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On the Lazarev–Lieb extension of the Hobby–Rice theorem

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Abstract

O. Lazarev and E.H. Lieb proved that, given $f_1, \ldots, f_n \in L^1([0,1];\mathbb{C})$, there exists a smooth function Φ that takes values on the unit circle and annihilates span$\{f_1, \ldots, f_n\}$. We give an alternative proof of that fact that also shows the $W^{1,1}$ norm of Φ can be bounded by $5\pi n + 1$. Answering a question raised by Lazarev and Lieb, we show that if $p > 1$ then there is no bound for the $W^{1,p}$ norm of any such multiplier in terms of the norms of $f_1, \ldots, f_n$.
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The Hobby–Rice Theorem [1] states

Theorem 1. *If $f_1, \ldots, f_n \in L^1([0,1];\mathbb{R})$ then there exists $\Phi : [0,1] \to \{-1, 1\}$ with at most n discontinuities such that for each k*

$$\int_0^1 f_k(t)\Phi(t)dt = 0.$$

The theorem has applications in L^1 approximation and in combinatorics, particularly necklace splitting problems [3]. An elegant proof of the Hobby–Rice Theorem was given by Pinkus [4] using the Borsuk–Ulam Theorem.

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Motivated by a problem in mathematical physics, Lazarev and Lieb [2] extended this result to obtain a smooth annihilator taking values on the unit circle, i.e.,

Theorem 2. *If* $f_1, \ldots, f_n \in L^1([0,1];\mathbb{C})$ *then there exists* $\theta \in C^\infty([0,1];\mathbb{R})$ *such that*

$$\forall k, \quad \int_0^1 f_k(t) e^{i\theta(t)} dt = 0. \tag{0.1}$$

Lazarev and Lieb suggested that there should be a simpler proof, and in this spirit, we offer the following proof. They also raised the question of calculating the $H^1 = W^{1,2}$ norm of $f_k e^{i\theta}$. Corollary 4 shows there is such θ with $\left\|e^{i\theta(\cdot)}\right\|_{W^{1,1}} \leq 5\pi n + 1$. We also show that for $p > 1$ there exists a large class of normed spaces $\mathcal{N} = \left\{(N, \|\cdot\|_N)\right\}$ including L^1 so that $\left\|e^{i\theta(\cdot)}\right\|_{W^{1,p}}$ cannot be bounded by $\|f_1\|_N, \ldots, \|f_n\|_N$.

Proof of Theorem 2. We may assume $f_1, \ldots, f_n$ are linearly independent in L^1 and thus choose $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$ so that

$$M := \begin{bmatrix} f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{bmatrix}$$

is invertible and each t_j is a Lebesgue point of all f_k (Lemma 9).

For each $u, v \in [-1,1]$, let $\theta_{u+iv} : [-1,1] \to \mathbb{R}$ be a step function that consecutively takes the values $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ on intervals of lengths $\frac{1+u}{2}, \frac{1+v}{2}, \frac{1-u}{2}, \frac{1-v}{2}$, respectively. Thus $\int_{-1}^1 e^{i\theta_{u+iv}(t)} dt = u + iv$.

Choose $\psi \in C^\infty(\mathbb{R};\mathbb{R}^+)$ supported on $[-1,1]$ such that $\int \psi(t)dt = 1$. Let $\psi_h(t) = \psi(t/h)/h$.

Also let I_S be the indicator function of S. Define $\theta^\#_{h,z} = \theta_z I_{(h-1,1-h)} + 2\pi I_{[1-h,\infty)}$ and

$$\theta_{h,z} = \begin{cases} \theta_z & \text{if } h = 0 \\ \psi_h * \theta^\#_{h,z} & \text{if } 0 < h < 1. \end{cases}$$

Note that if $h > 0$ then $\theta_{h,z}(-1) = 0$ and $\theta_{h,z}(1) = 2\pi$, while $\theta^{(m)}_{h,z}(\pm 1) = 0$ for all $m \geq 1$.

