta}_t \text{ if } \gamma_t = 1.$$

Proof. Let $t \in T$. Let a pair $x, \bar{x} \in X$ be such that $x^n = \bar{x}^n$ for each $n \in N^\ell \backslash N_t^\ell$. Given *ratio-scale invariance*, for each $\alpha > 0$, $x \mathrel{R} \bar{x}$ if and only if $\alpha x \mathrel{R} \alpha\bar{x}$. Using Steps 1 and 2, *ratio-scale invariance* implies that:

$$\sum_{n \in N_t^\ell} \pi^n r^n \left[v_t\left(\frac{x^n}{r^n}\right) - v_t\left(\frac{\bar{x}^n}{r^n}\right)\right] \geq 0$$

if and only if

$$\sum_{n \in N_t^\ell} \pi^n r^n \left[v_t\left(\frac{\alpha x^n}{r^n}\right) - v_t\left(\frac{\alpha\bar{x}^n}{r^n}\right)\right] \geq 0.$$

Since v_t is the same for each $n \in N_t^\ell$, Theorem 6 of Roberts (1980) applies: this directly implies that v_t is an increasing affine transformation of a power function. □

For each $t \in T$, let $q_t(x;r) \equiv q_t\left(\sum_{n \in N_t^\ell} \bar{v}^n(x^n)\right)$. Again due to *ratio-scale invariance*, $q_t(x;r)$ can be written as a product of a function ψ_t (to be identified in the subsequent step) and a specific positively linearly homogeneous function of x.

Step 4. *For each $t \in T$, there exists an increasing function $\psi_t : \mathbb{R} \to \mathbb{R}$ such that, for each $x \in X$,*

$$q_t(x;r) = \psi_t\left[(1-\gamma_t)\left(\frac{\eta_t}{1-\gamma_t}\sum_{n\in N_t^\ell}\pi^n r^n\left(\frac{x^n}{r^n}\right)^{1-\gamma_t}\right)^{\frac{1}{1-\gamma_t}}\right] \text{ if } \gamma_t \neq 1 \text{ and}$$

$$q_t(x;r) = \psi_t\left[\exp\left(\eta_t\sum_{n\in N_t^\ell}\pi^n r^n \ln\left(\frac{x^n}{r^n}\right)\right).\right] \text{ if } \gamma_t = 1.$$

Proof. Let $t \in T$. From Step 2,

$$q_t(x;r) = q_t\left(\sum_{n\in N_t^\ell}\pi^n r^n v_t\left(\frac{x^n}{r^n}\right) + \chi_t\right),$$

where $\chi_t \equiv \sum_{n\in N_t^\ell}\chi^n$. Let $x, \bar{x} \in X$ be such that $x^n = \bar{x}^n$ for each $n \in N^\ell \backslash N_t^\ell$. According to *ratio-scale invariance*, for each $\alpha > 0$, $V(x;r) \geq V(\bar{x};r)$ if and only if $V(\alpha x;r) \geq V(\alpha\bar{x};r)$. Since V is additive over time (Step 1), this is equivalent to $q_t(x;r) \geq q_t(\bar{x};r)$ if and only if $q_t(\alpha x;r) \geq q_t(\alpha\bar{x};r)$. Thus q_t is homothetic with respect to x. Thus, it can be written as $q_t(x;r) = \psi_t(\tilde{q}_t(x;r))$, where ψ_t is an increasing function and $\tilde{q}_t$ is positively linearly homogeneous such that:

$$\tilde{q}_t(x;r) = \overset{*}{q}_t\left(\sum_{n\in N_t^\ell}\pi^n r^n v_t\left(\frac{x^n}{r^n}\right) + \chi_t\right),$$

with $\overset{*}{q}_t$ being continuous and increasing.

