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Perfect Bayesian Equilibrium and Sequential Equilibrium*

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We introduce a formal definition of perfect Bayesian equilibrium (PBE) for multi-period games with observed actions. In a PBE, (P) the strategies form a Bayesian equilibrium for each continuation game, given the specified beliefs, and (B) beliefs are updated from period to period in accordance with Bayes rule whenever possible, and satisfy a "no-signaling-what-you-don't-know" condition. PBE is equivalent to sequential equilibrium if each player has only two types, or there are only two periods, but differs otherwise. Equivalence is restored by requiring that (B) apply to the relative probabilities of types with posterior probability zero. *Journal of Economic Literature* Classification Number: 026.

1. INTRODUCTION

Kreps and Wilson's [7] notion of consistent beliefs and sequential equilibrium provides one answer to the question, "What are reasonable inferences for a player to make if he sees an opponent play an action that has zero probability according to the equilibrium strategies?" Roughly, their answer is that after a deviation players infer (1) that all players will continue to follow the equilibrium strategies and (2) that the deviation was the result of a random mistake or "tremble," as in Selten [10], where the trembles are independent across information sets, and a player's probability of trembling is measurable with respect to his own information. Beliefs are "consistent" if they can be derived using Bayesian inference from arbitrarily small trembles. A combination of a strategy profile and a system of beliefs is a sequential equilibrium if the beliefs are consistent, and if the strategies are "sequentially rational" in the sense that at every information set the

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player's strategy maximizes his expected payoff given his beliefs and the strategies of his opponents.

In most applications of dynamic games of incomplete information, economists have used the weaker equilibrium concept "perfect Bayesian equilibrium" or "PBE" instead of sequential equilibrium. In a PBE, the strategies are required to be sequentially rational given the beliefs, but milder restrictions are imposed on the way that the beliefs are revised following zero-probability events. Of the several versions of PBE in the literature, the weakest places no restrictions at all on the beliefs off the equilibrium path. While this weak version may sometimes be appropriate, in other situations economists may prefer a more restrictive version of PBE that is closer to the sequential equilibrium concept. This paper develops several such versions.

The idea behind the restrictions we impose is that a player's deviation should not signal information that the player himself does not possess. We develop this theme in classes of games of increasing generality, in each case without reference to the sequences of trembles used to define sequential equilibrium.[1] As the games become more complex, the conditions we impose become more complex as well. All of our conditions are implied by consistency, and in some cases they are equivalent to it. Although we are agnostic about the plausibility of our various conditions, we think this paper may be of interest to those who apply equilibrium refinements to the study of particular economic situations. Our work can be viewed either as proposing new and less restrictive equilibrium concepts, or as providing further explanation of what the sequential equilibrium restriction entails.

We focus on multi-period games with observed actions, where the players move simultaneously in each period, and each period's play is revealed before the next period begins. The only asymmetry of information in these games is that each player knows his own "type" (Harsanyi [6]), which is chosen by nature at the start of play and revealed only to him; each player's payoff function depends on his type and possibly on the other players' types as well. This class of games includes many applications to economics, such as bargaining, reputation and predation games.

We begin by developing a definition of "PBE for games with independent types." This definition requires that:

(i) Bayes' rule should be used to update beliefs about player i's type from period t to period $(t+1)$ whenever player i's period-t action has positive probability conditional on the history of previous play;

(ii) the posterior beliefs at any date should be that the players' types are independently distributed; and

[1] Two other papers that develop equilibrium refinements without explicit use of trembles are Blume, Brandenberger, and Dekel [1] and Weibull [11].

(iii) the beliefs about a player at the beginning of period $(t+1)$ depend only on the history up to date t and on that player's date-t action, but not on the other players' date-t actions.

We prove that beliefs satisfying these conditions are consistent (and therefore that PBE is equivalent to sequential equilibrium) if there are only two types per player or only two periods.

When some players have three or more possible types and there are at least three periods, the definition of PBE given above is weaker than sequential equilibrium in the following way. After some sequences of actions, some of player i's types may be assigned zero probability by his opponents. As we show, sequential equilibrium requires that at every period there be a commonly agreed ranking of the *relative* probability of each player's zero-probability types. In particular, some of the zero-probability types may be "infinitely more likely" than others. This ranking follows from the "trembles" explanation of deviations, because the zero-probability types are assigned positive probabilities that converge to zero, and the limiting ratios of these probabilities defines the "infinitely more likely" relationship. This leads us to develop a more restrictive version of PBE that we call "perfect extended-Bayesian equilibrium," or "PEBE." First, we suppose that at each date, the players are prepared to assign relative probabilities to any pair of types of their opponents, even those that have probability zero. That is, the beliefs about each player are a "conditional probability system" in the terminology of Myerson [9]. Then we require the three conditions we gave above to hold for the beliefs about the *relative* probabilities of any two types of any player, and not only for absolute probabilities. These conditions imply that the players do have orderings over zero-probability types, and that these orderings are updated in the way that consistency requires. Thus PEBE is equivalent to sequential equilibrium with finitely many types and periods. The additional constraints implied by the extended version may or may not be attractive, but they seem different in kind and spirit from the desiderata Kreps and Wilson used to motivate the sequential equilibrium concept.

We then extend the PEBE definition in a straightforward way to games with correlated types. Once again, the set of PEBE coincides with the set of sequential equilibria. The generalization of the "no-signaling-what-you-don't-know" conditions is that the beliefs about other players than player i at the beginning of period $(t+1)$ *conditional* on player i's type depend only on the history up to date t and on date t actions by player i's rivals.

While our paper concentrates on multi-period games with observed actions, the reader may wonder how our ideas can be used to give a definition of PBE for general extensive form games. We provide such a definition in Section 6, and demonstrate its equivalence with sequential equilibrium.

Section 2 describes multi-period games of incomplete information, and Section 3 derives our equivalence result with two independent types per player or two periods. Section 4 shows why conditions on relative probabilities are needed when there are more than two types, and develops the equivalence result for that case. Section 5 provides a definition of PBE for correlated types, and shows that it is equivalent to sequential equilibrium. Section 6 gives a version of our conditions for general extensive forms.

2. Multi-period Games with Observed Actions

In Sections 2 through 5, we will consider only a restricted class of games; that of multi-period games with observed actions, as defined by Fudenberg and Levine [2]. Players are denoted by $i = 1, 2, ..., I$. Each player i has a type θ_i which is drawn from a finite set Θ_i. The vector of all players' types is denoted by $\theta \in \Theta = \times_{i=1}^{I} \Theta_i$. All players have the same prior distribution $\rho(\cdot)$ on Θ. We assume for the moment that the types are independent, so that $\rho = \prod \rho_i$, where ρ_i is the marginal distribution over player i's type. At the beginning of the game, each player is told his own type, but is not given any information about the types of his opponents. That is, player i's partition of Θ when his type is θ_i is $\mathscr{P}_i(\theta_i) = \{\theta' \mid \theta'_i = \theta_i\}$. For notational simplicity we identify the set of player i's partitions with the set Θ_i of his types. As the types are independent, player i's initial beliefs about the types of his opponents are given by the prior distribution $\rho_{-i} = \prod_{j \neq i} \rho_j$.

The game is played in "periods" $t = 1, 2, ...$, with the property that at each period t, all players simultaneously choose an action which is then revealed at the end of the period. (This specification is more general than it may appear, because the set of feasible actions can be time and history dependent, so that games with alternating moves are included.) We assume that players never receive additional observations of θ. For notational simplicity, we assume that each player's possible actions are independent of his type. Let $h^0 = 0$, and let $A_i(h^0)$ be the set of player i's possible first-period actions. If the history of moves (other than Nature's choice of types) before t is h^{t-1}, then player i's period-t action must belong to $A_i(h^{t-1})$; if $a^t \in \times_{i=1}^{I} A_i(h^{t-1})$ is played at time t, we set $h^t = (h^{t-1}, a^t)$. We assume that the action sets are finite, and that every player always has at least one feasible action. Since each player i knows θ_i but not θ_{-i}, the information set corresponding to player i's move in period t is identified with an element of $H^{t-1} \times \Theta_i$, where H^{t-1} is the set of all feasible histories at date t.

A (behavior) strategy π_i for player i is a sequence of maps from $\Theta_i \times H^{t-1}$ to $\Delta(A_i(h^{t-1}))$, where $\Delta(A_i(h^{t-1}))$ is the space of probability distributions over player i's date-t feasible actions given h^{t-1} and $\pi_i(a_i \mid \theta_i, h^{t-1})$ is the conditional probability that type θ_i plays a_i given history h^{t-1}. A strategy profile $\pi = (\pi_i)_{i=1}^{I}$ specifies a strategy for each player i: Player i's payoff $u_i(h^T, \theta_i, \theta_{i-1})$ depends on the final history h^T, his own type, and the types of his opponents θ_{-i}. (Note that the payoffs need not be separable over periods.) In a Bayesian Nash equilibrium (Harsanyi [6]) each player's strategy maximizes his expected payoff given his opponents' strategies and his prior beliefs about their types.

To extend the spirit of subgame perfection to these games, we would like to require that the strategies yield a Bayesian Nash equilibrium, not only for the whole game, but also for the "continuation games" starting in each period t after every possible history h^{t-1}. Of course, these continuation games are not "proper subgames," because they do not stem from a singleton information set. Thus to make the continuation games into true games we must specify the players' beliefs at the start of each continuation game. We denote player i's conditional beliefs that his opponents' types are θ_{-i} by $\mu_i(\theta_{-i} \mid \theta_i, h^{t-1})$, and assume that such beliefs are defined for all players i, dates t, histories h^{t-1}, and types θ_i.