Define $D = \{z \in \mathbb{C} : |z| \leq 1\}$, $d = \min_{j\in\{0\ldots n\}}\left(t_{j+1} - t_j\right)/2$, and $Q : [0,d] \times D^n \to \mathbb{C}^n$:

$$Q(h;\vec{z}) = \begin{cases} \left(\sum_{j=1}^n z_j \cdot f_k(t_j)\right)_{k=1\ldots n} & \text{if } h = 0 \\ \left(\sum_{j=1}^n \int_{-1}^1 f_k(th + t_j) e^{i\theta_{h,z_j}(t)} dt\right)_{k=1\ldots n} & \text{if } 0 < h \leq d \end{cases}$$

with $\vec{z} := (z_1, \ldots, z_n)$. Since $Q(0;\vec{z}) = M(\vec{z})$, Lemma 10 shows there is $\delta \in (0,d]$ such that for all $\vec{z} \in D^n$

$$\vec{z} - M^{-1}(Q(\delta;\vec{z})) \in \frac{1}{2} D^n.$$

Let $L_\delta = [0,1] \setminus \bigcup_{j=1}^n \left(t_j - \delta, t_j + \delta\right)$. By applying the Hobby–Rice Theorem[1] to $f_1 I_{L_\delta}$, $\ldots, f_n I_{L_\delta}$ and smoothing a finite set of discontinuities, we obtain $\phi \in C^\infty([0,1];\mathbb{R})$ supported

[1] The Riemann–Lebesgue Lemma also suffices, but the Hobby–Rice Theorem enables us to compute a bound of the $W^{1,1}$ norm in Corollary 4.

on L_δ so that

$$\vec{r} := \left(\int_{L_\delta} f_k(t) e^{i\phi(t)} dt\right)_{k=1\ldots n} \in \frac{\delta}{2} M\left(D^n\right).$$

Since ϕ vanishes together with all its derivatives at all $t_j \pm \delta$, for all $\vec{z} \in D^n$

$$\theta^*_{\vec{z}}(t) = \begin{cases} \theta_{\delta,z_j}\left(\left(t - t_j\right)/\delta\right) + 2\pi\,(j-1) & \text{if } t \in \left[t_j - \delta, t_j + \delta\right] \text{ and } 1 \le j \le n \\ \phi(t) & \text{if } t \in [0, t_1 - \delta) \\ \phi(t) + 2\pi j & \text{if } t \in \left(t_j + \delta, t_{j+1} - \delta\right) \\ & \text{ and } 1 \le j \le n-1 \\ \phi(t) + 2\pi n & \text{if } t \in (t_n + \delta, 1] \end{cases}$$

is in $C^\infty([0,1];\mathbb{R})$. Lemma 11 establishes the continuity of

$$T(\vec{z}) := \left(\int_0^1 f_k(t) e^{i\theta^*_{\vec{z}}(t)} dt\right)_{k=1\ldots n} = \delta Q(\delta; \vec{z}) + \vec{r}.$$

Since $\vec{z} - M^{-1}(Q(\delta;\vec{z}))$ and $\frac{1}{\delta}M^{-1}(\vec{r})$ are in $\frac{1}{2}D^n$ for all $\vec{z} \in D^n$, then $\vec{z} - \frac{1}{\delta}M^{-1}(T(\vec{z})) \in D^n$. By Brouwer's Fixed Point Theorem, there exists $\vec{z_0} \in D^n$ such that $\vec{z_0} - \frac{1}{\delta}M^{-1}(T(\vec{z_0})) = \vec{z_0}$, that is to say $T(\vec{z_0}) = 0$. $\square$

Definition 3. Let

$$\|g(\cdot)\|_{W^{1,p}} = \left(\int_0^1 |g(t)|^p\, dt + \int_0^1 \left|g'(t)\right|^p dt\right)^{\frac{1}{p}}$$

$$\text{and} \quad \|g(\cdot)\|_{\overset{\circ}{W}{}^{1,p}} = \left(\int_0^1 \left|g'(t)\right|^p dt\right)^{\frac{1}{p}}.$$

Corollary 4. *If* $f_1, \ldots, f_n \in L^1([0,1];\mathbb{C})$ *then there exists* $\theta \in C^\infty([0,1];\mathbb{R})$ *such that for each* k

$$\int_0^1 f_k(t) e^{i\theta(t)} dt = 0$$

and $\left\|e^{i\theta(\cdot)}\right\|_{W^{1,1}} \le 5\pi n + 1$.

