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K-stability of constant scalar curvature Kähler manifolds

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Abstract

We show that a polarised manifold with a constant scalar curvature Kähler metric and discrete automorphisms is K-stable. This refines the K-semistability proved by S.K. Donaldson.

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1. Introduction

Let (X, L) be a polarised manifold. One of the most striking realisations in Kähler geometry over the past few years is that if one can find a constant scalar curvature Kähler (cscK) metric g on X whose $(1,1)$-form ω_g belongs to the cohomology class $c_1(L)$ then (X, L) is *semistable*, in a number of senses. The seminal references are Yau [14], Tian [18], Donaldson [4,6].

In this paper we are concerned with Donaldson's *algebraic K-stability* [6]. This notion was inspired by Tian's K-stability for Fano manifolds [18] (we may refer to the latter as *analytic K-stability*). It should play a role similar to Mumford–Takemoto slope stability for bundles. The necessary general theory is recalled in Section 2.

Asymptotic Chow stability (which implies K-semistability, see e.g. [13] Theorem 3.9) for a cscK polarised manifold was first proved by Donaldson [5] in the absence of continuous automorphisms. Important work in this connection was also done by Mabuchi, see e.g. [10]. From

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the analytic point of view the fundamental result is the lower bound on the K-energy proved by Chen–Tian [3]. In the case of Kähler–Einstein manifolds with no nonzero holomorphic vector fields Tian [18] proved analytic K-stability. Much progress on the relationship between analytic K-stability (i.e. asymptotics of the K-energy) and algebraic K-stability (i.e. the Donaldson–Futaki invariant) has been made by Paul–Tian, e.g. [11], Phong–Sturm–Ross [12] and others.

A key feature of this paper is that we are able to avoid the hard analysis of the K-energy and rely instead on a perturbation argument and a well-known theorem of Arezzo–Pacard.

One of the neatest results in the algebraic context seems to be Donaldson's *lower bound on the Calabi functional*, which we now recall.

For a Kähler form ω let $S(\omega)$ denote the scalar curvature, $\widehat{S}$ its average (a topological quantity). Denote by F the Donaldson–Futaki invariant of a test configuration (Definitions 2.1, 2.2). The precise definition of the norm $\|\mathcal{X}\|$ appearing below will not be important for us.

Theorem 1.1. *(Donaldson [7].) For a polarised manifold* (X, L)

$$\inf_{\omega\in c_1(L)} \int_X \big(S(\omega) - \widehat{S}\big)^2 \omega^n \geqslant -\frac{\sup_{\mathcal{X}} F(\mathcal{X})}{\|\mathcal{X}\|},$$

where the supremum is taken with respect to all test configurations $(\mathcal{X}, \mathcal{L})$ *for* (X, L).

Thus, if $c_1(L)$ *admits a cscK representative,* (X, L) *is K-semistable (Definition* 2.5*).*

There is a strong analogy here with Hermite–Einstein metrics on holomorphic vector bundles. By the celebrated results of Donaldson and Uhlenbeck–Yau these are known to exist if and only if the bundle is slope polystable, namely a semistable direct sum of slope stable vector bundles.

In particular a simple vector bundle endowed with a Hermite–Einstein metric is slope stable. In this paper we will prove the corresponding result for polarised manifolds. Let us denote by $\mathrm{Aut}(X, L)$ the group of complex automorphisms of X which preserve the polarisation.

Theorem 1.2. *If* $c_1(L)$ *contains a cscK metric and* $\mathrm{Aut}(X, L)$ *is discrete then* (X, L) *is K-stable.*

Remark 1.3. Note that while one can construct slope semistable bundles which are simple but not stable, there is at present no known example of a K-semistable polarised variety (X, L) with $\mathrm{Aut}(X, L)$ discrete which is not K-stable.

Theorem 1.2 proves part of a more general, well-known conjecture.

Conjecture 1.4. *(Donaldson [6, p. 294].) If* $c_1(L)$ *contains a cscK metric then* (X, L) *is K-polystable (Definition* 2.7*).*

Remark 1.5. While we follow the definition of algebraic K-semistability given by Donaldson, we emphasise that our definition of K-stability differs from the original one of Donaldson in a way which is by now quite standard in the literature (see e.g. [13]). Namely we call K-polystable what Donaldson would simply call stable, and reserve the word K-stable for K-polystable polarised manifolds with no nontrivial holomorphic vector fields lifting to the polarisation (this is closer to geometric invariant theory, but complicates the terminology a bit).

Thus our result confirms this expectation when the group $\mathrm{Aut}(X, L)$ is discrete. From a differential-geometric point of view this means that X has no nontrivial Hamiltonian holomorphic vector fields, i.e. holomorphic vector fields that vanish somewhere.