Case 1. Assume $\gamma_t \neq 1$. For each $n \in N_t^\ell$, substitute $v_t(a) = \frac{\eta_t}{1-\gamma_t}a^{1-\gamma_t} + \bar{\eta}_t$ (obtained in Step 3):

$$\tilde{q}_t(x;r) = \overset{*}{q}_t\left(\frac{\eta_t}{1-\gamma_t}\sum_{n\in N_t^\ell}\pi^n r^n\left(\frac{x^n}{r^n}\right)^{1-\gamma_t} + r_t\bar{\eta}_t + \chi_t\right).$$

Since $\tilde{q}_t(x;r)$ is positively linearly homogeneous, $\tilde{q}_t(\alpha x;r) = \alpha\tilde{q}_t(x;r)$ for each $\alpha > 0$. Thus:

$$\begin{aligned}&\overset{*}{q}_t\left(\tfrac{\eta_t}{1-\gamma_t}\alpha^{1-\gamma_t}\textstyle\sum_{n\in N_t^\ell}\pi^n r^n\left(\tfrac{x^n}{r^n}\right)^{1-\gamma_t} + r_t\bar{\eta}_t + \chi_t\right) = \\ &\quad \alpha\overset{*}{q}_t\left(\tfrac{\eta_t}{1-\gamma_t}\textstyle\sum_{n\in N_t^\ell}\pi^n r^n\left(\tfrac{x^n}{r^n}\right)^{1-\gamma_t} + r_t\bar{\eta}_t + \chi_t\right).\end{aligned}$$

Since this holds for each $x \in X$, it follows that, for each $y \in \mathbb{R}$:

$$\overset{*}{q}_t(y) = (1-\gamma_t)(y - r_t\bar{\eta}_t - \chi_t)^{\frac{1}{1-\gamma_t}},$$

and, substituting:

$$\tilde{q}_t(x;r) = (1-\gamma_t)\left(\frac{\eta_t}{1-\gamma_t}\sum_{n\in N_t^\ell}\pi^n r^n\left(\frac{x^n}{r^n}\right)^{1-\gamma_t}\right)^{\frac{1}{1-\gamma_t}}.$$

Case 2. Assume $\gamma_t = 1$. For each $n \in N_t^\ell$, substitute $v_t(a) = \eta_t \ln a + \bar{\eta}_t$ (obtained in Step 3):

$$\tilde{q}_t(x; r) = \overset{*}{q}_t \left(\eta_t \sum_{n \in N_t^\ell} \pi^n r^n \ln\left(\frac{x^n}{r^n}\right) + r_t \bar{\eta}_t + \chi_t \right).$$

By the same reasoning as above, for each $y \in \mathbb{R}$:

$$\overset{*}{q}_t(y) = \exp\left(y - r_t \bar{\eta}_t - \chi_t\right)^{\frac{1}{1-\gamma_t}},$$

and, substituting:

$$\tilde{q}_t(x; r) = \exp\left(\eta_t \sum_{n \in N_t^\ell} \pi^n r^n \ln\left(\frac{x^n}{r^n}\right) \right). \quad \square$$

Next, by imposing *ratio-scale invariance*, the function ψ_t also needs to have a power form.

Step 5. *There exists $\rho \in \mathbb{R}$ and, for each $t \in T$, $\xi_t \in \mathbb{R}_+$, such that for each $a \in \mathbb{R}$ and each $t \in T$:*

$$\psi_t(a) = \frac{\xi_t}{1-\rho} a^{1-\rho} \text{ if } \rho \neq 1,$$
$$\psi_t(a) = \xi_t \ln a \text{ if } \rho = 1.$$

Proof. According to *ratio-scale invariance*, for each pair $x, \bar{x} \in X$ and each $\alpha > 0$, $x \, R \, \bar{x}$ if and only if $\alpha x \, R \, \alpha \bar{x}$. From Step 1 and substituting $q_t(x; r)$ for each $t \in T$, *ratio-scale invariance* implies that:

$$\sum_{t \in T} q_t(x; r) \geq 0 \text{ iff } \sum_{t \in T} q_t(\alpha x; r) \geq 0.$$

From Step 4, $q_t(x; r)$ is the product of a function ψ_t and a function $\tilde{q}_t(x; r)$ that is positively linearly homogeneous with respect to x. An immediate generalization of Theorem 6 in Roberts (1980) applies: for each $t \in T$, ψ_t is an increasing affine transformation of a power function. Since the function ψ_t can be different across time, each may be assigned a different positive weight ξ_t. $\square$

Next, I impose *ex-ante equity* to determine restrictions on ρ and the parameters ξ_t and η_t.

Step 6. The following parameter restrictions hold: $\rho \geq 0$ and, for each $t \in T$, $\xi_t = \eta_t^{-1}$ and $\eta_t = \frac{1-\gamma_t}{r_t}$ if $\gamma_t \neq 1$ and $\eta_t = \frac{1}{r_t}$ otherwise.

