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On the Dynkin index of a principal $\mathfrak{sl}_2$-subalgebra

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Abstract

Let $\mathfrak{g}$ be a simple Lie algebra over an algebraically closed field of characteristic zero. The goal of this note is to prove a closed formula for the Dynkin index of a principal $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$.

Keywords: Simple Lie algebra; Dynkin index; Freudenthal's strange formula

Introduction

The ground field $\Bbbk$ is algebraically closed and of characteristic zero. Let $\mathfrak{g}$ be a simple Lie algebra over $\Bbbk$. The goal of this note is to prove a closed formula for the Dynkin index of a principal $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$, see Theorem 3.2. The key step in the proof uses the "strange formula" of Freudenthal–de Vries. As an application, we (1) compute the Dynkin index any simple $\mathfrak{g}$-module regarded as $\mathfrak{sl}_2$-module and (2) obtain an identity connecting the exponents of $\mathfrak{g}$ and the dual Coxeter numbers of both $\mathfrak{g}$ and $\mathfrak{g}^\vee$, see Section 4.

1. The Dynkin index of representations and subalgebras

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra of rank n. Let $\mathfrak{t}$ be a Cartan subalgebra, and Δ the set of roots of $\mathfrak{t}$ in $\mathfrak{g}$. Choose a set of positive roots Δ^+ in Δ. Let Π be the set of

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simple roots and θ the highest root in Δ^+. As usual, $\rho = \frac{1}{2}\sum_{\gamma>0}\gamma$. The $\mathbb{Q}$-span of all roots is a ($\mathbb{Q}$-)subspace of $\mathfrak{t}^*$, denoted $\mathcal{E}$. Choose a non-degenerate invariant symmetric bilinear form $(,)_\mathfrak{g}$ on $\mathfrak{g}$ as follows. The restriction of $(,)_\mathfrak{g}$ to $\mathfrak{t}$ is non-degenerate, hence it induces the isomorphism of $\mathfrak{t}$ and $\mathfrak{t}^*$ and a non-degenerate bilinear form on $\mathfrak{t}^*$. We require that $(\theta,\theta)_\mathfrak{g} = 2$, i.e., $(\beta,\beta)_\mathfrak{g} = 2$ of any long root β in Δ.

Definition 1 *(E.B. Dynkin).*

(1) Let $\mathfrak{s}$ be a simple subalgebra of $\mathfrak{g}$. The *Dynkin index* of $\mathfrak{s}$ in $\mathfrak{g}$ is defined by

$$\mathsf{ind}(\mathfrak{s} \hookrightarrow \mathfrak{g}) = \frac{(x,x)_\mathfrak{g}}{(x,x)_\mathfrak{s}}, \quad x \in \mathfrak{s}.$$

(2) If $\nu : \mathfrak{g} \to \mathfrak{sl}(V)$ is a representation of $\mathfrak{g}$, then the *Dynkin index of the representation*, denoted $\mathsf{ind}_\mathrm{D}(\mathfrak{g}, V)$ or $\mathsf{ind}_\mathrm{D}(\mathfrak{g}, \nu)$, is defined by

$$\mathsf{ind}_\mathrm{D}(\mathfrak{g}, V) = \mathsf{ind}\big(\mathfrak{g} \hookrightarrow \mathfrak{sl}(V)\big).$$

It is not hard to verify that, for the simple Lie algebra $\mathfrak{sl}(V)$, the normalised bilinear form is given by $(x,x)_{\mathfrak{sl}(V)} = \mathrm{tr}(x^2)$, $x \in \mathfrak{sl}(V)$. Therefore, a more explicit expression for the Dynkin index of a representation $\nu : \mathfrak{g} \to \mathfrak{sl}(V)$ is

$$\mathsf{ind}_\mathrm{D}(\mathfrak{g}, V) = \frac{\mathrm{tr}(\nu(x)^2)}{(x,x)_\mathfrak{g}}. \tag{1.1}$$

Conversely, the index of a simple subalgebra can be expressed via indices of representations. Namely,

$$\mathsf{ind}(\mathfrak{s} \hookrightarrow \mathfrak{g}) = \frac{\mathsf{ind}_\mathrm{D}(\mathfrak{s}, \mathfrak{g})}{\mathsf{ind}_\mathrm{D}(\mathfrak{g}, \mathrm{ad}_\mathfrak{g})}. \tag{1.2}$$

The denominator in the right-hand side represents the index of the adjoint representation of $\mathfrak{g}$, and the numerator represents the index of the $\mathfrak{s}$-module $\mathfrak{g}$.