Conjecture 1.4 together with its converse form the Yau–Tian–Donaldson conjecture, sometimes called the Hitchin–Kobayashi correspondence for manifolds. There are no general results about the converse of Conjecture 1.4, although recently Donaldson gave a proof in the case of toric surfaces [8].

For the rest of this paper we will assume $\dim(X) \geqslant 2$. K-stability for Riemann surfaces is completely understood thanks to the work of Ross and Thomas [13, Section 6]. In particular Conjecture 1.4 is known for Riemann surfaces.

Our proof of Theorem 1.2 rests on the general principle that one should be able to *perturb* a semistable object (in the sense of geometric invariant theory) to make it unstable – although this necessarily involves perturbing the problem too (e.g. the linearisation), since the locus of semistable points for an action on a fixed variety is open. Conversely, in the absence of continuous automorphisms, the cscK property is open – at least in the sense of small deformations – so cscK should imply stability. This perturbation strategy for proving 1.4 was first pointed out to the author by Donaldson and G. Székelyhidi. Of course we need to make this rigorous; in particular testing small deformations is not enough to prove K-stability.

Thus suppose that (X, L) is strictly K-semistable (Definition 2.8). We will find a natural way to construct from this a family of K-unstable small perturbations $(X_\varepsilon, L_\varepsilon)$ for small $\varepsilon > 0$. Our choice for X_ε is only the blowup $\widehat{X} = \mathrm{Bl}_q X$ at a special point q, with exceptional divisor E. Only the polarisation changes, and quite naturally $L_\varepsilon = \pi^* L - \varepsilon \mathcal{O}(E)$. This would involve taking $\varepsilon \in \mathbb{Q}^+$ and working with $\mathbb{Q}$-divisors, but in fact we take tensor powers and work with $\widehat{X}$ polarised by $L_\gamma = \pi^* L^\gamma - \mathcal{O}(E)$ for integer $\gamma \gg 1$. K-(semi, poly, in)stability is unaffected (by Definition 2.2).

Proposition 1.6. *Let (X, L) be a strictly K-semistable polarised manifold. Then there exists a point $q \in X$ such that the polarised blowup*

$$\left(\mathrm{Bl}_q X, \pi^* L^\gamma \otimes \mathcal{O}(-E)\right)$$

is K-unstable for $\gamma \gg 1$.

Assume now that a strictly semistable (X, L) also admits a cscK metric $\omega \in c_1(L)$. If $\mathrm{Aut}(X, L)$ is discrete the blowup perturbation problem for ω is unobstructed by a theorem of Arezzo and Pacard [1], so we would get cscK metrics in $c_1(\pi^* L^\gamma \otimes \mathcal{O}(-E))$ for $\gamma \gg 1$, a contradiction. To sum up the main ingredients for our proof (besides Theorem 1.1) are:

1. A well-known embedding result for test configurations (Proposition 2.9), together with the algebro-geometric estimate Proposition 3.3;
2. A blowup formula for the Donaldson–Futaki invariant proved by the author [15, Theorem 1.3];
3. A special case of the results of Arezzo and Pacard on blowing up cscK metrics [1].

2. Some general theory

Let n denote the complex dimension of X.

Definition 2.1 *(Test configuration).* A *test configuration* with *exponent* r for a polarised manifold (X, L) is a polarised flat family $(\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ with $(\mathcal{X}_1, \mathcal{L}_1) \cong (X, L^r)$ endowed with a linearised $\mathbb{C}^*$-action which covers the natural action of $\mathbb{C}^*$ on $\mathbb{C}$.

Given a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) denote by A_k the matrix representation of the induced $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$. By (equivariant) Hirzebruch–Riemann–Roch we can find expansions

$$h^0\big(\mathcal{X}_0, \mathcal{L}_0^k\big) = a_0 k^n + a_1 k^{n-1} + O\big(k^{n-2}\big), \tag{2.1}$$

$$\operatorname{tr}(A_k) = b_0 k^{n+1} + b_1 k^n + O\big(k^{n-1}\big). \tag{2.2}$$

Definition 2.2 *(Donaldson–Futaki invariant).* This is the rational number

$$F(\mathcal{X}) = a_0^{-2}(b_0 a_1 - a_0 b_1) \tag{2.3}$$

which is independent of the choice of a lifting of the action to $\mathcal{L}_0$.