A priori it is not obvious that these beliefs must be common knowledge, but in accordance with most work on refinements we will assume that they are.[2] Under the common-knowledge assumption, for each h^{t-1} there is a single probability distribution on Θ, denoted $\mu(\cdot)$, such that

$$\mu_i(\theta_{-i} \mid \theta_i, h^{t-1}) \cdot \mu(\theta_i \mid h^{t-1}) = \mu(\theta \mid h^{t-1}),$$

where $\mu(\theta_i \mid h^{t-1})$ is the marginal probability of θ_i given h^{t-1}, so that the individual players' beliefs $\mu_i(\theta_{-i} \mid \theta_i, h^{t-1})$ correspond to the conditional probability $\mu(\theta_{-i} \mid \theta_i, h^{t-1})$ for all θ_i whose probability is positive given h^{t-1}. We will require that this equality hold for *all* types θ_i, and simplify the notation by setting $\mu_i(\theta_{-i} \mid \cdot, \cdot) = \mu(\theta_{-i} \mid \cdot, \cdot)$. Later we will require that the strategies following history h^{t-1} should yield a Bayesian equilibrium relative to the beliefs $\mu(\cdot \mid h^{t-1})$ for all histories h^{t-1}, including those which have zero probability according to the equilibrium strategies.

3. Reasonable Beliefs with Independent Types

Which systems of beliefs are reasonable? A minimal requirement is that beliefs should be those given by Bayes' rule where Bayes' rule is applicable, i.e., along the equilibrium path. [This weak requirement plus a twist

[2] One exception to this common-knowledge requirement is the notion of a "c-perfect equilibrium," which is developed in Fudenberg, Kreps, and Levine [3].

similar to the no-signaling condition defined below corresponds to the definition of perfect Bayesian equilibrium given in our [4] paper.] Kreps and Wilson [7] give a series of examples to motivate the idea that further restrictions may be natural. They then propose the notion of consistent beliefs, and note that it imposes all of the restrictions their examples suggest.

Recall that an assessment (μ, π) is *consistent* (Kreps and Wilson [7]) if there is a sequence of totally mixed strategy profiles $\pi^n \to \pi$ such that the beliefs μ^n computed from π^n using Bayes' rule converge to μ. We will say that (μ^n, π^n) "justifies" (μ, π). (A strategy profile is totally mixed if at every information set the associated behavioral strategy puts strictly positive probability on every action. Thus the beliefs associated with a totally mixed strategy profile are completely determined by Bayes' rule. Note that in our context, a totally mixed strategy profile is one in which at each period t for every history h^{t-1} every type of each player i assigns positive probability to every action in $A_i(h^{t-1})$. Remember also that in a totally mixed strategy profile the randomizations by different players are independent, as are the randomizations of a single player at different information sets).

We will reexamine the question of how to restrict the players' beliefs in the context of multi-period games with observed actions. The additional structure provided by these games may permit a sharper intuition about which restrictions to impose, and will allow us to give a simple characterization of the implications of consistency.

One additional restriction is that at any date t with beliefs $\mu(\cdot \mid h^{t-1})$, the beliefs at date $(t+1)$ should be consistent with Bayes' rule applied to the given strategies and the period-t beliefs, even if those strategies assign probability zero to history h^{t-1}. To motivate this restriction, consider a game where player 1 has two types, $\underline{\theta}$ and $\bar{\theta}$. Fix an equilibrium where no type of player 1 plays a certain action a in the first period. Since Bayes' law does not determine player 2's beliefs when this occurs, we can specify that following this deviation player 2 thinks player 1 is type $\underline{\theta}$. Now if player 1 does play action a in the first period, let the equilibrium strategies predict that he will play b in the second period regardless of his type. It might then seem natural that player 2's beliefs at the start of period 3 when player 1 has played a and then b should be the same as his beliefs at the start of period 2, i.e., that player 1 is type $\underline{\theta}$. However, since player 2's corresponding information set is off the equilibrium path, other beliefs for player 2 would not violate Bayes' law. For example, we could specify that player 2 is now certain that player 1 is type $\bar{\theta}$. To rule out this reversal of beliefs, we need to require that each player's beliefs in each period be consistent with Bayes' law applied to his beliefs of the previous period and the equilibrium strategies. This restriction is related to Kreps and Wilson's notion of structural consistency and also their conditions (5.3) and (5.4).

A more subtle condition is that no player i's deviation be treated as containing information about things that player i does not know. (If the deviations are thought of as "random errors," this corresponds to the assumption that each player's probability of error depends only on factors known to that player.) In a multi-period game with independent types, the condition is simply that the beliefs about different players' types be independent, and that each player's deviations be taken as signals only about that player's type. (Section 5 treats the case of correlated types, where one player's deviation can signal information about the type of another.) Given that the priors are independent and that players' deviations are thought of as uncorrelated, these conditions are a natural extension of the spirit of Bayesian updating.

DEFINITION 3.1. An assessment (μ, π) for a multi-period game with observed actions and independent types is *reasonable* if for all histories h^{t-1}:

(1) Bayes' rule is used to update beliefs whenever possible: For all players i, and for each $a_i \in A_i(h^{t-1})$, if $\exists \hat{\theta}_i$ with $\mu_i(\hat{\theta}_i \mid h^{t-1}) > 0$ and $\pi_i(a_i \mid \hat{\theta}_i, h^{t-1}) > 0$ (that is, a_i has positive probability conditional on h^{t-1}), then

$$\mu(\theta_i \mid h^{t-1}, a^t) = \frac{\mu(\theta_i \mid h^{t-1})\, \pi_i(a_i \mid \theta_i, h^{t-1})}{\sum_{\tilde{\theta}_i \in \Theta_i} \mu(\tilde{\theta}_i \mid h^{t-1})\, \pi_i(a_i \mid \tilde{\theta}_i, h^{t-1})};$$

(2) the posterior beliefs are independent:

$$\mu(\theta \mid h^t) = \prod_i \mu(\theta_i \mid h^t) \qquad \text{for all } \theta \text{ and } h^t\text{; and}$$

(3) the beliefs about player i at period $t+1$ depend only on h^{t-1} and player i's period-t action a_i^t:

$$\mu(\theta_i \mid h^{t-1}, a^t) = \mu(\theta_i \mid h^{t-1}, \tilde{a}^t) \text{ for all } \theta_i, \text{ and all } a^t, \tilde{a}^t \text{ with } a_i^t = \tilde{a}_i^t.$$

Conditions (2) and (3) combined are the "no-signaling-what-you-don't-know" condition: since the prior distribution is independent, and the past history of play is public information at the start of each period, each player i continues to form his beliefs about players j and k independently, and if player j unexpectedly deviates in period t, the interpretation of his deviation is independent of the simultaneous action of any other player. Note that condition (1) implies that (3) holds for actions whose conditional probability given h^{t-1} is positive. Note also that "reasonability" allows the beliefs about player i's type at time t to be *completely arbitrary* following a move by player i that has probability zero conditional on h^{t-1}. The only constraints are that player i's actions not change the beliefs about player j's type, and that, after i's first zero-probability move results in some new beliefs about his type, subsequent beliefs be determined by Bayes' rule and the strategy profile π until player i deviates again. Thus it is easy to check whether an assessment is reasonable.

While reasonability captures the three criteria we have discussed, those three do not exhaust the implications of sequential equilibria in general games. However, reasonability is equivalent to consistency if each player has at most two types, and the types are independent.

DEFINITION 3.2. A perfect Bayesian equilibrium (PBE) of a multi-period game with observed actions and independent types is an assessment (μ, π) satisfying

(B) (μ, π) is reasonable, and

(P) For each period t and history h^{t-1}, the continuation strategies are a Bayesian equilibrium for the continuation game given the beliefs $\mu(\cdot \mid h^{t-1})$.

PROPOSITION 3.1. *Consider a multi-period game with observed actions. Suppose that each player has only two possible types, that both types have nonzero prior probability, and that types are independent. Then an assessment (μ, π) is consistent iff it is reasonable.*

COROLLARY 3.1. *Under the hypotheses of Proposition* 3.1, *the sets of perfect Bayesian equilibria and sequential equilibria coincide.*

Remark. The condition that both types have positive prior probability is necessary because PBE permits a type with zero prior probability to have positive posterior probability following a zero-probability action. Kreps and Wilson exclude zero-probability types from the class of games they consider, but their definition extends immediately to such games, where it implies that any type with zero *ex ante* probability has probability zero throughout the game, as Nature's moves are not subject to trembles. For this reason, the set of sequential equilibria is not upper-hemicontinuous in prior beliefs at limit points that assign some types probability zero. The set of PBE can change when zero-probability types are added, while the set of sequential equilibria is not altered by the addition of such types. However, the set of sequential equilibria can change discontinuously when types are added whose prior probability is arbitrarily small, a point that is developed in Fudenberg, Kreps, and Levine [3].

Proof. If (μ, π) is consistent, fix a sequence of totally mixed strategy profiles $\pi^n \to \pi$ with associated beliefs $\mu^n \to \mu$. Since the π^n correspond to independent randomizations, each player's strategy π_i^n depends only on his type θ_i and the public history h^{t-1}, and the types are independent, the μ^n satisfy (1), (2), and (3). Since these properties are preserved in passing to the limit, consistent beliefs are reasonable.