Proof. Let a pair $x, \bar{x} \in X$ be such that for some $t, t' \in T$ and $a, b, \varepsilon \in \mathbb{R}_+$:

(i) $\dfrac{x^n}{r^n} = \dfrac{\bar{x}^n}{r^n} - \dfrac{\varepsilon}{\pi_t r^n} = a \geq 1$ for each $n \in N_t^\ell$;

(ii) $\dfrac{x^n}{r^n} = \dfrac{\bar{x}^n}{r^n} + \dfrac{\varepsilon}{\pi_{t'} r^n} = b \leq 1$for each $n \in N_{t'}^\ell$;

(iii) $x^n = \bar{x}^n$ for each $n \in N^\ell \setminus \left(N_t^\ell \bigcup N_{t'}^\ell\right)$.

Given *ex-ante equity*, $x \, R \, \bar{x}$ and, by Step 1 and assumption *(iii)*:

$$q_t(x; r) - q_t(\bar{x}; r) + q_{t'}(x; r) - q_{t'}(\bar{x}; r) \geq 0.$$

From Steps 4 and 5, if $\gamma_t \neq 1$, then:

$$q_t\left(x;r\right)=\frac{1-\gamma_t}{1-\rho}\xi_t\left(\frac{\eta_t}{1-\gamma_t}\sum_{n\in N_t^\ell}\pi^n r^n\left(a\right)^{1-\gamma_t}\right)^{\frac{1-\rho}{1-\gamma_t}};$$

$$q_t\left(\bar{x};r\right)=\frac{1-\gamma_t}{1-\rho}\xi_t\left(\frac{\eta_t}{1-\gamma_t}\sum_{n\in N_t^\ell}\pi^n r^n\left(a+\frac{\varepsilon}{\pi_t r^n}\right)^{1-\gamma_t}\right)^{\frac{1-\rho}{1-\gamma_t}}.$$

Thus, dividing $q_t\left(x;r\right)-q_t\left(\bar{x};r\right)$ by ε and taking the limit for $\varepsilon\to 0$, gives:

$$\xi_t\left(\left(a\right)^{1-\gamma_t}\frac{\eta_t}{1-\gamma_t}\sum_{n\in N_t^\ell}\pi^n r^n\right)^{\frac{1-\rho}{1-\gamma_t}-1}$$

$$\lim_{\varepsilon\to 0}\frac{q_t\left(x;r\right)-q_t\left(\bar{x};r\right)}{\varepsilon}=\xi_t a^{-\rho}\left(\frac{\eta_t}{1-\gamma_t}r_t\right)^{\frac{\gamma_t-\rho}{1-\gamma_t}}\eta_t.$$

Whereas, if $\gamma_t=1$, then:

$$q_t\left(x;r\right)=\frac{1}{1-\rho}\xi_t\left(\exp\left(\eta_t\sum_{n\in N_t^\ell}\pi^n r^n\ln\left(a\right)^{1-\gamma_t}\right)\right)^{1-\rho};$$

$$q_t\left(x;r\right)=\frac{1}{1-\rho}\xi_t\left(\exp\left(\eta_t\sum_{n\in N_t^\ell}\pi^n r^n\ln\left(a+\frac{\varepsilon}{\pi_t r^n}\right)^{1-\gamma_t}\right)\right)^{1-\rho}.$$

In this case, dividing $q_t\left(x;r\right)-q_t\left(\bar{x};r\right)$ by ε and taking the limit for $\varepsilon\to 0$, gives:

$$\lim_{\varepsilon\to 0}\frac{q_t\left(x;r\right)-q_t\left(\bar{x};r\right)}{\varepsilon}=\xi_t a^{-\rho}\left(\eta_t r_t\right)^{1-\rho}\eta_t.$$

Similarly,

$$\lim_{\varepsilon\to 0}\frac{q_{t'}\left(x;r\right)-q_{t'}\left(\bar{x};r\right)}{\varepsilon}=\begin{cases}-\xi_{t'}b^{-\rho}\left(\frac{\eta_{t'}}{1-\gamma_t}r_t\right)^{\frac{\gamma_{t'}-\rho}{1-\gamma_{t'}}}\eta_{t'} & \text{if } \gamma_{t'}\neq 1\\ -\xi_{t'}b^{-\rho}\left(\eta_{t'}r_{t'}\right)^{1-\rho}\eta_{t'} & \text{if } \gamma_{t'}=1.\end{cases}$$