Equivalently $F(\mathcal{X})$ is the coefficient F_1 in the expansion

$$\frac{\operatorname{tr}(A_k)}{k h^0(\mathcal{X}_0, \mathcal{L}_0^k)} = F_0 - F_1 \frac{1}{k} + O\big(k^{-2}\big).$$

Note moreover that F is invariant under taking tensor powers, i.e.

$$F(\mathcal{X}, \mathcal{L}) = F\big(\mathcal{X}, \mathcal{L}^r\big).$$

And of course $c_1(L)$ contains a cscK metric if and only if all its positive multiples do. Therefore for the rest of this paper we will assume without loss of generality that L is very ample and the exponent of the test configuration is 1.

Remark 2.3 *(Coverings).* Given a test configuration $(\mathcal{X}, \mathcal{L})$ we can construct a new one by pulling $\mathcal{X}$ and $\mathcal{L}$ back under the d-fold ramified covering of $\mathbb{C}$ given by $z \mapsto z^d$. This changes A_k to $d \cdot A_k$ and consequently F to $d \cdot F$.

Definition 2.4. A test configuration $(\mathcal{X}, \mathcal{L})$ is called a *product* if it is induced in the natural way by a $\mathbb{C}^*$-action on (X, L).

A product test configuration is called *trivial* if the associated action on (X, L) is trivial.

The Donaldson–Futaki invariant of a product test configuration coincides (up to a positive universal constant) with the classical Futaki invariant for holomorphic vector fields.

Definition 2.5 *(K-stability).* A polarised manifold (X, L) is *K-semistable* if for all test configurations $(\mathcal{X}, \mathcal{L})$

$$F(\mathcal{X}) \geqslant 0.$$

It is *K-stable* if the strict inequality holds for nontrivial test configurations.

Remark 2.6. It is important to point out that the Donaldson–Futaki invariant of a K-stable polarised manifold (X, L) can *never* be bounded away from 0. For example this can be seen by considering the test configuration given by degeneration to the normal cone with parameter c (see [13, Section 4]) and letting $c \to 0$.

In this connection a refinement of K-stability was proposed by G. Székelyhidi. If $\omega \in c_1(L)$ is cscK there should be a strictly positive lower bound for a suitable normalisation of F over all nonproduct test configurations. This condition is called *uniform K-polystability*. In [17, Section 3.1.1] it is shown that the correct normalisation in the case of algebraic surfaces coincides with that appearing in Theorem 1.1, namely $\frac{F(\mathcal{X})}{\|\mathcal{X}\|}$. For toric surfaces K-polystability implies uniform K-polystability with respect to torus-invariant test configurations; this is shown in [17, Section 4.2].

In Remark 3.5 we will offer an argument in favour of this refinement of K-stability.

In particular if (X, L) is K-stable then the group $\mathrm{Aut}(X, L)$ must be discrete. The correct notion to take care of continuous automorphisms is K-polystability.

Definition 2.7. A polarised manifold (X, L) is *K-polystable* if it is K-semistable, and moreover any test configuration $(\mathcal{X}, \mathcal{L})$ with $F(\mathcal{X}) = 0$ is a product.

Definition 2.8. A polarised manifold (X, L) is *strictly K-semistable* if it is K-semistable and it admits a nonproduct test configuration with vanishing Donaldson–Futaki invariant.

K-(semi)stability can by checked by 1-parameter flat families induced by projective embeddings.

Proposition 2.9. *(See e.g. Ross–Thomas [13, Theorem 3.7].) It is enough to check K-(semi)stability with respect to the test configurations induced by embeddings $X \hookrightarrow H^0(X, L^r)^*$ and 1-parameter subgroups of $\mathrm{GL}(H^0(X, L^r)^*)$ for all positive integers r.*

In [15] the author proved a blowup formula for the Donaldson–Futaki invariant. The statement involves some more terminology.

Definition 2.10 *(Hilbert–Mumford weight).* Let α be a 1-parameter subgroup of $\mathrm{SL}(N + 1)$, inducing a $\mathbb{C}^*$-action on $\mathbb{P}^N$. Choose projective coordinates $[x_0 : \ldots : x_N]$ such that α is given by $\mathrm{diag}(\lambda^{m_0}, \ldots, \lambda^{m_N})$. The Hilbert–Mumford weight of a closed point $q \in \mathbb{P}^N$ is defined by

$$\mu(q, \alpha) = -\min\{m_i \colon q_i \neq 0\}.$$

Note that this coincides with the weight of the induced action on the fibre of the hyperplane line bundle $\mathcal{O}(1)$ over the specialisation $\lim_{\lambda \to 0} \lambda \cdot q$.