Conversely, imagine that (μ, π) is reasonable. We establish the following claim by induction on T, the number of periods:

CLAIM. *In a T-period game with initial beliefs $\mu^0 = \mu(\cdot \mid h^0)$, if (μ, π) is reasonable then for any strictly positive prior assessment $\mu^{0,n} \to \mu^0$ there exists a sequence of totally mixed strategy profiles $\pi^n \to \pi$ such that the beliefs $\mu^{1,n} = \mu(\cdot \mid h^1)$ computed from $(\mu^{0,n}, \pi^n)$ using Bayes' rule converge to the specified beliefs μ at every information set. Moreover, if $\mu^{0,n}$ is strictly positive (i.e., has full support), we can take $\mu^{0,n} = \mu^0$.*

Note that proving this claim is sufficient for our result as the prior distribution is assumed to put positive probability on all types in Θ. The reason we consider sequences $\mu^{0,n}$ converging to μ^0 as opposed to simply μ^0 itself is that we will proceed by induction: First we will construct first-period trembles, then second-period trembles, and so on. In this process we will wish to treat view h^0 as the initial history in a continuation game from, say, period 2, and in so doing we will need to use the beliefs corresponding to first-period trembles.

Proof of Claim. I. We begin with a 2-period game, where each player i has two possible types $\underline{\theta}_i$ and $\bar{\theta}_i$. Here the only beliefs which are relevant are those following the first period's play h^1, which we denote $\mu^1 = \mu(\cdot \mid h^1)$. Because μ is reasonable, the beliefs μ_i^1 about player i depend on h^1 only through player i's choice of action a_i^1. In the obvious notation, we let $\underline{\mu}_i(a_i^1) = \mu(\underline{\theta}_i \mid h^1)$; we define $\bar{\mu}_i$, $\bar{\pi}_i$, and $\underline{\pi}_i$ analogously.

Choose a sequence $\varepsilon^n \to 0$ and let $\mu^{0,n} \to \mu(h^0)$ be such that for all θ_i and i, $\mu^{0,n}(\theta_i) > \varepsilon^n$. Without loss of generality we assume that $\bar{\mu}_i^0 > 0$ for all i. For each player i we define the sets $\bar{P}_i$ of actions in the support of $\bar{\pi}_i$ and $\bar{O}_i$ of actions that $\bar{\pi}_i$ assigns zero probability; the sets $\underline{P}_i$ and $\underline{O}_i$ are defined analogously. We will now construct totally mixed strategy profiles $\pi^n \to \pi$ such that the associated posterior beliefs computed from $\mu^{0,n}$ and π^n using Bayes' rule converge to the specified posterior μ^1. To do this we construct π_i^n separately for each player i, beginning with those pairs (a_i^1, θ_i) for which $\pi_i(a_i^1 \mid \theta_i) = 0$. That is, we first construct the trembles for type $\underline{\theta}_i$ and actions in $\underline{O}_i$ and type $\bar{\theta}_i$ and actions in $\bar{O}_i$; the strategies π_i^n for other action–type pairs are constructed by subtracting the trembles assigned to the zero-probability actions.

(a) Let us specify the probabilities that player i uses action $a_i^1 \in \underline{O}_i \cap \bar{O}_i$. If $\bar{\mu}_i(a_i^1)$ is positive, we choose $\bar{\pi}_i^n(a_i^1) \to 0$ and $\underline{\pi}_i^n(a_i^1) \to 0$ so that

$$\text{(i)} \qquad \underline{\mu}_i(a_i^1)/\bar{\mu}_i(a_i^1) = \lim_{n \to \infty} \underline{\mu}_i^{0,n} \underline{\pi}_i^n(a_i^1)/\bar{\mu}_i^{0,n} \bar{\pi}_i^n(a_i^1).$$

If $\bar{\mu}_i(a_i^1) = 0$, then since there are only two types, $\underline{\mu}_i(a_i^1) = 1$. This means we can choose the π_i^n so that

$$\text{(ii)} \qquad \bar{\mu}_i(a_i^1)/\underline{\mu}_i(a_i^1) = \lim_{n \to \infty} \bar{\mu}_i^{0,n} \bar{\pi}_i^n(a_i^1)/\underline{\mu}_i^{0,n} \underline{\pi}_i^n(a_i^1).$$

(b) Now we consider actions $a_i^1 \in \underline{P}_i \cap \bar{O}_i$ and construct $\bar{\pi}_i^n$ but do not yet specify $\underline{\pi}_i^n$. If $\underline{\mu}_i^0 > 0$, then (μ, π) reasonable implies that $\underline{\mu}_i(a_i^1) = 1$. Let $\bar{\pi}_i^n(a_i^1) \to 0$, and note for future reference that as long as $\underline{\pi}_i^n(a_i^1) \to \underline{\pi}_i(a_i^1)$, the beliefs $\mu_i^n(a_i^1)$ corresponding to action a_i^1 will converge to a point mass on $\underline{\theta}_i$ as desired. If $\underline{\mu}_i^0 = 0$, then we have two cases depending on whether $\bar{\mu}_i(a_i^1)$ is non-zero or zero. If it is non-zero, we choose $\bar{\pi}_i^n(a_i^1) \to 0$ so that

$$\text{(iii)} \qquad \underline{\mu}_i(a_i^1)/\bar{\mu}_i(a_i^1) = \lim_{n \to \infty} \underline{\mu}_i^{0,n} \underline{\pi}_i^n(a_i^1)/\bar{\mu}_i^{0,n} \bar{\pi}_i^n(a_i^1).$$

If $\bar{\mu}_i(a_i^1) = 0$, then since there are only two types, $\underline{\mu}_i(a_i^1) = 1$, and we choose the $\bar{\pi}_i^n$ so that

$$\text{(iv)} \qquad \bar{\mu}_i(a_i^1)/\underline{\mu}_i(a_i^1) = \lim_{n \to \infty} \bar{\mu}_i^{0,n} \bar{\pi}_i^n(a_i^1)/\underline{\mu}_i^{0,n} \underline{\pi}_i^n(a_i^1).$$

Note that as long as $\underline{\pi}_i^n \to \underline{\pi}_i$ eqs. (iii) and (iv) guarantee that $\mu_i^n(a_i^1) \to \mu_i(a_i^1)$.

(c) For actions a_i^1 in $\bar{P}_i \cap \underline{O}_i$ we know that $\bar{\mu}_i(a_i^1) = 1$. For these actions let $\underline{\pi}_i^n$ be any sequence converging to zero; then as long as $\bar{\pi}_i^n \to \bar{\pi}_i > 0$ the posteriors will have the appropriate limit.

(d) Finally we specify $\pi_i^n(a_i^1 \mid \theta_i)$ for actions in $P_i(\theta_i)$ by

$$\pi_i^n(a_i^1 \mid \theta_i) = \pi_i(a_i^1 \mid \theta_i) - \frac{\sum_{\hat{a}_i^1 \in O_i(\theta_i)} \pi_i^n(\hat{a}_i^1 \mid \theta_i)}{\# P_i(\theta_i)},$$

where $\# P_i(\theta_i)$ denotes the number of actions in $P_i(\theta_i)$. That is, we subtract the trembles on zero-probability actions from the positive probabilities. Note that, for all θ_i,

$$\sum_{a_i^1 \in A_i^1} \pi_i^n(a_i^1 \mid \theta_i) = \sum_{a_i^1 \in A_i^1} \pi_i(a_i^1 \mid \theta_i) = 1.$$

Since $\pi_i^n(a_i^1 \mid \theta_i) \to 0$ for all $a_i \in O_i(\theta_i)$, $\pi_i^n \to \pi_i$. By construction, the posteriors $\mu^{1,n}$ obtained by updating $\mu^{0,n}$ using μ^n converge to μ. Finally, note that if μ^0 assigns positive probability to both types of θ_i, we can take $\mu^{0,n} = \mu^0$. This proves our claim for two-period games.

II. Now we extend the claim to games of length T by induction. Assume that the claim is true for all games with $T-1$ or fewer periods, and consider a game G with T periods along with a reasonable assessment (μ, π). Let G_{T-1} be the game from period 1 through period $(T-1)$. By inductive hypotheses, there exist initial beliefs $\mu^{0,n}$ and totally mixed strategy profiles $\pi^{T-1,n}$ of G_{T-1} such that the associated posterior beliefs μ^n converge to μ at every information set through period $T-1$. Given the beliefs $\mu^{T-1,n}$ at the start of period T, we must show how to choose the period-T probabilities $\pi_i^n(a_i^T)$ so that the posterior beliefs at the start of the period $T+1$ converge to μ^T. To do so, we now apply the algorithm developed in Part I to initial beliefs $\hat{\mu}^{0,n} \equiv \mu^{T-1,n}$. If we then specify that players follow $\pi^{T-1,n}$ through period $T-1$ and $\pi_i^n(a_i^T)$ at period T, we will have constructed profiles π^n for the T-period game G such that the beliefs computed using prior $\mu^{0,n}$ and strategy profile π^n converge to μ at every information set. Q.E.D.

Proposition 3.1 assumes two types per player. Alternatively, we could allow an arbitrary number of types per player, but only two periods (that is, only one round of Bayesian updating) and preserve the equivalence between PBE and sequential equilibria.

PROPOSITION 3.2. *Consider a two-period game with observed actions and independent types. Then an assessment (μ, π) is consistent iff it is reasonable. Hence the sets of perfect Bayesian equilibria and sequential equilibria coincide.*

Proof. This is a consequence of the proof of Proposition 4.1: The trembles constructed there can be constructed in the first period of any game, because the initial beliefs μ^0 are totally mixed and thus generate a conditional probability system in the sense of Definition 4.1, and for totally mixed beliefs conditions 1′, 2′, and 3′ reduce to 1, 2, and 3. Q.E.D.