By imposing *ex-ante equity*,

$$\lim_{\varepsilon\to 0}\frac{q_t\left(x;r\right)-q_t\left(\bar{x};r\right)}{\varepsilon}\leq-\lim_{\varepsilon\to 0}\frac{q_{t'}\left(x;r\right)-q_{t'}\left(\bar{x};r\right)}{\varepsilon}$$

for each $a\geq 1\geq b$ and independently of γ_t and $\gamma_{t'}$. This requires that $\rho\geq 0$;

$$\xi_t\left(\frac{\eta_t}{1-\gamma_t}r_t\right)^{\frac{\gamma_t-\rho}{1-\gamma_t}}\eta_t=\xi_t\left(\eta_t r_t\right)^{1-\rho}\eta_t=1, \text{ and}$$

$$\xi_{t'}\left(\frac{\eta_{t'}}{1-\gamma_t}r_t\right)^{\frac{\gamma_{t'}-\rho}{1-\gamma_{t'}}}\eta_{t'}=\xi_{t'}\left(\eta_{t'}r_{t'}\right)^{1-\rho}\eta_{t'}=1,$$

which are satisfied when $\xi_t = \eta_t^{-1}$, $\xi_{t'} = \eta_{t'}^{-1}$, and

$$\eta_t = \begin{cases} \frac{1-\gamma_t}{r_t} & \text{if } \gamma_t \neq 1 \\ \frac{1}{r_t} & \text{if } \gamma_t = 1, \text{ and} \end{cases}$$

$$\eta_{t'} = \begin{cases} \frac{1-\gamma_{t'}}{r_{t'}} & \text{if } \gamma_{t'} \neq 1 \\ \frac{1}{r_{t'}} & \text{if } \gamma_{t'} = 1. \quad \square \end{cases}$$

The last step combines the previous ones.

Step 7. *Steps 1-6 imply that the social ranking is reference-dependent utilitarian with disentanglement.*

Proof. Substitute the restrictions on parameters and the functional forms obtained in Steps 2-6 into the additive representation obtained in Step 1. $\square$

Finally, I impose *separability* to characterize the *reference-dependent utilitarian* criterion.

Step 8. Assume that the social ranking satisfies *monotonicity, continuity, ex-ante equity, ex-post equity, ratio-scale invariance,* and *separability.* Then, the social ranking is reference-dependent utilitarian.

Proof. Since *separability* implies *intergenerational* and *intragenerational separability*, the social ranking is a member of the family of reference-dependent utilitarian criteria with disentanglement (12). Following Step 1, given monotonicity, continuity, and separability and using Gorman's (1968) theorem, there exist continuous functions q^n (one for each $n \in N^\ell$) such that the social ranking can be represented by $V(x;r) = \sum_{n \in N^\ell} q^n(x^n)$. This additive representation is consistent with reference-dependent utilitarianism with disentanglement (12) if and only if $v = w$. Thus, the family of reference-dependent utilitarian criteria (11) emerge. $\square$

Part 2. *The reference-dependent utilitarian criterion (with disentanglement) satisfies the axioms.*

Since both families of welfare criteria are increasing in the assigned utilities, they satisfy *monotonicity*. Since they are continuous, they satisfy *continuity*. Since they are homogeneous with respect to the allocation, they satisfy *ratio-scale invariance*.

Since the reference-dependent utilitarian criterion is additively separable over nodes, it satisfies *separability*.

For each $t \in T$, let the welfare of generation t be measured by $G_t(x_t;r) \equiv v^{-1}\left(E\left[r_t v\left(\frac{x_t}{r_t}\right)\right]\right)$. Social welfare (12) can be written as $W(x;r) = \sum_{t \in T} \pi_t w(G_t)$. Since this is additive with respect to each generation's welfare, the reference-dependent utilitarian criteria with disentanglement satisfy *intergenerational separability*. For each $t \in T$, the transformed generation's welfare $v(G_t(x_t;r)) = E\left[r_t v\left(\frac{x_t}{r_t}\right)\right]$ is an "expected utility" type of preferences and is thus additive with respect to the assignment at each node $n \in N_n^\ell$. Thus, it also satisfies *intragenerational separability*.