Proof. The calculation of the bound follows from a careful selection of ϕ in the preceding proof. The Hobby–Rice Theorem applied to the n real parts and n imaginary parts of $f_k I_{L_\delta}$ implies that there exists $\phi^\# : \mathbb{R} \to \{0, \pi\}$ with at most $2n$ discontinuities such that for each k

$$\int f_k(t) I_{L_\delta} e^{i\phi^\#(t)} dt = 0.$$

Since this equation still holds if $\phi^\#$ is replaced with $\pi - \phi^\#$, choose such $\phi^\#$ that is non-zero on at most n points at the boundary of L_δ. Thus $\phi^\# I_{L_\delta}$ has at most $3n$ discontinuities. Choose $\eta > 0$ so that by selecting

$$\phi = \left(\phi^\# I_{L_{\delta+\eta}}\right) * \psi_\eta$$

then

$$\vec{r_\eta} := \left[\int_{L_\delta} f_k(t)e^{i\phi(t)}dt\right]_{k=1\ldots n} \in \frac{\delta}{2}M\left(D^n\right).$$

Note that ϕ vanishes together with all its derivatives at all $t_j \pm \delta$. Also note that $\phi^{\#}I_{L_{\delta+\eta}}$ has no more discontinuities than $\phi^{\#}I_{L_\delta}$, which is at most $3n$. Thus there exist $m \leq 3n$ and $0 < y_1 < \cdots < y_m < 1$ such that $\phi^{\#}I_{L_{\delta+\eta}}(t)$ or $\pi - \phi^{\#}I_{L_{\delta+\eta}}(t)$ for all $t \in [0,1] \setminus \{y_1, \ldots, y_m\}$ is equal to $\pi \sum_{j=1}^{m} (-1)^{j+1} I_{[y_j,\infty)}(t)$. Consequently,

$$\begin{aligned}
\int_{L_\delta} \left|\left(\theta^*_{\vec{z}}\right)'(t)\right| dt &= \int_{L_\delta} \left|\phi'(t)\right| dt \\
&= \int_0^1 \left|\phi'(t)\right| dt \\
&= \int_0^1 \left|\left(\phi^{\#}I_{L_{\delta+\eta}} * \psi_\eta\right)'(t)\right| dt \\
&\leq \pi \sum_{j=1}^{m} \int_0^1 \left(I_{[y_j,\infty)} * \psi_\eta\right)'(t)dt \\
&\leq 3\pi n.
\end{aligned}$$

Recall that $\theta_{\delta,z}$ is an increasing function with $\theta_{\delta,z}(-1) = 0$ and $\theta_{\delta,z}(1) = 2\pi$ for all $z \in D$. Thus $\int_{t_j-\delta}^{t_j+\delta} \left|\left(\theta^*_{\vec{z}}\right)'(t)\right| = 2\pi$ for $1 \leq j \leq n$ and so $\int_0^1 \left|\left(\theta^*_{\vec{z}}\right)'(t)\right| dt \leq 5\pi n$. Consequently, $\left\|e^{i\theta^*_{\vec{z}}(\cdot)}\right\|_{\overset{\circ}{W}{}^{1,1}} \leq 5\pi n$ and $\left\|e^{i\theta^*_{\vec{z}}(\cdot)}\right\|_{W^{1,1}} \leq 5\pi n + 1$. Since $\max_{t\in[0,1]} \left|\theta^*_{\vec{z}}(t)\right| \leq (2n+1)\pi$, it follows that $\left\|\theta^*_{\vec{z}}(\cdot)\right\|_{W^{1,1}} \leq (7n+1)\pi$. $\square$

Clearly, if $f_1, \ldots, f_n$ are real valued, they may be combined into $\left\lceil \frac{n}{2} \right\rceil$ complex valued functions and the bounds reduce accordingly.

For $p > 1$ the situation is different.