4. EXTENDED REASONABLE BELIEFS

When there are more than two types, reasonable beliefs need not be consistent. This is illustrated in the fragment of a game in Fig. 1. This figure depicts a situation where player 1 has three possible types, θ_1', θ_1'', and θ_1^*, but where at time t Bayesian inference given the previous play has led to the conclusion that player 1 must be type θ_1^*. The equilibrium strategies at

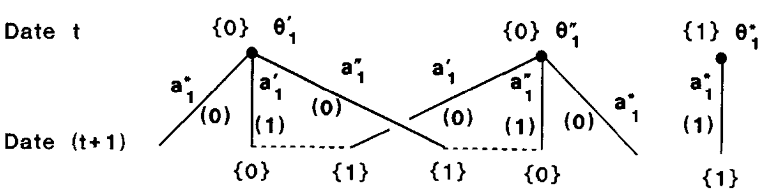

FIGURE 1

this point, which are written in curved brackets in the figure, are for type θ_1' to play a_1', type θ_1'' to play a_1'', and type θ_1^* to play a_1^*. Since the first two types have zero probability, player 2 expects to see player 1 play a_1^*. What should he believe if he sees one of the other two actions? The beliefs in the figure (given in the square brackets) are that if player 2 sees a_1' he concludes he is facing type θ_1'', while a_1'' is taken as a signal that player 1 is type θ_1'. Since our definition of PBE places no constraints on the beliefs about a player who has just deviated, the situation in Fig. 1 is compatible with PBE.

However, the situation of Fig. 1 cannot be part of a sequential equilibrium. To see this, imagine that there were trembles π^n that converged to the given strategies π and such that the associated beliefs μ^n converged to the given beliefs μ. Let the probability that μ^n assigns to type θ_1' at period t be ε_n' and let the probability of type θ_1'' be ε_n''. Since μ^n converges to μ, both ε_n' and ε_n'' converge to 0, and $\pi^n(a_1' \mid \theta_1'')$ and $\pi^n(a_1' \mid \theta_1^*)$ converge to zero as well. Since $\mu^n(\theta_1'' \mid a_1') = \mu^n(\theta_1'')\,\pi^n(a_1' \mid \theta_1'')/\sum_{\theta_1} \mu^n(\theta_1)\,\pi^n(a_1' \mid \theta_1)$, in order to have $\mu^n(\theta_1'' \mid a_1')$ converge to 1 it must be that $\varepsilon_n'/\varepsilon_n''$ converges to zero. In order for the beliefs following a_1' to be concentrated on type θ_1'' when θ_1' plays a_1' with probability one while θ_1'' assigns it probability zero, the prior beliefs must be that θ_1'' is infinitely more likely than θ_1'. On its own this requirement is compatible with sequential equilibrium. However, considering the beliefs following a_1'' leads to the conclusion that $\varepsilon_n''/\varepsilon_n'$ converges to zero, i.e., that θ_1' is infinitely more likely than θ_1'', and these two conditions are jointly incompatible with the beliefs' being consistent.

The example shows that in order to have an equivalence between PBE and sequential equilibrium when there are more than two types, the definition of beliefs must be extended to capture the relative probabilities of zero-probability types, and that restrictions must be imposed on the way that these relative probabilities are updated.

For any finite set Ω, let 2^Ω denote the set of all subsets of Ω.

DEFINITION 4.1 (Myerson [9]). A *conditional probability system* on a finite sample space Ω is a collection of functions $v(A \mid B)$ from $2^\Omega \times 2^\Omega$ to $[0, 1]$ such that

(i) for each $A \in 2^\Omega$, $v(\omega \mid A)$ is a probability distribution on A, and

(ii) for $A \subseteq B \subseteq C \in 2^{\Omega}$ with $B \neq \varnothing$, $\nu(A \mid B) \cdot \nu(B \mid C) = \nu(A \mid C)$.

Myerson shows that ν is a conditional probability system if and only if it corresponds to the limits of the conditional probabilities associated with a distribution that assigns positive probability to every $\omega \in \Omega$. Thus conditional probability systems over the whole game tree are related to consistent beliefs. This characterization differs from the definition of consistent beliefs, for the latter requires that the "trembles" be interpretable as mixed strategies of the game, which imposes many constraints on the way the trembles are constructed.

Let H denote the space of all possible partial histories h^t at any time t: $H \equiv \bigcup_{\{t, h^t\}} h^t$.

DEFINITION 4.2. An *extended belief system* for a multi-period game with observed actions and independent types is a collection of $(I+1)$ functions,

$$\eta_i \colon 2^{\Theta_i} \times 2^{\Theta_i} \times H \to [0, 1], \qquad i \in I,$$

and

$$\mu \colon \Theta \times H \to [0, 1],$$

such that for all t and h^t, $\eta_i(\cdot \mid \cdot, h^t)$ is a conditional probability system on Θ_i, and the marginal of μ on Θ_i satisfies

$$\mu(\theta_i \mid h^t) = \eta_i(\theta_i \mid \Theta_i, h^t)$$

for all i, θ_i, and h^t.

This definition requires that the joint distribution μ be compatible with the conditional probability systems η_i on each player i's type. In particular, the relative probability of any two types is well-defined after each h^{t-1}: θ_i is infinitely more likely than $\hat{\theta}_i$ if and only if $\eta_i(\theta_i \mid (\theta_i, \hat{\theta}_i), h^{t-1}) = 1$; θ_i is "as likely" as $\hat{\theta}_i$ if $0 < \eta_i(\theta_i \mid (\theta_i, \hat{\theta}_i), h^{t-1}) < 1$. It is easily verified that with a conditional probability system, the relationship "infinitely more likely than" is transitive on Θ_i.

The assumption of a coherent system of beliefs about types is weaker than the assumption that there is a complete conditional probability system over all of the nodes of the tree. This latter assumption, which is satisfied by beliefs generated from "trembles" *à la* Kreps and Wilson, requires that the relative probabilities of any two nodes of the tree be well-defined, so that player 3 is prepared to evaluate the relative probability of player 1 being type θ_1 after history h^t and of player 2 being type θ_2 after a different history $\hat{h}^{t'}$. However, comparisons of relative probabilities at nodes corresponding to different observed histories through period t are not relevant for the play of the game, because such nodes are never in the same information set. Moreover, an extended belief system does not even generate a conditional probability system over all of Θ for a given h^t, as the relative probabilities of the zero-probability types of different players need not be well defined. Once again, in our class of games this additional structure need not be assumed to ensure that beliefs are consistent.

DEFINITION 4.3. An extended assessment (μ, η, π) for a multi-period game with observed actions and independent types is *reasonable* if for all histories h^{t-1}, players i, and pairs $(\theta_i, \hat{\theta}_i)$:

(1′) Bayes' rule is used to update beliefs conditional on h^{t-1} and $(\theta_i, \hat{\theta}_i)$ wherever possible: For each $a_i \in A_i(h^{t-1})$, and all a^t with $a_i^t = a_i$,

$$\eta_i(\theta_i \mid (\theta_i, \hat{\theta}_i), h^{t-1}, a^t)\, \pi_i(a_i \mid \hat{\theta}_i, h^{t-1})\, \eta_i(\hat{\theta}_i \mid (\theta_i, \hat{\theta}_i), h^{t-1})$$
$$= \eta_i(\hat{\theta}_i \mid (\theta_i, \hat{\theta}_i), h^{t-1}, a^t)\, \pi_i(a_i \mid \theta_i, h^{t-1})\, \eta_i(\theta_i \mid (\theta_i, \hat{\theta}_i), h^{t-1}).$$

(2′) The posterior beliefs are independent:

$$\mu(\theta \mid h^t) = \prod_i \mu(\theta_i \mid h^t) \qquad \text{for all } \theta \text{ and } h^t.$$

(3′) The beliefs about the relative probabilities of player i's types at period $t+1$ depend only on h^{t-1} and player i's period-t action a_i^t:

$$\eta_i(\theta_i \mid (\theta_i, \hat{\theta}_i), h^{t-1}, a^t) = \eta_i(\theta_i \mid (\theta_i, \hat{\theta}_i), h^{t-1}, \tilde{a}^t) \qquad \text{if} \quad a_i^t = \tilde{a}_i^t.$$

Note that conditons (1′), (2′), and (3′) are the same as (1), (2), and (3), except that they apply more generally to relative probabilities, and that they reduce to (1), (2), and (3) when μ has full support.

The pairwise Bayes rule condition (1′) implies that if θ_i is infinitely more likely than $\hat{\theta}_i$ given h^{t-1}, and $\pi_i(a_i \mid \theta_i, h^{t-1}) > 0$, then after a_i is observed in period t, θ_i is still infinitely more likely than $\hat{\theta}_i$. Similarly, if two types are as likely given h^{t-1}, and both play action a_i with positive probability, the two types remain as likely. This will allow us to define a numerical ordering over all types in Θ_i for any history in H so that the order of a type stays constant whenever the observed action is one that its strategy assigns positive probability, and the order increases whenever the type's strategy assigns the observed action probability zero. This ordering plays a key role in the proof that reasonable extended beliefs are consistent.[3]

[3] This is reminiscent of Kreps and Wilson's [7] idea of a labeling, but it is simpler, as we only need to label the types, as opposed to all of the nodes of the game. The more general concept we develop in Section 6 does generate an ordering over all nodes.