The implications for *ex-post equity* and *ex-ante* are presented as lemmas. As mentioned, the reference-dependent utilitarian criterion (11) is a special case of the family of reference-dependent utilitarian criteria with disentanglement (12) when $w = v$. Thus, the following results for the reference-dependent utilitarian criteria with disentanglement also imply that the reference-dependent utilitarian social ranking satisfies these axioms.

Lemma 1. *If a social ordering is reference-dependent utilitarian with disentanglement, then it satisfies ex-post equity.*

Proof. Let a pair $x, \bar{x} \in X$ be such that, for some $t \in T$, a pair $n, n' \in N_t^{\ell}$, and $\varepsilon \in \mathbb{R}_+$, the following conditions hold: *(i)* $x^n = \bar{x}^n - \frac{\varepsilon}{\pi^n} \geq r^n$; *(ii)* $x^{n'} = \bar{x}^{n'} + \frac{\varepsilon}{n'} \leq r^{n'}$; *(iii)* $x^{\tilde{n}} = \bar{x}^{\tilde{n}}$ for each $\tilde{n} \in N^{\ell} \setminus \{n, n'\}$. I need to prove that $x \, R \, \bar{x}$.

Define $a \equiv \frac{x^n}{r^n}$, $\bar{a} \equiv \frac{\bar{x}^n}{r^n}$, $b \equiv \frac{x^{n'}}{r^{n'}}$, and $\bar{b} \equiv \frac{\bar{x}^{n'}}{r^{n'}}$. From *(i)* and *(ii)*, it follows that $\bar{a} > a \geq b > \bar{b}$. Condition *(iii)* implies that:

$$W(x; r) - W(\bar{x}; r) \geq 0 \;\Leftrightarrow W_t(x; r) - W_t(\bar{x}; r) \geq 0.$$

Case $\boldsymbol{\gamma_t \neq 1}$. First, let $\zeta_t \equiv 1 - \gamma_t$. By condition *(iii)*, $W_t(x; r) - W_t(\bar{x}; r) \geq 0$ if only if:

$$\frac{1}{\zeta_t}\left[\pi^n r^n \left(a^{\zeta_t} - \bar{a}^{\zeta_t}\right) + \pi^{n'} r^{n'} \left(b^{\zeta_t} - \bar{b}^{\zeta_t}\right)\right] \geq 0.$$

Define $\Delta \equiv \frac{\bar{a}^{\zeta_t} - \bar{b}^{\zeta_t}}{\bar{a} - \bar{b}}$. Two sub-cases emerge: if $\zeta_t \in (0, 1]$, then $\Delta > 0$; if $\zeta_t < 0$, then $\Delta < 0$.

Sub-case $\boldsymbol{\zeta_t \in (0, 1]}$. By first-order approximation:

$$a^{\zeta_t} = \left(\bar{a} - \frac{\varepsilon}{\pi^n r^n}\right)^{\zeta_t} \geq \bar{a}^{\zeta_t} - \frac{\varepsilon}{\pi^n r^n} \Delta \text{ and}$$
$$b^{\zeta_t} = \left(\bar{b} + \frac{\varepsilon}{\pi^{n'} r^{n'}}\right)^{\zeta_t} \geq \bar{b}^{\zeta_t} + \frac{\varepsilon}{\pi^{n'} r^{n'}} \Delta.$$

Premultiply the first by $\pi^n r^n$ and the second by $\pi^{n'} r^{n'}$. The sum of the resulting inequalities gives:

$$\pi^n r^n \left(a^{\zeta_t} - \bar{a}^{\zeta_t}\right) + \pi^{n'} r^{n'} \left(b^{\zeta_t} - \bar{b}^{\zeta_t}\right) \geq 0.$$

Since $\zeta_t \in (0, 1]$, this proves that $W_t(x; r) - W_t(\bar{x}; r)$ and $x \, R \, \bar{x}$.