The following properties easily follow from the definition:

Multiplicativity: If $\mathfrak{h} \subset \mathfrak{s} \subset \mathfrak{g}$ are simple Lie algebras, then $\mathsf{ind}(\mathfrak{h} \subset \mathfrak{s}) \cdot \mathsf{ind}(\mathfrak{s} \subset \mathfrak{g}) = \mathsf{ind}(\mathfrak{h} \subset \mathfrak{g})$.

Additivity: $\mathsf{ind}_\mathrm{D}(\mathfrak{g}, V_1 \oplus V_2) = \mathsf{ind}_\mathrm{D}(\mathfrak{g}, V_1) + \mathsf{ind}_\mathrm{D}(\mathfrak{g}, V_2)$. It is therefore sufficient to determine the indices for the irreducible representations.

Theorem 1.1. *(Dynkin [2, Theorem 2.5].) Let V_λ be a simple finite-dimensional $\mathfrak{g}$-module with highest weight λ. Then*

$$\mathsf{ind}_\mathrm{D}(\mathfrak{g}, V_\lambda) = \frac{\dim V_\lambda}{\dim \mathfrak{g}}(\lambda, \lambda + 2\rho)_\mathfrak{g}.$$

Although it is not obvious from the definition, the Dynkin index of a representation is an integer. This was proved by E.B. Dynkin [2, Theorem 2.2] using lengthy classification results.

Later, he gave a better proof that is based on a topological interpretation of the index. A short algebraic proof is given in [5, Chapter I, §3.10].

Example 1.2.

(1) Let R_d be the simple $\mathfrak{sl}_2$-module of dimension $d+1$. Then $\mathsf{ind}_{\mathrm{D}}(\mathfrak{sl}_2, \mathsf{R}_d) = \binom{d+2}{3}$.
(2) Recall that θ is the highest root in Δ^+. By Theorem 1.1,

$$\mathsf{ind}_{\mathrm{D}}(\mathfrak{g}, \mathrm{ad}) = (\theta, \theta + 2\rho)_{\mathfrak{g}} = (\theta, \theta)_{\mathfrak{g}}\big(1 + (\rho, \theta^\vee)_{\mathfrak{g}}\big) = 2\big(1 + (\rho, \theta^\vee)_{\mathfrak{g}}\big).$$

Note that the value $(\rho, \theta^\vee)_{\mathfrak{g}}$ does not depend on the normalisation of the bilinear form. The integer $1 + (\rho, \theta^\vee)$ is customary called the *dual Coxeter number* of $\mathfrak{g}$, and we denote it by $h^*(\mathfrak{g})$. Thus, $\mathsf{ind}_{\mathrm{D}}(\mathfrak{g}, \mathrm{ad}) = 2h^*(\mathfrak{g})$. In the simply-laced case, $h^*(\mathfrak{g}) = h(\mathfrak{g})$—the usual Coxeter number. For the other simple Lie algebras, we have $h^*(\mathbf{B}_n) = 2n-1$, $h^*(\mathbf{C}_n) = n+1$, $h^*(\mathbf{F}_4) = 9$, $h^*(\mathbf{G}_2) = 4$.

Andreev, Vinberg, and Elashvili applied the Dynkin index of representations to some invariant-theoretic problem [1]. To this end, they adjusted the index so that it does not depend on the choice of a bilinear form on $\mathfrak{g}$.