DEFINITION 4.4. A *perfect extended-Bayesian equilibrium* (PEBE) of a multiperiod game with observed actions and independent types is an extended assessment (μ, η, π) satisfying

(B) (μ, η, π) is reasonable, and

(P) For each period t and each history h^{t-1}, the continuation strategies are a Bayesian equilibrium for the continuation game given the beliefs $\mu(\cdot \mid h^{t-1})$.

PROPOSITION 4.1. *A reasonable extended system of beliefs is consistent, i.e., it is the limit of a sequence of beliefs computed from totally mixed strategies of the extensive form.*

Proof. See Appendix.

COROLLARY 4.1. *The sets of PEBE and sequential equilibria coincide.*

5. CORRELATED TYPES

When the players' types are not drawn from independent distributions, player i's actions in general signal not only his type, but also those of players whose types are correlated with his. We now generalize the definition of PEBE and the equivalence result of Section 4 to this situation.

When we allow correlation, the prior joint distribution $\rho(\theta)$ may now differ from the product of the prior marginals. To simplify the analysis, we assume a form of imperfect correlation, namely that all combinations of types in Θ have positive probability.

Assumption 5.1. For all $\theta \in \Theta$, $\rho(\theta) > 0$.

This implies in particular that all types of player i agree about which opponents' types have positive probability. As before, let $\mu(\theta \mid h^{t-1})$ denote the prior probability of θ conditional on h^{t-1}. We will let $\mu(\theta_{-i} \mid \theta_i, h^{t-1})$ denote the *conditional* probability of θ_{-i} given θ_i and h^{t-1} and let $\mu(\theta_i \mid h^{t-1})$ and $\mu(\theta_i \mid h^{t-1}, a_i^t)$ be the *marginal* probability of θ_i conditional on h^{t-1} and on h^{t-1} and player i's date-t action a_i^t, respectively. In contrast with the independent case, $\mu(\theta_i \mid h^{t-1}, a_i^t)$ need not equal $\mu(\theta_i \mid h^{t-1}, a^t)$ because the other players' actions can be correlated with player i's type. Note also that we define these distributions even for histories that are off the equilibrium path. We will require that $\mu(\theta_{-i} \mid \theta_i, h^{t-1})\, \mu(\theta_i \mid h^{t-1}) = \mu(\theta \mid h^{t-1})$ for all $\theta = (\theta_i, \theta_{-i})$ and h^{t-1}.

As suggested by an associate editor of this journal, an easy way to work with correlated distributions is to transform the game into one with independent types, and then map the resulting equilibrium strategies and beliefs to strategies and beliefs in the original game. Specifically, let $\hat{\rho}$ be the product of independent uniform marginal distributions $\hat{\rho}_i$ on Θ_i, so that

$$\hat{\rho}(\theta) = \left[\prod_{i=1}^{I} (\#\Theta_i)\right]^{-1} \quad \text{for all} \quad \theta \in \Theta.$$

Let the transformed utility functions be

$$\hat{u}_i(h^T, \theta_i, \theta_{-i}) = \rho(\theta_{-i} \mid \theta_i)\, u_i(h^T, \theta_i, \theta_{-i}). \tag{5.1}$$

(The conditional probabilities are well-defined because ρ has full support.) Myerson [8] observes that the original and transformed games have the same Bayesian *Nash* equilibria: A strategy maximizes the expected value of $u_i(\cdot, \theta_i, \theta_{-i})$ with respect to beliefs $\rho(\cdot \mid \theta_i)$ if and only if it maximizes the expected value of $\hat{u}_i(\cdot, \theta_i, \theta_{-i}) = \rho(\theta_{-i} \mid \theta_i)\, u_i(\cdot, \theta_i, \theta_{-i})$ with respect to uniform beliefs $\hat{\rho}(\cdot \mid \theta_i)$.

It is straightforward to check that this observation extends to the set of sequential equilibria. We give a proof for completeness.

PROPOSITION 5.1. *Assessment* $(\hat{\mu}, \hat{\pi})$ *is a sequential equilibrium of the transformed (independent types) game if and only if the assessment* (μ, π) *defined by* $\mu(\theta_{-i} \mid \theta_i, h^t) \equiv \rho(\theta_{-i} \mid \theta_i)\, \hat{\mu}(\theta_{-i} \mid h^t) / \sum_{\theta'_{-i}} \rho(\theta'_{-i} \mid \theta_i)\, \hat{\mu}(\theta'_{-i} \mid h^t)$ *and* $\pi = \hat{\pi}$ *is a sequential equilibrium of the original (correlated types) game.*

Proof. First we claim that $(\hat{\mu}, \hat{\pi})$ is consistent if and only if (μ, π) is. To see this, note that for any full-support prior ρ on Θ, and any totally mixed strategy π^n, player i's posterior beliefs about his opponents' types given history $h^t = (a^1, ..., a^t)$ are

$$\mu^n(\theta_{-i} \mid \theta_i, h^t) = \frac{\rho(\theta_{-i} \mid \theta_i) \prod_{\tau=1}^{t} \pi^n(a^\tau_{-i} \mid \theta_{-i}, h^{\tau-1})}{\sum_{\theta'_{-i} \in \Theta_{-i}} \rho(\theta'_{-i} \mid \theta_i) \prod_{\tau=1}^{t} \pi^n(a^\tau_{-i} \mid \theta'_{-i}, h^{\tau-1})}, \tag{5.2}$$

while player i's posterior beliefs in the transformed game with prior $\hat{\rho}$ are just

$$\hat{\mu}^n(\theta_{-i} \mid \theta_i, h^t) = \frac{\prod_{\tau=1}^{t} \pi^n(a^\tau_{-i} \mid \theta_{-i}, h^{\tau-1})}{\sum_{\theta'_{-i} \in \Theta_{-i}} \prod_{\tau=1}^{t} \pi^n(a^\tau_{-i} \mid \theta'_{-i}, h^{\tau-1})}, \tag{5.3}$$

so that

$$\mu^n(\theta_{-i} \mid \theta_i, h^t) = \frac{\rho(\theta_{-i} \mid \theta_i)\, \hat{\mu}^n(\theta_{-i} \mid \theta_i, h^t)}{\sum_{\theta'_{-i} \in \Theta_{-i}} \rho(\theta_{-i} \mid \theta_i)\, \hat{\mu}^n(\theta'_{-i} \mid \theta_i, h^t)}. \tag{5.4}$$

Taking limits in (5.4) shows that (μ, μ) is consistent iff $(\hat{\mu}, \hat{\pi})$ is.

Next we claim that for each player i and every h^t, maximizing utility u_i given beliefs μ and opponents' strategy π_{-i} is equivalent to maximizing utility $\hat{u}_i$ given beliefs $\hat{\mu}$, and $\hat{\pi}_{-i} = \pi_{-i}$. To verify this, let $P^\pi(h^T \mid h^{t-1}, \theta)$ be the probability under π of reaching h^T given observed history h^t and types θ, and note that player i's expected utility conditional on reaching h^t is

$$\sum_{\theta_{-i}} \mu(\theta_{-i} \mid \theta_i, h^t)\, P^\pi(h^T \mid h^t, \theta)\, u_i(h^T, \theta_i, \theta_{-i}),$$

which is proportional to

$$\sum_{\theta_{-i}} \hat{\mu}(\theta_{-i} \mid \theta_i, h^t)\, P^\pi(h^T \mid h^t, \theta)\, \rho(\theta_{-i} \mid \hat{\theta}_i)\, u_i(h^T, \theta_i, \theta_{-i})$$

$$= \sum_{\theta_{-i}} \hat{\mu}(\theta_{-i} \mid \theta_i, h^t)\, P^\pi(h^T \mid h^t, \theta)\, \hat{u}_i(h^T, \theta_i, \theta_{-i}). \qquad \text{Q.E.D.}$$

Proposition 5.1 gives a simple method for checking whether an assessment is consistent when types are correlated. One transforms beliefs into independent beliefs and then applies the conditions developed in Sections 3 and 4. Conversely, one may wonder what these conditions on the beliefs for independent types imply for the beliefs μ. We conclude this section by mentioning *some* of these implications.

Suppose that given some history h^{t-1}, only player i has possibly (but not necessarily) deviated: There exists $\hat{\theta}_{-i}$ and $\hat{\theta}_i$ such that $\mu(\hat{\theta}_{-i} \mid \hat{\theta}_i, h^{t-1}) > 0$ (types $\hat{\theta}_{-i}$ had positive probability given $\hat{\theta}_i$ at the beginning of period t) and $\pi_{-i}(a^t_{-i} \mid \hat{\theta}_{-i}, h^{t-1}) > 0$ (actions a^t_{-i} are played with positive probability by types $\hat{\theta}_{-i}$). It can be shown that given our assumption of full support for the prior, $\mu(\hat{\theta}_{-i} \mid \theta_i, h^{t-1}) > 0$ for all θ_i; that is, which types θ_{-i} have positive probability conditional on θ_i does not depend on θ_i: see Lemma 5.1 in Fudenberg and Tirole [5].