Sub-case $\boldsymbol{\zeta_t < 0}$. By first-order approximation:

$$a^{\zeta_t} = \left(\bar{a} - \frac{\varepsilon}{\pi^n r^n}\right)^{\zeta_t} \leq \bar{a}^{\zeta_t} - \frac{\varepsilon}{\pi^n r^n} \Delta \text{ and}$$
$$b^{\zeta_t} = \left(\bar{b} + \frac{\varepsilon}{\pi^{n'} r^{n'}}\right)^{\zeta_t} \leq \bar{b}^{\zeta_t} + \frac{\varepsilon}{\pi^{n'} r^{n'}} \Delta.$$

Premultiply the first by $\pi^n r^n$ and the second by $\pi^{n'} r^{n'}$. The sum of the resulting inequalities gives:

$$\pi^n r^n \left(a^{\zeta_t} - \bar{a}^{\zeta_t}\right) + \pi^{n'} r^{n'} \left(b^{\zeta_t} - \bar{b}^{\zeta_t}\right) \leq 0.$$

Since $\zeta_t < 0$, this proves that $W_t(x; r) - W_t(\bar{x}; r)$ and $x \, R \, \bar{x}$.

Case $\boldsymbol{\gamma_t = 1}$. By condition *(iii)*, $W_t(x; r) - W_t(\bar{x}; r) \geq 0$ if only if:

$$\pi^n r^n (\ln a - \ln \bar{a}) + \pi^{n'} r^{n'} \left(\ln b - \ln \bar{b}\right) \geq 0.$$

Define $\Delta \equiv \frac{\ln \bar{a} - \ln \bar{b}}{\bar{a} - \bar{b}}$. Since $\bar{a} > \bar{b}$, $\Delta > 0$. By first order linear approximation:

$$\ln a = \ln\left(\bar{a} - \frac{\varepsilon}{\pi^n r^n}\right) \geq \ln \bar{a} - \frac{\varepsilon}{\pi^n r^n}\Delta \text{ and}$$
$$\ln b = \ln\left(\bar{b} + \frac{\varepsilon}{\pi^{n'} r^{n'}}\right) \geq \ln \bar{b} + \frac{\varepsilon}{\pi^{n'} r^{n'}}\Delta.$$

Premultiply the first by $\pi^n r^n$ and the second by $\pi^{n'} r^{n'}$. The sum of the results again gives the required inequality, proving that $W_t(x;r) - W_t(\bar{x};r)$ and $x\,R\,\bar{x}$. □

Lemma 2. *If a social ranking is reference-dependent utilitarian with disentanglement, then it satisfies ex-ante equity.*

Proof. Let a pair $x, \bar{x} \in X$ be such that for some $t, t' \in T$, with $t' > t$, and some $a \in \mathbb{R}_+$ the following conditions hold: *(i)* $\frac{x^n}{r^n} = \frac{\bar{x}^n}{r^n} - \frac{a}{\pi_t r^n} \geq 1$ for each $n \in N_t^\ell$; *(ii)* $\frac{x^n}{r^n} = \frac{\bar{x}^n}{r^n} + \frac{a}{\pi_{t'} r^n} \leq 1$ for each $n \in N_{t'}^\ell$; *(iii)* $x^{\tilde{n}} = \bar{x}^{\tilde{n}}$ for each $\tilde{n} \in N^\ell \setminus \left(N_t^\ell \bigcup N_{t'}^\ell\right)$. I need to prove that $x\,R\,\bar{x}$.

Define $\varepsilon \equiv \frac{a}{\bar{k}}$ for $\bar{k} \in \mathbb{N}_+$. Let $\left(\{x_k\}_{k\in[1,\bar{k}]}\right) \in X^{\bar{k}}$ be such that: *(I)* $x_1 = x$ and $x_{\bar{k}} = \bar{x}$; *(II)* for each $k \in \left[1, \bar{k}-1\right]$, $\frac{x_k^n}{r^n} = \frac{x_{k+1}^n}{r^n} - \frac{\varepsilon}{\pi_t r^n}$ for each $n \in N_t^\ell$ and $\frac{x_k^n}{r^n} = \frac{x_{k+1}^n}{r^n} + \frac{\varepsilon}{\pi_{t'} r^n}$ for each $n \in N_{t'}^\ell$; *(III)* $x_k^{\tilde{n}} = x^{\tilde{n}}$ for each $\tilde{n} \in N^\ell \setminus \left(N_t^\ell \bigcup N_{t'}^\ell\right)$ and each $k \in \left[1, \bar{k}\right]$. The proof consists of showing that in the limit where $\bar{k} \to \infty$ (and thus $\varepsilon \to 0$), $W\left(x^k; r\right) - W\left(x^{k+1}; r\right) \geq 0$; then, by transitivity, the result follows. This is done first for $\rho \neq 1$ and then for $\rho = 1$.