Definition 2.11 *(Chow weight).* Let (Y, L) be a polarised scheme, $y \in Y$ a closed point, and α a $\mathbb{C}^*$-action on (Y, L). Suppose that L is very ample and $\alpha \hookrightarrow \mathrm{SL}(H^0(Y, L)^*)$. The Chow weight $ch_{(Y,L)}(y, \alpha)$ is defined to be the Hilbert–Mumford weight of $y \in \mathbb{P}(H^0(Y, L)^*)$ with respect to the induced action. This definition extends to 0-dimensional cycles on Y.

Theorem 2.12. *(See [15, Theorem 1.3].) For points $q_i \in X$ and integers $a_i > 0$ let $Z \subset X$ be the 0-dimensional closed subscheme $Z = \bigcup_i a_i q_i$. Let Λ be the 0-cycle on X given by $\sum_i a_i^{n-1} q_i$.*

Any 1-parameter subgroup $\alpha \hookrightarrow \mathrm{Aut}(X, L)$ induces a test configuration $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}})$ for $(\mathrm{Bl}_Z X, \pi^* L^{\gamma} \otimes \mathcal{O}_{\mathrm{Bl}_Z X}(1))$, where $\mathcal{O}_{\mathrm{Bl}_Z X}(1)$ denotes the exceptional invertible sheaf. More precisely let $O(Z)^-$ be the closure of the orbit of Z. Then $\widehat{\mathcal{X}} = \mathrm{Bl}_{O(Z)^-} \mathcal{X}$ and $\widehat{\mathcal{L}} = \pi^* \mathcal{L}^{\gamma} \otimes \mathcal{O}_{\widehat{\mathcal{X}}}(1)$.

Suppose that α acts through $\mathrm{SL}(H^0(X, L)^*)$ with Futaki invariant $F(X)$. Then the following expansion holds as $\gamma \to \infty$,

$$F(\widehat{\mathcal{X}}) = F(X) - ch_{(X,L)}(\Lambda, \alpha) \frac{\gamma^{1-n}}{2(n-1)!} + O(\gamma^{-n}).$$

We will need a slight generalisation of this result, covering blowups of non-product test configurations.

Proposition 2.13. *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration for (X, L), $Z = \bigcup_i a_i q_i$ as above. There is a test configuration $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}})$ for $(\mathrm{Bl}_Z X, \pi^* L^{\gamma} \otimes \mathcal{O}_{\mathrm{Bl}_Z X}(1))$ with total space $\widehat{\mathcal{X}}$ given by the blowup of $\mathcal{X}$ along $O(Z)^-$. The linearisation is the natural one induced on $\widehat{\mathcal{L}} = \pi^* \mathcal{L}^{\gamma} \otimes \mathcal{O}_{\widehat{\mathcal{X}}}(1)$.*

Let $q_{i,0} = \lim_{\lambda \to 0} \lambda \cdot q_i$ be the specialisation, Λ_0 the 0-cycle on $\mathcal{X}_0$ given by $\sum_i a_i^{n-1} q_{i,0}$. Let α denote the induced action on $(\mathcal{X}_0, \mathcal{L}_0)$ and suppose that α acts through $\mathrm{SL}(H^0(\mathcal{X}_0, \mathcal{L}_0)^*)$. Then the expansion

$$F(\widehat{\mathcal{X}}) = F(\mathcal{X}) - ch_{(\mathcal{X}_0, \mathcal{L}_0)}(\Lambda_0, \alpha) \frac{\gamma^{1-n}}{2(n-1)!} + O(\gamma^{-n})$$

holds as $\gamma \to \infty$.

We emphasise that the relevant Chow weight is computed *on the central fibre* $(\mathcal{X}_0, \mathcal{L}_0)$ with its induced $\mathbb{C}^*$-action.

Proof. For a unified proof of these results see [16, Sections 3.1, 3.2]. In fact the proof of Theorem 2.12 presented in [15, Section 4] carries over without any change to the case of non-product test configurations, with only two exceptions:

1. The proof of flatness of the composition $\widehat{\mathcal{X}} \to \mathcal{X} \to \mathbb{C}$;
2. The identification of the weight $ch_{(\mathcal{X}_0, \mathcal{L}_0)}(\Lambda_0)$ (with respect to the induced action on $\mathcal{X}_0$) with $ch_{(X,L)}(\Lambda, \alpha)$.

Here we do not care for the latter identification, and indeed it does not make sense for non-product test configurations since the general fibre is not preserved by the $\mathbb{C}^*$-action.

To prove flatness we use the criterion [9, III, Proposition 9.7]. Thus we need to prove that all the associated points of $\widehat{\mathcal{X}} = \mathrm{Bl}_{O(Z)^-} \mathcal{X}$ map to the generic point of the base $\mathbb{C}$. By flatness this is true for the morphism $\mathcal{X} \to \mathbb{C}$, and the blowup $\pi : \widehat{\mathcal{X}} \to \mathcal{X}$ does not contribute new associated points, only the Cartier exceptional divisor $E = \pi^{-1}(O(Z)^-)$.