Definition 2 *(Andreev–Vinberg–Elashvili, 1967).* Let $\nu : \mathfrak{g} \to \mathfrak{sl}(V)$ be a finite-dimensional representation of a simple Lie algebra. Then

$$\mathsf{ind}_{\mathrm{AVE}}(\mathfrak{g}, V) := \frac{\mathsf{ind}_{\mathrm{D}}(\mathfrak{g}, V)}{\mathsf{ind}_{\mathrm{D}}(\mathfrak{g}, \mathrm{ad})} = \frac{\mathrm{tr}(\nu(x)^2)}{\mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(x)^2)}, \quad x \in \mathfrak{g}.$$

It follows that $\mathsf{ind}_{\mathrm{AVE}}(\mathfrak{g}, \mathrm{ad}_{\mathfrak{g}}) = 1$ and

$$\mathsf{ind}_{\mathrm{AVE}}(\mathfrak{g}, V_\lambda) = \frac{\dim V_\lambda}{\dim \mathfrak{g}} \cdot \frac{(\lambda, \lambda + 2\rho)_{\mathfrak{g}}}{(\theta, \theta + 2\rho)_{\mathfrak{g}}}.$$

2. The "strange formula"

Let $\mathcal{K}$ be the Killing form on $\mathfrak{g}$, i.e., $\mathcal{K}(x, x) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(x)^2)$, $x \in \mathfrak{g}$. The induced bilinear form on $\mathfrak{t}^*$ (and $\mathcal{E}$) is denoted by $\langle , \rangle$. It is the so-called *canonical* bilinear form on $\mathcal{E}$. The canonical bilinear form is characterised by the following property:

$$\langle v, v\rangle = \sum_{\gamma \in \Delta} \langle v, \gamma\rangle\langle v, \gamma\rangle = 2\sum_{\gamma > 0} \langle v, \gamma\rangle\langle v, \gamma\rangle \quad \text{for any } v \in \mathcal{E}. \tag{2.1}$$

The "strange formula" of Freudenthal–de Vries (see [3, 47.11]) is

$$\langle \rho, \rho\rangle = \frac{\dim \mathfrak{g}}{24}.$$

Using our normalisation of $(,)_{\mathfrak{g}}$, the "strange formula" reads

$$(\rho, \rho)_{\mathfrak{g}} = \frac{\dim \mathfrak{g}}{12} h^*(\mathfrak{g}). \tag{2.2}$$

Indeed, it is well known that $\langle\theta,\theta\rangle = 1/h^*(\mathfrak{g})$ (see e.g. [6, Lemma 1.1]). Therefore, the transition factor between two forms $\langle,\rangle$ and $(,)_{\mathfrak{g}}$ (considered as forms on $\mathcal{E}$) equals $2h^*(\mathfrak{g})$. Using the transition factor, we can also rewrite Eq. (2.1) in terms of $(,)_{\mathfrak{g}}$:

$$h^*(\mathfrak{g})(v,v)_{\mathfrak{g}} = \sum_{\gamma>0}(v,\gamma)_{\mathfrak{g}}(v,\gamma)_{\mathfrak{g}}. \tag{2.3}$$

3. The index of a principal $\mathfrak{sl}_2$-subalgebra

If $e\in\mathfrak{g}$ is nilpotent, then the exists a subalgebra $\mathfrak{a}\subset\mathfrak{g}$ such that $\mathfrak{a}\simeq\mathfrak{sl}_2$ and $e\in\mathfrak{a}$ (Morozov, Jacobson). If e is a *principal* nilpotent element, then the corresponding $\mathfrak{sl}_2$-subalgebra is also called principal. (See [2, §9] and [4, Section 5] for properties of principal $\mathfrak{sl}_2$-subalgebras.) Let $(\mathfrak{sl}_2)^{\mathrm{pr}}$ be a principal $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$. In this section, we obtain a uniform expression for $\mathsf{ind}((\mathfrak{sl}_2)^{\mathrm{pr}}\hookrightarrow\mathfrak{g})$.

Recall that Δ has at most two root lengths. Let θ_s denote the short dominant root in Δ^+. (Hence $\theta=\theta_s$ if and only if Δ is simply-laced.) Set $r=\|\theta\|^2/\|\theta_s\|^2\in\{1,2,3\}$. Along with $\mathfrak{g}$, we also consider the Langlands dual algebra $\mathfrak{g}^\vee$, which is determined by the dual root system $\Delta^\vee$. Since the Weyl groups of $\mathfrak{g}$ and $\mathfrak{g}^\vee$ are isomorphic, we have $h(\mathfrak{g})=h(\mathfrak{g}^\vee)$. However, the dual Coxeter numbers can be different (cf. $\mathbf{B}_n$ and $\mathbf{C}_n$).