Definition 5. Let

$$A(f) = \left\{\theta \in C^\infty([0,1];\mathbb{R}) : \int_0^1 f(t)e^{i\theta(t)}dt = 0\right\}.$$

Definition 6. Let

$$\rho(f) = \inf\left\{\int_0^1 \left|\theta'(t)\right|^p dt : \theta \in A(f)\right\}.$$

Definition 7. Let

$$(\Upsilon_n f)(t) = \begin{cases} f\left(2^n t\right) & \text{if } 0 \leq t \leq 2^{-n} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 8. *Assume N is a norm for which there exists $f \in L^1([0,1];\mathbb{C})$ such that $0 < \|\Upsilon_n f(\cdot)\|_N < \infty$ for all $n \geq 1$ and $\rho(f) > 0$.*

Then, given any $l, K \in \mathbb{R}^+$, *there exists* $g \in L^1([0,1];\mathbb{C})$ *such that* $\|g(\cdot)\|_N = l$ *and* $\rho(g) > K$.

Proof. Choose $\epsilon > 0$ and $\theta \in A(\Upsilon_n f)$ such that $\int_0^1 |\theta'(t)|^p\, dt < \rho(\Upsilon_n f) + \epsilon$. Then $(\Upsilon_{-n}\theta)|_{[0,1]} \in A(f)$, and so

$$\begin{aligned}\rho(\Upsilon_n f) + \epsilon &> \int_0^1 |\theta'(t)|^p\, dt \\ &\geq \int_0^{2^{-n}} |\theta'(t)|^p\, dt \\ &= 2^{-n}\int_0^1 |\theta'(2^{-n}t)|^p\, dt \\ &= 2^{n(p-1)}\int_0^1 \left|\left(\theta(2^{-n}t)\right)'\right|^p dt \\ &\geq 2^{n(p-1)}\rho(f)\end{aligned}$$

proving $\rho(\Upsilon_n f) \geq 2^{n(p-1)}\rho(f)$.

Also, since $A(g) = A(cg)$ for all $c \neq 0$ then

$$2^{n(p-1)}\rho(f) \leq \rho(\Upsilon_n f) = \rho\left(l\,\Upsilon_n f / \|(\Upsilon_n f)(\cdot)\|_N\right).$$

Consequently, if n is large enough so that $2^{n(p-1)}\rho(f) > K$ then $g := l\,\Upsilon_n f / \|(\Upsilon_n f)(\cdot)\|_N$ has the property that $\rho(g) > K$ and $\|g(\cdot)\|_N = l$. □

The $W^{1,p}$ norms of $f_k(t)e^{i\theta(t)}$ fare no better, since if $f_1(t) = 1$, then $\left\|f_1(\cdot)\, e^{i\theta(\cdot)}\right\|_{W^{1,p}} = \left\|e^{i\theta(\cdot)}\right\|_{W^{1,p}} \geq (\max_k \rho(f_k))^{\frac{1}{p}}$.

Lemmas

We include the lemmas that were used above, some or all of which may be familiar to the reader.

Lemma 9 (*Infinite Gaussian Elimination*)**.** *If* $f_1, \ldots, f_n \in L^1([0,1];\mathbb{C})$ *are linearly independent in* L^1*, then there exist* $t_1, \ldots, t_n \in (0,1)$ *so that*

$$M := \begin{bmatrix} f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{bmatrix}$$

is invertible and t_j *is a Lebesgue point of* f_k *for each* $j, k \in 1 \ldots n$.

Proof. Let P be the set of points in $(0,1)$ that are Lebesgue points for all f_k.

The case $n = 1$ is clear. If $n > 1$, let us assume inductively that there are $t_1, \ldots, t_{n-1} \in P$ such that $M' := \left[f_k(t_j)\right]_{(n-1)\times(n-1)}$ is invertible. Thus there exist $\beta_1, \ldots, \beta_{n-1} \in \mathbb{C}$ such that

$$\left[\beta_1 \ldots \beta_{n-1}\right] M' = \left[f_n(t_1) \ldots f_n(t_{n-1})\right].$$

Furthermore, since $f_1, \ldots, f_n$ are linearly independent in L^1, there exists $t_n \in P$ such that

$$y_n := f_n(t_n) - \beta_1 f_1(t_n) - \cdots - \beta_{n-1} f_{n-1}(t_n) \neq 0.$$