To make this precise let $y \in \widehat{\mathcal{X}}$ be an associated point of the blowup and denote by $\{y\}^-$ be its closure, a closed subscheme of $\widehat{\mathcal{X}}$. Suppose that $\{y\}^- \subseteq E$. This would imply that the maximal ideal $\mathfrak{m}_y$ of the local ring $\mathcal{O}_{\widehat{\mathcal{X}},y}$ contains the image of a local defining equation f for E. But y being an associated point means precisely that every element of $\mathfrak{m}_y$ is a zero divisor (see the definition following [9, III, Corollary 9.6]). Since E is a Cartier divisor f is not a zero divisor, a contradiction.

Therefore $\{y\}^- \nsubseteq E$, or in other words $\mathfrak{m}_y$ contains some nontrivial zero divisor orthogonal to E. Thus the projection $\pi(y)$ is an associated point of $\mathcal{X}$ which in turn maps to the generic point of $\mathbb{C}$ by flatness of $\mathcal{X} \to \mathbb{C}$. $\square$

Remark 2.14. The assumption that α acts through SL is not restrictive for our purposes. Starting with a test configuration $(\mathcal{X}, \mathcal{L})$ we can construct a new one $(\mathcal{X}', \mathcal{L}')$ for which the induced action on $H^0(\mathcal{X}'_0, \mathcal{L}'_0)^*$ is special linear and such that the Donaldson–Futaki invariant $F(\mathcal{X}')$ is a positive multiple of $F(\mathcal{X})$. To see this start by scaling the linearisation of $\mathbb{C}^*$ on $\mathcal{L}$ by a character ξ, i.e. some integer weight. Then take

$$(\mathcal{X}', \mathcal{L}') = \rho_d^*(\mathcal{X}, \mathcal{L})$$

for some positive integer d, where $\rho_d : \mathbb{C} \to \mathbb{C}$ is taking dth powers. The trace of the induced action on $H^0(\mathcal{X}'_0, \mathcal{L}'_0)$ is given by

$$d \cdot \operatorname{tr}(A_1) + \xi h^0(\mathcal{X}_0, \mathcal{L}_0)$$

and we can always choose the integers ξ and $d > 0$ to make this vanish. Finally by Remark 2.3 we have $F(\mathcal{X}') = d \cdot F(\mathcal{X})$.

3. Proof of Theorem 1.2

It will be enough to prove Proposition 1.6 and to apply the result of Arezzo and Pacard recalled as Theorem 3.1 below.

For a fixed $q \in X$ let

$$(\widehat{X}, L_\gamma) = \big(\mathrm{Bl}_q X, \pi^* L^\gamma \otimes \mathcal{O}(-E)\big).$$

We need to show that when (X, L) is strictly semistable there exists q such that $(\widehat{X}, L_\gamma)$ is K-unstable for $\gamma \gg 1$. For some special choice of q, we will construct test configurations $(\widehat{\mathcal{X}}, \mathcal{L}_\gamma)$ for $(\widehat{X}, L_\gamma)$ which have strictly negative Donaldson–Futaki invariant for $\gamma \gg 1$.

By assumption (X, L) is strictly semistable, so it admits a nontrivial test configuration $(\mathcal{X}, \mathcal{L})$ with $F(\mathcal{X}) = 0$.

Moreover by Remark 2.14 we can assume that the induced $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}_0)^*$ is special linear.

We blow $\mathcal{X}$ up along the closure $O(q)^-$ of the orbit $O(q)$ of $q \in \mathcal{X}_1$ under the $\mathbb{C}^*$-action on $\mathcal{X}$, i.e. define

$$\widehat{\mathcal{X}} = \mathrm{Bl}_{O(q)^-} \mathcal{X}. \tag{3.1}$$

Let $\mathcal{O}_{\widehat{\mathcal{X}}}(1)$ denote the exceptional invertible sheaf on $\widehat{\mathcal{X}}$. We endow $\widehat{\mathcal{X}}$ with the polarisation

$$\mathcal{L}_\gamma = \pi^*\mathcal{L}^\gamma \otimes \mathcal{O}_{\widehat{\mathcal{X}}}(1). \tag{3.2}$$