Using conditions (1) and (2) of Definition 3.1 for the independent beliefs $\hat{\mu}$ and transforming back to the correlated beliefs μ yields

$$\mu(\theta_{-i} \mid \theta_i, h^{t-1}, a^t_{-i}) = \frac{\mu(\theta_{-i} \mid \theta_i, h^{t-1})\, \pi_{-i}(a^t_{-i} \mid \theta_{-i}, h^{t-1})}{\sum_{\tilde{\theta}_{-i}} \mu(\tilde{\theta}_{-i} \mid \theta_i, h^{t-1})\, \pi_{-i}(a^t_{-i} \mid \tilde{\theta}_{-i}, h^{t-1})} \tag{5.5}$$

and

$$\mu(\theta \mid h^t) = \mu(\theta_{-i} \mid \theta_i, h^t)\, \mu(\theta_i \mid h^t)$$

$$= \frac{\mu(\theta_{-i} \mid \theta_i, h^{t-1})\, \pi_{-i}(a^t_{-i} \mid \theta_{-i}, h^{t-1})\, \mu(\theta_i \mid h^{t-1}, a^t_i)}{\sum_{(\tilde{\theta}_{-i}, \tilde{\theta}_i)} \mu(\tilde{\theta}_{-i} \mid \tilde{\theta}_i, h^{t-1})\, \pi_{-i}(a^t_{-i} \mid \tilde{\theta}_{-i}, h^{t-1})\, \mu(\tilde{\theta}_i \mid h^{t-1}, a^t_i)}. \tag{5.6}$$

A few comments are in order. Condition (5.5) is the familiar Bayes formula for updating beliefs on player i's rivals given their observed behavior and player i's type. Note that the denominator of the right-hand side of (5.5) is positive for $\hat{\theta}_i$ by assumption. As we observed, it is therefore positive for each θ_i.

Condition (5.6) would be more transparent in the form

$$\mu(\theta \mid h^t) = \frac{\mu(\theta_{-i} \mid \theta_i, h^{t-1})\, \pi_{-i}(a^t_{-i} \mid \theta_{-i}, h^{t-1})\, \mu(\theta_i \mid h^{t-1})\, \pi_i(a^t_i \mid \theta_i, h^{t-1})}{\sum_{(\tilde{\theta}_{-i}, \tilde{\theta}_i)} \mu(\tilde{\theta}_{-i} \mid \tilde{\theta}_i, h^{t-1})\, \pi_{-i}(a^t_{-i} \mid \tilde{\theta}_{-i}, h^{t-1})\, \mu(\tilde{\theta}_i \mid h^{t-1})\, \pi_i(a^t_i \mid \tilde{\theta}_i, h^{t-1})}, \tag{5.7}$$

where we use the assumption $\mu(\theta \mid h^{t-1}) = \mu(\theta_{-i} \mid \theta_i, h^{t-1})\, \mu(\theta_i \mid h^{t-1})$. However, the right-hand side of (5.7) is not well defined if its denominator is equal to 0. This is the case in particular if a^t_i has zero probability conditional on h^{t-1} (that is, $\mu(\tilde{\theta}_i \mid h^{t-1})\, \pi_i(a^t_i \mid \tilde{\theta}_i) = 0$ for all $\tilde{\theta}_i$). When a^t_i has zero probability, one can still preserve the power of Bayes' rule by "short circuiting" the Bayesian updating for player i and writing the condition directly in terms of the posterior beliefs $\mu(\theta_i \mid h^{t-1}, a^t_i)$.

6. General Conditions

The definitions of reasonable beliefs in the previous sections exploited the special structure of games with observed actions. In these games, two nodes in the same information set always correspond to identical histories of play through the end of the last period, so that the only uncertainty relates to the opponents' types and to their simultaneously chosen actions in the current period. Since beliefs about the actions are pinned down by the equilibrium strategies, we had only to concern ourselves with undertainty about the types.

While this structure let us develop simple definitions, those definitions do not apply to more general games. This section develops a definition of reasonable beliefs for general games, which turns out to be equivalent to consistency.

We now consider extensive form games with nodes $x \in X$, information sets $h \in H$, and actions $a \in A(h)$.

For each $x \in X$ and $a \in A(h(x))$, there is a unique *successor* $\mathscr{s}(x, a)$. Thus the nodes of X are partially ordered by precedence: x is *before* y and y is *after* x if there is a succession of nodes and actions $x' = \mathscr{s}(x, a)$, $x'' = \mathscr{s}(x', a')$, ..., $y = \mathscr{s}(x^n, a^n)$. We require that this partial order extend to information sets, so that if $x \in h$, $x' \in h'$ and x' is after x, then no $x'' \in h'$ is before x.

If x' is after x, we define $\pi(x' \mid x)$ to be the probability under π of the succession of actions that lead from x to x'.

Let Z denote the set of terminal nodes. Given a subset Y of nodes of the tree, $Z(Y)$ is the subset of terminal nodes that are preceded by some element in Y.

As suggested by the associate editor, we now assume that there exists a conditional probability system $\nu(\cdot \mid \cdot)$ on the set of all terminal nodes (see Definition 4.1). This conditional probability system induces conditional beliefs $\mu(\cdot \mid \cdot)$: For any subsets A and B of X, such that $Z(A) \subseteq Z(B)$, $\mu(A \mid B) \equiv \nu(Z(A) \mid Z(B))$.

Note that this is stronger than the existence of separate conditional probability systems for each information set, which we assumed in Sections 4 and 5. It is also stronger than assuming some conditional probability system over nodes of the tree, as this latter assumption on its own does not require that the probability of a node be at least as large as that of the node's successors. That is, we can define conditional probability systems on nodes that do not respect the tree's structure. In contrast, when the relative probabilities of nodes are derived from the relative probabilities of their terminal successors, the conditional probability system already incorporates a great deal of the tree's structure. Indeed, Myerson [9] shows that these conditional probability systems correspond to the limits of conditional probabilities computed from strictly positive assignments of probabilities to every action at every node. These assignments need not be mixed strategies, as they need not respect the information sets of the game: For example, a given action may have different probabilities at two nodes in the same information set.

This section develops "no-signaling" conditions in the spirit of our definition of PEBE, and shows that they imply the distributions generating the conditional probability system are indeed mixed strategies, so that the associated beliefs are consistent.

DEFINITION 6.1. (i) An *extended assessment* (ν, π) is a profile of strategies π and a conditional probability system ν on the terminal nodes.

(ii) Let μ denote the conditional beliefs associated with ν. The extended assessment (ν, π) is *generally reasonable* if

(1) for all information sets h, actions $a \in A(h)$, and nodes $x \in h$,

$$\pi(a \mid h) = \mu(s(x, a) \mid x);$$

(2) for all information sets h, nodes x and y in h, and actions $a \in A(h)$,

$$\mu(s(x, a) \mid s(x, a), s(y, a)) = \mu(x \mid x, y).$$

Condition (1) incorporates Bayes' rule as well as "no-signaling-what-you-don't-know for positive probability actions": The probability of action $a \in A(h)$ is the same at all nodes in h. Condition (2), which is implied by condition (1) when $\pi(a \mid h) > 0$,[4] guarantees no-signaling-what-you-don't-know even when a has zero probability at h. Conditions (1) and (2) which require that μ respects the information structure, are very weak. As we will see, the strong assumption is that of the existence of a conditional probability system.

PROPOSITION 6.1. *An extended assessment* (v, π) *is generally reasonable only if* $(\mu(\cdot \mid h), \pi)$ *is consistent, where* $\mu(\cdot \mid h)$ *is the restriction of* μ *to nodes in* h. *Conversely, if* (μ, π) *is consistent, it can be extended to a generally reasonable assessment* (v, π).

Proof of Proposition 6.1. If (μ, π) is consistent, then the totally mixed strategies $\pi^n \to \pi$ generate a conditional probability system v on terminal nodes whose conditional probabilities at each information set converge to μ. Hence, a consistent assessment can be extended to a generally reasonable extended assessment.

Conversely, if (v, π) is a generally reasonable extended assessment, then from Myerson [9] there is a sequence of strictly positive probability distributions P^n on Z such that for $Z_1 \subseteq Z_2 \subseteq Z$, $v(Z_1 \mid Z_2) = \lim_{n \to \infty}(P^n(Z_1)/P^n(Z_2))$.

For any two subsets A and B of X with $Z(A) \subseteq Z(B)$ define $\mu^n(A \mid B) = P^n(Z(A))/P^n(Z(B))$.

Now arbitrarily select a single node x_h at each information set h, and define

$$\pi^n(a \mid h) = \mu^n(\jmath(x_h, a) \mid x_h) = P^n(Z(\jmath(x_h, a)))/P^n(Z(x_h)).$$

$\pi^n(a \mid h)$ is strictly positive for all a and h because P^n has full support, and $\sum_{a' \in A(h)} \pi^n(a' \mid h) = 1$ by construction. So the π^n are totally mixed strategies. It remains to show that $\pi^n \to \pi$ and that the beliefs $\hat{\mu}^n$ generated

[4] We have

$$\mu(\jmath(x, a) \mid \jmath(x, a), \jmath(y, a)) = \frac{v(Z(\jmath(x, a)))}{v(Z(\jmath(x, a))) + v(Z(\jmath(y, a)))}$$

$$= \frac{v(Z(x))\, \pi(a \mid h)}{v(Z(x))\, \pi(a \mid h) + v(Z(y))\, \pi(a \mid h)}$$

$$= \mu(x \mid x, y),$$

where the first and the third equalities come from the definition of μ and the second results from (i) and the definition of μ.

by π^n converge to the conditional beliefs μ generated by ν. That $\pi^n \to \pi$ results from condition (1) of Definition 6.1. Next, from the definition of π^n, for any h and for any x' in h,

$$\lim_{n\to\infty} \pi^n(a \mid x') = \lim \mu^n(\jmath(x_h, a) \mid x_h) = \nu(Z(\jmath(x_h, a))/Z(x_h) = \pi(a \mid h),$$

where the last equality follows from condition (1) of Definition 6.1.