The half-sum of positive roots for $\mathfrak{g}^\vee$ is

$$\rho^\vee := \frac{1}{2}\sum_{\gamma>0}\gamma^\vee = \sum_{\gamma>0}\frac{\gamma}{(\gamma,\gamma)_{\mathfrak{g}}}.$$

It is well known (and easily verified) that $(\rho^\vee,\gamma)_{\mathfrak{g}}=\mathsf{ht}(\gamma)$ for any $\gamma\in\Delta^+$. (This equality does not depend on the normalisation of a bilinear form.) It follows that $h^*(\mathfrak{g}^\vee)=(\rho^\vee,\theta_s)=\mathsf{ht}(\theta_s)$.

Proposition 3.1. *For a simple Lie algebra $\mathfrak{g}$ with the corresponding root system Δ, we have*

$$\sum_{\gamma>0}\mathsf{ht}^2(\gamma) = \frac{\dim\mathfrak{g}}{12}h^*(\mathfrak{g})h^*(\mathfrak{g}^\vee)r. \tag{3.1}$$

Proof. The equality in (3.1) is essentially equivalent to the "strange formula."

Applying Eq. (2.3) to $v=\rho^\vee$, we obtain

$$h^*(\mathfrak{g})(\rho^\vee,\rho^\vee)_{\mathfrak{g}} = \sum_{\gamma>0}(\rho^\vee,\gamma)_{\mathfrak{g}}(\rho^\vee,\gamma)_{\mathfrak{g}} = \sum_{\gamma>0}\mathsf{ht}^2(\gamma). \tag{3.2}$$

For $\mathfrak{g}^\vee$, the strange formula says that $(\rho^\vee,\rho^\vee)_{\mathfrak{g}^\vee}=\frac{\dim\mathfrak{g}}{12}h^*(\mathfrak{g}^\vee)$. Although the normalised bilinear forms $(,)_{\mathfrak{g}}$ and $(,)_{\mathfrak{g}^\vee}$ are proportional upon restriction to $\mathcal{E}$, they are not equal in general. Indeed, the square of the length of a long root in $\Delta^\vee$ with respect to $(,)_{\mathfrak{g}}$ equals $2r$. Hence the transition factor is r and

$$(\rho^\vee,\rho^\vee)_{\mathfrak{g}} = r(\rho^\vee,\rho^\vee)_{\mathfrak{g}^\vee} = \frac{\dim\mathfrak{g}}{12}h^*(\mathfrak{g}^\vee)r. \tag{3.3}$$

Then the assertion follows from (3.2) and (3.3). □

Theorem 3.2. $\mathsf{ind}((\mathfrak{sl}_2)^{\mathrm{pr}} \hookrightarrow \mathfrak{g}) = \frac{\dim \mathfrak{g}}{6} h^*(\mathfrak{g}^\vee) r$.

Proof. Combining Eq. (1.2), Example 1.2(2), and Definition 2 yields the following formula for the index of a simple subalgebra $\mathfrak{s}$ in $\mathfrak{g}$:

$$\mathsf{ind}(\mathfrak{s} \hookrightarrow \mathfrak{g}) = \frac{h^*(\mathfrak{s})}{h^*(\mathfrak{g})} \cdot \mathsf{ind}_{\mathrm{AVE}}(\mathfrak{s}, \mathfrak{g}). \tag{3.4}$$

We use this formula with $\mathfrak{s} = (\mathfrak{sl}_2)^{\mathrm{pr}}$. Let h be the semisimple element of a principal $\mathfrak{sl}_2$-triple. Without loss of generality, we may assume that h is dominant. Then $\alpha(h) = 2$ for any $\alpha \in \Pi$. Put $\tilde{h} = h/2$. Then $\gamma(\tilde{h}) = \mathsf{ht}(\gamma)$ for any $\gamma \in \Delta$ and $\mathrm{ad}\,\tilde{h}$ has the eigenvalues $-1, 0, 1$ in $(\mathfrak{sl}_2)^{\mathrm{pr}}$. Hence

$$\mathsf{ind}_{\mathrm{AVE}}\big((\mathfrak{sl}_2)^{\mathrm{pr}}, \mathfrak{g}\big) = \frac{\mathrm{tr}(\mathrm{ad}_{\mathfrak{g}} \tilde{h})^2}{\mathrm{tr}(\mathrm{ad}_{\mathfrak{s}} \tilde{h})^2} = \frac{\sum_{\gamma \in \Delta} \mathsf{ht}^2(\gamma)}{2} = \sum_{\gamma > 0} \mathsf{ht}^2(\gamma).$$

Since $h^*(\mathfrak{sl}_2) = 2$, the theorem follows from Proposition 3.1 and Eq. (3.4). □

Below, we tabulate the values of index for all simple Lie algebras.