Define the closed point $q_0 \in \mathcal{X}_0$ to be the specialisation

$$q_0 = \lim_{\lambda \to 0} \lambda \cdot q.$$

Applying the blowup formula 2.13 in this case gives

$$\begin{aligned} F(\widehat{\mathcal{X}}, \mathcal{L}_\gamma) &= F(\mathcal{X}, \mathcal{L}) - ch_{(\mathcal{X}_0, \mathcal{L}_0)}(q_0)\frac{\gamma^{1-n}}{2(n-2)!} + O\left(\gamma^{-n}\right) \\ &= -ch_{(\mathcal{X}_0, \mathcal{L}_0)}(q_0)\frac{\gamma^{1-n}}{2(n-2)!} + O\left(\gamma^{-n}\right). \end{aligned}$$

In Proposition 3.3 below we will prove that for some special $q \in \mathcal{X}_1 \cong X$,

$$ch_{(\mathcal{X}_0, \mathcal{L}_0)}(q_0) > 0.$$

This holds thanks to the assumption $F(\mathcal{X}) = 0$, or more generally $F(\mathcal{X}) \leqslant 0$. This is enough to prove Proposition 1.6.

The final step for Theorem 1.2 is to show that the perturbation problem is unobstructed provided $\mathrm{Aut}(X, L)$ is discrete. This is precisely the content of a beautiful result of C. Arezzo and F. Pacard.

Theorem 3.1. *(Arezzo–Pacard [1].) Let (X, L) be a polarised manifold with a cscK metric in the class $c_1(L)$. Suppose* $\mathrm{Aut}(X, L)$ *is discrete and let $q \in X$ be any point. Then the blowup* $\mathrm{Bl}_q X$ *with exceptional divisor E admits a cscK metric in the class $\gamma\pi^* c_1(L) - c_1(\mathcal{O}(E))$ for $\gamma \gg 1$.*

Remark 3.2. The Arezzo–Pacard theorem also holds in the non-projective case and, more importantly, even in the presence of nontrivial Hamiltonian holomorphic vector fields, provided a suitable stability condition is satisfied. We refer to [2,15] for further discussion.

Thus the following proposition will complete our proof(s). We believe it may also be of some independent interest.

Proposition 3.3. *Let $(\mathcal{X}, \mathcal{L})$ be a nonproduct test configuration for a polarised manifold (X, L) with nonpositive Donaldson–Futaki invariant and suppose the induced $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}_0)^*$ is special linear. Then there exists $q \in \mathcal{X}_1 \cong X$ such that $ch_{(\mathcal{X}_0, \mathcal{L}_0)}(q_0) > 0$.*

Proof. By the embedding Theorem 2.9 we reduce to the case of a nontrivial $\mathbb{C}^*$ acting on $\mathbb{P}^N$ for some N, of the form $\mathrm{diag}(\lambda^{m_0}, \dots, \lambda^{m_N})$, with

$$m_0 \leqslant m_1 \leqslant \cdots \leqslant m_N.$$

Let $\{Z_i\}_{i=0}^{k}$ be the distinct projective weight spaces, where Z_i has weight m_i (i.e. the induced action on Z_i is trivial with weight m_i). Each Z_i is a projective subspace of $\mathbb{P}^N$, and the central fibre with its reduced induced structure $\mathcal{X}_0^{\mathrm{red}}$ is contained in $\mathrm{Span}(Z_{i_1}, \dots, Z_{i_l})$, $0 = i_1 < i_2 < \cdots < i_l$, for some minimal l.

Case $1 < l$. In this case the induced action on closed points of $\mathcal{X}_0$ is nontrivial. Let $q \in \mathcal{X}_1$ be any point with

$$\lim_{\lambda \to 0} \lambda \cdot q = q_0 \in Z_{i_l}.$$

Such a point exists by minimality of i_l and because the specialisation of every point must lie in some Z_j. Since the action on $\mathcal{X}_0$ is induced from that on $\mathbb{P}^N$, q_0 belongs to the totally repulsive fixed locus $R = \mathcal{X}_0 \cap Z_{i_l} \subset \mathcal{X}_0$. By this we mean that every closed point in $\mathcal{X}_0 \setminus R$ specialises to a closed point in $\mathcal{X}_0 \setminus R$. In particular the natural birational morphism $\mathcal{X}_0 \dashrightarrow \mathrm{Proj}(\bigoplus_d H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes d})^{\mathbb{C}^*})$ blows up along R. So $q_0 \in R$ is an unstable point for the $\mathbb{C}^*$-action in the sense of geometric invariant theory. By the Hilbert–Mumford criterion the weight of the induced action on the line $\mathcal{L}_0|_{q_0}$ must be strictly positive. Since we are assuming that the induced action on $H^0(\mathcal{X}_0, \mathcal{L}_0)^*$ is special linear this weight coincides with the Chow weight, so $ch_{(\mathcal{X}_0, \mathcal{L}_0)}(q_0) > 0$.