Case $\boldsymbol{\rho \neq 1}$. Define $\zeta \equiv 1 - \rho$. From Condition *(III)*,

$$W\left(x^k; r\right) - W\left(x^{k+1}; r\right) = \frac{1}{\zeta} r_t \left[W_t\left(x^k; r\right)^\zeta - W_t\left(x^{k+1}; r\right)^\zeta\right] + \frac{1}{\zeta} r_{t'} \left[W_{t'}\left(x^k; r\right)^\zeta - W_{t'}\left(x^{k+1}; r\right)^\zeta\right]. \tag{B.4}$$

From Condition *(II)*, x^{k+1} can be written as a function of x^k and ε. Define the following functions by setting for each $\varepsilon > 0$:

$$e_t(\varepsilon) = W_t\left(x^{k+1}; r\right),$$
$$e_{t'}(\varepsilon) = W_{t'}\left(x^{k+1}; r\right).$$

Let $e_t(0) \equiv \lim_{\varepsilon\to 0} e_t(\varepsilon) = W_t\left(x^k; r\right)$ and $e_{t'}(0) \equiv \lim_{\varepsilon\to 0} e_{t'}(\varepsilon) = W_{t'}\left(x^k; r\right)$. Thus, (B.4) can be written as:

$$W\left(x^k; r\right) - W\left(x^{k+1}; r\right) = \frac{1}{\zeta} r_t \left[e_t(0)^\zeta - e_t(\varepsilon)^\zeta\right] + \frac{1}{\zeta} r_{t'} \left[e_{t'}(\varepsilon)^\zeta - e_{t'}(0)^\zeta\right].$$

Divide by ε, and take the limit for $\varepsilon \to 0$ (or equivalently $\bar{k} \to \infty$). As $\varepsilon \to 0$, $\frac{1}{\zeta} r_t \frac{e_t(0)^\zeta - e_t(\varepsilon)^\zeta}{\varepsilon}$ tends to:

$$\frac{1}{\zeta} r_t \left.\frac{\partial}{\partial \varepsilon} e_t(\varepsilon)^\zeta\right|_{\varepsilon=0} = r_t e_t(0)^{\zeta-1} \left.\frac{\partial e_t(\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}, \tag{B.5}$$

while $\frac{1}{\zeta} r_{t'} \frac{e_{t'}(0)^\zeta - e_{t'}(\varepsilon)^\zeta}{\varepsilon}$ tends to:

$$\frac{1}{\zeta}r_{t'}\left.\frac{\partial}{\partial\varepsilon}e_{t'}(\varepsilon)^{\zeta}\right|_{\varepsilon=0}=r_{t'}e_{t'}(0)^{\zeta-1}\left.\frac{\partial e_{t'}(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}. \tag{B.6}$$

Computing the derivatives of e_t and $e_{t'}$, yields:

$$\left.\frac{\partial e_t(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}=-\frac{1}{\pi_t r_t}e_t(0)^{1-\gamma_t}\sum_{n\in N_t^{\ell}}\pi^n\left(\frac{x_k^n}{r^n}\right)^{\gamma_t-1}; \tag{B.7}$$

$$\left.\frac{\partial e_{t'}(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}=\frac{1}{\pi_{t'}r_{t'}}e_{t'}(0)^{1-\gamma_{t'}}\sum_{n\in N_{t'}^{\ell}}\pi^n\left(\frac{x_k^n}{r^n}\right)^{\gamma_{t'}-1}. \tag{B.8}$$

Substituting (B.7) into (B.5) leads to:

$$\frac{1}{\zeta}r_t\left.\frac{\partial}{\partial\varepsilon}e_t(\varepsilon)^{\zeta}\right|_{\varepsilon=0}=-e_t(0)^{\zeta-1}\frac{\sum_{n\in N_t^{\ell}}\pi^n\left(\frac{x_k^n}{r^n}\right)^{\gamma_t-1}}{\sum_{n\in N_t^{\ell}}\pi^n\left(e_t(0)\right)^{\gamma_t-1}}.$$