Thus $M := \left[f_k(t_j)\right]_{n\times n}$ has a non-zero determinant, namely $y_n \det M'$. □

Lemma 10. *If $t_0 \in (0,1)$ is a Lebesgue point of $f \in L^1([0,1];\mathbb{C})$, then, uniformly in $z \in D$,*

$$\lim_{h\to 0^+}\int_{-1}^{1} f(th+t_0)\cdot\theta_{h,z}(t)dt = f(x)\cdot z.$$

Proof. Given $\epsilon > 0$, let $\delta_1 < \epsilon/20\pi(|f(t_0)|+1)$. If $0 < h < \delta_1$, then, since $\theta_z(t)$ is a step function, for all values of $t \in (2h-1, 1-2h)$ that are not within distance h of a discontinuity of θ_z, $\theta_{h,z}(t) = \theta_z(t)$. Since there are at most three discontinuities of θ_z in $(2h-1, 1-2h)$ and $\theta_{h,z}(t), \theta_z(t) \in [0, 2\pi]$,

$$2\pi\cdot 6h \geq \int_{2h-1}^{1-2h}\left|\theta_{h,z}(t)-\theta_z(t)\right|dt$$

and so

$$\begin{aligned}\int_{-1}^{1}\left|e^{i\theta_{h,z}(t)}-e^{i\theta_z(t)}\right|dt &\leq \int_{-1}^{1}\left|\theta_{h,z}(t)-\theta_z(t)\right|dt\\ &\leq 2\pi\cdot(6+4)h\\ &< \epsilon/(|f(t_0)|+1).\end{aligned}$$

Choose $\delta_2 > 0$ so that if $0 < h < \delta_2$ then

$$\int_{-1}^{1}|f(th+t_0)-f(t_0)|\,dt < \epsilon/2$$

and let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$\begin{aligned}&\left|\int_{-1}^{1} f(th+t_0)\cdot e^{i\theta_{h,z}(t)}dt - \overbrace{\int_{-1}^{1} f(t_0)\cdot e^{i\theta_z(t)}dt}^{=f(t_0)\cdot z}\right|\\ &\leq \left|\int_{-1}^{1}(f(th+t_0)-f(t_0))\,e^{i\theta_{h,z}(t)}dt\right| + \left|\int_{-1}^{1} f(t_0)\left(e^{i\theta_{h,z}(t)}-e^{i\theta_z(t)}\right)dt\right|\\ &\leq \int_{-1}^{1}|f(th+t_0)-f(t_0)|\,dt + |f(t_0)|\int_{-1}^{1}\left|e^{i\theta_{h,z}(t)}-e^{i\theta_z(t)}\right|dt\\ &< \epsilon.\quad\square\end{aligned}$$

Lemma 11. *If $f \in L^1([0,1];\mathbb{C})$, $t_0 \in (0,1)$, and $0 < h < \min\{t_0, 1-t_0\}$, then*

$$q(z) := \int_{-1}^{1} f(th+t_0)\,e^{i\theta_{h,z}(t)}dt$$

is a continuous function of $z \in D$.

Proof. Since $\theta^{\#}_{h,z}$ is a step function whose intervals of constancy vary continuously with z, for $\epsilon > 0$, there exists $\delta > 0$ such that if $|z_1 - z_2| < \delta$ then

$$\int_{-1}^{1}\left|\theta^{\#}_{h,z_1}(t)-\theta^{\#}_{h,z_2}(t)\right|dt < \epsilon/\left(\|f\|_{L^1}+1\right)h\,\|\psi(t)\|_{L^\infty}.$$

Then

$$\left\| e^{\theta_{h,z_1}(t)} - e^{\theta_{h,z_2}(t)} \right\|_{L^\infty} \le \left\| \theta_{h,z_1}(t) - \theta_{h,z_2}(t) \right\|_{L^\infty} < \epsilon / \left(\| f \|_{L^1} + 1 \right)$$

and so

$$|q(z_1) - q(z_2)| \le \int_{-1}^{1} \left| f(th + t_0) \left(e^{i\theta_{h,z_1}(t)} - e^{i\theta_{h,z_2}(t)} \right) \right| dt < \epsilon. \quad \square$$

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