Degenerate case. In the rest of the proof we will show that in the degenerate case $\mathcal{X}_0^{\mathrm{red}} \subset Z_0$ the Donaldson–Futaki invariant is strictly positive. Note that since by assumption the original $\mathbb{C}^*$-action on $\mathbb{P}^N$ is nontrivial, $Z_0 \subset \mathbb{P}^N$ is a proper projective subspace.

We digress for a moment to make the following observation: for any $\mathbb{C}^*$-action on $\mathbb{P}^N$ with ordered weights $\{m_i\}$, and a smooth nondegenerate manifold $Y \subset \mathbb{P}^N$, the map $\rho : Y \ni y \mapsto y_0 = \lim_{\lambda \to 0} \lambda \cdot y$ is *rational*, defined on the open dense set $\{y \in Y\colon\ \mu(y) = m_0\}$ of points with minimal Hilbert–Mumford weight. Indeed, in the above notation, generic points specialise to some point in the lowest fixed locus Z_0. In any case the map ρ blows up exactly along loci where the Hilbert–Mumford weight jumps.

Going back to our discussion of the case $\mathcal{X}_0^{\mathrm{red}} \subset Z_0$, we see that this means precisely that all points of $\mathcal{X}_1$ have minimal Hilbert–Mumford weight m_0, so there is a well-defined *morphism*

$$\rho : \mathcal{X}_1 \to Z_0.$$

Moreover ρ is a finite map: the pullback of $\mathcal{L}_0$ under ρ is L which is ample, therefore ρ cannot contract a positive dimensional subscheme. If ρ were an isomorphism on its image, it would fit in a $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X} \cong X \times \mathbb{C}$. Therefore ρ cannot be injective, either on closed points or tangent vectors. If, say, ρ identifies distinct points x_1, x_2, this means that the x_i specialise to the same x under the $\mathbb{C}^*$-action; by flatness then the local ring $\mathcal{O}_{\mathcal{X}_0,x}$ contains a nontrivial nilpotent pointing outwards of Z_0, i.e. the sheaf $\mathcal{I}_{\mathcal{X}_0 \cap Z_0}/\mathcal{I}_{\mathcal{X}_0}$ is nonzero. In other words $\mathcal{X}_0$ is not a closed subscheme of Z_0. The case when ρ annihilates a tangent vector produces the same kind of nilpotent in the local ring of the limit, by specialisation.

To sum up, the central fibre $\mathcal{X}_0$ is nonreduced, containing nontrivial Z_0-orthogonal nilpotents. Moreover the induced action on the closed subscheme $\mathcal{X}_0 \cap Z_0 \subset \mathcal{X}_0$ is trivial. The proof will be completed by the following weight computation.

Donaldson–Futaki invariant in the degenerate case. Suppose $Z_0 \subset \mathbb{P}^N$ has projective coordinates $[x_0 : \ldots : x_r]$, i.e. it is cut out by $\{x_{r+1} = \cdots = x_N = 0\}$. We change the linearisation by changing the representation of the $\mathbb{C}^*$-action, to make it of the form

$$[x_0 : \ldots : x_r : x_{r+1} : \ldots : x_N] \mapsto \left[x_0 : \ldots : x_r : \lambda^{m_{r+1}-m_0} x_{r+1} : \ldots : \lambda^{m_N - m_0} x_N\right], \tag{3.3}$$

and recall $m_{r+i} > m_0$ for all $i > 0$. It is possible that the induced action on $H^0(\mathcal{X}_0, \mathcal{L}_0)^*$ will not be special linear anymore, however this does not affect the Donaldson–Futaki invariant.

Note that for all large k,

$$H^0\big(\mathbb{P}^N, \mathcal{O}(k)\big) \to H^0\big(\mathcal{X}_0, \mathcal{L}_0^k\big) \to H^1\big(\mathcal{I}_{\mathcal{X}_0}(k)\big) = 0. \tag{3.4}$$

By (3.4), our geometric description of $\mathcal{X}_0$ and the choice of linearisation (3.3) we see that any section $\xi \in H^0(\mathcal{X}_0, \mathcal{L}_0^k)$ has *nonpositive* weight under the induced $\mathbf{G}_m$-action. The section ξ can only have *strictly negative* weight if it is of the form $x_{r+i} \cdot f$ for some $i > 0$ (x_{r+i} is now regarded as a linear form and the sign is opposite to that of the action on $\mathbb{P}^N$ by duality). Moreover we know there exists an integer $a > 0$ such that $x_{r+i}^a|_{\mathcal{X}_0} = 0$ for all $i > 0$. Let $w(k)$ denote the total weight of the action on $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$, i.e. the induced weight on the line $\Lambda^{P(k)} H^0(\mathcal{X}_0, \mathcal{L}_0^k)$, where $P(k) = h^0(\mathcal{X}_0, \mathcal{L}_0^k)$ is the Hilbert polynomial for $k \gg 0$. Our discussion implies the upper bound