Since $\frac{x_k^n}{r^n}\geq 1$ for each $n\in N_t^{\ell}$, $e_t(0)\geq 1$. Moreover $\zeta\leq 1$. Thus, $e_t(0)^{\zeta-1}\leq 1$. For the same reasons and since $\gamma_t\leq 1$, it follows that $\sum_{n\in N_t^{\ell}}\pi^n\left(e_t(0)\right)^{\gamma_t-1}\geq\sum_{n\in N_t^{\ell}}\pi^n\left(\frac{x_k^n}{r^n}\right)^{\gamma_t-1}$. Together, these imply that:

$$\frac{1}{\zeta}r_t\left.\frac{\partial}{\partial\varepsilon}e_t(\varepsilon)^{\zeta}\right|_{\varepsilon=0}\geq -1.$$

Similarly, substitute (B.8) into (B.6) to obtain:

$$\frac{1}{\zeta}r_{t'}\left.\frac{\partial}{\partial\varepsilon}e_{t'}(\varepsilon)^{\zeta}\right|_{\varepsilon=0}=e_{t'}(0)^{\zeta-1}\frac{\sum_{n\in N_{t'}^{\ell}}\pi^n\left(\frac{x_k^n}{r^n}\right)^{\gamma_{t'}-1}}{\sum_{n\in N_{t'}^{\ell}}\pi^n\left(e_{t'}(0)\right)^{\gamma_{t'}-1}}.$$

As above (but with opposite signs), since $\frac{x_k^n}{r^n}\leq 1$ for each $n\in N_{t'}^{\ell}$, $e_{t'}(0)\leq 1$; moreover, $\zeta\leq 1$. Thus, $e_{t'}(0)^{\zeta-1}\geq 1$. For the same reasons and since $\gamma_{t'}\leq 1$, it follows that $\sum_{n\in N_{t'}^{\ell}}\pi^n\left(e_{t'}(0)\right)^{\gamma_{t'}-1}\leq\sum_{n\in N_{t'}^{\ell}}\pi^n\left(\frac{x_k^n}{r^n}\right)^{\gamma_{t'}-1}$. Together, these imply that $\frac{1}{\zeta}r_{t'}\left.\frac{\partial}{\partial\varepsilon}e_{t'}(\varepsilon)^{\zeta}\right|_{\varepsilon=0}\geq 1$.

Substituting into (B.4), this shows that when $\bar{k}\to\infty$:

$$\lim_{\varepsilon\to 0}\frac{W(x_k;r)-W(x_{k+1};r)}{\varepsilon}=\frac{1}{\zeta}r_t\left.\frac{\partial}{\partial\varepsilon}e_t(\varepsilon)^{\zeta}\right|_{\varepsilon=0}+\frac{1}{\zeta}r_{t'}\left.\frac{\partial}{\partial\varepsilon}e_{t'}(\varepsilon)^{\zeta}\right|_{\varepsilon=0}\geq 0.$$

Since this inequality is true for each $k\in\left[1,\bar{k}\right]$, transitivity implies that $W(x_1;r)\geq W\left(x_{\bar{k}};r\right)$ or, equivalently, $W(x;r)\geq W(\bar{x};r)$ and $x\,R\,\bar{x}$.

Case $\rho=1$. Similar steps lead to:

$$r_t\left.\frac{\partial}{\partial\varepsilon}\ln e_t(\varepsilon)\right|_{\varepsilon=0}=-e_t(0)^{-1}\frac{\sum_{n\in N_t^{\ell}}\pi^n\left(\frac{x_k^n}{r^n}\right)^{\gamma_t-1}}{\sum_{n\in N_t^{\ell}}\pi^n\left(e_t(0)\right)^{\gamma_t-1}}\geq -1,$$

$$r_{t'} \frac{\partial}{\partial \varepsilon} \ln e_{t'}(\varepsilon)\bigg|_{\varepsilon=0} = e_{t'}(0)^{-1} \frac{\sum_{n \in N_{t'}^{\ell}} \pi^n \left(\frac{x_k^n}{r^n}\right)^{\gamma_{t'}-1}}{\sum_{n \in N_{t'}^{\ell}} \pi^n \left(e_{t'}(0)\right)^{\gamma_{t'}-1}} \geq 1.$$

Thus, when $\bar{k} \to \infty$, $\lim_{\varepsilon \to 0} \frac{W(x_k;r)-W(x_{k+1};r)}{\varepsilon}$ and, by transitivity, $x \, R \, \bar{x}$ follows. □

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