Finally, we check that the conditional beliefs $\hat{\mu}^n$ computed from π^n converge to the conditional beliefs μ generated by ν. By definition, for all h and $x \in h$, $\mu^n(x \mid x, x_h) = P^n(Z(x))/[P^n(Z(x)) + P^n(Z(x_h))]$ and from condition (2) of Definition 6.1 and $\mu^n \to \mu$ we have

$$\begin{aligned}\mu(x \mid x, x_h) &= \lim_{n\to\infty} P^n(Z(\jmath(x, a))/[P^n(Z(\jmath(x, a))) + P^n(Z(\jmath(x_h, a)))] \\ &= \lim_{n\to\infty} P^n(Z(x))/[P^n(Z(x)) + P^n(Z(x_h))].\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{n\to\infty} P^n(Z(\jmath(x, a)))/P^n(Z(x)) &= \lim_{n\to\infty} P^n(Z(\jmath(x_h, a))/P^n(Z(x_h)) \\ &= \lim_{n\to\infty} \pi^n(a \mid h),\end{aligned}$$

and so it does not matter which x_h is chosen as reference node in information set h to define the $\pi^n(\cdot \mid h)$. Therefore (as one can check) the beliefs $\hat{\mu}^n$ generated by the strategies π^n yield in the limit the same first-order beliefs μ as the converging sequence μ^n obtained from ν. Thus, (ν, π) defines a consistent assessment. Q.E.D.

Appendix

Proof of Proposition 4.1. We will say an extended belief system (μ^n, η_i^n) converges to (μ, η_i) when n tends to infinity if for each history h^t, $\mu^n(\theta_i \mid h^t) \to \mu(\theta_i \mid h^t)$ and $\eta_i^n(\theta_i \mid \tilde{\Theta}_i, h^t) \to \eta_i(\theta_i \mid \tilde{\Theta}_i, h^t)$ for all $\tilde{\Theta}_i \subseteq \Theta_i$ and all θ_i.

Since each $\eta_i(\cdot \mid \cdot, h^t)$ is a conditional probability system, the "infinitely more likely than" relation defines a finite partition, or relative ordering, of Θ_i. (This is proved by Myerson [9].) A labelling $k_i^t\colon \Theta_i \to \mathbb{R}$ is compatible with η_i if $k_i^t(\theta_i) < k_i^t(\hat{\theta}_i)$ if and only if $\eta_i(\theta_i \mid (\theta_i, \hat{\theta}_i), h^t) = 1$. (We will fix the history h^t so that the notation $k_i^t(\cdot)$ should not be confusing.)

As in the proof of Proposition 3.1, we proceed by induction, beginning with a two-period game whose prior beliefs $(\mu(\cdot \mid \cdot, h^0), \eta_i(\cdot \mid \cdot, h^0))$ are reasonable (but need not be strictly positive).

Fix h^0 *and* $h^1 = (h^0, a^0)$, and suppose we are given an ordering $k_i^0(\theta_i)$, compatible with $\eta_i(\cdot \mid \cdot, h^0)$.

Consider a sequence of strictly positive reasonable conditional belief systems $(\mu^n(\cdot \mid h^0), \eta_i^n(\cdot \mid \cdot, h^0))$

(a) that converges to $(\mu(\cdot \mid h^0), \eta_i(\cdot \mid \cdot, h^0))$ and

(b) such that letting $k_i^0 \equiv \min_{\Theta_i} k_i^0(\theta_i)$, there exists a function $\lambda_i^0(\theta_i, n) \to \lambda_i^0(\theta_i)$ with $0 < \lambda_i^0(\theta_i) < +\infty$ for all θ_i and $\mu^n(\theta_i \mid h^0) = \lambda_i^0(\theta_i, n)(1/n)^{k_i^0(\theta_i) - k_i^0}$. (That is, $\mu^n(\theta_i \mid h^0)$ is $O(k_i^0(\theta_i) - k_i^0)$, where $O(k)$ is the class of functions converging to zero at rate $(1/n)^k$).

We claim that conditions (1′), (2′), and (3′) imply that there exists a sequence of totally mixed strategies $\pi_i^n(a_i \mid \theta_i, h^0) \to \pi_i(a_i \mid \theta_i, h^0)$ such that (a) the conditional belief system $(\mu^n(\cdot \mid h^1), \eta_i^n(\cdot \mid \cdot, h^1))$ obtained from Bayes' rule converges to $(\mu(\cdot \mid h^1), \eta_i(\cdot \mid \cdot, h^1))$, (b) there exists $\lambda_i^1(\theta_i, n)$, such that $\mu^n(\theta_i \mid h^1) = \lambda_i^0(\theta_i, n)(1/n)^{k_i^1(\theta_i) - \min_{\theta_i'} k_i^1(\theta_i')}$ and $\lambda_i^1(\theta_i, n) \to \lambda_i^1(\theta_i)$ with $0 < \lambda_i^1(\theta_i) < +\infty$.

Note that condition (3′) implies that $\eta_i(\theta_i \mid \cdot, h^1) = \eta_i(\theta_i \mid \cdot, (h^0, a_i))$, where $a_i \equiv a_i^0$. Note also that the conditional probability system at date 1 imposes only a relative ordering of the types θ_i; that is, many different functions k_i^1 are compatible with the same ordering. We use this degree of freedom to specify a particular choice of k_i^1 that allows us to find trembles such that conditions (a) and (b) are satisfied at date 1 if they are satisfied at date 0.

So assign a real number $k_i^1(\theta_i)$ as follows:

$$\begin{aligned} &\text{(i)} && \text{if } \pi_i(a_i \mid \theta_i, h^0) > 0, \text{ then } k_i^1(\theta_i) = k_i^0(\theta_i); \\ &\text{(ii)} && \text{if } \pi_i(a_i \mid \theta_i, h^0) = 0, \text{ then } k_i^1(\theta_i) > k_i^0(\theta_i); \qquad \text{(A.1)} \\ &\text{(iii)} && k_i^1(\theta_i) < k_i^1(\hat{\theta}_i) \text{ if and only if } \eta_i(\theta_i \mid (\theta_i, \hat{\theta}_i), h^1) = 1 \end{aligned}$$

(i.e., iff θ_i is infinitely more likely than $\hat{\theta}_i$ at date 1).

(A.1(iii)) is simply the condition that the labelling be compatible with the conditional probability system, which we know is feasible. That a compatible labelling exists that satisfies (i) and (ii) results from Bayes' rule (1′), which implies that the relative ordering of two types at date 1 which play a_i with positive probability is the same as that at date 0, and that if $\eta_i(\theta_i \mid (\theta_i, \hat{\theta}_i), h^0) > 0$ and $\pi_i(a_i \mid \theta_i, h^0) > 0$, then $k_i^1(\theta_i) \leq k_i^1(\hat{\theta}_i)$, with strict inequality if $\pi_i(a_i \mid \hat{\theta}_i, h^0) = 0$.

Now we construct "preliminary trembles" $\hat{\pi}^n$, which generate the desired relative probabilities, but do not necessarily add up to 1 at each information set. Later we will modify these to obtain the totally mixed strategies π^n.

First, for all θ_i with $\pi_i(a_i \mid \theta_i, h^0) > 0$, set $\hat{\pi}_i^n(a_i \mid \theta_i, h^0) = \pi_i(a_i \mid \theta_i, h^0)$. Second, pick a number k in the range of $k_i^1(\cdot)$.

(α) If $\exists \theta_i^k$ with $\pi_i(a_i \mid \theta_i^k, h^0 > 0$, and $k_i^1(\theta_i^k) = k$, then for any θ_i with $k_i^1(\theta_i) = k$ and $\pi_i(a_i \mid \theta_i, h^0) = 0$, set

$$\hat{\pi}_i^n(a_i \mid \theta_i, h^0) \equiv \frac{\eta_i^n(\theta_i^k \mid (\theta_i^k, \theta_i), h^0)\, \pi_i(a_i \mid \theta_i^k, h^0)\, \eta_i(\theta_i \mid (\theta_i^k, \theta_i), h^1)}{\eta_i^n(\theta_i \mid (\theta_i^k, \theta_i), h^0)\, \eta_i(\theta_i^k \mid (\theta_i^k, \theta_i), h^1)} \tag{A.2}$$

(note that all terms in the numerator and denominator of the right-hand side of (A.2) are strictly positive).

(β) If for all θ_i with $k_i^1(\theta_i) = k$, $\pi_i(a_i \mid \theta_i, h^0) = 0$, then pick an arbitrary such θ_i^k and set

$$\hat{\pi}_i^n(a_i \mid \theta_i^k, h^0) = \left(\frac{1}{n}\right)^{k_i^1(\theta_i^k) - k_i^0(\theta_i^k)}. \tag{A.3}$$

Then for all θ_i with $k_i^1(\theta_i) = k$, set

$$\hat{\pi}_i^n(a_i \mid \theta_i, h^0) \equiv \frac{\eta_i^n(\theta_i^k \mid (\theta_i^k, \theta_i), h^0)\, \hat{\pi}_i^n(a_i \mid \theta_i^k, h^0)\, \eta_i(\theta_i \mid (\theta_i^k, \theta_i), h^1)}{\eta_i^n(\theta_i \mid (\theta_i^k, \theta_i), h^0)\, \eta_i(\theta_i^k \mid (\theta_i^k, \theta_i), h^1)} \tag{A.4}$$

(again, all terms in the numerator and denominator of the right-hand side of (A.4) are strictly positive).