$$\big|w(k)\big| \leqslant C\big(P(k-1) + \cdots + P(k-a)\big) \tag{3.5}$$

for some $C > 0$, independent of k. In particular,

$$w(k) = O\big(k^n\big). \tag{3.6}$$

On the other hand by the presence of Z_0-orthogonal nilpotents there exists a section x_{r+i}, $i > 0$, with $x_{r+i}|_{\mathcal{X}_0} \neq 0$. Multiplying by $H^0(\mathcal{X}_0^{\mathrm{red}}, \mathcal{L}_0^{k-1}|_{\mathcal{X}_0^{\mathrm{red}}})$ and writing $Q(k) = h^0(\mathcal{X}_0^{\mathrm{red}}, \mathcal{L}_0^k|_{\mathcal{X}_0^{\mathrm{red}}})$ gives the upper bound

$$w(k) \leqslant -C_1 Q(k-1) \tag{3.7}$$

for some $C_1 > 0$ independent of $k \gg 0$. Then

$$\frac{w(k)}{kP(k)} = \frac{w(k)}{kQ(k)} \frac{Q(k)}{P(k)} \leqslant -\frac{C_2}{k} \tag{3.8}$$

holds for $k \gg 0$ and some $C_2 > 0$ independent of k. Together with

$$\frac{w(k)}{kP(k)} = O\big(k^{-1}\big) \tag{3.9}$$

which follows from (3.6) the upper bound we have just proved implies

$$\frac{w(k)}{kP(k)} = -\frac{C_3}{k} + O\big(k^{-2}\big) \tag{3.10}$$

for some $C_3 > 0$ independent of k.

By definition of the Donaldson–Futaki invariant, this immediately implies

$$F(\mathcal{X}) \geqslant C_3 > 0,$$

a contradiction. $\square$

Example 3.4. Consider the test configurations for $\mathbb{P}^1$ given by the families of conics $\mathcal{X} = \{xz = \varepsilon y^2\}$, $\mathcal{X}' = \{\varepsilon^2 xz = y^2\} \subset \mathbb{P}^2 \times \mathbb{C}$. They are induced by the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ given by $[s:t] \mapsto [s^2 : st : t^2]$ and the $\mathbb{C}^*$ actions on $\mathbb{P}^2$ given respectively by $\mathrm{diag}(0, 0, \varepsilon)$, $\mathrm{diag}(0, \varepsilon, 0)$. Thus $\mathcal{X}$ is nondegenerate (meaning that the induced action on $\mathcal{X}_0^{\mathrm{red}} = \mathcal{X}_0$ is nontrivial) and the unique repulsive fixed point for the action on the central fibre $\{xz = 0\} \subset \mathbb{P}^2$ is given by $[0:0:1]$ (in fact $\mathcal{X}$ is isomorphic to the degeneration to the normal cone of $[0:1] \in \mathbb{P}^1$ with parameter $c = 1$). On the other hand $\mathcal{X}'$ is degenerate, since the action on the reduced central fibre $\{y = 0\} \subset \mathbb{P}^2$ is trivial. The action is not trivial on the nonreduced central fibre $\{y^2 = 0\} \subset \mathbb{P}^2$, however. One checks that, for $\mathcal{X}$, $h^0(\mathcal{L}_0^k) = 2k + 1$, $\mathrm{tr}(A_k) = -\frac{k(k+1)}{2}$, so $F(\mathcal{X}) = \frac{1}{8}$. Similarly, for $\mathcal{X}'$, $h^0(\mathcal{L}_0) = 2k + 1$, $\mathrm{tr}(A_k) = -k + 1$ (note that $\mathrm{tr}(A_k)$ has smaller degree in the degenerate case, as expected). So we find $F(\mathcal{X}') = \frac{1}{2}$.

Remark 3.5. Suppose we wish to prove an algebraic analogue Theorem 3.1. For simplicity let us consider the following statement:

If (X, L) is K-stable then the blowup of X along any 0-cycle with the now familiar polarisations is K-stable.

This is predicted by the Yau–Tian–Donaldson conjecture. By the blowup formula Proposition 2.13 it is clear that we would need a uniform lower bound for the Futaki invariant of nontrivial test configurations to prove this algebraically. But as we explained in Remark 2.6 this never holds.

We believe this observation gives an interesting argument in favour of the uniform notion of K-stability recalled in Remark 2.6.

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