$\mathfrak{g}$	$\mathbf{A}_n$	$\mathbf{B}_n$	$\mathbf{C}_n$	$\mathbf{D}_n$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$	$\mathbf{F}_4$	$\mathbf{G}_2$
$\mathsf{ind}((\mathfrak{sl}_2)^{\mathrm{pr}} \hookrightarrow \mathfrak{g})$	$\binom{n+2}{3}$	$\frac{n(n+1)(2n+1)}{3}$	$\binom{2n+1}{3}$	$\frac{(n-1)n(2n-1)}{3}$	156	399	1240	156	28

Remark 3.3. For the exceptional Lie algebras, Dynkin computed the indices of *all* $\mathfrak{sl}_2$-subalgebras, see [2, Tables 16–20].

Note that the index of a principal $\mathfrak{sl}_2$ is preserved under the unfolding procedure $\mathfrak{g} \rightsquigarrow \tilde{\mathfrak{g}}$ applied to multiply laced Dynkin diagram. Namely, $\mathsf{ind}((\mathfrak{sl}_2)^{\mathrm{pr}} \hookrightarrow \mathfrak{g}) = \mathsf{ind}((\mathfrak{sl}_2)^{\mathrm{pr}} \hookrightarrow \tilde{\mathfrak{g}})$, where the four pairs $(\mathfrak{g}, \tilde{\mathfrak{g}})$ are: $(\mathbf{C}_n, \mathbf{A}_{2n-1})$, $(\mathbf{B}_n, \mathbf{D}_{n+1})$, $(\mathbf{F}_4, \mathbf{E}_6)$, $(\mathbf{G}_2, \mathbf{D}_4)$. This is, of course, explained by the multiplicativity of the index of subalgebras and the fact that $\mathsf{ind}(\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}) = 1$.

Remark 3.4. Proposition 3.1 provides a uniform expression for $\sum_{\gamma>0} \mathsf{ht}^2(\gamma)$. One might ask for a similar formula for $\sum_{\gamma>0} \mathsf{ht}(\gamma)$. However, such a formula seems to only exist in the simply-laced case. Indeed, for any $\mathfrak{g}$ we have $2(\rho, \rho^\vee)_{\mathfrak{g}} = \sum_{\gamma>0} (\gamma, \rho^\vee)_{\mathfrak{g}} = \sum_{\gamma>0} \mathsf{ht}(\gamma)$. If Δ is simply-laced, then $\rho^\vee = 2\rho/(\theta, \theta)_{\mathfrak{g}} = \rho$, and using the "strange formula" one obtains

$$\sum_{\gamma>0} \mathsf{ht}(\gamma) = 2(\rho, \rho)_{\mathfrak{g}} = \frac{\dim \mathfrak{g}}{6} h(\mathfrak{g}).$$

Question. Consider the function $s \mapsto f(s) = \sum_{\gamma>0} \mathsf{ht}^s(\gamma)$. Are there some other values of s such that $f(s)$ has a nice closed expression?

4. Some applications

(A) Let $\nu : \mathfrak{g} \to \mathfrak{sl}(V_\lambda)$ be an irreducible representation. Our first observation is that using Theorems 1.1 and 3.2 we can immediately compute the Dynkin index of V_λ as $(\mathfrak{sl}_2)^{\rm pr}$-module:

