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Book Review

JOHN D. DOLLARD AND CHARLES N. FRIEDMAN, *Product Integration*, with an Appendix by Pesi Masani, Addison–Wesley, Reading, Mass., 1979.

Product integration got off to a brilliant start in 1887–1888, the creation of Vito Volterra and Giuseppe Peano.[1] Essentially by treating matrices as hypercomplex numbers, one can obtain an elegant and suggestive fresh conceptualization of much of the theory of linear ordinary differential equations. Given the system

$$dz_i/dt = \sum_{j=1}^{n} a_{ij}(t)\, z_j, \qquad i = 1,\dots, n, \tag{1}$$

where the independent variable t may be real or complex, and initial conditions $z_i(0) = c_i$, we can write the solution as a vector-valued function $\mathbf{z}(t)$. Moreover

$$\mathbf{z}(t) = B(t)\,\mathbf{c}, \tag{2}$$

where $B(t)$ is a matrix-valued function called the *matrizant* of (1). Specifically, $B(t)$ is the (definite) left *product integral* $\int_0^t [I + A(\tau)]\, d\tau$, in the sense that

$$B(0) = I, \qquad \text{the identity } n \times n \text{ matrix}, \tag{3a}$$

and (correspondingly) that it is the limit of products

$$B_\pi(t) = \prod_{j=1}^{n} [I + A(\tau_j)\,\Delta t_j] \tag{3b}$$

(multiplied from right to left), for partitions π by mesh points $\tau_0 = 0 < \tau_1 < \cdots < \tau_n = t$, as $\max |\Delta \tau_j| \downarrow 0$.

For t on the real line, this can be considered as defining a Riemann "product integral"; in the complex t-plane domain, one must pick a path Γ and show that the integral is independent the path chosen (up to classes of homotopic paths).

The last systematic treatise on the subject in book form was co-authored by Volterra and B. Hostinsky ("Opérations Infinitésimales Linéaires," Gauthier–Villars, 1938) just 50 years after its creation. Although this reviewer and Pesi Masani have extended the theory to linear and nonlinear operators on Banach spaces, these extensions were not systematically followed up or applied. Likewise, the theory of product integration was nearly dormant from 1940 to 1970, being invoked only sporadically and occasionally by Masani, H. S. Wall, J. S. MacNerney, and a few others.

During the past decade the subject has had a revival, and so a book reporting on this work and the classical theory is very timely. A brief summary of its contents and level of sophistication follows.

The first chapter deals with the product integration of matrix-valued functions $A(t)$ of a real variable t. Taken with Appendix I, it is very elementary. It can be read by college undergraduates who have had linear algebra and the advanced calculus. The second chapter, only a third as long, treats matrix functions of a complex variable. It derives an analogue of

[1] Although seldom given credit for being a co-discoverer of product integration, Peano's publications (*Atti. Roy. Accad. Sci. Torino* **20** (Feb. 1887), and *Math. Ann.* **32** (1888), 450–456) were as early as Volterra's.

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Cauchy's integral formula in a more rigorous form than that of Volterra and Hostinsky, but barely touches the monodromy group. For this one must skip to Masani's Appendix II.

Chapter 3, entitled "Strong product integration," is necessarily more abstract, and introduces the additive Bochner integral. This reviewer thinks that a more direct treatment, carried out in essentially the same spirit in Chapters 1 and 3, would be more in the spirit of the subject. A special and very worthwhile feature of this chapter is its treatment of *unbounded* infinitesimal unitary operators.

Chapter 4 on "Applications" begins with the Schrödinger equation and ends with a version of the Hille–Yosida theorem on contraction semigroups and an integration of an idea of Feynman. It is written in the style of a research monograph, as is Chapter 5 on "Product integration of measures." This concept, attributed to H. S. Wall, seems closely related to the Chapman–Kolmogorov equation treated in a very different notation and terminology by Masani in Appendix II. Improper integrals, and another application to the Schrödinger equation are included. Chapter 6 surveys briefly a number of additional topics and generalizations. This is supplemented by an extensive bibliography, and makes the book a very useful reference.

Finally, the book is rounded out by the 30–page Appendix II, by Masani. This relates product integration to the theory of Lie groups acting on manifolds, a very important application not mentioned elsewhere. It then takes up succinctly a number of topics from classical analysis, including Riemann's monodromy problem, Cousin's multiplicative problem, Cartan's factorization lemma, G. D. Birkhoff's factorization lemma, and a 1955 Theorem of Potapov, which valuably supplement Chapter 2.

In summary, the book by Dollard and Friedman, after an elementary introduction written for novices, quickly changes to a "Bericht" covering an impressive if somewhat kaleidoscopic range of topics on a much more advanced level. With its extensive bibliography and its Appendix by Masani, it constitutes a valuable survey of what is known about an interesting and somewhat neglected subject.

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