Note that these definitions of the $\hat{\pi}$ insure that the relative posterior probabilities of θ_i and $\hat{\theta}_i$ are those given by Bayes' rule for θ_i and $\hat{\theta}_i$ of the same posterior order.

After we have constructed the $\hat{\pi}^n$ as above for each k in the range of $k_i^1(\cdot)$, we adjust the $\hat{\pi}^n$ to assure that they sum to one. Let $Z(\theta_i) = \{\tilde{a}_i \mid \pi_i(\tilde{a}_i \mid \theta_i, h^0) = 0\}$ and let $\# Z(\theta_i)$ be the number of actions in $Z(\theta_i)$. Set

$$\pi_i^n(a_i \mid \theta_i, h^0) \equiv \hat{\pi}_i^n(a_i \mid \theta_i, h^0) \qquad \text{if} \quad \pi_i(a_i \mid \theta_i, h^0) = 0 \tag{A.5}$$

and

$$\pi_i^n(a_i \mid \theta_i, h^0) \equiv \pi_i(a_i \mid \theta_i, h^0) - \frac{\sum_{\tilde{a}_i \in Z(\theta_i)} \hat{\pi}_i^n(\tilde{a}_i \mid \theta_i, h^0)}{\#(A_i(h^0) \backslash Z(\theta_i))}, \tag{A.6}$$

if $\pi_i(a_i \mid \theta_i, h^0) > 0$. Clearly, for n large enough, those trembles are positive.

This ensures that $\sum_{a_i} \pi_i^n(a_i \mid \theta_i, h^0) = 1$.

Now we check that $\pi_i^n(a_i \mid \theta_i, h^0) \to \pi_i(a_i \mid \theta_i, h^0)$. This is obvious if $\pi_i(a_i \mid \theta_i, h^0) > 0$. If $\pi_i(a_i \mid \theta_i, h^0) = 0$, there is a θ_i^k with $k_i^1(\theta_i^k) = k_i^1(\theta_i)$. There are two possible cases:

In case (α), the facts that $k_i^1(\theta_i^k) = k_i^1(\theta_i)$, $\pi_i(a_i \mid \theta_i^k, h^0) > 0$, $\pi_i(a_i \mid \theta_i, h^0) = 0$, and (A.1(i) and (ii)) imply that $k_i^0(\theta_i^k) > k_i^0(\theta_i)$, or equivalently, $\eta_i(\theta_i^k \mid (\theta_i^k, \theta_i), h^0) = 0$ and $\eta_i(\theta_i \mid (\theta_i^k, \theta_i), h^0) = 1$. Now consider (A.2). By assumption, $\eta_i^n(\cdot \mid \cdot, h^0)$ converges to $\eta_i(\cdot \mid \cdot, h^0)$. Furthermore, the three terms on the right-hand side of (A.2) that are not indexed by n are strictly positive. We conclude that $\hat{\pi}_i^n(a_i \mid \theta_i,, h^0) = \pi_i^n(a_i \mid \theta_i, h^0)$ converges to $0 = \pi_i(a_i \mid \theta_i, h^0)$. Note further that, from (A.2) and

$$\frac{\eta_i^n(\theta_i^k \mid (\theta_i^k, \theta_i), h^0)}{\eta_i^n(\theta_i \mid (\theta_i^k, \theta_i), h^0)} \equiv \frac{\mu^n(\theta_i^k \mid h^0)}{\mu^n(\theta_i \mid h^0)} = \frac{\lambda_i^0(\theta_i^k, n)}{\lambda_i^0(\theta_i, n)} \left(\frac{1}{n}\right)^{k_i^0(\theta_i^k) - k_i^0(\theta_i)}, \tag{A.7}$$

there exists $\gamma_i^0(\theta_i, n) \to \gamma_i^0(\theta_i)$ such that $0 < \gamma_i^0(\theta_i) < +\infty$ and

$$\hat{\pi}_i^n(a_i \mid \theta_i, h^0) = \gamma_i^0(\theta_i, n) \left(\frac{1}{n}\right)^{k_i^1(\theta_i) - k_i^0(\theta_i)}. \tag{A.8}$$

Thus the tremble constructed for type θ_i belongs to $O(k_i^1(\theta_i) - k_i^0(\theta_i))$.

In case (β), (A.3), (A.4), the induction hypothesis, and the fact that $k_i^1(\theta_i^k) = k_i^1(\theta_i)$ imply that there exists $\gamma_i^0(\theta_i, n) \to \gamma_i^0(\theta_i)$ such that $0 < \gamma_i^0(\theta_i) < +\infty$ and

$$\hat{\pi}_i^n(a_i \mid \theta_i, h^0) = \gamma_i^0(\theta_i, n) \left(\frac{1}{n}\right)^{k_i^1(\theta_i) - k_i^0(\theta_i)}. \tag{A.9}$$

But from (ii), $k_i^1(\theta_i) > k_i^0(\theta_i)$, so that $\hat{\pi}_i^n(a_i \mid \theta_i, h^0)$ converges to $0 = \pi_i(a_i \mid \theta_i, h^0)$. Furthermore, the tremble belongs to $O(k_i^1(\theta_i) - k_i^0(\theta_i))$.

Note that in both cases (α) and (β), induction hypothesis (b) at date 0 together with (A.8) or (A.9) implies that induction hypothesis (b) is satisfied at date 1.

Finally, we check that the totally mixed strategies π_i^n together with the full-support priors μ^n give rise to conditional probability systems $\eta_i^n(\cdot \mid \cdot, h^1)$ that converge to $\eta_i(\cdot \mid \cdot, h^1)$.

Fix θ_i and θ_i'. First suppose that $0 < \eta_i(\theta_i \mid (\theta_i, \theta_i'), h^1) < 1$ (that is, $k_i^1(\theta_i) = k_i^1(\theta_i')$). Bayes' rule implies that

$$\begin{aligned}
&\eta_i^n(\theta_i \mid (\theta_i, \theta_i'), h^1) \\
&\quad \equiv \frac{\mu^n(\theta_i \mid h^1)}{\mu^n(\theta_i \mid h^1) + \mu^n(\theta_i' \mid h^1)} \\
&\quad = \frac{\mu^n(\theta_i \mid h^0)\, \pi_i^n(a_i \mid \theta_i, h^0)}{\mu^n(\theta_i \mid h^0)\, \pi_i^n(a_i \mid \theta_i, h^0) + \mu^n(\theta_i' \mid h^0)\, \pi_i^n(a_i \mid \theta_i', h^0)} \\
&\quad = \frac{\eta_i^n(\theta_i \mid (\theta_i, \theta_i'), h^0)\, \pi_i^n(a_i \mid \theta_i, h^0)}{\eta_i^n(\theta_i \mid (\theta_i, \theta_i'), h^0)\, \pi_i^n(a_i \mid \theta_i, h^0) + \eta_i^n(\theta_i' \mid (\theta_i, \theta_i'), h^0)\, \pi_i^n(a_i \mid \theta_i', h^0)}.
\end{aligned} \tag{A.10}$$

(A.10), together with (A.2) or (A.4) (depending on whether (α) or (β) applies), implies that $\eta_i^n(\theta_i \mid (\theta_i, \theta_i'), h^1)$ converges to $\eta_i(\theta_i \mid (\theta_i, \theta_i'), h^1) = \eta_i(\theta_i \mid (\theta_i, \theta_i^k), h^1)/(\eta_i(\theta_i \mid (\theta_i, \theta_i^k), h^1) + \eta_i(\theta_i' \mid (\theta_i', \theta_i^k), h^1))$.

Last, note that the induction hypothesis (b) at date 0 and (A.7) (or (A.8)) imply that for any θ_i and θ_i',

$$\eta_i^n(\theta_i \mid (\theta_i, \theta_i'), h^1)$$
$$= \frac{\lambda_i^0(\theta_i, n)\,\gamma_i^0(\theta_i, n)(1/n)^{k_i^1(\theta_i)}}{\lambda_i^0(\theta_i, n)\,\gamma_i^0(\theta_i, n)(1/n)^{k_i^1(\theta_i)} + \lambda_i^0(\theta_i', n)\,\gamma_i^0(\theta_i', n)(1/n)^{k_i^1(\theta_i')}}. \tag{A.11}$$

Hence if $k_i^1(\theta_i) > k_i^1(\theta_i')$, $\eta_i^n(\theta_i \mid (\theta_i, \theta_i'), h^1)$ converges to $0 = \eta_i(\theta_i \mid (\theta_i, \theta_i'), h^1)$.

To conclude, the trembles $\pi^n(a_i \mid \theta_i, h^0)$ lead to a strictly positive reasonable belief system that (a) converges to $(\mu(\cdot \mid h^1), \eta_i(\cdot \mid \cdot, h^1))$ for all h^1 and satisfies (b) as well. We now proceed by induction as in the proof of Proposition 3.1. By inductive hypothesis, there are totally mixed beliefs $\mu^n(\cdot \mid h^{t-1}) \to \mu(\cdot \mid h^{t-1})$, whose associated conditional beliefs converge to $\eta_i(\cdot \mid \cdot, h^{t-1})$ in a way consistent with (b). Thus we can use the algorithm defined above to construct trembles that yield strictly positive belief systems that converge to $(\mu(\cdot \mid h^t), \eta(\cdot \mid \cdot, h^t))$ for any h^t. This proves in particular that the beliefs $\mu(\cdot \mid h^t)$ are consistent. Q.E.D.

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