$$\begin{aligned}\mathsf{ind}_{\rm D}\big((\mathfrak{sl}_2)^{\rm pr}, V_\lambda\big) &= \mathsf{ind}\big((\mathfrak{sl}_2)^{\rm pr} \hookrightarrow \mathfrak{sl}(V_\lambda)\big) = \mathsf{ind}\big((\mathfrak{sl}_2)^{\rm pr} \hookrightarrow \mathfrak{g}\big) \cdot \mathsf{ind}\big(\mathfrak{g} \hookrightarrow \mathfrak{sl}(V_\lambda)\big)\\ &= \mathsf{ind}\big((\mathfrak{sl}_2)^{\rm pr} \hookrightarrow \mathfrak{g}\big) \cdot \mathsf{ind}_{\rm D}(\mathfrak{g}, V_\lambda)\\ &= \frac{\dim\mathfrak{g}}{6} h^*\big(\mathfrak{g}^\vee\big) r \cdot \frac{\dim V_\lambda}{\dim\mathfrak{g}} (\lambda, \lambda + 2\rho)_{\mathfrak{g}}\\ &= \frac{\dim V_\lambda}{6} \cdot h^*\big(\mathfrak{g}^\vee\big) \cdot r \cdot (\lambda, \lambda + 2\rho)_{\mathfrak{g}}.\end{aligned}$$

Furthermore, we have

$$\mathsf{ind}_{\rm D}\big((\mathfrak{sl}_2)^{\rm pr}, V_\lambda\big) = \mathsf{ind}_{\rm D}(\mathfrak{sl}_2, {\rm ad}) \cdot \mathsf{ind}_{\rm AVE}\big((\mathfrak{sl}_2)^{\rm pr}, V_\lambda\big) = 4 \cdot \mathsf{ind}_{\rm AVE}\big((\mathfrak{sl}_2)^{\rm pr}, V_\lambda\big) \tag{4.1}$$

and

$$\mathsf{ind}_{\rm AVE}\big((\mathfrak{sl}_2)^{\rm pr}, V_\lambda\big) = \frac{{\rm tr}(\nu(\tilde h)^2)}{{\rm tr}(({\rm ad}\,\tilde h)^2)} = \frac{\sum_{\mu \dashv V_\lambda} \mu(\tilde h)^2}{2},$$

where notation $\mu \dashv V_\lambda$ means that μ is a weight of V_λ, and the sum runs over all weights according to their multiplicities. Since $\mu(\tilde h) = (\mu, \rho^\vee)_{\mathfrak{g}}$, we finally obtain

$$\sum_{\mu \dashv V_\lambda} \big(\mu, \rho^\vee\big)^2_{\mathfrak{g}} = \frac{\dim V_\lambda}{12} \cdot h^*\big(\mathfrak{g}^\vee\big) \cdot r \cdot (\lambda, \lambda + 2\rho)_{\mathfrak{g}}. \tag{4.2}$$

This can be compared with the formula of Freudenthal–de Vries (see [3, 47.10.2]):

$$\sum_{\mu \dashv V_\lambda} \langle \mu, \rho\rangle^2 = \frac{\dim V_\lambda}{24} \langle \lambda, \lambda + 2\rho\rangle. \tag{4.3}$$

One can verify that Eqs. (4.2) and (4.3) agree in the simply-laced case, where ρ is proportional to $\rho^\vee$.

(B) Let $m_1, \dots, m_n$ be the exponents of $\mathfrak{g}$. Regarding $\mathfrak{g}$ as $(\mathfrak{sl}_2)^{\rm pr}$-module, one has $\mathfrak{g} = \bigoplus_{i=1}^n \mathsf{R}_{2m_i}$ [4, Corollary 8.7]. Then using Example 1.2(1), Eqs. (3.4), (4.1), and the additivity of the index of representations, we obtain the identity

$$\begin{aligned}\frac{\dim\mathfrak{g}}{6} h^*\big(\mathfrak{g}^\vee\big) r &= \mathsf{ind}\big((\mathfrak{sl}_2)^{\rm pr} \hookrightarrow \mathfrak{g}\big) = \frac{h^*(\mathfrak{sl}_2)}{h^*(\mathfrak{g})} \sum_{i=1}^n \mathsf{ind}_{\rm AVE}(\mathfrak{sl}_2, \mathsf{R}_{2m_i})\\ &= \frac{1}{2h^*(\mathfrak{g})} \sum_{i=1}^n \mathsf{ind}_{\rm D}(\mathfrak{sl}_2, \mathsf{R}_{2m_i}) = \frac{1}{2h^*(\mathfrak{g})} \sum_{i=1}^n \binom{2m_i + 2}{3}.\end{aligned}$$

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